Homework 3: Dimitri Lezcano

Wednesday, October 7, 2020 12:08 AM

1) $\dot{x} = ax - bu$ $\quad x(t_0)$ given

$$J = \frac{1}{2} c [x_f]^2 + \frac{1}{2} \int_{t_0}^{t_f} [u(t)]^2 dt$$

$x, u, a, b, c \in \mathbb{R}$ $\quad \phi = \frac{1}{2} c (x(t_f))^2$

$\psi = 0$ $\quad L = \frac{1}{2} u^2$

$\Phi = \phi$

Define

$$H = \frac{1}{2} u^2 + \lambda (ax - bu)$$

E-L Equations:

1) $\dot{x} = ax - bu$

2) $\dot{\lambda} = -\nabla_x = -a\lambda$

$$\Rightarrow \frac{\dot{\lambda}}{\lambda} = -a \Rightarrow \ln(\lambda) = -\int_{t_0}^{t} a + \ln(\lambda(t_0))$$

$$\lambda(t) = \lambda(t_0) e^{-a(t - t_0)}$$

Optimality:

$$\nabla_u H = 0$$

$$\nabla_u H = u - b\lambda = 0 \Rightarrow u(t) = b\lambda(t)$$

$$\Rightarrow u(t) = \underbrace{b\lambda(t_0)}\, e^{-a(t-t_0)}$$

$$\Rightarrow u(t) = \underbrace{u(t_0)}_{\text{combine}} e^{-a(t-t_0)}$$

Transversality Conds.:

Transversality Conds.:

1) $\psi(t_f) = 0$

2) $\lambda(t_f) = \nabla_x \phi \big|_{t=t_f}$

$$\lambda(t_f) = (c x)\big|_{t=t_f}$$

$$\lambda(t_f) = c x(t_f) = \lambda(t_0) e^{-a(t_f - t_0)}$$

$$\Rightarrow \quad \lambda(t_0) = c x(t_f) e^{a(t_f - t_0)}$$

$$\Rightarrow \lambda(t) = c x(t_f) e^{a(t_f - t_0)} e^{-a(t - t_0)}$$

$$= c x(t_f) e^{a(t_f - t_0 + t_0 - t)}$$

$$\lambda(t) = c x(t_f) e^{a(t_f - t)}$$

$$\Rightarrow \underline{u(t) = bc x(t_f) e^{a(t_f - t)}}$$

$$\dot{x} = ax - bu$$

$$\dot{x} = ax - b^2 c x_f e^{a(t_f - t)}$$

$$\dot{x} - ax = -b^2 c x_f e^{a(t_f - t)}$$

$$x = x_H + x_p$$

$$x_H: \quad \dot{x}_H - a x_H = 0$$

$$x_H = C_1 e^{a(t - t_0)}$$

$$x_p = C_2 e^{a(t_f - t)}$$

$$\dot{x}_p - a x_p = -2 C_2 a e^{a(t_f - t)} = -b^2 c x_f e^{a(t_f - t)}$$

$$\Rightarrow C_2 = \frac{b^2 c}{2a} x_f$$

$$x_p(t) = \frac{b^2 c}{2a} x_f e^{a(t_f - t)}$$

$$x(t) = C_1 e^{a(t - t_0)} + \frac{b^2 c}{2a} x_f e^{a(t_f - t)}$$

$$x(t_0) = x_0 = C_1 + \frac{b^2 c}{2a} x_f e^{a(t_f - t_0)}$$

$$x(t_0) = x_0 = C_1 + \frac{b^2c}{2a} x_f e$$

$$C_1 = x_0 - \frac{b^2c}{2a} x_f e^{a(t_f - t_0)}$$

$$x(t) = \left(x_0 - \frac{b^2c}{2a} x_f e^{a(t_f - t_0)} \right) e^{a(t - t_0)} + \frac{b^2c}{2a} x_f e^{a(t_f - t)}$$

$$x_f = x(t_f) = \left(x_0 - \frac{b^2c}{2a} x_f e^{a(t_f - t_0)} \right) e^{a(t_f - t_0)} + \frac{b^2c}{2a} x_f$$

$$x_f = x_0 e^{a(t_f - t_0)} + \left[-\frac{b^2c}{2a} e^{2a(t_f - t_0)} + \frac{b^2c}{2a} \right] x_f$$

$$\left[1 - \frac{b^2c}{2a} \left(1 - e^{2a(t_f - t_0)} \right) \right] x_f = x_0 e^{a(t_f - t_0)}$$

$$x_f = \frac{x_0 e^{a(t_f - t_0)}}{1 - \frac{b^2c}{2a} \left(1 - e^{2a(t_f - t_0)} \right)}$$

