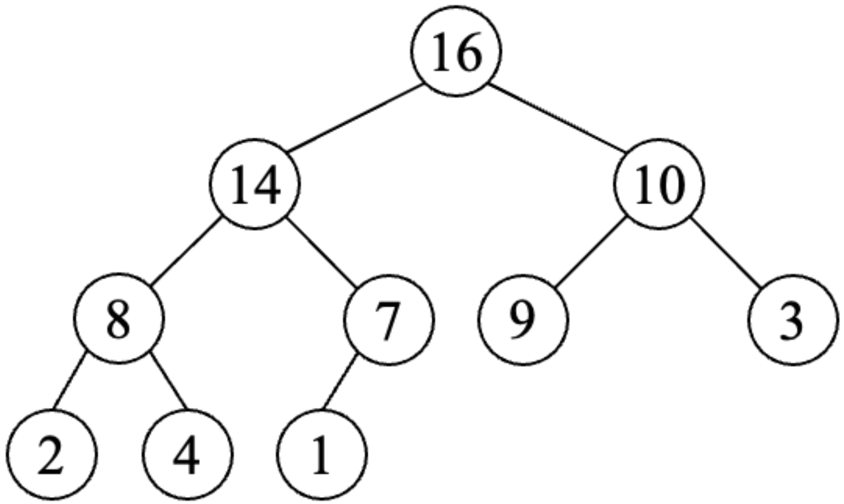
# Heapsort

Slides from Heejin Park

# Contents

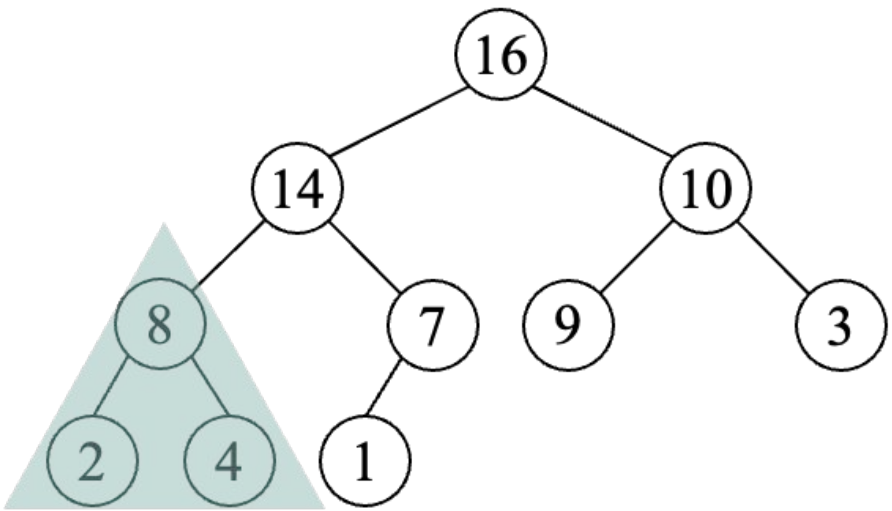
- Like merge sort
  - Running time is $O(n \lg n)$
- Like insertion sort
  - Heapsort sorts in place.

- The shape of a (binary) heap
  – A nearly *complete binary tree*.
  – Complete binary tree is in which all leaves have the same depth and all internal nodes have degree 2.



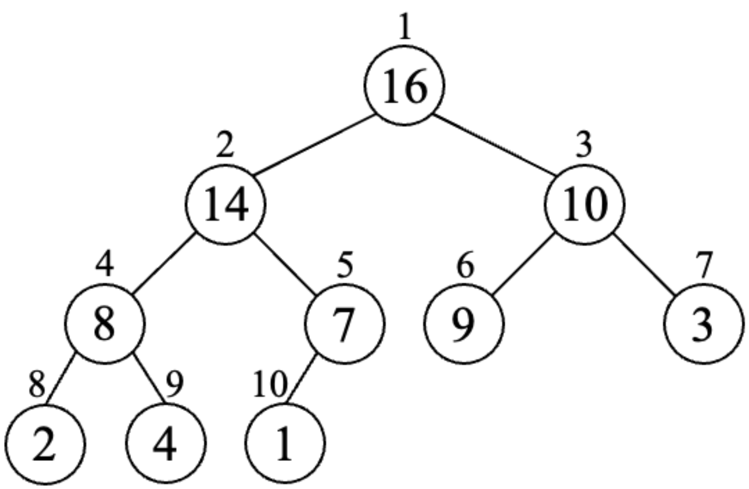
- Heap property
  - 2 kinds of binary heaps
    - *max-heaps* and *min-heaps*

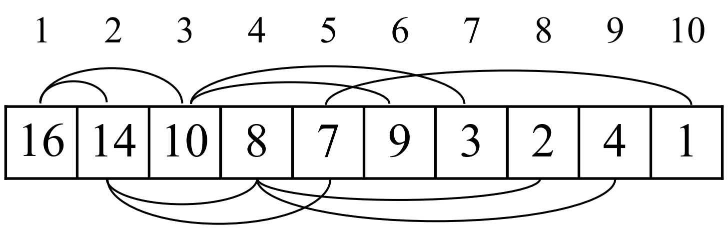
- ***max-heap property***
  - $A[\mathrm{PARENT}(i)] \geq A[i]$
    - The parent is bigger than or equal to its child.
    - The root node has the largest element.
    - The root of any subtree has the largest element among the subtree.



- ***min-heap property***
  - $A[\mathrm{PARENT}(i)] \leq A[i]$
    - A child is bigger than or equal to its parent.
    - The root node has the smallest element.

- A heap can be stored in an array.
  - The root is stored in $A[1]$.
  - All elements are stored in level order.



