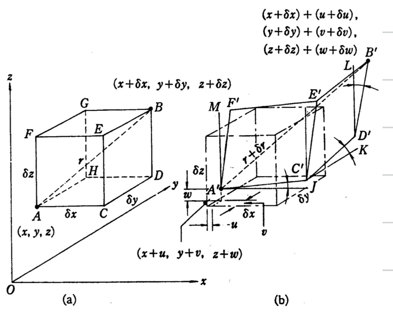
3.1 변위와 무한소 변형률

· 변형률 (strain) : 변형하는 물체에서 물체내의 점의 상대적 변위와 관련

cf) 강체적운동에서 변형률 관계 x



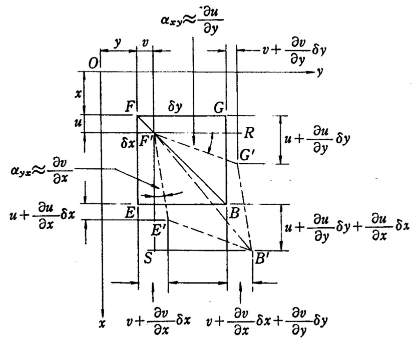
· 무한소의 변형률 증분 → 유한 변형률에서의 변형률 증분

· A (x, y, z) ⟶ A' ($x+u$, $y+v$, $z+w$) u : A점의 변위의 x, y, z 좌표 성분

B ($x+\delta x$, $y+\delta y$, $z+\delta z$) ⟶ B' ($x+\delta x+u+\delta u$, $y+\delta y+v+\delta v$, $z+\delta z+w+\delta w$) u : B점의 변위의 x, y, z 좌표 성분

· u (x, y, z)의 연속함수 $u=f(x,y,z)$로 표시 & δx, δy, δz 의 2차이상의 항 무시

$\delta u=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y+\frac{\partial u}{\partial z}\delta z$



· $\frac{\partial u}{\partial x}\delta x$: 최초의 길이 δx의 변화량

$\frac{\partial u}{\partial x}$: A점에서의 O_x 방향의 수직 변형률 (direct strain 또는 normal strain) ($=e_{xx}$)

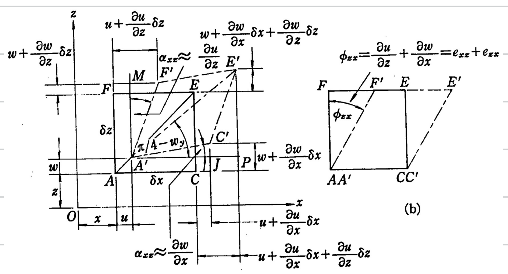
$\frac{\partial u}{\partial y}$: 변위 u로 생긴 xO_y 면에 평행한 면내에서의 회전 ∴ 각변형률 (angular strain)

O_y에 수직한 두 면 사이의 O_x 방향에 평행한 미끄러짐 ∴ 전단의 비율 ($=e_{xy}$)

$\frac{\partial u}{\partial z}$: 변위 u로 생긴 xO_z 면에 평행한 면내에서의 회전 ∴ 각변형률 (angular strain)

O_z에 수직한 두 면 사이의 O_x 방향에 평행한 미끄러짐 ∴ 전단의 비율 ($=e_{xz}$)

· e_{ij} ⇒ i : 운동의 방향 (O_x) / j : O_j 에 평행한 선분상에 있음



· $\delta v=\frac{\partial v}{\partial x}\delta x+\frac{\partial v}{\partial y}\delta y+\frac{\partial v}{\partial z}\delta z$ / $\delta w=\frac{\partial w}{\partial x}\delta x+\frac{\partial w}{\partial y}\delta y+\frac{\partial w}{\partial z}\delta z$

