Amortized Analysis

Slides from Heejin Park

Contents

- **Aggregate analysis**
- **Accounting method**
- **Potential method**
- **Dynamic Table**

- Aggregate analysis
  - A sequence of n operations takes *worst-case* time $T(n)$ in total.
  - Average cost per operation is $T(n)/n$ in the worst case
    = amortized cost
  - Amortized cost is the same to each operation
    - Even when there are several types of operations in the sequence

- Example of stack operation
  - Stack operations
    - PUSH(S, x)
    - POP(S)
    - MULTIPOP(S, k)
  - PUSH and POP run in $O(1)$ time.
    - Thus the cost of each is 1.
  - Actual running time for n operations is $\Theta(n)$.
    - Total cost of a sequence of these n operations is n

Aggregate analysis

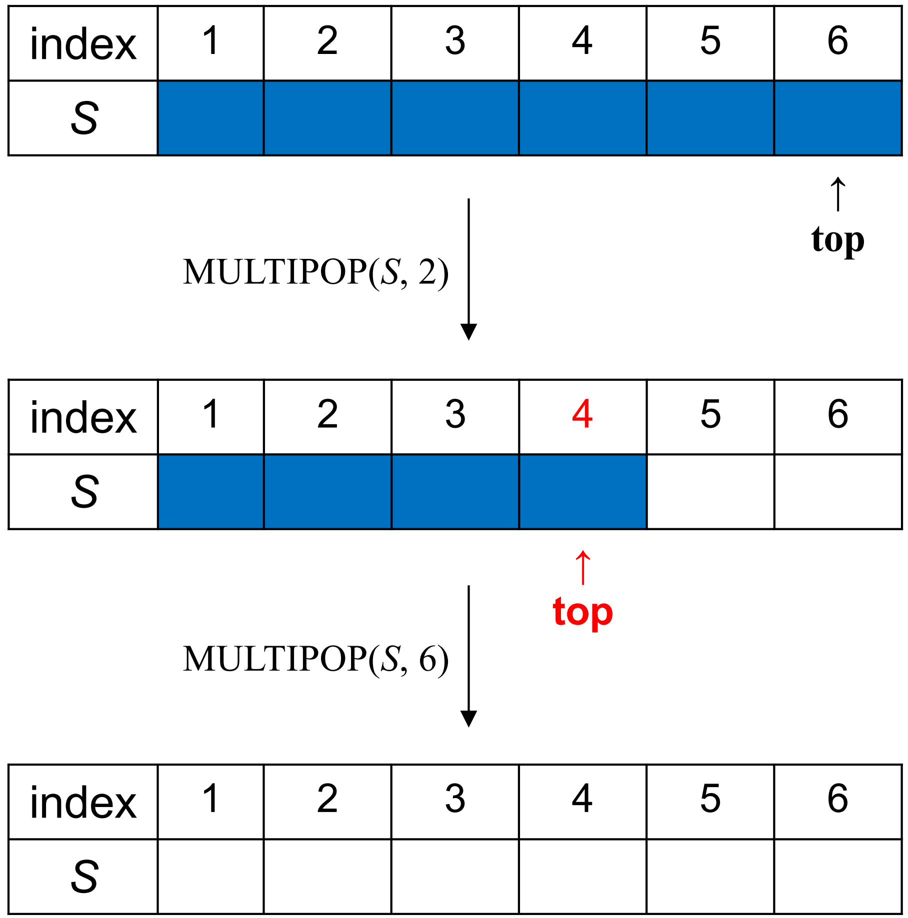
- Example of stack operation
  - MULTIPOP(S, k)
    - Actual running time is linear in the number of POP operations actually executed.

```
MULTIPOP(S, k)
    while not STACK-EMPTY(S) and k > 0
        POP(S)
        k = k - 1
```

$T(1) * k$

- So, cost of MULTIPOP(S, k) is $O(k)$.

- Example of stack operation
  - MULTIPOP(S, k)
    - Remove the k top objects of stack
    - If objects of stack are less than k, it removes the objects in the stack.
      - So, stack is empty



- Example of stack operation
  - Analysis of a sequence of n PUSH, POP and MULTIPOP operations
    - on an initially empty stack
  - Intuitive analysis of time complexity (wrong way)
    - The worst-case cost of one MULTIPOP: $O(n)$
    - Stack size: at most n

➔ Total cost : $O(n^2)$

- This cost isn't tight

Aggregate analysis

- Example of stack operation
  - Using Aggregate analysis
    - Can obtain a better upper bound the entire sequence of n operations
  - Any sequence of n PUSH, POP and MULTIPOP operations
    - on an initially empty stack
    - [Push, push, pop, push, push, push, multipop(2), …]

    = [Push, push, pop, push, push, push, {pop, pop}, …]

    $$n \geq \#(\text{push}) \geq \#(\text{pop})$$

    $$2n \geq \#(\text{push}) + \#(\text{pop})$$

    ➔ Total cost : $O(n)$

    - Amortized cost is $O(n) / n = O(1)$

- Example of incrementing binary counter
  - Consider the problem of implementing a k-bit binary counter that counts upward from 0
    - Use an array $A[0..k\text{-}1]$ of bits

***A*[k-1]**	**…**	***A*[2]**	***A*[1]**	***A*[0]**

Aggregate analysis

- Example of incrementing binary counter
  - Cost of INCREMENT operation is proportional to the number of bits flip

```
INCREMENT (A)
    i = 0
    while i < A.length and A[i] == 1
        A[i] = 0
        i = i + 1
    if i < A.length
        A[i] = 1
```

Bit flip, $T(1)$

Bit flip, $T(1)$

Aggregate analysis

- Example of incrementing binary counter
  - Cost of INCREMENT operation is proportional to the number of bits flip

Example: INCREMENT(A)

Counter value	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	cost	Total cost
0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1

- Example of incrementing binary counter
  - Cost of INCREMENT operation is proportional to the number of bits flip

Example: INCREMENT(A)

Counter value	$A[4]$	$A[3]$	$A[2]$	$A[1]$	$A[0]$	cost	Total cost
0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1
2	0	0	0	1	0	2	3

- Example of incrementing binary counter
  - Cost of INCREMENT operation is proportional to the number of bits flip

Example: INCREMENT(A)

Counter value	A[4]	A[3]	A[2]	A[1]	A[0]	cost	Total cost
0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1
2	0	0	0	1	0	2	3
3	0	0	0	1	1	1	4

- Example of incrementing binary counter
  - Cost of INCREMENT operation is proportional to the number of bits flip

Example: INCREMENT(A)

Counter value	A[4]	A[3]	A[2]	A[1]	A[0]	cost	Total cost
0	0	0	0	0	0	0	0
1	0	0	0	0	1	1	1
2	0	0	0	1	0	2	3
3	0	0	0	1	1	1	4
4	0	0	1	0	0	3	7

- Example of incrementing binary counter
  - A single execution of INCREMENT takes time $\Theta(k)$ in the worst case
    - In which array A contains all 1s.

$A[k\text{-}1]$	…	$A[2]$	$A[1]$	$A[0]$	cost
1	…	1	1	1	-
0	…	0	0	0	k

  - Thus, a sequence of n INCREMENT operations on an initially zero counter takes time $O(nk)$ in the worst case.

- Example of incrementing binary counter
  - Aggregate Analysis
    - can tighten our analysis to yield a worst-case cost of $O(n)$ for a sequence of n INCREMENT operations by observing that not all bits flip each time INCREMENT is called

- Example of incrementing binary counter
  - Compute bit flip of Array A
    - Time of flip of $A[0]$: n
    - Time of flip of $A[1]$:
    - Time of flip of $A[2]$:
  - The total number of flip in the sequence
    - $\sum_{i=0}^{k-1} \lfloor n/2^i \rfloor < \sum_{i=0}^{\infty} n/2^i = 2n$

    ➔ Total cost $O(n)$
  - Amortized cost $= O(n)/n = O(1)$

	$A[3]$	$A[2]$	$A[1]$	$A[0]$
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
7	0	1	1	1
8	1	0	0	0
	↑	↑	↑	↑
	$n/8$	$n/4$	$n/2$	n

Aggregate analysis

- Running time
  - $O(n)$ time in total
- Amortized cost
  - $O(n) / n = O(1)$

Contents

- *Aggregate analysis*
- **Accounting method**
- **Potential method**
- **Dynamic Table**

- Accounting method
  - assign differing charges to different operations, with some operations charged more or less than they actually cost
  - Amortized cost

    The amount of charged operation
  - Credit
    - assign the difference to specific objects in the data structure

      can help pay for later operations whose amortized cost is less than their actual cost

- Accounting method
  - We want to show that in the worst case the average cost per operation is small by analyzing with amortized costs,
    - c_i : actual cost of the ith operation
    - $\hat{c}_i$: amortized cost of the ith operation
    - $\sum_{i=0}^{n} \hat{c}_i \geq \sum_{i=0}^{n} c_i$ for all sequences of n operations
  - The total credit stored in the data structure is the difference between the total amortized cost and the total actual cost
    - $\sum_{i=0}^{n} \hat{c}_i - \sum_{i=0}^{n} c_i$

