

All-Pairs Shortest Paths

Slides from Heejin Park

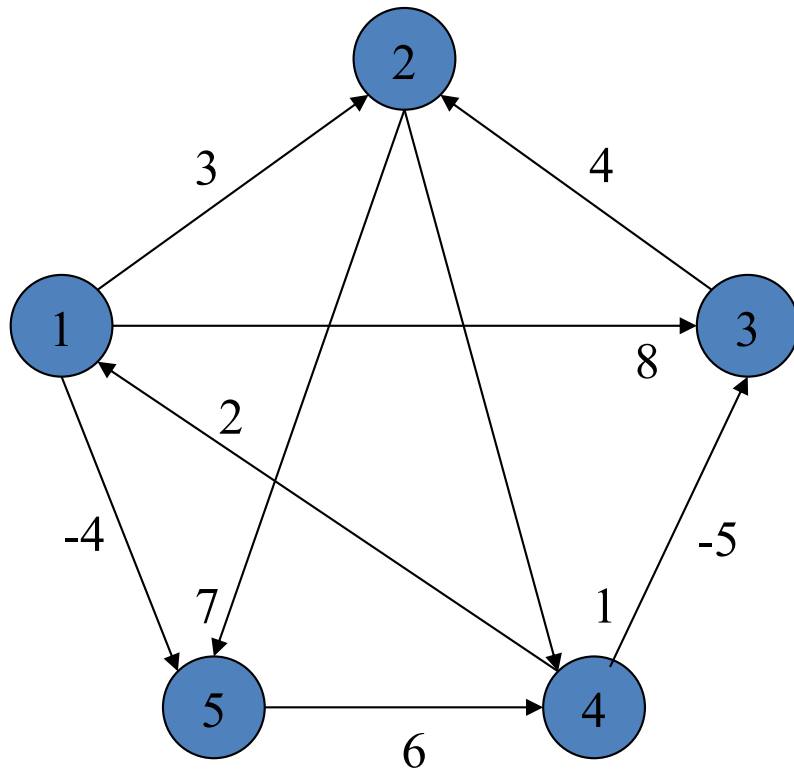
- **Using SSSP (single source shortest path) algorithms**
- **Floyd-Warshall algorithm**
- **Transitive closure of a directed graph**

- We can solve an all-pairs shortest-paths problem by running a *single-source shortest-paths algorithm* $|V|$ *times*, once for each vertex as the source.
- **Nonnegative-weight** edges
 - Dijkstra's algorithm
 - The linear-array implementation
 $O(V \cdot V^2) = O(V^3)$.
 - The binary min-heap implementation
 $O(V \cdot (V \lg V + E \lg V)) = O(V^2 \lg V + V E \lg V)$

- ***Negative-weight edges***
 - Bellman-Ford algorithm
 - $O(V \cdot VE) = O(V^2E)$
 - $O(V^4)$ on a dense graph