· 회전의 부호 : 오른나사의 법칙

· 공학적 전단변형률 (engineering shear strain) : 최초에 직각이였던 각 FAC의 변화

ϕ_{zx} ⟵ 2개의 미끄러짐으로 정의 $\phi_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}=e_{xz}+e_{zx}$ / $e_{xx}=\frac{\partial u}{\partial x}$ $e_{yy}=\frac{\partial v}{\partial y}$ $e_{zz}=\frac{\partial w}{\partial z}$ $\phi_{yz}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}$ $\phi_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$ $\phi_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$

· 변형률 텐서

$$e_{ij}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}$$

· O_x 축의 둘레로의 회전 $w_x\approx\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)=\frac{1}{2}(e_{zy}-e_{yz})$

O_y 축의 둘레로의 회전 $w_y\approx\frac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)=\frac{1}{2}(e_{xz}-e_{zx})$

O_z 축의 둘레로의 회전 $w_z\approx\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\frac{1}{2}(e_{yx}-e_{xy})$

· 비회전 (irrotational) : $w_x=w_y=w_z=0$ ∴ $e_{yz}=e_{zy}=\phi_{zy}/2$ $e_{xz}=e_{zx}=\phi_{zx}/2$ $e_{xy}=e_{yx}=\phi_{yx}/2$

· 변형률의 접합조건식 (strain compability equation) 변위 3개의 성분 → 변형률 6개의 성분 ∴ 독립적 x

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2\phi_{xy}}{\partial x\,\partial y}\qquad 2\frac{\partial^2 e_{zz}}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\phi_{yz}}{\partial x}+\frac{\partial\phi_{zx}}{\partial y}-\frac{\partial\phi_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2\phi_{xy}}{\partial x\,\partial y}\qquad 2\frac{\partial^2 e_{zz}}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\phi_{yz}}{\partial x}+\frac{\partial\phi_{zx}}{\partial y}-\frac{\partial\phi_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 e_{xx}}{\partial y^2}+\frac{\partial^2 e_{yy}}{\partial x^2}=\frac{\partial^2\phi_{xy}}{\partial x\,\partial y}\qquad 2\frac{\partial^2 e_{zz}}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\phi_{yz}}{\partial x}+\frac{\partial\phi_{zx}}{\partial y}-\frac{\partial\phi_{xy}}{\partial z}\right)$$

3.2 변형률 텐서

- $e_{ij} = \frac{\partial u_i}{\partial x_j}$ u_i : i 방향에서의 변위 x_j : 축

  $e_{ij} = \varepsilon_{ij} + \omega_{ij}$

- $\varepsilon_{x_i x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \varepsilon_{x_i x_j}$ if) $i=j \rightarrow \varepsilon_{ij}$ $i \neq j \rightarrow \tau_{ij}$

- $$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{pmatrix}$$

- $\delta x(1+e_x)\delta y(1+e_y)\delta z(1+e_z) = \delta x\,\delta y\,\delta z\,(1+e_x)(1+e_y)(1+e_z) = \delta x\,\delta y\,\delta z\,(1+e_x+e_y+e_z+e_x e_y+e_y e_z+e_z e_x+e_x e_y e_z)$

  $\delta V = \delta x\,\delta y\,\delta z\,(e_x+e_y+e_z)$

  $\therefore \Delta = e_x + e_y + e_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

3.3 선소의 미소변형

- $r^2 = \overline{AB}^2 = (\delta x)^2 + (\delta y)^2 + (\delta z)^2$

$(r+\delta r)^2 = \overline{A'B'}^2 = (\delta x + \delta u)^2 + (\delta y + \delta v)^2 + (\delta z + \delta w)^2$

$= (\delta x)^2 + (\delta y)^2 + (\delta z)^2 + (\delta u)^2 + (\delta v)^2 + (\delta w)^2 + 2(\delta x \delta u + \delta y \delta v + \delta z \delta w)$

$(r+\delta r)^2 - r^2 = (\delta u)^2 + (\delta v)^2 + (\delta w)^2 + 2(\delta x \delta u + \delta y \delta v + \delta z \delta w)$

