Homework 4: Dimitri Lezcano

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1) $\dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}}_{A} x + u \underbrace{e_2}_{B} = f(x,u)$

$a \geq 0$, $|u| \leq 1$, $x(t_f) = 0$

$$\min \int_{t_0}^{t_f} \underbrace{[\gamma + |u(t)|]}_{L} dt$$

$\gamma \geq 0$ constant, $t_f$ free.

a) Adjoint equations

$H = L + \lambda^T f$

$H = \gamma + u + \lambda^T [Ax + Bu]$

$\dot{\lambda} = \nabla_x H = -A^T \lambda$

$\dot{\lambda} = \begin{pmatrix} 0 & 0 \\ -1 & a \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$

$\dot{\lambda}_1 = 0 \rightarrow \lambda_1 = c_1$

$\dot{\lambda}_2 = -\lambda_1 + a\lambda_2$

$\dot{\lambda}_2 - a\lambda_2 = -c_1$

$\lambda_2 = \frac{c_1}{a} + c_2 e^{at}$

$\lambda_1 = c_1$

$\lambda_2 = \frac{c_1}{a} + c_2 e^{at}$ $c_1, c_2$ constants

$H(u) = u + \lambda^T B u$ ("u" terms of u only)

$= (1 + \lambda^T B) u$

$dH/du = 1 + \lambda^T B = 1 + \lambda_2$

So we have min H when:

$u = \begin{cases} +1 & 1 + \lambda^T B < 0 \\ -1 & 1 + \lambda^T B > 0 \end{cases}$

$$u = \begin{cases} +1 & 1+\lambda^T B < 0 \\ -1 & 1+\lambda^T B > 0 \\ \text{undefined} & 1+\lambda^T B = 0 \end{cases}$$

$$u = \begin{cases} +1 & \lambda_2 < -1 \\ -1 & \lambda_2 > -1 \\ \text{undefined} & \lambda_2 = -1 \end{cases}$$

b) Possible control sequences

Possible maneuvers

$$\lambda_2(t) = \frac{c_1}{a} + c_2 e^{at}$$

↳ here, we have that there is a maximum of 1 switch

$$u^*(t) = \begin{cases} +1 \\ -1 \\ h(t-t_0) - 2h(t-t_1) \\ -h(t-t_0) + 2h(t-t_1) \end{cases}$$

for $t \in [t_0, t_f]$ where

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$ heaviside function

and $t_1 = \frac{1}{a} \ln\left(-\frac{c_1}{a c_2}\right)$ if exists. (i.e. if there is a 0-crossing)

c) Show that there are no singular intervals

Given $u = \pm 1$, Dynamics turn into

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_2 \pm 1$$

$$\dot{x}_2 + a x_2 = \pm 1$$

$$x_2 = \pm \frac{1}{a} + c_3 e^{-at}$$

$$\dot{x}_1 = \pm \frac{1}{a} + c_3 e^{-at}$$

$\int(\cdot)dt$

$$\int(\cdot)\,dt \left( \begin{array}{l} x_1 = \pm \frac{1}{a} + c_3 e \\ \quad x_1 = \frac{\pm(t-t_0)}{a} - \frac{c_3}{a} e^{-at} + c_4 \end{array} \right.$$

$$x_1 = \pm \frac{t}{a} - \frac{c_3}{a} e^{-at} + c_4$$

$$x_2 = \pm \frac{1}{a} + c_3 e^{-at}$$

$$a x_1 + x_2 = \pm t \pm \frac{1}{a} - c_3 e^{-at} + c_3 e^{-at} + c_4$$

$$a x_1 + x_2 = \pm\left( t + \frac{1}{a} \right) + c_4 \quad (\text{line?,})$$

Suppose at $t_1$ we switch from
$u = +1 \rightarrow u = -1$
so we have

$$a x_1(t_1^-) + x_2(t_1^-) = t_1 + \frac{1}{a} + c_4$$

$$\downarrow$$

$$a x_1(t_1^+) + x_2(t_2^+) = -t_1 - \frac{1}{a} + c_4$$

And by continuity in $x$ we have

$$0 = 2t_1 + \frac{2}{a} \Rightarrow t_1 = \frac{-1}{a} < 0 \quad \because a > 0.$$

There is no switch $\because$ "switch" happens for $t_1 < 0 \le t_0$ (start time).

d) Optimal Control Law

So from c) we only have 2 conditions:

$u^* = \Big\{ \; +1$

conditions:

