Homework 5: Dimitri Lezcano

Tuesday, October 27, 2020 11:51 PM

1) $J = \frac{1}{2} x(1)^2 + \int_0^1 \underbrace{\frac{1}{2}\left[x(t)u(t)\right]^2}_{L} dt$

s.t.

$\dot{x} = xu$

$x(0) = 1$

HJB

$$-\partial_t V = \min_u \left\{ L(x,u,t) + \partial_x V \cdot f(x,u) \right\}$$

$$\min_u \left\{ \frac{1}{2} x^2 u^2 + \partial_x V \cdot xu \right\}$$

$$\frac{d}{du}\left(\frac{1}{2} x^2 u^2 + \partial_x V \cdot xu \right) = 0$$

$$x^2 u^* + V_x \cdot x = 0$$

$$u^* = -\frac{V_x}{x}$$

$$-\partial_t V = \frac{1}{2} x^2 u^{*2} + V_x \cdot x u^*$$

$$= \frac{1}{2} x^2 \frac{V_x^2}{x^2} + V_x \cdot x\left(-\frac{V_x}{x}\right)$$

$$-\partial_t V = -\frac{1}{2} V_x^2$$

$$\frac{1}{2} V_x^2 = \partial_t V \Rightarrow$$

$$\frac{1}{2} V_x^2 = \partial_t V$$

$$V(x,t) = X(x)\,T(t)$$

$$\frac{1}{2} X'^2 T^2 = X T'$$

$$\frac{1}{2} \frac{X'^2}{X} = \frac{T'}{T^2} = K \quad K \text{ is a scalar}$$

$$\frac{1}{2} X'^2 = K X$$

$$X(x) = c x^2$$

$$X'(x) = 2 c x$$

$$\downarrow$$

$$\frac{1}{2} (2cx)^2 = K (c x^2)$$

$$2 c^2 x^2 = K c x^2$$

$$c = K/2$$

$$\text{If } X(x) = \frac{1}{2} K x^2$$

$$X'(x) = K x$$

$$\frac{1}{2} (X')^2 = \frac{1}{2} (Kx)^2 = \frac{1}{2} K^2 x^2 = K X \checkmark$$

$$\text{So} \quad \underline{V(x,t) = \left(\frac{1}{2} K x^2\right) T(t)}$$

$$u^* = -\underline{\nabla_x V}_{x} = -\underline{K x\, T(t)}_{x} = -K T(t)$$

$$2) \; J = \frac{1}{2} x_f^T P_f x_f + \underline{\frac{1}{2}} \int_{t_0}^{t_f} \underbrace{(x^T Q x + u^T R u)}_{L} dt$$

$$R > 0$$

$$\underbrace{\qquad\qquad}_{t_0}\quad\underbrace{\qquad\qquad\qquad\qquad}_{L}$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \qquad R > 0 \quad Q, P_f \geq 0$$

$w(t)$ is known disturbances

a) C/S time of HJB

HJB

$$-\partial_t V = \min_u \left\{ L(x,u,t) + \nabla_x V^T f(x,u,t) \right\}$$

$$\nabla_u \left(L(x,u,t) + \nabla_x V^T f(x,u,t) \right)$$

$$= \nabla_u \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \nabla_x V^T (Ax + Bu + w) \right)$$

$$\Rightarrow R u^* + B^T \nabla_x V = 0$$

$$① \quad u^* = -R^{-1} B^T \nabla_x V$$

So then

$$-\partial_t V = \frac{1}{2} x^T Q x + \frac{1}{2} \nabla_x V^T B R B^T \nabla_x V + \nabla_x V^T A x - \nabla_x V^T B R^{-1} B^T \nabla_x V + \nabla_x V^T w$$

$$= \frac{1}{2} x^T Q x - \frac{1}{2} \nabla_x V^T B R B^T \nabla_x V + \nabla_x V^T (Ax + w)$$

Suppose $V = \frac{1}{2} x^T P(t) x + b(t)^T x + c(t)$

Then we have

$$\partial_t V = \frac{1}{2} x^T \dot{P} x + \dot{b}^T x + \dot{c}$$

$$\nabla_x V = P x + b(t)$$

HJB becomes

$$-\frac{1}{2} x^T \dot{P} x - \dot{b}^T x - \dot{c} =$$

$$-\frac{1}{2}x^T\dot{P}x - \dot{b}^T x - \dot{c} =$$

$$\frac{1}{2}x^T Q x - \frac{1}{2}(Px+b)^T B R B^T (Px+b) + (Px+b)^T (Ax+w)$$

$$= \frac{1}{2}x^T Q x - \frac{1}{2}x^T P B R B^T P x - x^T P B R B^T b$$

$$-\frac{1}{2} b^T B R B^T b + x^T P A x + b^T A x + x^T P w + b^T w$$

Group Like Terms!

Quadratics

$$\frac{1}{2}x^T\dot{P}x = -\frac{1}{2}x^T Q x + \frac{1}{2}x^T P B R B^T P x - x^T P A x$$

Single x

$$\dot{b}^T x = x^T P B R B^T b - b^T A x - x^T P w$$

$$x^T \dot{b} = x^T (P B R B^T b) - x^T (A^T b) - x^T (P w)$$

No x

$$\dot{c} = \frac{1}{2} b^T B R B^T b - b^T w$$

So Quadratics are satisfied if

$$\underline{\dot{P} = -Q + P B R B^T P - P A - A^T P}$$

The "single x" are satisfied if

$$\underline{\dot{b} = P B R B^T b - A^T b - P w}$$

And "no x" is satisfied if

And "no x" is satisfied if

$$\dot{c} = \frac{1}{2} b^T B R B^T b - b^T w$$

Now we want
$$V(t_f) = \frac{1}{2} x_f^T P_f x_f$$

So $V(t_f) = \frac{1}{2} x_f^T P(t_f) x_f + b(t_f)^T x_f + c(t_f) = \frac{1}{2} x_f^T P_f x_f$

so boundary conditions are

$$\begin{cases} \bullet\ P(t_f) = P_f \\ \bullet\ b(t_f) = 0 \\ \bullet\ c(t_f) = 0 \end{cases}$$

$$\Rightarrow u^* = -R^{-1} B^T \nabla_x V$$
$$= -R^{-1} B^T (P x + b)$$

$$u^*(t) = \underbrace{-R^{-1}(t) B^T P(t)}_{K(t)} x(t) \underbrace{- R^{-1}(t) B^T b(t)}_{h(t)}$$

$$u^*(t) = K(t)\, x(t) + h(t)$$

b) Discrete-time

$$x_{i+1} = A_i x_i + B_i u_i + w_i$$

$$L_i(x,u) = \frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u_i^T R_i u_i$$

$$\phi_f = \frac{1}{2} x_N^T P_f x_N$$

By Bellman eqs.

$$V_i = \min_u \left\{ L_i(x,u) + V_{i+1}\, f_i(x,u) \right\}$$

and

$$V_N = \phi_f = \frac{1}{2} x_N^T P_f x_N$$

$$\min_u \left\{ \frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u_i^T R_i u_i + V_{i+1}(A_i x_i + B_i u_i + w_i) \right\}$$

$$\Rightarrow \nabla_u \left(\frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u^T R_i u + V_{i+1}(A x_i + B u + w_i) \right) = 0$$

$$R u^* + \nabla_u \left[V_{i+1} (A_i x_i + B_i u + w) \right] \Big|_{u = u^*} = 0$$

Suppose $V_i(x) = \frac{1}{2} x^T P_i x + b_i^T x + c_i$

Then $V_{i+1}(A_i x + B_i u + w_i) =$

$$\frac{1}{2}(A_i x + B_i u + w_i)^T P_{i+1} (A_i x + B_i u + w_i) + b_{i+1}^T (A_i x + B_i u + w_i) + c_{i+1}$$

$$= \frac{1}{2}(A_i x + w_i)^T P_{i+1} (A_i x + w_i) + u^T B_i^T P_{i+1} (A_i x + w_i)$$

$$+ \frac{1}{2} u^T B_i^T P_{i+1} B_i u + b_{i+1}^T (A_i x + w_i) + b_{i+1}^T B_i u + c_{i+1}$$

$\therefore$

$$\nabla_u V_{i+1} = B_i^T P_{i+1}(A_i x + w_i) + B_i^T P_{i+1} B_i u + B_i^T b_{i+1}$$

$$\downarrow$$

$$R_i u^* + B_i^T P_{i+1}(A_i x + w_i) + B_i^T P_{i+1} B_i u^* + B_i^T b_{i+1} = 0$$

$$\Rightarrow \left[R_i + B_i^T P_{i+1} B_i\right] u^* = - B_i^T P_{i+1}(A_i x + w_i) - B_i^T b_{i+1}$$

$$u^* = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T \left(P_{i+1} A_i x + P_{i+1} w_i + b_i \right)$$

$$\underline{u^* = K_i x + k_i}$$

$$\underline{K_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T P_{i+1} A_i}$$

and

$$\underline{k_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T \left(P_{i+1} w_i + b_{i+1}\right)}$$

Finally conditions for P_i, b_i, c_i ..

