Piecewise Linear with Decay

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1 Introduction

A piecewise defined function is a function whose formula varies on different regions of the domain. Typically those regions are continuous intervals, i.e. sets of the form

$$[a, b] \triangleq \{x | a \leq x \leq b\}$$

or

$$(a, b) \triangleq \{x | a < x < b\}$$

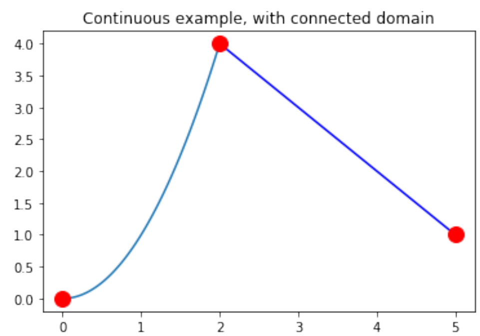
or some mixture of half-closed and half-opened intervals.

The best way to get a feel for piecewise defined functions is to see an example and the corresponding graph of the function. Consider the following piecewise defined function examples:

Note 1. *In the graphs below We use the red dots to indicate the beginning and end points of the domains of definition of the function.*

Example 1.

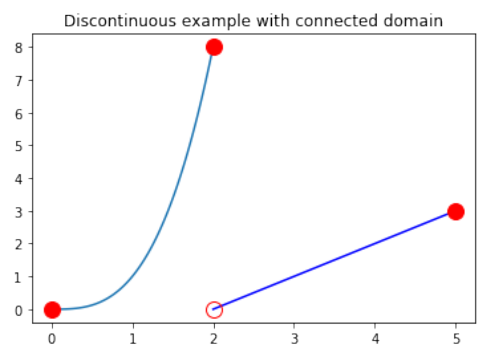
$$f(x) = \begin{cases} x^2 & \text{if } x \in [0, 2] \\ -x + 6 & \text{if } x \in [2, 5] \end{cases}$$



The function $f(x)$ is only defined on the closed interval $[0, 5]$, that is we can only calculate $f(x)$ for values of x between 0 and 5. The function is continuous, that is we can draw the graph without picking up our pencil. Also the domain of the function, namely $x \in [0, 5]$ is connected.

Example 2.

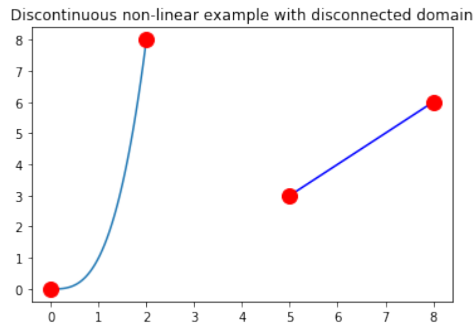
$$f(x) = \begin{cases} x^3 & \text{if } x \in [0, 2] \\ x - 2 & \text{if } x \in (2, 5] \end{cases}$$



This is an example of a discontinuous piecewise defined function because the graph has a break over $x = 2$. Note that the first definition covers the closed interval $[0, 2]$ whereas the second definition only covers the half-opened interval $(2, 5]$. However this function has a domain of $x \in [0, 5]$ which is a connected domain (there are no gaps in it).

Example 3.

$$f(x) = \begin{cases} x^3 & \text{if } x \in [0, 2] \\ x - 2 & \text{if } x \in [5, 8] \end{cases}$$



This is an example of a discontinuous piecewise defined function because the graph has a break from $x = 2$ to $x = 5$. Also his function has a domain of $x \in [0, 2] \cup [5, 8]$ which is a disconnected domain, i.e. there is a gap in the domain from 2 to 5.

For the purposes of this paper we only consider piecewise defined functions that are continuous and have a connected domain. In other words we only consider the case of Example 1. We do not consider discontinuous piecewise defined functions like Example 2. Neither do we consider piecewise defined functions whose domain is disconnects, as in Example 3.

2 Piecewise linear functions

A piecewise linear function is simply a piecewise defined function whose every piece is a linear function. There are several ways we could formally define this. The following is a more abstract definition.

