Homework 5: Dimitri Lezcano

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1) $J = \frac{1}{2} x(1)^2 + \int_0^1 \underbrace{\frac{1}{2}[x(t)u(t)]^2}_{L} dt$

s.t.

$\dot{x} = xu$

$x(0) = 1$

HJB

$-\partial_t V = \min_u \{ L(x,u,t) + \partial_x V \cdot f(x,u) \}$

$\min_u \{ \frac{1}{2} x^2 u^2 + \partial_x V \cdot xu \}$

$\frac{d}{du}\left(\frac{1}{2} x^2 u^2 + \partial_x V \cdot xu \right) = 0$

$x^2 u^* + V_x \cdot x = 0$

$u^* = -\frac{V_x}{x}$

$-\partial_t V = \frac{1}{2} x^2 u^{*2} + V_x \cdot x u^*$

$= \frac{1}{2} x^2 \frac{V_x^2}{x^2} + V_x \cdot x \left(-\frac{V_x}{x} \right)$

$-\partial_t V = -\frac{1}{2} V_x^2$

$\frac{1}{2} V_x^2 = \partial_t V \Rightarrow$

$\frac{1}{2} V_x^2 = \partial_t V$

$V(x,t) = X(x) T(t)$

$$\frac{1}{2} X'^2 T^2 = X T'$$

$$\frac{1}{2}\frac{X'^2}{X} = \frac{T'}{T^2} = K \quad K \text{ is a scalar}$$

$$\frac{1}{2} X'^2 = K X$$

$$X(x) = c x^2$$

$$X'(x) = 2 c x$$

$$\downarrow$$

$$\frac{1}{2}(2cx)^2 = K(cx^2)$$

$$2c^2x^2 = Kcx^2$$

$$c = K/2$$

$$\text{If } X(x) = \frac{1}{2} K x^2$$

$$X'(x) = Kx$$

$$\frac{1}{2}(X')^2 = \frac{1}{2}(Kx)^2 = \frac{1}{2}K^2x^2 = KX \checkmark$$

$$\text{So} \quad \underline{V(x,t) = \left(\frac{1}{2}Kx^2\right)T(t)}$$

$$u^* = -\frac{\nabla_x V}{x} = -\frac{Kx\,T(t)}{x} = -K\,T(t)$$

$$2)\ J = \frac{1}{2} x_f^T P_f x_f + \frac{1}{2}\int_{t_0}^{t_f} \underbrace{\left(x^T Q x + u^T R u\right)}_{L} dt$$

$$R > 0$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w(t) \quad R > 0 \quad Q, P_f \geq 0$$

$w(t)$ is known disturbances

a) C/S time of HJB

HJB

$$-\partial_t V = \min_u \left\{ L(x,u,t) + \nabla_x V^T f(x,u,t) \right\}$$

$$\nabla_u \left(L(x,u,t) + \nabla_x V^T f(x,u,t) \right)$$

$$= \nabla_u \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^T R u + \nabla_x V^T (Ax + Bu + w) \right)$$

$$\Rightarrow R u^* + B^T \nabla_x V = 0$$

$$① \quad u^* = -R^{-1} B^T \nabla_x V$$

So then

$$-\partial_t V = \frac{1}{2} x^T Q x + \frac{1}{2} \nabla_x V^T B R B^T \nabla_x V + \nabla_x V^T A x - \nabla_x V^T B R^{-1} B^T \nabla_x V + \nabla_x V^T w$$

$$= \frac{1}{2} x^T Q x - \frac{1}{2} \nabla_x V^T B R B^T \nabla_x V + \nabla_x V^T (Ax + w)$$

Suppose $V = \frac{1}{2} x^T P(t) x + b(t)^T x + c(t)$

Then we have

$$\partial_t V = \frac{1}{2} x^T \dot{P} x + \dot{b}^T x + \dot{c}$$

$$\nabla_x V = P x + b(t)$$

HJB becomes

$$-\frac{1}{2} x^T \dot{P} x - \dot{b}^T x - \dot{c} =$$

$$-\frac{1}{2}x^T\dot{P}x - \dot{b}^Tx - \dot{c} =$$

$$\frac{1}{2}x^TQx - \frac{1}{2}(Px+b)^T BRB^T(Px+b) + (Px+b)^T(Ax+w)$$

$$= \frac{1}{2}x^TQx - \frac{1}{2}x^TPBRB^TPx - x^TPBRB^Tb$$

$$-\frac{1}{2}b^TBRB^Tb + x^TPAx + b^TAx + x^TPw + b^Tw$$

Group Like Terms!