$r\delta r = \delta x \delta u + \delta y \delta v + \delta z \delta w$

$r\delta r = \delta x \left[\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z\right] + \delta y \left[\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y + \frac{\partial v}{\partial z}\delta z\right] + \delta z \left[\frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z\right]$

$= (\delta x)^2 \frac{\partial u}{\partial x} + (\delta y)^2 \frac{\partial v}{\partial y} + (\delta z)^2 \frac{\partial w}{\partial z} + \delta x \delta y\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + \delta x \delta z\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + \delta y \delta z\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$

- $\delta x = rl \quad \delta y = rm \quad \delta z = rn$

$\therefore \ \delta r = r\left[l^2 \frac{\partial u}{\partial x} + m^2 \frac{\partial v}{\partial y} + n^2 \frac{\partial w}{\partial z} + lm\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + ln\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) + mn\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right]$

- AB 방향의 변형률

$e_r = \frac{\delta r}{r} = e_x l^2 + e_y m^2 + e_z n^2 + \phi_{xy} lm + \phi_{yz} mn + \phi_{zx} nl$

$= \varepsilon_x l^2 + \varepsilon_y m^2 + \varepsilon_z n^2 + 2\gamma_{xy} lm + 2\gamma_{yz} mn + 2\gamma_{zx} nl$

3.4 유한변형

- $\xi = x + \delta x + u + \delta u - (x + u) = \delta x + \delta u$

  $\eta = \delta y + \delta v$

  $\psi = \delta z + \delta w$

- $\xi = \left(1 + \frac{\partial u}{\partial x}\right)\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z$

  $\eta = \frac{\partial v}{\partial x}\delta x + \left(1 + \frac{\partial v}{\partial y}\right)\delta y + \frac{\partial v}{\partial z}\delta z$

  $\psi = \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \left(1 + \frac{\partial w}{\partial z}\right)\delta z$

- $A'B' = r + \delta r$

  $\frac{\xi}{r + \delta r} = l_1 = \left[\left(1 + \frac{\partial u}{\partial x}\right) l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n\right] \frac{r}{r + \delta r}$

  $\frac{\eta}{r + \delta r} = m_1 = \left[\frac{\partial v}{\partial x} l + \left(1 + \frac{\partial v}{\partial y}\right) m + \frac{\partial v}{\partial z} n\right] \frac{r}{r + \delta r}$

  $\frac{\psi}{r + \delta r} = n_1 = \left[\frac{\partial w}{\partial x} l + \frac{\partial w}{\partial y} m + \left(1 + \frac{\partial w}{\partial z}\right) n\right] \frac{r}{r + \delta r}$

3.5 주변형률

- 응력에 관하여 공부할 때 서로 직교하는 3개의 면이 존재하며 이를 주응력면이라고 부름

- 같은 방법으로 전단변형률이 발생하지 않는 면 존재

- 처음 수직이었던 선은 신장 or 수축 / 방향 변화 X

- 수직한 방향이 주방향 [illegible] 대응하는 변형률은 주변형률 (principal strain)

- $\varepsilon = \frac{\delta r}{r} = \left(\varepsilon_x = \frac{\delta u}{\delta x}\right) = \left(\varepsilon_y = \frac{\delta v}{\delta y}\right) = \left(\varepsilon_z = \frac{\delta w}{\delta z}\right)$

- $\delta u = \varepsilon \delta x \quad \delta v = \varepsilon \delta y \quad \delta w = \varepsilon \delta z$

- $$\left.\begin{aligned} \delta u &= \varepsilon_x \delta x + \varepsilon_{xy} \delta y + \varepsilon_{xz} \delta z \\ \delta v &= \varepsilon_{yx} \delta x + \varepsilon_y \delta y + \varepsilon_{yz} \delta z \\ \delta w &= \varepsilon_{zx} \delta x + \varepsilon_{zy} \delta y + \varepsilon_z \delta z \end{aligned}\right\}$$

