

## 0 Basics

Vector from point B to P in frame  $\mathcal{A}$ :  $\mathcal{A}r_{BP}$

Reference coordinate System  $\mathcal{A}$ :  $(e_x^{\mathcal{A}}, e_y^{\mathcal{A}}, e_z^{\mathcal{A}})$

### Cartesian Coordinates

Stacked parameters of Position:  $\chi_{Pc} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Position Vector:  $\mathcal{A}r = xe_x^{\mathcal{A}} + ye_y^{\mathcal{A}} + ze_z^{\mathcal{A}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

### Cylindric Coordinates

### Spherical Coordinates

$\chi_{Pz} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$ ;  $\mathcal{A}r = \begin{pmatrix} \rho \cos \theta \\ \rho \sin \theta \\ z \end{pmatrix}$   $\chi_{Ps} = \begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$ ;  $\mathcal{A}r = \begin{pmatrix} r \cos \theta \sin \phi \\ r \sin \theta \sin \phi \\ r \cos \phi \end{pmatrix}$

## 1 Kinematics

### 1.1 Linear Velocity

The Velocity of point B relative to point A is given by:  $\dot{r}_{AB}$ . There exists a linear mapping  $E_p(\chi)$  between velocities  $\dot{r}$  and the derivatives of the representation  $\dot{\chi}_P$ :

$$\dot{r} = E_P(\chi_P) \dot{\chi}_P$$

$$\dot{\chi}_P = E_P^{-1}(\chi_P) \dot{r}$$

For **Cartesian** this is the identity:  $E_{Pc}(\chi_{Pc}) = E_{Pc}^{-1}(\chi_{Pc}) = 1$   
In **Cylindrical** we get:

$$E_{Pz} = \frac{\partial r(\chi)}{\partial \chi} = \begin{pmatrix} c\theta & -\rho s\theta & 0 \\ s\theta & \rho c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad E_{Ps}^{-1} = \begin{pmatrix} -\frac{c\theta}{\rho} & \frac{s\theta}{\rho} & 0 \\ -\frac{1}{\rho} s\theta & \frac{1}{\rho} c\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For **Spherical**:

$$E_{Ps} = \begin{pmatrix} c\theta s\phi & -r s\theta s\phi & r c\theta c\phi \\ s\theta s\phi & r c\theta s\phi & r s\theta c\phi \\ c\phi & 0 & -r s\phi \end{pmatrix} \quad E_{Ps}^{-1} = \begin{pmatrix} \frac{c\theta s\phi}{r} & \frac{s\theta s\phi}{r} & \frac{c\phi}{r} \\ -s\theta/(r s\phi) & c\theta/(r s\phi) & 0 \\ \frac{1}{r} c\theta c\phi & \frac{1}{r} s\theta c\phi & -\frac{1}{r} s\phi \end{pmatrix}$$

### 1.2 Rotation

Orientation of frame  $\mathcal{B}$  with reference to frame  $\mathcal{A}$ :

$$\phi_{AB} \in SO(3)$$

**Important:** There is no numerical equivalent to a position such as "angular position". Instead the orientation can be parametrized in several ways.

**Passive Rotation:** Passive rotations are transformations between different coordinate frames.  $\mathcal{A}u = C_{AB} \cdot \mathcal{B}u$

**Active Rotations:** Active rotations contain an operator (i.e.  $R \in \mathbb{R}^{3 \times 3}$ ) to rotate a vector in the same frame.  $\mathcal{A}v = R \cdot \mathcal{A}u$

#### 1.2.1 Rotation Matrix

Mapping Coordinates P from frame  $\mathcal{B}$  to  $\mathcal{A}$ :

$$\mathcal{A}r_{AP} = \begin{pmatrix} \mathcal{A}e_x^{\mathcal{B}} & \mathcal{A}e_y^{\mathcal{B}} & \mathcal{A}e_z^{\mathcal{B}} \end{pmatrix} \cdot \mathcal{B}r_{AP} = C_{AB} \cdot \mathcal{B}r_{AP}$$

This matrix is orthogonal:  $C_{BA} = C_{AB}^{-1} = C_{AB}^T$

#### 1.2.2 Elementary Rotations

Around X-Axis      Around Y-Axis      Around Z-Axis

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

#### 1.2.3 Homogeneous Transformation

Combined Translation and Rotation for frames with an offset. The vector  $\mathcal{A}r_{AB}$  points from the origin of frame  $\mathcal{A}$  to the origin of the  $\mathcal{B}$  in frame  $\mathcal{A}$ .  $T_{AB}$  transforms a **point P** from frame  $\mathcal{B}$  to  $\mathcal{A}$ .

$$\begin{pmatrix} \mathcal{A}r_{AP} \\ 1 \end{pmatrix} = \underbrace{\begin{bmatrix} C_{AB} & \mathcal{A}r_{AB} \\ 0_{1 \times 3} & 1 \end{bmatrix}}_{T_{AB}} \begin{pmatrix} \mathcal{B}r_{BP} \\ 1 \end{pmatrix}$$

$$\text{With inverse: } T_{AB}^{-1} = \begin{bmatrix} \mathcal{B}r_{BA} \\ C_{AB}^T & -C_{AB}^T \mathcal{A}r_{AB} \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

### 1.3 Representation of Rotations

#### 1.3.1 Euler Angles

Rotation Matrix from Euler Angles:

$$C_{AD} = C_{AB}(z)C_{BC}(y)C_{CD}(x)$$

Euler Angles from Rotation Matrix:

$$\chi_{R, \text{eulerZYX}} = \begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} \text{atan2}(c_{21}, c_{11}) \\ \text{atan2}(-c_{31}, \sqrt{c_{32}^2 + c_{33}^2}) \\ \text{atan2}(c_{32}, c_{33}) \end{pmatrix}$$

#### 1.3.2 Angle Axis

$$\chi_{R, \text{AngleAxis}} = \begin{pmatrix} \theta \\ \vec{n} \end{pmatrix} \quad \text{Rotation Vector: } \varphi = \theta \cdot \vec{n}$$

Rotation Matrix from Angle Axis:

$$C_{AB} = \cos(\theta)I_{3 \times 3} - \sin(\theta)[\vec{n}]_{\times} + (1 - \cos(\theta))nn^T$$

Parameters from Rotation Matrix:

$$\theta = \cos^{-1} \left( \frac{c_{11} + c_{22} + c_{33} - 1}{2} \right); \quad n = \frac{1}{2 \sin(\theta)} \begin{pmatrix} c_{32} - c_{23} \\ c_{13} - c_{31} \\ c_{21} - c_{12} \end{pmatrix}$$

#### 1.3.3 Unit Quaternion

A non-minimal representation which doesn't suffer from singularities  
From rotation vector  $\varphi \in \mathbb{R}^3$  to unit quaternion:

$$\chi_{R, quat} = \xi = \begin{pmatrix} \xi_0 \\ \xi \end{pmatrix} = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \vec{n} \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

Hamiltonian conventions:

$$\xi = \xi_0 + \xi_1 i + \xi_2 j + \xi_3 k \quad i^2 = j^2 = k^2 = ijk = -1;$$

$$ij = -ji = -ijk^2 = k \quad jk = -kj = i \quad ki = -ik = j$$

The first parameter  $\xi_0$  is the real part of the quaternion,  $\xi$  the imaginary part.  $\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2 = 1$

Given a rotation matrix, the corresponding quaternion is:

$$\chi_{R, quat} = \xi_{AD} = \frac{1}{2} \begin{pmatrix} \sqrt{c_{11} + c_{22} + c_{33} + 1} \\ \text{sgn}(c_{32} - c_{23}) \sqrt{c_{11} - c_{22} - c_{33} + 1} \\ \text{sgn}(c_{13} - c_{31}) \sqrt{c_{22} - c_{33} - c_{11} + 1} \\ \text{sgn}(c_{21} - c_{12}) \sqrt{c_{33} - c_{11} - c_{22} + 1} \end{pmatrix}$$

with  $\text{sgn}(x) = 1$  for  $x \geq 0$  and  $\text{sgn}(x) = -1$  for  $x < 0$