$$u^* = \begin{cases} +1 \\ -1 \end{cases}$$

and looking at the conditions

$$a x_1 + x_2 = \pm(t + 1/c) + c_4$$

for optimal final time $t_f^*$,

$$a x_1(t_f^*) + x_2(t_f^*) = 0 = \pm(t_f^* + 1/c) + c_4$$

$$\Rightarrow c_4 = \mp(t_f^* + 1/c)$$

$$\Downarrow$$

$$a x_1 + x_2 = \pm(t + 1/c) \mp (t_f^* + 1/c)$$

$$= \pm(t - t_f^*)$$

$$a x_1 + x_2 = \mp(t_f^* - t) \quad \text{for } u = \pm 1$$

$$t_f^* - t > 0$$

$$\forall t \in [t_0, t_f^*)$$

so for

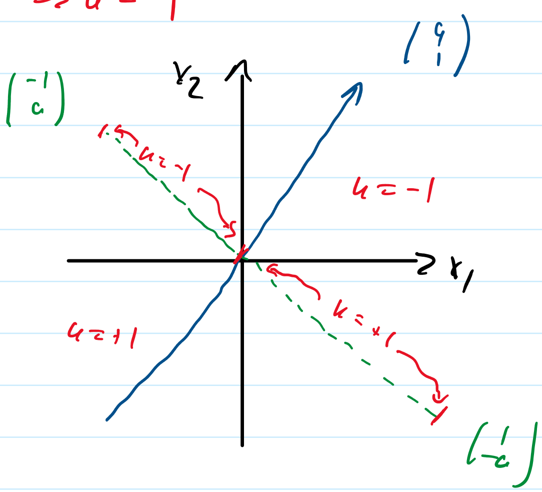
$$a x_1 + x_2 = -(t_f^* - t) < 0$$

$$\Rightarrow u = +1$$

and

$$a x_1 + x_2 = (t_f^* - t) > 0$$

$$\Rightarrow u = -1$$




So we have 3 conditions so far:

$$\begin{pmatrix} a \\ 1 \end{pmatrix}^T x < 0 \rightarrow u = +1$$

$$\begin{pmatrix} a \\ 1 \end{pmatrix}^T x > 0 \rightarrow u = -1$$

$$x = 0 \rightarrow u = 0$$

(goal)

But what about perpendicular conditions to $\begin{pmatrix} a \\ 1 \end{pmatrix}$? Look @ dynamics:

Case 1: $x = \gamma \begin{pmatrix} -1 \\ a \end{pmatrix}$ (upper-left quadrant) $\gamma \geq 0$

$\dot{x}_1 = x_2$
$\dot{x}_2 = -a x_2 + u$

$\because x_2 = \gamma a \geq 0 \Rightarrow \dot{x}_1 \geq 0$ so $x_1$ approaches closer to goes to 0.

also
$\dot{x}_2 = -\gamma a^2 + u$ when $-\gamma a^2 < 0$. $\because$ we we want $x_2$ to decrease ($\dot{x}_2 < 0$) to approach $x_2 = 0$, we let $u = -1$.

so where
$x = \gamma \begin{pmatrix} -1 \\ a \end{pmatrix}$ $\gamma > 0 \to u = -1$

And for
$x = \gamma \begin{pmatrix} 1 \\ -a \end{pmatrix}$ $\gamma \geq 0$, where

$\dot{x}_1 = x_2 = -\gamma a < 0$
$\dot{x}_2 = +\gamma a^2 + u$ where we want $\dot{x}_2 \geq 0 \Rightarrow$ let $u = +1$

$$u^*(t) = \begin{cases} +1 & \begin{pmatrix} 1 \\ a \end{pmatrix}^T x < 0 \\ -1 & \begin{pmatrix} 1 \\ a \end{pmatrix}^T x \geq 0 \\ +1 & x = \gamma \begin{pmatrix} 1 \\ -a \end{pmatrix};\ \gamma > 0 \\ -1 & x = \gamma \begin{pmatrix} -1 \\ a \end{pmatrix};\ \gamma > 0 \\ 0 & x = 0 \end{cases}$$

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2) $\min \ J = \|x(t_f)\|^2$

$\dot{x} = Ax + Bu = f(x,u)$
$x \in \mathbb{R}^n$, $A \in M_n(\mathbb{R})$, $x(0) = x_0$
$u \in \mathbb{R}$, $B \in \mathbb{R}^n$

$t_f$ fixed / given

constraint : $|u(t)| \le 1$

Suppose $J_{min} > 0$. $(\Rightarrow x(t_f) \neq 0)$

$L = 0$

$\phi(x_f, t_f) = x_f^T x_f$

$$J = \phi + \int_{t_0}^{t_f} L \, dt$$

$$H = L + \lambda^T f$$

$$H = \lambda^T (Ax + Bu)$$

$$\dot{\lambda} = -D_x H = -A^T \lambda$$

$\hookrightarrow \lambda(t) = \exp(-A^T t)\lambda(t_0)$

$H_u(u, \lambda) = (\lambda^T B) u$ ($u$ terms of $H$)

Transversality Conds. :

$$\lambda(t_f) = D_x \phi \big|_{t = t_f}$$

$\lambda(t_f) = 2x(t_f) \neq 0 \quad \because J_{min} < 0.$

$\hookrightarrow \lambda(t) = 2\exp(-A(t - t_f))\, x(t_f) \neq 0$

$\because |\exp(M)| \neq 0 \quad \forall M \in \mathbb{R}^{n \times n}$ and

$x(t_f) \neq 0.$

So we have $H$ is minimized when

$H_u = (\lambda^T B) u$ is minimized

$\because \lambda \neq 0,$

$H_u$ is minimized when

$$u = \begin{cases} +1 & \lambda^T B < 0 \\ -1 & \lambda^T B > 0 \\ \text{undefined} & \lambda^T B = 0 \end{cases}$$

which means the optimal control is "bang-bang".

