REPORT FOR THE ELE2024 COURSEWORK

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1. Part A: Control Theory

Given Equations. The following equations were given by Dr P. Soposakis as part of the coursework brief:

$$L = L_0 + L_1 \exp(-\alpha y) \tag{1}$$

$$F_{mag} = c\frac{I^2}{y^2} \tag{2}$$

where y = δ - x

Problem A1.

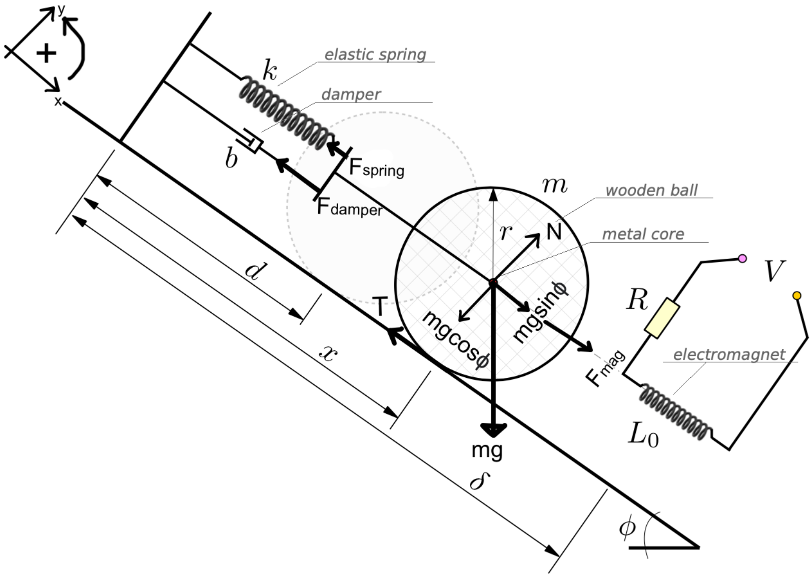

Figure 1. System of a wooden ball on n inclined plane that can be attracted by an electromagnet, controlled by a voltage V, with forces applied labelled.

$$F_{spring} = k(x - d) \tag{3}$$
$$F_{damper} = b\dot{x} \tag{4}$$

$$Tr = -I\ddot{\theta} \therefore T = -\frac{I\ddot{\theta}}{r} \tag{5}$$

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Some note goes here. Version 0.0.1. Last updated: January 21, 2021.

$$I = \frac{2}{5} m r^2 \tag{6}$$

$$\ddot{\theta} = \frac{\ddot{x}}{r} \tag{7}$$

Sub (6) and (7) into (5):

$$T = -\frac{2m\ddot{x}}{5} \tag{8}$$

Apply Newtons 2nd Law of Motion to the system:

$$F_{mag} + mg\sin\phi - T - F_{spring} - F_{damper} = m\ddot{x} \tag{9}$$

Sub (2), (3), (4) and (8) into (9):

$$\frac{cI^2}{y^2} + mg\sin\phi + \frac{2m\ddot{x}}{5} - k(x-d) - b\dot{x} = m\ddot{x}$$

$$(\text{ODE})\frac{cI^2}{(\delta - x)^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\ddot{x} \tag{10}$$

Apply Kirchoff's Voltage Law to system circuit:

$$V = V_R + V_L \tag{11}$$

$$V_R = IR \tag{12}$$

$$V_L = L\dot{I} \tag{13}$$

Sub (12), (13) into (11):

$$V = IR + L\dot{I} \tag{14}$$

Sub (1) into (14):

$$(\text{ODE})V = IR + (L_0 + L_1\exp(-\alpha y))\dot{I} \tag{15}$$

Equations (10) and (15) are the ordinary differential equations for the system shown in Figure 1.