The Corresponding Rotation Matrix can be calculated with:

$$\begin{aligned} C_{AD} &= I_{3 \times 3} + 2\xi_0 [\xi]_{\times} + 2[\xi]_{\times}^2 \\ &= (2\xi_0^2 - 1) I_{3 \times 3} + 2\xi_0 [\xi]_{\times} + 2\xi \xi^T \\ &= \begin{pmatrix} \xi_0^2 + \xi_1^2 - \xi_2^2 - \xi_3^2 & 2\xi_1 \xi_2 - 2\xi_0 \xi_3 & 2\xi_0 \xi_2 + 2\xi_1 \xi_3 \\ 2\xi_0 \xi_3 + 2\xi_1 \xi_2 & \xi_0^2 - \xi_1^2 + \xi_2^2 - \xi_3^2 & 2\xi_2 \xi_3 - 2\xi_0 \xi_1 \\ 2\xi_1 \xi_3 - 2\xi_0 \xi_2 & 2\xi_0 \xi_1 - 2\xi_2 \xi_3 & \xi_0^2 - \xi_1^2 - \xi_2^2 + \xi_3^2 \end{pmatrix} \end{aligned}$$

It holds that:

$$\xi = \begin{pmatrix} \xi_0 \\ \xi \end{pmatrix} \xrightarrow{\text{inverse}} \xi^{-1} = \xi^T = \begin{pmatrix} \xi_0 \\ -\xi \end{pmatrix}$$

Multiplication:  $\xi_{AB} \otimes \xi_{BC} = M_I(\xi_{AB}) \xi_{BC}$

$$\underbrace{\begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & -\xi_3 & \xi_2 \\ \xi_2 & \xi_3 & \xi_0 & -\xi_1 \\ \xi_3 & -\xi_2 & \xi_1 & \xi_0 \end{pmatrix}}_{M_I(\xi_{AB})} ; \underbrace{\begin{pmatrix} \xi_0 & -\xi_1 & -\xi_2 & -\xi_3 \\ \xi_1 & \xi_0 & \xi_3 & -\xi_2 \\ \xi_2 & -\xi_3 & \xi_0 & \xi_1 \\ \xi_3 & \xi_2 & -\xi_1 & \xi_0 \end{pmatrix}}_{M_r(\xi_{BC})}$$

Direct way of rotating a Vector: (with  $p(\mathcal{B}r) = \begin{pmatrix} 0 \\ \mathcal{B}r \end{pmatrix}$ )

$$p(\mathcal{A}r) = \xi_{AB} \otimes p(\mathcal{B}r) \otimes \xi_{AB}^T = M_I(\xi_{AB}) M_r(\xi_{AB}^T) p(\mathcal{B}r)$$

### 1.4 Angular Velocity

Consider Frame  $\mathcal{B}$  which is moving with respect to fixed Frame  $\mathcal{A}$ . The angular velocity of the rotation of  $\mathcal{B}$  w.r.t  $\mathcal{A}$  is  $\mathcal{A}\omega_{AB}$ . It can be shown that:

$$[\mathcal{A}\omega_{AB}]_{\times} = \dot{C}_{AB} \cdot C_{AB}^T \quad \omega_{AB} = -\omega_{BA}$$

with  $[\mathcal{A}\omega_{AB}]_{\times} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$  and  $\mathcal{A}\omega_{AB} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$

Angular Velocities can be transformed as normal vectors:

$$\mathcal{B}\omega_{AB} = C_{BA} \cdot \mathcal{A}\omega_{AB}$$

the product matrix is transformed by:

$$[\mathcal{B}\omega_{AB}]_{\times} = C_{BA} \cdot [\mathcal{A}\omega_{AB}]_{\times} \cdot C_{AB}$$

Angular Velocities in the same Frame can be added:

$$\mathcal{I}\omega_{IE} = \mathcal{I}\omega_{I0} + \dots + \mathcal{I}\omega_{nE}$$

#### 1.4.1 Time Derivative of Rotation Parameters

Similar to lin. vel. we can define:  $\mathcal{A}\omega_{AB} = E_R(\chi_R) \cdot \dot{\chi}_R$

quaternions:  $E_R = 2H(\xi) = 2 \begin{bmatrix} -\xi & [\xi]_{\times} + \xi_0 I_{3 \times 3} \end{bmatrix}$

Angle Axis:  $E_R = [n \quad \sin \theta I_{3 \times 3} + (1 - \cos \theta)[n]_{\times}]$

Rotation Vector:  $E_R = \left[ I_{3 \times 3} + [\varphi]_{\times} \left( \frac{1 - \cos \|\varphi\|}{\|\varphi\|^2} \right) + [\varphi]_{\times}^2 \left( \frac{\|\varphi\| - \sin \|\varphi\|}{\|\varphi\|^3} \right) \right]$

### 1.5 Velocity in Moving Bodies

$v_P$  the absolute Velocity of point P  
 $a_P = \dot{v}_P$  (absolute) acceleration of P  
 $\Omega_{\mathcal{B}} = \omega_{AB}$  (absolute) angular velocity of body B  
 $\Psi_{\mathcal{B}} = \dot{\Omega}_{\mathcal{B}}$  (absolute) angular acceleration of body B

We can write the Position P as:

$$\mathcal{A}r_{AP} = \mathcal{A}r_{AB} + C_{AB} \cdot \mathcal{B}r_{BP}$$

Which can be differentiated to:

$$\mathcal{A}\dot{r}_{AP} = \mathcal{A}\dot{r}_{AB} + \mathcal{A}\omega_{AB} \times \mathcal{A}r_{BP}$$

This is the famous **Rigid Body Formulation** for velocities:

$$v_P = v_B + \Omega \times r_{BP}$$

For Accelerations:  $a_P = a_B + \Psi \times r_{BP} + \Omega \times (\Omega \times r_{BP})$

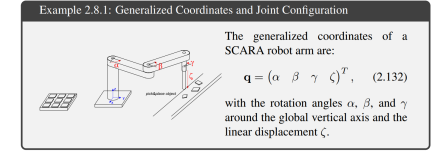
In case a moving system  $\mathcal{B}$  is used for representation, the **Euler Differentiation rule** must be applied (with a non moving system  $\mathcal{A}$ ):

$$\begin{aligned} \mathcal{B}v_P &= C_{BA} \cdot \frac{d}{dt} (C_{AB} \cdot \mathcal{B}r_{AP}) \\ &= \mathcal{B}\dot{r}_{AP} + \mathcal{B}\omega_{AB} \times \mathcal{B}r_{AP} \end{aligned}$$

## 1.6 Kinematics of Systems of Bodies

### 1.6.1 Generalized Coordinates and Joint Configurations

The configuration of a robot such as a manipulator can be described by the **generalized coordinate vector**:  $q = (q_1 \dots q_n)^T$ . The choice of  $q$  isn't unique, but it has to completely describe the configuration of the system ( $q$  const.  $\Rightarrow$  robot can't move).



### 1.6.2 Task-Space Coordinates

#### End-Effector Configuration Parameters

The position  $r_e \in \mathbb{R}^3$  and rotation  $\phi_e \in SO(3)$  of a frame w.r.t a base can be parametrized by:

$$\chi_e = \begin{pmatrix} \chi_{eP} \\ \chi_{eR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_m \end{pmatrix} \in \mathbb{R}^m$$

**Operational Space Coordinates** The end-effector operates in the operational space, which depends on the geometry and structure of the arm. It can be described with:

$$\chi_o = \begin{pmatrix} \chi_{oP} \\ \chi_{oR} \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{m_o} \end{pmatrix}$$

where  $\chi_1 \dots \chi_{m_o}$  are *independent* operational space coordinates. It can be seen as a **minimal selection** of the above end-effector parameters.

