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Ionospheric Elementary Current Systems in Spherical Coordinates and Their Application

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Two sets of basis functions in spherical coordinates are presented, in terms of which any given ionospheric current system, consisting of horizontal sheet currents and their accompanying field-aligned currents, can be expanded, regardless of any considerations on the ionospheric conductances or the electric field. The single basis functions are called elementary current systems. One basis function set is curl-free and poloidal, and causes a toroidal magnetic field that is restricted to the area above the ionosphere. The other one is divergence-free and toroidal, and causes a poloidal magnetic field which is solely responsible for the magnetic effect of ionospheric currents below the ionosphere. The field-aligned currents are assumed to flow radially. The expansion presented is used on a model of a Cowling channel to decompose its Hall and Pedersen currents into their total divergence-free and curl-free parts. This application example shows how the analysis technique based on the elementary current expansion resolves the physically relevant primary and secondary currents inside the Cowling channel.

1. Introduction

Several earlier works have been carried out to study the magnetic effect of special ionospheric current configurations (e.g., Tamao, 1964, 1986; Kern, 1966; Fukushima, 1976, and references therein). However, these authors used assumptions on the ionospheric conductances (mostly, uniform conductances were required) and on the electric field configuration which reduced the generality of their findings. Tamao (1964) and Kern (1966) showed that in case of uniform conductances, the equivalent currents are identical with the Hall currents. The most extensive studies in this field have been carried out by Fukushima, summarised in Fukushima (1976). He found that a current system, consisting of one field-aligned current (FAC), being compensated by opposite FACs that are uniformly distributed over the whole ionospheric sphere, and accompanying horizontal ionospheric currents that flow radially to or away from the first FAC (depending on its sign), causes no magnetic field below the ionosphere and is suitable for forming realistic current systems by superposition of several systems of the type described. However, he did not use a second set of basis functions to describe the divergence-free part of the ionospheric currents (i.e., the part that is not connected with FACs). Moreover, since he attributed the ionospheric currents in his system to Pedersen currents, he was restricted to the case of uniform conductances. For certain special cases with nonuniform conductances, he gave corrections for the equivalent currents compared to the uniform case, thereby also pointing out how different the true ionospheric currents may look from the equivalent ones. Also assuming uniform conductances, Tamao (1986) extended his analysis by considering the magnetic effect of straight, but oblique field-aligned currents, whereas in all other studies mentioned FACs are assumed to flow radially.

For the case of a plane cartesian geometry, Untiedt and Baumjohann (1993) discussed the decomposition of any ionospheric current system into its curl-free and divergence-free parts, without making assumptions on the conductances or the electric field. They also showed one example for this decomposition, but did not give equations for the basis functions used. An example for the application of this technique

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on an omega band model has been given by Amm (1996).

In this paper, expressions for curl-free and divergence-free basis functions (called "elementary current systems") in spherical geometry will be derived, and it will be shown how any ionospheric current system together with its FACs can be expanded by means of these elementary systems, regardless of the prevailing conductances or electric fields. Finally, as an application example we show the decomposition of a current system into its total curl-free and divergence-free parts for a model of a Cowling channel and discuss the physical meaning of the results.

2. Elementary Current Systems and Their Superposition to Real Ionospheric Current Systems

This chapter deals with two topics: The first main one is the purely mathematical operation of constructing curl-free and divergence-free vector basis functions on the sphere, our elementary current systems, and expand a given vector field $\vec{J}$, our currents, into them. Whereas for that no further physical specifications would be necessary, when we discuss the magnetic field effect of our elementary current systems as a second topic, we need to specify how the divergences of $\vec{J}$, i.e., the field-aligned currents, flow out of the ionosphere. We will do that by assuming that they flow radially through the magnetosphere. Since our study is intended to be applied to physical problems, we will also discuss the first topic in physical terms.