Problem A2.
For Equation (10), let $x_1 = x, x_2 = \dot{x}$ and $\dot{x_2} = \ddot{x}$

$$\frac{cI^2}{(\delta - x_1)^2} + mg\sin\phi - k(x_1 - d) - bx_2 = \frac{3m\dot{x_2}}{5}$$

$$\frac{5cI^2}{3m(\delta - x_1)^2} + \frac{5g\sin\phi}{3} - \frac{5k}{3m}(x_1 - d) - \frac{5bx_2}{3m} = \dot{x_2}$$

For Equation (15), let $I_1 = I$ and $\dot{I_1} = \dot{I}$:

$$V = I_1 R + (L_0 + L_1 \exp -\alpha y)\,\dot{I_1}$$

State Space Representation:

$$\therefore \dot{\boldsymbol{x}} = \begin{bmatrix} x_2 \\ \frac{5cI^2}{3m(\delta - x_1)^2} + \frac{5g\sin\phi}{3} - \frac{5k}{3m}(x_1 - d) - \frac{5bx_2}{3m} \\ \frac{V - I_1 R}{(L_0 + L_1\exp -\alpha y} \end{bmatrix} \tag{16}$$

The State variable of Equation (16) is $x = (x_1, x_2, I)$.
The input signal of Equation (16) is V, the voltage applied across the electromagnet.

Problem A3.
A point (x_1^e, x_2^e, I^e, V^e) is an equilibrium point of the system if:

$$\begin{cases} x_2^e & = 0 \\ \frac{5cI^{e2}}{3m(\delta - x_1^e)^2} + \frac{5g\sin\phi}{3} - \frac{5k}{3m}(x_1^e - d) & = 0 \\ \frac{V^e - I_1^e R}{(L_0 + L_1\exp -\alpha y)} & = 0 \end{cases} \tag{17}$$

Problem A4.
Subtract by parts Equations (16) and (17):

$$
\begin{aligned}
\dot{x_1} &= x_2 \\
- \qquad 0 &= x^e \\
\hline
\therefore \dot{\bar{x_1}} &= x_2 - x_2^e \\
&= \bar{x_2}
\end{aligned}
$$

$$
\begin{aligned}
\dot{x_2} &= \frac{5cI^2}{3m(\delta - x_1)^2} + \frac{5g\sin\phi}{3} - \frac{5k}{3m}(x_1 - d) - \frac{5bx_2}{3m} \\
- \qquad 0 &= \frac{5cI^{e2}}{3m(\delta - x_1^e)^2} + \frac{5g\sin\phi}{3} - \frac{5k}{3m}(x_1^e - d)
\end{aligned}
$$

$$
\dot{\bar{x_2}} = \frac{5c}{3m}\left(\frac{I^2}{(\delta - x_1)^2} - \frac{I^{e2}}{(\delta - x_1^e)^2}\right) - \frac{5k}{3m}(x_1 - x_1^e) - \frac{5b}{3m}x_2 \tag{18}
$$

Apply Taylor's Theorem to Equation (18):

$$
g(I, x_1) \approx g(I^e, x_1^e) + \frac{\partial g}{\partial I_{I^e, x_1^e}} \cdot (I - I^e) + \frac{\partial g}{\partial x_{I^e, x_1^e}} \cdot (x_1 - x_1^e) \tag{19}
$$

$$
g(I, x_1) = \frac{I^2}{(\delta - x_1)^2} \tag{20}
$$

$$
\frac{\partial g}{\partial I} = \frac{2I}{(\delta - x_1)^2} \tag{21}
$$

$$
\frac{\partial g}{\partial x_1} = \frac{2I^2}{(\delta - x_1)^3} \tag{22}
$$

Sub Equations (20),(21) and (22) into (19):

$$
\frac{I^2}{(\delta - x_1)^2} = \frac{I^{e2}}{(\delta - x_1^e)^2} + \frac{2I^e}{(\delta - x_1^e)^2} \cdot (I - I^e) + \frac{2I^{e2}}{(\delta - x_1^e)^3} \cdot (x_1 - x_1^e)
$$

$$
\frac{I^2}{(\delta - x_1)^2} - \frac{I^{e2}}{(\delta - x_1^e)^2} = \frac{2I^e}{(\delta - x_1^e)^2} \cdot (I - I^e) + \frac{2I^{e2}}{(\delta - x_1^e)^3} \cdot (x_1 - x_1^e) \tag{23}
$$

Sub Equation (23) into Equation (18):

$$
\dot{\bar{x_2}} = \frac{5c}{3m}\left(\frac{2I^e}{(\delta - x_1^e)^2} \cdot (I - I^e) + \frac{2I^{e2}}{(\delta - x_1^e)^3} \cdot (x_1 - x_1^e)\right) - \frac{5k}{3m}(x_1 - x_1^e) - -\frac{5b}{3m}(x_2) \tag{24}
$$

