Priorities to Pairwise

Bill Adams

Aug 2017

Abstract

Given a prioritiy vector $\vec{p}$ there is a method to create a pairwise comparison matrix $M_{\vec{p}}$ that has $\vec{p}$ as its priority vector. In this paper we provide context for why this calculation could be useful, the definition of the calculation itself, and a few example calculations.

1 Introduction

Given n items whose $n \times n$ pairwise comparison matrix is M, standard AHP theory uses the largest eigenvector calculation to find the priority vector $\vec{p}_M$ for those n items. In addition, standard AHP theory provides a way to work in reverse; i.e. given a priority vector $\vec{p}$ we can create a pairwise comparison matrix $M_{\vec{p}}$ whose largest eigenvector is $\vec{p}$.

This method of working in reverse can be useful in the following circumstance. If one knows the priority vector for a group of users, but the software that gives that priority does not provide the group pairwise comparison matrix, one can reconstruct a pairwise comparison matrix for the group that results in the given priority vector. This circumstance is what prompted the writing of this paper.

2 Calculation and proof

Given a priority vector $\vec{p} = (p_1, \ldots, p_n)$, we construct the matrix M whose priority vector is $\vec{p}$ by taking the matrix made of the ratios of the p_i's. We state this result as the following lemma.

Note 1. *The reason we use the ratio matrix can be easily seen with an example. If the priority vector is alt1=0.1, alt2=0.3, and alt3=0.6 we can easily see that alt3 is 6 times better than alt1, because alt3 has a score of 0.6 and alt1 has a score of 0.1 and the value of 6 is simply* $0.6/0.1$ *the ratio of alt3's score to that of alt1's score.*

Lemma 1. *Let $\vec{p} = (p_1, \ldots, p_n)$ be the priority vector of n items. If we define the $n \times n$ matrix $M_{\vec{p}} = (m_{ij})$ by the formula*

$$m_{ij} = \frac{p_i}{p_j}$$

then the priority vector for $M_{\vec{p}}$ is $\vec{p}$.

Proof. Let largest eigenvector of $M_{\vec{p}}$ can be calculated from the following formula

$$\lim_{n\to\infty} \frac{b^n}{\sum_{i=1}^n b_i^n}$$

where b^i is an n dimensional vector defined using the following recursive formula:

$$\begin{aligned} b^0 &= (1, \ldots, 1) \\ b^{i+1} &= M_{\vec{p}} \cdot b^i \end{aligned}$$

Thus we need only understand b^i. Let us begin by calculating b^1.

$$\begin{aligned}
b^1 &= M_{\vec{p}} \cdot b^0 \\
&= \begin{bmatrix} 1 & p_1/p_2 & \ldots & p_1/p_n \\ p_2/p_1 & 1 & \ldots & p_2/p_n \\ \vdots & vdots & \ddots & \vdots \\ p_n/p_1 & p_n/p_2 & \ldots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i=1}^n p_1/p_i \\ \sum_{i=1}^n p_2/p_i \\ \ldots \\ \sum_{i=1}^n p_n/p_i \end{bmatrix} \\
&= \begin{bmatrix} p_1 \sum_{i=1}^n 1/p_i \\ p_2 \sum_{i=1}^n 1/p_i \\ \vdots \\ p_n \sum_{i=1}^n 1/p_i \end{bmatrix} \\
&= \sum_{i=1}^n 1/p_i \begin{bmatrix} p_1 \\ p_2 \\ \ldots \\ p_n \end{bmatrix}
\end{aligned}$$

Since we are dividing by the sum of the elements of b^i in the limit, we can multiply or divide b^i by any constant and not change the limit. Thus effectively $b^1 = (p_1, \ldots, p_n)$.

Next we claim that $b^2 = (p_1, \ldots p_n)$. If that is the case, then we know that all of the b^i's for $i \geq 1$ are that as well, and we will have shown that the largest eigenvector of $M_{\vec{p}}$ is $\vec{p}$. Let us calculate the value of b^2 to complete the proof.

$$
\begin{aligned}
b^2 &= M_{\vec{p}} \cdot b^1 \\
&= \begin{bmatrix} 1 & p_1/p_2 & \dots & p_1/p_n \\ p_2/p_1 & 1 & \dots & p_2/p_n \\ \vdots & vdots & \ddots & \vdots \\ p_n/p_1 & p_n/p_2 & \dots & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix} \\
&= \begin{bmatrix} np_1 \\ np_2 \\ \vdots \\ np_n \end{bmatrix}
\end{aligned}
$$

Again, since we are dividing by the sum of the elements of b^2 in the limit, we can multiply or divide by any constant and not effect the limit calculation. Thus, effectively, b^2 is effectively $(p_1, \dots, p_n)$. $\square$

3 Examples

We calculate the matrix $M_{\vec{p}}$ for several priority vectors $\vec{p}$ below.

1. If $\vec{p} = (1/2, 1/3, 1/6)$, the ratio matrix $M_{\vec{p}}$ is:

$$
\begin{aligned}
M_{\vec{p}} &= \begin{bmatrix} 1 & \frac{1/2}{1/3} & \frac{1/2}{1/6} \\ \frac{1/3}{1/2} & 1 & \frac{1/3}{1/6} \\ \frac{1/6}{1/2} & \frac{1/6}{1/3} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 3/2 & 3 \\ 2/3 & 1 & 2 \\ 1/3 & 1/2 & 1 \end{bmatrix}
\end{aligned}
$$

2. If $\vec{p} = (3, 2, 1, 4)$ we can calculate the ratio matrix as below. Notice that the priority vector need not be normalized (i.e. its components need not sum to 1).

$$
\begin{aligned}
M_{\vec{p}} &= \begin{bmatrix} 1 & \frac{3}{2} & \frac{3}{1} & \frac{3}{4} \\ \frac{2}{3} & 1 & \frac{2}{1} & \frac{2}{4} \\ \frac{1}{3} & \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{4}{3} & \frac{4}{2} & \frac{4}{1} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \frac{3}{2} & 3 & \frac{3}{4} \\ \frac{2}{3} & 1 & 2 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & \frac{1}{4} \\ \frac{4}{3} & 2 & 4 & 1 \end{bmatrix}
\end{aligned}
$$