### 1.6.3 Forward Kinematics

Forward kinematics map from joint coordinates  $q$  to the end-effector configuration  $\chi_e$ :  $\chi_e = \chi_e(q)$   
This relation can be obtained through the transformations of each link:

$$T_{IE}(q) = T_{I0} \left( \prod_{k=1}^{n_j} T_{k-1,k} \right) T_{n_j E} = \begin{bmatrix} C_{IE}(q) & I^{rIE}(q) \\ 0_{1 \times 3} & 1 \end{bmatrix}$$

### 1.6.4 Differential Kinematics and Analytical Jacobian

To linearise the forward kinematics we use a first order approximation:

$$\Delta \chi_e \approx J_{eA}(q) \Delta q, \quad J_{eA}(q) = \begin{bmatrix} \frac{\partial \chi_1}{\partial q_1} & \dots & \frac{\partial \chi_1}{\partial q_{n_j}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \chi_m}{\partial q_1} & \dots & \frac{\partial \chi_m}{\partial q_{n_j}} \end{bmatrix}$$

It results in an exact relation between velocities:

$$\dot{\chi}_e = J_{eA}(q) \dot{q}$$

Literature often talk about **position** and **rotation** Jacobians:

$$J_{eA} = \begin{bmatrix} J_{eA_P} \\ J_{eA_R} \end{bmatrix} = \begin{bmatrix} \frac{\partial \chi_{eP}}{\partial q} \\ \frac{\partial \chi_{eR}}{\partial q} \end{bmatrix}$$

### 1.6.5 Geometric or Basic Jacobian

The geometric or basic Jacobian relates the generalized velocity  $\dot{q}$  to the end-effector velocity (linear  $v_e$  and angular  $\omega_e$ ):

$$w_e = \begin{pmatrix} v_e \\ \omega_e \end{pmatrix} = J_{e0}(q) \cdot \dot{q}$$

Note: In the most general cases  $J_{e0}$  has dimension  $6 \times n_j$  and has Frame  $\mathcal{A}$  as a basis (like the velocity). From the velocities  $w_C = w_B + w_{BC}$  we can derive, that geometric Jacobians can simply be added (in the same reference):

$${}^{\mathcal{A}}J_C = {}^{\mathcal{A}}J_B + {}^{\mathcal{A}}J_{BC}$$

#### Geometric Jacobian:

$${}^{\mathcal{I}}J_{e0} = \begin{bmatrix} {}^{\mathcal{I}}J_{e0P} \\ {}^{\mathcal{I}}J_{e0R} \end{bmatrix} = \begin{bmatrix} {}^{\mathcal{I}}n_1 \times {}^{\mathcal{I}}r_{1(n+1)} & \dots & {}^{\mathcal{I}}n_n \times {}^{\mathcal{I}}r_{n(n+1)} \\ {}^{\mathcal{I}}n_1 & \dots & {}^{\mathcal{I}}n_n \end{bmatrix}$$

where  $n_k$  represents the rotation axis of joint  $k$  such that:

$$\omega_{(k-1)k} = n_k \dot{q}_k$$

and  $r_{1(n+1)} \dots r_{n(n+1)}$  represent the position vector from the joint  $1 \dots n$  to the end-effector.

**Don't forget to transform to inertial frame!**

For **prismatic Joints** the Position part ( $n \times r$ ) is an unit-vector  $\xi_k$  in joint direction. The Rotational part ( $n_i$ ) is obviously zero.

Mapping from Analytic to Geometric Jacobian, it holds that:

$$J_{e0}(q) = E_e(\chi) J_{eA}(q)$$

$$\text{with } E_e(\chi) = \begin{bmatrix} E_p(\chi) & 0 \\ 0 & E_R(\chi) \end{bmatrix} \in \mathbb{R}^{6 \times m_e}$$

## 1.7 Kinematic Control Methods

### 1.7.1 Inverse Differential Kinematics

The Jacobian  $J_{e0}(q)$  performs a simple mapping from joint space to end-effector velocity.

$$w_e = J_{e0} \dot{q}$$

To solve the inverse problem, we use take the **pseudo-inverse**  $J_{e0}^+$  of the Jacobian.

$$\dot{q} = J_{e0}^+ \cdot w_e^*$$

By taking the Moore-Penrose pseudo inverse, the solution  $\dot{q} = J_{e0}^+ \cdot w_e^*$  minimises the least square error  $\|w_e^* - J_{e0} \dot{q}\|^2$

#### Moore-Penrose Inverse

$$A^+ = A^T (A A^T)^{-1} \text{ right inverse (full row rank) } \square$$

$$A^+ = (A^T A)^{-1} A^T \text{ left inverse (full col rank) } \square$$

**Note:** For close to singular configurations,  $J_{e0}$  becomes badly conditioned, what causes large joint velocities for just a small end-effector velocity. This can be handled by using a **damped solution**.  $\dot{q} = J_{e0}^T (J_{e0} J_{e0}^T + \lambda^2 \mathbb{1})^{-1} w_e^*$

#### Redundancy

For a Robot that has more joints than DOF ( $\text{rank}(J_{e0}) < n$ ), the configuration is called **redundant**. Like previous, we can take the pseudo inverse:

$$\dot{q} = J_{e0}^T (J_{e0} J_{e0}^T)^{-1} \cdot w_e^* = J_{e0}^+ \cdot w_e^*$$

redundancy implies, that there are infinite additional solutions:

$$\dot{q} = J_{e0}^+ \cdot w_e^* + N \dot{q}_0$$

with  $N = \mathcal{N}(J_{e0})$  as null-space projection matrix, fulfilling  $J_{e0} N = 0$ . Thus, we can choose arbitrary  $\dot{q}_0$  without changing the velocity  $w_e^*$ .

The simplest method for the **null-space projection** is:

$$N = \mathbb{1} - J_{e0}^+ J_{e0}$$

### 1.7.2 Multi-task Inverse Differential Kinematic Control

For multiple tasks (same priority)  $task_i := \{J_i, w_i^*\}$  we can calculate the velocity:

$$\dot{q} = \underbrace{\begin{bmatrix} J_1 \\ \vdots \\ J_n \end{bmatrix}}_J^+ \cdot \underbrace{\begin{bmatrix} w_1^* \\ \vdots \\ w_n^* \end{bmatrix}}_{\tilde{w}}$$

For weighted tasks we could use a weighted pseudo inverse:  $\tilde{J}^+ W = (\tilde{J}^T W \tilde{J})^{-1} \tilde{J}^T W$  with weight  $W = \text{diag}(w_1, \dots, w_m)$

#### Multitask Prioritisation

An approach for prioritisation tasks (descending priority) is to use consecutive null-space projections.

Using the solution for task 1  $\dot{q} = J_1^+ w_1^* + N_1 \dot{q}_0$ , we can derive a term for  $q_0$ :

$$w_2 = J_2 \dot{q} = J_2 (J_1^+ w_1^* + N_1 \dot{q}_0)$$

$$\iff \dot{q}_0 = (J_2 N_1)^+ (w_2^* - J_2 J_1^+ w_1^*)$$

Substitution in the first solution for task 1 gives:

$$\dot{q} = J_1^+ w_1^* + N_1 (J_2 N_1)^+ (w_2^* - J_2 J_1^+ w_1^*)$$

For  $n_t$  tasks this can be written recursively:

$$\dot{q} = \sum_{i=1}^{n_t} \tilde{N}_i \dot{q}_i \text{ with } \dot{q}_i = (J_i \tilde{N}_i)^+ \left( w_i^* - J_i \sum_{k=1}^{i-1} \tilde{N}_k \dot{q}_k \right)$$

with  $\tilde{N}_i$  the null space projection of the stacked  $J$ ,  $\tilde{J}_i = [J_1^T \dots J_{i-1}^T]^T$

### 1.7.3 Inverse Kinematics (Numerical Solution)

The goal of inverse Kinematics is to find the joint configuration for a given end-effector configuration  $\chi_e^*$ :  $q = q(\chi_e^*)$

We can solve this problem iteratively by using:  $\Delta \chi_e = J_{eA} \Delta q$

#### Algorithmus 1 : Numerical Inverse Kinematics

```

q ← q0; // Start Configuration
while ||χe* − χe(q)|| > tol do
    JeA ← JeA(q) = ∂χe/∂q; // Evaluate (local) Jacobian
    JeA+ ← (JeA)+; // Calculate Pseudo Inverse
    Δχe ← χe* − χe(q); // Find Error Vector
    q ← q + JeA+ Δχe; // Update generalized Coordinates
end

```

To  $\Delta \chi_e^i$  the linearization (Jacobian) becomes inaccurate, scale:  $q \leftarrow q + k J_{eA}^+ \Delta \chi_e$ ,  $0 < k < 1$

For badly conditioned (singular) Jacobians, we use either the damped inverse or use the Jacobi-transposed method:

$$q \leftarrow q + \alpha J_{eA}^T \Delta \chi_e$$

For small enough  $\alpha$  convergence can be guaranteed.