Introduce deviation variables:

$$
\begin{aligned}
\bar{I} &= I - I^e \\
\bar{x_1} &= x_1 - x_1^e \\
\bar{x_2} &= x_2 - x_2^e = x_2 - 0 = x_2
\end{aligned}
$$

to Equation (24):

$$\dot{\bar{x_2}} = \frac{5c}{3m}\left(\frac{2I^e}{(\delta - x_1^e)^2}\cdot \bar{I} + \frac{2I^{e2}}{(\delta - x_1^e)^3}\cdot \bar{x_1}\right) - \frac{5k}{3m}(\bar{x_1}) - \frac{5b}{3m}(\bar{x_2})$$
$$\dot{\bar{x_2}} = \frac{5c2I^e}{(3m\delta - x_1^e)^2}\cdot \bar{I} + \frac{10cI^{e2}}{3m(\delta - x_1^e)^3}\cdot \bar{x_1} - \frac{5k}{3m}(\bar{x_1}) - \frac{5b}{3m}(\bar{x_2})$$
$$\dot{\bar{x_2}} = \frac{5c2I^e}{(3m\delta - x_1^e)^2}\cdot \bar{I} + \left(\frac{10cI^{e2}}{3m(\delta - x_1^e)^3} - \frac{5k}{3m}\right)(\bar{x_1}) - \frac{5b}{3m}(\bar{x_2}) \tag{25}$$

For Equation (25), let :

$$a = \frac{5c2I^e}{(3m\delta - x_1^e)^2}, b = \left(\frac{10cI^{e2}}{3m(\delta - x_1^e)^3} - \frac{5k}{3m}\right) c = \frac{5b}{3m}$$

Subtract by parts (**??**) and (**??**) :

$$\dot{I} = \frac{V - I_1 R}{(L_0 + L_1 \exp -\alpha y)}$$
$$— \; 0 = \frac{V^e - I_1^e R}{(L_0 + L_1 \exp -\alpha (d - x))}$$
$$\overline{}$$
$$\dot{\bar{I}} = \frac{V - V^e - I_1 R - I_1^e R}{(L_0 + L_1 \exp -\alpha y)} \tag{26}$$

Introduce deviation variables,

$$\bar{V} = V - V^e$$
$$\bar{I} = I_1 - I_1^e$$

to Equation (26):

$$\dot{\bar{I}} = \frac{\bar{V} - \bar{I_1} R}{(L_0 + L_1 \exp -\alpha y)} \tag{27}$$

Let $f = L_0 + L_1 \exp(-\alpha y)$ for Equation (27)

The Linearised system equations are:

$$\dot{\bar{x_1}} = \bar{x_2} \tag{28}$$
$$\dot{\bar{x_2}} = a\bar{I} + b\bar{x_1} - c\bar{x_2} \tag{29}$$
$$\dot{\bar{I}} = \frac{1}{f}(\bar{V} - \bar{I}R) \tag{30}$$

Problem A5.

Apply Laplace Transform to Equations (28),(29) and (30):

$$s\bar{X}_1 = \bar{X}_2 \tag{31}$$
$$s\bar{X}_2 = a\bar{I} + b\bar{X}_1 - c\bar{X}_2 \tag{32}$$
$$s\bar{I} = \frac{\bar{V}}{f} - \frac{\bar{I}R}{f} \tag{33}$$

Sub Equation (31) into (32):

$$\begin{aligned} s(s\bar{X}_1) &= a\bar{I} + b\bar{X}_1 - c(s\bar{X}_1) \\ s^2\bar{X}_1 &= a\bar{I} + b\bar{X}_1 - cs\bar{X}_1 \\ a\bar{I} &= s^2\bar{X}_1 - b\bar{X}_1 + cs\bar{X}_1 \end{aligned} \tag{34}$$

Rearrange Equation (33):

$$\begin{aligned} s\bar{I} &= \frac{\bar{V}}{f} - \frac{\bar{I}R}{f} \\ sf\bar{I} + \bar{I}R &= \bar{V} \\ \bar{I}(sf+R) &= \bar{V} \\ \bar{I} &= \frac{\bar{V}}{(sf+R)} \end{aligned} \tag{35}$$