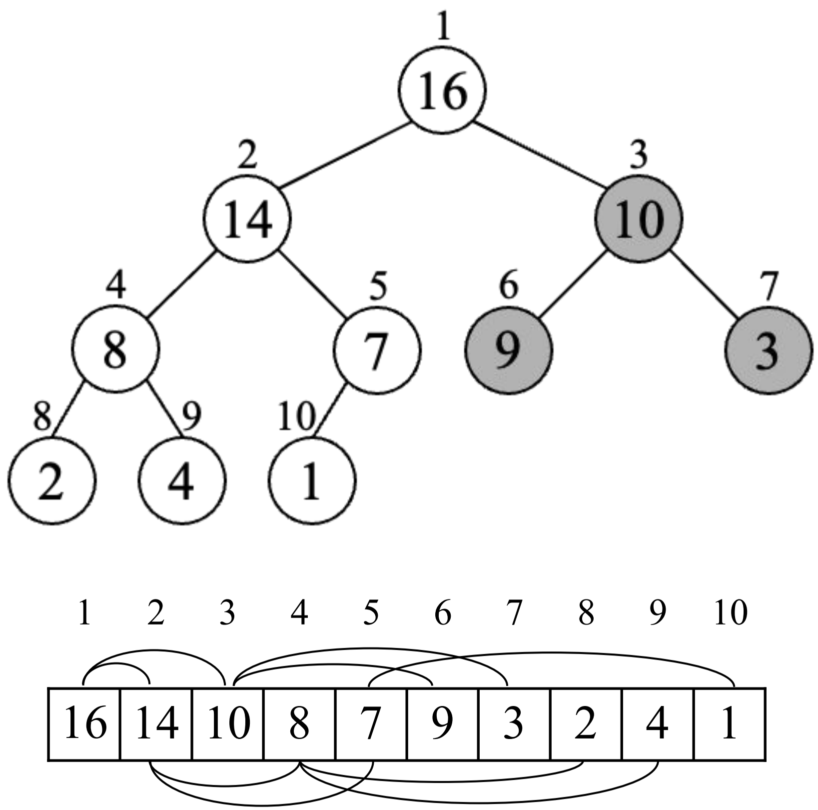



- PARENT($i$)

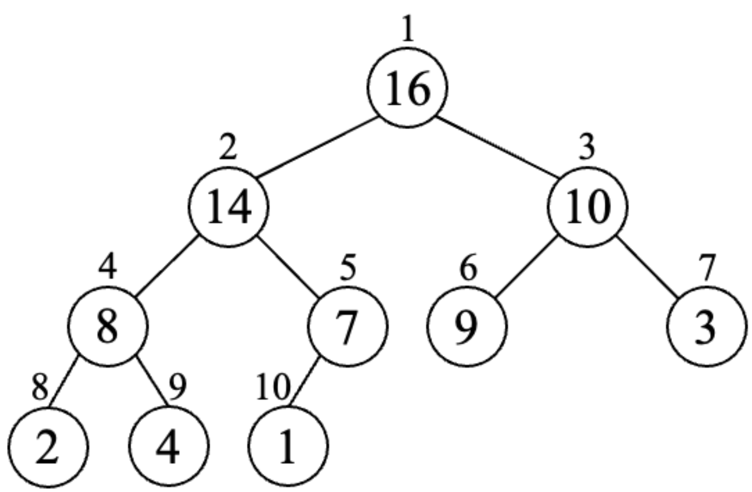
  **return** $\left\lfloor \frac{i}{2} \right\rfloor$
- LEFT($i$)

  **return** $2i$
- RIGHT($i$)

  **return** $2i + 1$

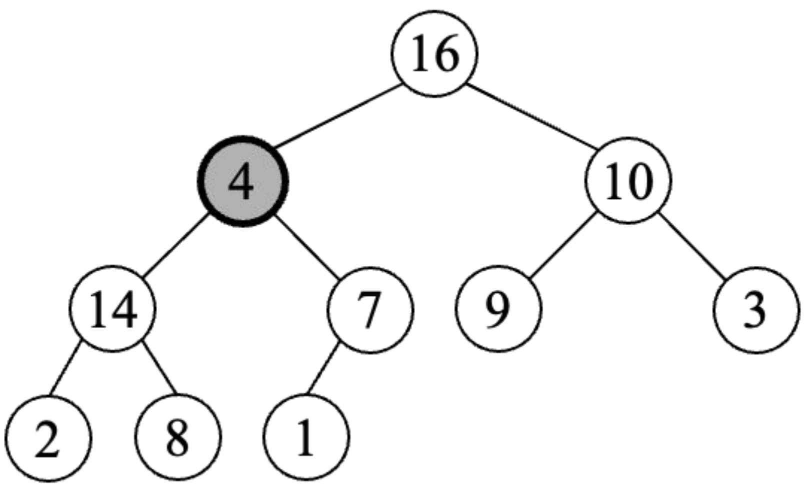


- The height of a node
  - The number of edges on the longest simple downward path from the node to a leaf.