For

$\dot{x}_1 = x_2 \qquad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = e_2$

$$\dot{x}_1 = x_2 \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = e_2$$
$$\dot{x}_2 = u$$

↓ (by class)

$$x_1 = \pm \frac{1}{2} t^2 + c_3 t + c_4$$

$$x_2 = \pm t + c_3 \qquad t \in [t_0, t_f]$$
$$u = \pm 1$$

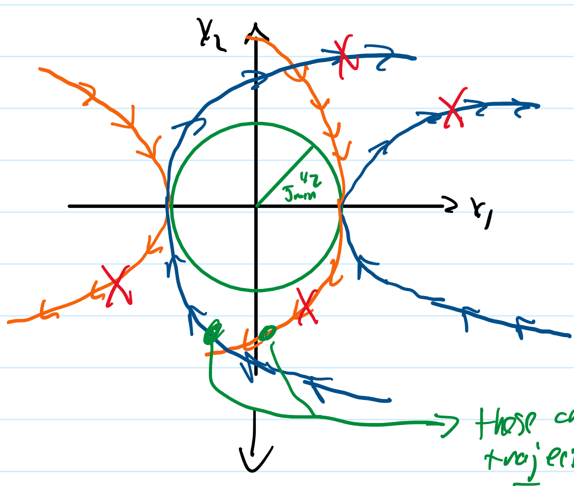
$$x_2^2 = t^2 \pm 2tc_3 + c_3^2$$

$$x_1 \mp \frac{1}{2} x_2^2 = \left( \cancel{\pm \frac{1}{2} t^2} + \cancel{c_3 t} + c_4 \right) \mp \left( \cancel{\frac{1}{2} t^2} \cancel{\pm t c_3} + c_3^2 \right)$$
$$= c_4 \mp c_3^2$$
$$= c_5, c_6$$
$$x_1 = \pm 1/2\, x_2^2 + c_{5,6} \qquad u = \pm 1$$

$$\left.\begin{aligned} x_1 &= +1/2\, x_2^2 + c_5 \longrightarrow u = +1 \\ x_1 &= -1/2\, x_2^2 + c_6 \longrightarrow u = -1 \end{aligned}\right\}$$ family of parabolas in $x_1$ direction

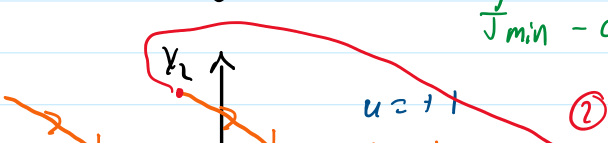
$\because J_{min} > 0$, $(x_1(t_f), x_2(t_f)) \neq 0 \Rightarrow$
$c_5, c_6 \neq 0$

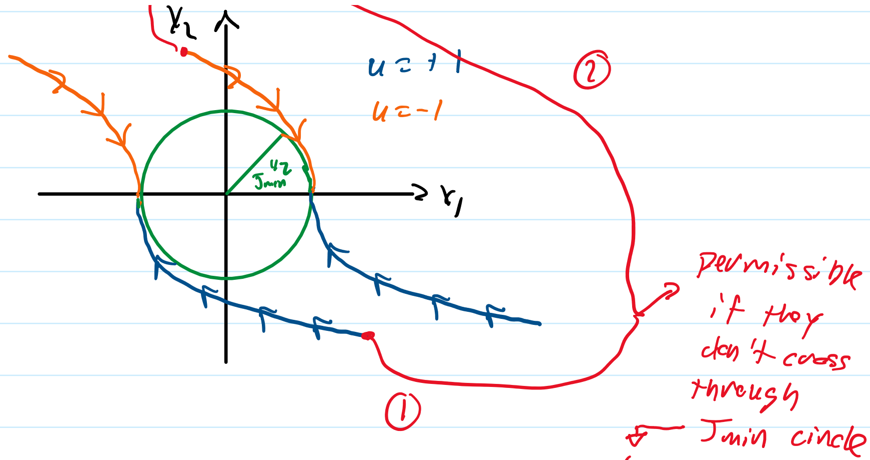
Since all of these trajectories are in the $x_1$ direction, we can control $x_2 \to 0$ and then we get
$x_1 = c_5 = \pm\sqrt{J_{min}}$ (on positive $x_1$-side generally)
$x_1 = c_6 = \mp\sqrt{J_{min}}$ (on negative $x_1$-side generally)




$u = +1$
$u = -1$

→ these only viable if trajectories don't cross through $J_{min}$ - circle






① $x_1 = +\frac{1}{2} x_2^2 - J_{min}^{1/2}$

only valid for $J_{min}^{1/2} \leq 1$ or $J_{min} \leq 1$

cond: $\forall x_1, x_2$

$$x_1^2 + x_2^2 \geq J_{min}$$

$$\frac{1}{4} x_2^4 - x_2^2 J_{min}^{1/2} + J_{min} + x_2^2 \geq J_{min}$$

$$\frac{1}{4} x_2^4 + (1 - J_{min}^{1/2}) x_2^2 \geq 0$$

$$\equiv \frac{1}{4} x_2^2 + (1 - J_{min}^{1/2}) \geq 0$$

$\forall x_2 = 0$

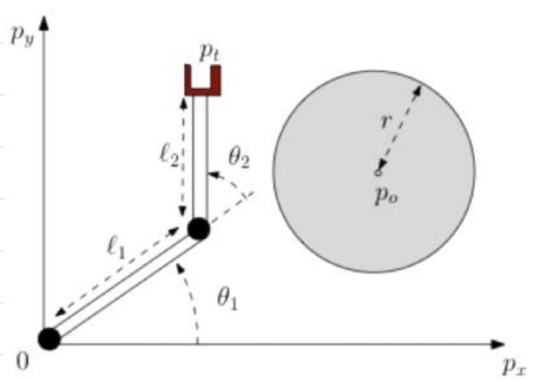
$$\equiv \frac{1}{4} (0)^2 + (1 - J_{min}^{1/2}) \geq 0$$

$$\Rightarrow (1 - J_{min}^{1/2}) \geq 0$$

$$J_{min}^{1/2} \leq 1$$

Similarly for ② by symmetry.