Accounting method

- Example of stack operation
  - The actual costs of the operations
    - PUSH 1
    - POP 1
    - MULTIPOP $\min(k,s)$
  - The amortized costs of the operations
    - PUSH 2
    - POP 0
    - MULTIPOP 0

- Example of stack operation

index	1	2	3	4
S				

cost				

S	**1**	**2**	**3**	**4**
credit				

- Example of stack operation
  - PUSH

index	1	2	3	4
S				

↑
top

cost	1			

S	**1**	**2**	**3**	**4**
credit	**1**			

  - PUSH : actual cost 1 + prepaid credit 1
  - Amortized cost : actual cost + credit = 2

- Example of stack operation
  - PUSH

index	1	2	3	4
S				

↑
top

cost	1	1		

S	1	2	3	4
credit	1	1		

  - PUSH : actual cost 1 + prepaid credit 1
  - Amortized cost : actual cost + credit = 2

- Example of stack operation
  - PUSH

index	1	2	3	4
S				

↑
top

cost	1	1	1	

S	**1**	**2**	**3**	**4**
credit	**1**	**1**	**1**	

  - PUSH : actual cost 1 + prepaid credit 1
  - Amortized cost : actual cost + credit = 2

- Example of stack operation
  - POP

index	1	2	3	4
S				

↑
top

cost	1	1	1	

S	**1**	**2**	**3**	**4**
credit	**1**	**1**	**0**	

  - POP and MULTIPOP : pay credit 1
  - Amortized cost : actual cost - credit = 0

- Example of stack operation
  - PUSH

index	1	2	3	4
S				

↑
top

cost	1	1	1	1

S	**1**	**2**	**3**	**4**
credit	**1**	**1**	**1**	

  - PUSH : actual cost 1 + prepaid of credit 1
  - Amortized cost : actual cost + credit = 2

- Example of stack operation
  - POP and MULTIPOP must execute after PUSH operation
    - Charging the PUSH operation a little bit more (= credit)

      So, credit pay actual cost of POP and MULTIPOP operation
  - The amount of credit is always nonnegative
    - Because the stack always has nonnegative objects.
    - Thus, the total amortized cost is an upper bound on the total actual cost
  - Total amortized cost : $O(n)$
  - Total actual cost : $O(n)$

Accounting method

- Example of incrementing binary counter
  - The actual costs
    - Bit set (0 → 1) : 1
    - Bit reset (1 → 0) : 1
  - The amortized costs
    - Bit set : 2
    - Bit reset : 0

Accounting method

- Example of incrementing binary counter

$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	1

cost	1				

A	$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$
credit	1				

- Example of incrementing binary counter

$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	1
			0

cost	1				

A	$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$
credit					

- Example of incrementing binary counter

A[3]	*A*[2]	*A*[1]	*A*[0]
0	0	0	1
0	0	1	0

cost	1	1			

A	*A*[0]	*A*[1]	*A*[2]	*A*[3]	*A*[4]
credit		1			

- Example of incrementing binary counter

A[3]	*A*[2]	*A*[1]	*A*[0]
0	0	0	1
0	0	1	0
0	0	1	1

cost	1	1	1		
A	*A*[0]	*A*[1]	*A*[2]	*A*[3]	*A*[4]
credit	1	1			

- Example of incrementing binary counter

$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	1
0	0	1	0
0	0	1	1
			0

cost	1	1	1		

A	$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$
credit		1			

- Example of incrementing binary counter

A[3]	*A*[2]	*A*[1]	*A*[0]
0	0	0	1
0	0	1	0
0	0	1	1
		0	0

cost	1	1	1		
A	***A*[0]**	***A*[1]**	***A*[2]**	***A*[3]**	***A*[4]**
credit					

- Example of incrementing binary counter

$A[3]$	$A[2]$	$A[1]$	$A[0]$
0	0	0	1
0	0	1	0
0	0	1	1
	1	0	0

cost	1	1	1	1	

A	$A[0]$	$A[1]$	$A[2]$	$A[3]$	$A[4]$
credit			1		

- Example of incrementing binary counter

A[3]	*A*[2]	*A*[1]	*A*[0]
0	0	0	1
0	0	1	0
0	0	1	1
0	1	0	0
0	1	0	1

cost	1	1	1	1	1

A	*A*[0]	*A*[1]	*A*[2]	*A*[3]	*A*[4]
credit	1		1		

Accounting method

- Example of incrementing binary counter
  - Bit reset must execute after each bit set
    - Charging the bit set in credit

      So, credit pay for actual cost of reset operation

  - The amount of credit is always nonnegative
    - Because the number of 1s in the counter never becomes negative
    - Thus, the total amortized cost is an upper bound on the total actual cost

  - Total amortized cost : $O(n)$

  - Total actual cost : $O(n)$

Accounting method

- Amortized cost
  - $O(n)$ time in total

- Running time
  - $O(n)$ time in total

Contents

- ***Aggregate analysis***
- ***Accounting method***
- **Potential method**
- **Dynamic Table**

Potential method

- Potential method
  - Similar accounting method

    Credit → “potential energy” or just “potential”

  - The potential with the data structure as a whole rather than with specific objects within the data structure.

A[3]	*A*[2]	*A*[1]	*A*[0]
0	**0**	**0**	**1**

Credit →

cost	**1**				

A	*A*[0]	*A*[1]	*A*[2]	*A*[3]	*A*[4]
credit	**1**				

Potential →

1	**1**
cost	Potential

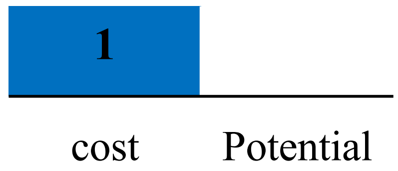
A[3]	*A*[2]	*A*[1]	*A*[0]
0	**0**	**0**	**1**
			0

Credit

cost	**1**				

A	***A*[0]**	***A*[1]**	***A*[2]**	***A*[3]**	***A*[4]**
credit					

Potential



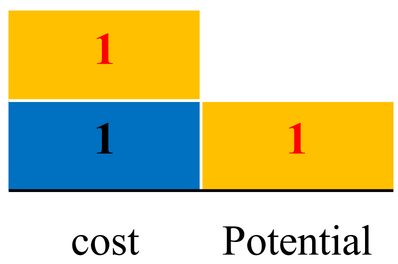
A[3]	*A*[2]	*A*[1]	*A*[0]
0	0	0	1
0	0	1	0

Credit

cost	1	1			

A	*A*[0]	*A*[1]	*A*[2]	*A*[3]	*A*[4]
credit		1			

Potential



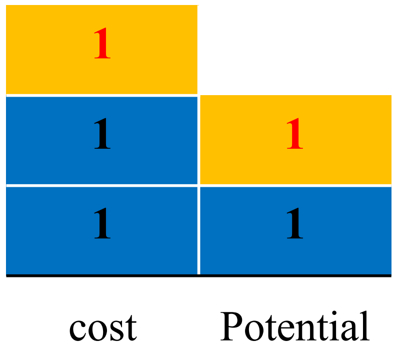
A[3]	A[2]	A[1]	A[0]
0	0	0	1
0	0	1	0
0	0	1	1

Credit

cost	1	1	1		

A	A[0]	A[1]	A[2]	A[3]	A[4]
credit	1	1			

Potential



- Potential method
  - will perform n operations,

    D_0 : an initial data structure

    D_i : the data structure that results after applying the ith operation to data structure $D_{i\text{-}1}$

    $\Phi(D_i)$: the potential associated with data structure D_i

  - Potential difference ($\Phi(D_i) - \Phi(D_{i-1})$)
    - positive

      The potential of the data structure increases
    - negative

      The decrease in the potential pays for the actual cost of the operation

- Potential method
  - Amortized cost
    - $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
  - The total amortized cost of the n operations
    - $\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n}(c_i + \Phi(D_i) - \Phi(D_{i-1}))$
      $= \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$
  - We require $\Phi(D_i) \geq \Phi(D_0)$ for all i
    - So that $\sum_{i=1}^{k} \hat{c}_i \geq \sum_{i=1}^{k} c_i$ for all $1 \leq k \leq n$

- Example of stack operation
  - Potential function Φ
    - the number of objects in the stack
    - $\Phi(D_0) = 0$
  - The stack D_i that results after the ith operation has nonnegative potential
    - $\Phi(D_i) \geq 0$
      $= \Phi(D_0)$

- Example of stack operation
  - Amortized cost analysis of each operation
    - PUSH operation
      - If the ith operation on a stack containing s objects is a PUSH operation,
        - » $\Phi(D_i) - \Phi(D_{i-1}) = (s + 1) - s = 1$
        - » So, the amortized cost is $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
          $= 1 + (s + 1) - s$
          $= 2$
    - POP operation
      - If the ith operation on a stack containing s objects is a POP operation,
        - » $\Phi(D_i) - \Phi(D_{i-1}) = (s - 1) - s = -1$
        - » So, the amortized cost is $\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
          $= 1 + (s - 1) - s$
          $= 0$

