Homework 2

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Question 5

1.12.2: Proving arguments are valid using rules of inference.

(1b)
$$\begin{array}{l} p \to (q \land r) \\ \underline{\neg q \qquad\qquad} \\ \therefore \neg p \end{array}$$

1.	$p \to (q \land r)$	Hypothesis
2.	$\neg p \lor (q \lor r)$	Conditional Identity, 1
3.	$(\neg p \lor q) \land (\neg p \lor r)$	Distributive Laws, 2
4.	$(\neg p \lor q)$	Simplification, 3
5.	$q \lor \neg p$	Commutative Laws, 4
6.	$\neg q$	H2
7.	$\neg p$	Disjunctive Syllogism, 5, 6

(1e)
$$\begin{array}{l} p \lor q \\ \neg p \lor r \\ \underline{\neg q \qquad} \\ \therefore r \end{array}$$

1.	$p \lor q$	H1
2.	$\neg p \lor r$	H2
3.	$q \lor r$	Resolution, 1, 2
4.	$\neg q$	H3
5.	r	Disjunctive Syllogism, 3, 4

1.12.3: Proving the rules of inference using other rules.

(2c)
$$\begin{array}{l} p \lor q \\ \underline{\neg p \qquad} \\ \therefore q \end{array}$$

1.	$p \lor q$	H1
2.	$\neg\neg p \lor r$	Double Negation, 1
3.	$\neg p \to q$	Conditional Identity 2
4.	$\neg p$	H2
5.	q	Modus Ponens, 3, 4

1.12.5: Proving arguments in English are valid or invalid.

(3c)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.

$\therefore$ I will not buy a new car.

- j: I will get a job
- c: I will buy a new car
- h: I will buy a new house

The form of the argument is
$$\begin{array}{l} (c \wedge h) \to j \\ \neg j \\ \hline \therefore \neg c \end{array}$$

The argument is not valid. When c = T and h = j = F, the hypotheses are both true and the conclusion $\neg c$ is false.

(3d)

I will buy a new car and a new house only if I get a job.
I am not going to get a job.
I will buy a new house

$\therefore$ I will not buy a new car.

The form of the argument is
$$\begin{array}{l} (c \wedge h) \to j \\ \neg j \\ h \\ \hline \therefore \neg c \end{array}$$

The argument is valid:

1.	$(c \wedge h) \to j$	H1
2.	$\neg r$	H2
3.	$\neg(c \wedge h)$	Modus Tollens 2, 1
4.	$\neg c \wedge \neg h$	De Morgan's Law
5.	$\neg h \wedge \neg c$	Commutative Law, 4
6.	h	H3
7.	$\neg\neg q$	Double Negation, 6
8.	$\neg p$	Disjunctive Syllogism, 5, 7

1.13.3: Show an argument with quantified statements is invalid.

(1b)

$$\begin{array}{l} \exists x(P(x) \vee Q(x)) \\ \exists x \neg Q(x) \\ \hline \therefore \exists x P(x) \end{array}$$

	P	Q
a	F	T
b	F	F

$\exists x(P(x) \lor Q(x))$ is true because $Q(a)$ is true.

$\exists x \neg Q(x)$ is true because when x = a, $\neg Q(a)$ is true.

However, since $P(a)$ = P(b) = F, $\exists x P(X)$ is false.

$\therefore$ both hypotheses are true and the conclusion is false.

1.13.5: Determine and prove whether an argument in English is valid or invalid.

(2d)

- D(x): x got a detention.
- M(x): x missed class.

Every student who missed class got a detention.
Penelope is a student in the class.
Penelope did not miss class.

$\therefore$ Penelope did not get a detention.

The form of the argument is:

$\forall x(M(x) \to D(x))$
$Penelope$, a student in the class
$\neg M(Penelope)$

$\therefore \neg D(Penelope)$

The argument is not valid. $Penelope$ is a student in the class (H2). Consider $M(Penelope) = F$ and $D(Penelope) = T$, the hypotheses are all true regardless of the truth value of $D(Penelope)$ and the conclusion is false.

(2e)

- M(x): x missed class.
- D(x): x got a detention.
- A(x): x got an A.

Every student who missed class or got a detention did not get an A.
Penelope is a student in the class.
Penelope got an A.

$\therefore$ Penelope did not get a detention.

The form of the argument is:

$\forall x((M(x) \vee D(x)) \to \neg A(x))$
$Penelope$, a student in the class
$A(Penelope)$

$\therefore \neg D(Penelope)$

The argument is valid:

1.	$(c \wedge h) \to j$	H1
2.	$\neg r$	H2
3.	$\neg(c \wedge h)$	Modus Tollens 2, 1
4.	$\neg c \wedge \neg h$	De Morgan's Law
5.	$\neg h \wedge \neg c$	Commutative Law, 4
6.	h	H3
7.	$\neg\neg h$	Double Negation, 6
8.	$\neg c$	Disjunctive Syllogism, 5, 7

Question 6

2.2.1: Proving conditional statements with direct proofs.

(d) The product of two odd integers is an odd integer.