#### Shortest Path rotation

For a straight rotation along the "shortest path", we rotate along the rotation vector  $\Delta \varphi$ .

The rotation Matrix is given by:

$$C_{AB}(\Delta \varphi) = C_{\mathcal{I}A}(\varphi^t)^T C_{\mathcal{I}B}(\varphi^*)$$

(Note that  $\Delta \varphi \neq \varphi^* - \varphi^t$ )

The rotation vector is the same in both frames A & B.

$${}^{\mathcal{A}}\Delta \varphi = {}^{\mathcal{B}}\Delta \varphi = \text{rotVec}(C_{AB})$$

Instead of mapping this vector into  $\mathcal{I}$ , we can derive it directly:

$${}^{\mathcal{I}}\Delta \varphi = \text{rotVec}(C_{\mathcal{I}B} C_{\mathcal{I}A}^T)$$

Now we can change the update step 6 of the algorithm to:

$$q \leftarrow q + k_{PR} {}^{\mathcal{I}}J_{e0R}^+ {}^{\mathcal{I}}\Delta \varphi$$

### 1.7.4 Trajectory Control

Pure inverse differential kinematics often drift away from the predefined path. Hence, we introduce a feedback.

For predefined position  $r_e^*(t)$  and velocity  $\dot{r}_e^*(t)$ :

$$\dot{q}^* = J_{e0P}^+(q^t) \cdot (\dot{r}_e^*(t) + k_{PP} \Delta r_e^t) \quad \text{with } \Delta r_e^t = r_e^*(t) - r_e(q^t)$$

Similar for orientation  $\chi_R^*(t)$  and angular velocity  $\omega^*(t)$ :

$$\dot{q}^* = J_{e0R}^+(q^t) \cdot (\omega_e^*(t) + k_{PR} \Delta \varphi) \quad \text{with } \Delta \varphi \text{ from above.}$$

## 2 Dynamics

We formulate multi-body dynamics as:

$$M(q) \ddot{q} + b(q, \dot{q}) + g(q) = \tau + J_c(q)^T F_c$$

consisting of the following elements:

$M(q)$	Generalized mass (or inertia) matrix (orthogonal)
$\dot{q}, \ddot{q}, \ddot{\ddot{q}}$	Generalized position, velocity and acceleration vectors
$b(q, \dot{q})$	Coriolis and centrifugal terms
$g(q)$	Gravitational terms
$\tau$	External generalized forces
$F_c$	External cartesian forces (e.g. from contacts)
$J_c(q)$	Geometric Jacobian corresponding to external forces

### 2.1 Principle of virtual Work

$$\delta W \int_{\mathcal{B}} \delta r^T \cdot (\ddot{r} dm - dF_{ext}) = 0, \quad \forall \delta r$$

$dm$	infinitesimal mass element
$dF_{ext}$	external Forces acting on element dm
$\ddot{r}$	acceleration of element dm
$\delta r$	virtual displacement of dm
$\mathcal{B}$	Body System containing infinitesimal particles dm

### 2.2 Newton-Euler Method

$$m \cdot \ddot{x} = \sum F_i \quad \text{and} \quad \Theta \cdot \ddot{\varphi} = \sum T_i$$

For Multi-Body System we need to cut every joint free and introduce constraining forces for every piece. This results in a system of equations with additional kinematic constraints.

### 2.3 Lagrange Method

The **Lagrangian Function** for mech. sys. is exactly the difference between the total kinetic energy  $\mathcal{T}$  and total potential energy  $\mathcal{U}$ .

$$\mathcal{L} = \mathcal{T} - \mathcal{U}$$

**Euler-Lagrange (of the second kind):**  $\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \left( \frac{\partial \mathcal{L}}{\partial q} \right) = \tau \Rightarrow$

$$\frac{d}{dt} \left( \frac{\partial \mathcal{T}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{T}}{\partial q} + \frac{\partial \mathcal{U}}{\partial q} = \tau \text{ with total gen. ext. forces } \tau$$

The **Hamiltonian** states the total energy:  $\mathcal{H} = \mathcal{T} + \mathcal{U}$

#### 2.3.1 Kinetic Energy

The kinetic energy is defined as (recall the basic formulas  $E = \frac{1}{2} m v^2$  &  $\frac{1}{2} J \omega^2$ ):

$$\mathcal{T} = \sum_{i=1}^{n_b} \left( \frac{1}{2} m_i {}^{\mathcal{A}}\dot{r}_{Si}^T {}^{\mathcal{A}}\dot{r}_{Si} + \frac{1}{2} {}^{\mathcal{B}}\Omega_{Si}^T \cdot {}^{\mathcal{B}}\Theta_{Si} \cdot {}^{\mathcal{B}}\Omega_{Si} \right)$$

With the Jacobian relations  $\dot{r}_{Si} = J_{Si} \dot{q}$ ,  $\Omega_{Si} = J_{Ri} \dot{q}$  we can rewrite this:

$$\mathcal{T}(q, \dot{q}) = \frac{1}{2} \dot{q}^T \underbrace{\left( \sum_{i=1}^{n_b} (J_{Si}^T m_i J_{Si} + J_{Ri}^T \Theta_{Si} J_{Ri}) \right)}_{M(q)} \dot{q}$$

### 2.3.2 Potential Energy

Knowing  $r_{Si}$  to the Center of Mass of each body, we can calculate gravitational forces (Zero energy level can be chosen arbitrarily):

$$F_{g_i} = m_i \cdot g \cdot \mathcal{I} e_g \Rightarrow \mathcal{U}_g = - \sum_{i=1}^{n_b} r_{Si}^T F_{g_i}$$

Potential energy for elastic elements:  $\mathcal{U}_{E_j} = \frac{1}{2} k_j \underbrace{(d(q) - d_0)}_{\text{deflection}}^2$   
with i.e., the current length of the spring  $d(q)$  & resting length  $d_0$ .

### 2.4 Projected Euler Method

$$\begin{bmatrix} v_s \\ \Omega \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix} \dot{q} \quad \begin{bmatrix} a_s \\ \Psi \end{bmatrix} = \begin{bmatrix} J_P \\ J_R \end{bmatrix} \ddot{q} + \begin{bmatrix} j_P \\ j_R \end{bmatrix} \dot{q}$$

Rotate Inertia Tensor:  $\mathcal{I}\Theta = C_{\mathcal{I}B} \cdot {}^{\mathcal{B}}\Theta \cdot C_{\mathcal{I}B}^T$

$$M = \sum_{i=1}^{n_b} ({}^{\mathcal{A}}J_{Si}^T \cdot m_i \cdot {}^{\mathcal{A}}J_{Si} + {}^{\mathcal{B}}J_{Ri}^T \cdot {}^{\mathcal{B}}\Theta_{Si} \cdot {}^{\mathcal{B}}J_{Ri})$$

$$b = \sum_{i=1}^{n_b} ({}^{\mathcal{A}}J_{Si}^T m_i {}^{\mathcal{A}}j_{Si} \dot{q} + {}^{\mathcal{B}}J_{Ri}^T ({}^{\mathcal{B}}\Theta_{Si} \cdot {}^{\mathcal{B}}j_{Ri} \cdot \dot{q} + \underbrace{{}^{\mathcal{B}}\Omega_{Si} \times {}^{\mathcal{B}}\Theta_{Si} \cdot {}^{\mathcal{B}}\Omega_{Si}}_{J_{Ri} \dot{q} \times {}^{\mathcal{B}}\Theta_{Si} J_{Ri} \dot{q}=0 \text{ in 2D}}))$$

$$g = \sum_{i=1}^{n_b} (-{}^{\mathcal{A}}J_{Si}^T \cdot {}^{\mathcal{A}}F_{g,i})$$