- Example of stack operation
  - Amortized cost analysis of each operation
    - MULTIPOP(S, k) operation
      - If the ith operation on a stack containing s objects is a MULTIPOP operation,
      - $k' = \min(k, s)$: The number of objects to be popped off the stack
        - » $\Phi(D_i) - \Phi(D_{i-1}) = -\min(k, s) = -k'$

          The amortized cost is $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$

          $= k' - k'$

          $= 0$

- Example of stack operation
  - Amortized cost : $O(1)$
  - Total amortized cost : $O(n)$
  - Total actual cost : $O(n)$

- Example of incrementing binary counter
  - Potential function Φ
    - The number of 1s in the array
    - b_i : The number of 1s in the counter after the ith INCREMENT operation
    - t_i : The number of bits reset in the ith INCREMENT operation
  - Actual cost of the ith operation
    - $c_i \leq t_i + 1$
      - since in addition to resetting t_i bits, it sets at most one bit to 1

```
INCREMENT (A)
    i = 0
    while i < A.length and A[i] == 1
        A[i] = 0
        i = i + 1
    if i < A.length
        A[i] = 1
```

- Example of incrementing binary counter
  - Case of $b_i = 0$
    - the ith operation resets all k bits
    - $b_{i\text{-}1} = t_i = k$
  - Case of $b_i > 0$
    - $b_i = b_{i\text{-}1} - t_i + 1$
  - In either case
    - $b_i \leq b_{i\text{-}1} - t_i + 1$

Ex) 1111 -> 0000

Counter value	$A[k]$	…	$A[2]$	$A[1]$	$A[0]$	b_i
i-1	1	…	1	1	1	k
i	0	…	0	0	0	0

Ex) 0111 -> 1000

Counter value	$A[k]$	…	$A[2]$	$A[1]$	$A[0]$	b_i
i-1	0	…	1	1	1	k-1
i	1	…	0	0	0	1

- Example of incrementing binary counter
  - Potential difference
    - $\Phi(D_i) - \Phi(D_{i-1}) = b_i - b_{i\text{-}1}$
      $\leq (b_{i\text{-}1} - t_i + 1) - b_{i\text{-}1}$
      $= 1 - t_i$
  - Amortized cost
    - $\widehat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1})$
      $\leq (t_i + 1) + (1 - t_i) = 2$
    - ➔ O(1)

- Example of incrementing binary counter
  - If the counter starts at zero, $\Phi(D_0) = 0$ and since $\Phi(D_i) \geq 0$ for all i
    - The total amortized cost of a sequence of n INCREMENT operations is an upper bound on the total actual cost
    - The worst-case cost of n INCREMENT operations is $O(n)$

- Example of incrementing binary counter
  - If does not start at zero
    - $0 \leq b_0, b_n \leq k$ (k : the number of bits in the counter)
    - $\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{n} c_i + \Phi(D_n) - \Phi(D_0)$
      - $\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} \hat{c}_i - \Phi(D_n) + \Phi(D_0)$ ($\hat{c}_i \leq 2$ for all $1 \leq i \leq n$)

        $\leq \sum_{i=1}^{n} 2 - b_n + b_0$ ($\Phi(D_n) = b_n$, $\Phi(D_0) = b_0$)

        $= 2n - b_n + b_0$
    - The total actual cost is $O(n)$ (as long as $k = O(n)$, since $b_0, b_n \leq k$)

Contents

Dynamic tables

- Table allocation problem
- We do not always know in advance how many objects some applications will store in a table
  - insertion
    - So allocate space for a table and reallocate the table when new item is added.
  - deletion
    - Similarly, if many objects have been deleted from the table, it may be worthwhile to reallocate the table with a smaller size
  - Using amortized analysis, we shall show that the amortized cost of insertion and deletion is only $O(1)$

- INSERT
  - When inserting an item into a full table, we can expand the table by allocating a new table with more slots than the old table had.
  - A common heuristic allocates a new table with **twice** as many slots as the old one.

- INSERT
  - *T.table* : a pointer to the block of storage representing the table.
  - *T.num* : the number of items in the table
  - *T.size* : the total number of slots in the table.

Aggregate analysis

```
TABLE-INSERT(T, x)
    if T.size == 0
        allocate T.table with 1 slot
        T.size = 1
    if T.num == T.size
        allocate new-table with 2 * T.size slots
        insert all items in T.table into new-table
        free T.table
        T.table = new-table
        T.size = 2 * T.size
    insert x into T.table
    T.num = T.num +1
```

elementary insertion

expansion

- Let us analyze a sequence of n TABLE-INSERT operations on an initially empty table.
  - If the current table has room for the new item, then cost $c_i = 1$.
  - If the current table is full, an expansion occurs, then $c_i = i$.
    - 1 for insert new item, $i - 1$ for move for extend

- Let us analyze a sequence of n TABLE-INSERT operations on an initially empty table.
  - If the current table has room for the new item, then cost $c_i = 1$.
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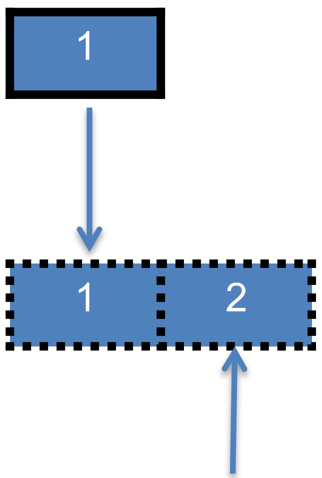

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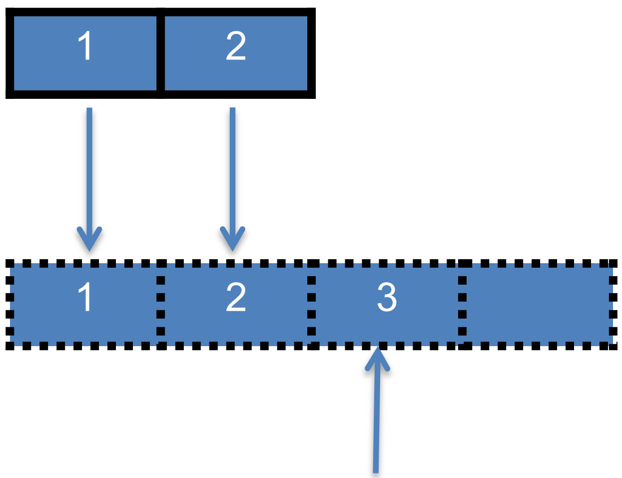

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  - If the current table is full, an expansion occurs, then $c_i = i$.
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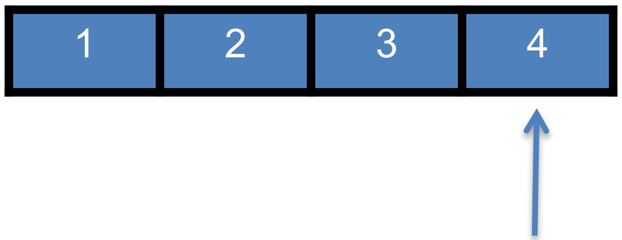


- Let us analyze a sequence of n TABLE-INSERT operations on an initially empty table.

$$c_i = \begin{cases} i & \text{if } i - 1 \text{ is an exact power of 2} \\ 1 & \text{otherwise} \end{cases}$$

Aggregate analysis

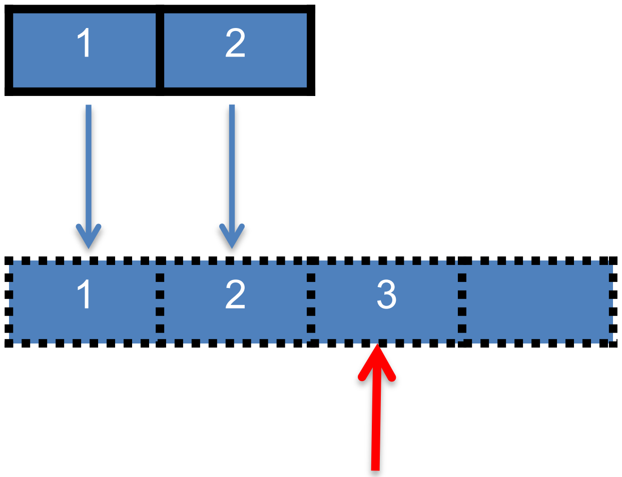
- The total cost of n TABLE-INSERT operations Is therefore

$$\sum_{i=1}^{n} c_i = n + \sum_{j=0}^{\lfloor lg n \rfloor} 2^j$$

$$< n + 2n$$

$$= 3n$$

- Let us analyze a sequence of n TABLE-INSERT operations on an initially empty table.
  - For 1 to n, when item inserted in table, its cost is 1.
    - It requires $1 * n = n$ cost.