3)




$$x = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \dot\theta_1 \\ \dot\theta_2 \end{pmatrix} \qquad \dot x = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + u_1 e_3 + u_2 e_4$$

$$\dot{\theta}_2 \qquad\qquad 0 \; 0 \; 0 \; 0$$

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ u_1 \\ u_2 \end{pmatrix} \qquad P_0 = \begin{pmatrix} P_{0x} \\ P_{0y} \end{pmatrix} \in \mathbb{R}^2$$

$$P_t = \begin{pmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

c) $c(x(t), t) \leq 0$, derive $q$-th order state control inequality

we want

$$\| P_t(t) - P_0 \|^2 \geq r^2$$

$$r^2 - (P_t - P_0)^T (P_t - P_0) \leq 0$$

$$c(x) = r^2 - P_t^T P_t + 2 P_t^T P_0 - P_0^T P_0 \leq 0$$

Note: $P_t = l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1$

where $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$\dot{R}(\theta) = R(\theta) R^T(\theta) \dot{R}(\theta)$$

$$= R(\theta) \hat{\dot{\theta}} \quad \text{where}$$

$$\hat{\dot{\theta}} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}$$

Short hand: $R(\theta_1) = R_1$, $R_{12} = R(\theta_1 + \theta_2) = R(\theta_1) R(\theta_2)$

$R(\theta_2) = R_2$, $\theta_{12} = \theta_1 + \theta_2$

$$P_t = l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1$$

$$\dot{P}_t = l_1 \dot{R}_1 e_1 + l_2 \dot{R}_{12} e_1$$

$$= l_1 R_1 R_1^T \dot{R}_1 e_1 + l_2 R_{12} R_{12}^T \dot{R}_{12} e_1$$

$$= l_1 R_1 \hat{\dot{\theta}}_1 e_1 + l_2 R_{12} \hat{\dot{\theta}}_{12} e_1$$

$$\hat{\dot{\theta}} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}$$

$$\hat{\dot{\theta}} e_1 = \dot{\theta} e_2$$

$$= l_1 R_1 (\dot{\theta}_1 e_2) + l_2 R_{12} (\dot{\theta}_{12} e_2)$$

$$\dot{P}_t = (l_1 \dot{\theta}_1) R_1 e_2 + (l_2 \dot{\theta}_{12}) R_{12} e_2$$

$$= l_1 R_1(\theta_1 e_2) + l_2 R_{12}(\theta_{12} e_2)$$

$$\dot p_t = (l_1 \dot\theta_1) R_1 e_2 + (l_2 \dot\theta_{12}) R_{12} e_2$$

$$\ddot p_t = (l_1 \ddot\theta_1) R_1 e_2 + (l_1 \dot\theta_1) \dot R_1 e_2 + l_2 \ddot\theta_{12} R_{12} e_2 + l_2 \dot\theta_{12} \dot R_{12} e_2$$

$$= l_1 \ddot\theta_1 R_1 e_2 + l_2 \ddot\theta_{12} R_{12} e_2 + l_1 \dot\theta_1 R_1 \hat{\dot\theta}_1 e_2 + l_2 \dot\theta_{12} R_{12} \hat{\dot\theta}_{12} e_2$$

$$= l_1 \ddot\theta_1 R_1 e_2 + l_2 \ddot\theta_{12} R_{12} e_2 - l_1 \dot\theta_1^2 R_1 e_1 - l_2 \dot\theta_{12}^2 R_{12} e_1$$

$$\because \hat{\dot\theta} e_2 = \begin{pmatrix} 0 & -\dot\theta \\ \dot\theta & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dot\theta \\ 0 \end{pmatrix}$$

$$\dot p_t = (l_1 \dot\theta_1) R_1 e_2 + (l_2 \dot\theta_{12}) R_{12} e_2$$

$$\ddot p_t = (l_1 \ddot\theta_1) R_1 e_2 + (l_2 \ddot\theta_{12}) R_{12} e_2 - (l_1 \dot\theta_1^2) R_1 e_1 - (l_2 \dot\theta_{12}^2) R_{12} e_1$$