Quadratics

$$\frac{1}{2}x^T\dot{P}x = -\frac{1}{2}x^TQx + \frac{1}{2}x^TPBRB^TPx - x^TPAx$$

Single x

$$\dot{b}^Tx = x^TPBRB^Tb - b^TAx - x^TPw$$

$$x^T\dot{b} = x^T(PBRB^Tb) - x^T(A^Tb) - x^T(Pw)$$

No x

$$\dot{c} = \frac{1}{2}b^TBRB^Tb - b^Tw$$

So Quadratics are satisfied if

$$\underline{\dot{P} = -Q + PBRB^TP - PA - A^TP}$$

The "single x" are satisfied if

$$\underline{\dot{b} = PBRB^Tb - A^Tb - Pw}$$

And "no x" is satisfied if

And "no x" is satisfied i.f.

$$\dot{c} = \frac{1}{2} b^T B R B^T b - b^T w$$

Now we want

$$V(t_f) = \frac{1}{2} x_f^T P_f x_f$$

So $V(t_f) = \frac{1}{2} x_f^T P(t_f) x_f + b(t_f)^T x_f + c(t_f) = \frac{1}{2} x_f^T P_f x_f$

so boundary conditions are

$$\begin{cases} \bullet \; P(t_f) = P_f \\ \bullet \; b(t_f) = 0 \\ \bullet \; c(t_f) = 0 \end{cases}$$

$$\Rightarrow u^* = -R^{-1} B^T \nabla_x V$$
$$= -R^{-1} B^T (P x + b)$$

$$u^*(t) = \underbrace{-R^{-1}(t) B^T P(t)}_{K(t)} x(t) \underbrace{- R^{-1}(t) B^T b(t)}_{h(t)}$$

$$u^*(t) = K(t)\, x(t) + h(t)$$

b) Discrete-time

$$x_{i+1} = A_i x_i + B_i u_i + w_i$$

$$L_i(x,u) = \frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u_i^T R_i u_i$$

$$\phi_f = \frac{1}{2} x_N^T P_f x_N$$

By Bellman eqs.

$$V_i = \min_u \left\{ L_i(x,u) + V_{i+1}\left(f_i(x,u)\right) \right\}$$

and

$$V_N = \phi_f = \frac{1}{2} x_N^T P_f x_N$$

$$\min_u \left\{ \frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u_i^T R_i u_i + V_{i+1}(A_i x_i + B_i u_i + w_i) \right\}$$

$$\Rightarrow \nabla_u \left(\frac{1}{2} x_i^T Q_i x_i + \frac{1}{2} u^T R_i u + V_{i+1}(A x_i + B u + w_i) \right) = 0$$

$$R u^* + \nabla_u \left[V_{i+1}(A_i x_i + B_i u + w) \right] \Big|_{u=u^*} = 0$$

Suppose $V_i(x) = \frac{1}{2} x^T P_i x + b_i^T x + c_i$

Then $V_{i+1}(A_i x + B_i u + w_i) =$

$$\frac{1}{2}(A_i x + B_i u + w_i)^T P_{i+1}(A_i x + B_i u + w_i) + b_{i+1}^T(A_i x + B_i u + w_i) + c_{i+1}$$

$$= \frac{1}{2}(A_i x + w_i)^T P_{i+1}(A_i x + w_i) + u^T B_i^T P_{i+1}(A_i x + w_i)$$

$$+ \frac{1}{2} u^T B_i^T P_{i+1} B_i u + b_{i+1}^T(A_i x + w_i) + b_{i+1}^T B_i u + c_{i+1}$$

$\therefore$

$$\nabla_u V_{i+1} = B_i^T P_{i+1}(A_i x + w_i) + B_i^T P_{i+1} B_i u + B_i^T b_{i+1}$$

$$\downarrow$$

$$R_i u^* + B_i^T P_{i+1}(A_i x + w_i) + B_i^T P_{i+1} B_i u^* + B_i^T b_{i+1} = 0$$

$$\Rightarrow \left[R_i + B_i^T P_{i+1} B_i\right] u^* = - B_i^T P_{i+1}(A_i x + w_i) - B_i^T b_{i+1}$$

$$u^* = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T \left(P_{i+1} A_i x + P_{i+1} w_i + b_i \right)$$

$$\underline{u^* = K_i x + k_i}$$

$$\underline{K_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T P_{i+1} A_i}$$

and

$$\underline{k_i = -\left(R_i + B_i^T P_{i+1} B_i\right)^{-1} B_i^T \left(P_{i+1} w_i + b_{i+1}\right)}$$

Finally conditions for P_i, b_i, c_i..

$$V_i = L_i(x, u^*) + V_{i+1}\left(f_i(x, u^*)\right)$$

$$\frac{1}{2} x^T P_i x + b_i^T x + c_i = \frac{1}{2} x^T Q_i x + \frac{1}{2} u^{*T} R_i u^*$$
$$+ \frac{1}{2}(A_i x + B_i u^* + w_i)^T P_{i+1} (A_i x + B_i u^* + w_i)$$
$$+ b_{i+1}^T (A_i x + B_i u^* + w_i)$$
$$+ c_{i+1}$$

$$= \frac{1}{2}\left[(A_i + B_i K_i) x + (w_i + B_i k_i) \right]^T P_{i+1} \left[(A_i + B_i K_i) x + (w_i + B_i k_i) \right]$$
$$+ b_{i+1}^T \left[(A_i + B_i K_i) x + (w_i + B_i k_i) \right] + c_{i+1}$$
$$+ \frac{1}{2} x^T Q_i x + \frac{1}{2} (K_i x + k_i)^T R_i (K_i x + k_i)$$