To construct the curl-free elementary current system, we put a divergence of $\vec{J}$ of magnitude $I_{0,\mathrm{cf}}$—in physical terms a FAC of that strength flowing into the ionosphere—at a point which we call "pole" of the elementary current system and assign with $\vartheta' = 0$, i.e., we use a spherical coordinate system whose pole is at that point. We then require for the horizontal ionospheric elementary current system $\vec{J}_{\mathrm{cf,el}}(\vec{r}')$ connected with that FAC that

$$\left[\mathrm{curl}\vec{J}_{\mathrm{cf,el}}(\vec{r}')\right]_r \equiv 0 \tag{1}$$

$$\mathrm{div}_{\mathrm{h}}\vec{J}_{\mathrm{cf,el}}(\vec{r}') \equiv \mathrm{const.}, \quad \text{for } \vartheta' \neq 0$$

where the subscript h denotes the horizontal part of the divergence operator. Obviously, since the currents have to flow out of the ionosphere again, in the spherical case the const. in Eq. (1) cannot be chosen to zero, as it can be in the cartesian case where the currents can be assumed to flow back at infinity. Since we selected the current system such that the current flows homogeneously out of the ionosphere, const. $= -I_{0,\mathrm{cf}}/4\pi R_{\mathrm{I}}^2$ results where R_{I} is the radius of the ionosphere. Furthermore, no horizontal current can be left on the opposite side of the pole of the elementary current system, i.e., $\vec{J}_{\mathrm{cf,el}}\ (\vartheta' = 180°) = 0$ is required. A straightforward integration then yields

$$\vec{J}_{\mathrm{cf,el}}(\vec{r}') = \frac{I_{0,\mathrm{cf}}}{4\pi R_{\mathrm{I}}}\cot(\vartheta'/2)\underline{e}_{\vartheta'}. \tag{2}$$

This is in fact the same ionospheric current system that Fukushima (1976) attributed to Pedersen currents in his study. In our study, however, we use it as a mere vector basis function, and thus no specification of how the current is physically produced is needed. When interpreted with an additional electric field model, our elementary current systems will in general contain both Hall and Pedersen current parts.

The current system given in Eq. (2) is curl-free and poloidal, and together with its accompanying FACs (given by the divergence of Eq. (2)) it does not cause any magnetic field below the ionosphere, for radially flowing FACs. Fukushima (1976) gave already two different proofs for this. The deviation of the

real flow direction of FACs from a straight line normal to the ionosphere produces toroidal currents in the magnetosphere which lead to a poloidal magnetic field that "leaks" into the region below the ionosphere. However, this effect has been discussed and shown to be weak by several authors (Richmond, 1974; Fukushima, 1976; Tamao, 1986; Untiedt and Baumjohann, 1993; Amm, 1995) and is not subject of this paper.

We construct the divergence-free elementary system completely analogous to the curl-free one and result in

$$\vec{J}_{\mathrm{df,el}}(\vec{r}\,') = \frac{I_{0,\mathrm{df}}}{4\pi R_{\mathrm{I}}}\cot(\vartheta'/2)\underline{e}_{\varphi'}. \tag{3}$$

Besides the fact that it is divergence-free, this current system is toroidal and thus causes a poloidal magnetic field which stretches over the region above and below the ionosphere. Except for its pole, it has a constant curl of $-(I_{0,\mathrm{df}}/4\pi R_{\mathrm{I}}^2)\underline{e}_r$ everywhere.

That our two elementary current systems are in fact vector basis functions on the two-dimensional sphere follows directly from their linear independence and their completeness. The former is seen from their orthogonality and the later from the fact that they can uniquely represent any curl and divergence distribution on the sphere (e.g., Cushing, 1975). Although the basis functions show axial symmetry with respect to their pole, superpositions of them with different positions of the pole are not restricted to any special symmetry.

Note that for small υ', with $\cot(\upsilon'/2) \approx (2/\upsilon')$ and $R_{\mathrm{I}}\upsilon' \approx \rho'$, Eqs. (2) and (3) reduce to the cartesian case elementary systems

$$\begin{aligned} \vec{J}_{\mathrm{cf,el,cart}}(\vec{r}\,') &= \frac{I_{0,\mathrm{cf}}}{4\pi\rho'}\underline{e}_{\rho'} \\ \vec{J}_{\mathrm{df,el,cart}}(\vec{r}\,') &= \frac{I_{0,\mathrm{df}}}{4\pi\rho'}\underline{e}_{\varphi'} \end{aligned} \tag{4}$$

written in a cylindrical coordinate system (ρ, φ, z), z upward, with $\rho = 0$ equal to the pole of the elementary system. These elementary systems have zero divergence or curl, respectively, outside their poles (corresponding to $R_{\mathrm{I}} \to \infty$ if seen from the spherical case).

