REPORT FOR THE ELE2024 COURSEWORK

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1. Part A: Control Theory

Given Equations. The following equations were given by Dr P. Soposakis as part of the coursework brief:

$$L = L_0 + L_1 e(-\alpha y) \tag{1}$$

$$F_{mag} = c\frac{I^2}{y^2} \tag{2}$$

$$y = \delta - x \tag{3}$$

This is the equation that we have used:

Report/figures/main_diagram.png

Problem A1. Use first principles from physics, as well as Equations (1) and (2), to derive a system of ordinary differential equations that describes how the input voltage, V , affects the position, x, of the ball on the inclined plane. Note: introduce an inertial frame of reference where counterclockwise rotations are positive.

$$\begin{aligned} F_{spring} &= k(x-d) \quad (4)\\ F_{damper} &= b\dot{x} \quad (5)\\ F_{mag} &= c\frac{I^2}{y^2}, \text{ where } y = d - x \quad (6)\end{aligned}$$

$$\begin{aligned} -Tr &= I\ddot{\theta} \quad (7)\\ a = \ddot{x} &= \ddot{\theta}r\\ \therefore T &= \frac{I\ddot{\theta}}{r} \quad (8)\\ \ddot{\theta} &= \frac{\ddot{x}}{r} \quad (9)\\ I &= \frac{2}{5}mr^2 \quad (10)\end{aligned}$$

Using Equations (**??**), (**??**) and (**??**):

$$\begin{aligned} T &= -\frac{2mr^2\ddot{x}}{5r^2}\\ T &= -\frac{2m\ddot{x}}{5} \quad (11)\end{aligned}$$

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Some note goes here. Version 0.0.1. Last updated: January 9, 2021.

$$F = m\ddot{x}$$
$$F_{mag} + mg\sin\phi - T - F_{spring} - F_{damper} = m\ddot{x}$$

Using Equations (??), (??), (??) and (??):

$$\frac{cI^2}{y^2} + mg\sin\phi + \frac{2m\ddot{x}}{5} - k(x-d) - b\dot{x} = m\ddot{x} \tag{12}$$

$$V_R = IR \tag{13}$$
$$V_L = L\dot{I} \tag{14}$$
$$L = L_0 + L_1^{-\alpha y}, \text{ where } y = d - x \tag{15}$$

$$V = V_R + V_L$$

Using (??) and (??):

$$\therefore V = IR + L\dot{I}$$
$$V = IR + (L_0 + L_1^{-\alpha y})\dot{I}$$

$$V = IR + (L_0 + L_1^{-\alpha(d-x)})\dot{I} \tag{16}$$

Now going back to the system described in Equation (??):

$$\frac{cI^2}{y^2} + mg\sin\phi + \frac{2m\ddot{x}}{5} - k(x-d) - b\dot{x} = m\ddot{x}$$
$$\frac{cI^2}{y^2} + mg\sin\phi - k(x-d) - b\dot{x} = m\ddot{x} - \frac{2m\ddot{x}}{5}$$
$$\frac{cI^2}{y^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\ddot{x}$$

$$\frac{cI^2}{(d-x)^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\ddot{x} \tag{17}$$

Problem A2. Write the system you determined in Problem A1 in a state space representation. What are the states and inputs of the system?

$$V = IR + (L_0 + L_1^{-\alpha(d-x)})\dot{I} \tag{18}$$

$$I_1 = I$$
$$\therefore \dot{I} = \dot{I}_1$$
$$V = IR + (L_0 + L_1^{-\alpha(d-x)})\dot{I}$$
$$V = IR + (L_0 + L_1^{-\alpha(d-x)})\dot{I}_1$$
$$V - IR = (L_0 + L_1^{-\alpha(d-x)})\dot{I}_1$$
$$V - IR = (L_0 + L_1^{-\alpha(d-x)})\dot{I}_1$$
$$\therefore \dot{I} = \frac{V - IR}{(L_0 + L_1^{-\alpha(d-x)})} \tag{19}$$