#### 2.4.1 External Forces & Actuation

For known Forces  $F_j$  acting on the system, we can calculate the generalized forces  $\tau_{F,ext}$  (due to the external force).

$$\tau_{F,ext} = \sum J_{P,j}^T F_j$$

with the translational (geometric) Jacobian of Point  $j$  (i.e.  $J_e$  for end effector)

Similar for external Torques  $T_j$ :  $\tau_{T,ext} = \sum J_{R,j}^T T_j$

### 2.5 Joint-Space Dynamic Control

#### Joint Impedance Regulation

In case of torque controlled actuators, we can get a simple PD control law for the desired(\*) actuator torque:

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

This ends in a steady state offset of:  $g(q) = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$

**Gravity Compensation:** To compensate for the gravity offset, we simply add an estimated value  $\hat{g}(q)$  to the control law:

$$\tau^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q}) + \hat{g}(q)$$

**Note:**  $k_d$  and  $k_p$  are constant for all configurations ( $q$ ), which reduces the overall performance.

**Inverse Dynamics Control:** A simple way to get dynamic decoupling and motion control is to get estimates  $\hat{M}$ ,  $\hat{b}$  and  $\hat{g}$  and select the torque with:

$$\tau^* = \hat{M}(q) \ddot{q}^* + \hat{b}(q, \dot{q}) + \hat{g}(q)$$

Then, a common approach selects the desired acceleration according to:

$$\ddot{q}^* = k_p(q^* - q) + k_d(\dot{q}^* - \dot{q})$$

which has eigenfrequency  $\omega = \sqrt{k_p}$  and Damping  $D = \frac{k_d}{2\sqrt{k_p}}$  ( $D = 1$  critical-,  $D < 1$  under-,  $D > 1$  over-damped)

## 2.6 Task-Space Dynamic Control

To move to a specific point in Task-Space (Fixed Frame) we need the linear and rotational acceleration of the end-effector:

$$\dot{w}_e = \begin{pmatrix} \ddot{r} \\ \ddot{\omega} \end{pmatrix} = J_e \dot{q} + \dot{J}_e \dot{q}$$

### 2.6.1 Multi-task

Similar to kinematics, we can fulfill multiple tasks:

$$\ddot{q} = \begin{bmatrix} J_1 \\ \vdots \\ J_{n_t} \end{bmatrix}^+ \left( \begin{bmatrix} \dot{w}_1 \\ \vdots \\ \dot{w}_{n_t} \end{bmatrix} - \begin{bmatrix} \dot{J}_1 \\ \vdots \\ \dot{J}_{n_t} \end{bmatrix} \dot{q} \right)$$

and the recursive algorithm:

$$\dot{q} = \sum_{i=1}^{n_t} \bar{N}_i \dot{q}_i \text{ with } \dot{q}_i = (J_i \bar{N}_i)^+ \left( w_i^* - \dot{J}_i \dot{q} - J_i \sum_{k=1}^{i-1} \bar{N}_k \dot{q}_k \right)$$

### 2.6.2 End-Effector Dynamics

With  $\tau = J_e^T F_e$  we can formulate the end-effector Dynamics:

$$\Lambda_e \dot{w}_e + \mu + p = F_e$$

with  $\Lambda_e = (J_e M^{-1} J_e^T)^{-1}$ ,  $\mu = \Lambda_e J_e M^{-1} b - \Lambda_e \dot{J}_e \dot{q}$ ,  
 $p = \Lambda_e J_e M^{-1} g$

as the end-effector inertia, centrifugal and gravitational terms in task-space.

#### End-Effector Motion Control

From the above Dynamics, we can get an inversion motion control, like in the joint space:

$$\tau^* = \hat{J}^T (\hat{\Lambda}_e \dot{w}_e^* + \hat{\mu} + \hat{p})$$

together with a control law:

$$\dot{w}_e^* = k_p \begin{pmatrix} r_e^* - r_e \\ \Delta \phi_e \end{pmatrix} + k_d (w_e^* - w_e)$$

For small errors we can approximate:

$$\begin{pmatrix} r_e^* - r_e \\ \Delta \phi_e \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & E_R \end{pmatrix} \begin{pmatrix} r^* - r \\ \chi_R^* - \chi_R \end{pmatrix}$$

### 2.6.3 Operational Space Control

**Note:** We need to extend the end effector dynamics with a contact Force  $F_c$ :

$$F_c + \Lambda_e \dot{w}_e + \mu + p = F_e$$

In some situations the robot has to either apply a force or move in a direction. This can be described by two specification matrices for position and orientation:

$$\Sigma_p = \begin{pmatrix} \sigma_{px} & 0 & 0 \\ 0 & \sigma_{py} & 0 \\ 0 & 0 & \sigma_{pz} \end{pmatrix} \quad \Sigma_r = \begin{pmatrix} \sigma_{rx} & 0 & 0 \\ 0 & \sigma_{ry} & 0 \\ 0 & 0 & \sigma_{rz} \end{pmatrix}$$

with  $\sigma_i$  either 1(move) or 0(don't).

$$\tau^* = \hat{J}^T (\hat{\Lambda}_e S_M \dot{w}_e^* + S_F F_c + \hat{\mu} + \hat{p})$$

$$S_M = \begin{pmatrix} C^T \Sigma_p C & 0 \\ 0 & C^T \Sigma_r C \end{pmatrix} \quad S_F = \begin{pmatrix} C^T (1 - \Sigma_p) C & 0 \\ 0 & C^T (1 - \Sigma_r) C \end{pmatrix}$$

## 2.7 Least Square Optimisation

I Slides 6.5

So far we only considered the optimisation of  $\min \|\ddot{q}\|_2$  as result of the Pseudoinverse  $J^+$ . To optimize another objective, we can formulate the problem in multiple tasks:

$$\left| \begin{array}{l} \tau = M \ddot{q} + b + g \\ \dot{w} = J \ddot{q} + \dot{J} \dot{q} \end{array} \right| \Rightarrow \left| \begin{array}{l} [M \quad -1] \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} + b + g = 0 \\ [J_e \quad 0] \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} + \dot{J}_e \dot{q} = \dot{w}^* \end{array} \right|$$

This has always to be fulfilled, and can be extended by additional objectives.

It can be solved as single (stacked) tasks:

$$\min_{q, \tau} \left\| \begin{bmatrix} M & -1 \\ J_e & 0 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - \begin{pmatrix} -b - g \\ \dot{w}_e^* - \dot{J}_e \dot{q} \end{pmatrix} \right\|_2$$

Or with different priorities:

$$\min_{q, \tau} \left\| [J_e \quad 0] \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - (\dot{w}_e^* - \dot{J}_e \dot{q}) \right\|_2$$

such that:  $[M \quad -1] \begin{pmatrix} \ddot{q} \\ \tau \end{pmatrix} - (-b - g) = 0$

This will exploit the nullspace of the higher priority task to minimize the solution.  $\Rightarrow$  Solve with numeric solver

## 3 Floating Base Systems

### 3.1 FB Kinematics

Free floating robots are described by  $n_b$  **unactuated** base coordinates  $q_b$  and  $n_j$  actuated joint coordinates  $q_j$ .

$$q = \begin{pmatrix} q_b \\ q_j \end{pmatrix} \quad \text{with } q_b = \begin{pmatrix} q_{bP} \\ q_{bR} \end{pmatrix} \in \mathbb{R}^3 \times SO(3)$$

The minimal number of generalized coordinates for the base is  $n_{b0} = 6$  (3D).