Sub Equation (35) into (34):

$$\begin{aligned} s^2\bar{X}_1 - b\bar{X}_1 + cs\bar{X}_1 &= a\frac{\bar{V}}{(sf+R)} \\ (s^2 - b + cs)\bar{X}_1 &= \bar{V}\frac{a}{(sf+R)} \end{aligned} \tag{36}$$

The Transfer Function for Equation (36) where $\bar{V}$ is the input,voltage across the circuit and $\bar{X}_1$ is the output of the system, the position of the ball on the inclined plane.

$$G(s) = \frac{\bar{X}_1(s)}{\bar{V}(s)} = \frac{a}{(sf+R)(s^2 - b + cs)} \tag{37}$$

Poles of Equation (37):

$$\frac{a}{(sf+R)(s^2 - b + cs)} = 0$$

$$\begin{aligned} sf + R &= 0 \\ s &= -\frac{R}{f} \end{aligned} \tag{38}$$

$$\begin{aligned} s^2 - b + cs &= 0 \\ s = \frac{-c \pm \sqrt{c^2 - 4(1)(-b)}}{2} & \\ &= \frac{-c \pm \sqrt{c^2 + 4b}}{2} \\ s = \frac{-c + \sqrt{c^2 + 4b}}{2}, &\frac{-c - \sqrt{c^2 + 4b}}{2} \end{aligned} \tag{39}$$

As shown in Equations (38) & (39), Equation (37) has 3 poles. Furthermore the presence of complex poles indicates that the impulse response is oscillatory.

Part B: Analysis and Controller Design

Problem B1.
The wooden ball is limited to where it can equilibrate by the physical limits of the system. x_{max} is the point furthest down the slope where the ball can equilibrate. This cannot be any further down the slope than the location of the electromagnet. This location is denoted on Figure 1 with δ.
$\therefore$ We can say that x_{max} can be no greater than δ.
The limit of where the ball could equilibriate on the upper end of the slope is denoted by x_{min}. d is the natural length of the spring, so x_{min} can be no less than that, but we also need to take into account the downward force of the ball. This downward force is denoted in Figure 1 with $mg\sin\phi$.
However, the ball is not free to roll unobstructed. Taking into account the stiffness of the spring k, we can see that x_{min} can be no smaller than $d + \frac{mg\sin\phi}{k}$.
$\therefore$ We can say that the system can only equilibrate at those positions x^e that satisfy:

$$d + \frac{mg\sin\phi}{k} < x^e < \delta \tag{40}$$

To calculate the equilibrium voltage and current we use Equation (10).

$$\frac{cI^2}{(\delta - x)^2} + mg\sin\phi - k(x-d) - b\dot{x} = \frac{3m}{5}\ddot{x}$$

To equilibriate, velocity and acceleration are equal to zero:

$$\begin{aligned} \therefore \frac{cI^2}{(\delta - x)^2} + mg\sin\phi - k(x-d) &= 0 \\ \frac{-cI^2}{(\delta - x)^2} &= mg\sin\phi - k(x-d) \\ \therefore I^e &= \sqrt{\frac{(mg\sin\phi) - k(x_1^e - d)}{-c}(\delta - x_1^e)^2} \end{aligned} \tag{41}$$

$$\begin{aligned} V^e &= I^e \cdot R \\ \therefore V^e &= \left(\sqrt{\frac{(mg\sin\phi) - k(x_1^e - d)}{-c}(\delta - x_1^e)^2}\right) R \end{aligned} \tag{42}$$

Problem B2.
The linearisation of the system was tested by writing simulating both systems in Python. The implementation of the simulations can be found here.