$$\frac{cI^2}{(d-x)^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\ddot{x} \tag{20}$$

$$x_1 = x, x_2 = \dot{x_1}, \therefore \ddot{x} = \dot{x_2}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ I \end{bmatrix} \therefore \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{I} \end{bmatrix}$$

$$\frac{cI^2}{(d-x)^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\dot{x_2}$$

$$\frac{5(\frac{cI^2}{(d-x)^2} + mg\sin\phi - k(x-d) - b\dot{x})}{3m} = \dot{x_2}$$

$$\therefore \dot{\boldsymbol{x}} = \begin{bmatrix} x_2 \\ \frac{5(\frac{cI^2}{(d-x)^2} + mg\sin\phi - k(x-d) - b\dot{x})}{3m} \\ \frac{V-IR}{(L_0+L_1^{-\alpha(d-x)})} \end{bmatrix} \tag{21}$$

Problem A3. *Characterise the equilibrium points of the system using Proposition 3.4 in the textbook.*
A pair of equilibrium points $(x^{\mathrm{eq}}, F^{\mathrm{eq}})$ can be defined for the system $f(x, F)$ if $f(x^{\mathrm{eq}}, F^{\mathrm{eq}}) = 0$

$$\therefore \text{Equilibrium points} = \begin{bmatrix} x_2^{\mathrm{eq}} & = 0 \\ \frac{5(\frac{cI^2}{(d-x_1)^2} + mg\sin\phi - k(x_1^{\mathrm{eq}} - d))}{3m} & = 0 \end{bmatrix} \tag{22}$$

Problem A4. *Linearise the system at an equilibrium point (use deviation variables).*

$$\dot{x_2} = \frac{CI^2}{(d-x_1^{\mathrm{eq}})^2} + \frac{5}{3}g\sin\phi - \frac{5k}{3m}(x_1 - d) - \frac{5b}{3m}x_2$$

$$\dot{x_2} = \frac{5c}{3m}(\frac{I^2}{(d-x_1)^2} - \frac{I^{\mathrm{eq}2}}{(d-x_1^{\mathrm{eq}})^2}) - \frac{5k}{3m}(x_1 - x_1^{\mathrm{eq}}) - \frac{5b}{3m}x_2$$

$$\dot{x_2} = \frac{5c}{3m}(\frac{2I^2}{(d-x_1)^2}(I - I^{\mathrm{eq}}) - \frac{2I^{\mathrm{eq}2}}{(d-x_1^{\mathrm{eq}})^3}(x_1 - x_1^{\mathrm{eq}})) - \frac{5k}{3m}(x_1 - x_1^{\mathrm{eq}}) - \frac{5b}{3m}x_2$$

$$\begin{aligned} \bar{I} &= I - I^{\mathrm{eq}} \\ \bar{x_1} &= x_1 - x_1^{\mathrm{eq}} \\ \bar{x} &= x_2 - x_2^{\mathrm{eq}} \\ &= x_2 \end{aligned}$$

$$\dot{x_2} = \frac{5c}{3m}(\frac{2I^{\mathrm{eq}}}{(d-x_1)^2}\bar{I} - \frac{2I^{\mathrm{eq}2}}{(d-x_1^{\mathrm{eq}})^3}\bar{x_1}) - \frac{5k}{3m}\bar{x_1} - \frac{5b}{3m}\bar{x_2}$$

$$\begin{aligned} g(I) &= I^2 & \qquad g(I^{\mathrm{eq}}) &= I^{\mathrm{eq}2} \\ g'(I) &= 2I & \qquad g'(I^{\mathrm{eq}}) &= 2I^{\mathrm{eq}} \end{aligned}$$

