

Homework 4: Dimitri Lezcano

Saturday, October 10, 2020 8:24 PM

$$1) \dot{x} = \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & -a \end{pmatrix}}_A x + u \underbrace{e_2}_B = f(x, u)$$

$$a > 0, |u| \leq 1, x(t_f) = 0$$

$$\min \int_{t_0}^{t_f} \underbrace{[\gamma + |u(t)|]}_L dt$$

$\gamma > 0$ constant, t_f free.

a) Adjoint equations

$$H = L + J^T f$$

$$H = \gamma + u + J^T [Ax + Bu]$$

$$\dot{J} = \nabla_x H = -A^T J$$

$$\dot{J} = \begin{pmatrix} 0 & 0 \\ -1 & a \end{pmatrix} \begin{pmatrix} J_1 \\ J_2 \end{pmatrix}$$

$$J_1 = 0 \rightarrow J_1 = c_1$$

$$J_2 = -J_1 + aJ_2$$

$$J_2 - aJ_2 = -c_1$$

$$J_2 = \frac{c_1}{a} + c_2 e^{at}$$

$$\frac{J_1 = c_1}{J_2 = \frac{c_1}{a} + c_2 e^{at}} \quad c_1, c_2 \text{ constants}$$

$$H(u) = u + J^T B u \quad ("u" \text{ terms of } u \text{ only})$$

$$= (I + J^T B) u$$

$$\frac{dH}{du} = I + J^T B = I + J_2$$

So we have $\min H$ when:

$$u = \begin{cases} +1 & I + J^T B < 0 \\ -1 & I + J^T B > 0 \end{cases}$$

$$u = \begin{cases} +1 & 1 + \lambda^T B < 0 \\ -1 & 1 + \lambda^T B > 0 \\ \text{undefined} & 1 + \lambda^T B = 0 \end{cases}$$

$$\underline{u} = \begin{cases} +1 & \lambda_2 < -1 \\ -1 & \lambda_2 > -1 \\ \text{undefined} & \lambda_2 = -1 \end{cases}$$

b) Possible control sequences

Possible maneuvers

$$\lambda_2(t) = \frac{c_1}{a} + c_2 e^{at}$$

↳ here, we have that there is
a maximum of 1 switch

$$u^*(t) = \begin{cases} +1 \\ -1 \\ h(t-t_0) - 2h(t-t_1) \\ -h(t-t_0) + 2h(t-t_1) \end{cases}$$

for $t \in [t_0, t_f]$ where

$$h(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases} \quad \text{heaviside function}$$

$$\text{and } t_1 = \frac{1}{a} \ln \left(\frac{-c_1}{a c_2} \right) \text{ if exists.} \quad \left(\text{i.e. if there is a } 0\text{-crossing} \right)$$

c) Show that there are no singular intervals

Given $a = \pm 1$, Dynamics turn into

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -ax_2 \pm 1$$

$$\ddot{x}_2 + ax_2 = \pm 1$$

$$+ x_2 = \pm \frac{1}{a} + c_3 e^{-at}$$

$$\int (c \cdot) dt + \quad \dot{x}_1 = \pm \frac{1}{a} + c_3 e^{-at}$$

$$\int c(t)dt + \begin{cases} x_1 = \pm \frac{1}{q} + c_3 e^{-at} \\ x_2 = \pm \frac{(t-t_0)}{q} - \frac{c_3}{q} e^{-at} + c_4 \end{cases}$$

$$x_1 = \pm \frac{1}{q} - \frac{c_3}{q} e^{-at} + c_4$$

$$x_2 = \pm \frac{1}{q} + c_3 e^{-at}$$

$$q x_1 + x_2 = \pm + \pm \frac{1}{q} - c_3 e^{-at} + c_3 e^{-at} + c_4$$

$$a x_1 + x_2 = \pm \left(+ \pm \frac{1}{q} \right) + c_4 \quad (\text{live?})$$

Suppose $a < t_1$, we switch from

$$u = +1 \rightarrow u = -1$$

so we have

$$a x_1(t_1^-) + x_2(t_1^-) = t_1 + \frac{1}{q} + c_4$$

↓

$$a x_1(t_1^+) + x_2(t_1^+) = -t_1 - \frac{1}{q} + c_4$$

And by continuity in x we have

$$0 = 2t_1 + \frac{2}{q} \Rightarrow t_1 = -\frac{1}{q} < 0 \quad \because c > 0;$$

There is no switch \because "switch" happens from

$t_1 < 0 \leq t_0$ (start time).

d) Optimal Control Law

So from c) we only have 2 conditions:

$$u^* = \begin{cases} +1 \\ -1 \end{cases}$$

Conditions:

$$u^* = \begin{cases} +1 \\ -1 \end{cases}$$

and looking at the conditions

$$\alpha x_1 + x_2 = \pm(t + t_c) + c_4$$

for optimal final time t_f^* ,

$$\alpha x_1(t_f^*) + x_2(t_f^*) = 0 = \pm(t_f^* + t_c) + c_4$$
$$\Rightarrow c_4 = \mp(t_f^* + t_c)$$

↓

$$\alpha x_1 + x_2 = \pm(t + t_c) = (t_f^* + t_c)$$
$$= \pm(t - t_f^*)$$

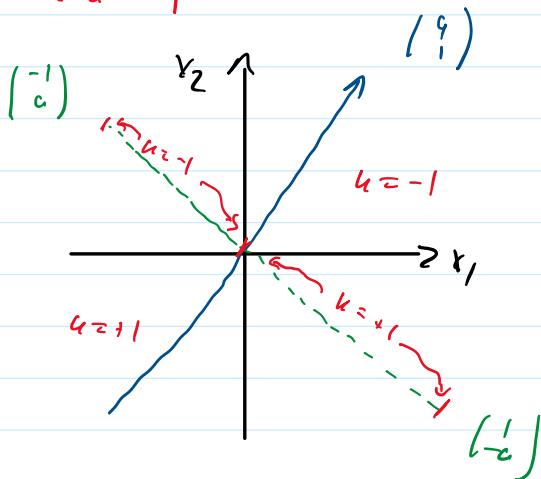
$$\alpha x_1 + x_2 = \mp(t_f^* - t) \quad \text{for } u = \pm 1$$
$$t_f^* - t > 0$$
$$\nexists t \in [t_c, t_f^*]$$

so for

$$\alpha x_1 + x_2 = -(t_f^* - t) < 0$$
$$\Rightarrow u = +1$$

and

$$\alpha x_1 + x_2 = (t_f^* - t) > 0$$
$$\Rightarrow u = -1$$



So up to 3 conditions so far:

$$\begin{pmatrix} q \\ 1 \end{pmatrix}^T x < 0 \rightarrow u = +1$$

$$\begin{pmatrix} q \\ 1 \end{pmatrix}^T y > 0 \rightarrow u = -1$$

$$x = 0 \rightarrow u = 0$$

(goal)

But what about perpendicular condition
to \dot{x}_1 ? Look @ dynamics:

Case 1: $x = \begin{pmatrix} -1 \\ u \end{pmatrix}$ (upper-left quadrant)
 $\dot{x} \geq 0$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\gamma x_2 + u$$

$$\because x_2 = \gamma u \geq 0 \Rightarrow \dot{x}_1 \geq 0 \text{ so}$$

x_1 approaches closer to
also goes to 0.

$$\dot{x}_2 = -\gamma u^2 + u \text{ where } -\gamma u^2 \leq 0. \therefore$$

We want $x_2 \rightarrow$
decrease ($x_2 < 0$) to
approach $x_2 \geq 0$, we
let $u = -1$.

so why

$$x = \begin{pmatrix} -1 \\ u \end{pmatrix} \quad \dot{x} \geq 0 \rightarrow u = -1$$

And for

$$x = \begin{pmatrix} 1 \\ -u \end{pmatrix} \quad \dot{x} \geq 0, \text{ where}$$

$$\dot{x}_1 = x_2 = -\gamma u \leq 0$$

$$\dot{x}_2 = +\gamma u^2 + u \text{ where we want } \dot{x}_2 \geq 0 \Rightarrow$$

$$\text{let } u = +1$$

$$u^*(t) = \begin{cases} +1 & \begin{pmatrix} 1 \\ u \end{pmatrix}^T x \leq 0 \\ -1 & \begin{pmatrix} 1 \\ u \end{pmatrix}^T x \geq 0 \\ +1 & x = \begin{pmatrix} 1 \\ -u \end{pmatrix}; \dot{x} \geq 0 \\ -1 & x = \begin{pmatrix} -1 \\ u \end{pmatrix}; \dot{x} \geq 0 \\ 0 & x = 0 \end{cases}$$

2) min $J = \|x(t_f)\|^2$

$$\dot{x} = Ax + Bu = f(x, u)$$

 $x \in \mathbb{R}^n, A \in M_n(\mathbb{R}), x(0) = x_0$
 $u \in \mathbb{R}, B \in \mathbb{R}^n$

t_f fixed / given

constraint: $|u(t)| \leq 1$

Suppose $J_{\min} > 0$. ($\Rightarrow x(t_f) \neq 0$)

$L = 0$

$$\phi(x_f, t_f) = x_f^T x_f$$

$$J = \phi + \int_{t_0}^{t_f} L dt$$

$$H = L + \lambda^T f$$

$$H = \lambda^T (Ax + Bu)$$

$$\dot{\lambda} = -D_x H = -A^T \lambda$$

$$\hookrightarrow \lambda(t) = \exp(-A^T t) \lambda(t_0)$$

$$H_u(u, \lambda) = (\lambda^T B) u \quad (\text{u terms of } H)$$

Transversality (conds.):

$$\lambda(t_f) = D_x \phi|_{t=t_f}$$

$$\lambda(t_f) = 2x(t_f) \neq 0 \quad \because J_{\min} < 0,$$

$$\hookrightarrow \lambda(t) = \exp(-A(t-t_f)) x(t_f) \neq 0$$

$$\therefore |\exp(\lambda)| \neq 0 \quad \forall \lambda \text{ non-zero}$$

$$x(t_f) \neq 0.$$

So we have H is minimized when

$$H_u = (\lambda^T B) u \quad \text{is minimized}$$

$$\because \lambda \neq 0,$$

H_u is minimized when

$$u = \begin{cases} +1 & \lambda^T B < 0 \\ -1 & \lambda^T B > 0 \\ \text{undefined} & \lambda^T B = 0 \end{cases}$$

which means the optimal control
is "bang-bang".

For

$$\begin{aligned} x_1 &= x_2 \\ \vdots &\quad \dots \end{aligned} \quad A = \begin{pmatrix} 0 & 1 \\ \ddots & \ddots \end{pmatrix} \quad B = c_2$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned}$$

$$A = \begin{pmatrix} 0 & 1 \\ c_3 & c_4 \end{pmatrix} \quad B \in \mathbb{C}^2$$

\downarrow (by class)

$$x_1 = \pm \frac{1}{2} t^2 + c_3 t + c_4$$

$$x_2 = t + c_3 \quad t \in [t_0, t_f] \\ u = \pm 1$$

$$x_1^2 = t^2 \pm 2c_3 t + c_3^2$$

$$\begin{aligned} x_1 \mp \frac{1}{2} x_2^2 &= (\frac{1}{2} t^2 + c_3 t + c_4) \mp (\frac{1}{2} t^2 \pm c_3 t + c_3^2) \\ &= c_4 \mp c_3^2 \\ &= c_5, c_6 \\ x_1 &= \pm \frac{1}{2} x_2^2 + c_{5,6} \quad u = \pm 1 \end{aligned}$$

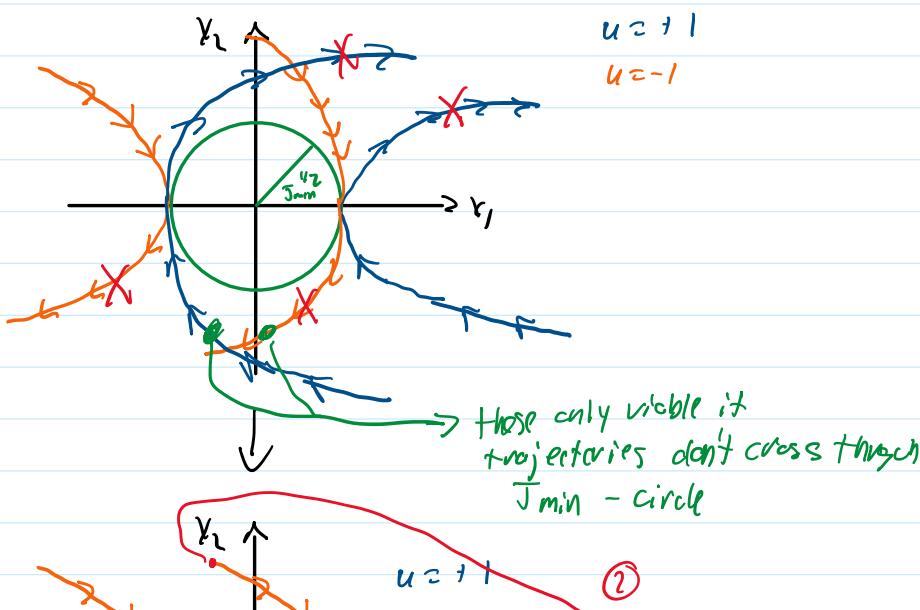
$$\begin{aligned} x_1 &= \pm \frac{1}{2} x_2^2 + c_5 \quad \rightarrow u = +1 \\ x_1 &= -\frac{1}{2} x_2^2 + c_6 \quad \rightarrow u = -1 \end{aligned} \quad \left. \begin{array}{l} \text{family of} \\ \text{parabolas in} \\ x_1 \text{ direction} \end{array} \right\}$$