$\Rightarrow$

$$u(t) = b c x_f e^{a(t_f - t)}$$

$$\underline{u(t) = \frac{x_0 b c e^{a(t_f - t_0)} e^{a(t_f - t)}}{1 - \frac{b^2c}{2a} \left(1 - e^{2a(t_f - t_0)} \right)}}$$

2) $\dot{x} = ax - bu = f(x, u)$

$$J = \frac{1}{2} c x^2(t_f) + \int_{t_0}^{t_f} \frac{1}{2} (u(t))^2 \, dt$$

$x, u, a, b, c \in \mathbb{R}$

$A = a \qquad B = -b$

$x, u, a, b, c \in \mathbb{R}$

$A = a \qquad B = -b$

$P_f = c \qquad (\frac{1}{2} x_f \cdot c \cdot x_f)$

$Q = 0$

$R = 1 \qquad (\frac{1}{2} u \cdot 1 \cdot u)$

Assume $\lambda(t) = P(t)\, x(t)$ "$\lambda = Px$"

Riccati:

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P \cancel{-Q}^{\,0}$$

$$\dot{P} = -2aP + P(-b)(1^{-1})(-b)P$$

$$\dot{P} = -2aP + b^2P^2$$

s.t. $P(t_f) = P_f = c$

Let $P = -b^2 \dfrac{\dot{v}}{v}$

$$\dot{P} = -b^2\frac{\ddot{v}}{v} + b^2\frac{\dot{v}^2}{v^2} = -b^2\frac{\ddot{v}}{v} + b^2P^2$$

$$\dot{P} - b^2P^2 = -b^2\frac{\ddot{v}}{v} = -2aP$$

$$+\frac{b^2\ddot{v}}{\cancel{v}} = +2aP = +2a\frac{\dot{v}}{\cancel{v}}$$

$$b^2\ddot{v} = 2a\dot{v}$$

$$b^2\ddot{v} - 2a\dot{v} = 0$$

integrate $\hookrightarrow$ $b^2\dot{v} - 2av = C_1$

$$\dot{v} - \frac{2a}{b^2}v = C_1$$

$$v(t) = v_H(t) + v_p(t)$$

$$v_p(t) = -\frac{b^2}{2a}C_1$$

$$v_p(t) = -\frac{b^2}{2a} c_1$$

$$v_H(t) = c_2 e^{-\frac{2a}{b^2}(t-t_0)}$$

$$v(t) = -\frac{b^2 c_1}{2a} + c_2 e^{-\frac{2a}{b^2}(t-t_0)}$$

$$\dot{v}(t) = -c_2\, 2a/b^2\, e^{-2a/b^2(t-t_0)}$$

$$P = \frac{\dot{v}}{v} = \frac{c_2 2a/b^2\, e^{-\frac{2a}{b^2}(t-t_0)}}{\frac{b^2 c_1}{2a} + c_2 e^{-2a/b^2(t-t_0)}}$$

$$P(t_f) = P_f = \frac{c_2 4a^3 e^{\frac{-2a(t_f-t_0)}{b^2}}}{b^4 c_1 + 2ab^2 c_2 e^{-\frac{2a}{b^2}(t_f-t_0)}} = c$$

$$\frac{4a^2 e^{-\frac{2a}{b^2}(t_f-t_0)}}{\frac{c_1}{c_2} b^4 + 2ab^2 e^{-\frac{2a}{b^2}(t_f-t_0)}} = c \qquad c' = \frac{c_1}{c_2}$$

$$\frac{4a^2}{c} e^{-\frac{2a}{b^2}(t_f-t_0)} = c' b^4 + 2ab^2 e^{-\frac{2a}{b^2}(t_f-t_0)}$$

$$\frac{\left(\frac{4a^3}{c} - 2ab^2\right)}{b^4} e^{-\frac{2a}{b^2}(t_f-t_0)} = c'$$

$$\Rightarrow P(t) = \left(\frac{4a^2 - 2ab^2 c}{cb^4}\right) e^{-\frac{2a}{b^2}(t_f-t_0)}\, e^{-\frac{2a}{b^2}(t-t_0)}$$

$$P(t) = \left(\frac{4a^2 - 2ab^2 c}{cb^4}\right) e^{-\frac{2a}{b^2}(t+t_f-2t_0)}$$