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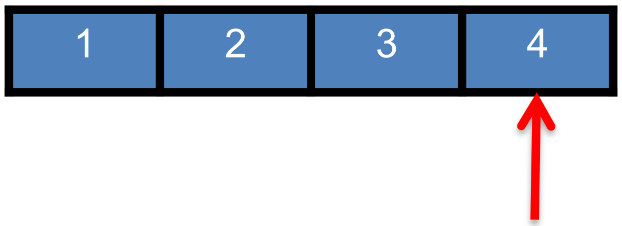

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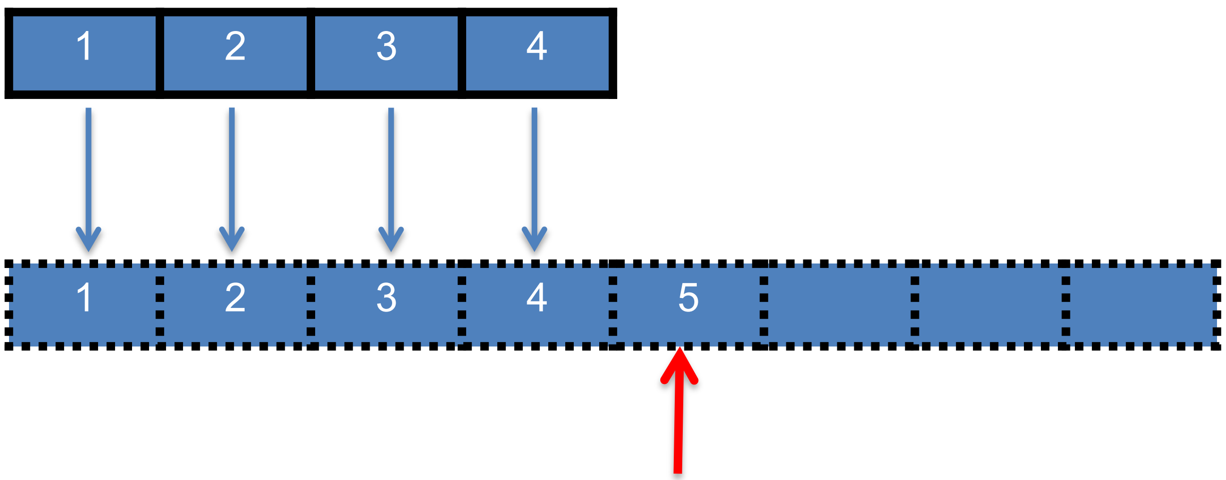

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Aggregate analysis

- Let us analyze a sequence of n TABLE-INSERT operations on an initially empty table.
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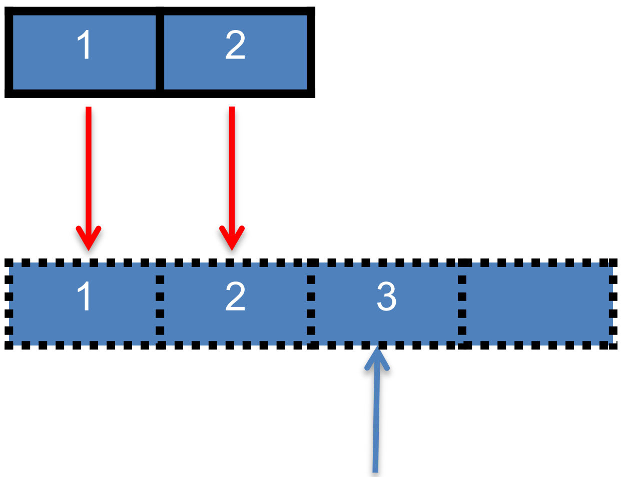
    - 2^j insert is occurred.
    - And it occurred $\lfloor \lg n \rfloor$ times.



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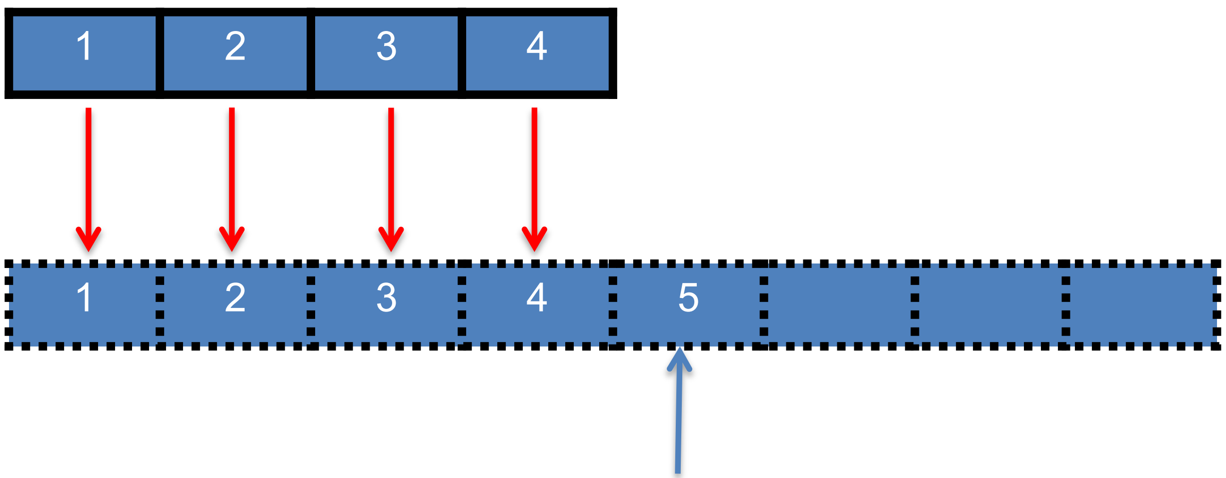

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Aggregate analysis

- The total cost of n TABLE-INSERT operations Is therefore

$$\sum_{i=1}^{n} c_i = n + \sum_{j=0}^{\lfloor lg n \rfloor} 2^j$$
$$< n + 2n$$
$$= 3n$$

  – Since the total cost of n TABLE-INSERT operations is bounded above by $3n$, the amortized cost of single operation is at most 3 ($3n$ / n).

Accounting method

- By using the accounting method, we can get some feel for why the amortized cost of a TABLE-INSERT operation should be 3.

Accounting method

```
TABLE-INSERT(T, x)
    if T.size == 0
        allocate T.table with 1 slot
        T.size = 1
    if T.num == T.size
        allocate new-table with 2 * T.size slots
        insert all items in T.table into new-table
        free T.table
        T.table = new-table
        T.size = 2 * T.size
    insert x into T.table
    T.num = T.num +1
```

elementary insertion

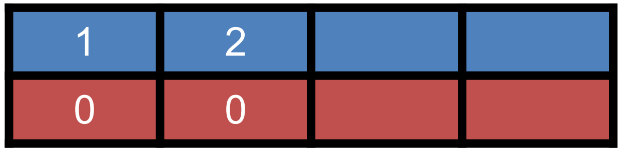
- By using the accounting method, we can get some feel for why the amortized cost of a TABLE-INSERT operation should be 3.
  - There are two types of elementary insertion:
    - 6 insert all items in $T.table$ into new-table
    - 10 insert x into $T.table$

Accounting method

- By using the accounting method, we can get some feel for why the amortized cost of a TABLE-INSERT operation should be 3.
  - each item pays for 3 elementary insertions:
    - 1 **cost** for line 10,
    - 2 **credit** for line 6.

    - Credit is used to move items when expansion occurred

- By using the accounting method, we can get some feel for why the amortized cost of a TABLE-INSERT operation should be 3.
  - each item pays for 3 elementary insertions:
    - inserting itself into the current table
    - moving itself when the table expands
    - moving another item that has already been moved once when the table expands



Credit for move

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1	2	3	
0	0	0	

Credit for move

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1	0	1		Credit for move

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  - each item pays for 3 elementary insertions:
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    - moving another item that has already been moved once when the table expands

1	2	3	4	
1	1	1	1	Credit for move

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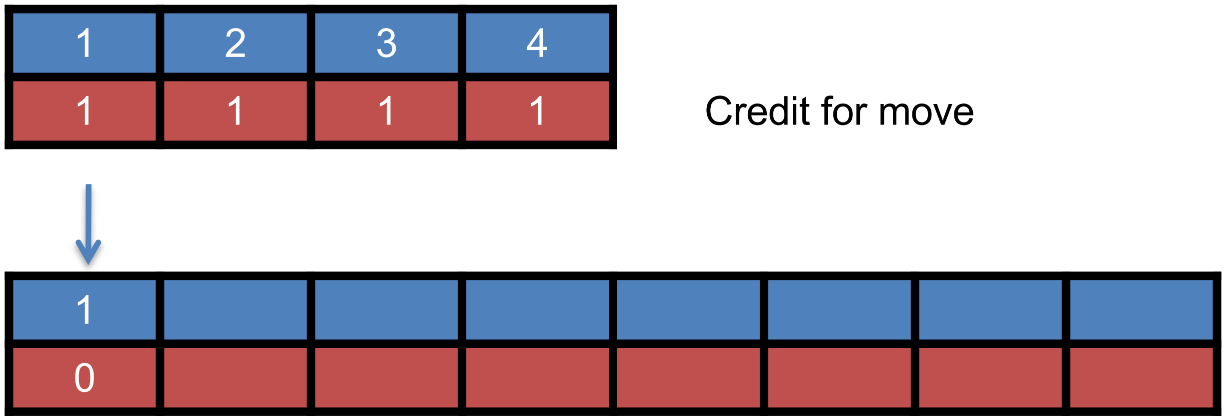
  - each item pays for 3 elementary insertions:
    - inserting itself into the current table
    - moving itself when the table expands
    - moving another item that has already been moved once when the table expands