Using the elementary systems (2) and (3), and the well-known fact that any current system $\vec{J}$—as any vector field—can be uniquely decomposed into a curl-free part $\vec{J}_{\mathrm{cf}}$ and a divergence-free part $\vec{J}_{\mathrm{df}}$ (e.g., Lindell, 1992), we can write for an arbitrary $\vec{J}$

$$\vec{J}(\vec{r}) = \vec{J}_{\mathrm{cf}}(\vec{r}) + \vec{J}_{\mathrm{df}}(\vec{r}) \tag{5a}$$

$$= \iint_{\mathrm{Ionosph.}} \frac{\mathrm{div_h}\vec{J}(\vec{r}\,')}{4\pi R_{\mathrm{I}}}\cot\left(\tilde{\vartheta}/2\right)\underline{e}_{\tilde{\vartheta}}d^2r' + \iint_{\mathrm{Ionosph.}} \frac{\left[\mathrm{curl}\vec{J}(\vec{r}\,')\right]_r}{4\pi R_{\mathrm{I}}}\cot\left(\tilde{\vartheta}/2\right)\underline{e}_{\tilde{\varphi}}d^2r' \tag{5b}$$

where $\tilde{\vartheta}$ and $\tilde{\varphi}$ denote the coordinates of $\vec{r}$ in the spherical coordinate system with its pole at $\vec{r}\,'$, and $\underline{e}_{\tilde{\vartheta}}$ and $\underline{e}_{\tilde{\varphi}}$ are the unit vectors according to this coordinate system. For Eq. (5b) we made use of Gauß and Stokes laws to yield $\iint_{K_r, r\to 0}\mathrm{div_h}\vec{J}(\vec{r}\,')d^2r' = I_{0,\mathrm{cf}}(\vec{r})$ and $\iint_{K_r, r\to 0}\left[\mathrm{curl}\vec{J}(\vec{r}\,')\right]_r d^2r' = I_{0,\mathrm{df}}(\vec{r})$ (where K_r is a circular ionospheric area with radius r around $\vec{r}$, and $I_{0,\left\{\begin{smallmatrix}\mathrm{cf}\\ \mathrm{df}\end{smallmatrix}\right\}}(\vec{r})$ denote the respective constants of the

elementary current systems with pole at $\vec{r}$). The constant curls and divergences of the elementary systems outside their poles vanish in the superposition Eq. (5b) since to yield physically reasonable current systems, $\iint_{\text{Ionosph.}} I_{0,\left\{\begin{smallmatrix}\text{cf}\\ \text{df}\end{smallmatrix}\right\}}(\vec{r})d^2r' = 0$ is required (compare Fukushima, 1976, for the curl-free type of elementary system). For a practical calculation, Eq. (5b) means summing up the elementary current systems with a weight corresponding to the local horizontal divergence and the r component of the curl of $\vec{J}$.

3. Calculation of the Total Curl-Free and Divergence-Free Parts of a Current System

A main application of the decomposition into elementary current systems is to use Eq. (5b) in order to calculate the total curl-free and divergence-free parts $\vec{J}_{\text{cf}}$ and $\vec{J}_{\text{df}}$ (as in Eq. (5a)) of a given current system $\vec{J}$. It may also be useful to apply this operation to the Hall and Pedersen currents, $\vec{J}_{\text{H}}$ and $\vec{J}_{\text{P}}$, separately (see our model of a Cowling channel below). Note that Eq. (5b) allows to calculate equivalent currents, too, since in case of radial FACs any superposition of the curl-free elementary systems causes no magnetic effect below the ionosphere, i.e., the equivalent currents immediately below the ionospheric current sheet are equal to $\vec{J}_{\text{df}}$ (cf. Untiedt and Baumjohann, 1993).

As an input for an analysis of total curl-free and divergence-free parts of a current system, two-dimensional distributions of actual ionospheric currents are to be used, as they can be obtained, e.g., from ground-based measurements by the method of characteristics (Inhester *et al.*, 1992; Amm, 1995), the KRM (Kamide *et al.*, 1981) or AMIE (Richmond and Kamide, 1988) methods, or by modeling results.