$$\begin{aligned} h(x_1) &= (d-x_1)^2 & \qquad h(x_1^{\mathrm{eq}}) &= (d-x_1^{\mathrm{eq}})^2 \\ h'(x_1) &= 2(d-x_1) & \qquad h'(x_1^{\mathrm{eq}}) &= 2(d-x_1^{\mathrm{eq}}) \end{aligned}$$

$$
\begin{aligned}
g(I) &\approx g(I^{\text{eq}}) + g'(I^{\text{eq}}) \cdot (I - I^{\text{eq}}) \\
I^2 &= I^{\text{eq}2} + 2I^{\text{eq}}(I - I^{\text{eq}}) \\
\therefore I^2 - I^{\text{eq}2} &= 2I^{\text{eq}}(I - I^{\text{eq}})
\end{aligned}
$$

$$
\begin{aligned}
h(x_1) &\approx h(x_1^{\text{eq}}) + h'(x_1^{\text{eq}}) \cdot (x_1 - x_1^{\text{eq}}) \\
(d - x_1^{\text{eq}})^2 &= (d - x_1^{\text{eq}})^2 + 2(d - x_1^{\text{eq}})^2(x_1 - x_1^{\text{eq}}) \\
\therefore (d - x_1^{\text{eq}})^2 - (d - x_1^{\text{eq}})^2 &= 2(d - x_1^{\text{eq}})^2(x_1 - x_1^{\text{eq}})
\end{aligned}
$$

$$
g(I, x_1) \approx g(I^{\text{eq}}, x_1^{\text{eq}}) + \frac{\delta g}{\delta I_{I^{\text{eq}}, x_1^{\text{eq}}}} \cdot (I - I^{\text{eq}}) + \frac{\delta g}{\delta x_{I^{\text{eq}}, x_1^{\text{eq}}}} \cdot (x_1 - x_1^{\text{eq}})
$$

$$
g(I, x_1) = \frac{I^2}{(d - x_1)^2} \qquad \frac{\delta g}{\delta I} = \frac{2I}{(d - x_1)^2} \qquad \frac{\delta g}{\delta x} = I^2 \cdot (d - x_1)^{-2} = \frac{2I^2}{(d - x_1)^3}
$$

$$
\begin{aligned}
g(I, x_1) &= g(I^{\text{eq}}, x_1^{\text{eq}}) + \frac{2I^{\text{eq}}}{(d - x_1^{\text{eq}})^2} \cdot (I - I^{\text{eq}}) + \frac{2I^{\text{eq}2}}{(d - x_1^{\text{eq}})^3} \cdot (x_1 - x_1^{\text{eq}}) \\
&= \frac{I^{\text{eq}2}}{(d - x_1^{\text{eq}})^2} + \frac{2I^{\text{eq}2}}{(d - x_1^{\text{eq}})^2} \cdot (I - I^{\text{eq}}) + \frac{2I^{\text{eq}2}}{(d - x_1^{\text{eq}})^3} \cdot (x_1 - x_1^{\text{eq}})
\end{aligned}
$$

$$
\begin{aligned}
\dot{x_2} &= \frac{5c}{3m}\left(\frac{2I^{\text{eq}2}}{(d - x_1^{\text{eq}})^2}\bar{I} + \frac{2I^{\text{eq}2}}{(d - x_1^{\text{eq}})^3}\bar{x_1}\right) - \frac{5k}{3m}\bar{x_1} - \frac{5b}{3m}\bar{x_2} \\
&= \frac{5c2I^{\text{eq}2}}{3m(d - x_1^{\text{eq}})^2}\bar{I} + \left(\frac{5c2I^{\text{eq}2}}{3m(d - x_1^{\text{eq}})^3} - \frac{5k}{3m}\right)\bar{x_1} - \frac{5b}{3m}\bar{x_2}
\end{aligned}
$$

