Homework 3: Dimitri Lezcano

Wednesday, October 7, 2020 12:08 AM

1) $\dot{x} = ax - bu$ $\quad x(t_0)$ given

$$J = \frac{1}{2} c [x_f]^2 + \frac{1}{2} \int_{t_0}^{t_f} [u(t)]^2 dt$$

$x, u, a, b, c \in \mathbb{R}$ $\quad \phi = \frac{1}{2} c (x(t_f))^2$

$\psi = 0$ $\quad \Phi = \phi$ $\quad L = \frac{1}{2} u^2$

Define

$H = \frac{1}{2} u^2 + \lambda (ax - bu)$

E-L Equations:

1) $\dot{x} = ax - bu$
2) $\dot{\lambda} = -\nabla_x = -a\lambda$

$\Rightarrow \frac{\dot{\lambda}}{\lambda} = -a \Rightarrow \ln(\lambda) = -\int_{t_0}^{t} a + \ln(\lambda(t_0))$

$$\lambda(t) = \lambda(t_0) e^{-a(t-t_0)}$$

Optimality:

$\nabla_u H = 0$

$\nabla_u H = u - b\lambda = 0 \Rightarrow u(t) = b\lambda(t)$

$\Rightarrow u(t) = \underbrace{b\lambda(t_0)} e^{-a(t-t_0)}$

$\Rightarrow u(t) = \underbrace{u(t_0)}_{\text{combine}} e^{-a(t-t_0)}$

Transversality Conds.:

Transversality Conds.:

1) $\psi(t_f) = 0$

2) $\lambda(t_f) = \nabla_x \phi \big|_{t=t_f}$

$$\lambda(t_f) = (c x)\big|_{t=t_f}$$

$$\lambda(t_f) = c x(t_f) = \lambda(t_0) e^{-a(t_f - t_0)}$$

$$\Rightarrow \lambda(t_0) = c x(t_f) e^{a(t_f - t_0)}$$

$$\Rightarrow \lambda(t) = c x(t_f) e^{a(t_f - t_0)} e^{-a(t - t_0)}$$

$$= c x(t_f) e^{a(t_f - t_0 + t_0 - t)}$$

$$\lambda(t) = c x(t_f) e^{a(t_f - t)}$$

$$\Rightarrow \underline{u(t) = bc x(t_f) e^{a(t_f - t)}}$$

$$\dot{x} = ax - bu$$

$$\dot{x} = ax - b^2 c x_f e^{a(t_f - t)}$$

$$\dot{x} - ax = -b^2 c x_f e^{a(t_f - t)}$$

$$x = x_H + x_p$$

$$x_H: \ \dot{x}_H - a x_H = 0$$

$$x_H = C_1 e^{a(t - t_0)}$$

$$x_p = C_2 e^{a(t_f - t)}$$

$$\dot{x}_p - a x_p = -2 C_2 a e^{a(t_f - t)} = -b^2 c x_f e^{a(t_f - t)}$$

$$\Rightarrow C_2 = \frac{b^2 c}{2a} x_f$$

$$x_p(t) = \frac{b^2 c}{2a} x_f e^{a(t_f - t)}$$

$$x(t) = C_1 e^{a(t - t_0)} + \frac{b^2 c}{2a} x_f e^{a(t_f - t)}$$

$$x(t_0) = x_0 = C_1 + \frac{b^2 c}{2a} x_f e^{a(t_f - t_0)}$$

$$x(t_0) = x_0 = C_1 + \frac{b^2 c}{2c} x_f e^{\dots}$$

$$C_1 = x_0 - \frac{b^2 c}{2a} x_f e^{a(t_f - t_0)}$$

$$x(t) = \left(x_0 - \frac{b^2 c}{2a} x_f e^{a(t_f - t_0)} \right) e^{a(t - t_0)} + \frac{b^2 c}{2a} x_f e^{a(t_f - t)}$$

$$x_f = x(t_f) = \left(x_0 - \frac{b^2 c}{2a} x_f e^{a(t_f - t_0)} \right) e^{a(t_f - t_0)} + \frac{b^2 c}{2a} x_f$$

$$x_f = x_0 e^{a(t_f - t_0)} + \left[-\frac{b^2 c}{2a} e^{2a(t_f - t_0)} + \frac{b^2 c}{2a} \right] x_f$$

$$\left[1 - \frac{b^2 c}{2a} \left(1 - e^{2a(t_f - t_0)} \right) \right] x_f = x_0 e^{a(t_f - t_0)}$$

$$x_f = \frac{x_0 e^{a(t_f - t_0)}}{1 - \frac{b^2 c}{2a} \left(1 - e^{2a(t_f - t_0)} \right)}$$