So then $u^* = -R^{-1} B^T P x$

$$= +(1^{-1})(+b)\, P\, x$$

$$= + (1^{-1})(+b)\, P\, x$$

$$u^*(t) = \left(\frac{4a^2 - 2ab^2c}{cb^3}\right) e^{\frac{-2a}{b^2}(t + t_f - 2t_0)}\, x(t)$$

3) $$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x + u e_2$$

$$J = \int_0^{t_f} \left[\left(x_1^2 + \tfrac{1}{2} x_2^2\right) + \tfrac{1}{2} u^2\right] dt$$

$$x(0) = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \qquad t_f = 20$$

$$P_f = 0_{2\times 2}$$

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \qquad B = e_2$$

$$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

$$R = 1$$

Assume $\lambda = P x$

Riccati $\qquad P(t_f) = P_f = 0_{2\times 2}$

$$\dot{P} = -A^T P - PA + P B R^{-1} B^T P - Q$$

$$= -A^T P - PA + P e_2 (1^{-1}) e_2^T P - Q$$

$$\dot{P} = -A^T P - PA + P e_2 e_2^T P - Q$$

4) $$\dot{x} = f(x, u) = \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\theta} \\ \dot{v} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} v \cos\theta \\ v \sin\theta \\ v \tan\delta \\ u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ \dot{\delta} \end{pmatrix} \quad \begin{vmatrix} u_1 \\ u_2 \end{vmatrix}$$

$$x_d(t) = \begin{pmatrix} t \\ 2t \\ \tan^{-1}(2) \\ \sqrt{5} \\ 0 \end{pmatrix} \qquad u_d(t) = \vec{0}$$

$\cos\theta \equiv c\theta$

$\sin\theta \equiv s\theta$

$\tan\delta \equiv t\delta$

a) $e = x(t) - x_d(t)$

$$\dot{e} = \dot{x} - \dot{x}_d$$

$$= \begin{pmatrix} v c\theta \\ v s\theta \\ v t\delta \\ u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{e} = \begin{pmatrix} v c\theta \\ v s\theta \\ v t\delta \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 \\ u_2 \end{pmatrix}$$

---

b) $\dot{e} = f(x, u) - \dot{x}_d$

$$\dot{e} = f(\cancel{x_d}, u_d) + \partial_x f\big|_{x=x_d} \cdot (x - x_d) + \partial_u f (u - u_d) + O(\|x\|, \|u\|) - \cancel{\dot{x}_d}$$

$$\dot{e} \approx \partial_x f \cdot e + \partial_u f \cdot s$$

(1st order)

$$A = \partial_x f = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -v_d s\theta_d & c\theta_d & 0 \\ v_d c\theta_d & s\theta_d & 0 \\ 0 & t\delta_d & v_d \sec^2\delta_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

$$B = \partial_u f = \begin{pmatrix} e_4 & e_5 \end{pmatrix}$$ e_i = i-th colum of Identity matrix

$$\dot{e} = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -v_d s\theta_d & c\theta_d & 0 \\ v_d c\theta_d & s\theta_d & 0 \\ 0 & t\delta_d & v_d \sec^2\delta_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix} e + \begin{pmatrix} e_4 & e_5 \end{pmatrix} s$$