- $$\begin{aligned} &(\varepsilon_x - \varepsilon) \delta x + \varepsilon_{xz} \delta z = 0 \\ &\varepsilon_{yx} \delta x + (\varepsilon_y - \varepsilon) \delta y + \varepsilon_{yz} \delta z = 0 \\ &\varepsilon_{zx} \delta x + \varepsilon_{zy} \delta y + (\varepsilon_z - \varepsilon) \delta z = 0 \end{aligned}$$

- $\varepsilon \delta x_i = \varepsilon_{ij} \delta x_j$

- $(\varepsilon_{ij} - \delta_{ij} \varepsilon) \delta x_j = 0$

- $$\begin{vmatrix} (\varepsilon_x - \varepsilon) & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & (\varepsilon_y - \varepsilon) & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & (\varepsilon_z - \varepsilon) \end{vmatrix} = 0$$

- $\varepsilon^3 - (\varepsilon_x + \varepsilon_y + \varepsilon_z) \varepsilon^2 =$

- $$\begin{aligned} J_1 &= \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ J_2 &= -(\varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1) \\ J_3 &= \varepsilon_1 \varepsilon_2 \varepsilon_3 \end{aligned}$$

3.6 주 전단 변형률과 최대전단 변형률

· 변형체 내의 한 점에서 전단응력이 최대가 되는 방향이 있으므로 전단변형률이 최대가 됨

· 주변형률방향 O_1, O_2 및 O_3로 취하고 단위길이 선분 OP를 세각 방향에서 l, m, n

· 길이 방향 또는 수직변형률 $\varepsilon = RQ/QR$

· $\varepsilon = \overline{RQ}/\overline{OR} \approx \varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2$

$\therefore \varepsilon^2 + \theta^2 = \dfrac{\overline{RQ}^2 + \overline{RP}^2}{\overline{OR}^2} = \dfrac{\overline{PQ}^2}{\overline{OR}^2} \approx \overline{PR}^2$

$\therefore \theta^2 \approx \varepsilon_1^2 l^2 + \varepsilon_2^2 m^2 + \varepsilon_3^2 n^2 - (\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2)^2$

$\theta^2 \approx \varepsilon_1^2 l^2 + \varepsilon_2^2 m^2 + \varepsilon_3^2 n^2 - (\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2)^2$

$\theta^2 \approx \varepsilon_1^2 l^2 + \varepsilon_2^2 m^2 + \varepsilon_3^2 n^2 - (\varepsilon_1 l^2 + \varepsilon_2 m^2 + \varepsilon_3 n^2)^2$

· $\gamma_1 = \pm\frac{1}{2}(\varepsilon_2 - \varepsilon_3)$
$\gamma_2 = \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_3)$
$\gamma_3 = \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_2)$

$\gamma_{max} = \pm\frac{1}{2}(\varepsilon_1 - \varepsilon_3)$

3.7 팔면체 변형률

- $\varepsilon_{oct} = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)/3 = J_1/3 = \varepsilon_m$

- $\gamma_{oct}^2 = \frac{1}{3}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2) - \frac{1}{9}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)^2$

$$= \frac{1}{9}\left[(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2\right]$$

$$\gamma_{oct} = \frac{1}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

$$\gamma_{oct} = \frac{\sqrt{2}}{3}\left[J_1^2 + 3J_2\right]^{\frac{1}{2}}$$

$$\gamma_{oct} = \frac{1}{3}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_y - \varepsilon_z)^2 + (\varepsilon_z - \varepsilon_x)^2 + 6(\varepsilon_{xy}^2 + \varepsilon_{yz}^2 + \varepsilon_{zx}^2)}$$

3.8 변형률

$$\varepsilon_{ij} = e_{mij} + \varepsilon_{ij}'$$

$$e_{mij} = \varepsilon_m \delta_{ij} = \begin{pmatrix} \varepsilon_m & 0 & 0 \\ 0 & \varepsilon_m & 0 \\ 0 & 0 & \varepsilon_m \end{pmatrix}$$