← controls appear here

$\ddot\theta_1 = u_1$

$\ddot\theta_{12} = u_2$

$$\frac{d}{dt} c = \nabla_{p_t} c^T \dot p_t$$

$$= (2p_o - 2p_t)^T \dot p_t$$

$$\dot c = 2p_o^T \dot p_t - 2p_t^T \dot p_t$$

$$\ddot c = 2p_o^T \ddot p_t - 2\dot p_t^T \dot p_t - 2p_t^T \ddot p_t$$

$$= (2p_o - 2p_t)^T \ddot p_t - 2\dot p_t^T \dot p_t$$

$$\dot p^T \dot p = l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_{12}^2 + 2(l_1 \dot\theta_1 R_1 e_2)^T (l_2 \dot\theta_{12} R_{12} e_2)$$

$$= l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_{12}^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_{12} \, e_2^T R_1^T R_{12} e_2$$

$$= l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_{12}^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_{12} \, e_2^T R(\theta_2) e_2$$

$$= l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_{12}^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_{12} \sin\theta_2$$

$$p_t^T \ddot p_t = \left[ (l_1 R_1 + l_2 R_{12}) e_1 \right]^T \left[ (l_1 u_1 R_1 + l_2 u_2 R_{12}) e_2 - (l_1 \dot\theta_1^2 R_1 + l_2 \dot\theta_{12}^2 R_{12}) e_1 \right]$$

$$= e_1^T (l_1 R_1^T + l_2 R_{12}^T)(l_1 u_1 R_1 + l_2 u_2 R_{12}) e_2$$
$$- e_1^T (l_1 R_1^T + l_2 R_{12}^T)(l_1 \dot\theta_1^2 R_1 + l_2 \dot\theta_{12}^2 R_{12}) e_1$$
$$= e_1^T \left( \cancel{l_1^2 u_1 I_3} + \cancel{l_2^2 u_2 I_3} + l_1 l_2 u_1 R_{12}^T R_1 + l_1 l_2 u_2 R_1^T R_{12} \right) e_2$$
$$- e_1^T \left( l_1^2 \dot\theta_1^2 I_3 + l_2^2 \dot\theta_{12}^2 I_3 + l_1 l_2 \dot\theta_1^2 R_{12}^T R_1 + l_1 l_2 \dot\theta_{12}^2 R_1^T R_{12} \right) e_1$$

$$= e_1^T\left(l_1^2 \cancel{u_1} I_3 + l_2^2 \cancel{u_2} I_3 + l_1 l_2 u_1 R_{12}^T R_1 + l_1 l_2 u_2 R_1^T R_{12}\right) e_2$$
$$- e_1^T\left(l_1^2 \dot\theta_1^2 I_3 + l_2^2 \dot\theta_{12}^2 I_3 + l_1 l_2 \dot\theta_1^2 R_{12}^T R_1 + l_1 l_2 \dot\theta_{12}^2 R_1^T R_{12}\right) e_1$$

$$= e_1^T\left(l_1 l_2 u_1 R_2^T + l_1 l_2 u_2 R_2\right) e_2$$
$$- l_1^2 \dot\theta_1^2 - l_2^2 \dot\theta_2^2$$
$$- e_1^T\left(l_1 l_2 \dot\theta_1^2 R_2^T + l_1 l_2 \dot\theta_{12}^2 R_2\right) e_1$$

$$p_t^T \ddot p_t = l_1 l_2 e_1^T\left(u_1 R_2^T + u_2 R_2\right) e_2$$
$$- l_1^2 \dot\theta_1^2 - l_2^2 \dot\theta_2^2 - l_1 l_2 e_1^T\left(\dot\theta_1^2 R_2^T + \dot\theta_{12}^2 R_2\right) e_1$$

$$a R_2^T + b R_2 = \begin{pmatrix} (a+b)\cos\theta_2 & (a-b)\sin\theta_2 \\ (b-a)\sin\theta_2 & (a+b)\cos\theta_2 \end{pmatrix}$$

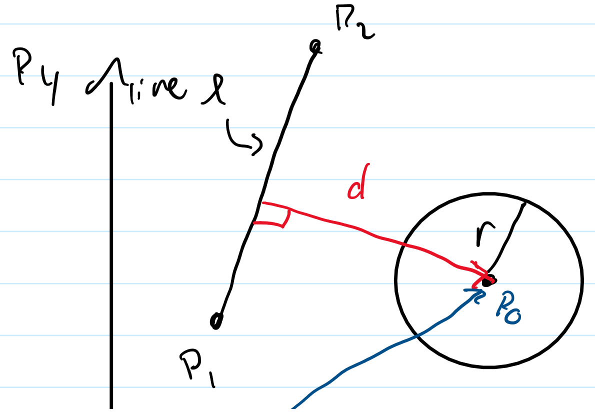
$$p_t^T \ddot p_t = l_1 l_2\left(u_1 R_2^T + u_2 R_2\right)_{12}$$
$$- l_1^2 \dot\theta_1^2 - l_2^2 \dot\theta_{12}^2$$
$$- l_1 l_2\left(\dot\theta_1^2 R_2^T + \dot\theta_{12}^2 R_2\right)_{11}$$
$$= l_1 l_2 (u_1 - u_2)\sin\theta_2 - l_1^2 \dot\theta_1^2 - l_2^2 \dot\theta_{12}^2$$
$$- l_1 l_2\left(\dot\theta_1^2 + \dot\theta_{12}^2\right)\cos\theta_2$$

$$\ddot c = 2(p_o - p_t)^T \ddot p_t - 2 \dot p_t^T \dot p_t$$

$$= 2 p_o^T \ddot p_t - 2 p_t^T \ddot p_t - 2 \dot p_t^T \dot p_t$$

$$\ddot c = 2 p_o^T \ddot p_t - 2\begin{pmatrix} l_1 l_2 (u_1 - u_2)\sin\theta_2 - l_1^2 \dot\theta_1^2 - l_2^2(\dot\theta_1 + \dot\theta_2)^2 \\ - l_1 l_2\left(\dot\theta_1^2 + (\dot\theta_1 + \dot\theta_2)^2\right)\cos\theta_2 \end{pmatrix} \le 0$$
$$-2\left(l_1^2 \dot\theta_1^2 + l_2^2(\dot\theta_1 + \dot\theta_2)^2 + 2 l_1 l_2 (\dot\theta_1^2 + \dot\theta_1 \dot\theta_2)\sin\theta_2\right)$$