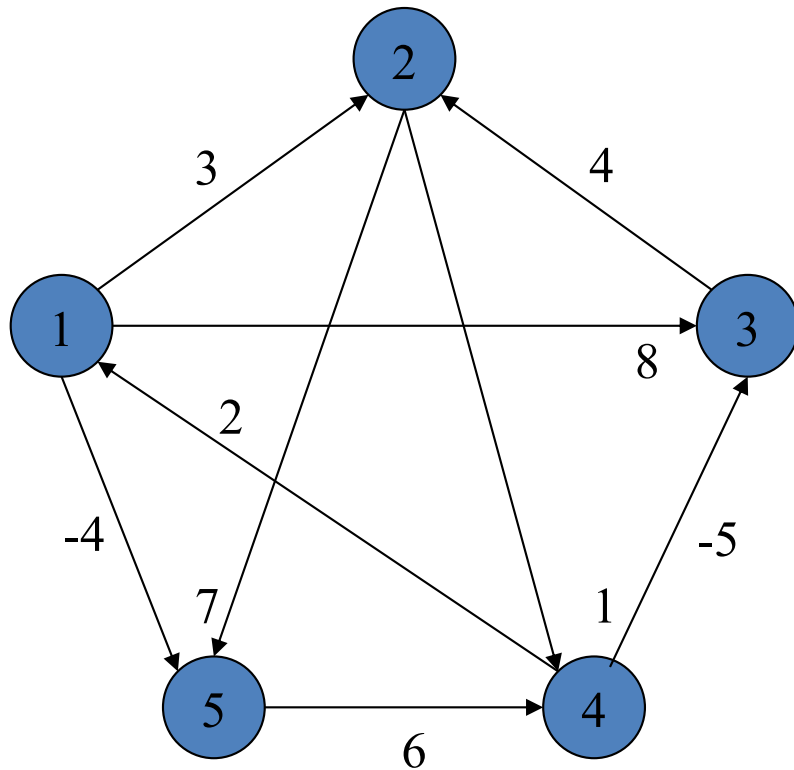
- *Using SSSP (single source shortest path) algorithms*
- Floyd-Warshall algorithm
 - $\Theta(V^3)$ -time
- Transitive closure of a directed graph

- Adjacency Matrix W
 - $w_{ij} = w(i,j)$



$$\begin{pmatrix}
 0 & 3 & 8 & \infty & -4 \\
 \infty & 0 & \infty & 1 & 7 \\
 \infty & 4 & 0 & \infty & \infty \\
 2 & \infty & -5 & 0 & \infty \\
 \infty & \infty & \infty & 6 & 0
 \end{pmatrix}$$

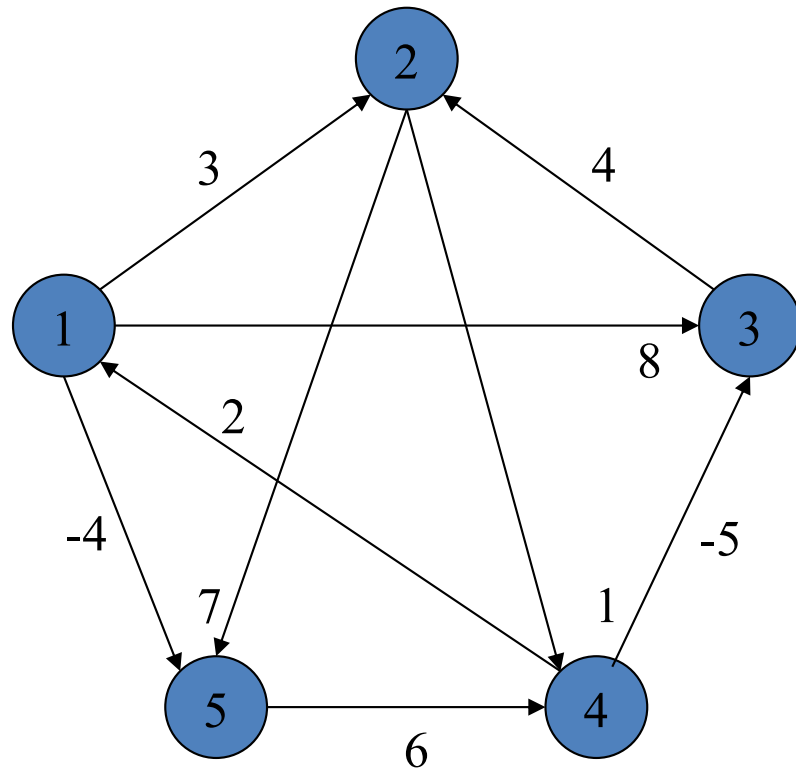
- Shortest Distance Matrix D
 - $d_{ij} = \delta(i,j)$



$$\begin{pmatrix} 0 & 1 & -3 & 2 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

• Predecessor Matrix Π

- $\pi_{ij} = \text{NIL}$ if either $i = j$ or there is not path from i to j .
- π_{ij} is the predecessor of j on some shortest path from i to j .



$$\begin{pmatrix} \text{NIL} & 3 & 4 & 5 & 1 \\ 4 & \text{NIL} & 4 & 2 & 1 \\ 4 & 3 & \text{NIL} & 2 & 1 \\ 4 & 3 & 4 & \text{NIL} & 1 \\ 4 & 3 & 4 & 5 & \text{NIL} \end{pmatrix}$$

- The following procedure prints a shortest path from i to j due to the optimal substructure of the shortest-paths problem.