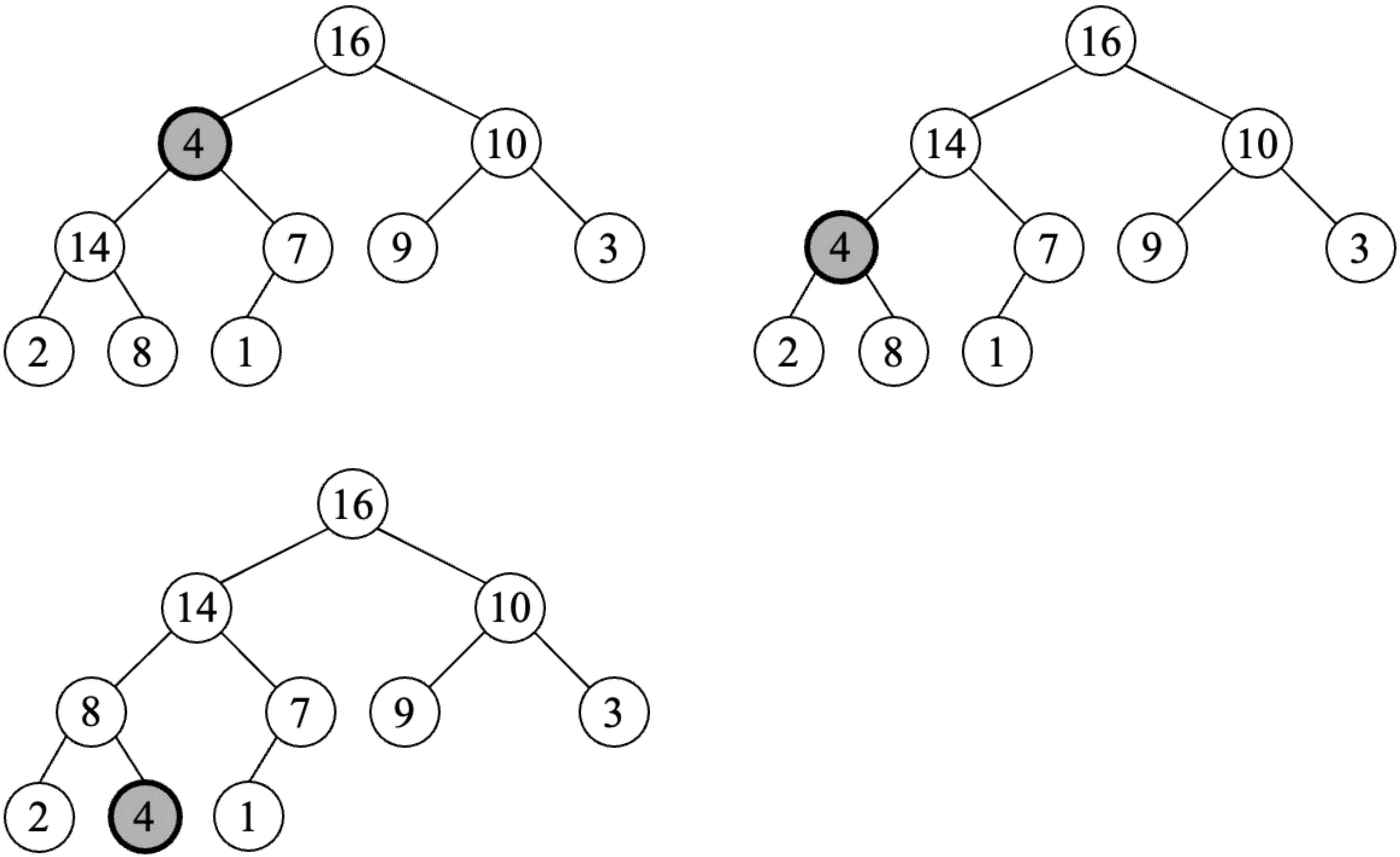


- The height of a heap
  - The height of the root.
  - $\Theta(\lg n)$
  - Since a heap of $n$ elements is based on a complete binary tree.

- Max-Heapify
  - Input : A node whose left and right subtrees are max-heaps, but the value at the node may be smaller than those of its children, thus violating the max-heap property.
  - Let the value at the node "float down" in the max-heap so that the subtree rooted at the node becomes a max-heap.



# Maintaining the heap property



- The running time of MAX-HEAPIFY
  - $T(n)$ where $n$ is the number of nodes in the subtree.
    - $\Theta(1)$ time to exchange values
    - $O(h) = O(\lg n)$ time in total

- BUILD-MAX-HEAP

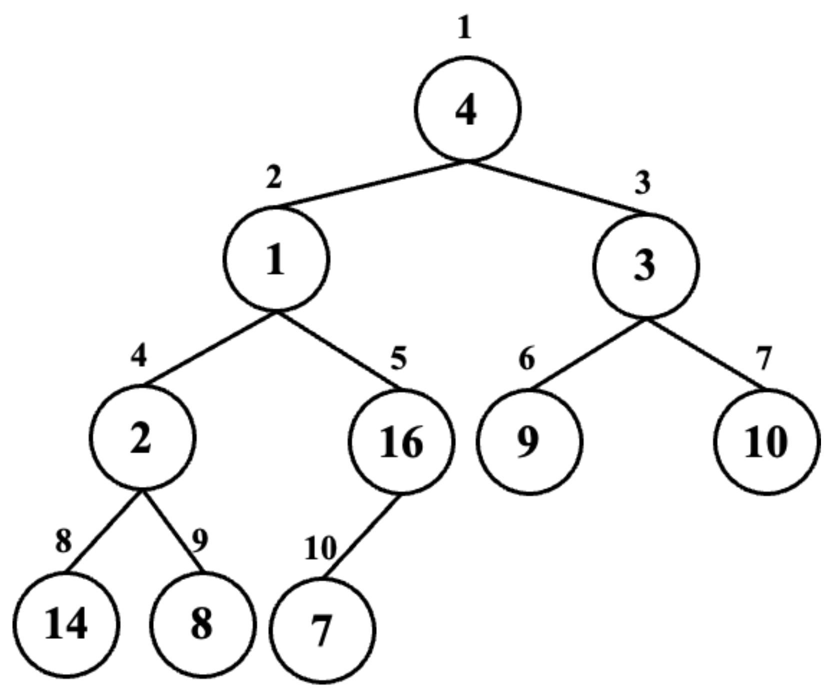
**BUILD-MAX-HEAP(*A*)**

1. *A.heap-size* = *A.length*
2. **for** *i* = $\lfloor$*A.length* / 2$\rfloor$ **downto 1**
3.     MAX-HEAPIFY(*A*, *i*)