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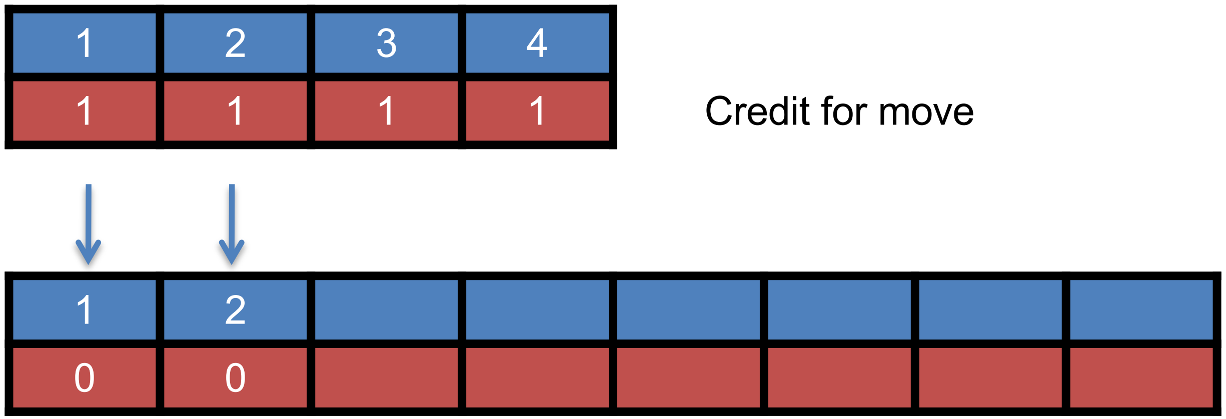

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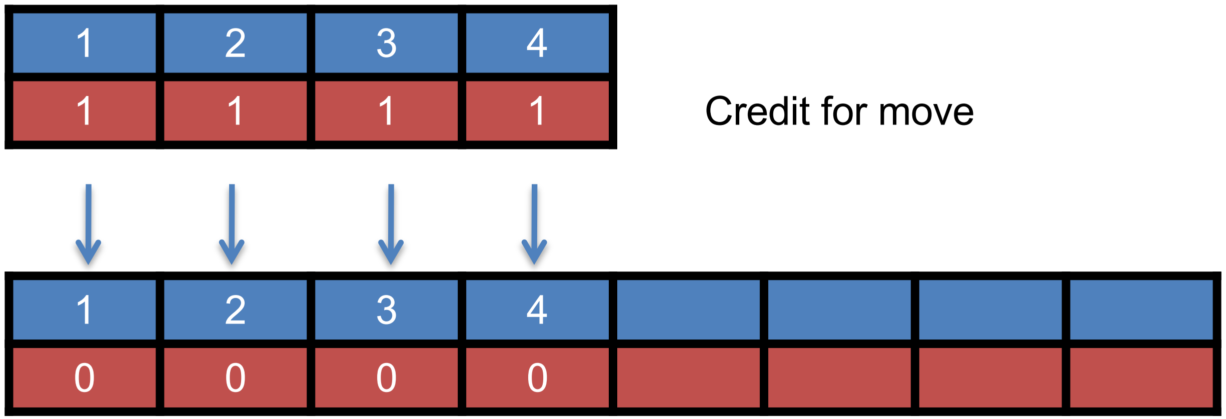


- We can use the potential method to analyze a sequence of n TABLE-INSERT operations.
  - and we shall use it in Section 17.4.2 to design a TABLE-DELETE operation that has an $O(1)$ amortized cost as well

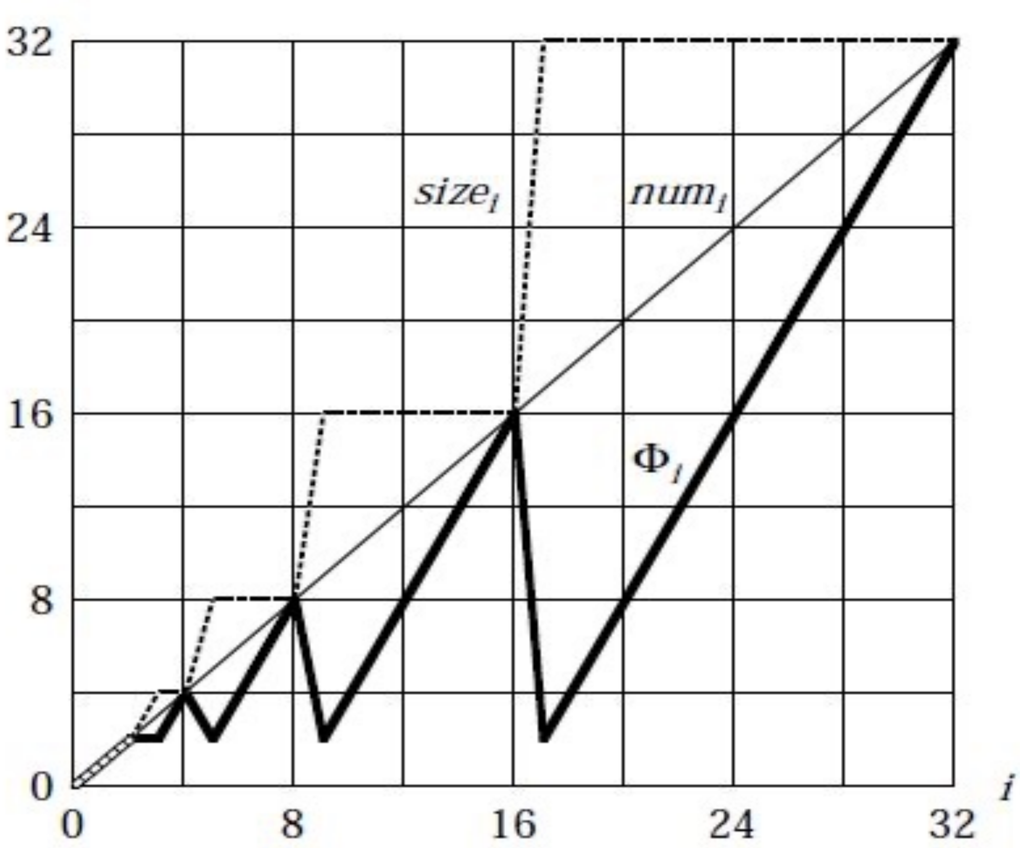
- We can use the potential method to analyze a sequence of n TABLE-INSERT operations.
  - and we shall use it in Section 17.4.2 to design a TABLE-DELETE operation that has an $O(1)$ amortized cost as well.
  - We start by defining a potential function Φ
    - 0 immediately after an expansion
    - table size by the time the table is full

- $\Phi(T) = 2*T.num - T.size \qquad (17.5)$

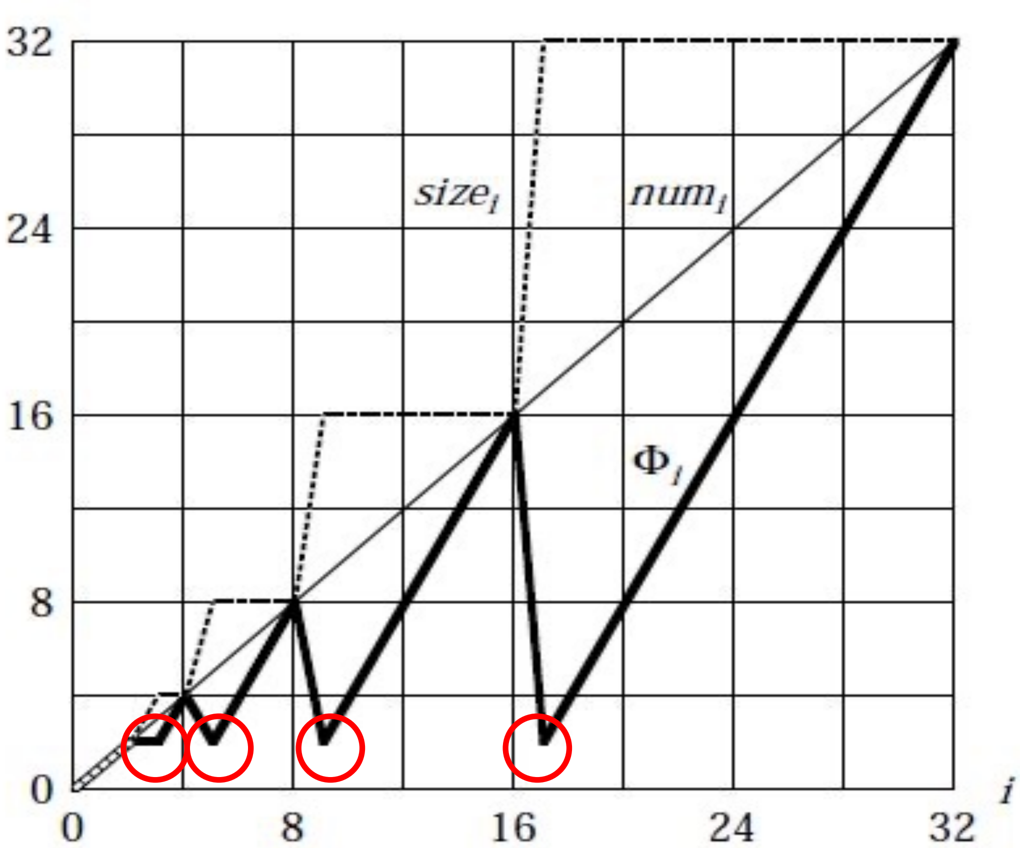
- Immediately before an expansion, we have $T.num = T.size$ and thus $\Phi(T) = T.num$

