Data Structures for Disjoint Sets

Slides from Heejin Park

Contents

- Disjoint-sets
- Disjoint-set operations
- An application of disjoint-set data structures
- Disjoint-set data structures

- *Disjoint sets*
  - Two sets A and B are disjoint if $A \cap B = \{\}$.

    Ex> $A = \{1, 2\}, B = \{3, 4\}$

  - Sets $S_1, S_2, \ldots, S_k$ are disjoint if every two distinct sets S_i and S_j are disjoint.

    Ex> $S_1 = \{1, 2, 3\},\ S_2 = \{4, 8\},\ S_3=\{5,7\}$

- A *collection* of disjoint sets
  - A set of disjoint sets is called a collection of disjoint sets.

    Ex> {{1, 2, 3}, {4, 8}, {5,7}}

  - Each set in a collection has a ***representative member*** and the set is identified by the member.

    Ex> {{1, 2, 3}, {4, 8}, {5,7}}

- A collection of *dynamic disjoint sets*
  - **Dynamic**: Sets are changing.
    - New sets are created.
      - {{1, 2, 3}, {4, 8}, {5,7}} ➔ {{1, 2, 3}, {4, 8}, {5,7}, {9}}
    - Two sets are united.
      - {{1, 2, 3}, {4, 8}, {5,7} } ➔ {{1, 2, 3}, {4, 8, 5, 7}}

- Disjoint-set operations
  - **MAKE-SET(*x*)**
  - **UNION(*x*, *y*)**
  - **FIND-SET(*x*)**

- MAKE-SET(x)
  - Given a member x, generate a set for x.
  - MAKE-SET(9)

    {{1, 2, 3}, {4, 8}, {5,7}} ➔ {{1, 2, 3}, {4, 8}, {5,7}, {9} }

- UNION(x, y)
  - Given two members x and y, unite the set containing x and another set containing y.
  - UNION(1,4)
  - {{1, 2, 3}, {4, 8}, {5,7}} → {{1, 2, 3, 4, 8}, {5,7}}
- FIND-SET(x)
  - Find the representative of the set containing x.
  - FIND-SET(5): 7

- Problem
  - ***Developing data structures*** to maintain a collection of dynamic disjoint sets supporting disjoint-set operations, which are MAKE-SET(x), UNION(x, y), FIND-SET(x).

- Parameters for running time analysis
  - #Total operations: m
  - #MAKE-SET ops: n
  - #UNION ops: u
  - #FIND-SET ops: f
  - $m = n + u + f$

- $u \leq n - 1$
  - n is the number of sets are generated by MAKE-SET ops.
  - Each UNION op reduces the number of sets by 1.
  - So, after $n - 1$ UNION ops, we have only 1 set and then we cannot do UNION op more.

Assumption

- The first n operations are MAKE-SET operations.

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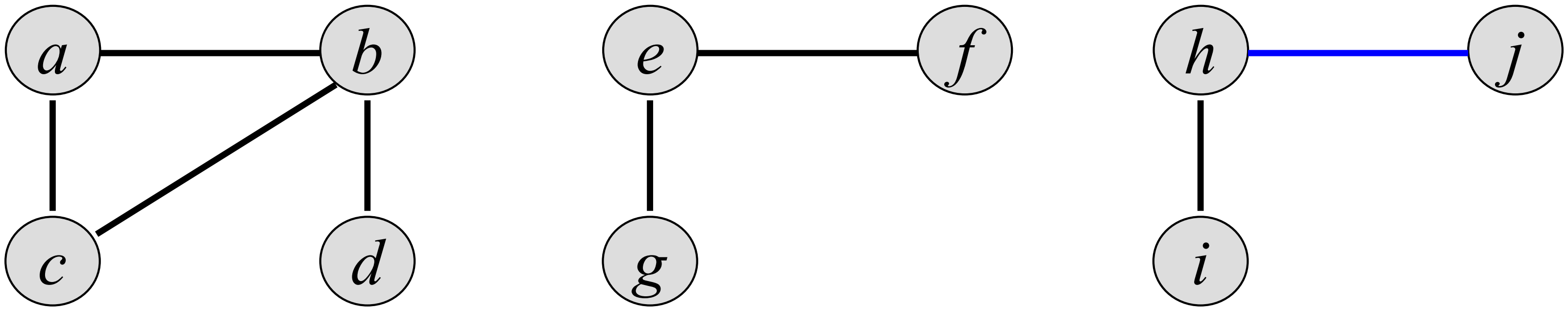
- Computing connected components (CC)
  - Static graph
    - Depth-first search: $\Theta(V+E)$
  - Dynamic graph
    - Depth-first search is inefficient.
    - Maintaining a disjoint-set data structure is more efficient.