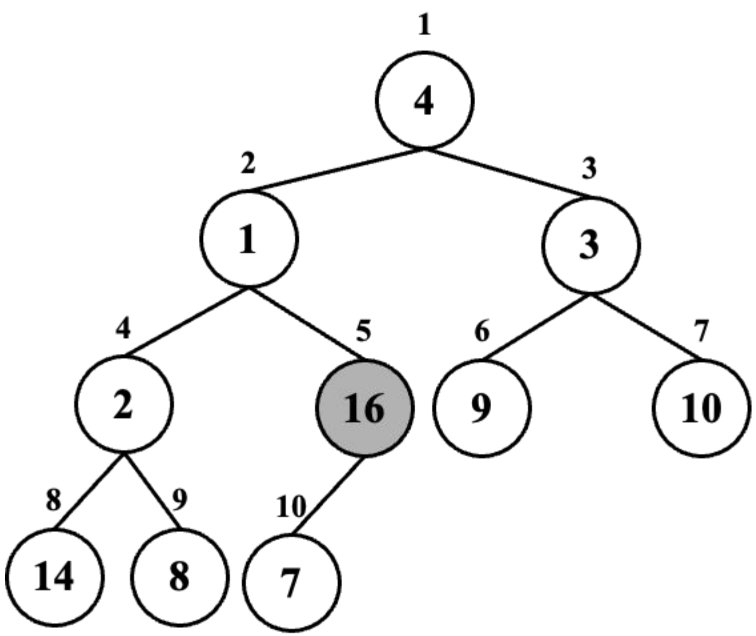
# Building a heap

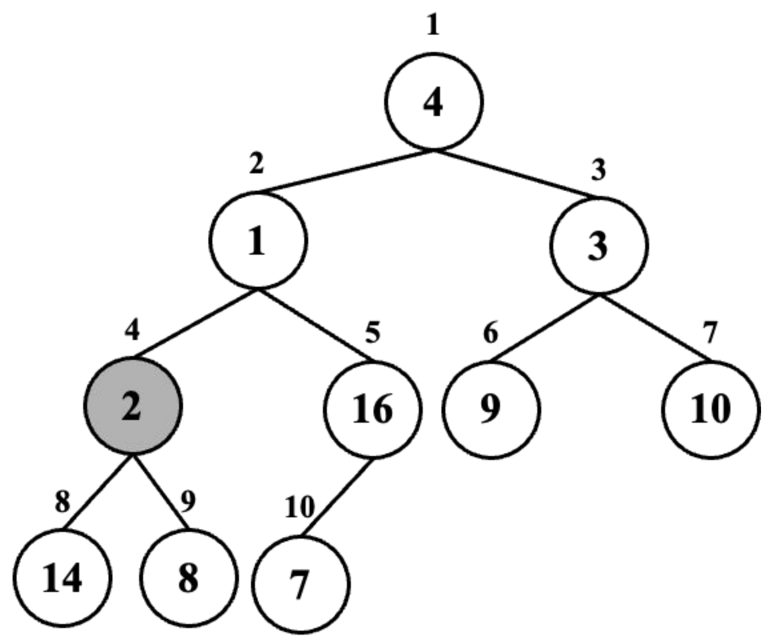
- BUILD-MAX-HEAP
  - The input array with 10 elements and its binary tree representation.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 1 | 3 | 2 | 16 | 9 | 10 | 14 | 8 | 7 |

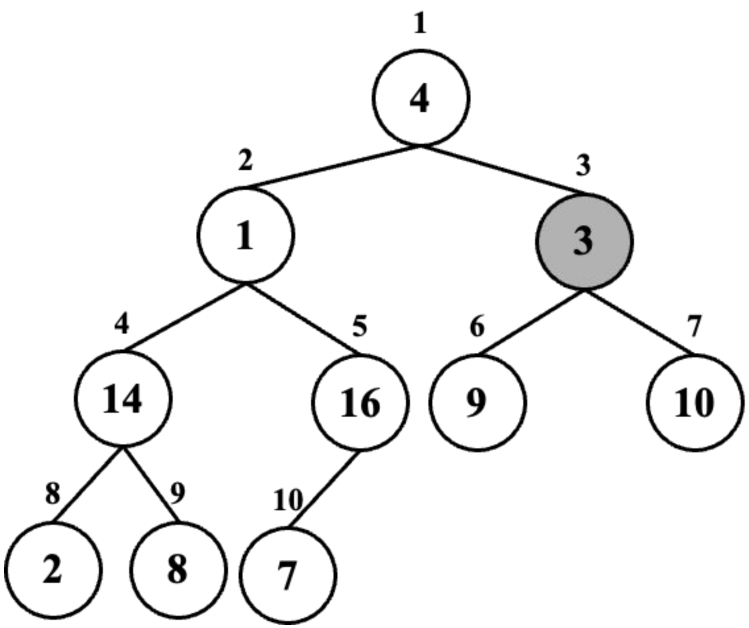


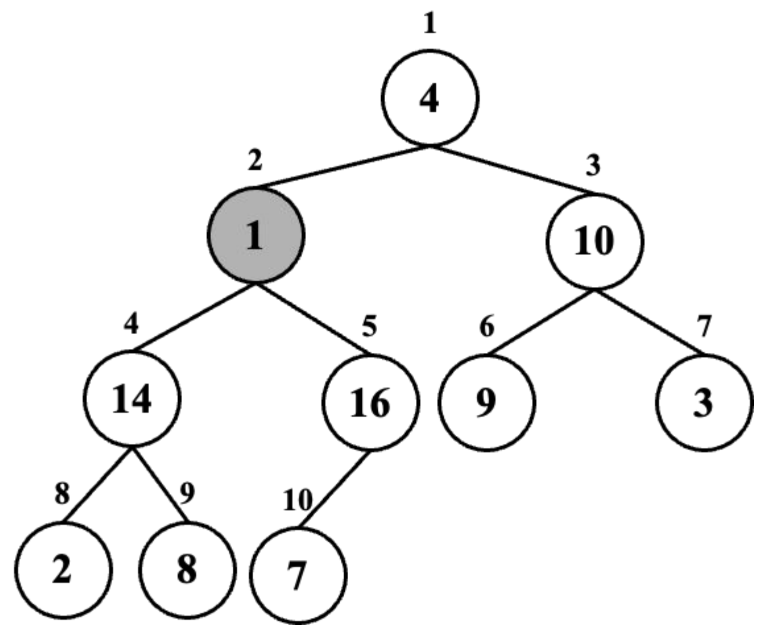
- Call MAX-HEAPIFY($A$, $i$) at the rightmost node that has the child from the bottom. $i = \lfloor A.length / 2 \rfloor$