**Generalized Velocity** (often simply written as  $\dot{q}$ )

$$u = \begin{pmatrix} \mathcal{I}^V B \\ \mathcal{B} \omega_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \quad \text{with mapping} \quad u = E_{fb} \cdot \dot{q}, \quad E_{fb} = \begin{bmatrix} 1 \times 3 \times 3 & 0 & 0 \\ 0 & E_{\chi_R} & 0 \\ 0 & 0 & 1_{n_j \times n_j} \end{bmatrix}$$

#### 3.1.1 Forward Kinematics

The position vector of point  $Q$  can be expressed via the Base  $\mathcal{B}$

$$\mathcal{I}^r I_Q(q) = \mathcal{I}^r I_B(q) + C_{IB}(q) \cdot \mathcal{B}^r B_Q(q)$$

#### 3.1.2 Differential Kinematics

The spacial Jacobian maps  $u$  to  $v$  and  $\omega$ :

$$\begin{pmatrix} \mathcal{I}^v Q \\ \mathcal{I}^{\omega} I_Q \end{pmatrix} = \mathcal{I} J_Q(q) \cdot u$$

$$\mathcal{I} J_Q(q) = \begin{bmatrix} 1 \times 3 \times 3 & -C_{IB} \cdot [\mathcal{B}^r B_Q] \times & C_{IB} \cdot \mathcal{B} J_{P_{q_j}}(q_j) \\ 0 \times 3 \times 3 & C_{IB} & C_{IB} \cdot \mathcal{B} J_{R_{q_j}}(q_j) \end{bmatrix}$$

## 3.1.3 Contacts & Constraints

Every Point  $C_i$  in contact with the environment imposes **constant** position and **zero** velocity and acceleration.  $\rightarrow$  Contact Jac.  $J_{C_i}$

$$\mathcal{I} J_{C_i} u = 0, \quad \mathcal{I} J_{C_i} \dot{u} + \mathcal{I} \dot{J}_{C_i} u = 0$$

where multiple  $J_{C_i}$  can be stacked for multiple contact points.

The  $rank(J_c)$  indicates the number of independent contact constraints. The stacked  $J_c$  can be split in a body and joint part:

$$J_c = [J_{c,b} \quad J_{c,j}]$$

If the rank of  $J_{c,b}$  has full rank ( $=6$  in 3D), the joints can move the body in every direction. The difference  $rank(J_c) - rank(J_{c,b})$  is the number of **internal kinematic constraints** (i.e. legs can move in respect to each other).

### 3.1.4 Inverse Kinematics

We apply inverse kinematics, where the ground contact  $J_c u = 0$  has the highest priority:

$$J_c u = 0 \Rightarrow u = J_c^+ 0 + \mathcal{N}(J_c) u_0 = \mathcal{N}_c u_0$$

Given a demanded motion  $w_t$  we can calculate the required velocity:  $w_t = J_t u \Rightarrow u = N_c (J_t N_c)^+ w_t$

### 3.2 FB Dynamics

We need to extend the known dynamics with a selection for the torques  $\tau$ , since the body is unactuated:

$$M(q) \dot{u} + b(q, u) + g(q) = S^T \tau + J_{ext}(q)^T F_{ext}$$

with  $u_j = S u = S \begin{pmatrix} u_b \\ u_j \end{pmatrix} = [0_{n_j \times 6} \quad 1_{n_j \times n_j}] \begin{pmatrix} u_b \\ u_j \end{pmatrix}$

**Note:** If we have the forces that the robot exerts **on its environment**, we need them to switch the side in the equation:

$$M(q) \dot{u} + b(q, u) + g(q) + J_c(q)^T F_c = S^T \tau$$

Together with the contact constraints we can calculate the contact Force  $F_c$ :

$$F_c = (J_c M^{-1} J_c^T)^{-1} (J_c M^{-1} (S^T \tau - b - g) + \dot{J}_c u)$$

#### 3.2.1 Constraint Dynamics

We can define a Null-space matrix for the contact constraints:

$$N_c = 1 - M^{-1} J_c^T (J_c M^{-1} J_c^T)^{-1} J_c$$

This gives the following equations of motion, which are reduced, but **consistent with the constraints** (contact forces):

$$N_c^T (M \dot{u} + b + g) = N_c^T S^T \tau$$

#### 3.2.2 FB Inverse Dynamics

With a desired  $\dot{u}_{consistent}^*$  We can invert the equation of motion of above:

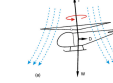
$$\tau^* = (N_c^T S^T)^+ N_c^T (M \dot{u}^* + b + g) + \mathcal{N}(N_c^T S^T) \tau_0^*$$

When taking only the first part without a Nullspace ( $\tau_0^* = 0$ ), the solution is the least square minimal torque  $\tau^*$  that fulfils the EoM.

## 4 Rotorcrafts

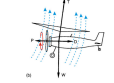
### Overview

Helicopter



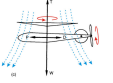
Has a power driven main rotor, which can be tilted

Autogyro



Passive main rotor and a forward facing active propeller. Can't hover.

Gyrodyne

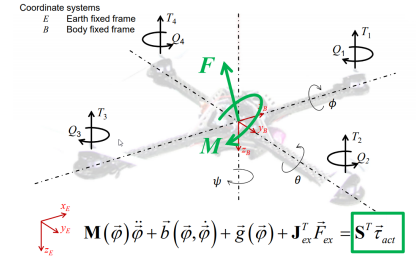


Active main rotor, but can't be tilted. Additional front facing active propeller.

Typical rotorcrafts are: Single Rotors, Multi rotors, Coaxial, Ducted Fan, Omnidirectional Multicopter (movable rotors).

### 4.1 Modelling of Quadrotor

Modelling and simulations are important, but they must be validated in reality.



#### Structural Properties:

Arm length  $l$ , Rotor height  $h$ , Mass  $m$ , Inertia  $I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$

Hub force & rolling moments depend on flight regime and can be neglected in hovering.

#### Rotation:

$\begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix} = \begin{pmatrix} Roll \\ Pitch \\ Yaw \end{pmatrix} \rightarrow \begin{pmatrix} x Axis \\ y Axis \\ z Axis \end{pmatrix}$  with the known rot. matrices.

This can be used for an equation for the rotational speed:

$$\mathcal{B} \omega = E_r \dot{\chi}_r = E_r \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}, \text{ with } E_r = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \sin \phi \cos \theta \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix}$$

Linearization for small Roll and Pitch ( $\phi \approx \theta \approx 0$ ) results in a **unity matrix**  $E_r = 1$

#### 4.1.1 Body Dynamics

$$\begin{bmatrix} m \mathbb{1}_{3 \times 3} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{B} \dot{v} \\ \mathcal{B} \dot{\omega} \end{bmatrix} + \begin{bmatrix} \mathcal{B} \omega \times m \mathcal{B} v \\ \mathcal{B} \omega \times I \mathcal{B} \omega \end{bmatrix} = \begin{bmatrix} \mathcal{B} F \\ \mathcal{B} M \end{bmatrix}$$

with Forces

$$\mathcal{B} F = \mathcal{B} F_G + \mathcal{B} F_{Aero} = C_{EB}^T \begin{bmatrix} 0 \\ 0 \\ mg \end{bmatrix} + \sum_{i=1}^4 \begin{pmatrix} 0 \\ 0 \\ -T_i \end{pmatrix}$$

and Hover Moments

$$\mathcal{B} M_{Aero} = \mathcal{B} \begin{pmatrix} l(T_4 - T_2) \\ l(T_1 - T_3) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 Q_i (-1)^{(i-1)} \end{pmatrix}$$

with Thrust forces  $T_i = b_i \omega_{p,i}^2$  and Drag Forces  $Q_i = d_i \omega_{p,i}^2$

**Rotational Dynamics** (1st row of Dynamics)

$$\begin{aligned} I_{xx} \dot{\omega}_x &= \omega_y \cdot \omega_z (I_{yy} - I_{zz}) + l \cdot b (\omega_{p,4}^2 - \omega_{p,2}^2) \\ I_{yy} \dot{\omega}_y &= \omega_z \cdot \omega_x (I_{zz} - I_{xx}) + l \cdot b (\omega_{p,1}^2 - \omega_{p,3}^2) \\ I_{zz} \dot{\omega}_z &= d (\omega_{p,1}^2 - \omega_{p,2}^2 + \omega_{p,3}^2 - \omega_{p,4}^2) \end{aligned}$$

with  $\omega_{xyz}$  as entries of the body rotation  $\mathcal{B}\omega$

$\Rightarrow$  We have full control over all rotational speeds (every equation depends on rotor speeds  $\omega_{p,i}$ ). Note that  $(I_{xx} - I_{yy}) = 0$  (x-y symmetry), which dropped out of the 3rd equation.