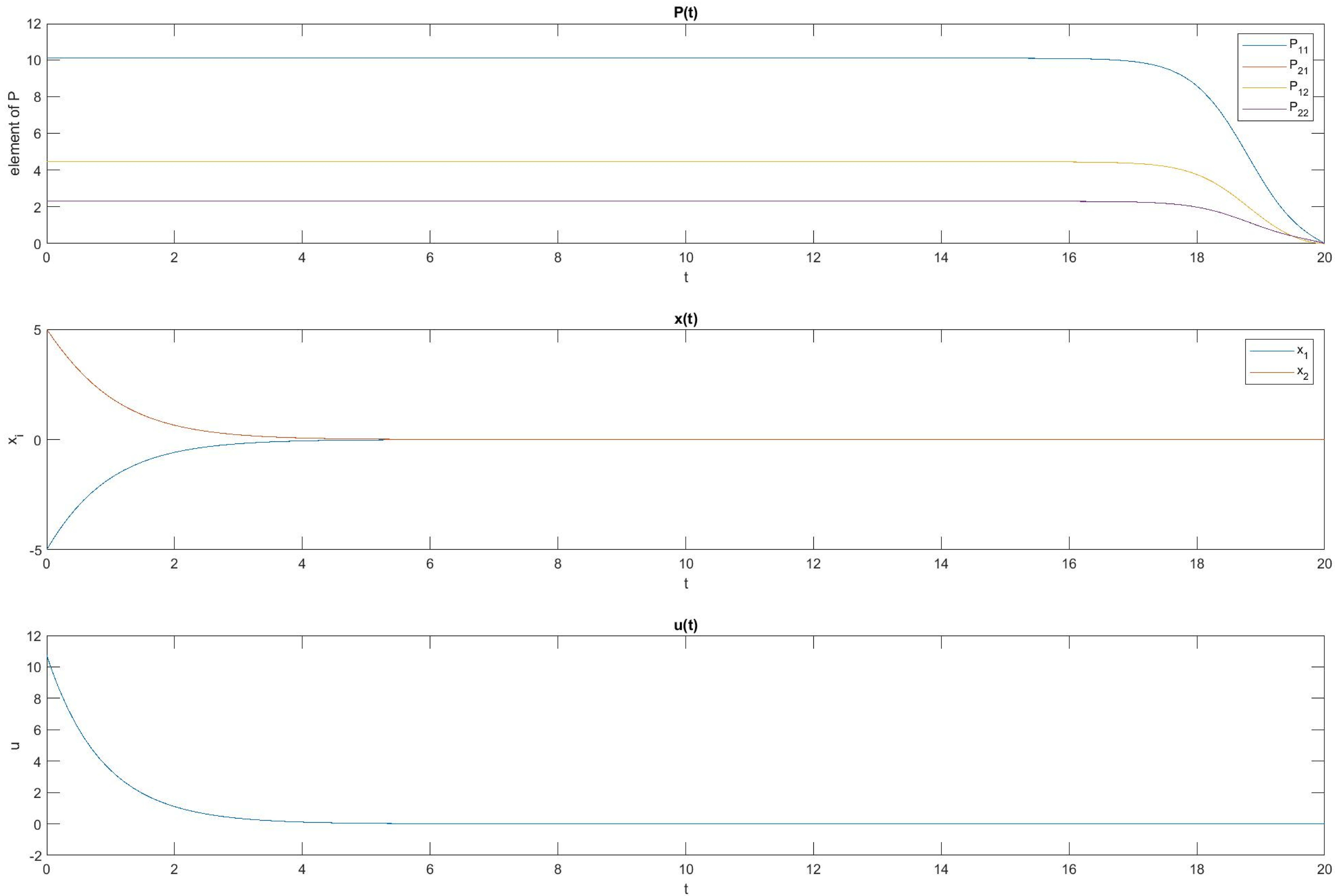
$\sin(\tan^{-1}(2)) = \frac{2}{\sqrt{5}}$ $\cos(\tan^{-1}(2)) = \frac{1}{\sqrt{5}}$

$$A = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -2 & 1/\sqrt{5} & 0 \\ 1 & 2/\sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \\ \vec{0} & \vec{0} & \vec{0} \end{matrix} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```matlab
%% prob_3.m
%
% this script is to solve problem 3's part of solving the Ricatti equation for
% optimal control
%
% - written by: Dimitri Lezcano

%% Set-up system params
global A B R Q Pf Rinv

% system parameters
A = [0 1; 2 -1];
B = [0; 1];
R = 1;
Rinv = inv(R);
Q = diag([2, 1]);
Pf = zeros(2);
tf = 20;

%% calculate P(t)
tspan_P = [tf, 0];
[tP, Pv] = ode45(@(t, Pv) riccati(t, Pv), tspan_P, reshape(Pf, [], 1));

Pv = Pv'; % transpose Pv to be 4 x N
P = reshape(Pv, 2, 2, []);

%% calculate the dynamics
tspan = [0, tf];
x_0 = [-5; 5];
[tx, x] = ode45(@(t, x) dynamics(t, x, P, tP), tspan, x_0);

x = x'; % transpose it to be 2 x N

%% Calculate the control
u = zeros(1, length(tx));
for i = 1:length(x)
    t_i = tx(i); % time at this instance
    x_i = x(:,i);

    % calculate the control
    u(i) = control_law(t_i, x_i, P, tP);

end

%% Plotting
fig = figure(1);