- $\Phi(T)$ is always nonnegative
  - The initial value of the potential is 0
  - and since the table is always at least half full, $T.num \geq T.size/2$

Potential method



- Before expansion, $\Phi_i = num_i$



- After expansion, $\Phi_i = 0$ but immediately increased by 2

Potential method

- The amortized cost of the ith TABLE-INSERT operation
  - num_i : the number of items in the table after the ith operation
  - $size_i$: the total size of the table after the ith operation
  - Φ_i : the potential after the ith operation
  - $\widehat{c_i}$: its amortized cost with respect to Φ

  - Initially, we have $num_0 = 0$, $size_0 = 0$, *and* $\Phi_0 = 0$.

- The amortized cost of the ith TABLE-INSERT operation
  - If the ith TABLE-INSERT operation does not trigger an expansion, then we have $size_i = size_{i-1}$ and the amortized cost of the operation is
    - $\Phi(T) = 2{*}T.num - T.size$

$$\begin{aligned}\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\ &= 1 + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\ &= 1 + (2 * num_i - size_i) - (2 * (num_i - 1) - size_i) \\ &= 3\end{aligned}$$

- The amortized cost of the ith TABLE-INSERT operation

If the ith TABLE-INSERT operation does trigger an expansion, then we have

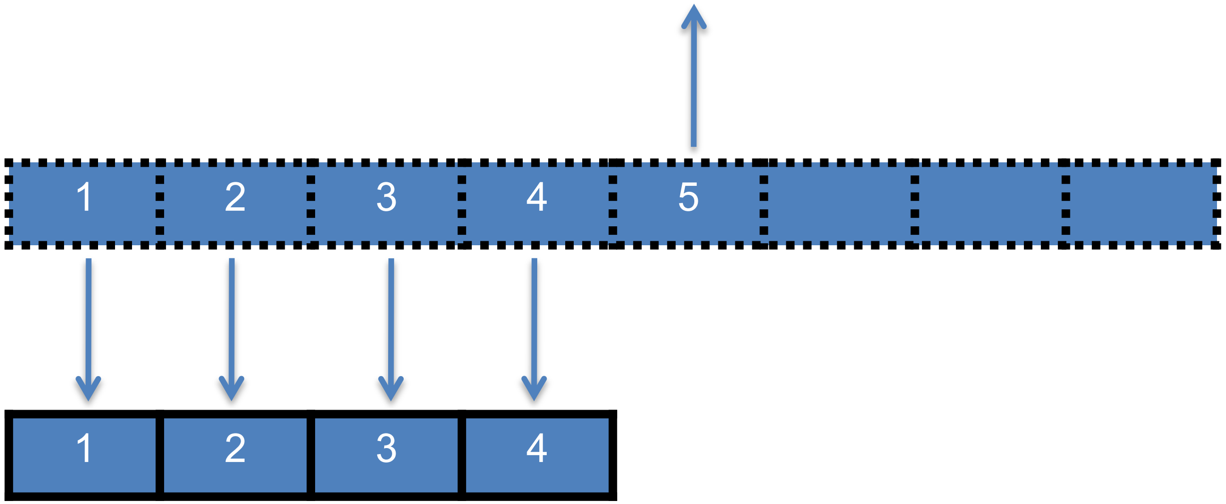
$$size_i = 2 * size_{i-1}$$
$$size_{i-1} = num_{i-1} = num_i - 1$$
$$size_i = 2 * (num_i - 1)$$

Thus, the amortized cost of the operation is

$$\begin{aligned}
\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\
&= num_i + (2 * num_i - size_i) - (2 * num_{i-1} - size_{i-1}) \\
&= num_i + (2 * num_i - 2 * (num_i - 1)) - (2 * (num_i - 1) - (num_i - 1)) \\
&= num_i + 2 - (num_i - 1) \\
&= 3
\end{aligned}$$

Table expansion and contraction

- TABLE-DELETE operation.
  - Table contraction is analogous to table expansion:
    - when the number of items in the table drops too low, we allocate a new, smaller table and then copy the items from the old table into the new one.



- TABLE-DELETE operation.
  - load factor : $\alpha(T) = T.num \,/\, T.size$

- TABLE-DELETE operation.
  - load factor : $\alpha(T) = T.num \,/\, T.size$
  - we would like to preserve two properties:
    - the load factor of the dynamic table is bounded below by a positive constant
    - the amortized cost of a table operation is bounded above by a constant.

```
TABLE-DELETE(T, x)
    if T.size == 0
        break
    if T.num == T.size / 2
        allocate new-table with T.size / 2 slots
        insert all items in T.table into new-table
        free T.table
        T.table = new-table
        T.size = T.size / 2
    delete x into T.table
    T.num = T.num – 1
```

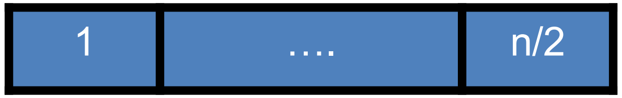
- Table expansion and contraction
  - double the table size upon inserting an item into a full table
  - halve the size when a deleting an item would cause the table to become less than half full
  - This strategy would guarantee that the load factor of the table never drops below 1/2, but have a **problem**

- Table expansion and contraction
  - We perform n operations on a table T , where n is an exact power of 2.
  - The first $n/2$ operations are insertions,
    - cost a total of $\Theta(n)$.
  - At the end of this sequence of insertions, $T.num = T.size = n/2$.
  - For the second $n/2$ operations, we perform the following sequence:
    - insert, delete, delete, insert, insert, delete, delete, insert, insert, ...

- Table expansion and contraction
  - First $n/2$ insertion.

1	….	n/2

- Table expansion and contraction
  - First $n/2$ insertion.



  - And insert, delete, delete, insert, insert, delete, delete, insert, insert, . . .

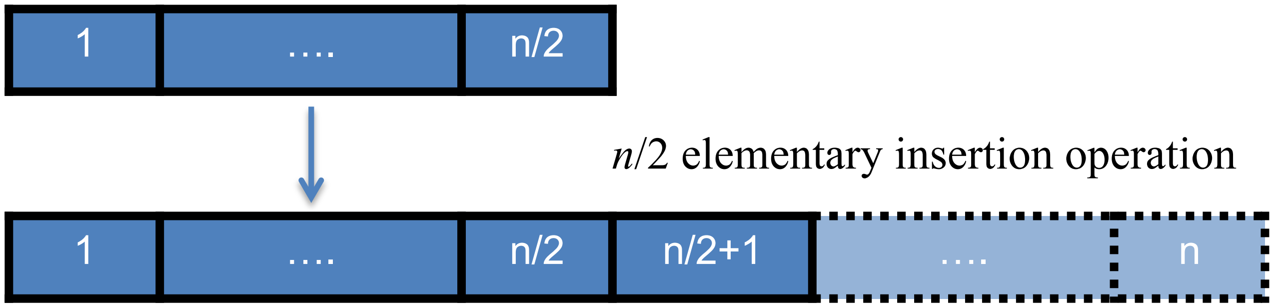
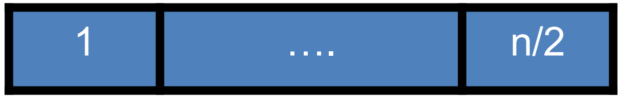


Table expansion and contraction

- Table expansion and contraction
  - First $n/2$ insertion.



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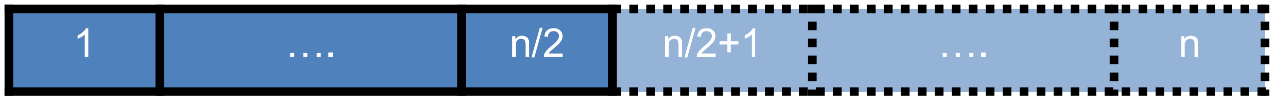
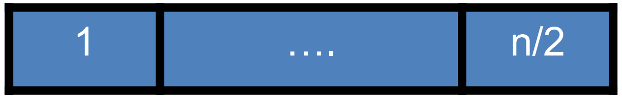


- Table expansion and contraction
  - First $n/2$ insertion.