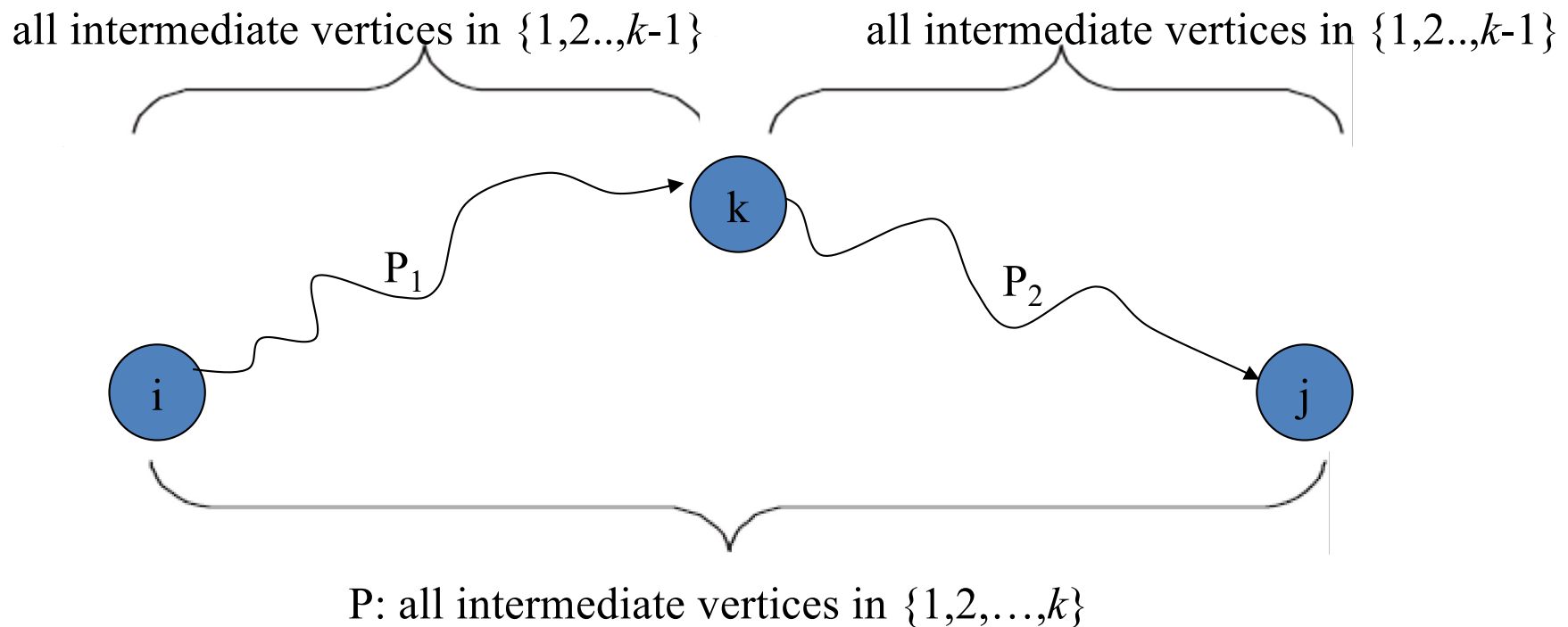
```
PRINT-ALL-PAIRS-SHORTEST-PATH( $\Pi, i, j$ )  
1  if  $i == j$   
2      print  $i$   
3  elseif  $\pi_{ij} == \text{NIL}$   
4      print “no path from”  $i$  “to”  $j$  “exists”  
5  else PRINT-ALL-PAIRS-SHORTEST-PATH( $\Pi, i, \pi_{ij}$ )  
6      print  $j$ 
```

- **Intermediate Vertex**

- An intermediate vertex of a simple path $p = \langle v_1, v_2, \dots, v_l \rangle$ is any vertex of p between v_1 and v_l .