$$V_i = L_i(x, u^*) + V_{i+1}(f_i(x, u^*))$$

$$\frac{1}{2} x^T P_i x + b_i^T x + c_i = \frac{1}{2} x^T Q_i x + \frac{1}{2} u^{*T} R_i u^*$$
$$+ \frac{1}{2}(A_i x + B_i u^* + w_i)^T P_{i+1} (A_i x + B_i u^* + w_i)$$
$$+ b_{i+1}^T (A_i x + B_i u^* + w_i)$$
$$+ c_{i+1}$$

$$= \frac{1}{2}\left[(A_i + B_i K_i) x + (w_i + B_i k_i)\right]^T P_{i+1} \left[(A_i + B_i K_i) x + (w_i + B_i k_i)\right]$$
$$+ b_{i+1}^T \left[(A_i + B_i K_i) x + (w_i + B_i k_i)\right] + c_{i+1}$$
$$+ \frac{1}{2} x^T Q_i x + \frac{1}{2}(K_i x + k_i)^T R_i (K_i x + k_i)$$

(Quadratics)

$$\frac{1}{2}x^T P_i x = \frac{1}{2}x^T (A_i + B_i K_i)^T P_{i+1} (A_i + B_i K_i) x + \frac{1}{2} x^T Q_i x + \frac{1}{2} x^T K_i^T R_i K_i x$$

$$\downarrow$$

$$\underline{P_i = (A_i + B_i K_i)^T P_{i+1} (A_i + B_i K_i) + Q_i + K_i^T R_i K_i}$$

(Single x)

$$x^T b_i = x^T (A_i + B_i K_i)^T P_{i+1} (w_i + B_i k_i) + x^T (A_i + B_i K_i)^T b_{i+1} + x^T K_i^T R_i k_i$$

$$\downarrow$$

$$\underline{b_i = (A_i + B_i K_i)^T \left[P_{i+1} (w_i + B_i k_i) + b_{i+1} \right] + K_i^T R_i k_i}$$

(No x)

$$\underline{c_i = \frac{1}{2} (w_i + B_i k_i)^T P_{i+1} (w_i + B_i k_i) + b_{i+1}^T (w_i + B_i k_i) + c_{i+1} + \frac{1}{2} k_i^T R_i k_i}$$

With Boundary conditions

$$V_N = \frac{1}{2} x_N^T P_N x_N + b_N^T x_N + c_N = \frac{1}{2} x_N^T P_f x_N$$

$$\mapsto \begin{cases} \cdot\ P_N = P_f \\ \cdot\ b_N = 0 \\ \cdot\ c_N = 0 \end{cases}$$

3)
$$p_{i+1} = p_i + \Delta t\, v_i$$
$$v_{i+1} = v_i + \Delta t (-0.5 v_i + 0.2 p_i + u_i + 0.1)$$

$$x_{i+1} = \begin{pmatrix} p_{i+1} \\ v_{i+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \Delta t \\ 0.2\Delta t & 1 - 0.5\Delta t \end{pmatrix}}_{A_i} x_i + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_i} u_i + \underbrace{\begin{pmatrix} 0 \\ 0.1 \end{pmatrix}}_{w_i}$$

$$J = \frac{1}{2}\left(p_N^2 + v_N^2\right) + \sum_{i=0}^{N} \frac{1}{2}\left(R u_i^2\right)$$

$$= \frac{1}{2} x_N^T P_f x_N + \sum_{i=0}^{N} \frac{1}{2} R u_i^2$$

$$P_f = I_2 \qquad Q_i = 0_{2\times 2} \qquad R_i = R$$

So we have that

$$V_i(x) = \frac{1}{2} x^T P_i x + b_i^T x + c_i$$

$$K_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T P_{i+1} A$$

* $K_i = -\left(R + e_2^T P_{i+1} e_2\right)^{-1} e_2^T P_{i+1} A$ (1×2 - row vector)

$$k_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T \left(P_{i+1} w_i + b_{i+1}\right)$$

* $k_i = -\left(R + e_2^T P_{i+1} e_2\right)^{-1} e_2^T \left(P_{i+1} w_i + b_{i+1}\right)$ (scalar)

where

$$P_i = \left(A_i + B_i K_i\right)^T P_{i+1} \left(A_i + B_i K_i\right) + Q_i + K_i^T R_i K_i$$

* $P_i = \left(A + e_2 K_i\right)^T P_{i+1} \left(A + e_2 K_i\right) + K_i^T R K_i$

$$P_N = P_f = I_2$$

$$b_i = \left(A_i + B_i K_i\right)^T \left[P_{i+1}\left(w_i + B_i k_i\right) + b_{i+1}\right] + K_i^T R_i k_i$$

* $b_i = \left(A + e_2^T K_i\right)^T \left[P_{i+1}\left(w + e_2 k_i\right) + b_{i+1}\right] + K_i^T R k_i$

$b_N = 0$ (2x1 col. vector)

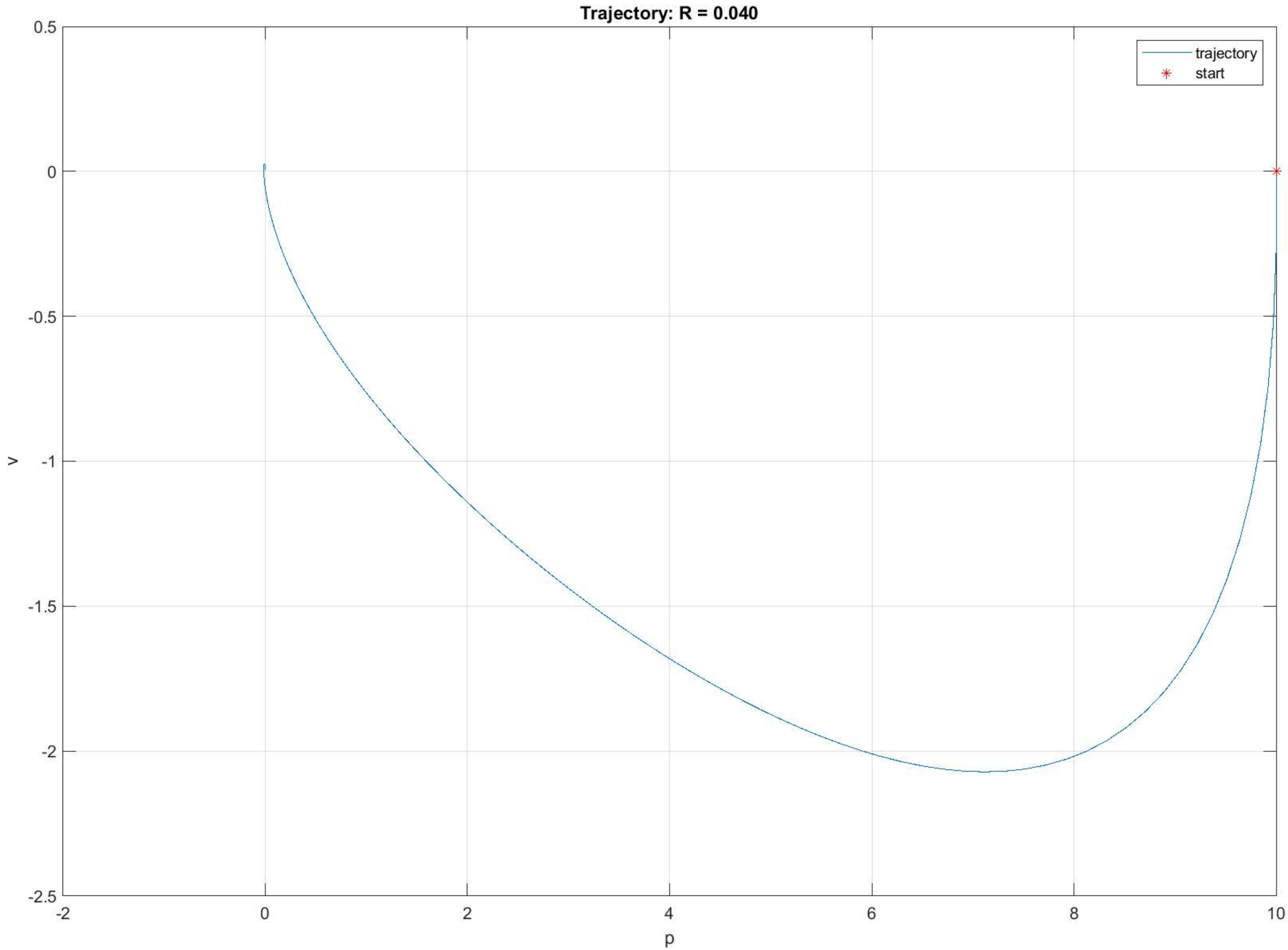
$$c_i = \frac{1}{2}(w_i + B_i k_i)^T P_{i+1}(w_i + B_i k_i) + b_{i+1}^T(w_i + B_i k_i) + c_{i+1} + \tfrac{1}{2} k_i^T R_i k_i$$

$$c_i = \frac{1}{2}(w + e_2 k_i)^T P_{i+1}(w + e_2 k_i) + b_{i+1}^T(w + e_2 k_i) + c_{i+1} + \tfrac{1}{2} k_i R k_i$$