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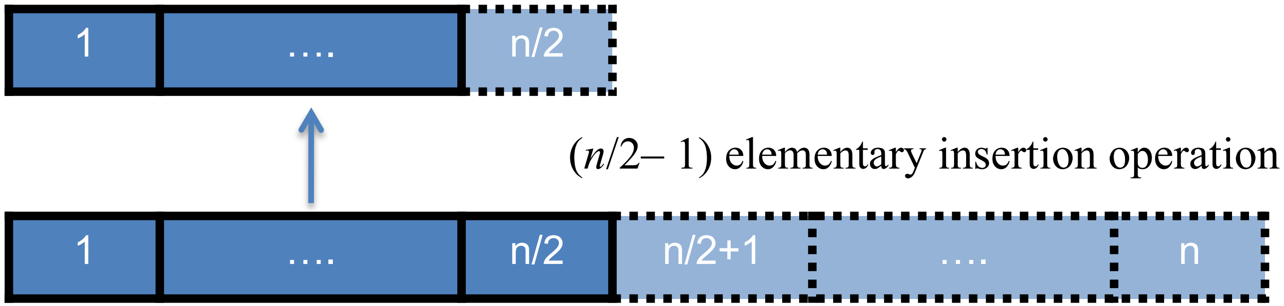

Table expansion and contraction

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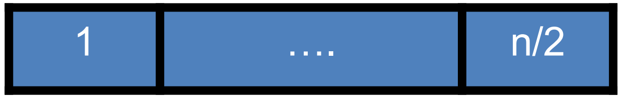

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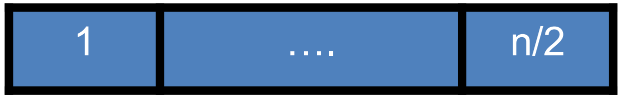

Table expansion and contraction

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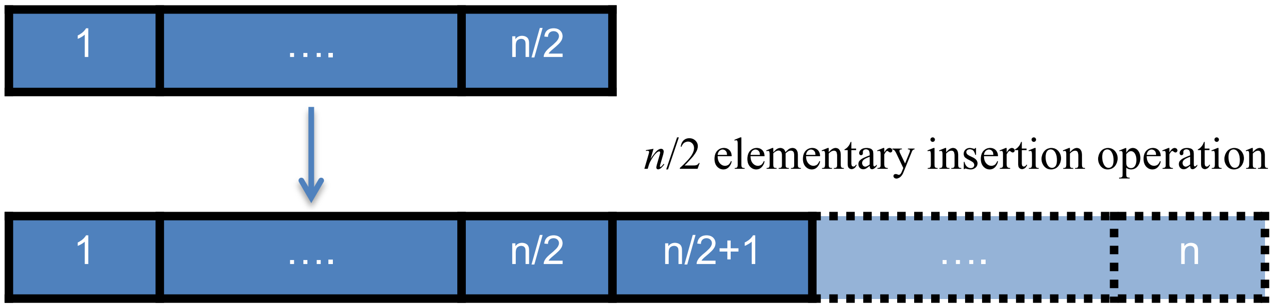


Table expansion and contraction

- Table expansion and contraction
  - First $n/2$ insertion.

1	….	n/2

  - And insert, delete, delete, insert, insert, delete, delete, insert, insert, . . .
    - $n/2$ number of insertion occur in 2 operation!

- Table expansion and contraction
  - The cost of each expansion and contraction is $\Theta(n)$, and there are $\Theta(n)$ of them.
    - After $n/2$-th operation, cost of each 2 operation is $n/2$
  - Thus, the total cost of the n operations is $\Theta(n^2)$, making the amortized cost of an operation $\Theta(n)$.

Table expansion and contraction

- Improve upon this strategy
  - Specifically, we continue to double the table size upon inserting an item into a full table,
  - but we halve the table size when deleting an item causes the table to become less than 1/4 full, rather than 1/2 full as before.
  - The load factor of the table is therefore bounded below by the constant 1/4.

Table expansion and contraction

- potential method to analyze the cost of a sequence of n TABLE-INSERT and TABLE-DELETE operations
  - Let us denote the load factor of a nonempty table T by $\alpha(T) = T.num \,/\, T.size$
  - Since for an empty table, $T.num = T.size = 0$ *and* $\alpha(T) = 1$
  - We shall use as our potential function

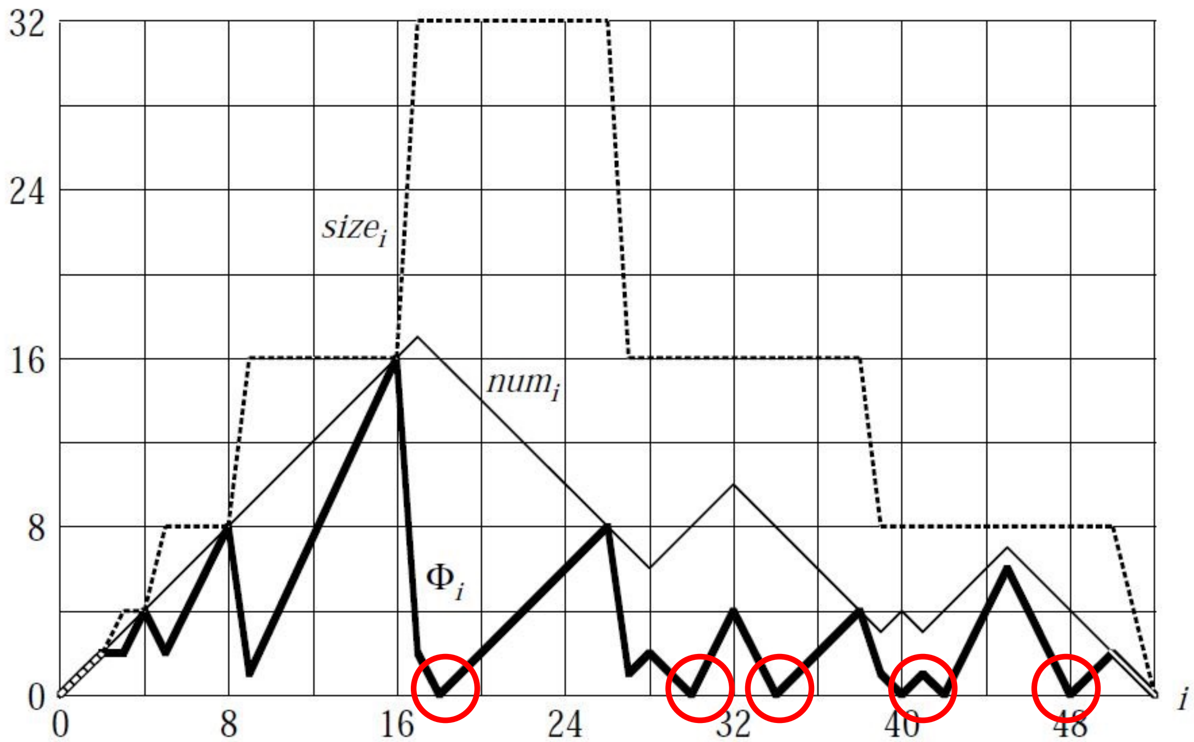
$$\Phi_i = \begin{cases} 2 * num_i - size_i & \text{if } \alpha(T) \geq 1/2 \\ size_i/2 - num_i & \text{if } \alpha(T) < 1/2 \end{cases}$$

- potential method to analyze the cost of a sequence of n TABLE-INSERT and TABLE-DELETE operations