b) No point on robot collides w/ obstacle




$l : \lambda p_1 + (1-\lambda) p_2 \qquad \lambda \in [0, 1]$

$p_\lambda \in l$

$p_\lambda$ intersects circle cntrd $p_0$ if

$$\| p_\lambda - p_0 \| = r$$

$$(p_\lambda - p_0)^T (p_\lambda - p_0) = r^2$$

$$p_\lambda^T p_\lambda + p_0^T p_0 - 2 p_0^T p_\lambda = r^2$$

$$(\lambda^2 p_1^T p_1 + (1-\lambda)^2 p_2^T p_2 + 2\lambda(1-\lambda) p_1^T p_2) + p_0^T p_0 - 2 p_0^T (\lambda p_1 + (1-\lambda) p_2) = r^2$$

$$\lambda^2 p_1^T p_1 + (\lambda^2 - 2\lambda + 1) p_2^T p_2 + 2(\lambda - \lambda^2) p_1^T p_2 + p_0^T p_0 - 2\lambda p_0^T p_1 - 2(1-\lambda) p_0^T p_2 = r^2$$

$$(p_1^T p_1 + p_2^T p_2 - 2 p_1^T p_2)\lambda^2 + (2 p_1^T p_2 - 2 p_2^T p_2 - 2 p_0^T p_1 + 2 p_0^T p_2)\lambda + (p_2^T p_2 + p_0^T p_0 - 2 p_0^T p_2 - r^2) = 0$$

$$A(p_0, p_1, p_2) \doteq p_1^T p_1 - 2 p_1^T p_2 + p_2^T p_2 = (p_2 - p_1)^T (p_2 - p_1)$$

$$B(p_0, p_1, p_2) \doteq 2(p_1^T p_2 - p_2^T p_2 - p_0^T p_1 + p_0^T p_2)$$

$$C(p_0, p_1, p_2) \doteq p_2^T p_2 + p_0^T p_0 - 2 p_0^T p_2 - r^2 = (p_0 - p_2)^T (p_0 - p_2) - r^2$$

$A\lambda^2 + B\lambda + C = 0$, solve for $\lambda$.

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We have there are intersections if $\lambda \in [0, 1]$.

So $B^2 - 4AC \geq 0$ for real roots

so we want:

1) $B^2 - 4AC \leq 0$ (Non-real roots / 1 root)

OR

2) $B^2 - 4AC \geq 0$ w/ $\lambda \leq 0$ or $\lambda \geq 1$

(have to account for Bounds: intersection does not occur on line segment $P_1 \to P_2$)

$\lambda \leq 0$ or $\lambda \geq 1$ $-1/2$

$\lambda \leq 0 \Rightarrow c \leq 0 \Rightarrow 1-\lambda \geq 0$

$\lambda(\cdots)$

$$\frac{-B \pm \sqrt{B^2-4AC}}{2A} \leq 0$$

$$\frac{-B \pm \sqrt{B^2-4AC}}{2A} \geq 1 \to 1 + \frac{B \mp \sqrt{B^2-4AC}}{2A} \leq 0$$

- $$\frac{-B + \sqrt{B^2-4AC}}{2A} \leq 0 \quad \text{OR} \quad \left(1 + \frac{B - \sqrt{B^2-4AC}}{2A}\right) \leq 0$$

$$\equiv \left(\frac{-B + \sqrt{B^2-4AC}}{2A}\right)\left(1 - \frac{-B + \sqrt{B^2-4AC}}{2A}\right) \leq 0$$

Only happens for one or the other, never Both.

$$= (-B + \sqrt{B^2-4AC})(2A + B - \sqrt{B^2-4AC}) \leq 0$$

similarly

- $$-\left(\frac{-B - \sqrt{B^2-4AC}}{2A}\right)\left(1 - \frac{-B - \sqrt{B^2-4AC}}{2A}\right) \leq 0$$
- $$-(-B - \sqrt{B^2-4AC})(2A + B + \sqrt{B^2-4AC}) \leq 0$$

So for each root for $\lambda$ we have either there is an imaginary root $B^2 - 4AC \leq 0$,

$$C_\pm \doteq \begin{cases} B^2 - 4AC & \text{if } B^2 - 4AC \leq 0 \\ -(-B \pm \sqrt{B^2-4AC})(2A + B \mp \sqrt{B^2-4AC}) & \text{else} \end{cases}$$

2 constraints

2 equations

$$c = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

using the above we have this for the 2 joints

So for joint 1 $\Big( \underset{\text{"}p_1\text{"}}{(0,0)} \to \underset{\text{"}p_2\text{"}}{l_1 R_1 e_1} \Big)$

we have

$$A_1 = A(p_0,\ 0,\ l_1 R(\theta_1) e_1)$$
$$B_1 = B(p_0,\ 0,\ l_1 R(\theta_1) e_1)$$
$$C_1 = C(p_0,\ 0,\ l_1 R(\theta_1) e_1)$$