Definition 1. A function $f(x)$ is a piecewise linear, continuous function, with finite connected domain iff:

1. $f(x)$ is continuous
2. The domain of $f(x)$, which we write as $\mathrm{Dom}(f)$, is a finite connected interval. In other words, it does not go to infinity on either side.
3. For every point $x_0 \in \mathrm{Dom}(f)$ there is an interval $[a, b]$ so that $x \in [a, b]$ and $f(x)$ restricted to $[a, b]$ is a linear function.

Note 2. *We only consider piecewise linear, continuous functions with connected domain. Thus we shorten that long phrase to piewewise linear function.*

A more concrete definition of a piecewise linear function is the following.

Definition 2 (Concrete definition of piecewise linear)**.** A function $f(x)$ is a piecewise linear function iff it can be written as

$$f(x) = \begin{cases} m_1 x + b_1 & \text{if } x \in [a_1, a_2] \\ m_2 x + b_2 & \text{if } x \in [a_2, a_3] \\ \vdots & \vdots \\ m_{n-1} x + b_{n-1} & \text{if } x \in [a_{n-1}, a_n] \end{cases}$$

where $a_1 < a_2 < \ldots < a_n$.

Note 3. *These two definitions are equivalent. The second definition is much more useful for computational purposes, the first would be more useful for the purposes of proving things.*

Note 4. *In Definition 2, the slope m_i and y intercept b_i in the function definition on the interval $[a_i, a_{i+1}]$ can be calculated by knowing $y_i = f(a_i)$ and $y_{i+1} = f(a_{i+1})$. We can show the formula by noting that the function is linear from the point (a_i, y_i) to the point (a_{i+1}, y_{i+1}). Then the slope m_i is:*

$$m_i = \frac{y_{i+1} - y_i}{a_{i+1} - a_i}$$

and b_i can be found by plugging either point into the equation of the line $y = mx + b$ and solving for b. If we use (a_i, y_i) we get:

$$b_i = y_i - ma_i$$

Based Note 4 we see that a piecewise linear function can be defined by giving a list of a_i's and y_i's. Therefore we have a simply computational definition of a piecewise linear function.

Definition 3 (Computational)**.** A piecewise linear function is defined by $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{y} = (y_1, y_2, \ldots y_n)$, where $a_1 < a_2 < \ldots < a_n$. The formula for the piecewise linear function defined by $\vec{a}$ and $\vec{y}$ is given by

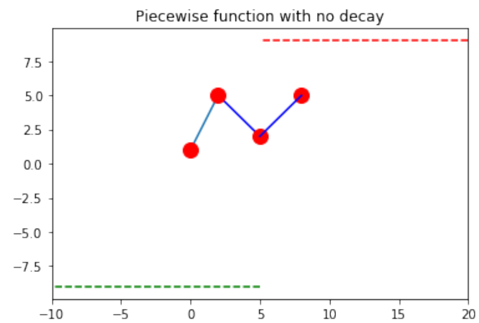
$$f_{\vec{a},\vec{y}}(x) \triangleq \begin{cases} m_1 x + b_1 & \text{if } x \in [a_1, a_2] \\ m_2 x + b_2 & \text{if } x \in [a_2, a_3] \\ \vdots & \vdots \\ m_{n-1} x + b_{n-1} & \text{if } x \in [a_{n-1}, a_n] \end{cases}$$

where:

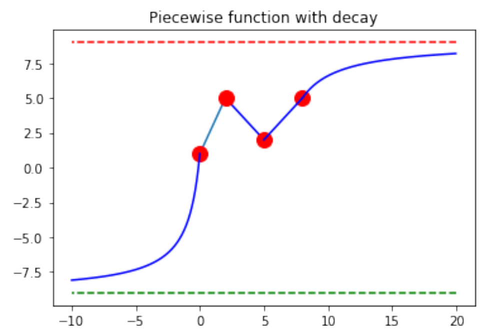
$$\begin{aligned} m_i &= \frac{y_{i+1} - y_i}{a_{i+1} - a_i} \\ b_i &= y_i - mx_i \end{aligned}$$

3 Piecewise linear with decay

Given a piecewise linear function (that is continuous with connected finite domain), is there a natural way to extend it to positive and negative infinity given asymptotes on both sides? For example consider the graph shown below:



The red dotted line on top represents the asymptote on the right and the green dotted line on the bottom represents the asymptote to the left. Can we extend our function continuously so that the graph approaches the given asymptotes? There are many ways we can, consider:



This graph extends the previous one with the desired asymptotic behavior. However there are many other choices as well. Looking at the graph above notice that the slope of the graph where the left hand decay begins does not match the slope of the piecewise linear piece beside it. If we want the slope of the decay function to match the slope of the linear piece the decay connects with, this limits our choices more. In fact if we:

1. specify the decay function to use (e.g. linear, exponential, etc), and
2. specify the resulting function must be continuous, and
3. specify that the slopes of the decays must match the slope of the piecewise linear function on the edges of the domain ...

then we uniquely determine the formula for the decay function to use. Let's collect that idea in Definition 5. However first we need to define the formulas for our decay functions.

Definition 4 (Decay functions)**.** We define the exponential decay function as:

$$\mathfrak{e}_B(x) \triangleq Ae^{Bx} + C$$

and we define the power decay function of degree n as:

$$\mathfrak{p}_{k,B}(x) \triangleq \frac{A}{(x-B)^k} + C$$

Note 5. *The parameter C in both decay functions represents the limiting value the function is decaying towards.*

Definition 5. Given $f(x)$ a piecewise linear function (continuous on a finite connected domain $[a,b]$) and

1. a left hand asymptotic value $y_{-\infty}$
2. a right hand asymptotic value y_{∞}
3. a decay function type (either exponential for fixed b, or power for fixed b and k)

an <u>extension with decay</u> of $f(x)$ with decay is a function $\overline{f}(x)$ that:

1. the extension $\overline{f}$ agrees with f on the domain $[a,b]$
2. the extension restricted to $(-\infty, a]$ matches the decay function type
3. the extension restricted to $[b, \infty)$ matches the decay function type
4. the extension is continuous
5. the slope of the extension at the points $x = a$ and $x = b$ are well defined.

We now have our main result:

Theorem 1. *Given a piecewise linear function (continuous on a finite connected domain $[a,b]$) such that the slope at $x = a$ is not zero, and similarly for $x = b$, and*

1. *a left hand asymptotic value $y_{-\infty}$*
2. *a right hand asymptotic value y_{∞}*
3. *a decay function type (either exponential for fixed B, or power for fixed B and k)*

then there exists a unique extension with decay $\overline{f}$ of f.

We prove this theorem by proving the two cases separately

Proof. Proven in 2 cases, Theorem 2 and 3. □

Theorem 2 (Power Decay Version of Theorem 1)**.** *Theorem 1 is true when the decay function is a power function, i.e.* $\mathfrak{p}_{k,B}(x)$*. Furthermore the parameters for the decay function on the left hand side decay function is given by:*

$$\begin{aligned} B &= a_1 + \frac{k}{m_1}(y_1 - B) \\ A &= (y_1 - C)(a_1 - B)^k \end{aligned}$$

and the parameters for the right hand side decay function are given by:

$$\begin{aligned} B &= a_n + \frac{k}{m_{n-1}}(y_n - B) \\ A &= (y_n - C)(a_n - B)^k \end{aligned}$$

Where $\vec{a} = (a_1, \ldots, a_n)$*,* $\vec{y} = (y_1, \ldots, y_n)$ *and* m_i *are as Definition 3.*

Proof. We need only derive the formula for B and A given that we know the value of the decay function at point x_0 and the value of its derivative at x_0 and the limiting value, which is C. So given our decay function:

$$\mathfrak{p}_{k,B}(x) = \frac{A}{(x-B)^k} + C \tag{1}$$

and assume we know:

$$\begin{aligned} \mathfrak{p}_{k,B}(x_0) &= y_0 && (2) \\ \mathfrak{p}'_{k,B}(x_0) &= m_0 && (3) \\ C &= \text{the limiting value} && (4) \end{aligned}$$

we want to use those equations to solve for A, B. Notice that:

$$\mathfrak{p}'_{k,B}(x) = \frac{-kA}{(x-B)^{k+1}} \tag{5}$$

Using Equation 2 and the definition in Equation 1 we get:

$$\begin{aligned} \mathfrak{p}_{k,B}(x_0) &= y_0 && (6) \\ \frac{A}{(x_0-B)^k} + C &= y_0 && (7) \\ \frac{A}{(x_0-B)^k} &= y_0 - C && (8) \end{aligned}$$