$\{\{a,b,c,d\},\{e,f,g\}, \{h,i\}, \{j\}\}$

$\rightarrow$ $\{\{a,b,c,d\}, \{e,f,g\}, \{h, i, j\}\}$

Depth first search: $\Theta(V + E)$

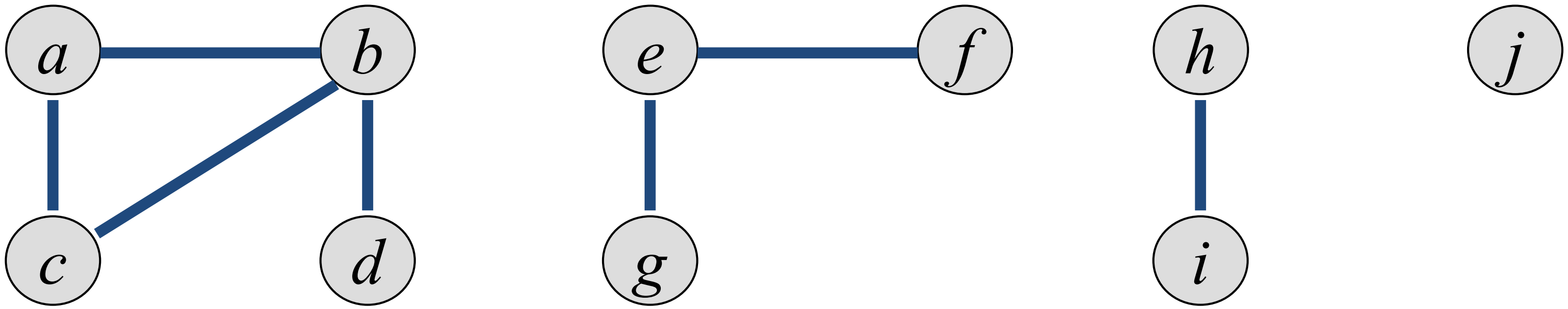
Disjoint-set data structures: $\mathrm{UNION}(h, j)$

- Computing CC using disjoint set operations

CONNECTED-COMPONENTS(G)

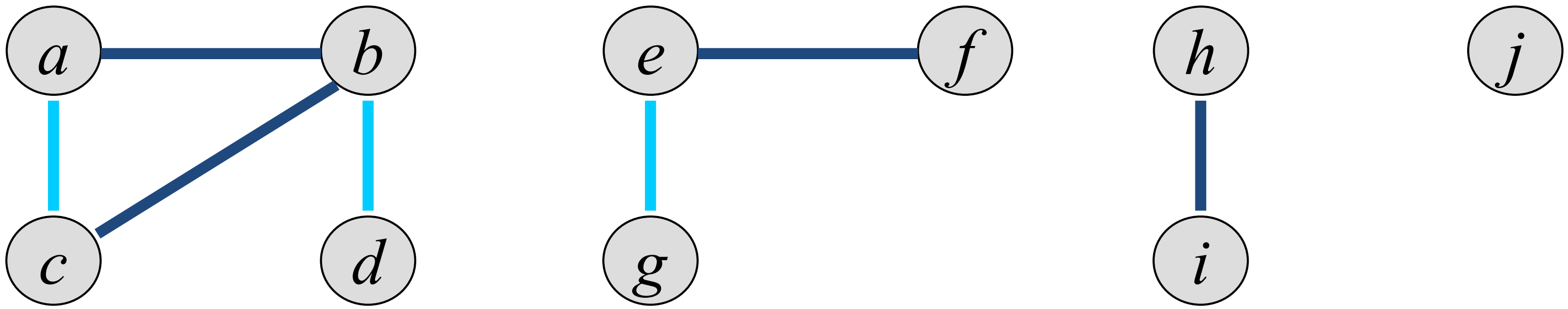
```
for each vertex v ∈ G.V
    MAKE-SET(v)

for each edge (u, v) ∈ G.E
    if FIND-SET(u) ≠ FIND-SET(v)
        UNION(u, v)
```