Some points should be noted on that analysis: 1) The decomposition into curl-free and divergence-free parts just consists of the superposition of two sets of elementary systems (basis functions), as seen in Eq. (5b). Thus, it is unaffected by the question of how $\vec{J}$ was produced, i.e., no assumptions on the Hall and Pedersen conductance Σ_{H} and Σ_{P} nor on the electric field $\vec{E}$ are made. 2) The resulting current systems $\vec{J}_{\text{cf}}$ and $\vec{J}_{\text{df}}$ do not need to be physical current systems in the sense that Ohm's law can be applied to them. However, many physical vector fields naturally split up into curl-free and divergence-free parts. E.g., for the ionospheric currents this can seen by the fact that only the divergence-free part produces a magnetic field below the ionosphere (another example is shown with our Cowling channel model below), and for the seismic displacement field this subdivision leads to the decoupling of s and p waves. 3) Although Eq. (5a) is valid locally, the decomposition into $\vec{J}_{\text{cf}}$ and $\vec{J}_{\text{df}}$ can only be performed globally as in Eq. (5b). If $\vec{J}$ is known on a subarea of the ionosphere only, but has curls or divergences outside that subarea, then Eq. (5b) with the integrals restricted to that area cannot reproduce the complete $\vec{J}$. In this case, a third current system $\vec{J}_{\text{lap}}$ which includes the remainder of $\vec{J}$ has to be inserted in Eqs. (5a) and (5b). These currents are curl- and divergence-free in the subarea and therefore have a scalar potential that satisfies Laplace's equation there.

4. Application Example: Model of a Cowling Channel

Finally, we shall demonstrate an example of the technique of decomposing a current system into its total divergence- and curl-free parts by the superposition of the elementary systems as in Eq. (5b), on a model of a Cowling channel (e.g., Boström, 1974):

If the ionospheric conductances are largely enhanced inside an elongated strip of the ionosphere, and the (primary) electric field points parallel to the longer sides of the strip (this direction we call "channel direction" here), the (primary) Hall currents will build up space charges at the borders of the strip perpendicular to the channel direction, thus invoking a secondary electric field perpendicular to the primary one. The secondary Pedersen currents caused by that field will compensate the primary Hall

currents such that no further space charges are accumulated at the flanks of the channel, and a stationary situation is reached. Hence, the total current perpendicular to the channel direction is zero. The secondary Hall currents will enhance the primary Pedersen currents in the channel direction, implying an effective conductance in that direction of $\Sigma_C = \Sigma_P + \Sigma_H^2/\Sigma_P$, called "Cowling conductance".

Figure 1 shows a sketch of our model of the final situation, with the channel direction from north (top) to south (bottom). The electric field has a total strength of 11.2 mV/m (10 mV/m in eastward, 5 mV/m in southward direction), the Hall and Pedersen conductances inside the channel are 10 S and 5 S, respectively. For simplicity, we set the conductances to zero outside the channel. At the northern and southern border of the channel, there are FACs to feed and draw off the channel currents. Such a configuration may occur inside an omega band (Amm, 1996), and a similar one was found to apply to an auroral breakup event by Baumjohann *et al.* (1981).

The divergence- and curl-free parts of the Hall and Pedersen currents, i.e., the separate results of the two integrals in Eq. (5b) when applied to either current type, are shown in Fig. 2. The channel stretches over the region between 66 to 70 degrees of latitude and 20 to 22 degrees of longitude. It can clearly be seen that inside the channel, the curl-free parts of the Hall and Pedersen currents (Figs. 2(a) and 2(b)) are identical with the primary Hall and secondary Pedersen currents, respectively, pointing in opposite direction perpendicular to the channel direction and compensating each other in their east-west component. The slight southward deviation of both currents is due to end effects caused by the finite north-south extension of the channel. Likewise, the divergence-free parts of the Hall and Pedersen currents (Figs. 2(c) and 2(d)) correspond to the secondary Hall and primary Pedersen currents, respectively, both pointing southward (again, with slight deviations due to end effects) and adding up to the "Cowling current" inside the channel. Besides, these two current systems are together equal to the equivalent currents immediately below the ionosphere (Fig. 2(e)). Hence, the main part of the equivalent currents is here caused by the divergence-free part of the Hall currents which is also responsible for the typical "equivalent backflow" seen outside the channel.