So we have

$$c_1 = \begin{pmatrix} c_{1+} \\ c_{1-} \end{pmatrix}$$

where

$$c_{1\pm} = \begin{cases} B_1^2 - 4A_1C_1 & \text{if } B_1^2 - 4A_1C_1 \geq 0 \\ \left(-B_1 \pm \sqrt{B_1^2 - 4A_1C_1}\right)\left(2A_1 + B_1 \mp \sqrt{B_1^2 - 4A_1C_1}\right) & \text{else} \end{cases}$$

And for joint 2 $\Big( \underset{\text{"}p_1\text{"}}{l_1 R_1 e_1} \to \underset{\text{"}p_2\text{"}}{l_1 R_1 e_1 + l_2 R_{12} e_1} \Big)$

$$p_2 - p_1 = l_2 R_{12} e_1$$

$$A_2 = A(p_0,\ l_1 R(\theta_1) e_1,\ l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1)$$
$$B_2 = B(p_0,\ l_1 R(\theta_1) e_1,\ l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1)$$
$$C_2 = C(p_0,\ l_1 R(\theta_1) e_1,\ l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1)$$

$$c_2 = \begin{pmatrix} c_{2+} \\ c_{2-} \end{pmatrix}$$

where

$$c_{2\pm} = \begin{cases} B_2^2 - 4A_2C_2 & \text{if } B_2^2 - 4A_2C_2 \geq 0 \\ \left(-B_2 \pm \sqrt{B_2^2 - 4A_2C_2}\right)\left(2A_2 + B_2 \mp \sqrt{B_2^2 - 4A_2C_2}\right) & \text{else} \end{cases}$$

and finally

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_{1+} \\ c_{1-} \end{pmatrix} \leq 0$$

$$c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_{1+} \\ c_{1-} \\ c_{2+} \\ c_{2-} \end{pmatrix} \leq 0$$

4) $$\dot{x}_1 = \frac{cu}{x_2} - \frac{D}{x_2}$$
$$\dot{x}_2 = -u \qquad 0 \leq u \leq u_{max}$$

$x_1 \to$ horizontal velocity, $x_2$ mass,
$c$ exhaust gass speed, $D$ aerodynamic drag

maximize rocket range, $x_f$ specified, $t_f$ free.

First, what is the rocket range?
$x_1 \to$ horizontal velocity $\therefore$
$$J = \int_0^{t_f} x_1(t)\, dt$$
$$\max J = \int_0^{t_f} x_1(t)\, dt \quad \text{s.t. } \dot{x} = f(x, u).$$
$$L = x_1$$

a) assume $D$ constant.
i) Adjoint equations

$$H \doteq L + \lambda^T f = x_1 + \lambda_1 \left( \frac{cu}{x_2} - \frac{D}{x_2} \right) - \lambda_2 u$$

Adjoint:
1) $\dot{x} = f(x, u) \to \nabla_\lambda H = 0$
2) $\dot{\lambda} = -\nabla_x H =$
$$\frac{\partial H}{\partial x_1} = 1 \qquad \frac{\partial H}{\partial x_2} = -\frac{\lambda_1 (cu - D)}{x_2^2}$$

$$\dot{\lambda} = -\begin{pmatrix} 1 \\ \frac{\lambda_1 (D - cu)}{x_2^2} \end{pmatrix}$$
$\Rightarrow \lambda_1 = -t + c_1$

$$\Rightarrow \lambda_1 = -t + c_1$$

$$\dot{\lambda}_2 = -\frac{\lambda_1 (D - cu)}{x_2^2} = -\frac{(-t + c_1)(D - cu)}{x_2^2}$$

TC:

3) $(\phi_t + \nu^T \psi + L + \lambda^T f)\big|_{t=t_f} = 0$

$H(t_f) = 0$

And by Poyntresjin's Maximum principle

$H(x, u^* + \delta u, \lambda^*) \leq H(x, u^*, \lambda^*)$

$H_u = \frac{\lambda_1 c u}{x_2} - \lambda_2 u$ (H terms of u

$H_u = \left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) u$ ← want to maximize

ii) Possible Singular Control Intervals

We have when $\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) < 0$, $u = 0$,

when $\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) > 0$, $u = u_{max}$

and when $\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = 0 \Rightarrow u$ undefined.

Undefined control interval where

$$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0$$

and $\frac{d}{dt}\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = 0$

$$\rightarrow \frac{\dot{\lambda}_1 c}{x_2} - \frac{\lambda_1 c \dot{x}_2}{x_2^2} - \dot{\lambda}_2 = 0$$

$$\rightarrow \frac{-c}{x_2} - \cancel{\frac{\lambda_1 c (-u)}{x_2^2}} - \left(-\frac{\lambda_1 (D \cancel{- cu})}{x_2^2}\right) = 0$$

$$\rightarrow \frac{-c}{x_2} - \frac{\lambda_1 D}{x_2^2} = 0$$

$$\rightarrow c + \frac{\lambda_1 D}{x_2} = 0$$

$\Rightarrow c + \frac{\lambda_1 D}{x_2} = 0$

$\Rightarrow c + \frac{(-t+c_1) D}{x_2}$

If $u^* = 0$

$x_2 =$ constant

$\Rightarrow \quad c + \frac{(-t+c_1)\, D}{x_2} = 0$

has 1 crossing (linear function)