**Translational Dynamics** (2nd row of dynamics)

$$\begin{aligned} m \dot{v}_x &= m(\omega_z \cdot v_y - \omega_y \cdot v_z) - \sin \theta mg \\ m \dot{v}_y &= m(\omega_x \cdot v_z - \omega_z \cdot v_x) + \sin \phi \cos \theta mg \\ m \dot{v}_z &= m(\omega_y \cdot v_x - \omega_x \cdot v_y) + \cos \phi \cos \theta mg \\ &\quad - b(\omega_{p,1}^2 + \omega_{p,2}^2 + \omega_{p,3}^2 + \omega_{p,4}^2) \end{aligned}$$

with gravitational Terms in blue,  $\mathcal{B}FG = C_{EB}^T \cdot E \vec{n}_z mg$

$\Rightarrow$  Only z-Axis can be controlled directly with  $\omega_{p,i}$

**Note:** To be consistent with the lecture notation:  $(\omega_x, \omega_y, \omega_z) = (p, q, r)$  and  $(v_x, v_y, v_z) = (u, v, w)$

## 4.2 Control of a Quadrotor

The system has 6 DoF, but only 4 Inputs (Motors)  $\rightarrow$  **Under-actuated!**  
 $\Rightarrow$  Forward Motion requires tipping around Roll and Pitch.

**Define Virtual control inputs:**

By defining a new set of inputs, we can decouple the Dynamic equations

Total Thrust                      Moments along axis

$$\begin{aligned} U_1 &= b(\omega_{p1}^2 + \omega_{p2}^2 + \omega_{p3}^2 + \omega_{p4}^2) \\ U_2 &= l \cdot b(\omega_{p4}^2 - \omega_{p2}^2) \\ U_3 &= l \cdot b(\omega_{p1}^2 - \omega_{p3}^2) \\ U_4 &= d(\omega_{p1}^2 - \omega_{p2}^2 + \omega_{p3}^2 - \omega_{p4}^2) \end{aligned}$$

which simplify the dynamics from above.

**Linearize Attitude Dynamics**

Linearization around the Equilibrium  $(\omega_x, \omega_y, \omega_z = \phi = \theta = U_{2,3,4} = 0; U_1 = mg)$  gives further simplification (Recall:  $E_r = \mathbb{1} \Rightarrow \dot{\chi}_r = [\ddot{\phi} \ \ddot{\theta} \ \ddot{\psi}]^T$ ):  $\Rightarrow$  Can use 3 individual controller

$$\dot{\omega}_x = \ddot{\phi} = \frac{1}{I_{xx}} U_2 \quad \dot{\omega}_y = \ddot{\theta} = \frac{1}{I_{yy}} U_3 \quad \dot{\omega}_z = \ddot{\psi} = \frac{1}{I_{zz}} U_4$$

**Attitude Control:**

Dynamics from above, but in *Inertial Frame*:

$$\dot{v}_z = \ddot{z} = g - \cos \phi \cos \theta \frac{1}{m} U_1 =: g - \frac{1}{m} T_z$$

Now we can derive the input  $U_1$  for a chosen controller  $T_z$  (representing the Thrust in the inertial frame):

$$T_z = -k_p(z_{des} - z) + k_d \dot{z} - mg \Rightarrow U_1 = \frac{T_z}{\cos \phi \cos \theta}$$

**Position Control:**

We can use 3 separate PD Controller to get the trust for x,y and z in the inertial frame.

$$\text{Dynamics} \Rightarrow [\ddot{x} \ \ddot{y} \ \ddot{z}]^T = \frac{1}{m} [T_x \ T_y \ T_z]^T + [0 \ 0 \ g]^T$$

This must be transformed to get the desired Total thrust as well as roll and pitch angles.

$$T = \sqrt{T_x^2 + T_y^2 + T_z^2} \quad \& \quad \frac{1}{T} C_{E1}^T(z, \psi) \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi \\ -\sin \phi \\ \cos \theta \cos \phi \end{bmatrix}$$

**Remark:**  $C_{E1}(z, \psi)$  is the rotation matrix around z with angle  $\psi$ .

**Remark2:**  $\begin{bmatrix} \sin \theta \cos \phi \\ -\sin \phi \\ \cos \theta \cos \phi \end{bmatrix}$  describes the Thrust of the Quadrotor in the

Inertial frame (can only apply thrust perpendicular to the rotors). It's calculated with  $C_{12}(y, \theta) \cdot C_{2B}(x, \phi) \cdot (0, 0, 1)^T$

## 4.3 Propeller Aerodynamics

There are 4 main forces generated by the rotor:

**For a Rotor in Hover:**

Thrust Force T

Aerodynamic force perpendicular to rotor plane

$$|T| = \frac{\rho}{2} A_P C_T (\omega_p R_P)^2$$

**For a Rotor in forward flight:**

Hub Force H

Opposite to horizontal flight direction  $V_H$

$$|H| = \frac{\rho}{2} A_P C_H (\omega_p R_P)^2$$

Drag Torque Q

Torque around rotor plane

$$|Q| = \frac{\rho}{2} A_P C_Q (\omega_p R_P)^2 R_P$$

Rolling Moment R

Around flight direction

$$|R| = \frac{\rho}{2} A_P C_R (\omega_p R_P)^2 R_P$$

### 4.3.1 Momentum Theory

$\rho$ : fluid density,  $\vec{V}$ : flow speed,  $\vec{n}$ : surface normal,

$dA$ : surface Area patch,  $p$ : surface pressure,  $E$ : Energy,  $P$ : Power

Conservation of fluid mass

Mass flow inside and outside (closed) control Volume must be equal

$$\iint \rho \vec{V} \cdot \vec{n} dA = 0$$

Conservation of fluid Momentum:

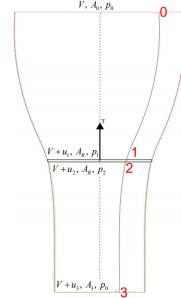
The net Force is the change of momentum of the fluid

$$\iint p \cdot \vec{n} dA + \iint (\rho \vec{V}) \vec{V} \cdot \vec{n} dA = \vec{F}$$

Conservation of energy:

Work done on the fluid results in a gain of kinetic energy

$$\iint \frac{1}{2} \rho V^2 \vec{V} \cdot \vec{n} dA = \frac{dE}{dt} = P$$



**1D Analysis:**

The formulas lead to the following results:

$$\begin{aligned} \rho A_0 V &= \rho A_R (V + u_1) \\ &= \rho A_R (V + u_2) = \rho A_R (V + u_3) \\ \Rightarrow u_1 &= u_2 \end{aligned}$$

$$\begin{aligned} F_{Thrust} &= \rho A_R (V + u_1) u_3 \\ P_{Thrust} &= F_{Thrust} (V + u_1) \\ &= \frac{1}{2} \rho A_R (V + u_1) (2V + u_3) u_3 \\ \Rightarrow u_3 &= 2u_1 \end{aligned}$$

**In the Hover case ( $V = 0$ ):**

Thrust Force:  $F_{Thrust} = 2 \rho A_R u_1^2$

Slipstream Tube:  $A_0 = \infty \quad A_3 = \frac{A_R}{2}$

Combining  $P = F_{Thrust} (\overset{=0}{V} + u_1)$  with  $F_{Thrust} = 2 \rho A_R u_1^2$  gives the Ideal Power to Hover:

$$P = \frac{F_{Thrust}^{3/2}}{\sqrt{2 \rho A_R}} = \frac{(mg)^{3/2}}{\sqrt{2 \rho A_R}} \quad \text{with } F_{Thrust} = mg$$

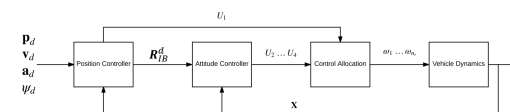
The Power depends on the **Disc Loading**  $F_{Thrust}/A_R$

**Defining the rotor efficiency, Figure of Merit, FM**

$$FM = \frac{\text{Ideal power to hover}}{\text{Actual power to hover}} < 1$$

$\rightarrow$  compare different propellers with the same disc loading.