Connected component computation



Initial sets	$\{a\}$ $\{b\}$ $\{c\}$ $\{d\}$ $\{e\}$ $\{f\}$ $\{g\}$ $\{h\}$ $\{i\}$ $\{j\}$
(b,d)	$\{a\}$ $\{b,d\}$ $\{c\}$ $\{e\}$ $\{f\}$ $\{g\}$ $\{h\}$ $\{i\}$ $\{j\}$
(e,g)	$\{a\}$ $\{b,d\}$ $\{c\}$ $\{e,g\}$ $\{f\}$ $\{h\}$ $\{i\}$ $\{j\}$
(a,c)	$\{a,c\}$ $\{b,d\}$ $\{e,g\}$ $\{f\}$ $\{h\}$ $\{i\}$ $\{j\}$

Connected component computation



(a,c)	$\{a,c\}$	$\{b,d\}$	$\{e,g\}$	$\{f\}$	$\{h\}$	$\{i\}$	$\{j\}$
(h,i)	$\{a,c\}$	$\{b,d\}$	$\{e,g\}$	$\{f\}$	$\{h,i\}$		$\{j\}$
(a,b)	$\{a,b,c,d\}$		$\{e,g\}$	$\{f\}$	$\{h,i\}$		$\{j\}$
(e,f)	$\{a,b,c,d\}$		$\{e,f,g\}$		$\{h,i\}$		$\{j\}$
(b,c)	$\{a,b,c,d\}$		$\{e,f,g\}$		$\{h,i\}$		$\{j\}$

SAME-COMPONENT(*u*, *v*)

```
if  FIND-SET(u) == FIND-SET(v)
        return TRUE
else return FALSE
```

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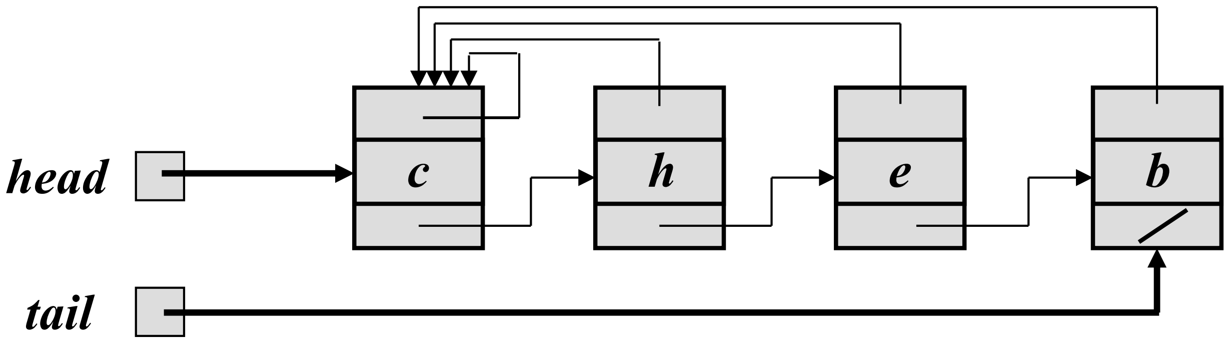
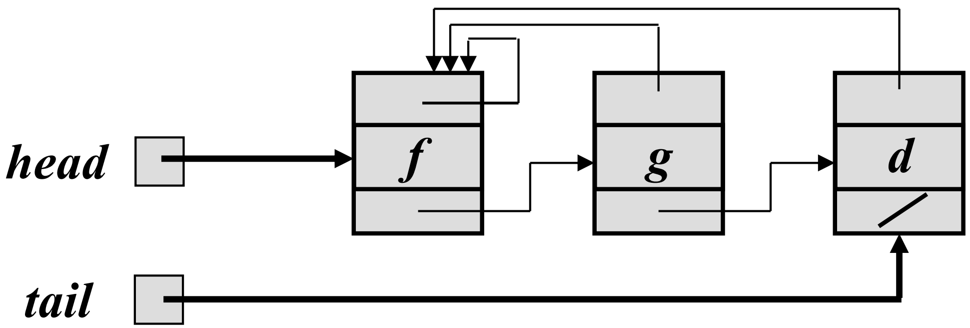
- Disjoint-set data structures
  - Linked-list representation
  - Forest representation

