Growth of Functions

Slides from Heejin Park

Contents

- **Asymptotic notation**
  - Θ-notation
  - O-notation
  - Ω-notation

Simple examples

- $\Theta(n^2) = 3n^2 + 2n - 1$
- $\Theta(n) = 3n - 1$
- $\Theta(n^2) \neq 3n - 1$

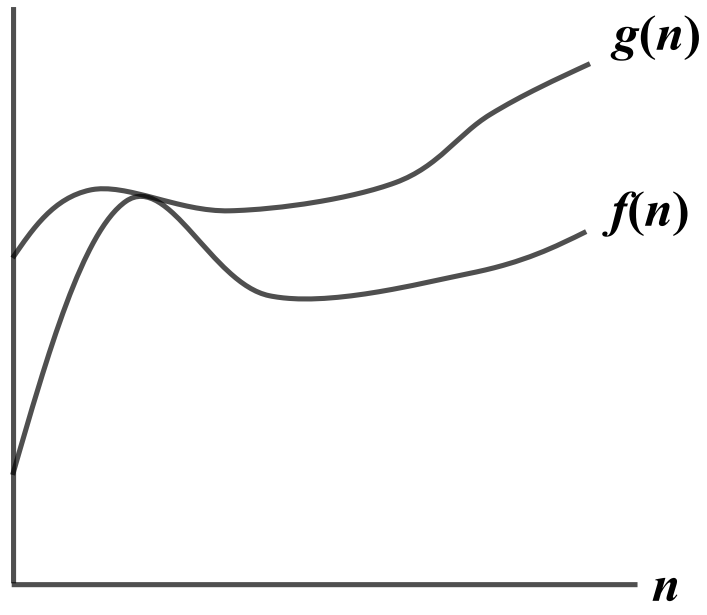
- $O(n^2) = 3n^2 + 2n - 1$
- $O(n) = 3n - 1$
- $O(n^2) = 3n - 1$

- $\Omega(n) = 3n - 1$
- $\Omega(n) = 3n^2 - 1$

Analogy

- $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
- $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
- $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.

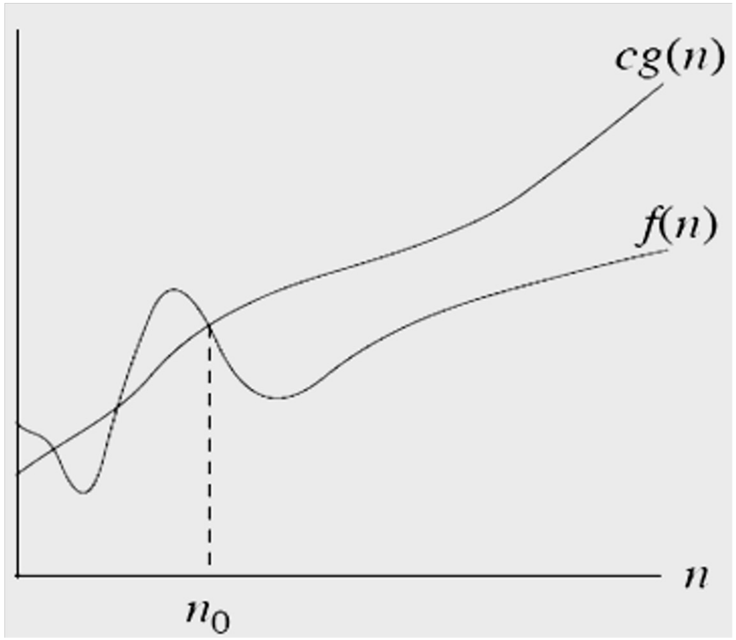
- **Upper bound**
  - $g(n)$ is an *upper bound* of $f(n)$.



- $g(n)$ is an ***asymptotic*** ***upper bound*** of $f(n)$.
  - $f(n) = O(g(n))$

There exist positive constants c and n_0 such that

$0 \le f(n) \le cg(n)$ for all $n \ge n_0$.



- Example

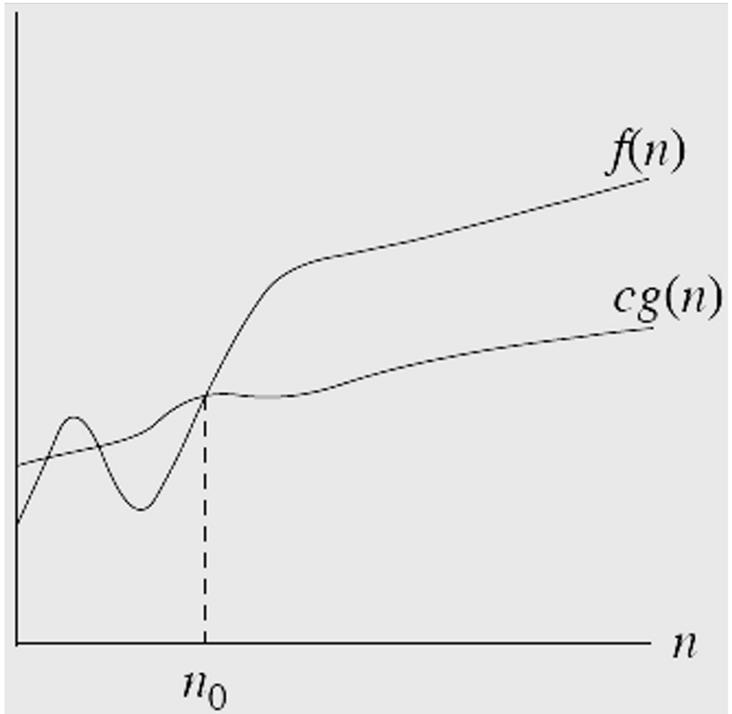
$$3n+1 = \mathbf{O}(n^2)$$

- Show there are c and n_o such that $3n+1 \le cn^2$ for all $n \ge n_0$.
- Dividing by n^2 yields $\frac{3}{n} + \frac{1}{n^2} \le c$.
- The inequality holds for any $n \ge 1$ ($n_0 = 1$) and $c = 4$.