## 5 Case Study: Micro Aerial Vehicles



## 5.1 Control for MAVs

**Virtual Control Input (allocation):**

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = A \begin{pmatrix} \omega_1^2 \\ \dots \\ \omega_n^2 \end{pmatrix}, \quad A \in \mathbb{R}^{4 \times n_r}$$

### 5.1.1 Trajectory Tracking Controller

Error definitions: Position  $e_p = p - p_d$ , Velocity  $e_v = v - v_d$

$$z_B^d = \frac{-K_p e_p - K_v e_v - m(g - a_d)}{|| -K_p e_p - K_v e_v - m(g - a_d) ||}$$

$$I z_{temp}^d = \begin{pmatrix} \cos \psi_d \\ \sin \psi_d \\ 0 \end{pmatrix} \quad I \dot{z}_B^d = \frac{I z_B^d \times I x_{temp}^d}{|| I z_B^d \times I x_{temp}^d ||}$$

From this we can construct the Rotation Matrix  $R_{IB}^d = [I y_B^d \times I z_B^d, I y_B^d, I z_B^d]$

**Note:**  $I z_{temp}^d$  is generally not perpendicular to  $I z_B^d$ , so we use some orthogonal projections for the first entry of  $R_{IB}^d$

**Attitude Control**

$$\begin{pmatrix} U_2 \\ U_3 \\ U_4 \end{pmatrix} = -K_R e_R - K_\omega e_\omega + \omega \times J \omega$$

with errors:

$$e_R = \frac{1}{2} \left( (R_{IB}^d)^T R_{IB} - R_{IB}^T R_{IB}^d \right)^\vee, \quad e_\omega = \omega - R_{IB}^T R_{IB}^d \omega_d$$

where  $(\cdot)^\vee$  maps the cross product matrix to a vector.

**Trajectory Tracking**

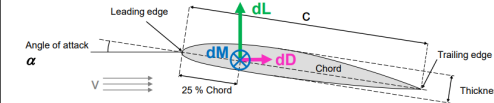
Project Thrust onto Body Z-axis:

$$U_1 = (-K_p e_p - K_v e_v - m(g - a_d)) \cdot I \vec{n}_{z,B}$$

## 6 Fixed Wing UAVs

### 6.1 Basic Principles

For incompressible and non viscous fluids, we can use the Bernoulli Equation:  $\frac{v^2}{2} + gh + \frac{p}{\rho} = \text{const.}$  This results in a **Lift** and a **Drag** Force with an additional **Moment** on the Airfoil. Lift/Drag is always perpendicular/parallel to the air velocity!



**Stall Point:** The angle of attack at which the maximum lift occurs. (Flow will separate for higher  $\alpha$ )

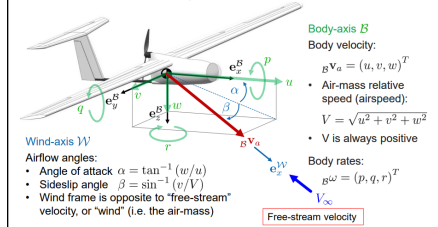
### 6.2 Fixed Wing Kinematics

**Assumptions:** Aerodynamics where we don't enter Stall, neglect side-forces and interference effects

**Inertial frame:** X-Axis: North; Y-Axis: East; Z-Axis: Downwards

**Body-fixed frame:** X-Axis: out of the nose; Y-Axis: out of the right wing; Z-Axis: Downwards; The Orientation of the Body Frame is describes as usual in Euler Angles with roll, pitch and yaw  $\rightarrow \phi, \theta, \psi$ .

**Control Surfaces:** The standard Control surfaces are **Elevator** (pitch), **Aileron** (roll), **Rudder** (yaw).



**Flight Path Angle:**  $\gamma$ , defined from horizon to  $I v_a$

**Heading Angle:**  $\xi$ , defined from North to  $I v_a$

**Course Angle:**  $\chi$ , defined from North to  $I v = I v_a + I v_{wind}$

## 6.3 Fixed Wing Dynamics

### 6.3.1 Forces

**Non Aerodynamics:** Weight, Propeller Thrust. **Aerodynamic:**

$$\text{Lift: } L = \frac{1}{2} \rho V^2 S c_L, \quad \text{Drag: } D = \frac{1}{2} \rho V^2 S c_D$$

with surface  $S$ .

Note:  $c_L$  and  $c_D$  are dependent on  $\alpha$ , Side-Forces: assumed zero

### 6.3.2 Equations of Motion, ( $I x_z$ is typically small)

$$\mathcal{B} \dot{v}_a = \frac{1}{m} \mathcal{B} F - \mathcal{B} \omega \times \mathcal{B} v_a, \quad \mathcal{I} \dot{r} = C_{IB} \mathcal{B} v_a + \mathcal{I} v_w$$

$$\mathcal{B} \dot{\omega} = \mathcal{B} I^{-1} \left( \sum \mathcal{B} M - \mathcal{B} \omega \times (\mathcal{B} I \mathcal{B} \omega) \right), \quad \mathcal{B} I = \begin{bmatrix} I_{xx} & 0 & I_{xz} \\ 0 & I_{yy} & 0 \\ I_{xz} & 0 & I_{zz} \end{bmatrix}$$

## 6.4 Cascaded Control Loops

### 6.4.1 Steady Level Turning

We need  $\mathcal{B} \dot{v}_a = \mathcal{B} \dot{\omega} = 0$ ;  $\theta = \alpha \rightarrow \gamma = 0$  and  $\phi = \text{const} \neq 0$ .

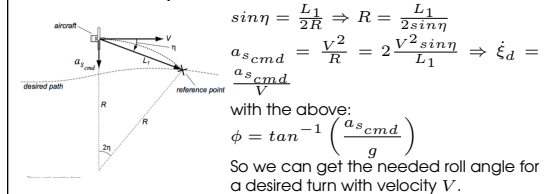
- The Lift  $L$  increases with  $\frac{1}{\cos \phi} \leftarrow (L = \frac{mg}{\cos \phi})$

- The air speed  $V_{min}$  increases with  $\sqrt{1/\cos \phi} \leftarrow (\frac{1}{2} \rho V^2 S c_L = L)$

The Heading rate  $\xi$  can be found with a force balance with the centripetal force (and  $\dot{\psi} \approx \xi$ ). Note:  $\xi = V/R \Leftrightarrow v = r \cdot \omega$

$$F_{cent} = m \frac{V_G^2}{R} = L \cdot \sin \phi \quad \& \quad L \cdot \cos \phi = mg \Rightarrow \dot{\psi} = \xi = \frac{g}{V} \tan \phi$$

**Lateral-directional path**



### 6.4.2 TECS - Total Energy Control System

$$\dot{E}_{spec} = \frac{\dot{V}}{g} + \sin \gamma \approx \frac{\dot{V}}{g} + \gamma, \quad \dot{E}_{dist} = \gamma - \frac{\dot{V}}{g}$$

'Potential energy rate minus kinetic energy rate'