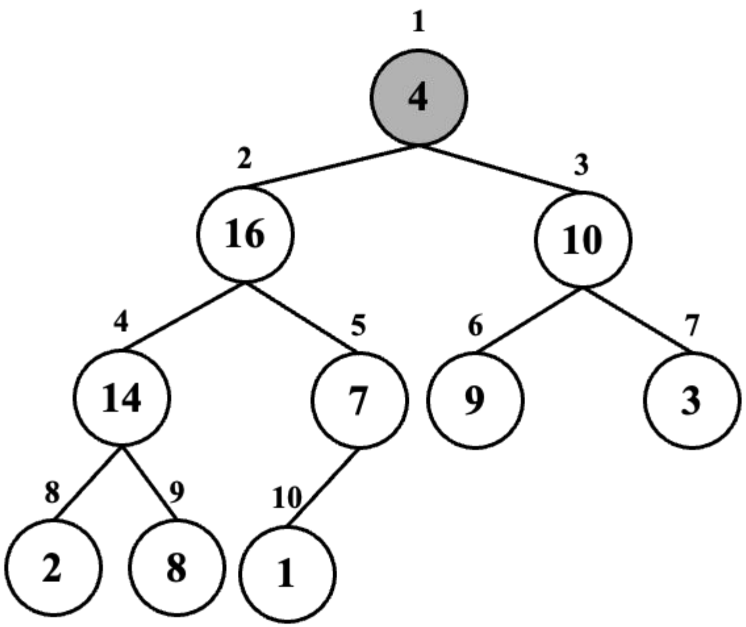


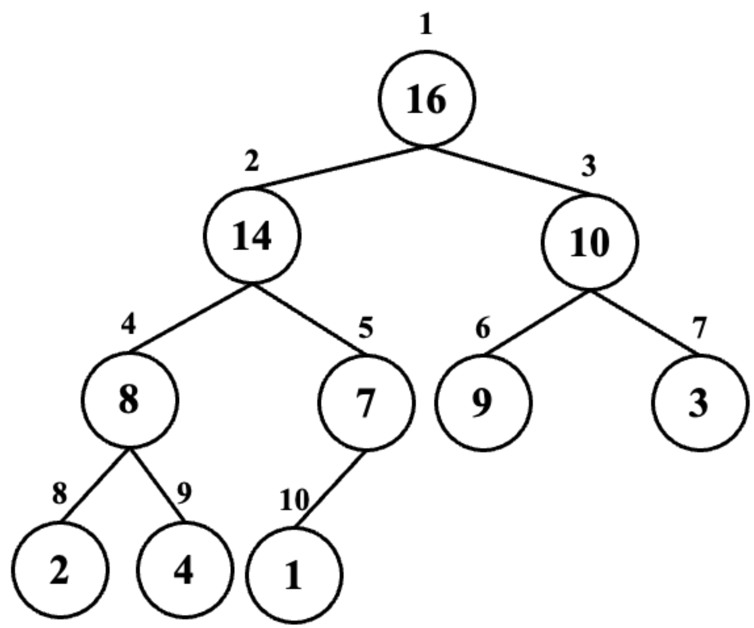



# Building a heap




1
4
2
16
3
10
4
14
5
7
6
9
7
3
8
2
9
8
10
1


1
16
2
14
3
10
4
8
5
7
6
9
7
3
8
2
9
4
10
1

- Running time
  - Upper bound
    - Each call to MAX-HEAPIFY costs $O(\lg n)$ time, and there are $\Theta(n)$ such calls, Thus, the running time is $O(n \lg n)$.

- Running time
  - Tighter bound
    - The time for MAX-HEAPIFY to run at a node varies with the height of the node in the tree, and the heights of most nodes are small.
    - Our tighter analysis relies on the properties that an $n$-element heap has height $\lfloor \lg n \rfloor$ and at most $\lceil n/2^{h+1} \rceil$ nodes of any height $h$.

- Tighter bound
  - The running time of MAX-HEAPIFY on a node of height $h$ is $O(h)$, so the total cost of BUILD-MAX-HEAP is

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}\right)$$

  - The last summation can be evaluated as follows.

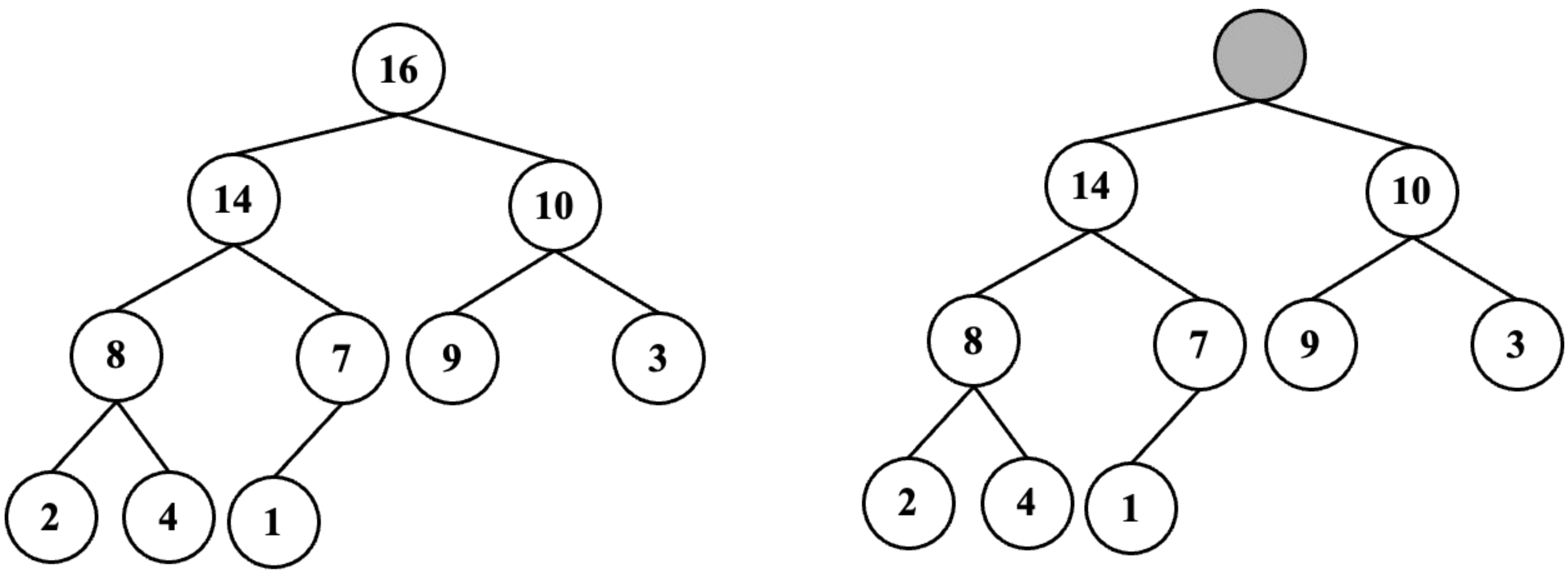
$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2$$