***Proof* :**
Direct Proof. Assume p and q are two odd integers. We will show that $p \cdot q$ is an odd integer.

Since both p and q are odd integers:
$p = 2n + 1$ for some integer n
$q = 2k + 1$ for some integer k

$$\begin{aligned}(2n+1)\cdot(2k+1) &= 2n \cdot 2k + 2k + 2n + 1 \\ &= 4nk + 2k + 2n + 1 \\ &= 2(2nk + k + n) + 1\end{aligned}$$

Since n and k are both odd integers, then $2nk + k + n$ is also an integer. Now we can write that $p \cdot q = 2m + 1$ where $m = 2nk + k + n$ is an odd integer.

(c) If x is a real number and $x \leq 3$, then $12 - 7x + x2 \geq 0$.

***Proof* :**
Direct proof. Assume $x \leq 3$ and x is a real number. We must isolate x to get $0 \leq 3 - x$ then add 1 to get $4 - x \geq 1$ which completes the polynomial.

$$\begin{aligned}&= xx + x(-3) + (-4)x + (-4)(-3) \\ &= xx - 3x - 4x + 4 \cdot\ 3 \\ &= x^2 - 7x + 12 \\ &12 - 7x + x2 \geq 0\end{aligned}$$

Question 7

2.3.1: Proving conditional statements by contrapositive.

(d) For every integer n, if $n^2 - 2n + 7$ is even, then n is odd.

Proof **:** Proof by contrapositive. If n is even we need to prove $n^2 - 2n + 7$ is odd.
Assume n is even, $n = 2k$, for some integer k.

Plug $n = 2k$ into $n^2 - 2n + 7$ which yields $4k^2 - 4k + 7$.

$$\begin{aligned} n &= 2k \\ n^2 - 2n + 7 &= (2k^2) - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(2k^2 - 2k + 3) + 1 \end{aligned}$$

Because k is an integer, then $(2k^2 - 2k + 3)$ is also an integer.
Since $(2k^2 - 2k + 3) + 1$ is $= 2c + 1$
$\therefore$ if n is even then n$^2 - 2n + 7$ is odd.

(f) For every non-zero real number x, if x is irrational, then $\frac{1}{x}$ is also irrational.

Proof **:** Proof by contrapositive. Assume $\frac{1}{x}$ is not irrational we need to prove that x is rational.

Because x is a non zero real number then $\frac{1}{x}$ must be a real number.

Since $\frac{1}{x}$ is NOT irrational, then $\frac{1}{x} = \frac{p}{q}$ for integers p and q

Reciprocal on both sides and you get $x = \frac{q}{p}$ where q and p are both $\neq 0$ and can be simplified.
$\therefore$ x is rational

(g)For every pair of real numbers x and y, if $x^3 + xy^2 \leq x^2y + y^3$, then $x \leq y$.

Proof **:**
Proof by contrapositive. We assume $x > y$ we need to show $x^3 + xy^2 > x^2y + y^3$.

Since $x > y$ then x and y cannot be both zero, however, $x^2 + y^2$ must be greater than 0.

$$\begin{aligned} &= (x > y) \cdot x^2 + y^2 \\ &= (x^2 + y^2) > y(x^2 + y^2) \end{aligned}$$

$$= x^3 + xy^2 > x^2y + y^3$$

(l) For every pair of real numbers x and y, if $x + y > 20$, then $x > 10$ or $y > 10$.

Proof **:**
Proof by contrapositive. We assume $x \leq 10$ and $y \leq 10$ we need to prove that $x + y \leq 20$

Since both $x \leq 10$ and $y \leq 10$, we add both and get $x + y \leq 20$

Question 8

2.4.1: Proofs by contradiction.

(c) The average of three real numbers is greater than or equal to at least one of the numbers

***Proof* :**
Proof by contradiction. Assume the average of x, y, z is smaller than the three real numbers.

$\frac{x+y+z}{3} < x$, $\frac{x+y+z}{3} < y$, $\frac{x+y+z}{3} < z$ Are three inequalities that when you add all three:

$$\begin{aligned} \frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} &< x+y+z \\ \frac{3x+3y+3z}{3} &< x+y+z \\ x+y+z &< x+y+z \end{aligned}$$

The contradiction in the assumption that the average of three numbers is less than all of the three numbers $3(x+y+z) < 3x+3y = 3z$. Therefore the original statement of the average of three real numbers is greater than or equal to at least one of the numbers must be true.

(e) There is no smallest integer.

***Proof* :**
Proof by contradiction. Assume there is a smallest integer x

Subtracting 1 from x to get $y = x - 1$ which is smaller than our smallest integer x. This contradicts the assumption, therefore, there is no smallest integer must be true.

Question 9

2.5.1: Proofs by cases.

(c) If integers x and y have the same parity, then $x + y$ is even.

***Proof* :**
Proof by cases.

***Case* 1 :** x and y are even.

$x = 2k$ for integer k
$y = 2j$ for integer j
$x + y = 2k + 2j$
$x + y = 2(k + j)$

Since k and j are both integers $k + j$ is also an integer, then $x + y = 2(k + j)$ must be even.

***Case* 2 :** x and y are odd.

$x = 2p + 1$ for some integer p
$y = 2q + 1$ for some integer q

Adding both x and y
$x + y = 2p + 1 + 2q + 1$
$x + y = 2p + 2q + 2$
$x + y = 2(p + q + 1)$

Since p and q are both integers $l = p + q + 1$ is also an integer. Since $x + y$ can be expressed as 2 times some integer, $x + y$ must be even.