As it can easily be verified from Fig. 2, not only the directions, but also the magnitudes of the respective primary and secondary current parts are resolved correctly. Hence, our analysis of the Cowling channel model in terms of the total curl-free and divergence-free part of its Hall and Pedersen currents is able to completely recover the physically relevant current subsystems inside the channel.

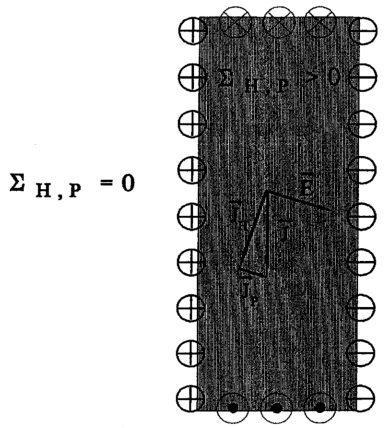


Fig. 1. Schematic view of the Cowling channel model. The shaded area marks the channel where the Hall and Pedersen conductances Σ_H and Σ_P are enhanced, whereas they are zero outside. The vectors illustrate the directions and magnitude of the total currents $\vec{J}$, the Hall and Pedersen currents $\vec{J}_H$ and $\vec{J}_P$, and the electric field $\vec{E}$ inside the channel. Plus and minus signs mark positive and negative space charges, circles with crosses and dots downward and upward flowing FACs, respectively (for details see text).