(Quadratics)

$$\frac{1}{2}x^T P_i x = \frac{1}{2} x^T (A_i + B_i K_i)^T P_{i+1} (A_i + B_i K_i) x + \frac{1}{2} x^T Q_i x + \frac{1}{2} x^T K_i^T R_i K_i x$$

$$\downarrow$$

$$\underline{P_i = (A_i + B_i K_i)^T P_{i+1} (A_i + B_i K_i) + Q_i + K_i^T R_i K_i}$$

(Single x)

$$x^T b_i = x^T (A_i + B_i K_i)^T P_{i+1} (w_i + B_i k_i) + x^T (A_i + B_i K_i)^T b_{i+1}$$
$$+ x^T K_i^T R_i k_i$$

$$\downarrow$$

$$\underline{b_i = (A_i + B_i K_i)^T \left[P_{i+1} (w_i + B_i k_i) + b_{i+1} \right] + K_i^T R_i k_i}$$

(No x)

$$c_i = \frac{1}{2} (w_i + B_i k_i)^T P_{i+1} (w_i + B_i k_i) + b_{i+1}^T (w_i + B_i k_i) + c_{i+1}$$
$$+ \frac{1}{2} k_i^T R_i k_i$$

With Boundary conditions

$$V_N = \frac{1}{2} x_N^T P_N x_N + b_N^T x_N + c_N = \frac{1}{2} x_N^T P_f x_N$$

$$\mapsto \begin{cases} \cdot\ P_N = P_f \\ \cdot\ b_N = 0 \\ \cdot\ c_N = 0 \end{cases}$$

3)
$$p_{i+1} = p_i + \Delta t\, v_i$$
$$v_{i+1} = v_i + \Delta t\,(-0.5 v_i + 0.2 p_i + u_i + 0.1)$$

$$x_{i+1} = \begin{pmatrix} p_{i+1} \\ v_{i+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & \Delta t \\ 0.2\Delta t & 1 - 0.5\Delta t \end{pmatrix}}_{A_i} x_i + \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{B_i} u_i + \underbrace{\begin{pmatrix} 0 \\ 0.1 \end{pmatrix}}_{w_i}$$

$$J = \frac{1}{2}\left(p_N^2 + v_N^2\right) + \sum_{i=0}^{N} \frac{1}{2}\left(R u_i^2\right)$$

$$= \frac{1}{2} x_N^T P_f x_N + \sum_{i=0}^{N} \frac{1}{2} R u_i^2$$

$$P_f = I_2 \qquad Q_i = 0_{2\times 2} \qquad R_i = R$$

So we have that

$$V_i(x) = \frac{1}{2} x^T P_i x + b_i^T x + c_i$$

$$K_i = -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T P_{i+1} A$$

$$* \; K_i = -(R + e_2^T P_{i+1} e_2)^{-1} e_2^T P_{i+1} A \qquad (1\times 2 \text{ - row vector})$$

$$k_i = -(R_i + B_i^T P_{i+1} B_i)^{-1} B_i^T (P_{i+1} w_i + b_{i+1})$$

$$* \; k_i = -(R + e_2^T P_{i+1} e_2)^{-1} e_2^T (P_{i+1} w_i + b_{i+1}) \qquad (\text{scalar})$$

where

$$P_i = (A_i + B_i K_i)^T P_{i+1} (A_i + B_i K_i) + Q_i + K_i^T R_i K_i$$

$$* \; P_i = (A + e_2 K_i)^T P_{i+1} (A + e_2 K_i) + K_i^T R K_i$$

$$P_N = P_f = I_2$$

$$b_i = (A_i + B_i K_i)^T \left[P_{i+1}(w_i + B_i k_i) + b_{i+1}\right] + K_i^T R_i k_i$$

$$* \; b_i = (A + e_2^T K_i)^T \left[P_{i+1}(w + e_2 k_i) + b_{i+1}\right] + K_i^T R k_i$$

$b_N = 0$ (2x1 col. vector)

$$c_i = \frac{1}{2}(w_i + B_i k_i)^T P_{i+1}(w_i + B_i k_i) + b_{i+1}^T (w_i + B_i k_i) + c_{i+1} + \tfrac{1}{2} k_i^T R_i k_i$$

$$c_i = \frac{1}{2}(w + e_2 k_i)^T P_{i+1}(w + e_2 k_i) + b_{i+1}^T (w + e_2 k_i) + c_{i+1} + \tfrac{1}{2} k_i R k_i$$

$c_N = 0$ (scaler)