% plot P(t)
subplot(3,1,1);
plot(tP, Pv);
xlabel('t'); ylabel('element of P');
legend('P_{11}', 'P_{21}', 'P_{12}', 'P_{22}');
title('P(t)');

% plot x(t)
subplot(3,1,2);
plot(tx, x);
xlabel('t'); ylabel('x_i');
legend('x_1', 'x_2');
title('x(t)');

% plot u(t)
subplot(3,1,3);
plot(tx, u);
xlabel('t'); ylabel('u');
title('u(t)');

%% Saving the figure
fig_save = 'prob_3.jpg';
```

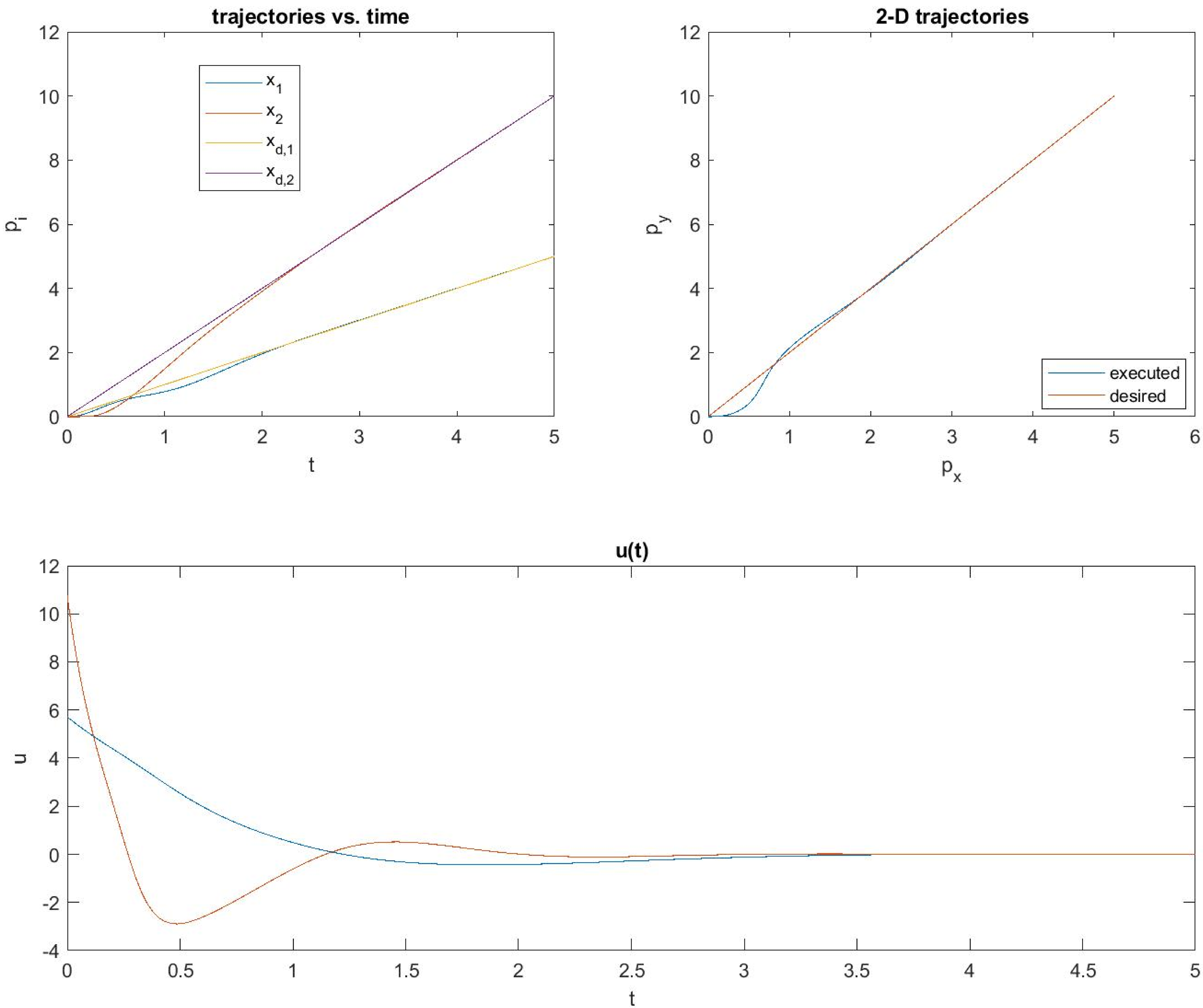
```matlab
saveas(fig, fig_save);
fprintf('Saved figure: %s\n\n', fig_save);