- Thus, the running time of BUILD-MAX-HEAP can be bounded as

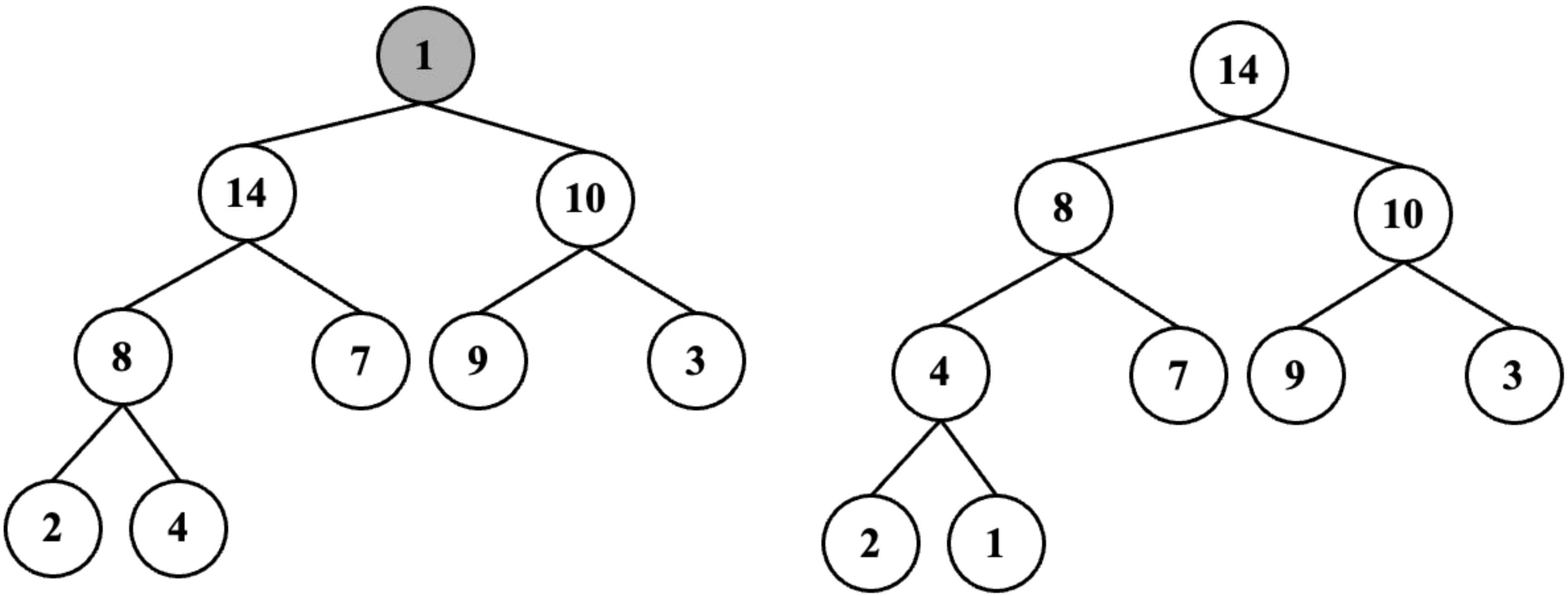
$$O\left(n\sum_{h=0}^{\lfloor \lg n \rfloor}\frac{h}{2^h}\right) = O\left(n\sum_{h=0}^{\infty}\frac{h}{2^h}\right) = O(n)$$

- Hence, we can build a max-heap in linear time.

- Extract-Max
  - Remove the maximum element from a heap
  - Restore the structure to a heap

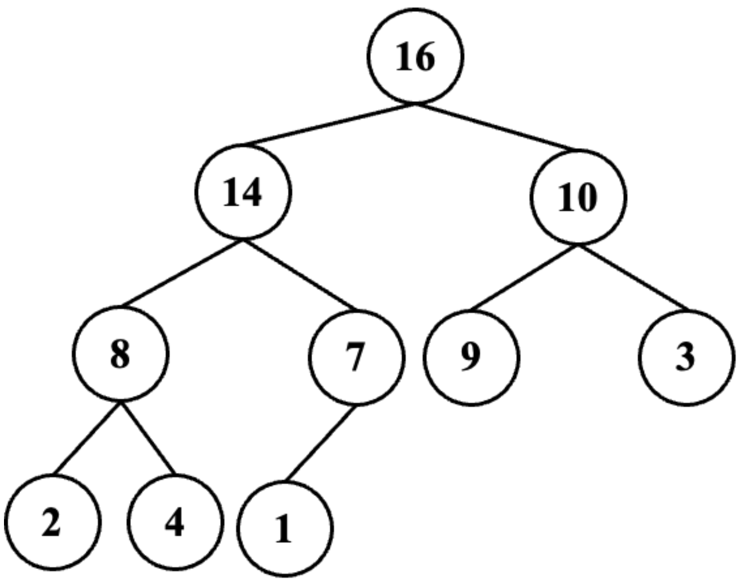


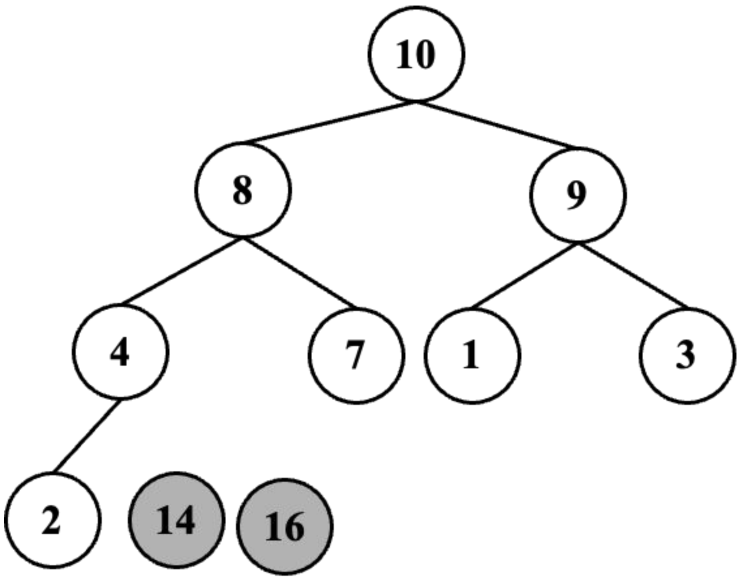
- Extract-Max
  - Restore the structure to a heap