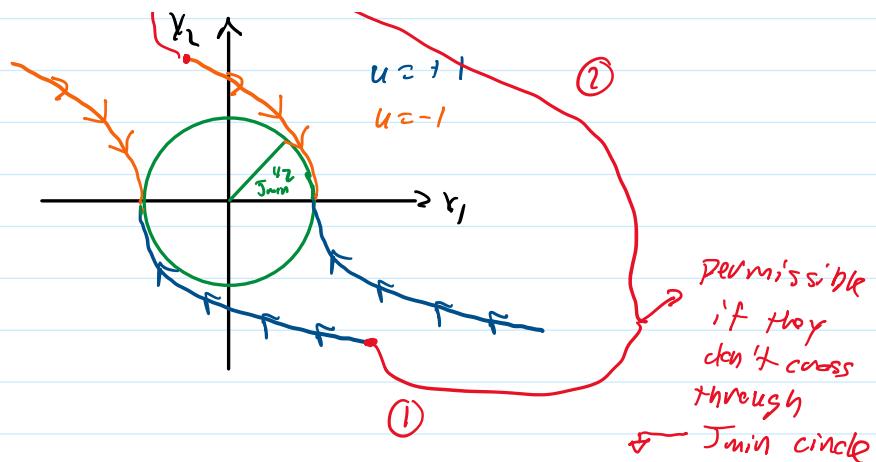
$$\because J_{\min} > 0, (x_1(t_f), x_2(t_f)) \neq 0 \Rightarrow \\ c_5, c_6 \neq 0$$

Since all of these trajectories are in the x_1 direction, we can control $x_2 \rightarrow 0$ and then

we get $x_1 = c_5 = \pm \sqrt{J_{\min}}$ (on positive x_1 -side generality)

$x_1 = c_6 = \pm \sqrt{J_{\min}}$ (on negative x_1 -side generality)





$$① \quad \dot{x}_1 = +\frac{1}{2}x_2^2 - J_{\min}^{1/2}$$

cond: \dot{x}_1, \dot{x}_2

$$x_1^2 + x_2^2 \geq J_{\min}$$

$$\frac{1}{4}x_2^4 - x_2^2 J_{\min}^{1/2} + J_{\min} + x_2^2 \geq J_{\min}$$

$$\frac{1}{4}x_2^4 + (1 - J_{\min}^{1/2})x_2^2 \geq 0$$

$$\equiv \frac{1}{4}x_2^2 + (1 - J_{\min}^{1/2}) \geq 0$$

$\forall x_2 \in \mathbb{R}$

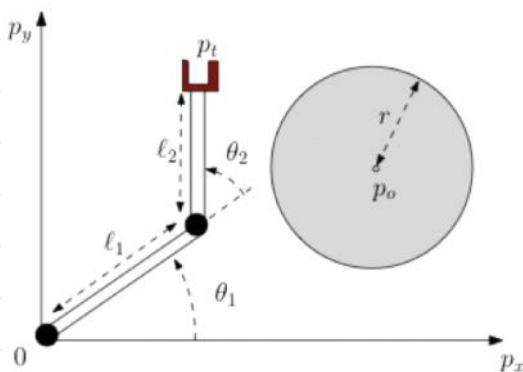
$$\equiv \frac{1}{4}(0)^2 + (1 - J_{\min}^{1/2}) \geq 0$$

$$\Rightarrow (1 - J_{\min}^{1/2}) \geq 0$$

$$J_{\min}^{1/2} \leq 1$$

Similarly for ② by symmetry.

3)



$$X = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

$$\dot{X} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} X + u_1 e_3 + u_2 e_4$$

$$|\dot{\theta}_1| \quad |\dot{\theta}_2| \quad - \quad u_1 \quad u_2$$

$$\begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P_0 = \begin{pmatrix} P_{0x} \\ P_{0y} \end{pmatrix} \in \mathbb{R}^2$$

$$P_+ = \begin{pmatrix} l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{pmatrix}$$

c) $c(x(t), t) \leq 0$, derive q -th order state control inequality

we want

$$\|P_+(t) - P_0\|^2 \geq r^2$$

$$r^2 - (P_+ - P_0)^T (P_+ - P_0) \leq 0$$

$$c(x) = r^2 - P_+^T P_+ + 2 P_+^T P_0 - P_0^T P_0 \leq 0$$

Note: $P_+ = l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1$

where $R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

$\tilde{R}(\theta) = R(\theta) R^T(\theta) \tilde{R}(\theta)$

$\hat{\theta} = R(\theta) \dot{\theta}$ where

$$\hat{\theta} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}$$

Short hand: $R(\theta_1) = R_1, R_{12} = R(\theta_1 + \theta_2) = R(\theta_1) R(\theta_2)$

$R(\theta_2) = R_2, \theta_{12} = \theta_1 + \theta_2$

$P_+ = l_1 R(\theta_1) e_1 + l_2 R(\theta_1 + \theta_2) e_1$

$$\begin{aligned} \dot{P}_+ &= l_1 \dot{R}_1 e_1 + l_2 \dot{R}_{12} e_1 \\ &= l_1 R_1 \dot{R}_1^T \dot{R}_1 e_1 + l_2 R_{12} \dot{R}_{12}^T \dot{R}_{12} e_1 \\ &= l_1 R_1 \hat{\theta}_1 e_1 + l_2 R_{12} \hat{\theta}_{12} e_1 \end{aligned}$$

$$\hat{\theta} = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix}$$

$$\hat{\theta} e_1 = \theta e_2$$

$$\begin{aligned} \dot{P}_+ &= l_1 R_1 (\hat{\theta}_1 e_2) + l_2 R_{12} (\hat{\theta}_{12} e_2) \\ P_+ &= (l_1 \hat{\theta}_1) R_1 e_2 + (l_2 \hat{\theta}_{12}) R_{12} e_2 \end{aligned}$$

$$\dot{P}_+ = l_1 K_1(\theta, e_2) + l_2 K_{12}(\theta_{12}, e_2)$$

$$\begin{aligned}\ddot{P}_+ &= (l_1 \ddot{\theta}_1) R_1 e_2 + (l_1 \dot{\theta}_1) \dot{R}_1 e_2 + l_2 \ddot{\theta}_{12} R_{12} e_2 \\ &\quad + l_2 \dot{\theta}_{12} \dot{R}_{12} e_2 \\ &= l_1 \ddot{\theta}_1 R_1 e_2 + l_2 \ddot{\theta}_{12} R_{12} e_2 \\ &\quad + l_1 \dot{\theta}_1 R_1 \dot{\theta}_1 e_2 + l_2 \dot{\theta}_{12} R_{12} \dot{\theta}_{12} e_2 \\ &= l_1 \ddot{\theta}_1 R_1 e_2 + l_2 \ddot{\theta}_{12} R_{12} e_2 \\ &\quad - l_1 \dot{\theta}_1^2 R_1 e_1 - l_2 \dot{\theta}_{12}^2 R_{12} e_1 \\ &\quad \therefore \ddot{\theta} e_2 = \begin{pmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\dot{\theta} \\ 0 \end{pmatrix}\end{aligned}$$

$$\dot{P}_+ = (l_1 \dot{\theta}_1) R_1 e_2 + (l_2 \dot{\theta}_{12}) R_{12} e_2$$

$$\begin{aligned}\ddot{P}_+ &= (l_1 \ddot{\theta}_1) R_1 e_2 + (l_2 \ddot{\theta}_{12}) R_{12} e_2 \quad \leftarrow \text{controls appear here} \\ &\quad - (l_1 \dot{\theta}_1^2) R_1 e_1 - (l_2 \dot{\theta}_{12}^2) R_{12} e_1, \quad \dot{\theta}_1 = u_1 \\ &\quad \dot{\theta}_{12} = u_2\end{aligned}$$

