Homework 2: Dimitri Lezcano

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1) $$J(x) = \int_0^1 \left(\frac{1}{2}\dot{x}^2 + 3x\dot{x} + 2x^2 + 3x \right) dt$$

$$\min_x J \quad \text{s.t. } x(0) = 0, \quad x(1) = 4$$

$$L(x, \dot{x}) := \frac{1}{2}\dot{x}^2 + 3x\dot{x} + 2x^2 + 3x$$

Using Euler-Lagrange's equations:

$$\nabla_x L = \frac{d}{dt} \nabla_{\dot{x}} L$$

$$\frac{d}{dt}(\nabla_{\dot{x}} L) = \frac{d}{dt}\left(\dot{x} + 3x \right) = \ddot{x} + 3\dot{x}$$

$$\nabla_x L = 3\dot{x} + 4x + 3$$

$\Rightarrow$

$$\ddot{x} + 3\dot{x} = 3\dot{x} + 4x + 3$$

$$\rightarrow \ddot{x} - 4x = 3$$

$$x^*(t) = x_H(t) + x_p(t)$$

Let $x_p(t) = \frac{-3}{4}$ ∴

$$\ddot{x}_p - 4x_p = 0 - 4\left(\frac{-3}{4}\right) = 3 \checkmark$$

$x_H(t)$ is soln,

$$\ddot{x}_H - 4x_H = 0$$

$$x_H = c_1 e^{2t} + c_2 e^{-2t}$$

$$x^*(t) = c_1 e^{2t} + c_2 e^{-2t} - 3/4$$

$$x^*(0) = c_1 + c_2 - 3/4 = 0$$

$$x^*(1) = c_1 e^2 + c_2 e^{-2} - 3/4 = 1$$

$$\begin{pmatrix} 1 & 1 \\ e^2 & e^{-2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 7/4 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ e^2 & e^{-2} \end{pmatrix}^{-1}\begin{pmatrix} 3/4 \\ 7/4 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cong \begin{pmatrix} 0.2273 \\ 0.5227 \end{pmatrix}$$

$$\underline{x^*(t) = 0.2273 e^{2t} + 0.5227 e^{-2t} - 3/4}$$

2) $J = \int^{t_f} (\dot{x}_1^2 + \dot{x}_2^2 + 3x_1 x_2)\, dt$

2) $J = \int_0^{t_f} (\dot{x}_1^2 + \dot{x}_2^2 + 3x_1 x_2)\, dt$

$L(x, \dot{x}) := \dot{x}_1^2 + \dot{x}_2^2 + 3x_1 x_2$

c) $x_1(0) = x_2(0) = 0$

$x_2(t_f) = 1$

$x_1(t_f)$ free (x_{1f})

$t_f = 1$

Using E-L equations

$$\nabla_x L = \begin{pmatrix} 3x_2 \\ 3x_1 \end{pmatrix} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$$

$$\frac{d}{dt}(\nabla_{\dot{x}} L) = \frac{d}{dt} \begin{pmatrix} 2\dot{x}_1 \\ 2\dot{x}_2 \end{pmatrix} = 2\ddot{x}$$

$$2\ddot{x} = 3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x$$

$$\ddot{x} = \frac{3}{2} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} x$$

$A = SDS^{-1}$

$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

and $\left.\frac{\partial}{\partial \dot{x}_1} L\right|_{t = t_f} = 0$

$$\frac{\partial \dot{x}_1}{} \Big|_{t=t_f} = 0 \qquad (1 \ 1 \ -1)$$

$$\Rightarrow \dot{x}_1(t_f) = 0$$

$$S^{-1}\ddot{x} = \frac{3}{2} D S^{-1} x$$

$$z = S^{-1} x$$

$$\ddot{z} = \frac{3}{2} D z$$

$$\hookrightarrow \Rightarrow \ddot{z}_1 - 3/2\, z_1 = 0$$

$$\ddot{z}_2 + 3/2\, z_2 = 0$$

$$z_1(t) = c_1 e^{+t\sqrt{3/2}} + c_2 e^{-t\sqrt{3/2}}$$

$$z_2(t) = c_3 \cos\left(t\sqrt{\tfrac{3}{2}}\right) + c_4 \sin\left(t\sqrt{\tfrac{3}{2}}\right)$$