- **Asymptotic lower bound**
  - $f(n) = \Omega(g(n))$

There exist positive constants c and n_0 such that

$0 \leq cg(n) \leq f(n)$ for all $n \geq n_0$.



- Example

$$3n^2 - 4n + 1 = \Omega(n)$$

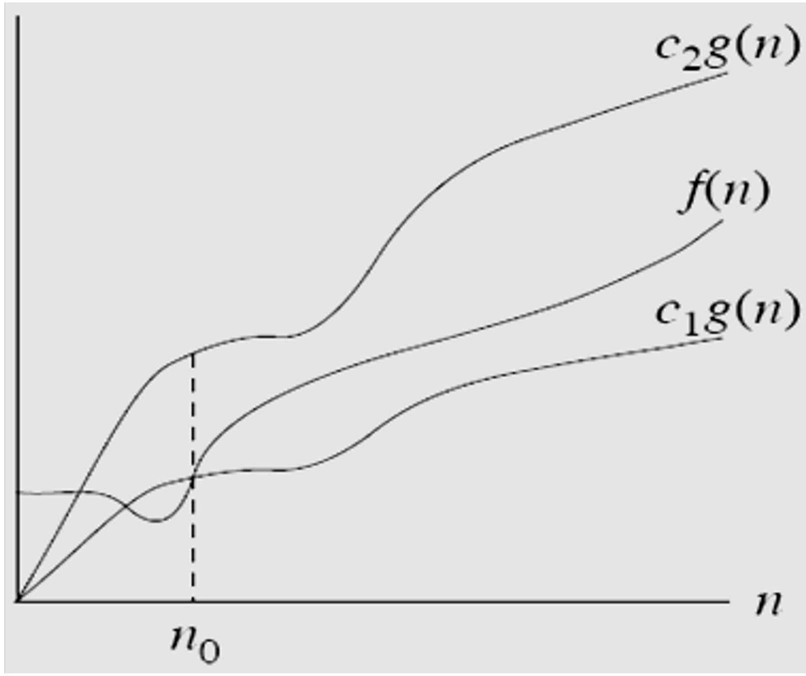
- Show there are c and n_o such that $3n^2 - 4n + 1 \geq cn$ for all $n \geq n_0$.
- Dividing by n yields $3n - 4 + \frac{1}{n} \geq c.$
- The inequality holds for any $n \geq 2$ ($n_0 = 2$) and $c = 2$.

- **Asymptotically tight bound**
  - $f(n) = \Theta(g(n))$

There exist positive constants c_1, c_2, and n_0 such that

$$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$$

for all $n \ge n_0$.



- Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

To show there exist positive constants $c_1,\ c_2$ and n_o such that

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Dividing by n^2 yields $c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$

- Example

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$$

- The right-hand inequality holds for $n \ge 1$ by choosing $c_2 \ge 1/2$.
- The left-hand inequality holds for $n \ge 7$ by choosing $c_1 \le 1/14$.
- Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$,
  we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

- Example
  - Consider any quadratic function $f(n) = an^2 + bn + c$, where a, b, and c are constants and $a > 0$.
  - Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$.
  - The reader may verify that $0 \leq c_1 n^2 \leq an^2 + bn + c \leq c_2 n^2$ for all $n \geq n_0$. (Self-study)
  - In general, for any polynomial $p(n) = \sum_{i=0}^{d} a_i n^i$ where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$.

Examples

- **Insertion sort**
  - $O(n^2),\ \Omega(n)$
- **Selection sort**
  - $\Theta(n^2)$
- **Merge sort**
  - $\Theta(n \lg n)$
- **Binary search**
  - $O(\lg n),\ \Omega(1)$

Analogy

- $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
- $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
- $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
- $f(n) = o(g(n)) \approx f(n) < g(n)$ in degree.
- $f(n) = \omega(g(n)) \approx f(n) > g(n)$ in degree.

Comparison of functions

- Transitivity
- Reflexivity
- Symmetry
- Transpose symmetry

- Transitivity ($=, \leq, \geq, <, >$)
- Reflexivity ($=, \leq, \geq$)
- Symmetry ($=$)
- Transpose symmetry ($\leq \leftrightarrow \geq, < \leftrightarrow >$)

- **Transitivity** (=, ≤, ≥, <, >)
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$,
  - $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$,
  - $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$,
  - $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$,
  - $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$.

- **Reflexivity** (=, ≤, ≥)
  - $f(n) = \Theta(f(n))$
  - $f(n) = O(f(n))$
  - $f(n) = \Omega(f(n))$

- **Symmetry** $(=)$
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

- **Transpose symmetry** $(\leq \leftrightarrow \geq,\ < \leftrightarrow >)$
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$,
  - $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

- **Trichotomy**
  - For any two real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, $a > b$.
  - That is, any two numbers are comparable.
  - Are any two functions asymptotically comparable?
    - Is it possible $f(n) \neq O(g(n))$ and $f(n) \neq \Omega\ (g(n))$?
    - n and $n^{1+\sin n}$

- **Exercise 3.1-1**
  - Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$

- **Exercise 3.1-4**
  - Is $2^{n+1} = O(2^n)$?
  - Is $2^{2n} = O(2^n)$?

- **Problem 3-2 for O, Θ, and Ω.**
  - Use $\lg(n!) = \Theta(n\lg n)$