$$\frac{d}{dt} C = D_{P_+} C^\top \dot{P}_+$$

$$= [2p_0 - 2p_+]^\top \dot{P}_+$$

$$\dot{C} = 2p_0^\top \dot{P}_+ - 2p_+^\top \dot{P}_+$$

$$\ddot{C} = 2p_0^\top \ddot{P}_+ - 2p_+^\top \dot{P}_+ - 2p_+^\top \ddot{P}_+$$

$$= [2p_0 - 2p_+]^\top \ddot{P}_+ - 2p_+^\top \dot{P}_+$$

$$\begin{aligned}\dot{P}^\top \ddot{P} &= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_{12}^2 + 2(l_1 \dot{\theta}_1 R_1 e_1)^\top (l_2 \dot{\theta}_{12} R_{12} e_2) \\ &= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_{12}^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_{12} e_1^\top R_1^\top R_{12}^\top R_{12} e_2 \\ &= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_{12}^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_{12} e_2^\top R(e_1) e_2 \\ &= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_{12}^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_{12} \sin \theta_2\end{aligned}$$

$$\begin{aligned}P_+^\top \ddot{P}_+ &= \left[(l_1 R_1 + l_2 R_{12}) e_1 \right]^\top \left[(l_1 u_1 R_1 + l_2 u_2 R_{12}) e_2 - (l_1 \dot{\theta}_1^2 R_1 + l_2 \dot{\theta}_{12}^2 R_{12}) e_1 \right] \\ &= e_1^\top (l_1 P_1^\top + l_2 P_{12}^\top) (l_1 u_1 R_1 + l_2 u_2 R_{12}) e_2 \\ &\quad - e_1^\top (l_1 R_1^\top + l_2 R_{12}^\top) (l_1 \dot{\theta}_1^2 R_1 + l_2 \dot{\theta}_{12}^2 R_{12}) e_1 \\ &= e_1^\top \left[l_1^2 u_1 I_3 + l_2^2 u_2 I_3 + l_1 l_2 u_1 R_{12}^\top P_1 + l_1 l_2 u_2 P_1^\top R_{12} \right] e_2 \\ &\quad - e_1^\top \left[l_1^2 \dot{\theta}_1^2 I_3 + l_2^2 \dot{\theta}_{12}^2 I_3 + l_1 l_2 \dot{\theta}_1^2 R_{12}^\top P_1 + l_1 l_2 \dot{\theta}_{12}^2 P_1^\top R_{12} \right] e_1\end{aligned}$$

$$= e_1^T (\cancel{x_1 u_1} \dot{u}_3 + \cancel{x_2 u_2} \dot{u}_3) + l_1 l_2 u_1 R_{12}^T P_1 + l_1 l_2 u_2 R_{12}^T P_{12}) e_2 \\ - e_1^T (l_1^2 \dot{\theta}_1^2 I_3 + l_2^2 \dot{\theta}_2^2 I_3 + l_1 l_2 \dot{\theta}_1^2 R_{12}^T P_1 + l_1 l_2 \dot{\theta}_2^2 R_{12}^T P_{12}) e_1$$

$$= e_1^T (l_1 l_2 u_1 R_2^T + l_1 l_2 u_2 R_2) e_2 \\ - l_1^2 \dot{\theta}_1^2 - l_2^2 \dot{\theta}_2^2 \\ - e_1^T (l_1 l_2 \dot{\theta}_1^2 R_2^T + l_1 l_2 \dot{\theta}_2^2 R_2) e_1$$

$$P_1^T \ddot{p}_+ = l_1 l_2 e_1^T (u_1 R_2^T + u_2 R_2) e_2 \\ - l_1^2 \dot{\theta}_1^2 - l_2^2 \dot{\theta}_2^2 - l_1 l_2 e_1^T (\dot{\theta}_1^2 R_2^T + \dot{\theta}_2^2 R_2) e_1$$

$$aR_2^T + bR_2 = \begin{pmatrix} (a+b)\cos\theta_2 & (a-b)\sin\theta_2 \\ (b-a)\sin\theta_2 & (a+b)\cos\theta_2 \end{pmatrix}$$

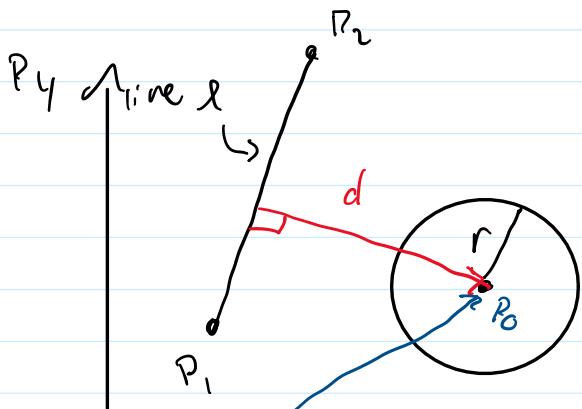
$$P_1^T \ddot{p}_+ = l_1 l_2 (u_1 R_2^T + u_2 R_2)_{12} \\ - l_1^2 \dot{\theta}_1^2 - l_2^2 \dot{\theta}_2^2 \\ - l_1 l_2 (\dot{\theta}_1^2 R_2^T + \dot{\theta}_2^2 R_2)_{11} \\ = l_1 l_2 (u_1 - u_2) \sin\theta_2 - l_1^2 \dot{\theta}_1^2 - l_2^2 \dot{\theta}_2^2 \\ - l_1 l_2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) \cos\theta_2$$

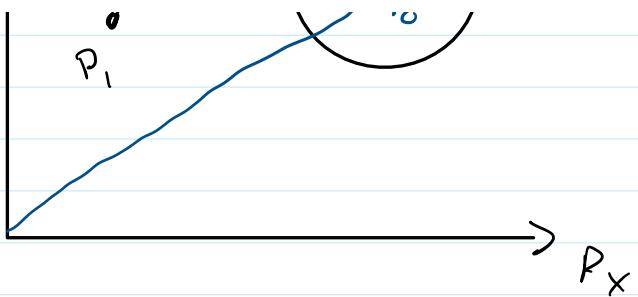
$$\ddot{c} = 2(p_0 - p_+)^T \ddot{p}_+ - 2 \dot{p}_+^T \dot{p}_+$$

$$= 2 p_0^T \ddot{p}_+ - 2 p_+^T \ddot{p}_+ - 2 \dot{p}_+^T \dot{p}_+$$

$$\ddot{c} = 2 p_0^T \ddot{p}_+ - 2 \left(l_1 l_2 (u_1 - u_2) \sin\theta_2 - l_1^2 \dot{\theta}_1^2 - l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) \right) \leq 0 \\ - 2 \left(l_1^2 \dot{\theta}_1^2 + l_2^2 (\dot{\theta}_1^2 + \dot{\theta}_2^2)^2 + 2 l_1 l_2 (\dot{\theta}_1^2 + \dot{\theta}_2^2) \sin\theta_2 \right)$$