- **The structure of a shortest path**
 - Floyd-Warshall algorithm is based on the observation of the intermediate vertices, which costs $\Theta(V^3)$ time.
 - Let $V = \{1, 2, \dots, n\}$.
 - For any pair of vertices $i, j \in V$, consider all paths from i to j whose intermediate vertices are all drawn from $\{1, 2, \dots, k\}$, and let p be a minimum weight path from among them.

- If k is not an intermediate vertex of path p , then all intermediate vertices of p are in $\{1, 2, \dots, k-1\}$.
- If k is an intermediate vertex of path p , then we break p down into $i \xrightarrow{p_1} k \xrightarrow{p_2} j$.



- A recursive solution to the all-pairs shortest-paths problem
 - Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex i to vertex j for which all intermediate vertices are in the set $\{1, 2, \dots, k\}$.
 - We have the following recurrence:

$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \text{if } k = 0, \\ \min \left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)} \right) & \text{if } k \geq 1. \end{cases} \quad (25.5)$$

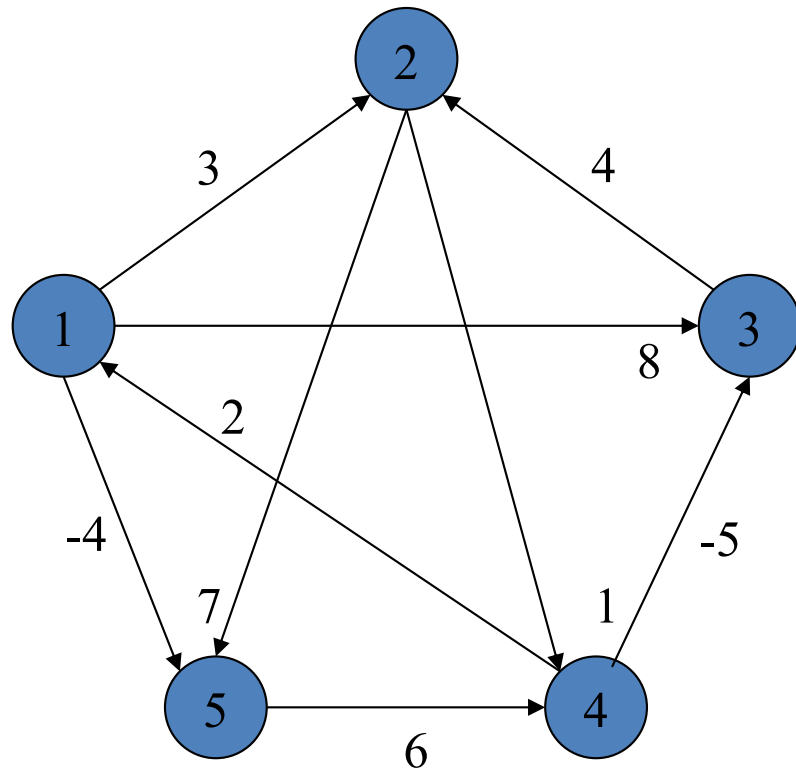
- Because for any path, all intermediate vertices are in the set $\{1, 2, \dots, n\}$, the matrix $D^{(n)} = d_{ij}^{(n)}$ gives the final answer:
 $d_{ij}^{(n)} = \partial(i, j)$ for all $i, j \in V$.