- $\varepsilon_m = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$

$$\varepsilon_{ij}' = \begin{pmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\varepsilon_x - \varepsilon_y - \varepsilon_z}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \frac{2\varepsilon_y - \varepsilon_z - \varepsilon_x}{3} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \frac{2\varepsilon_z - \varepsilon_x - \varepsilon_y}{3} \end{pmatrix}$$

- $$\varepsilon_{ij}' = \begin{pmatrix} \frac{2\varepsilon_1 - \varepsilon_2 - \varepsilon_3}{3} & 0 & 0 \\ 0 & \frac{2\varepsilon_2 - \varepsilon_3 - \varepsilon_1}{3} & 0 \\ 0 & 0 & \frac{2\varepsilon_3 - \varepsilon_1 - \varepsilon_2}{3} \end{pmatrix}$$

- $$\left.\begin{aligned} J_1' &= 0 \\ J_2' &= \frac{1}{3}(3J_2 + J_1^2) \\ J_3' &= \frac{1}{27}(2J_1^3 + 9J_1J_2 + 27J_3) \end{aligned}\right\}$$

- $J_2' = \frac{3}{2}\sqrt{\varepsilon_e^2}$

3.9 변형률증분과 변형률속도

· 큰 변형률 미소(무한소) 변형률 그대로 적용 X

· 큰 변형률 변형 중의 어떤 시각을 기준으로 하여 미소시간 dt 만큼 지났을 때 변형률증분 (strain increment)과 독립성분 생각

· 미소변형은 개념 그대로 답습 가능 미소 변형률 ε 이나 γ 대신 변형률 증분 $d\varepsilon$나 $d\gamma$로 대치 가능

· 변형률증분은 시각 t에서의 ε으로의 위치를 (x, y, z)라 하면

$$d\varepsilon_x = \frac{\partial(du)}{\partial x} \qquad d\gamma_{yz} = \frac{1}{2}\left[\frac{\partial(dv)}{\partial z} + \frac{\partial(dw)}{\partial y}\right]$$

$$d\varepsilon_y = \frac{\partial(dv)}{\partial y} \qquad d\gamma_{zx} = \frac{1}{2}\left[\frac{\partial(dw)}{\partial x} + \frac{\partial(du)}{\partial z}\right]$$

$$d\varepsilon_z = \frac{\partial(dw)}{\partial z} \qquad d\gamma_{xy} = \frac{1}{2}\left[\frac{\partial(du)}{\partial y} + \frac{\partial(dv)}{\partial x}\right]$$

· $d\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} du_i + \frac{\partial}{\partial x_i} du_j\right)$, $i = 1, 2, 3$

· $d\varepsilon_1 = \frac{dl}{l}$

$$\int_{l_0}^{l} \frac{dl}{l} = \ln\left(\frac{l}{l_0}\right)$$

· 일반적인 경우, 증분 $\int d\varepsilon_{ij}$ 계산 X 물리적 의미 X

- BUT 증분 $d\varepsilon_{ij}$가 어떤 파라미터(ex 시간)의 함수로 알려진 경우 계산 가능

∴ 주응력 고정된 경우의 적분으로서 범위 제한

· 속도성분 (v_x, v_y, v_z)라 하면

$$du = v_x\,dt \qquad dv = v_y\,dt \qquad dw = v_z\,dt$$

$$\dot{\varepsilon}_x = \frac{d\varepsilon_x}{dt} = \frac{\partial v_x}{\partial x} \qquad \dot{\gamma}_{yz} = \frac{d\gamma_{yz}}{dt} = \frac{1}{2}\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)$$

$$\dot{\varepsilon}_y = \frac{d\varepsilon_y}{dt} = \frac{\partial v_y}{\partial y} \qquad \dot{\gamma}_{zx} = \frac{d\gamma_{zx}}{dt} = \frac{1}{2}\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)$$

$$\dot{\varepsilon}_z = \frac{d\varepsilon_z}{dt} = \frac{\partial v_z}{\partial z} \qquad \dot{\gamma}_{xy} = \frac{d\gamma_{xy}}{dt} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)$$

· $\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$