b) No point on robot collides w/
obstacle





$$\lambda: \lambda p_1 + (1-\lambda) p_2 \quad \lambda \in [0, 1]$$

$p_1 \in \lambda$

p_1 intersects circle around p_0 if
 $\|p_1 - p_0\| = r$

$$(p_1 - p_0)^T (p_1 - p_0) = r^2$$

$$p_1^T p_1 + p_0^T p_0 - 2 p_0^T p_1 = r^2$$

$$(1^2 p_1^T p_1 + (1-1)^2 p_2^T p_2 + 2 \lambda (1-\lambda) p_1^T p_2)$$

$$+ p_0^T p_0 - 2 p_0^T (\lambda p_1 + (1-\lambda) p_2) = r^2$$

$$\underline{1^2 p_1^T p_1 + (1^2 - 2\lambda + 1) p_2^T p_2 + 2(1-\lambda^2) p_1^T p_2}$$

$$+ p_0^T p_0 - 2\lambda p_0^T p_1 - 2(1-\lambda) p_0^T p_2 = r^2$$

$$(p_1^T p_1 + p_2^T p_2 - 2 p_1^T p_2) \lambda^2$$

$$+ (2 p_1^T p_2 - 2 p_2^T p_2 - 2 p_0^T p_1 + 2 p_0^T p_2) \lambda$$

$$+ (p_2^T p_2 + p_0^T p_0 - 2 p_0^T p_2 - r^2) = 0$$

$$A(p_0, p_1, p_2) \doteq p_1^T p_1 - 2 p_1^T p_2 + p_2^T p_2 = (p_2 - p_1)^T (p_2 - p_1)$$

$$B(p_0, p_1, p_2) \doteq 2(p_1^T p_2 - p_2^T p_2 - p_0^T p_1 + p_0^T p_2)$$

$$C(p_0, p_1, p_2) \doteq p_2^T p_2 + p_0^T p_0 - 2 p_0^T p_2 - r^2 = (p_0 - p_2)^T (p_0 - p_2) - r^2$$

$$A\lambda^2 + B\lambda + C = 0, \text{ solve for } \lambda.$$

$$\lambda = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

We have there are intersections if

$$\lambda \in [0, 1].$$

So

$$B^2 - 4AC \geq 0 \quad \text{for real roots}$$

So we want:

$$1) B^2 - 4AC \leq 0 \quad (\text{Non-real roots} / 1 \text{ root})$$

$$2) B^2 - 4AC \geq 0 \quad w/ \quad \begin{matrix} \text{OR} \\ 1 < 0 \quad \text{or} \quad 1 \geq 1 \end{matrix}$$

(how to count for Bounds: intersection does not occur on line segment $P_1 \rightarrow P_2$)

$$-\frac{B \pm \sqrt{B^2 - 4AC}}{2A} \leq 0$$

$$\begin{aligned} 1 &\leq 0 \quad \text{or} \quad 1 \geq 1 \\ 1 \leq 0 \Rightarrow & 1 \leq 0 \Rightarrow 1 - 1 \geq 0 \\ 1 &\geq 1 \end{aligned}$$

$$-\frac{B \pm \sqrt{B^2 - 4AC}}{2A} \geq 1 \rightarrow 1 + \frac{B \mp \sqrt{B^2 - 4AC}}{2A} \leq 0$$

$$\bullet -\frac{B + \sqrt{B^2 - 4AC}}{2A} \leq 0 \quad \text{or} \quad \left(1 + \frac{B - \sqrt{B^2 - 4AC}}{2A} \right) \leq 0$$

$$\equiv \left(\frac{-B + \sqrt{B^2 - 4AC}}{2A} \right) \left| 1 - \frac{-B + \sqrt{B^2 - 4AC}}{2A} \right| \leq 0$$

Only happens for one or the other,
never both.

$$= (-B + \sqrt{B^2 - 4AC})(2A + B - \sqrt{B^2 - 4AC}) \leq 0$$

Similarly

$$\bullet -\left(\frac{-B - \sqrt{B^2 - 4AC}}{2A} \right) \left| 1 - \frac{-B - \sqrt{B^2 - 4AC}}{2A} \right| \leq 0$$

$$\bullet -(-B - \sqrt{B^2 - 4AC})(2A + B + \sqrt{B^2 - 4AC}) \leq 0$$

So for each root for λ we have either
there is an imaginary root $B^2 - 4AC \leq 0$,

$$c_{\pm} = \begin{cases} B^2 - 4AC & \text{if } B^2 - 4AC \leq 0 \\ -(-B \pm \sqrt{B^2 - 4AC})(2A + B \mp \sqrt{B^2 - 4AC}) & \text{else} \end{cases}$$

↑
2 constraints ↑
 2 equations

$$C = \begin{pmatrix} C_+ \\ C_- \end{pmatrix}$$

using file open we have this for the 2 joints

so for joint 1 $(O, O) \rightarrow l_1 R(\theta_1) e_1$
 $"P_1" "P_2"$

we have

$$\begin{aligned} A_1 &= A(P_0, O, l_1 R(\theta_1) e_1) \\ B_1 &= B(P_0, O, l_1 R(\theta_1) e_1) \\ C_1 &= C(P_0, O, l_1 R(\theta_1) e_1) \end{aligned}$$

So we have

$$C_1 = \begin{pmatrix} C_{1+} \\ C_{1-} \end{pmatrix}$$

where

$$C_{1\pm} = \begin{cases} B_1^2 - 4A_1C_1 & \text{if } B_1^2 - 4A_1C_1 \geq 0 \\ (-B_1 \pm \sqrt{B_1^2 - 4A_1C_1}) / (2A_1 + B_1 \mp \sqrt{B_1^2 - 4A_1C_1}) & \text{else} \end{cases}$$

and for joint 2 $l_1 R(\theta_1) e_1 \rightarrow l_1 R(\theta_1) e_1 + l_2 R(\theta_2) e_1$
 $"P_1" "P_2"$

$$P_2 - P_1 = l_2 R(\theta_2) e_1$$

$$A_2 = A(P_0, l_1 R(\theta_1) e_1, l_1 R(\theta_1) e_1 + l_2 R(\theta_2) e_1)$$

$$B_2 = B(P_0, l_1 R(\theta_1) e_1, l_1 R(\theta_1) e_1 + l_2 R(\theta_2) e_1)$$

$$C_2 = C(P_0, l_1 R(\theta_1) e_1, l_1 R(\theta_1) e_1 + l_2 R(\theta_2) e_1)$$

$$C_2 = \begin{pmatrix} C_{2+} \\ C_{2-} \end{pmatrix}$$

where

$$C_{2\pm} = \begin{cases} B_2^2 - 4A_2C_2 & \text{if } B_2^2 - 4A_2C_2 \leq 0 \\ (-B_2 \pm \sqrt{B_2^2 - 4A_2C_2}) / (2A_2 + B_2 \mp \sqrt{B_2^2 - 4A_2C_2}) & \text{else} \end{cases}$$

and finally

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} C_{1+} \\ C_{1-} \end{pmatrix} \subseteq O$$

$$C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_1+ \\ c_1- \\ c_2+ \\ c_2- \end{pmatrix} \subseteq O$$

4) $\dot{x}_1 = \frac{cu}{x_2} - \frac{D}{x_2}$
 $\dot{x}_2 = -u \quad 0 \leq u \leq u_{max}$

$x_1 \rightarrow$ horizontal velocity, x_2 mass,
 c exhaust gas speed, D aerodynamic drag

maximize rocket range, x_f specified, t_f free.