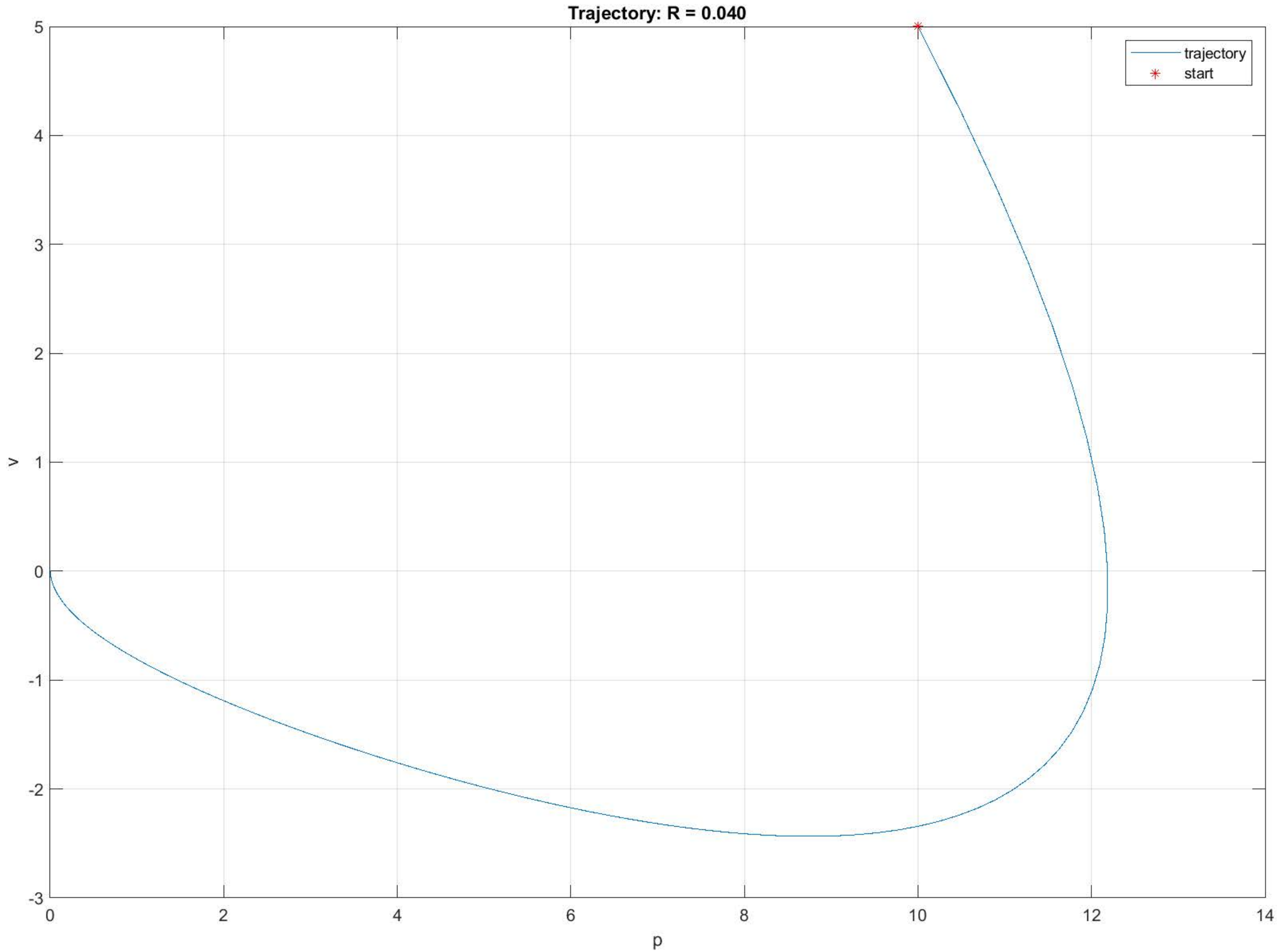
$c_N = 0$ (scaler)

```
%% prob_3.m
%
% this script is for HW5 problem 3
%
% - written by: Dimitri Lezcano

    ;

%% Set-up
 = 100;

% system
  = [10; -5];
.   =    (2);

.  = 0.04;
.   = 0.1;

.  = [1,  .  ; 0.2* .  , 1 - 0.5* .  ];
.  = [0; 1];
.  = [0; 0.1];

% Value function params
.  =     (1, );
.  =     (1, );
.  =     (1, );

. { } =  .  ;
. { } = [0;0];
. { } = 0;

% control law
.  =     (1, );
.  =     (1, );

% trajectory and control arrays
 =      (2, );
(:,1) =   ;

 =      (1, );


%% Back integrate to get P_i, b_i, c_i, K_i, k_i
for   =  -1:-1:1

    % get P_i+1, b_i+1, c_i+1
          =  . {  + 1};
          =  . {  + 1};
          =  . {  + 1};

    % determine K_i and k_i
             =    ( .  +  . ' *       *  . ); % helper inverse

        = -         *  . ' *       *  . ;
        = -         *  . ' * (      *  .  +      );

    % determine P_i, b_i, c_i
        = ( .  +  .  *    )'*     *( .  +  .  *    ) +    '* . *   ;
        = ( .  +  .  *    )'*(      * ( .   +  . *   ) +      ) +    '* . *   ;
        = 1/2 * ( .  +  .  *    )' *       * ( .  +  .  *    ) +      '*( .  +  .  *    ) + ...
              + 1/2 *    ' *  .  *    ;

    % assign the values
     . { } =    ;
     . { } =    ;

     . { } =    ;
     . { } =    ;
     . { } =    ;

end



%% Forward integrate using dynamics and new control law
```

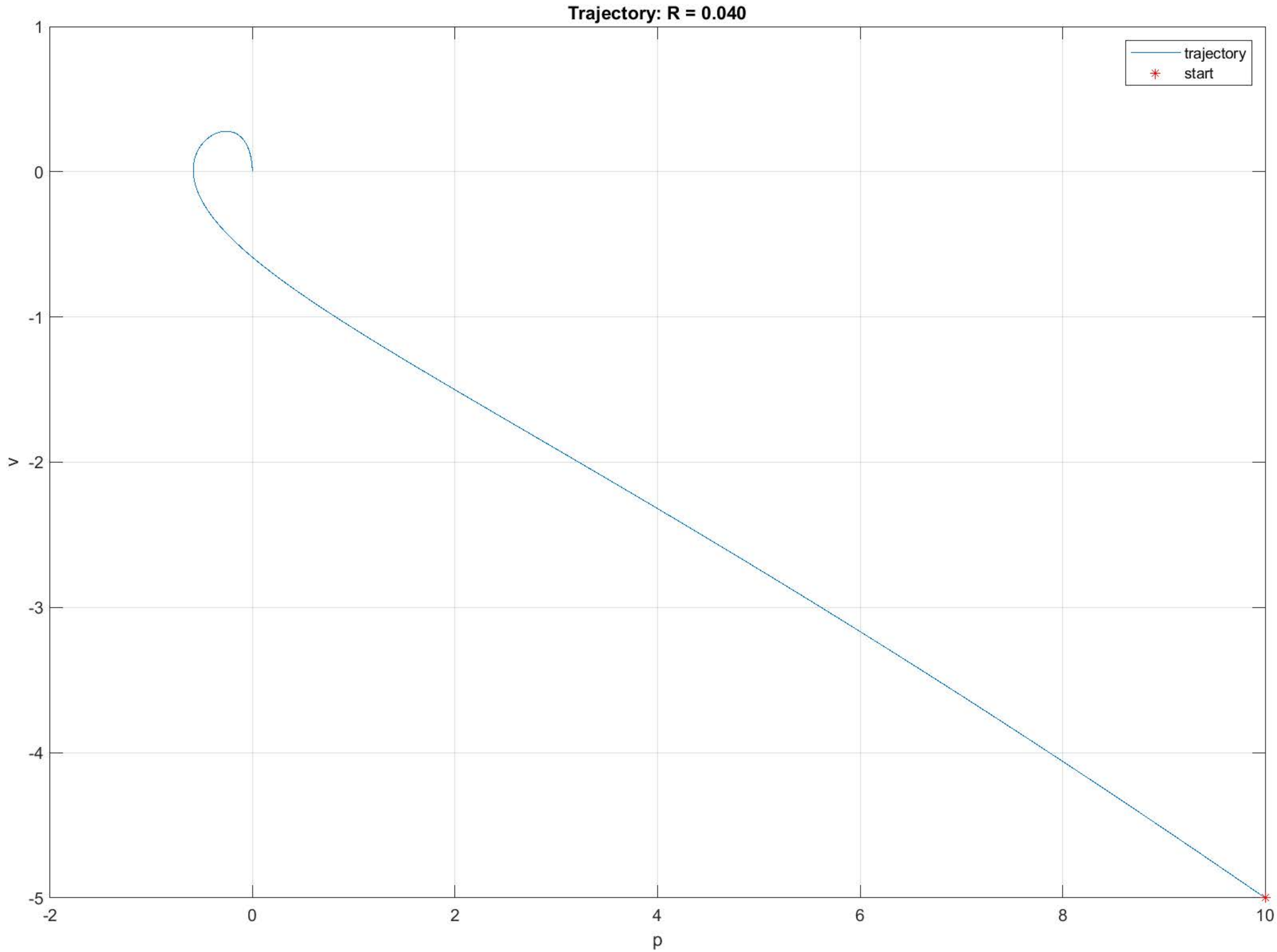
```
for   = 1: -1
    % get x_i to integrate to x_i+1
        =  (:, );  % i-th column

    % get the control law
        =         ( ,    ,  );

    % get x_i+1
          =         ( ,    ,    ,  );

    % add x_i+1 and u_i to the array
     (:, +1) =      ;
     ( ) =    ;

end

%% Plotting
       =       (1);
    ( (1,:),  (2,:), 'DisplayName', 'trajectory');        ;
    (  (1),   (2), 'r*', 'DisplayName', 'start');         ;
     (       ('Trajectory: R = %.3f',  . ));
      ('p');       ('v');
      ();
       ;

       =       (2);
    ( );
     (       ('Control: R = %.3f',  . ));
      ('i');       ('u');
       ;

%% Saving
          = "prob_3_%s_" +        ("R-%.3f_x0_%d_%d",  . ,   (1),   (2));

      (     ,        (         , 'traj') + ".png");
    ("Saved figure: " +        (         , 'traj') + ".png");

      (     ,        (         , 'ctrl') + ".png");
    ("Saved figure: " +        (         , 'ctrl') + ".png");

%% Functions
% dynamics
function       =         ( ,    ,    ,  )
          =  .  *     +  .  *     +  . ;

end

% control law
function     =         ( ,  ,  )
        =  . { };
        =  . { };

        =    *   +    ;

end
```