- Linked-list representation
  - Each set is represented by a linked list.
  - Members of a disjoint set are objects in a linked list.
  - The first object in the linked list is the representative.
  - All objects have pointers to the representative.

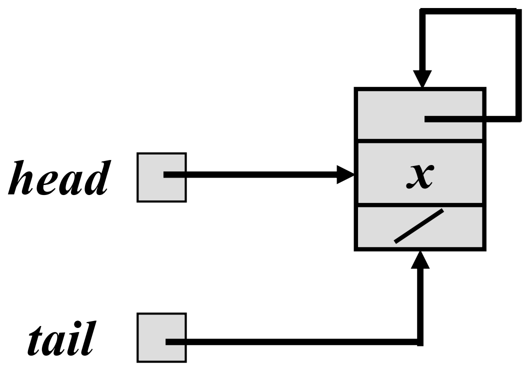
$\{\{b,c,e,h\}, \{d,f,g\}\}$: Two linked lists are needed.





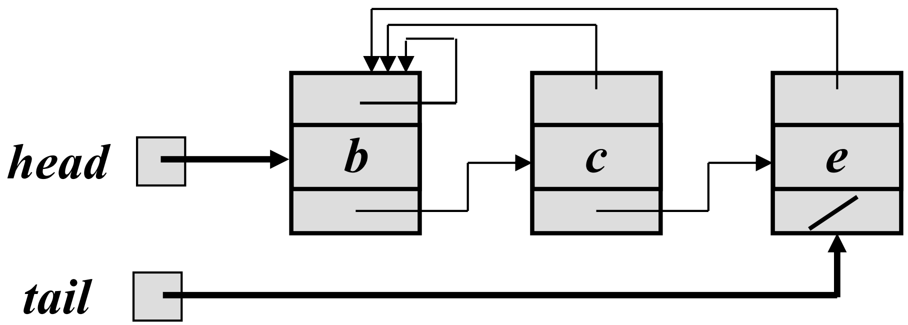
Linked-list representation

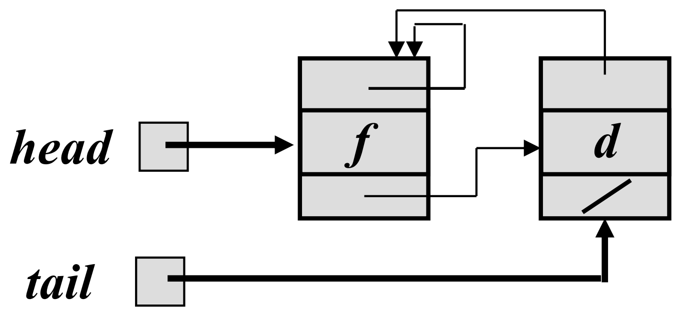
- MAKE-SET(x)
  - $\Theta(1)$

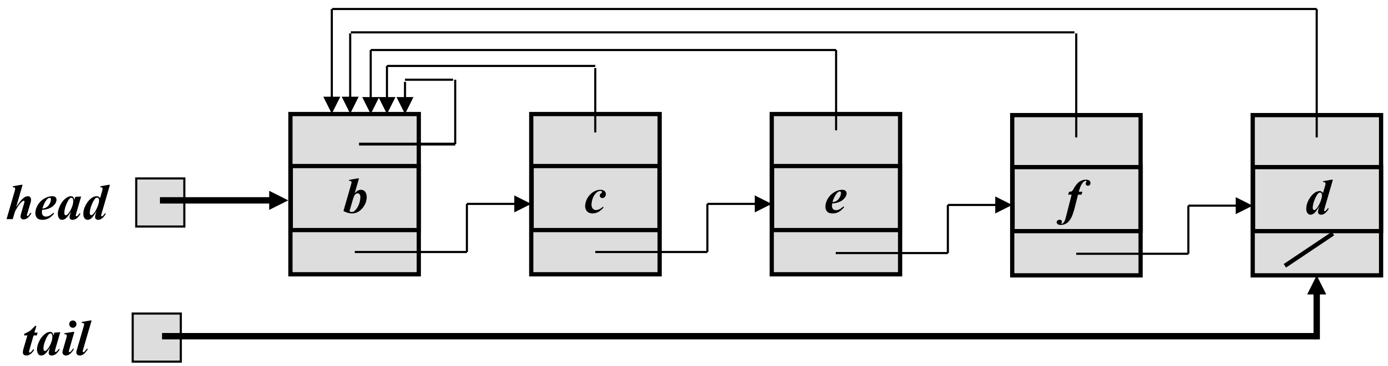


- FIND-SET(x)
  - $\Theta(1)$

- UNION(x,y): Attaching a linked list to the other

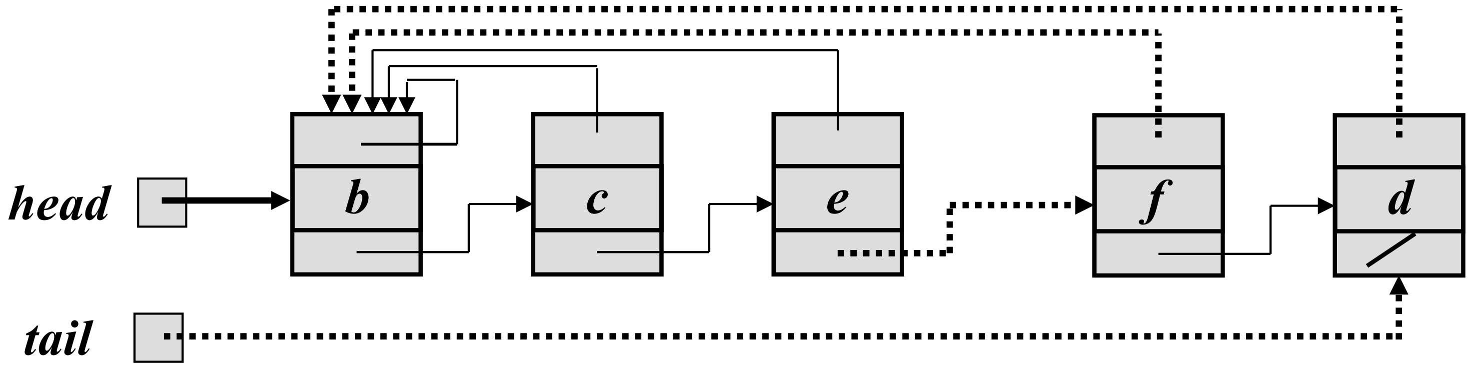






Linked-list representation