First, what is the rocket range?

$x_1 \rightarrow$ horizontal velocity \therefore

$$J = \int_0^{t_f} x_1(t) dt$$

$$\text{max } J = \int_0^{t_f} x_1(t) dt \text{ s.t. } \dot{x} = f(x, u).$$

$$L \approx x_1$$

a) assume D constant.

i) Adjoint equations

$$H \doteq L + \lambda^T f = x_1 + \lambda_1 \left(\frac{cu}{x_2} - \frac{D}{x_2} \right) - \lambda_2 u$$

Adjoint:

$$1) \dot{\lambda}_1 = f(x, u) \rightarrow \nabla_x H = 0$$

$$2) \dot{\lambda}_2 = -\nabla_u H =$$

$$\frac{\partial H}{\partial x_1} = 1 \quad \frac{\partial H}{\partial x_2} = -\frac{\lambda_1(cu - D)}{x_2^2}$$

$$\dot{\lambda}_1 = - \left(\frac{1}{\frac{\lambda_1(D-cu)}{x_2^2}} \right)$$

$$\Rightarrow \lambda_1 = - \cdot , +c_1 \cdot , \dots \cdot \cdot \cdot$$

$$\Rightarrow \dot{x}_1 = - + \frac{x_2^2}{x_2} /$$

$$\dot{x}_2 = -l_1(D - cu) = -\frac{(-t + c_1)(D - cu)}{x_2^2}$$

TC:

$$\Rightarrow (\mathcal{J}_f \Phi + V^T \Psi + L + J^T f) \Big|_{t_f=t_f} = 0$$

$$H(J_f) = 0$$

And by Pontryagin's Maximum principle

$$H(x_1^{*}, u^*, \lambda^*) \leq H(x_1, u^*, \lambda^*)$$

$$H_u = \frac{\lambda_1 c y}{x_2} - l_2 y \quad (\text{H terms of } y)$$

$$H_u = \left(\frac{\lambda_1 c}{x_2} - l_2 \right) y \quad \leftarrow \text{want to maximize}$$

(i) Possible Singular Control Intervals

when $\left(\frac{\lambda_1 c}{x_2} - l_2 \right) < 0, u = 0,$

when $\left(\frac{\lambda_1 c}{x_2} - l_2 \right) > 0, u = u_{\max}$

and when $\left(\frac{\lambda_1 c}{x_2} - l_2 \right) = 0 \Rightarrow u \text{ undetermined.}$

Understand control interval where

$$\frac{\lambda_1 c}{x_2} - l_2 = 0$$

and $\frac{d}{dt} \left(\frac{\lambda_1 c}{x_2} - l_2 \right) = 0$

$$\rightarrow \frac{\dot{\lambda}_1 c}{x_2} - \frac{\lambda_1 c \dot{x}_2}{x_2^2} - \dot{l}_2 = 0$$

$$\rightarrow -\frac{c}{x_2} - \frac{\lambda_1 c (1/u)}{x_2^2} - \left(-\frac{l_1 (D - cu)}{x_2^2} \right) \geq 0$$

$$\rightarrow -\frac{c}{x_2} - \frac{l_1 D}{x_2^2} = 0$$

$$\rightarrow c + \frac{l_1 D}{x_2} = 0$$

.....

$$\begin{aligned} & \rightarrow \dot{x}_1 = \frac{c_1 u - u}{x_2} \\ & \rightarrow \dot{x}_1 + \frac{(-t + c_1) D}{x_2} \end{aligned}$$

If $u^* = 0$

$x_2 = \text{constant}$

$$\Rightarrow c + \frac{(-t + c_1) D}{x_2} = 0$$

has 1 crossing (linear function)

If $u^* = u_{\max}$

$$x_2 = -u_{\max} t + c_2$$

$$\Rightarrow c + \frac{(-t + c_1) D}{-u_{\max} t + c_2} = 0$$

$$-c u_{\max} t + c \cdot c_2 + (-t + c_1) D = 0$$

$$-(D + c u_{\max}) t + (c \cdot c_2 + c_1 D) = 0$$

which again has a maximum of 1 zero-crossing

$$\therefore D, c, u_{\max} > 0, \quad D + c u_{\max} \neq 0.$$

so if there is a time where

$\frac{\lambda_1 c}{x_2} - t_2 = 0$, then it will be of measure zero because there is no time interval that will maintain

$$\frac{d}{dt} \left(\frac{\lambda_1 c}{x_2} - t_2 \right) = 0.$$

$$b) D(x_1, x_2) = \alpha x_1^2 + \beta \frac{x_2^2}{x_1^2} \geq 0$$

$$\alpha, \beta > 0$$

i) Adjoint

$$\begin{aligned} H &= x_1 + \lambda_1 \left(\frac{c u}{x_2} - \frac{D}{x_2} \right) + \lambda_2 (-u) \\ &= x_1 + \lambda_1 \left(\frac{c u}{x_2} - \frac{\alpha x_1^2}{x_2} - \beta \frac{x_2^2}{x_1^2} \right) - \lambda_2 u \end{aligned}$$

Adjoint:

$$1) \nabla_x H = 0 \rightarrow \dot{x} = f(x, u)$$

$$2) \dot{t} = -\nabla_x H$$

$$-\frac{\partial H}{\partial x_1} = -1 + 2\lambda_1 \frac{\alpha x_1}{x_2} - 2\lambda_1 \beta \frac{x_2}{x_1^3}$$

$$-\frac{\partial H}{\partial x_2} = -\lambda_1 \left(\frac{c u}{x_2^2} + \frac{\alpha x_1^3}{x_2^2} - \frac{\beta}{x_1^2} \right)$$

$$\dot{x}_1 = -1 + 2\lambda_1 \frac{\alpha x_1}{x_2} - 2\lambda_1 \beta \frac{x_2}{x_1^3}$$

$$\dot{x}_2 = \lambda_1 \left(\frac{c u}{x_2^2} - \frac{\alpha x_1^2}{x_2^2} + \frac{\beta}{x_1^2} \right)$$

And some Principle for Pontryagin's Maximum

$$H_u = \left(\frac{\lambda_1 c}{x_2} - \lambda_2 \right) u$$

$$\text{max } \frac{\lambda_1 c}{x_2} - \lambda_2 \leq 0 \Rightarrow u^* = 0$$

$$\bullet \frac{\lambda_1 c}{x_2} - \lambda_2 \geq 0 \Rightarrow u^* = c_{\max}$$

$$\bullet \frac{\lambda_1 c}{x_2} - \lambda_2 = 0 \Rightarrow u^* \text{ undefined.}$$

(ii) Singular Control Interval

\exists interval of t if $\exists [t_1, t_2]$ s.t.

