

3.1 변위와 무한소 변형률

· 변형률 (strain) : 변형하는 물체에서 물체내의 점의 상대적 변위와 관련

CF) 강체적운동에서 변형률 관계 x

· 무한소의 변형률 증분 → 유한 변형률에서의 변형률 증분

· $A(x, y, z) \rightarrow A'(x+u, y+v, z+w)$ u : A점의 변위의 x, y, z 좌표 성분

$B(x+\delta x, y+\delta y, z+\delta z) \rightarrow B'(x+\delta x+u+\delta u, y+\delta y+v+\delta v, z+\delta z+w+\delta w)$ u : B점의 변위의 x, y, z 좌표 성분

· $u(x, y, z)$ 의 연속함수 $u = f(x, y, z)$ 로 표시 & $\delta x, \delta y, \delta z$ 의 2차이상의 항 무시

$$\delta u = \frac{\partial u}{\partial x} \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z$$

· $\frac{\partial u}{\partial x} \delta x$: 최초의 길이 δx 의 변화량

$\frac{\partial u}{\partial x}$: A점에서의 O_x 방향의 수직 변형률 (direct strain 또는 normal strain) ($= e_{xx}$)

$\frac{\partial v}{\partial y}$: 변위 u 로 생긴 xO_z 면에 평행한 면내에서의 회전 \therefore 각변형률 (angular strain)

O_y 에 수직인 두 면 사이의 O_x 방향에 평행한 미끄러짐 \therefore 전단의 비율 ($= e_{xy}$)

$\frac{\partial w}{\partial z}$: 변위 u 로 생긴 xO_z 면에 평행한 면내에서의 회전 \therefore 각변형률 (angular strain)

O_z 에 수직인 두 면 사이의 O_x 방향에 평행한 미끄러짐 \therefore 전단의 비율 ($= e_{xz}$)

· $e_{ij} \Rightarrow i$: 운동의 방향 (O_i) / j : O_j 에 평행한 선분상에 있음

· $\delta v = \frac{\partial v}{\partial x} \delta x + \frac{\partial v}{\partial y} \delta y + \frac{\partial v}{\partial z} \delta z$ / $\delta w = \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + \frac{\partial w}{\partial z} \delta z$

· 회전의 부호 : 오른나사의 법칙

· 공학적 전단변형률 (engineering shear strain) : 최초에 직각이었던 각 FAC의 변화

$$\phi_{zx} \leftarrow 274 \text{의 미끄러짐으로 정의} \quad \phi_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = e_{xz} + e_{zx} \quad / \quad e_{xx} = \frac{\partial u}{\partial x} \quad e_{yy} = \frac{\partial v}{\partial y} \quad e_{zz} = \frac{\partial w}{\partial z} \quad \phi_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \phi_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \quad \phi_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

· 변형률 텐서

$$e_{ij} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix}$$

· O_x 축의 돌레로의 회전 $w_x \approx \frac{1}{2} \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \frac{1}{2} (e_{zy} - e_{yz})$

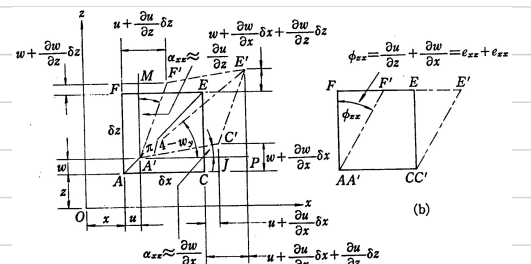
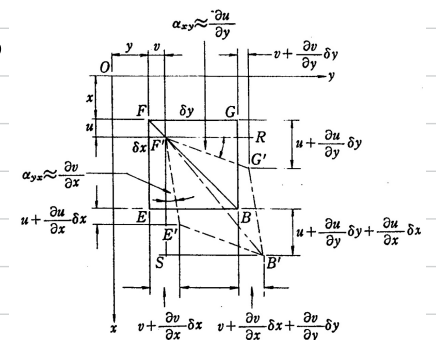
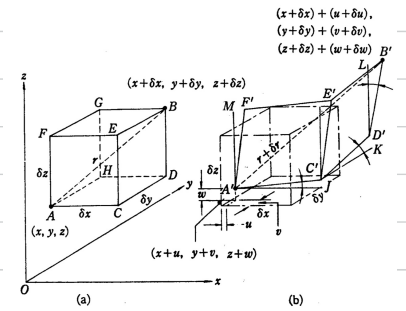
O_y 축의 돌레로의 회전 $w_y \approx \frac{1}{2} \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) = \frac{1}{2} (e_{xz} - e_{zx})$

O_z 축의 돌레로의 회전 $w_z \approx \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} (e_{yx} - e_{xy})$

· 비회전 (irrotational) : $w_x = w_y = w_z = 0 \quad \therefore e_{yz} = e_{zy} = \phi_{zy}/2 \quad e_{xz} = e_{zx} = \phi_{zx}/2 \quad e_{xy} = e_{yx} = \phi_{yx}/2$

· 변형률의 접합조건식 (strain compatibility equation) 변위 3개의 성분 → 변형률 6개의 성분 \therefore 독립적 x

$$\begin{aligned} \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} &= \frac{\partial^2 \phi_{xy}}{\partial x \partial y} & 2 \frac{\partial^2 e_{xz}}{\partial x \partial y} &= \frac{\partial}{\partial z} \left(\frac{\partial \phi_{yz}}{\partial x} + \frac{\partial \phi_{zx}}{\partial y} - \frac{\partial \phi_{xy}}{\partial z} \right) \\ \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} &= \frac{\partial^2 \phi_{xy}}{\partial x \partial y} & 2 \frac{\partial^2 e_{yz}}{\partial y \partial z} &= \frac{\partial}{\partial z} \left(\frac{\partial \phi_{yz}}{\partial y} + \frac{\partial \phi_{zy}}{\partial z} - \frac{\partial \phi_{xy}}{\partial z} \right) \\ \frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} &= \frac{\partial^2 \phi_{xy}}{\partial x \partial y} & 2 \frac{\partial^2 e_{yz}}{\partial y \partial z} &= \frac{\partial}{\partial z} \left(\frac{\partial \phi_{yz}}{\partial y} + \frac{\partial \phi_{zy}}{\partial z} - \frac{\partial \phi_{xy}}{\partial z} \right) \end{aligned}$$