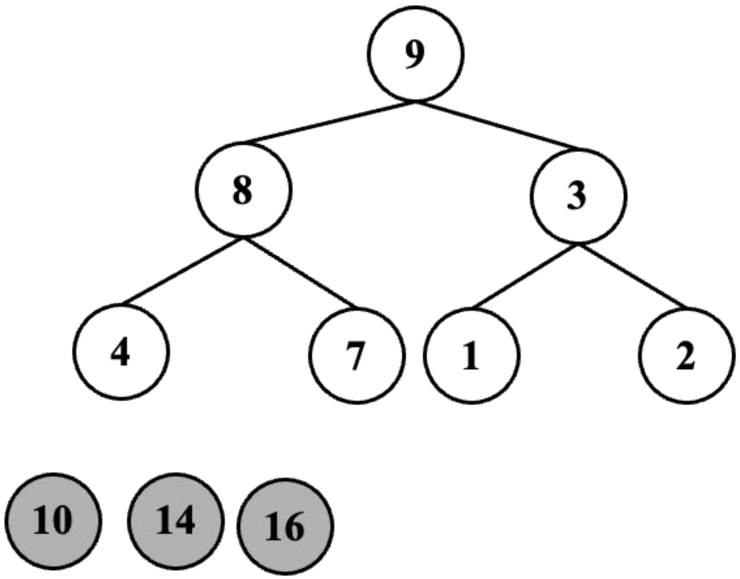


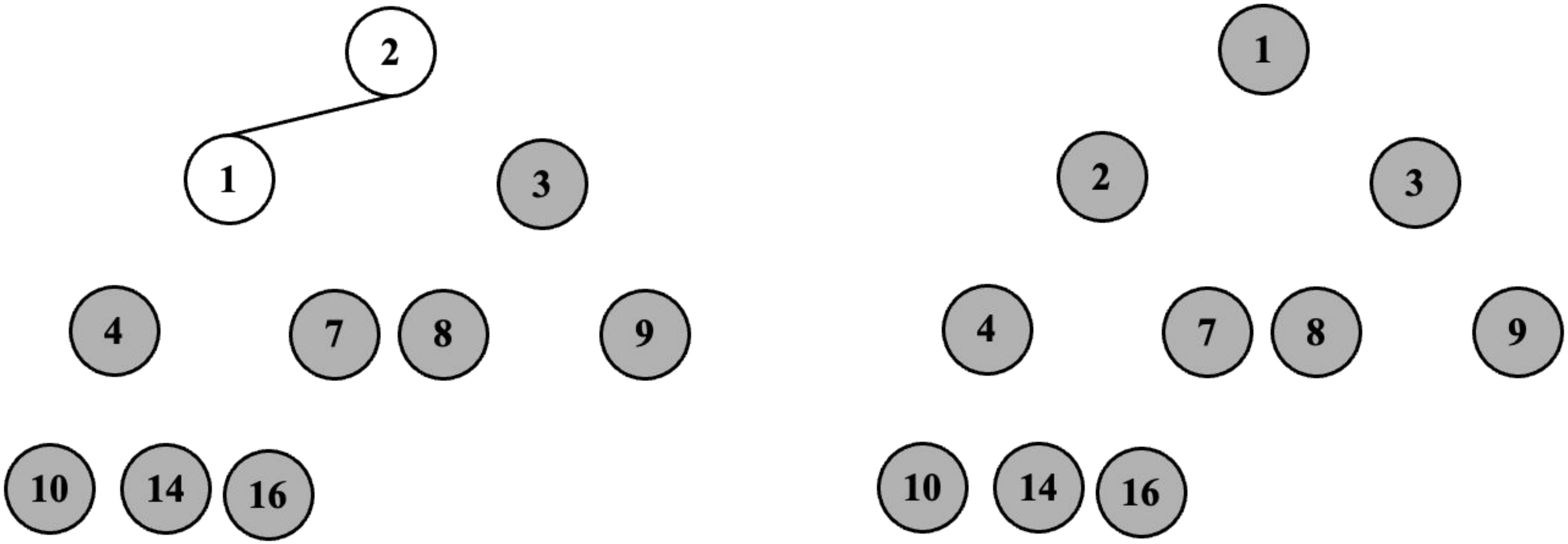

  - $O(\lg n)$

- The Heapsort algorithm
  - BUILD-MAX-HEAP on $A[1..n]$
    - $O(n)$ time.
  - Extract Max for $n$ times
    - $O(n\lg n)$ time.


16
14
10
8
7
9
3
2
4
1
14
8
10
4
7
9
3
2
1
16


10
8
9
4
7
1
3
2
14
16


9
8
3
4
7
1
2
10
14
16




| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 7 | 8 | 9 | 10 | 14 | 16 |

**HEAPSORT(*A*)**

```
BUILD-MAX-HEAP(A)
for i = A.length downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size - 1
    MAX-HEAPIFY(A, 1)
```

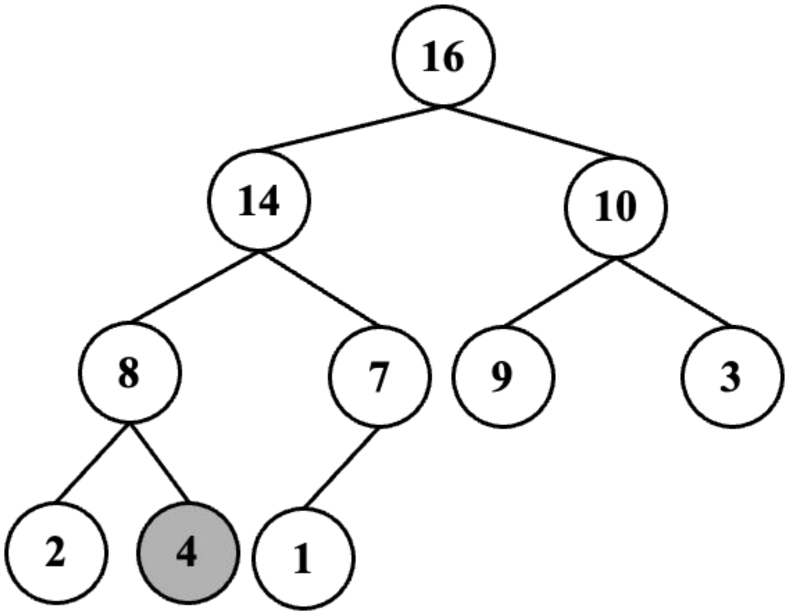
- The running time of Heapsort
  - $O(n\lg n)$
    - BUILD-MAX-HEAP: $O(n)$
    - MAX-HEAPFY: $O(\lg n)$