$$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0 \quad \text{and}$$

$$\frac{d}{dt} \left(\frac{\lambda_1 c}{x_2} - \lambda_2 \right) = 0$$

$$\frac{d}{dt} \left(\frac{\lambda_1 c}{x_2} - \lambda_2 \right) = \frac{\dot{\lambda}_1 c}{x_2} - \frac{\lambda_1 c \dot{x}_2}{x_2^2} - \dot{\lambda}_2 = 0$$

$$-\frac{c}{v_-} - \frac{\beta (2c + x_1) \lambda_1}{v_-^3} + \frac{\alpha x_1 \lambda_1 / 2c + x_1}{v_-^2} = 0$$

$$\frac{-c}{x_2} - \frac{\beta(2c+x_1)\lambda_1 + \alpha x_1 \lambda_1 / (2c+x_1)}{x_2^2} = 0$$

$$\rightarrow \underbrace{cx_1^3x_2 - 2\alpha cx_1^4\lambda_1 - \alpha x_1^5\lambda_1 + 2\beta cx_2^2\lambda_1}_{x_1^3x_2^2} + \beta x_1x_2^2\lambda_1 = 0$$

(Generally $x_1, x_2 \geq 0$, (velocity $\neq 0$ and mass non-zero)

$$cx_1^3x_2 - 2\alpha cx_1^4\lambda_1 - \alpha x_1^5\lambda_1 + 2\beta cx_2^2\lambda_1 + \beta x_1x_2^2\lambda_1 = 0$$

$$-cx_1^3x_2 + (2c+x_1)(\alpha x_1^4 - \beta x_2^2)\lambda_1 = 0$$

$$(2c+x_1)(\alpha x_1^4 - \beta x_2^2)\lambda_1 = cx_1^3x_2$$

when $\lambda_1 = \frac{cx_1^3x_2}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$

If $\alpha x_1^4 - \beta x_2^2 = 0$,
 $\Rightarrow x_1 = 0$ or $x_2 = 0$

is when $\frac{d}{dt} \left[\frac{\lambda_1 c}{x_2} - \lambda_2 \right] = 0$

$\frac{d}{dt} \left[\frac{\lambda_1 c}{x_2} - \lambda_2 \right] = 0$

and $\because c, x_1 \geq 0$
 $2c+x_1 \neq 0$

Now when

$$\frac{\lambda_1 c}{x_2} - \lambda_2 = 0$$

$$\rightarrow \lambda_2 = \frac{\lambda_1 c}{x_2} = \frac{c^2 x_1^3}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$$

This causes a singular control interval, unless

$$\lambda_1 = \frac{cx_1^3x_2}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$$

and $\alpha x_1^4 - \beta x_2^2 \neq 0$

$$\lambda_2 = \frac{c^2 x_1^3}{(2c+x_1)(\alpha x_1^4 - \beta x_2^2)}$$

```

1 %% prob_3c.m
2 %
3 % this is a script to plot the valid configuration space of 2-DoF robot arm
4 %
5 % - written by: Dimitri Lezcano
6
7 %% Set-Up
8 global po r l1 l2
9
10 % run params
11 check_fwd_kin = false;
12 check_constraints = false;
13
14 % physical parameters
15 l1 = 1;
16 l2 = 1;
17 po = 1/2*[1;1];
18 r = 1/4;
19
20
21 % possible angles to iterate over
22 N_angles = 150;
23 theta1_v = linspace(0, 2*pi, N_angles);
24 theta2_v = linspace(0, 2*pi, N_angles);
25
26
27 % Determine which angles to keep
28 % part a
29 theta_valid_a = zeros(2, N_angles^2);
30 num_valid_configs_a = 0;
31
32 % part b
33 theta_valid_b = zeros(2, N_angles^2);
34 num_valid_configs_b = 0;
35 for theta1 = theta1_v
36     for theta2 = theta2_v
37         % check if valid configuration: part a
38         c_a = constrainta(theta1, theta2);
39         if (c_a <= 0)
40             % increment num_valid_configs
41             num_valid_configs_a = num_valid_configs_a + 1;
42
43             % add [theta1; theta2] to theta_valid
44             theta_valid_a(:, num_valid_configs_a) = [theta1; theta2];
45
46     end
47
48     % check if valid configuration: part b
49     c_b = constraintb(theta1, theta2);
50     if all(c_b <= 0)
51         % increment num_valid_configs
52         num_valid_configs_b = num_valid_configs_b + 1;
53
54         % add [theta1; theta2] to theta_valid
55         theta_valid_b(:, num_valid_configs_b) = [theta1; theta2];
56
57     end
58 end
59 end
60
61 % remove unused points
62 theta_valid_a = theta_valid_a(:,1:num_valid_configs_a);
63 theta_valid_b = theta_valid_b(:,1:num_valid_configs_b);
64
65 %% Plotting
66 % plot the points: part b
67 fb = figure(1);
68 theta_valid_deg_b = rad2deg(theta_valid_b);
69
```

```

70 plot(theta_valid_deg_b(1,:), theta_valid_deg_b(2,:), '.', 'MarkerSize', 10);
71 xlabel('theta1 (deg)'); ylabel('theta2 (deg)');
72 xlim([0, 360]); ylim([0, 360]);
73 title('Problem 3.c | Part b)');
74 grid on;
75
76 % plot the points: part a
77 fa = figure(2);
78 theta_valid_deg_a = rad2deg(theta_valid_a);
79 plot(theta_valid_deg_a(1,:), theta_valid_deg_a(2,:), '.', 'MarkerSize', 10);
80 xlabel('theta1 (deg)'); ylabel('theta2 (deg)');
81 xlim([0, 360]); ylim([0, 360]);
82 title('Problem 3.c | Part a)');
83 grid on;
84
85 % check the forward kinematics
86 if check_fwd_kin
87     theta1 = pi/4; theta2 = 0;
88     p1 = calculate_joint1(theta1, theta2);
89     p2 = calculate_tool(theta1, theta2);
90     p = [zeros(2,1) p1 p2];
91
92     figure(4);
93     plot(p(1,:), p(2,:),'.-'); hold on;
94     % check the constraints
95     if check_constraints
96         pts_obs = plot_obstacle_pts(po, r);
97         plot(pts_obs(1,:), pts_obs(2,:), 'r'); hold off;
98         disp('constraint');
99         disp(constraintb(theta1, theta2));
100    end
101    hold off;
102    xlim([- (l1 + l2 + 1) (l1 + l1 + l2)])
103    ylim([- (l1 + l2+ 1) (l1 + l1 + l2)])
104    grid on;
105 end
106
107
108
109 %% Saving
110 % part a figure
111 saveas(fa, 'prob_3c-a.png');
112 disp('Saved: prob_3c-a.png');
113
114 % part b figure
115 saveas(fb, 'prob_3c-b.png');
116 disp('Saved: prob_3c-b.png');
117
118 %% Functions
119 % implementation of the constraint function: Part a
120 function c = constrainta(theta1, theta2)
121     global po r
122     p_t = calculate_tool(theta1, theta2); % tool position
123
124     c = r^2 - (p_t - po)'*(p_t - po);
125
126
127 end
128 % implementation of the constraint function: Part b
129 function c = constraintb(theta1, theta2)
130     global po r
131     % get the joint position
132     p_base = zeros(2, 1);
133     p_jt1 = calculate_joint1(theta1, theta2);
134     p_t = calculate_tool(theta1, theta2);
135
136     % get the lambda conditions
137     % joint 1
138     A1 = calculate_A(po, p_base, p_jt1, r);