- UNION(x,y): Attaching a linked list to the other

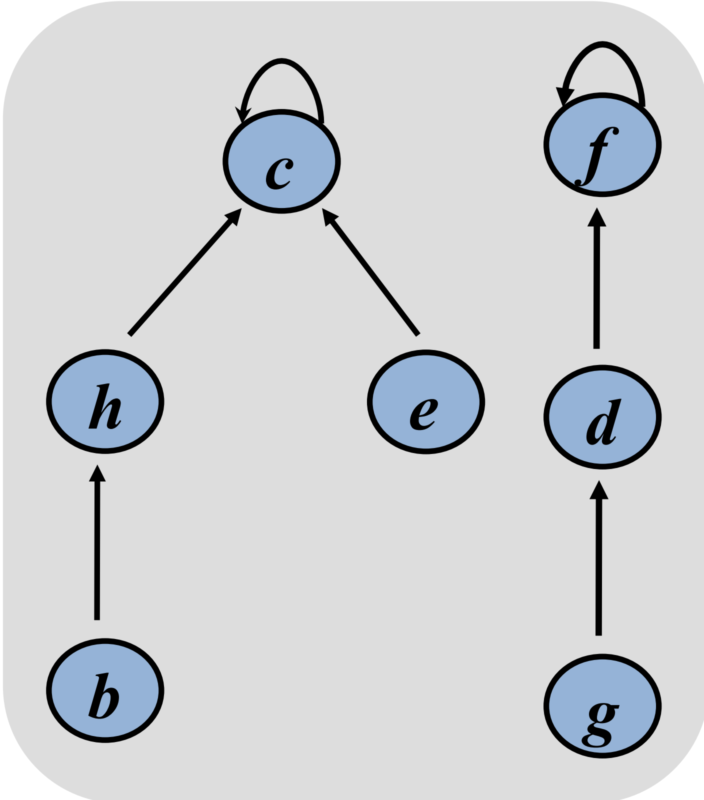


- $\Theta(m_2)$ time where m_2 is the number of objects in the linked list being attached.
  - Changing tail pointer & linking two linked lists: $\Theta(1)$
  - Changing pointers to the representative: $\Theta(m_2)$

- Running time for m $(= n + f + u)$ operations
  - Simple implementation of union
    - $O(n+f+n^2)$ time ➔ $O(m+n^2)$ time
      - Because $u < n$
  - A weighted-union heuristic
    - $O(n+f+n\lg n)$ time ➔ $O(m+n\lg n)$ time

- Forest representation
  - Each set is represented by a tree.
  - Each member points to its parent.
  - The root of each tree is the rep.