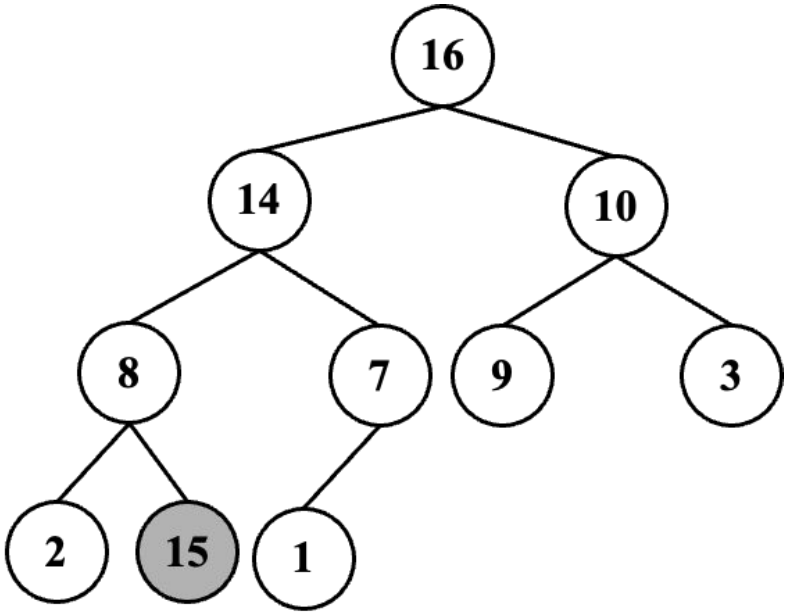
- Priority Queue
  - A data structure for maintaining a set $S$ of elements, each with an associated value called a key.
  - A max-priority queue operations.
    - INSERT($S$, $x$) inserts the element $x$ into the set $S$.
    - MAXIMUM($S$) returns the element of $S$ with the largest key.
    - EXTRACT-MAX($S$) removes and returns the element of $S$ with the largest key.
    - INCREASE-KEY($S$, $x$, $k$) increases the value of element $x$'s key to the new value $k$.

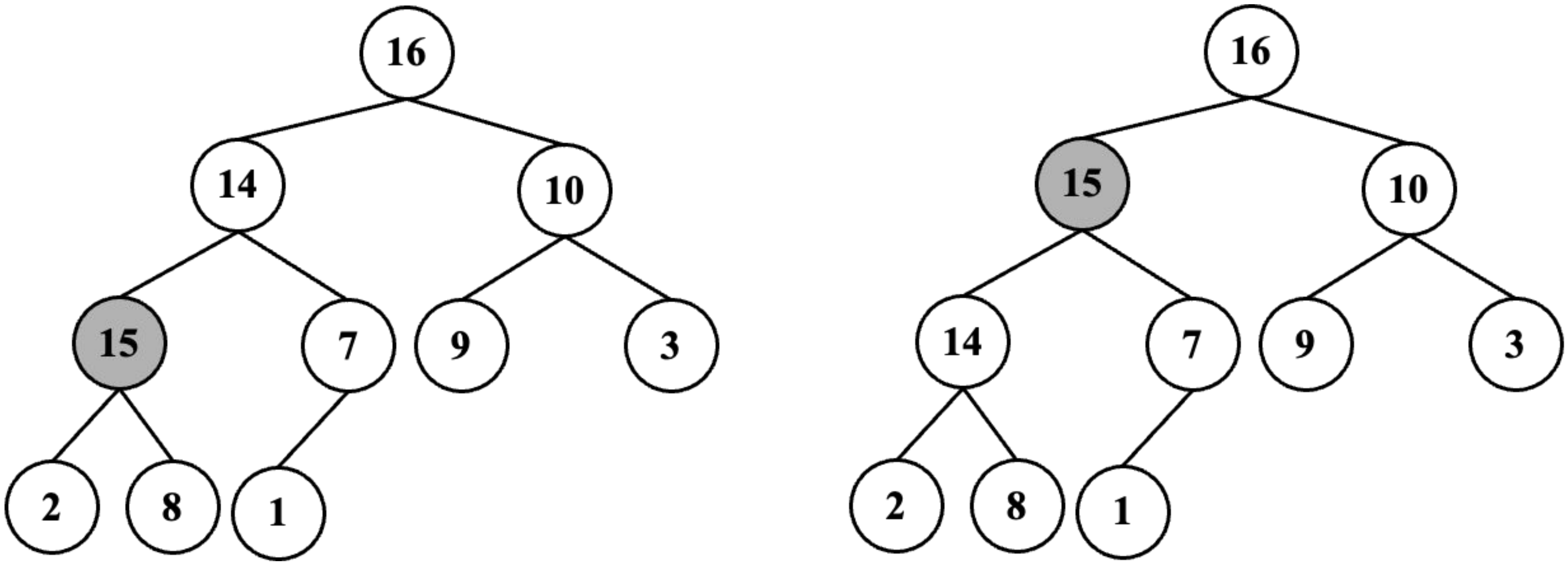
- MAXIMUM
  - Read the max value
  - $O(1)$ time

- EXTRACT-MAX
  - Remove the max value + MAX-HEAPIFY
  - $O(\lg n)$

- INCREASE-KEY




16
14
10
15
7
9
3
2
8
1
16
15
10
14
7
9
3
2
8
1

- HEAP-INCREASE-KEY
  - $O(\lg n)$ time.

- INSERT
  - $O(\lg n)$ time.

**MAX-HEAP-INSERT($A$, *key*)**
1. $A.heap\text{-}size = A.heap\text{-}size + 1$
2. $A[A.heap\text{-}size] = -\infty$
3. HEAP-INCREASE-KEY($A$, $A.heap\text{-}size$, *key*)

- **Exercise 6.3-1**
  - BUILD-MAX-HEAP on $A$ = <5, 3, 17, 10, 84, 19, 6, 22, 9>
- **Exercise 6.4-1**
  - HEAPSORT on $A$ = <5, 13, 2, 25, 7, 17, 20, 8, 4>
- **Exercise 6.5-8 (6.5-7 in the 2nd ed.)**
  - Give an algorithm for HEAP-DELETE($A$, $i$) in $O(\lg n)$ time.