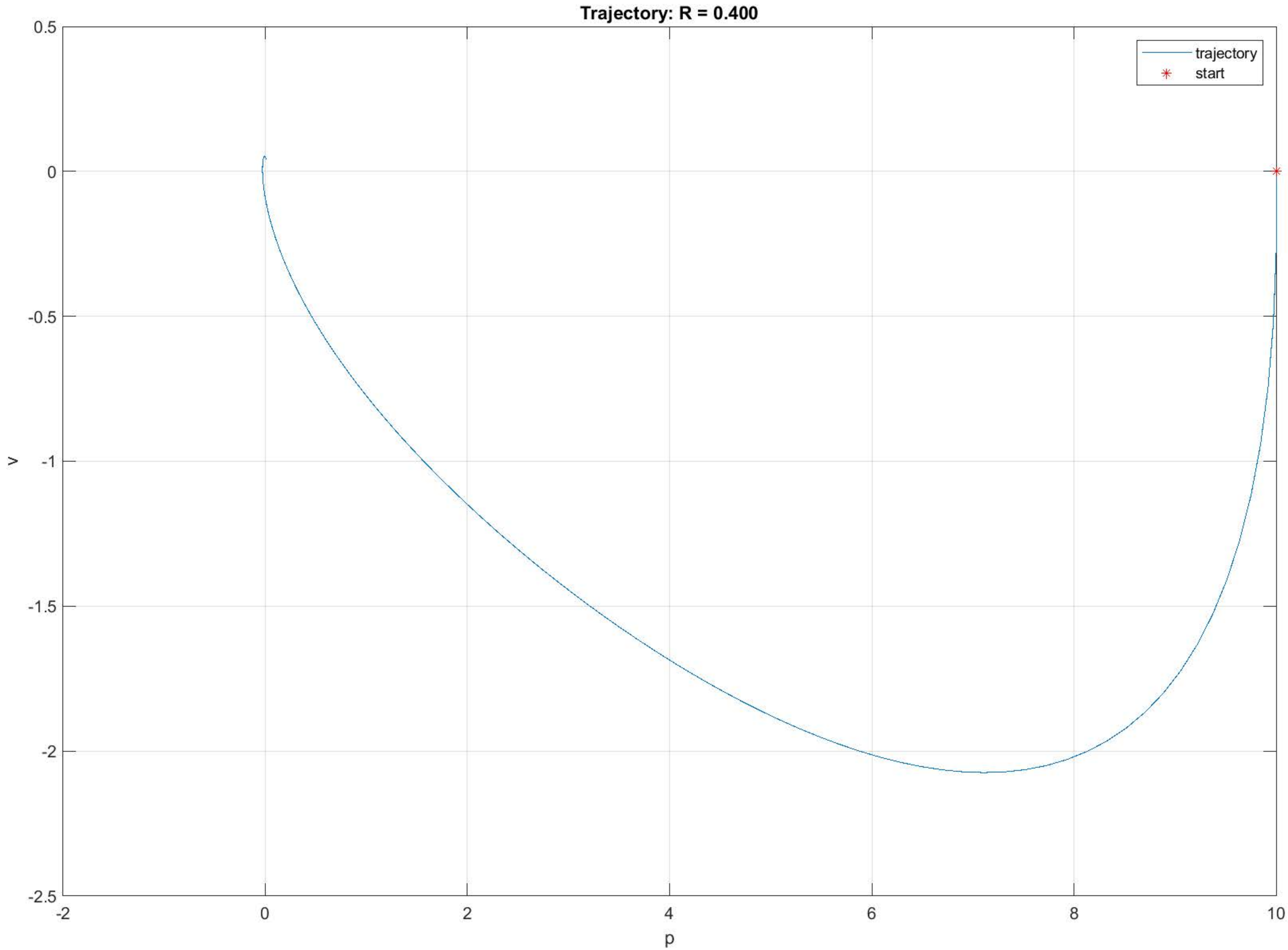
Trajectory: R = 0.040
trajectory
start
v
p
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
0
2
4
6
8
10