$$
\dot{x_1} = x_2 \qquad 0 = x_2^{\text{eq}}
$$

$$
\therefore \dot{x_1} = x_2 - x_2^{\text{eq}} = \bar{x_2}
$$

$$
a = \frac{5c2I^{\text{eq}2}}{3m(d - x_1^{\text{eq}})^2} \qquad b = \left(\frac{5c2I^{\text{eq}2}}{3m(d - x_1^{\text{eq}})^3} - \frac{5k}{3m}\right) \qquad c = \frac{5b}{3m}
$$

$$
\dot{\overline{x_1}} = \overline{x_2} \qquad \dot{\overline{x_2}} = a\bar{I} + b\bar{x_1} - c\bar{x_2}
$$

$$
\begin{aligned}
\dot{\bar{I}} &= \frac{\bar{V} - \bar{I}R}{L_0 - L_1 e^{-\alpha(d-x)}} \\
f &= L_0 - L_1 e^{-\alpha(d-x)} \\
\therefore \dot{\bar{I}} &= \frac{1}{f}(\bar{V} - \bar{I}R)
\end{aligned}
$$

A pair of equilibrium points $(x^{\text{eq}}, F^{\text{eq}})$ can be defined for the system $f(x, F)$ if $f(x^{\text{eq}}, F^{\text{eq}}) = 0$

$$
\text{Equilibrium points} = \begin{bmatrix} x_2^{\text{eq}} & = 0 \\ \frac{5\left(\frac{cI^2}{(d-x_1)^2} + mg\sin\phi - k(x_1^{\text{eq}} - d)\right)}{3m} & = 0 \end{bmatrix}
$$

$$\dot{\bar{I}} = \frac{V - I_1 R}{L_0 - L_1 e^{-\alpha(d-x)}} => \dot{I} = f(I, V)$$
$$f(I^{\text{eq}}, V^{\text{eq}}) = 0$$
$$\frac{V^{\text{eq}} - I^{\text{eq}} R}{L_0 - L_1 e^{-\alpha(d-x)}} = 0$$

$$\text{Equilibrium points} = \begin{bmatrix} V - I_1 R & = 0 \end{bmatrix}$$

$$\bar{V} = V - V^{\text{eq}} \qquad \bar{I} = I - I^{\text{eq}}$$

$$\dot{I} = \frac{(V - V^{\text{eq}}) - R(I - I^{\text{eq}})}{L_0 - L_1 e^{-\alpha(d-x)}}$$

$$\dot{I} = \frac{\bar{V} - \bar{I}R}{L_0 - L_1 e^{-\alpha(d-x)}}$$

Problem A5. *Determine the transfer function of the linearised system that you determined in Problem A4. The input of the system is the voltage across the circuit and the output is the position of the ball on the inclined plane. How many poles does this transfer function have? Derive sufficient conditions on the system parameters under which the impulse response of the transfer function is oscillatory.*

$$\dot{\bar{x}}_1 = \bar{x}_2$$
$$\dot{\bar{x}}_2 = a\bar{I} + b\bar{x}_1 - c\bar{x}_2$$
$$\dot{\bar{I}} = \frac{1}{f}(\bar{V} - \bar{I}R) = \frac{\bar{V}}{f} - \frac{\bar{I}R}{f}$$

$$s\bar{X}_1 = \bar{X}_2$$
$$s\bar{X}_2 = a\bar{I} + b\bar{X}_1 - c\bar{X}_2$$
$$s\bar{I} = \frac{\bar{V}}{f} - \frac{\bar{I}R}{f}$$
$$s\bar{I} + \frac{\bar{I}R}{f} = \frac{\bar{V}}{f}$$
$$\left(s + \frac{R}{f}\right)\bar{I} = \frac{\bar{V}}{f}$$
$$\bar{I} = \frac{\bar{V}}{f\left(s + \frac{R}{f}\right)}$$
$$= \frac{\bar{V}}{fs + R} \tag{23}$$