let $\lambda = \sqrt{3/2}$

$$z(0) = S^{-1} x(0) = 0$$

$$z(1) = S^{-1} x(1) = S^{-1} \begin{pmatrix} x_{1f} \\ 1 \end{pmatrix}$$

$$z(0) = \begin{pmatrix} c_1 + c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} c_1 = -c_2 \\ c_3 = 0 \end{matrix}$$

$$z(t) = \begin{pmatrix} c_1 e^{\lambda t} - c_1 e^{-\lambda t} \\ c_4 \sin \lambda t \end{pmatrix}$$

$$z(1) = \begin{pmatrix} c_1 (e^{\lambda} - e^{-\lambda}) \\ c_4 \sin \lambda \end{pmatrix} = S^{-1} \begin{pmatrix} x_{1f} \\ 1 \end{pmatrix}$$

$= (c_1 (e^{\lambda} - e^{-\lambda})) = \ldots$

$$= \begin{pmatrix} c_1(e^{\lambda}-e^{-\lambda}) \\ c_4 \sin\lambda \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_{1f} \\ 1 \end{pmatrix}$$

$$\Rightarrow c_1 = \frac{x_{1f}+1}{2(e^{\lambda}-e^{-\lambda})}$$

$$c_2 = \frac{x_{1f}-1}{2\sin\lambda}$$

$$z(t) = \begin{bmatrix} \frac{x_{1f}+1}{2}\left(\frac{e^{\lambda t}-e^{-\lambda t}}{e^{\lambda}-e^{-\lambda}}\right) \\ \frac{x_{1f}-1}{2}\left(\frac{\sin\lambda t}{\sin\lambda}\right) \end{bmatrix}$$

$$x(t) = S\,z(t)$$
$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{x_{1f}+1}{2}\left(\frac{e^{\lambda t}-e^{-\lambda t}}{e^{\lambda}-e^{-\lambda}}\right) \\ \frac{x_{1f}-1}{2}\left(\frac{\sin\lambda t}{\sin\lambda}\right) \end{pmatrix}$$

$$x(t) = \begin{bmatrix} \frac{x_{1f}+1}{2}\left(e^{\lambda t}-e^{-\lambda t}/e^{\lambda}-e^{-\lambda}\right) + \frac{x_{1f}-1}{2}\left(\sin\lambda t/\sin\lambda\right) \\ \frac{x_{1f}+1}{2}\left(e^{\lambda t}-e^{-\lambda t}/e^{\lambda}-e^{-\lambda}\right) - \frac{x_{1f}-1}{2}\left(\sin\lambda t/\sin\lambda\right) \end{bmatrix}$$

$$x(t) = \begin{bmatrix} \frac{x_{1f}+1}{2}\left(\frac{\sinh\lambda t}{\sinh\lambda}\right) + \frac{x_{1f}-1}{2}\left(\frac{\sin\lambda t}{\sin\lambda}\right) \\ \frac{x_{1f}+1}{2}\left(\frac{\sinh\lambda t}{\sinh\lambda}\right) - \frac{x_{1f}-1}{2}\left(\frac{\sin\lambda t}{\sin\lambda}\right) \end{bmatrix}$$

$$\dot{x}_1(1) = 0 \Rightarrow$$

$$\frac{x_{1f}+1}{\cancel{2}}\left(\cancel{\lambda}\tanh\lambda\right) + \frac{x_{1f}-1}{\cancel{2}}\left(\cancel{\lambda}\tan\lambda\right) = 0$$

$$\Leftrightarrow (\tanh\lambda + \tan\lambda)\,x_{1f} = \tan\lambda - \tanh\lambda$$

$$x_{1f} = \frac{\tan\lambda - \tanh\lambda}{\tan\lambda + \tanh\lambda} \qquad \lambda = \left(\frac{3}{2}\right)^{1/2}$$

$$\tan\lambda + \tanh\lambda \qquad (2)$$

b) $x(0)=0$ $\qquad L = \dot{x}_1^2 + \dot{x}_2^2 + 3x_1x_2$

$x(t_f)$ on

$$x_1 + 3x_2 + 5t = 15$$

$$\psi(x_f, t_f) = x_{1_f} + 3x_{2_f} + 5t_f - 15 = 0$$

t_f free

$$w = \gamma\psi$$

Conditions

① $\nabla_x(\gamma\psi)\big|_{t=t_f} + \nabla_{\dot{x}}L\big|_{t=t_f} = 0$

② $\frac{\partial}{\partial t_f}\left[\gamma\psi(x(t_f), t_f)\right] + L\big|_{t=t_f} - \nabla_{\dot{x}}L\big|_{t=t_f}^T \dot{x}(t_f) = 0$