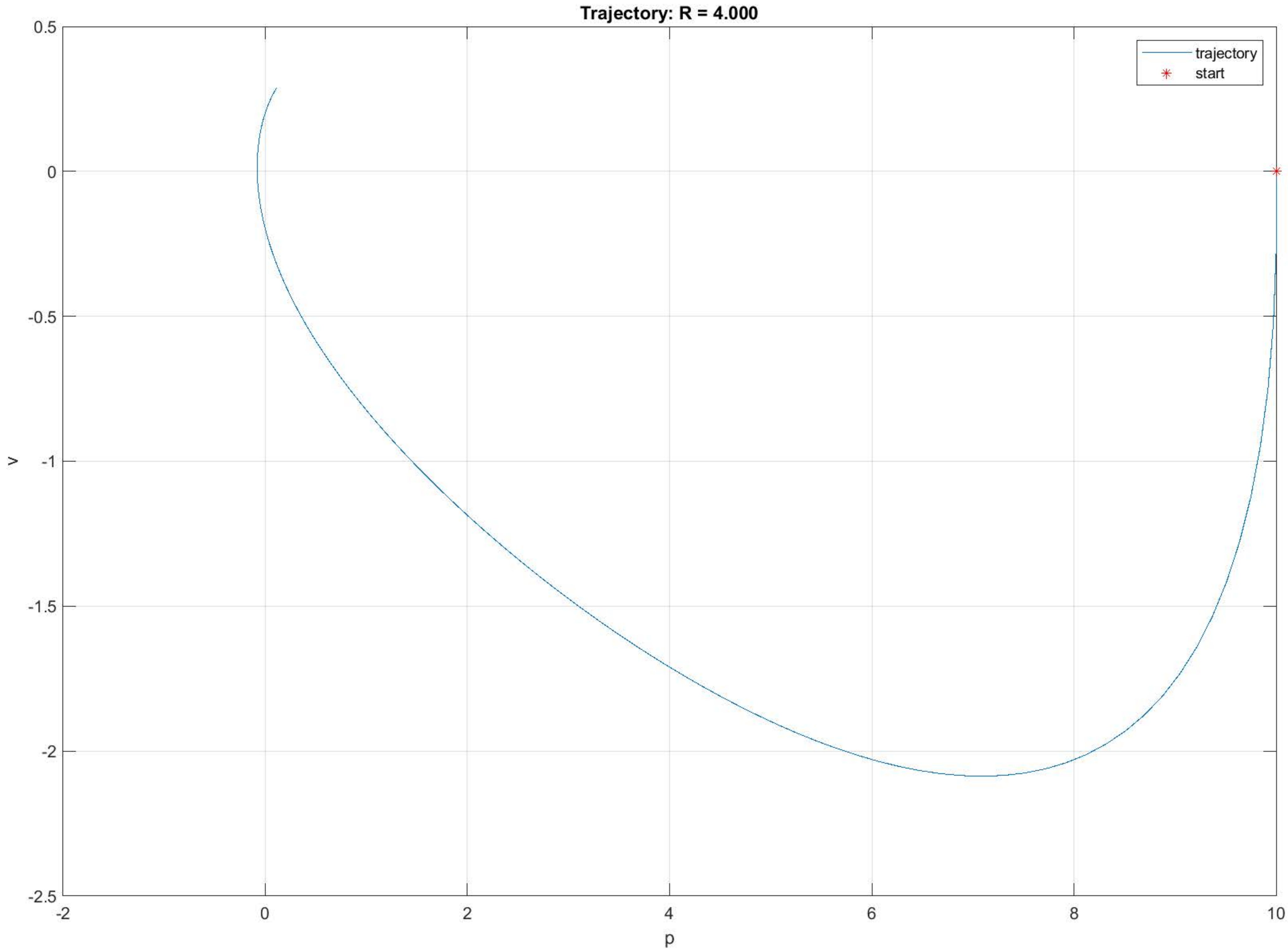


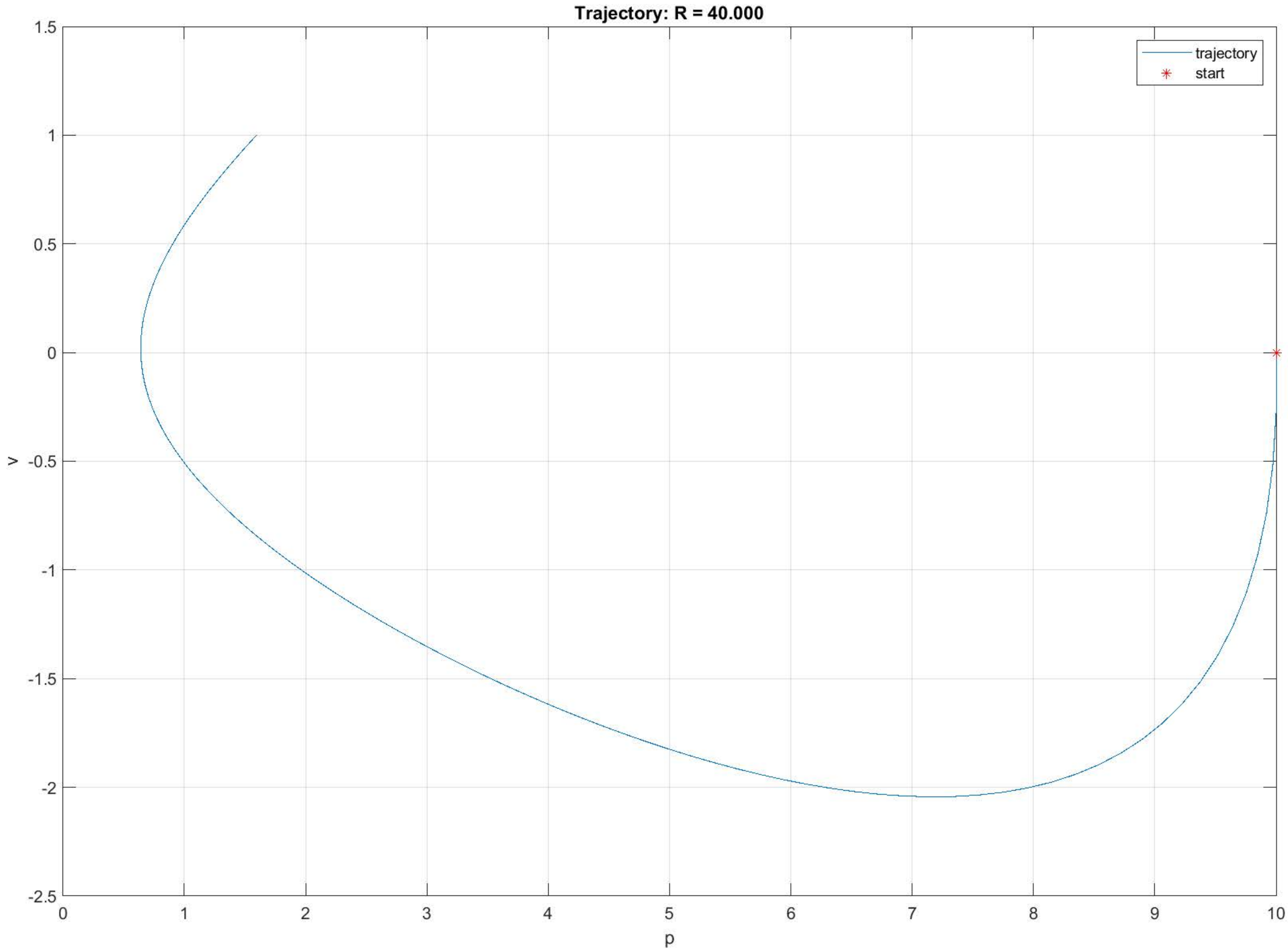
Trajectory: R = 0.040
trajectory
start
v
p
5
4
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14