%% Functions
% riccati differential equation

function dPv = riccati(t, Pv)
    global A B Q Rinv
    % turn P into a matrix again
    P = reshape(Pv, 2,2);

    % calculate matrix dP
    dP = -A'* P - P * A + P * B * Rinv * B' * P - Q;

    % vectorize dP
    dPv = reshape(dP, [], 1);

end

% function for computing the dynamics
function dx = dynamics(t, x, P, tP)
    % P is of shape 2x2xN : N is the number of time elements
    global A B
    u = control_law(t, x, P, tP);

    dx = A * x + B * u;


end

function u = control_law(t, x, P, tP)
    global Rinv B
    [~, t_idx] = min(abs(t - tP));
    u = -Rinv * B' * P(:,:,t_idx) * x;


end
```



P(t)
element of P
P_{11}
P_{21}
P_{12}
P_{22}
t
x(t)
x_i
x_1
x_2
t
u(t)
u
t

```matlab
%% prob_4.m
%
% this script is used to answer problem 4
%
% - written by: Dimitri Lezcano

%% Set-up parameters
x_0 = zeros(5,1); % initial state
Q = diag([5, 5, 0.01, 0.1, 0.1]);
R = diag([0.5, 0.1]);
tf = 5; % final time
ud = [0; 0];

%% Compute A and B matrices
x_0d = compute_xd(0);
A_d = compute_A(x_0d);
B_d = compute_B(x_0); % grab B since it is a constant


%% Optimize the lqr problem to get the K matrix
[K, S, e] = lqr(A_d, B_d, Q, R, 0);

%% Determine the trajectory
[t, x] = ode45(@(t, x) dynamics(t, x, K), [0,tf], x_0);

x = x'; % reshape to 5 x N matrix
xd = compute_xd(t); % desired trajectory

%% Get the control
u = zeros(length(ud), length(t));
for i = 1:length(t)
    t_i = t(i);
    x_i = x(:,i);

    u(:,i) = control_law(t_i, x_i, K);

end

%% Plotting
fig = figure(1);
% plot the trajectories
subplot(2,2,1);
plot(t, x(1:2,:)); hold on;
plot(t, xd(1:2,:)); hold off;
legend('x_1', 'x_2', 'x_{d,1}', 'x_{d,2}', 'location', 'best');
xlabel('t'); ylabel('p_i');
title('trajectories vs. time')

% plot the 2-d Trajectories
subplot(2,2,2);
plot(x(1,:), x(2,:), 'DisplayName', 'executed'); hold on;
plot(xd(1,:), xd(2,:), 'DisplayName', 'desired'); hold off;
legend('location', 'best');
xlabel('p_x'); ylabel('p_y');
title('2-D trajectories');

% plot the control
subplot(2,2,[3 4]);
plot(t, u);
xlabel('t'); ylabel('u');
title('u(t)');

%% Saving the figure
fig_save = 'prob_4.jpg';
saveas(fig, fig_save);
fprintf('Saved figure: %s\n\n', fig_save);

%% Functions
% function for computing A matrix
```

```
function A = compute_A(x)
    v = x(4); % velocity
    th = x(3); % theta
    delta = x(5); % delta

    A = zeros(5);
    % Set values
    A(1, 3) = - v * sin(th);
    A(2, 3) = v * cos(th);

    A(1,4) = cos(th);
    A(2,4) = sin(th);
    A(3,4) = tan(delta);

    A(3, 5) = v*sec(delta)^2;

end

% function for computing B matrix
function B = compute_B(x)
    B = zeros(5,2);

    % set values
    B(4,1) = 1;
    B(5,2) = 1;

end

% compute the desired trajectory
function xd = compute_xd(t)
    t = reshape(t, 1, []);
    xd = [t; 2*t; atan(2)*ones(size(t)); sqrt(5)*ones(size(t)); zeros(size(t))];

end

% the system dynamics
function dx = dynamics(t, x, K)
    v = x(4); % velocity
    th = x(3); % theta
    delta = x(5); % delta

    u = control_law(t, x, K);
    dx = [v*cos(th); v*sin(th); v*tan(delta); u];


end

% the control law
function u = control_law(t, x, K)
    xd = compute_xd(t);
    u = -K * (x - xd);

end
```


trajectories vs. time
x1
x2
xd,1
xd,2
pi
t
2-D trajectories
executed
desired
py
px
u(t)
u
t