```

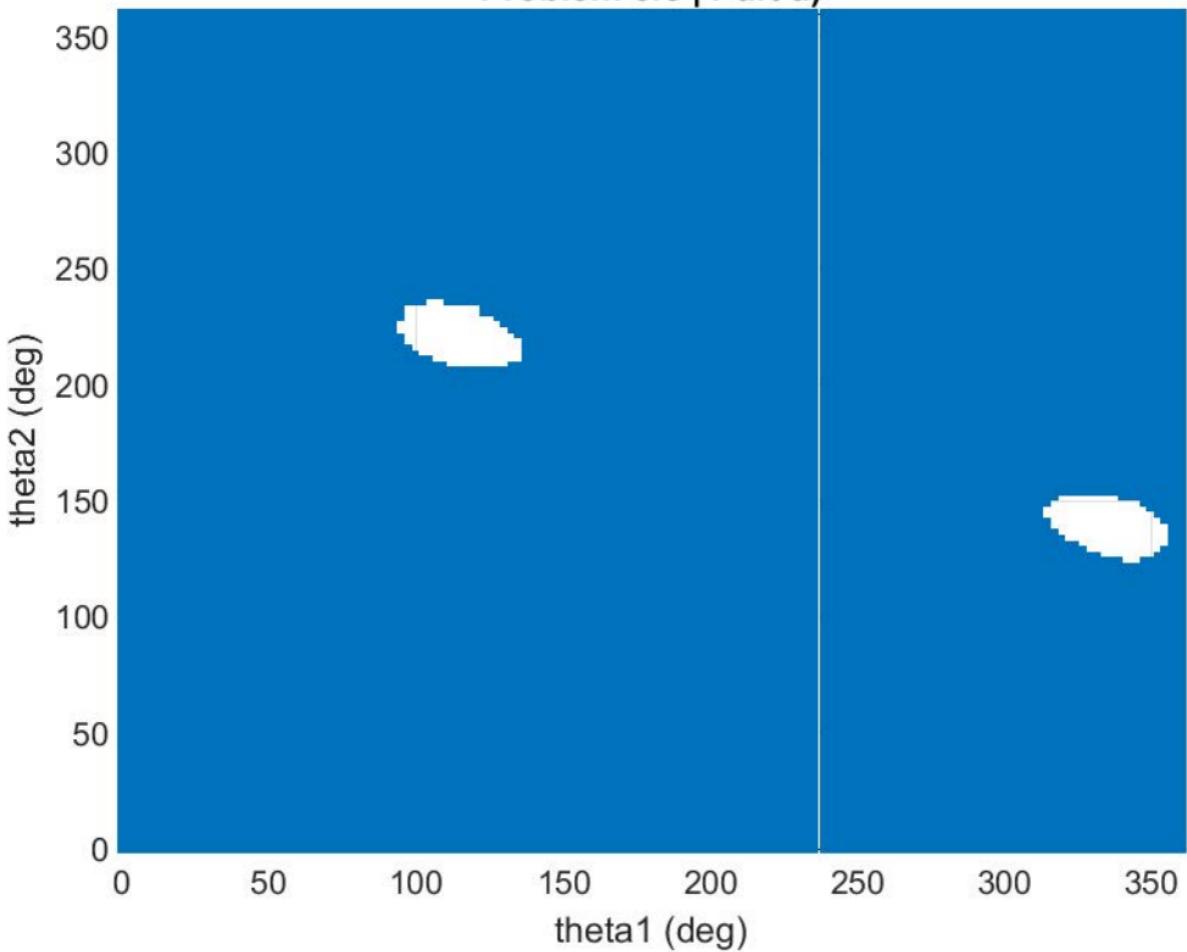
```

139     B1 = calculate_B(po, p_base, p_jt1, r);
140     C1 = calculate_C(po, p_base, p_jt1, r);
141
142     % joint2
143     A2 = calculate_A(po, p_jt1, p_t, r);
144     B2 = calculate_B(po, p_jt1, p_t, r);
145     C2 = calculate_C(po, p_jt1, p_t, r);
146
147     % perform analysis
148     % discriminant
149     discrim1 = B1^2 - 4 * A1 * C1;
150     discrim2 = B2^2 - 4 * A2 * C2;
151
152     c = zeros(4,1);
153     % joint1
154     if discrim1 <= 0
155         c(1:2) = discrim1;
156
157     else % 2 real roots
158         lambda_p = (-B1 + sqrt(discrim1))/(2*A1);
159         lambda_m = (-B1 - sqrt(discrim1))/(2*A1);
160         c(1) = (-B1 + sqrt(discrim1))*(2*A1 + B1 - sqrt(discrim1));
161         c(2) = (-B1 - sqrt(discrim1))*(2*A1 + B1 + sqrt(discrim1));
162     end
163
164     % joint 2
165     if discrim2 <= 0
166         c(3:4) = discrim2;
167
168     else % 2 real roots
169         lambda_p = (-B2 + sqrt(discrim2))/(2*A2);
170         lambda_m = (-B2 - sqrt(discrim2))/(2*A2);
171         c(3) = (-B2 + sqrt(discrim2))*(2*A2 + B2 - sqrt(discrim2));
172         c(4) = (-B2 - sqrt(discrim2))*(2*A2 + B2 + sqrt(discrim2));
173     end
174
175
176 end
177
178 % 2x2 rotation matrix
179 function R = rotate(theta)
180     R = [cos(theta), -sin(theta); sin(theta), cos(theta)];
181
182 end
183
184 % calculate A value for lambda root
185 function A = calculate_A(p0, p1, p2, r)
186     A = (p2 - p1)' * (p2 - p1);
187
188 end
189
190 % calculate B value for lambda root
191 function B = calculate_B(p0, p1, p2, r)
192     B = 2*(p1'*p2 - p2'*p2 - p0'*p1 + p0'*p2);
193
194 end
195
196 % calculate C value for lambda root
197 function C = calculate_C(p0, p1, p2, r)
198     C = (p0 - p2)' * (p0 - p2) - r^2;
199
200 end
201
202 % calculate joint 1 position
203 function p = calculate_joint1(theta1, theta2)
204     global l1 l2
205
206     p = l1 * rotate(theta1) * [1; 0];
207

```

```
208 end
209
210 % calculate tool position
211 function p = calculate_tool(theta1, theta2)
212     global l1 l2
213
214     p1 = calculate_joint1(theta1, theta2);
215     p = p1 + l2 * rotate(theta1 + theta2) * [1; 0];
216
217
218 end
219
220 % obstacle plot points
221 function pts = plot_obstacle_pts(po, r)
222     theta = linspace(0, 2*pi, 100);
223     pts = po + r * [cos(theta); sin(theta)];
224
225 end
226
227
228
```

Problem 3.c | Part a)



Problem 3.c | Part b)