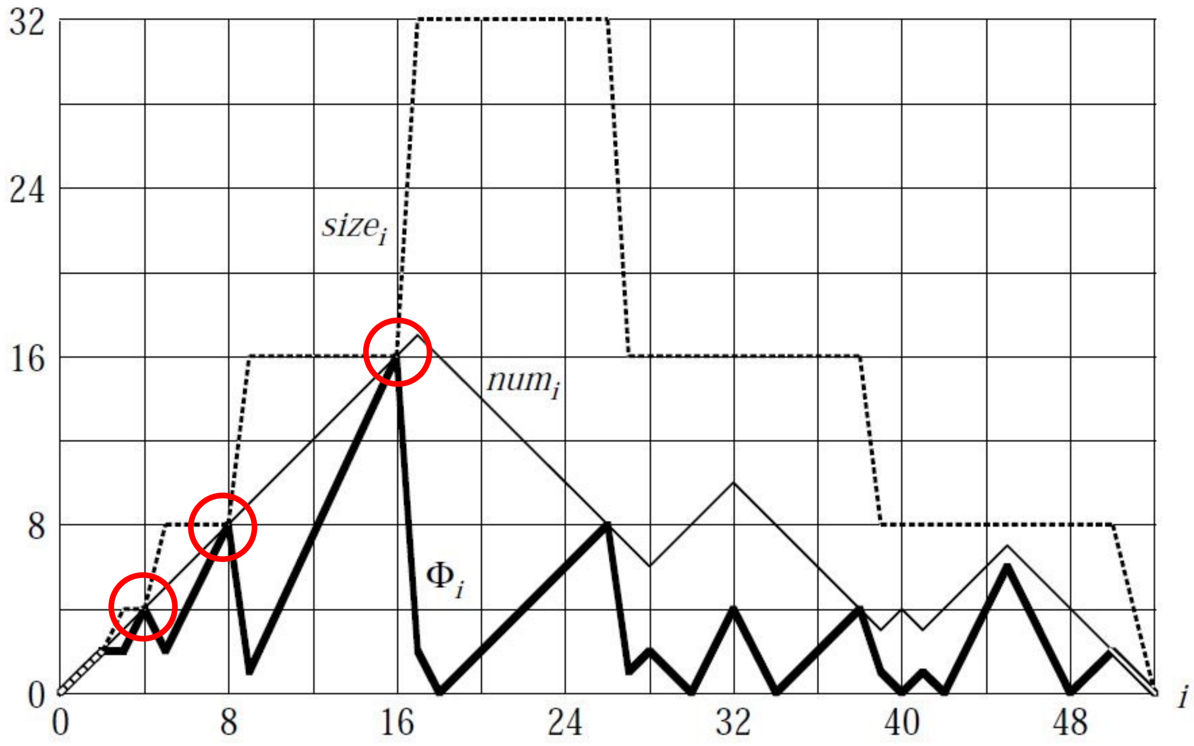
$$\Phi_i = \begin{cases} 2 * num_i - size_i & \text{if } \alpha(T) \geq 1/2 \\ size_i/2 - num_i & \text{if } \alpha(T) < 1/2 \end{cases}$$

  - When $\alpha(T) = 1/4$, and if ith operation is deletion
    - Contraction has occurred.
    - It needs num_i potential

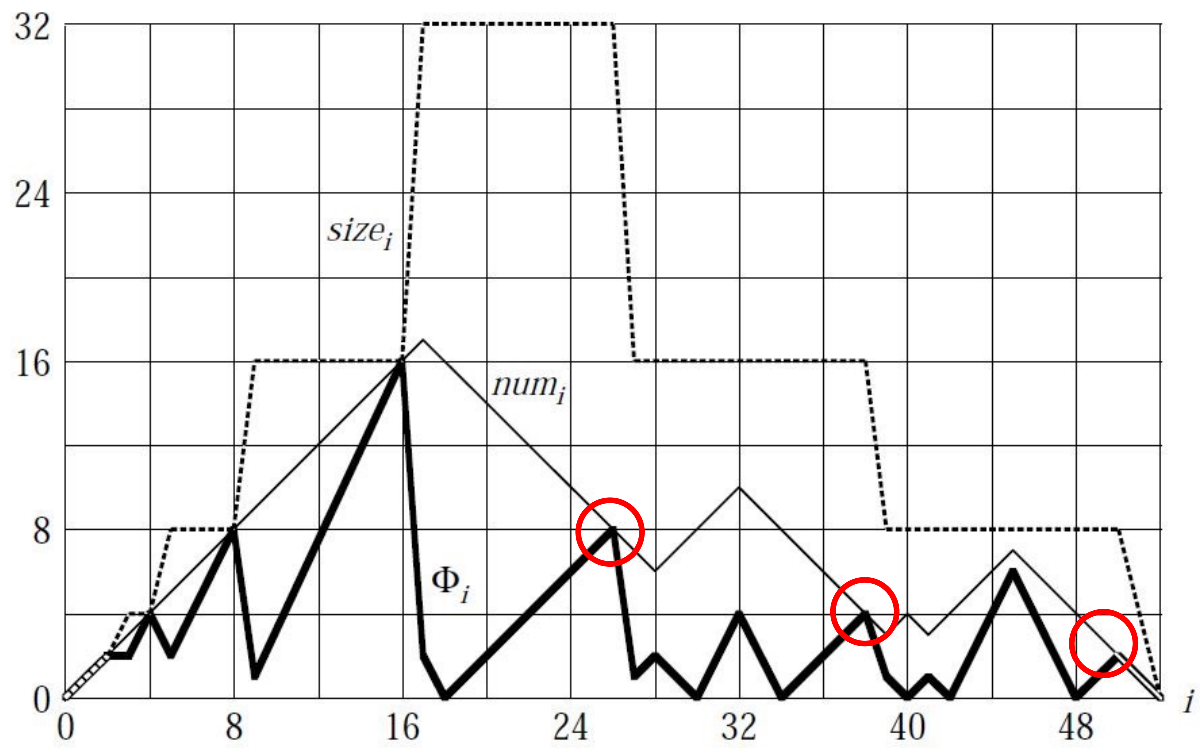
Table expansion and contraction



- When load factor is 1/2, the potential is 0.



- When the load factor is 1
  - we have $T.size = T.num$
    which implies $\Phi(T) = T.num$
  - Thus the potential can pay for an expansion if an item is inserted.



- When the load factor is 1/4,
  - we have $T.size = 4*T.num,$ which implies $\Phi(T) = T.num$
  - Thus the potential can pay for a contraction if an item is deleted.

- TABLE-INSERT and TABLE-DELETE
  - c_i : the actual cost of the ith operation
  - $\widehat{c}_i$: its amortized cost with respect to Φ
  - num_i : the number of items stored in the table after the ith operation
  - $size_i$: the total size of the table after the ith operation
  - α_i : the load factor of the table after the ith operation
  - Φ_i : the potential after the ith operation

  - Initially, $num_0 = 0$, $size_0 = 0$, $\alpha_0 = 1$, and $\Phi_0 = 0$

- TABLE-INSERT
  - The analysis is identical to that for table expansion in Section 17.4.1 if $\alpha_{i-1} \geq 1/2$.
    - Whether the table expands or not,
      the amortized cost of the operation is at most 3

- TABLE-INSERT
  - If $\alpha_{i-1} < 1/2$, table cannot expand.
    - Then $size_i = size_{i-1}$
    - $num_{i-1} = num_i - 1$

      Then the amortized cost of the ith operation is

$$
\begin{aligned}
\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\
&= 1 + (size_i/2 - num_i) - (size_i/2 - (num_i - 1)) \\
&= 0
\end{aligned}
$$

Table expansion and contraction

- TABLE-INSERT
  - If $\alpha_{i-1} < 1/2$ but $\alpha_i \geq 1/2$, then
    - $\Phi_{i-1} = T.size/2 - T.num$
    - $\Phi_i = 2 * T.num - T.size$
    - $num_i = num_{i-1} + 1 = \alpha_i * size_i$

- TABLE-INSERT
  - If $\alpha_{i-1} < \ 1/2$ but $\alpha_i \geq \ 1/2$, then

$$
\begin{aligned}
\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (2 * num_i - size_i) - (size_{i-1}/2 - num_{i-1}) \\
&= 1 + (2 * (num_{i-1} + 1) - size_{i-1}) - (size_{i-1}/2 - num_{i-1}) \\
&= 3 * num_{i-1} - 3 * size_{i-1}/2 + 3 \\
&= 3 * \alpha_{i-1} size_{i-1} - 3 * size_{i-1}/2 + 3 \\
&\quad < 3 * size_{i-1}/2 - 3 * size_{i-1}/2 + 3 \\
&\quad = 3
\end{aligned}
$$

  - Thus, the amortized cost of a TABLE-INSERT operation is at most 3.

- TABLE-DELETE
  - $num_i = num_{i-1} - 1$
  - If $\alpha_{i-1} < 1/2$

    1. operation causes the table to contract

    2. operation does trigger a contraction

- TABLE-DELETE
  - If $\alpha_{i-1} < 1/2$
    1. Operation does not cause the table to contract
       Then $size_i = size_{i-1}$

$$
\begin{aligned}
\widehat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\
&= 1 + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\
&= 1 + (size_i/2 - num_i) - (size_i/2 - (num_i+1)) \\
&= 2
\end{aligned}
$$

- TABLE-DELETE
  - If $\alpha_{i-1} < 1/2$

    2. operation does trigger a contraction
       - actual cost of the operation is $c_i = num_i + 1$
       - $size_i/2 = size_{i-1}/4 = num_{i-1} = num_i + 1$
       - the amortized cost of the operation is

$$\begin{aligned}\hat{c_i} &= c_i + \Phi_i - \Phi_{i-1} \\ &= (num_i + 1) + (size_i/2 - num_i) - (size_{i-1}/2 - num_{i-1}) \\ &= (num_i + 1) + \big((num_i + 1) - num_i\big) - \big((2 * num_i + 2) - (num_i + 1)\big) \\ &= 1\end{aligned}$$

- TABLE-DELETE
  - If $\alpha_{i-1} \geq 1/2$
    - Its amortize cost is constant.
    - If $\alpha_i < 1/2$, operation doesn't trigger contraction.

- In summary, since the amortized cost of each operation is bounded above by a constant, the actual time for any sequence of n operations on a dynamic table is $O(n)$.