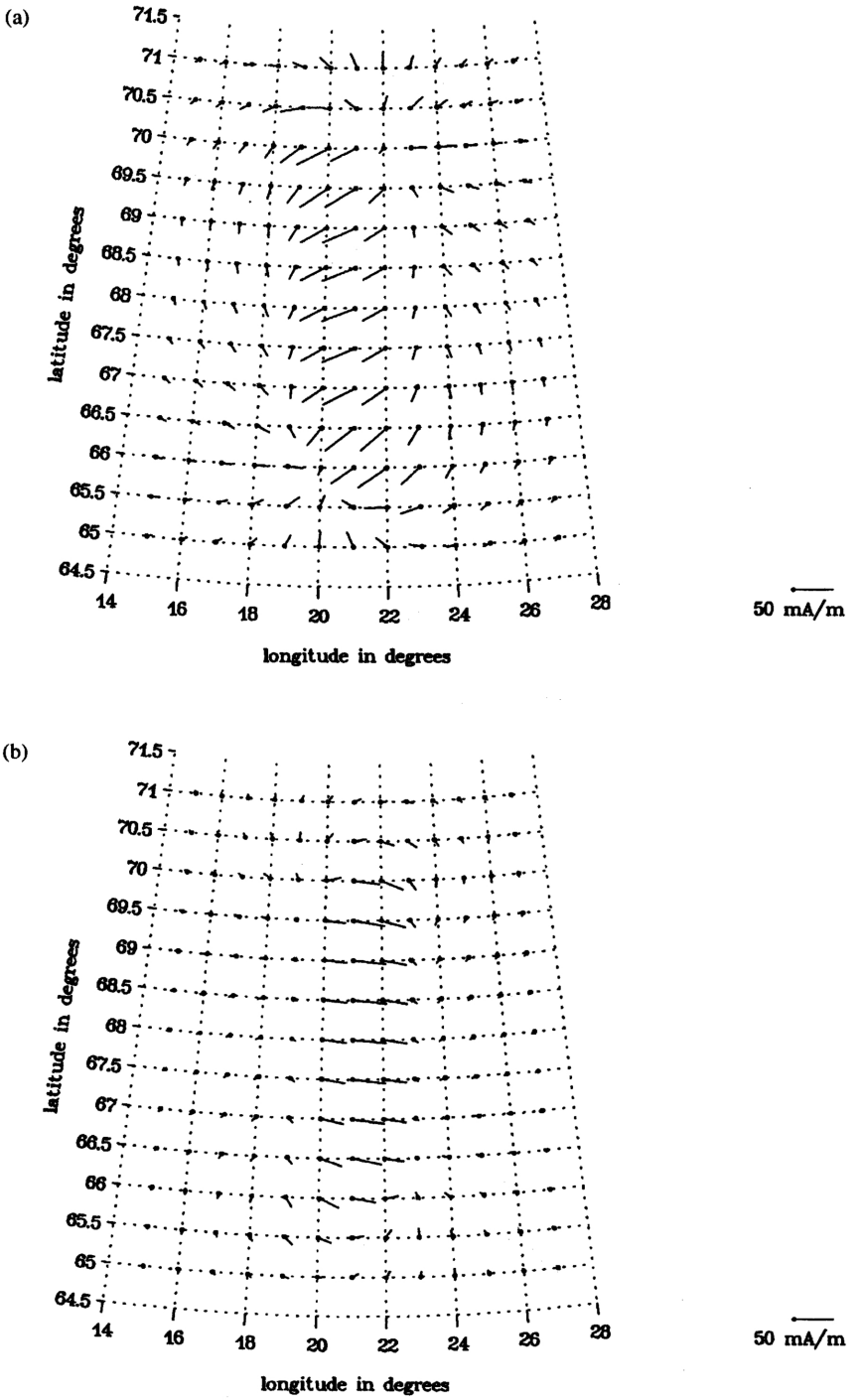
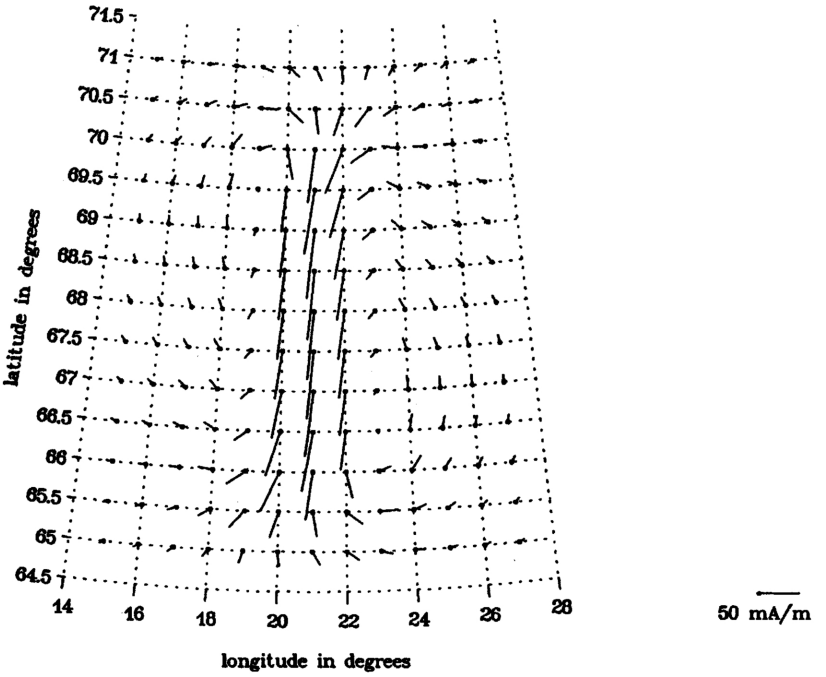


Fig. 2. Decomposition of the Hall and Pedersen currents into their total curl-free and divergence-free parts. (a) Curl-free part of the Hall currents, corresponding to primary Hall currents. (b) Curl-free part of the Pedersen currents, corresponding to secondary Pedersen currents. (c) Divergence-free part of the Hall currents, corresponding to secondary Hall currents. (d) Divergence-free part of the Pedersen currents, corresponding to primary Pedersen currents. (e) Equivalent currents immediately below the ionosphere, equal to the sum of the currents shown in (c) and (d).