$$\{\{b,c,e,h\},\ \{f,d,g\}\}$$



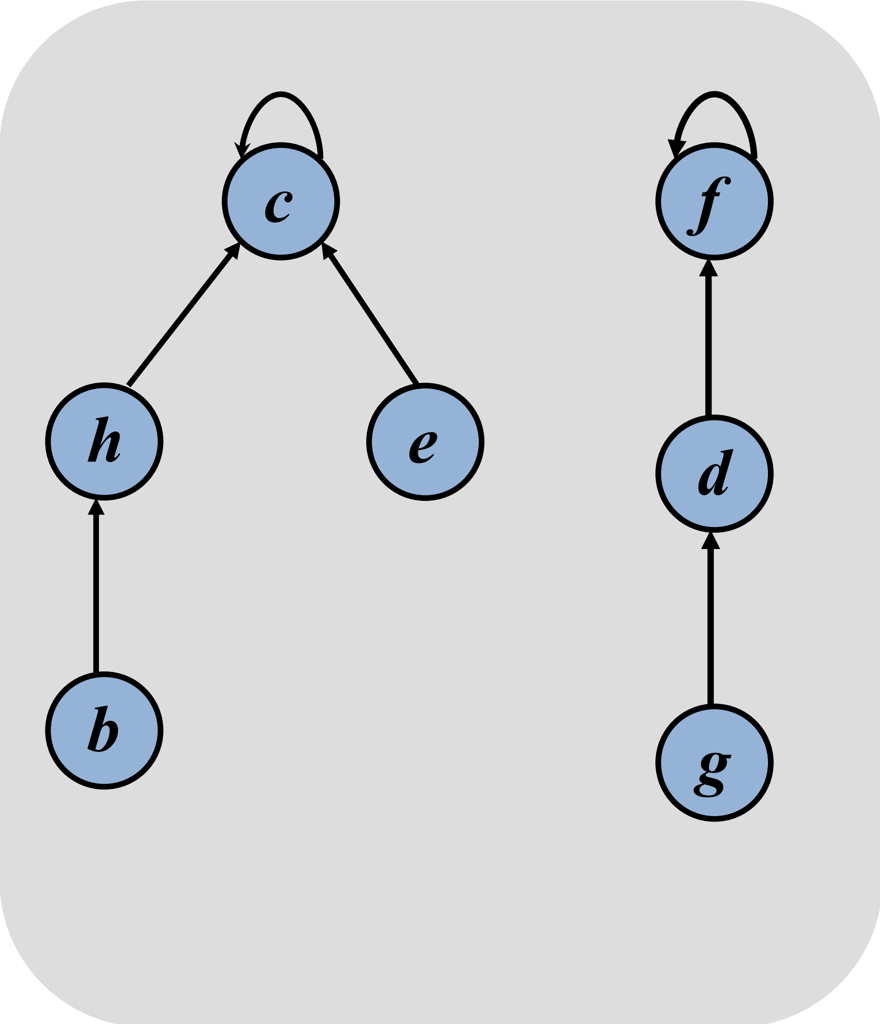
Forest representation

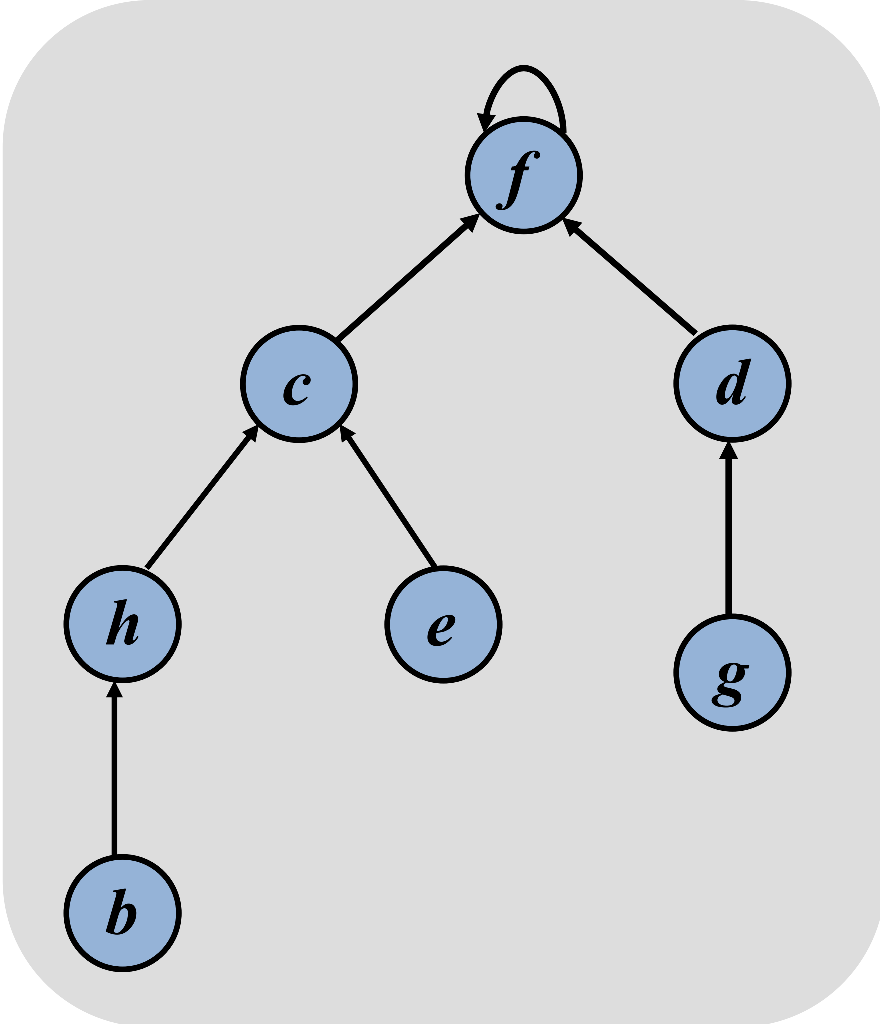
```
MAKE-SET(x)
    x.p = x
```

```
FIND-SET(x)
    if x == x.p
        return x
    else return FIND-SET(x.p)
```

- Union by rank
  - ***Idea*: **Attach the shorter tree to the higher tree.
  - Each node maintains a ***rank****,* which is an upper bound on the height of the node.
  - Compare the ranks of the two roots and attach the tree whose root's rank is smaller to the other.