③ $\psi(x(t_f), t_f) = 0$

④ $\frac{d}{dt}\left(\nabla_{\dot{x}}L\right) - \nabla_x L = 0$

- $\nabla_{\dot{x}}L = 2\dot{x}$ $\quad$ • $\nabla_x\psi = \begin{pmatrix}1\\3\end{pmatrix}$ $\quad$ • $\frac{\partial\psi}{\partial t_f} = 5$

By ④ and using notation above $\left(z = S^{-1}x,\ \lambda = (3/2)^{1/2}\right)$

$$\ddot{x} = \lambda^2 A x \quad\rightarrow\quad \ddot{z} = \lambda^2\begin{pmatrix} z_1 \\ -z \end{pmatrix}$$

$$x = T^{-1} A x \rightarrow z = \lambda \begin{pmatrix} z_1 \\ -\dot{z}_2 \end{pmatrix}$$

$$\Rightarrow z(t) = \begin{pmatrix} c_1 e^{\lambda t} + c_2 e^{-\lambda t} \\ c_3 \cos \lambda t + c_4 \sin \lambda t \end{pmatrix}$$

$$z(0) = 0 \quad (BC) \Rightarrow z(t) = \begin{pmatrix} c_1 \sinh \lambda t \\ c_4 \sin \lambda t \end{pmatrix} *$$

$$(1) \quad \nabla_x (\nu \psi) + \nabla_{\dot{x}} L \Big|_{t=t_f} = 0$$

$$\nu \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \dot{x}(t_f) = 0$$

$$* \quad \dot{x}(t_f) = -\frac{\nu}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\dot{z}(t_f) = S^{-1} \dot{x}(t_f) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left[-\frac{\nu}{2} \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right]$$

$$= -\frac{\nu}{4} \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

$$\cdot \quad \underbrace{\dot{z}_f = \nu \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}}$$

$$\Rightarrow \dot{z}(t_f) = \begin{pmatrix} c_1 \lambda \cosh \lambda t_f \\ c_4 \lambda \cos \lambda t_f \end{pmatrix} = \nu \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$$

$$\Rightarrow c_1 = -\nu \quad 1$$

$$\Rightarrow c_1 = -\frac{\gamma}{\lambda} \frac{1}{\cosh \lambda t_f}$$

$$c_4 = \frac{\gamma}{2\lambda} \frac{1}{\cos \lambda t}$$

$$\Rightarrow z(t_f) = \begin{pmatrix} -\gamma/\lambda \tanh \lambda t_f \\ \gamma/2\lambda \tan \lambda t_f \end{pmatrix} = z_f$$

$$2) \frac{\partial}{\partial t_f}\left(\gamma \psi\right) + L(t_f) - \nabla_{\dot{x}} L(t_f)^T \dot{x}_f = 0$$

$$5\gamma + \dot{x}_f^T \dot{x}_f + 3x_{1f} x_{2f} - 2\dot{x}_f^T \dot{x}_f = 0$$

$$5\gamma - \dot{x}_f^T \dot{x}_f + 3x_{1f} x_{2f} = 0$$

$$5\gamma - \gamma^2 \frac{(1+9)}{4} + 3x_{1f} x_{2f} = 0$$

$$5\gamma - \frac{5}{2}\gamma^2 + 3x_{1f} x_{2f} = 0$$

$$\because z = S^{-1}x \rightarrow x_f = S z_f = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z_{1f} \\ z_{2f} \end{pmatrix}$$

$$= \begin{pmatrix} z_{1f} + z_{2f} \\ z_{1f} - z_{2f} \end{pmatrix}$$

$$|z_{1f} - z_{2f}|$$

$$\Rightarrow 5\gamma - \frac{5}{2}\gamma^2 + 3(z_{1f} + z_{2f})(z_{1f} - z_{2f}) = 0$$