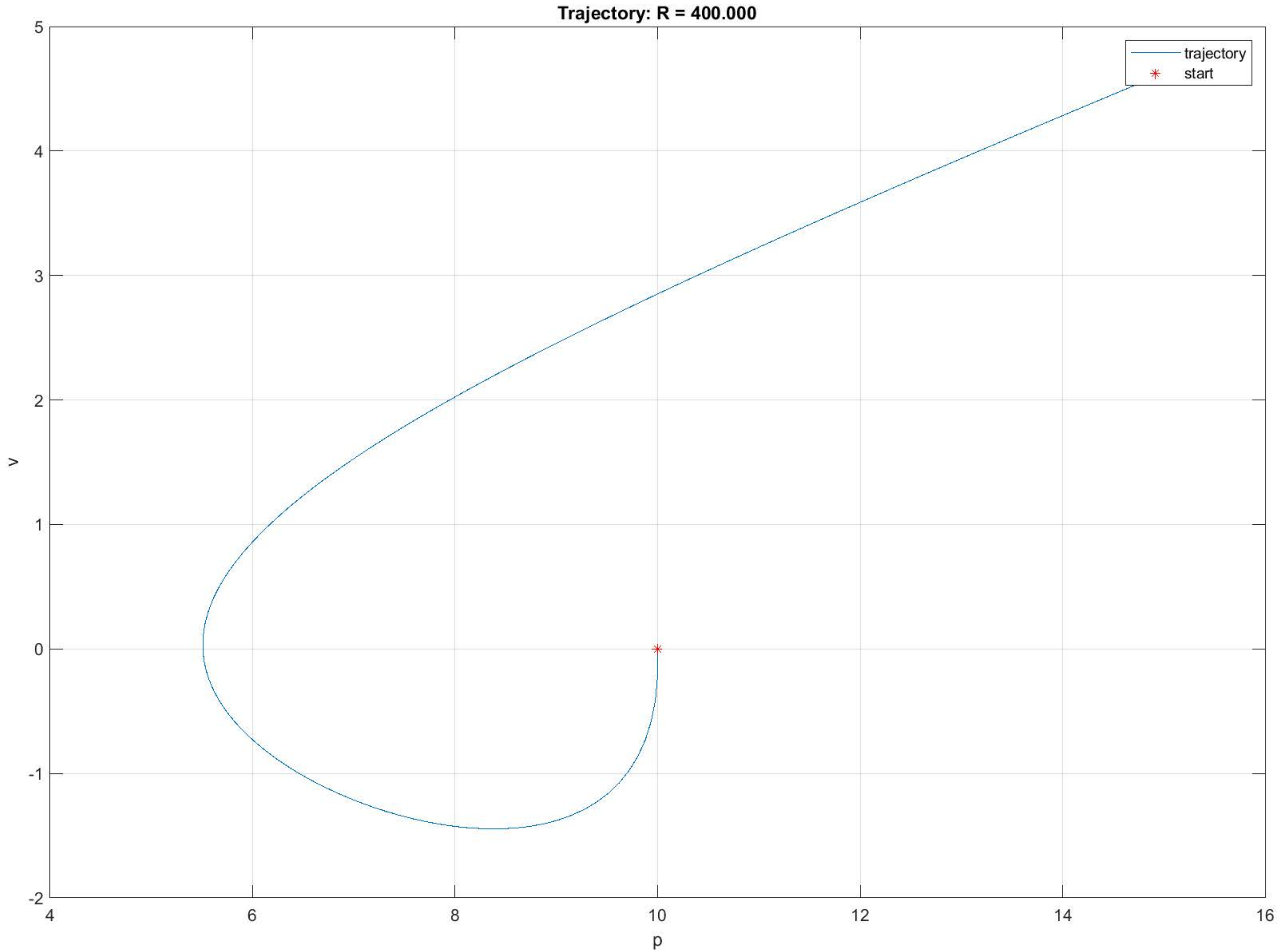


Trajectory: R = 0.040
trajectory
start
v
p
1
0
-1
-2
-3
-4
-5
-2
0
2
4
6
8
10

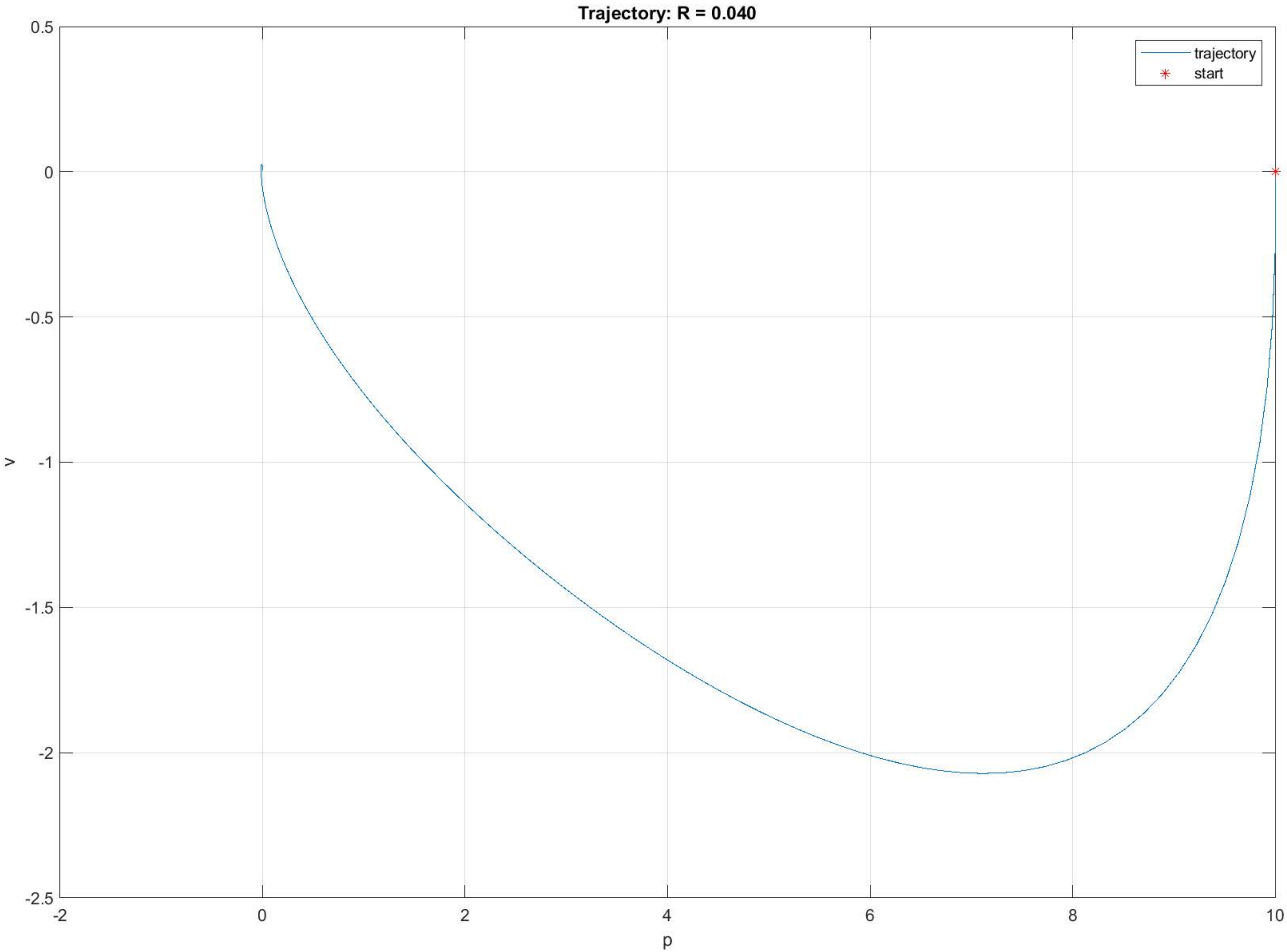

Trajectory: R = 0.400
trajectory
start
0.5
0
-0.5
-1
-1.5
-2
-2.5
v
-2
0
2
4
6
8
10
p

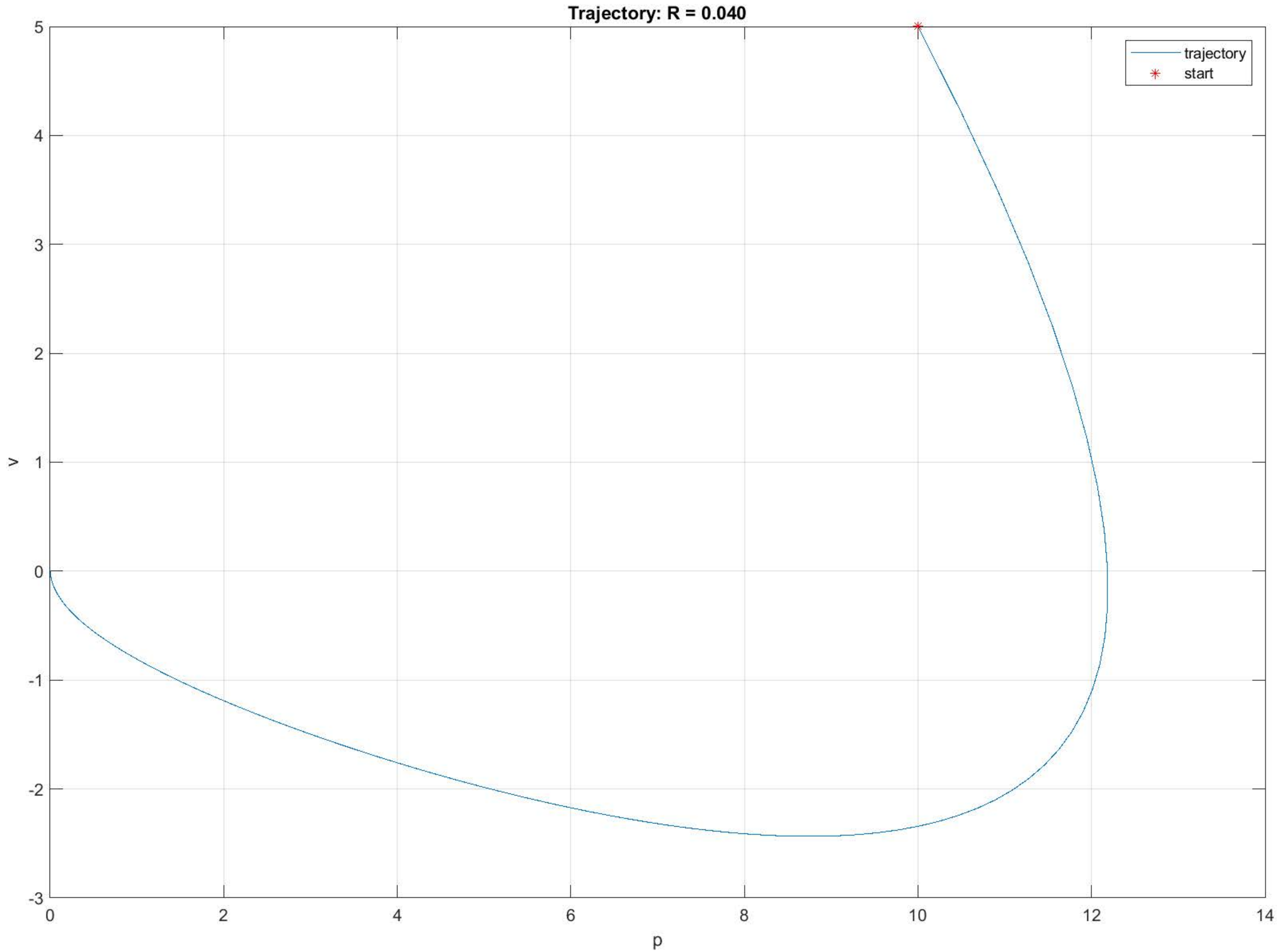

Trajectory: R = 4.000
trajectory
start
0.5
0
-0.5
-1
-1.5
-2
-2.5
v
-2
0
2
4
6
8
10
p


Trajectory: R = 40.000
trajectory
start
v
p
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
9
10