(c)



(d)

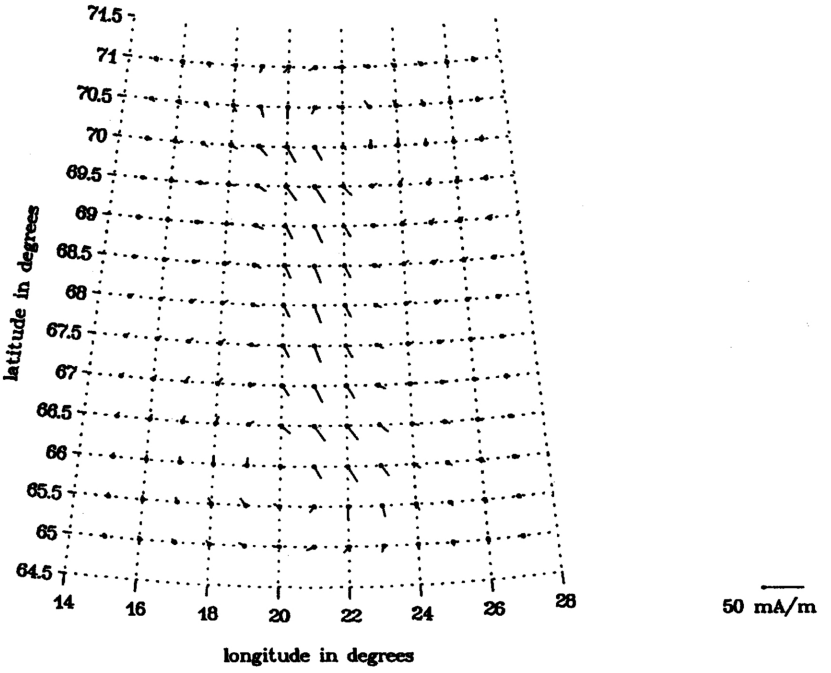


Fig. 2. (continued).

(e)

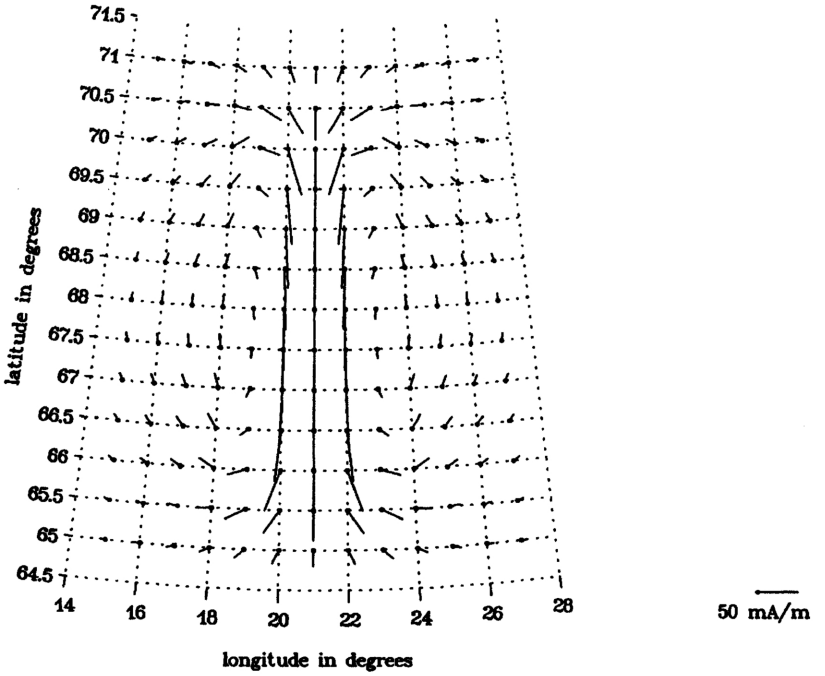


Fig. 2. (continued).

5. Summary

We have shown how any ionospheric current system given in spherical coordinates, together with its accompanying field-aligned currents that are assumed to flow radially, can be expressed as a superposition of two types of elementary current systems, one of which is curl-free and the other divergence-free. No assumptions on the ionospheric conductances or electric field were needed for that. We then pointed out how the superposition described can be used to extract the total curl-free and divergence-free part of any given current system. This analysis technique was then used on a model of a Cowling channel and shown to be able to resolve the primary and secondary current systems inside the channel. Besides, the equivalent currents of any current system immediately below the current sheet can be calculated by this technique, without the need to perform a Biot-Savart integration.

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