$\Rightarrow$

$$u(t) = b c x_f e^{a(t_f - t)}$$

$$\underline{u(t) = \frac{x_0 b c e^{a(t_f - t_0)} e^{a(t_f - t)}}{1 - \frac{b^2 c}{2a} \left(1 - e^{2a(t_f - t_0)} \right)}}$$

2) $\dot{x} = ax - bu = f(x, u)$

$$J = \frac{1}{2} c x^2(t_f) + \int_{t_0}^{t_f} \frac{1}{2} (u(t))^2 \, dt$$

$x, u, a, b, c \in \mathbb{R}$

$A = a \qquad B = -b$

$x, u, a, b, c \in \mathbb{R}$

$A = a \qquad B = -b$

$P_f = c \qquad \left(\frac{1}{2} x_f \cdot c \cdot x_f\right)$

$Q = 0$

$R = 1 \qquad \left(\frac{1}{2} u \cdot 1 \cdot u\right)$

Assume $\lambda(t) = P(t)\, x(t)$ "$\lambda = Px$"

Ricatti:

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P \cancel{-Q}^{\,0}$$

$$\dot{P} = -2aP + P(-b)(1^{-1})(-b)P$$

$$\dot{P} = -2aP + b^2 P^2$$

s.t. $P(t_f) = P_f = c$

Let $P = -b^2 \frac{\dot{v}}{v}$

$$\dot{P} = -b^2 \frac{\ddot{v}}{v} + \frac{b^2 \dot{v}^2}{v^2} = -b^2 \frac{\ddot{v}}{v} + b^2 P^2$$

$$\dot{P} - b^2 P^2 = -b^2 \frac{\ddot{v}}{v} = -2aP$$

$$+ \frac{b^2 \ddot{v}}{\cancel{v}} = +2aP = +2a \frac{\dot{v}}{\cancel{v}}$$

$$b^2 \ddot{v} = 2a\dot{v}$$

$$b^2 \ddot{v} - 2a\dot{v} = 0$$

integrate $\hookrightarrow b^2 \dot{v} - 2av = C_1$

$$\dot{v} - \frac{2a}{b^2} v = C_1$$

$$v(t) = v_H(t) + v_p(t)$$

$$v_p(t) = -\frac{b^2}{2a} C_1$$

$$v_p(t) = -\frac{b^2}{2a} c_1$$

$$v_H(t) = c_2 e^{-\frac{2a}{b^2}(t-t_0)}$$

$$v(t) = -\frac{b^2}{2a} c_1 + c_2 e^{-\frac{2a}{b^2}(t-t_0)}$$

$$\dot{v}(t) = -c_2\, 2a/b^2\, e^{-2a/b^2(t-t_0)}$$

$$P = \frac{\dot{v}}{v} = \frac{c_2\, 2a/b^2\, e^{-\frac{2a}{b^2}(t-t_0)}}{\frac{b^2 c_1}{2a} + c_2 e^{-2a/b^2(t-t_0)}}$$

$$P(t_f) = P_f = \frac{c_2\, 4a^3\, e^{\frac{-2a(t_f-t_0)}{b^2}}}{b^4 c_1 + 2ab^2 c_2 e^{-\frac{2a}{b^2}(t_f-t_0)}} = c$$

$$\frac{4a^2 e^{-\frac{2a}{b^2}(t_f-t_0)}}{\frac{c_1}{c_2} b^4 + 2ab^2 e^{-\frac{2a}{b^2}(t_f-t_0)}} = c \qquad c' = \frac{c_1}{c_2}$$

$$\frac{4a^2}{c} e^{-\frac{2a}{b^2}(t_f-t_0)} = c' b^4 + 2ab^2 e^{-\frac{2a}{b^2}(t_f-t_0)}$$

$$\frac{\left(\frac{4a^3}{c} - 2ab^2\right) e^{-\frac{2a}{b^2}(t_f-t_0)}}{b^4} = c'$$

$$\Rightarrow P(t) = \left(\frac{4a^2 - 2ab^2 c}{cb^4}\right) e^{-\frac{2a}{b^2}(t_f-t_0)}\, e^{-\frac{2a}{b^2}(t-t_0)}$$

$$P(t) = \left(\frac{4a^2 - 2ab^2 c}{cb^4}\right) e^{-\frac{2a}{b^2}(t+t_f-2t_0)}$$