$$\begin{aligned}
s\left(s\bar{X}_1\right) &= a\bar{I} + b\bar{X}_1 - c\left(s\bar{X}_1\right)\\
s^2\bar{X}_1 &= a\bar{I} + b\bar{X}_1 - cs\bar{X}_1\\
s^2\bar{X}_1 - b\bar{X}_1 + cs\bar{X}_1 &= a\bar{I}\\
\therefore \text{As an Equation (23)} => (s^2 - b + cs)\bar{X}_1 &= a\left(\frac{\bar{V}}{fs+R}\right)\\
&= \bar{V}\left(\frac{a}{fs+R}\right)
\end{aligned}$$

$$\begin{aligned}
G(s) &= \frac{X(s)}{U(s)}\\
X(s) = \text{displacement} &\qquad U(s) = \text{voltage}\\
\frac{\bar{X}_1}{\bar{V}} &= \frac{a}{(fs+R)(s^2-b+cs)}\\
fs+R &= 0\\
&\therefore s = \frac{-R}{f}
\end{aligned} \tag{24}$$

$$\begin{aligned}
s^2 + cs - b &= 0\\
s &= \frac{-c \pm \sqrt{c^2 - 4(1)(-b)}}{2(1)}\\
&= \frac{-c \pm \sqrt{c^2+4b}}{2}\\
\therefore s &= \frac{-c}{2} + \frac{\sqrt{c^2+4b}}{2}\\
\& \; s &= \frac{-c}{2} - \frac{\sqrt{c^2+4b}}{2}
\end{aligned}$$

(25)

(26)

Part B: Analysis and Controller Design

Problem B1.

Problem B2.

Problem B3.

Problem B4.

Problem B5.

Problem B6.

Part C: Bonus Questions

Problem C1. Determine the Laplace transform of $\mathrm{f(t)} = \log\left(t^3\right)$, $t > 0$.

The solution is:

$$\left(\frac{\sqrt{3}\left(-G_{5,2}^{2,3}\left(\begin{matrix}\frac{2}{3}, \frac{1}{3}, 0 & 1, 1\\ 0, 0 & \end{matrix}\middle|\frac{27}{s^3}\right) + G_{5,2}^{0,5}\left(\begin{matrix}\frac{2}{3}, \frac{1}{3}, 0, 1, 1 & \\ & 0, 0\end{matrix}\middle|\frac{27}{s^3}\right)\right)}{2\pi s}, \ 0, \ \mathrm{True}\right) \tag{27}$$

Problem C2. Determine the Laplace transform of $\mathrm{f(t)} = |\cos(\omega)|)$, $t \geq 0$, $\omega > 0$.

The solution is:

$$\left(\frac{|\cos(\omega)|}{\omega}, \ 0, \ \mathrm{True}\right) \tag{28}$$

Part D: Planning, Organisation & Collaboration

D1. We used a GitHub repository to collaborate on writing code. We used the git issue tracker to keep track of the tasks. This issue tracker was critical as it allowed us to see how the tasks were assigned and how those tasks were progressing. We distributed the tasks on an initial call where we took problems that we felt confident with. These changed as we went on and might have found some problems tougher than we expected. Regular communication was the key to ensuring we were all aware of the progress on projects. The link to the issue tracker can be found here.

D2.

Communication. We communicated clearly and regularly. This meant we were all aware of the progress of the coursework and what remained to be done.

Organisation. We began early and kept on top of our work. This was important as work from other modules began to pile up nearing the end of Semester 1. Having an issue tracker was key to the organisation behind the project.

Education. When one person was struggling with a part of the coursework, or didn't understand the method behind a solution, we were able to explain our understandings of it. This definitely helps with our understanding of the theory from this module going forward.

D3. The restrictions that exist due to Covid-19 was the main challenge presented to us. The fact that we were physically apart meant communication was limited to calls, texts and emails. Because these forms of communication are inherently more limited than face-to-face conversing it was important that we we communicate, we clearly define the goals of it. Things like the issue tracker in GitHub was key to ensuring we were all on the same page when it came to progress and outstanding tasks.