Using Equation 3 and the definition of the slope in Equation 5 we see:

$$\begin{aligned}
\mathfrak{p}'_{k,B}(x_0) &= m_0 && (9)\\
\frac{-kA}{(x_0-B)^{k+1}} &= m_0 && (10)\\
\left(\frac{A}{(x_0-B)^k}\right)\left(\frac{-k}{x_0-B}\right) &= m_0 && (11)\\
\text{plug Equation 8 in} & && (12)\\
(y_0-C)\left(\frac{-k}{x_0-B}\right) &= m_0 && (13)\\
y_0-C &= -(x_0-B)\frac{m_0}{k} && (14)\\
(y_0-C)\frac{k}{m_0} &= -x_0+B && (15)\\
(y_0-C)\frac{k}{m_0}+x_0 &= B && (16)
\end{aligned}$$

With this we have solved for B, now we need to solve for A. We can use Equation 8 to do this.

$$\begin{aligned}
\frac{A}{(x_0-B)^k} &= y_0-C && (17)\\
A &= (x_0-B)^k(y_0-C) && (18)
\end{aligned}$$

Using Equations 16 and 18 for $x_0=a_1, y_0=y_1, m_0=m_1$ we get the left side decay formula for A and B. Similarly using those equations for a_n, y_n, m_{n-1} we get the right side decay formula for A and and B. □

Theorem 3 (Exponential Decay Version of Theorem 1)**.** *Theorem 1 is true when the decay function is an exponential function, i.e. $\mathfrak{e}_B(x)$. Furthermore the parameters for the decay function on the left hand side decay function is given by:*

$$\begin{aligned}
A &= (y_1-C)e^{-Ba_1}\\
B &= \frac{m_1}{y_1-C}
\end{aligned}$$

and the parameters for the right hand side decay function are given by:

$$\begin{aligned}
A &= (y_n-C)e^{-Ba_n}\\
B &= \frac{m_{n-1}}{y_n-C}
\end{aligned}$$

Where $\vec{a}=(a_1,\ldots,a_n)$, $\vec{y}=(y_1,\ldots,y_n)$ and m_i are as Definition 3.

Proof. As with the proof of Theorem 2 we simply need to prove the formulas for A and C when we have the conditions

$$\begin{aligned}
\mathfrak{e}_B(x_0) &= y_0 && (19)\\
\mathfrak{e}'_B(x_0) &= m_0 && (20)
\end{aligned}$$

Notice that the exponential decay function and its derivative are given by:

$$\mathfrak{e}_B(x) = Ae^{Bx} + C \tag{21}$$
$$\mathfrak{e}'_B(x) = ABe^{Bx} \tag{22}$$

We can use the formula in Equation 21 and plug into Equation 19 and get:

$$\mathfrak{e}_B(x_0) = y_0 \tag{23}$$
$$Ae^{Bx_0} + C = y_0 \tag{24}$$
$$Ae^{Bx_0} = y_0 - C \tag{25}$$
$$A = (y_0 - C)e^{-Bx_0} \tag{26}$$

Using the formula from Equation 22 with Equation 20 and solving for A we get:

$$\mathfrak{e}'_B(x_0) = m_0 \tag{27}$$
$$ABe^{Bx_0} = m_0 \tag{28}$$
$$B\left(Ae^{Bx_0}\right) = m_0 \tag{29}$$
$$B(y_0 - C) = m_0 \text{ (plugin Equation 25)} \tag{30}$$
$$B = \frac{m_0}{y_0 - C} \tag{31}$$

We can use Equations 26 and 31 when $x_0 = a_1$, $y_0 = y_1$ and $m_0 = m_1$ to get the formulas for the left hand side decay formulas for A and C:

$$A = (y_1 - C)e^{-Ba_1}$$
$$B = \frac{m_1}{y_1 - C}$$

And similarly we get the equations for the right hand side decay formula. □