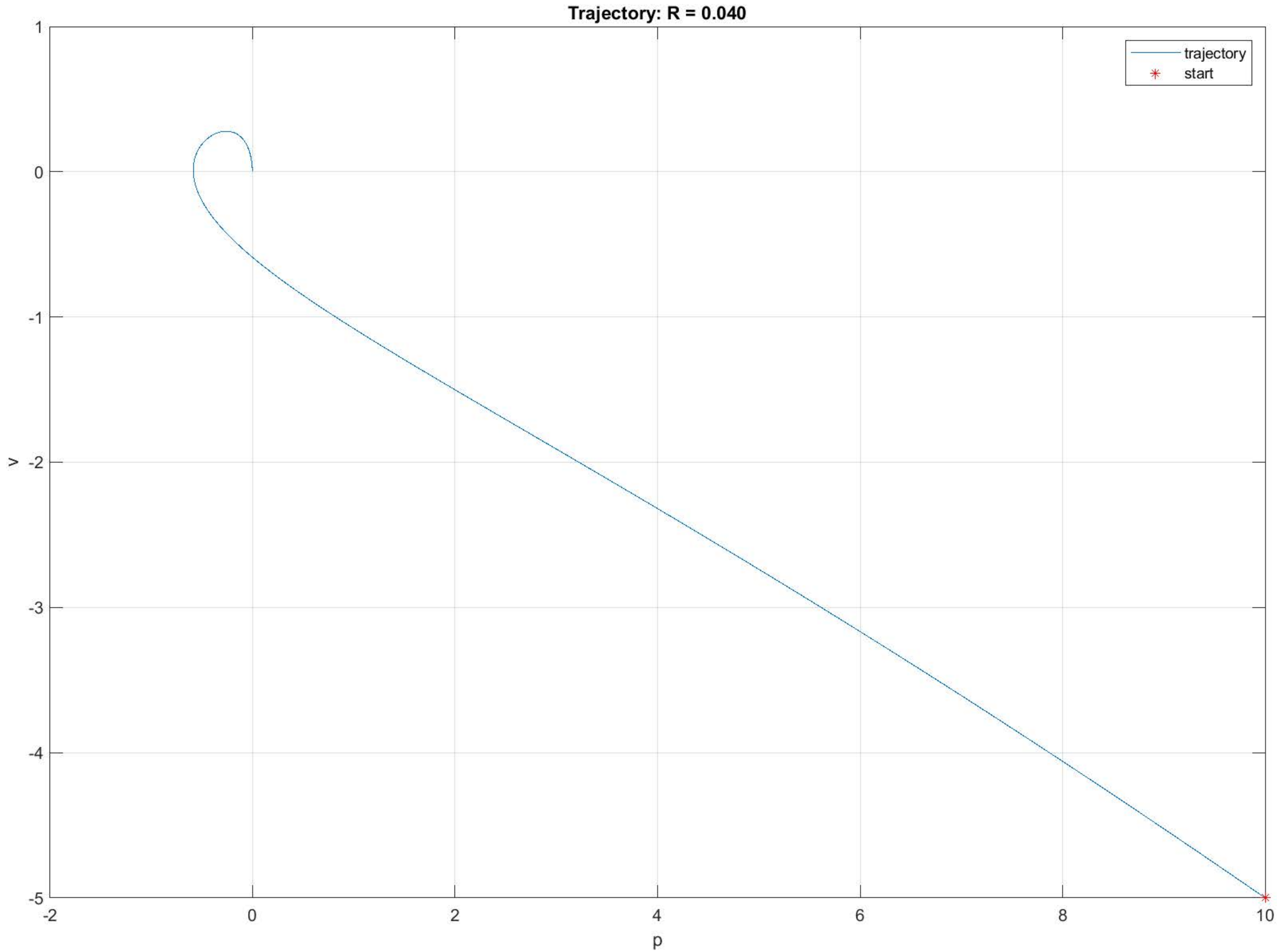


Trajectory: R = 400.000
trajectory
start
v
p
5
4
3
2
1
0
-1
-2
4
6
8
10
12
14
16

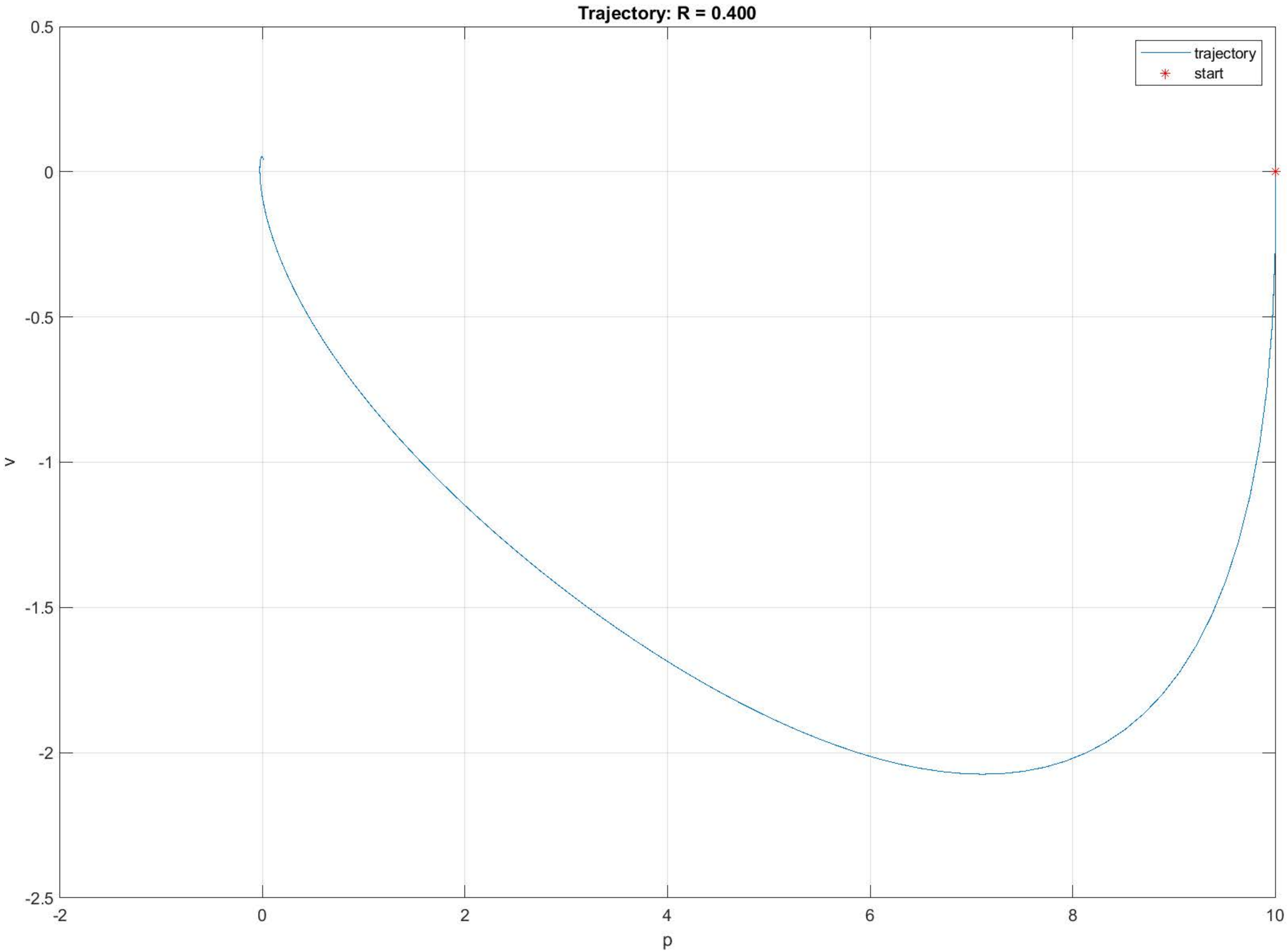

Trajectory: R = 0.040
trajectory
start
v
p
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
0
2
4
6
8
10

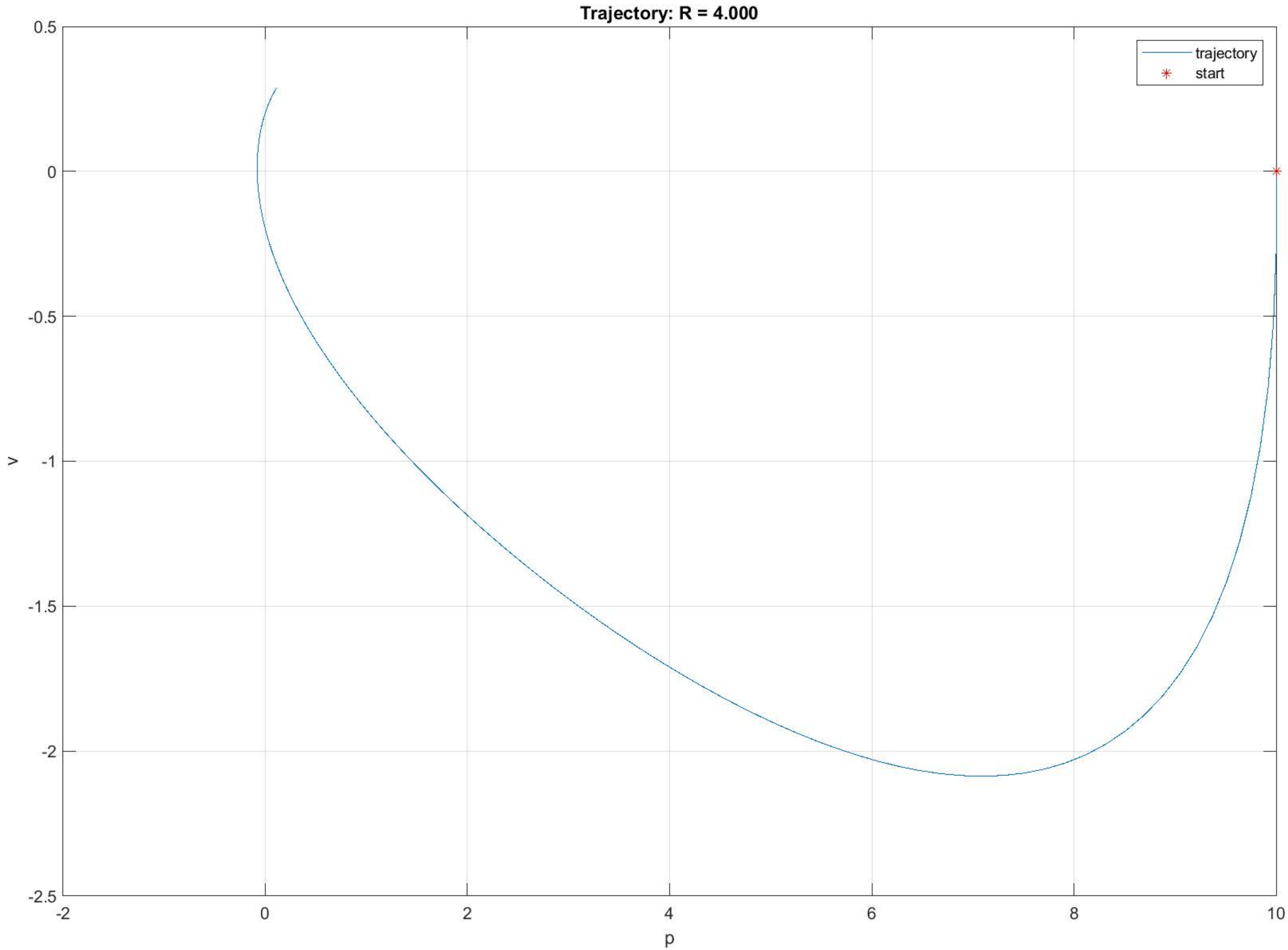


Trajectory: R = 0.040
trajectory
start
v
p
5
4
3
2
1
0
-1
-2
-3
0
2
4
6
8
10
12
14