FLOYD-WARSHALL(W)

```
1   $n = W.rows$ 
2   $D^{(0)} = W$ 
3  for  $k = 1$  to  $n$ 
4      let  $D^{(k)} = (d_{ij}^{(k)})$  be a new  $n \times n$  matrix
5      for  $i = 1$  to  $n$ 
6          for  $j = 1$  to  $n$ 
7               $d_{ij}^{(k)} = \min(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)})$ 
8  return  $D^{(n)}$ 
```

- costs $\Theta(n^3)$ time.

Floyd-Warshall algorithm



$$\begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

Floyd-Warshall algorithm

$$D^{(0)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & \infty & -5 & 0 & \infty \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\Pi^{(0)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & NIL & 4 & NIL & NIL \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$\Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

Floyd-Warshall algorithm

$$D^{(1)} = \begin{pmatrix} 0 & 3 & 8 & \infty & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & \infty & \infty \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\Pi^{(1)} = \begin{pmatrix} NIL & 1 & 1 & NIL & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & NIL & NIL \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$\Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(2)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & 5 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$\Pi^{(2)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 1 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$\Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$D^{(3)} = \begin{pmatrix} 0 & 3 & 8 & 4 & -4 \\ \infty & 0 & \infty & 1 & 7 \\ \infty & 4 & 0 & 5 & 11 \\ 2 & -1 & -5 & 0 & -2 \\ \infty & \infty & \infty & 6 & 0 \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\Pi^{(3)} = \begin{pmatrix} NIL & 1 & 1 & 2 & 1 \\ NIL & NIL & NIL & 2 & 2 \\ NIL & 3 & NIL & 2 & 2 \\ 4 & 3 & 4 & NIL & 1 \\ NIL & NIL & NIL & 5 & NIL \end{pmatrix}$$

$$\Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

$$D^{(4)} = \begin{pmatrix} 0 & 3 & -1 & 4 & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$D^{(5)} = \begin{pmatrix} 0 & \textcircled{1} & \textcircled{-3} & \textcircled{2} & -4 \\ 3 & 0 & -4 & 1 & -1 \\ 7 & 4 & 0 & 5 & 3 \\ 2 & -1 & -5 & 0 & -2 \\ 8 & 5 & 1 & 6 & 0 \end{pmatrix}$$

$$\Pi^{(4)} = \begin{pmatrix} NIL & 1 & 4 & 2 & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

$$\Pi^{(5)} = \begin{pmatrix} NIL & \textcircled{3} & 4 & \textcircled{5} & 1 \\ 4 & NIL & 4 & 2 & 1 \\ 4 & 3 & NIL & 2 & 1 \\ 4 & 3 & 4 & NIL & 1 \\ 4 & 3 & 4 & 5 & NIL \end{pmatrix}$$

- Constructing A Shortest Path

- Let Π_{ij}^k be the predecessor of vertex j on a shortest path from vertex i with all intermediate vertices in $\{1, 2, \dots, k\}$.

$$\Pi_{ij}^{(0)} = \begin{cases} \text{NIL} & \text{if } i = j \text{ or } w_{ij} = \infty, \\ i & \text{if } i \neq j \text{ and } w_{ij} < \infty. \end{cases}$$

$$\Pi_{ij}^{(k)} = \begin{cases} \Pi_{ij}^{(k-1)} & \text{if } d_{ij}^{(k-1)} \leq d_{ik}^{(k-1)} + d_{kj}^{(k-1)}, \\ \Pi_{kj}^{(k-1)} & \text{if } d_{ij}^{(k-1)} > d_{ik}^{(k-1)} + d_{kj}^{(k-1)}. \end{cases}$$

- Transitive Closure of Graph

- Given a directed graph $G = (V, E)$ with vertex set $V = \{1, 2, \dots, n\}$.
- The transitive closure of G is defined as the graph $G^* = (V, E^*)$, where $E^* = \{(i, j) : \text{there is a path from vertex } i \text{ to vertex } j \text{ in } G\}$.