$$5\gamma - \frac{5}{2}\gamma^2 + 3z_{1f}^2 - 3z_{2f}^2 = 0$$

$$5\gamma - \frac{5}{2}\gamma^2 - \frac{3\gamma^2}{\lambda^2}\tanh^2\lambda t_f - \frac{3\gamma^2}{4\lambda^2}\tan^2\lambda t_f = 0$$

$$\frac{1}{2}\gamma\left[10 - \left(5 + \frac{3}{\lambda^2}\tanh^2\lambda t_f + \frac{3}{4\lambda^2}\tan^2\lambda t_f\right)\gamma\right] = 0$$

$$\left(\gamma = 0\ ,\ \frac{40\lambda^2}{20\lambda^2 + 12\tanh^2\lambda t_f + 3\tan^2\lambda t_f}\right)$$

$\Rightarrow \gamma = 0$ is a solution to the above

$$\Rightarrow z(t) = 0$$

$$\Rightarrow x(t) = S\,z(t) = S\,0 = 0$$

$$\Rightarrow x_1(t_f) + 3x_2(t_f) + 5t_f = 0 + 3\cdot 0 + 5t_f = 15$$

$$\Rightarrow t_f = 3$$

so

$x(t) = 0$ for $t \in [0, 3]$

is a solution to this problem

for $\gamma \neq 0$, (3) becomes $\forall i = 1, 2$

we note $|z_i(t)| \propto \gamma \Rightarrow |x_i(t)| \propto \gamma$ and $|\dot{x}_i(t)| \propto \gamma \Rightarrow$

$L \propto \gamma^2 \Rightarrow J \propto \gamma^2$ $\therefore$ we have

that smallest $|\gamma|$ will yield the minimal solution

$\Rightarrow \gamma = 0$ is the optimal solution!

$\Rightarrow$ $\boxed{x(t) = 0 \quad t \in [0, 3] \text{ is the}}$

2) $\eta = 0$ is the optimal solution.

$\Rightarrow$ $\boxed{x(t) = 0 \quad t \in [0, 3] \text{ is the extremal!}}$

3) $$J = \int_{t_0}^{t_f} g(x, \dot{x}, t)\, dt$$

Let $x = x^* + \varepsilon\eta$ s.t. x^* is the extremal of J w/ fixed endpoints.

$$J = \int_{t_0}^{t_f} g(x^* + \varepsilon\eta, \dot{x}^* + \varepsilon\dot{\eta}, t)\, dt$$

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = \left[\frac{d}{d\varepsilon}\int_{t_0}^{t_f} g(x, \dot{x}, t)\, dt\right]\Bigg|_{\varepsilon=0}$$

$$= \int_{t_0}^{t_f} \frac{d}{d\varepsilon} g(x, \dot{x}, t)\, dt \qquad \because \varepsilon \perp \text{ of integral}$$

$$= \int_{t_0}^{t_f} \left[\nabla_x g^T \frac{dx}{d\varepsilon} + \nabla_{\dot{x}} g^T \frac{d\dot{x}}{d\varepsilon}\right]\Bigg|_{\varepsilon=0} dt = 0 \qquad \because \text{chain rule}$$

$$= \int_{t_0}^{t_f} \left[\nabla_x g^T \eta + \nabla_{\dot{x}} g^T \dot{\eta} \right] dt \Bigg|_{\varepsilon = 0} = 0$$

by integration by parts...

$$\Rightarrow \left[\nabla_x g^T \eta \right] \Big|_{t_0}^{t_f} + \int_{t_0}^{t_f} \left(\nabla_x g - \frac{d}{dt} (\nabla_{\dot{x}} g) \right)^T \eta \, dt = 0$$

$\because \eta(t_f) = \eta(t_0) = 0$

$$\Rightarrow \int_{t_0}^{t_f} \left(\nabla_x g - \frac{d}{dt} (\nabla_{\dot{x}} g) \right)^T \eta \, dt = 0$$

and $\because$ η is arbitrary

$$\Rightarrow \nabla_x g - \frac{d}{dt} (\nabla_{\dot{x}} g) = 0$$