3.2 변형률 텐서

$$\cdot e_{ij} = \frac{\partial u_i}{\partial x_j} \quad u_i : i \text{ 방향에서의 변위} \quad x_j : \text{축}$$

$$e_{ij} = \varepsilon_{ij} + \omega_{ij}$$

$$\cdot \varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = \varepsilon_{ij} \quad \text{if } i=j \rightarrow \varepsilon_{ij} \quad i \neq j \rightarrow \omega_{ij}$$

$$\cdot \varepsilon_{ij} = \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{pmatrix}$$

$$\cdot \partial_x (H e_x) \partial_y (H e_y) \partial_z (H e_z) = \partial_x \partial_y \partial_z (H e_x) (H e_y) (H e_z) = \partial_x \partial_y \partial_z (H e_x + e_y + e_z + e_x e_y + e_y e_z + e_z e_x + e_x e_y e_z)$$

$$\delta V = \partial_x \partial_y \partial_z (e_x + e_y + e_z)$$

$$\therefore \Delta = e_x + e_y + e_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

3.3 선의 미분

· $r^2 = \overline{AB}^2 = (\partial x)^2 + (\partial y)^2 + (\partial z)^2$

$(r + \partial r)^2 = \overline{A'B'}^2 = (\partial x + \partial x')^2 + (\partial y + \partial y')^2 + (\partial z + \partial z')^2$

$= (\partial x)^2 + (\partial y)^2 + (\partial z)^2 + (\partial x')^2 + (\partial y')^2 + (\partial z')^2 + 2(\partial x \partial x' + \partial y \partial y' + \partial z \partial z')$

$(r + \partial r)^2 - r^2 = (\partial x')^2 + (\partial y')^2 + (\partial z')^2 + 2(\partial x \partial x' + \partial y \partial y' + \partial z \partial z')$

$r \partial r = \partial x \partial x' + \partial y \partial y' + \partial z \partial z'$

$r \partial r = \partial x \left[\frac{\partial u}{\partial x} \partial x + \frac{\partial u}{\partial y} \partial y + \frac{\partial u}{\partial z} \partial z \right] + \partial y \left[\frac{\partial v}{\partial x} \partial x + \frac{\partial v}{\partial y} \partial y + \frac{\partial v}{\partial z} \partial z \right] + \partial z \left[\frac{\partial w}{\partial x} \partial x + \frac{\partial w}{\partial y} \partial y + \frac{\partial w}{\partial z} \partial z \right]$

$= (\partial x)^2 \frac{\partial u}{\partial x} + (\partial y)^2 \frac{\partial v}{\partial y} + (\partial z)^2 \frac{\partial w}{\partial z} + \partial x \partial y \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + \partial x \partial z \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \right) + \partial y \partial z \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right)$

· $\partial x = r l \quad \partial y = r m \quad \partial z = r n$

$\therefore \partial r = r \left[l^2 \frac{\partial u}{\partial x} + m^2 \frac{\partial v}{\partial y} + n^2 \frac{\partial w}{\partial z} + l m \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) + l n \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \right) + m n \left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) \right]$

· AB 방향의 변형률

$e_r = \frac{\partial r}{r} = e_x l^2 + e_y m^2 + e_z n^2 + \gamma_{xy} l m + \gamma_{xz} m n + \gamma_{yz} n l$

$= e_x l^2 + e_y m^2 + e_z n^2 + 2 \epsilon_{xy} l m + 2 \epsilon_{yz} m n + 2 \epsilon_{zx} n l$

3.4 유한변형

$$\cdot \xi = x + \delta x + u + \delta u - (x + u) = \delta x + \delta u$$

$$\eta = \delta y + \delta v$$

$$\psi = \delta z + \delta w$$

$$\cdot \xi = (1 + \frac{\partial u}{\partial x}) \delta x + \frac{\partial u}{\partial y} \delta y + \frac{\partial u}{\partial z} \delta z$$

$$\eta = \frac{\partial v}{\partial x} \delta x + (1 + \frac{\partial v}{\partial y}) \delta y + \frac{\partial v}{\partial z} \delta z$$

$$\psi = \frac{\partial w}{\partial x} \delta x + \frac{\partial w}{\partial y} \delta y + (1 + \frac{\partial w}{\partial z}) \delta z$$

$$\cdot A'B' = r + \delta r$$

$$\frac{\xi}{r + \delta r} = l_i = \left[\left(1 + \frac{\partial u}{\partial x} \right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n \right] \frac{r}{r + \delta r}$$

$$\frac{\eta}{r + \delta r} = m_i = \left[\frac{\partial v}{\partial x} l + \left(1 + \frac{\partial v}{\partial y} \right) m + \frac{\partial v}{\partial z} n \right] \frac{r}{r + \delta r}$$

$$\frac{\psi}{r + \delta r} = n_i = \left[\frac{\partial w}{\partial x} l + \frac{\partial w}{\partial y} m + \left(1 + \frac{\partial w}{\partial z} \right) n \right] \frac{r}{r + \delta r}$$