Forest representation





Forest representation

MAKE-SET(x)

$x.p = x$
$x.rank = 0$

UNION(x, y)

LINK(FIND-SET(x), FIND-SET(y))

LINK(x, y)

if $x.rank > y.rank$
    $y.p = x$
else $x.p = y$
    if $x.rank == y.rank$
        $y.rank = y.rank + 1$

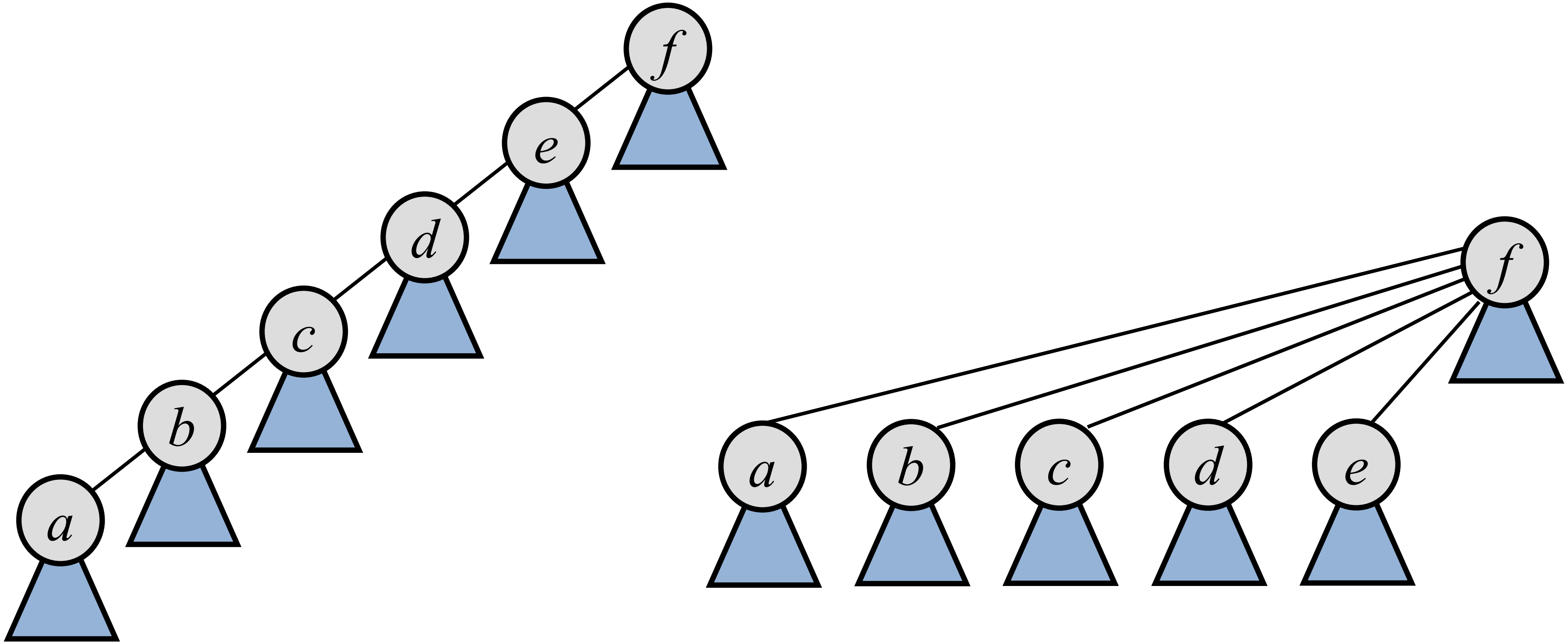
Forest representation

- Path compression
  - Change the parent to the root during FIND-SET(x).

FIND-SET(x)

```
if x ≠ x.p
    x.p = FIND-SET(x.p)
return x.p
```

Forest representation



Forest representation

- Worst case running time : $O(m\ \alpha(n))$
- $\alpha(n) \leq 4$:for all practical situations.