So then $u^* = -R^{-1} B^T P x$

$$= +(1^{-1})(+b) P x$$

$$= +(1^{-1})(+b)\,P\,x$$

$$u^*(t) = \left(\frac{4a^2 - 2ab^2c}{cb^3}\right) e^{\frac{-2a}{b^2}(t+t_f-2t_0)}\, x(t)$$

3) $$\dot{x} = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} x + u e_2$$

$$J = \int_0^{t_f} \left[\left(x_1^2 + \tfrac{1}{2} x_2^2\right) + \tfrac{1}{2} u^2\right] dt$$

$$x(0) = \begin{pmatrix} -5 \\ 5 \end{pmatrix} \quad t_f = 20$$

$P_f = 0_{2\times 2}$

$A = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix} \qquad B = e_2$

$Q = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

$R = 1$

Assume $\lambda = P x$

Riccati $\qquad P(t_f) = P_f = 0_{2\times 2}$

$$\dot{P} = -A^T P - PA + P B R^{-1} B^T P - Q$$

$$= -A^T P - PA + P e_2 (1^{-1}) e_2^T P - Q$$

$$\dot{P} = -A^T P - PA + P e_2 e_2^T P - Q$$

4) $$\dot{x} = f(x,u) = \begin{pmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{\theta} \\ \dot{v} \\ \dot{\delta} \end{pmatrix} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \\ v\tan\delta \\ u_1 \\ u_2 \end{pmatrix}$$

$$\begin{pmatrix} \vdots \\ \dot{\delta} \end{pmatrix} \quad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$x_d(t) = \begin{pmatrix} t \\ 2t \\ \tan^{-1}(2) \\ \sqrt{5} \\ 0 \end{pmatrix} \qquad u_d(t) = \vec{0}$$

$\cos\theta \equiv c\theta$

$\sin\theta \equiv s\theta$

$\tan\delta \equiv t\delta$

a) $e = x(t) - x_d(t)$

$$\dot{e} = \dot{x} - \dot{x}_d$$

$$= \begin{pmatrix} v c\theta \\ v s\theta \\ v t\delta \\ u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\dot{e} = \begin{pmatrix} v c\theta \\ v s\theta \\ v t\delta \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ u_1 \\ u_2 \end{pmatrix}$$

b) $\dot{e} = f(x, u) - \dot{x}_d$

$$\dot{e} = f(\cancel{x_d}, u_d) + \partial_x f\big|_{x = x_d} \cdot (x - x_d) + \partial_u f\,(u - u_d) + O(\|x\|, \|u\|) - \cancel{\dot{x}_d}$$

$$\dot{e} \approx \partial_x f \cdot e + \partial_u f \cdot s$$

(1st order)

$$A = \partial_x f = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -v_d s\theta_d & c\theta_d & 0 \\ v_d c\theta_d & s\theta_d & 0 \\ 0 & t\delta_d & v_d \sec^2 \delta_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

$$B = \partial_u f = \begin{pmatrix} e_4 & e_5 \end{pmatrix} \qquad e_i = \text{i-th colum of Identity matrix}$$

$$\dot{e} = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -v_d s\theta_d & c\theta_d & 0 \\ v_d c\theta_d & s\theta_d & 0 \\ 0 & t\delta_d & v_d \sec^2 \delta_d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{pmatrix} e + \begin{pmatrix} e_4 & e_5 \end{pmatrix} s$$

$$\sin(\tan^{-1}(2)) = \frac{2}{\sqrt{5}} \qquad \cos(\tan^{-1}(2)) = \frac{1}{\sqrt{5}}$$

$$A = \begin{pmatrix} \vec{0} & \vec{0} & \begin{matrix} -2 & 1/\sqrt{5} & 0 \\ 1 & 2/\sqrt{5} & 0 \\ 0 & 0 & \sqrt{5} \\ \vec{0} & \vec{0} & \vec{0} \end{matrix} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$