If $u^* = u_{max}$

$x_2 = -u_{max} t + c_2$

$$\Rightarrow \quad c + \frac{(-t+c_1) D}{-u_{max} t + c_2} = 0$$

$$-c\, u_{max} t + c \cdot c_2 + (-t + c_1) D = 0$$

$$-(D + c\,u_{max}) t + (c \cdot c_2 + c_1 D) = 0$$

which again has a maximum of 1 zero-crossing

$\because D, c, u_{max} > 0, \quad D + c\,u_{max} \neq 0.$

So if there is a time where

$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0$, then it will be of measure zero because there is no time interval that will maintain

$$\frac{d}{dt}\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = 0.$$

b) $D(x_1, x_2) = \alpha x_1^2 + \beta \frac{x_2^3}{x_1^2} \geq 0$

$\alpha, \beta > 0$

i) Adjoint

$$H = x_1 + \lambda_1\left(\frac{cu}{x_2} - \frac{D}{x_2}\right) + \lambda_2(-u)$$

$$= x_1 + \lambda_1\left(\frac{cu}{x_2} - \frac{\alpha x_1^2}{x_2} - \beta\frac{x_2}{x_1^2}\right) - \lambda_2 u$$

Adjoint:

1) $\nabla_\lambda H = 0 \rightarrow \dot{x} = f(x,u)$

2) $\dot{\lambda} = -\nabla_x H$

$$-\frac{\partial H}{\partial x_1} = -1 + \frac{2\lambda_1 \alpha x_1}{x_2} - \frac{2\lambda_1 \beta x_2}{x_1^3}$$

$$-\frac{\partial H}{\partial x_2} = -\lambda_1\left(-\frac{cu}{x_2^2} + \frac{\alpha x_1^2}{x_2^2} - \frac{\beta}{x_1^2}\right)$$

$$\dot{\lambda}_1 = -1 + 2\alpha\lambda_1\frac{x_1}{x_2} - 2\lambda_1\beta\frac{x_2}{x_1^3}$$

$$\dot{\lambda}_2 = \lambda_1\left(\frac{cu}{x_2^2} - \frac{\alpha x_1^2}{x_2^2} + \frac{\beta}{x_1^2}\right)$$

And same Principle for Pontryagin's Maximum

$$H_u = \left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) u$$

where
- $\frac{\lambda_1 c}{x_2} - \lambda_2 < 0 \Rightarrow u^* = 0$
- $\frac{\lambda_1 c}{x_2} - \lambda_2 > 0 \Rightarrow u^* = u_{max}$
- $\frac{\lambda_1 c}{x_2} - \lambda_2 = 0 \Rightarrow u^*$ undefined.

ii) Singular Control Interval

$\exists$ interval at $t$ if $\exists [t_1, t_2]$ s.t.

$$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0 \quad \text{and}$$

$$\frac{d}{dt}\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = 0$$

$$\frac{d}{dt}\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = \frac{\dot{\lambda}_1 c}{x_2} - \frac{\lambda_1 c \dot{x}_2}{x_2^2} - \dot{\lambda}_2 = 0$$

$$-\frac{c}{x_2} - \frac{\beta(2c+x_2)\lambda_1}{x_1^3} + \frac{\alpha x_1 \lambda_1 (2c+x_1)}{x_2^2} = 0$$

$$-\frac{c}{x_2} - \frac{\beta(2c+x_1)\lambda_1}{x_1^3} + \frac{\alpha x_1 \lambda_1 (2c+x_1)}{x_2^2} = 0$$

$$\rightarrow \frac{c x_1^3 x_2 - 2\alpha c x_1^4 \lambda_1 - \alpha x_1^5 \lambda_1 + 2\beta c x_2^2 \lambda_1 + \beta x_1 x_2^2 \lambda_1}{x_1^3 x_2^2} = 0$$

Generally $x_1, x_2 \neq 0$, (velocity $\neq 0$ and mass non-zero)

$$c x_1^3 x_2 - 2\alpha c x_1^4 \lambda_1 - \alpha x_1^5 \lambda_1 + 2\beta c x_2^2 \lambda_1 + \beta x_1 x_2^2 \lambda_1 = 0$$

$$-c x_1^3 x_2 + (2c+x_1)(\alpha x_1^4 - \beta x_2^2)\lambda_1 = 0$$

$$(2c+x_1)(\alpha x_1^4 - \beta x_1^2)\lambda_1 = c x_1^3 x_2$$

If $\alpha x_1^4 - \beta x_1^2 = 0$,

$\Rightarrow x_1 = 0$ or $x_2 = 0$

$\lightning$ and $\because c, x_1 \neq 0$, $2c + x_1 \neq 0$

when $\boxed{\lambda_1 = \frac{c x_1^3 x_2}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}}$

is when $\frac{d}{dt}\left(\frac{\lambda_1 c}{x_2} - \lambda_2\right) = 0$

Now when

$$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0$$

$$\rightarrow \boxed{\lambda_2 = \frac{\lambda_1 c}{x_2} = \frac{c^2 x_1^3}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}}$$

This causes a singular control interval, when

$$\lambda_1 = \frac{c x_1^3 x_2}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$$

and $\alpha x_1^4 - \beta x_2^2 \neq 0$

$$\lambda_2 = \frac{c^2 x_1^3}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$$