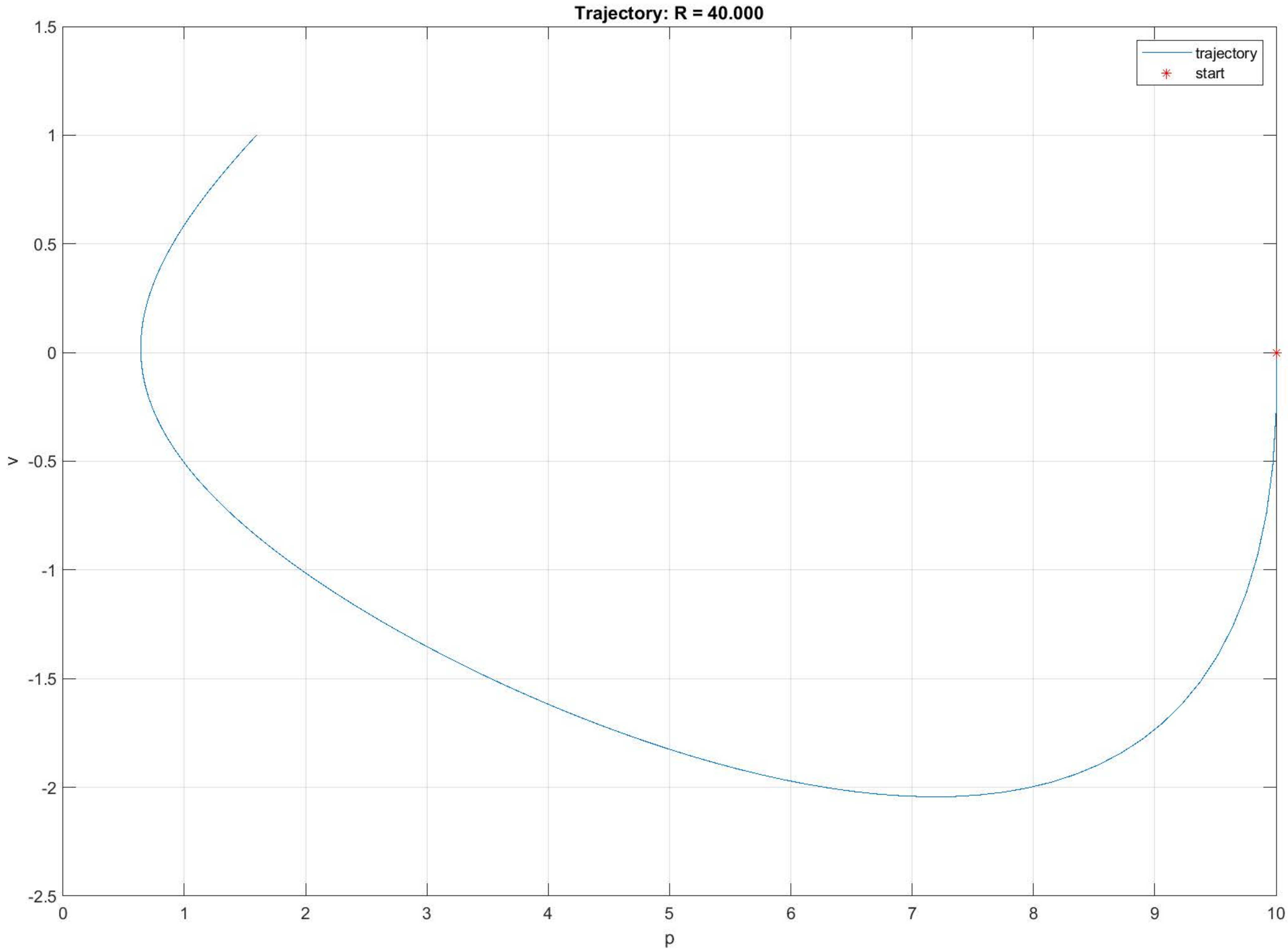


Trajectory: R = 0.040
trajectory
start
v
p
1
0
-1
-2
-3
-4
-5
-2
0
2
4
6
8
10

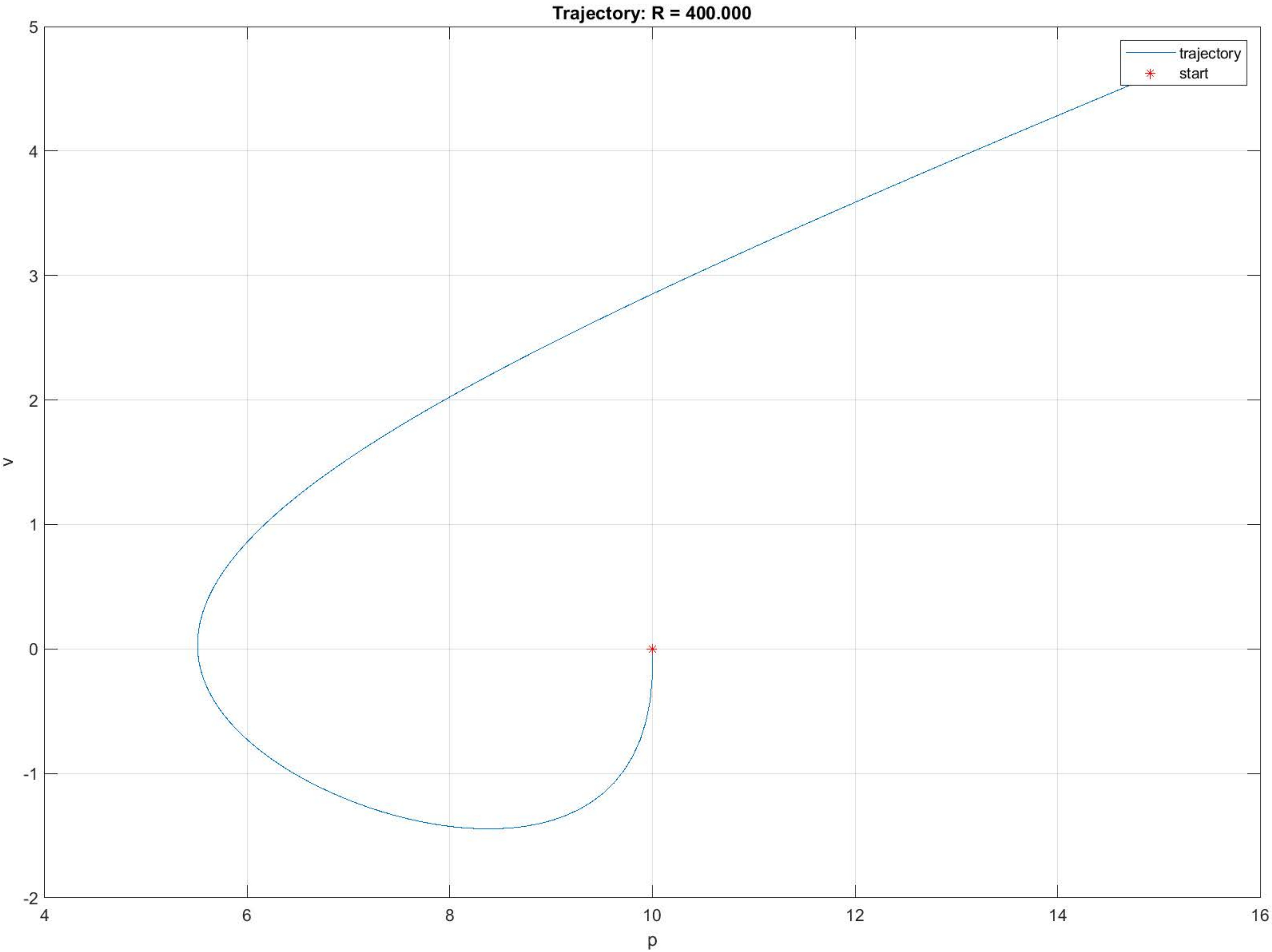

Trajectory: R = 0.400
trajectory
start
v
p
0.5
0
-0.5
-1
-1.5
-2
-2.5
-2
0
2
4
6
8
10



Trajectory: R = 4.000
trajectory
start
0.5
0
-0.5
-1
-1.5
-2
-2.5
v
-2
0
2
4
6
8
10
p



Trajectory: R = 40.000
trajectory
start
v
p
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
1
2
3
4
5
6
7
8
9
10



Trajectory: R = 400.000
trajectory
start
5
4
3
2
1
0
-1
-2
v
4
6
8
10
12
14
16
p