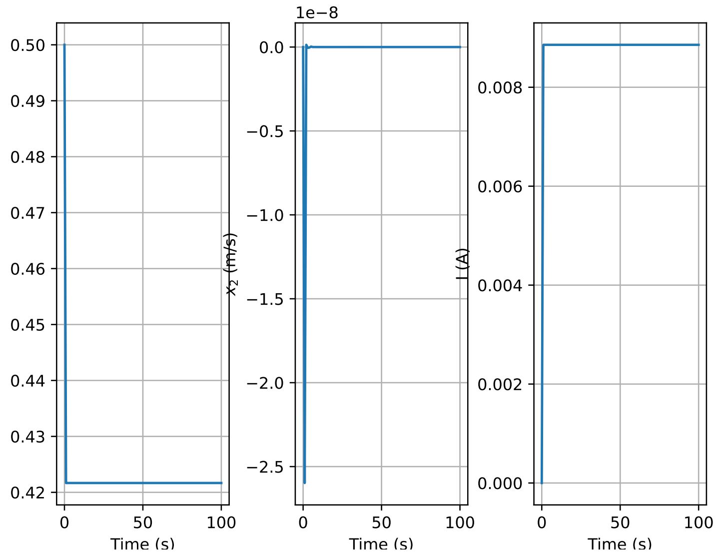

Figure 2. Three plots of the simulation of the non-linear system with the x position, $\dot{x}$ and I of the ball in respect to time.

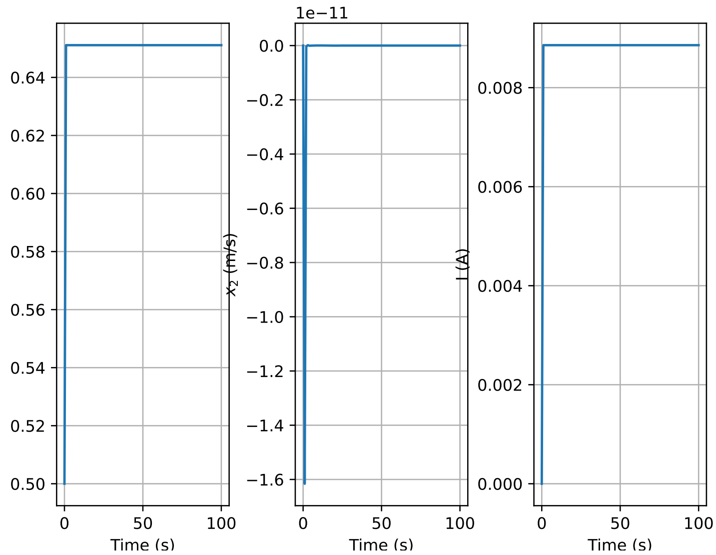

Figure 3. Three plots of the simulation of the linearised system with the x position, $\dot{x}$ and I of the ball in respect to time.

As can be seen in both Figure 2 and Figure 3 both simulations showed stable systems. However the non-linear system shown in Figure 2 does not behave close to the equilibrium point of the linearised system shown in Figure 3. The non-linear system shows an equilibrium point of x_1 at around 0.42m whereas the linearised system shows that the equilibrium point of x_1 is around 0.65m. It can therefore be determined, assuming that the simulation was programmed correctly, that there is an error in the linearisation. It is believed that this issue comes specifically from the linearisation of $\dot{I}$.

Problem B3.
A simulation was run to determine the impulse response and the step response of the transfer of the linearised system. This simulation was written in Python for which the code can be found here and which produced the Figure 4 below.

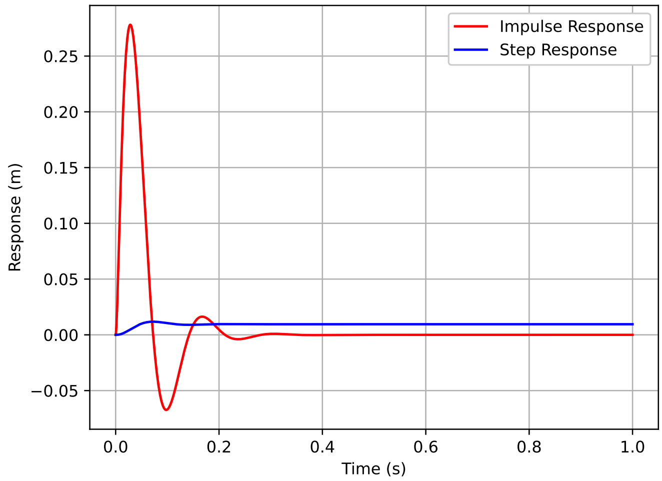

FIGURE 4. Graph showing both impulse response and step response of the transfer function of the linearised system.

Problem B4.
A simulation was run to determine the low-frequency and high-frequency asymptotes of the magnitude and phase lag of the transfer function of the linearised system. This simulation was written in Python for which the code can be found here and which produced the Figure 5 below.

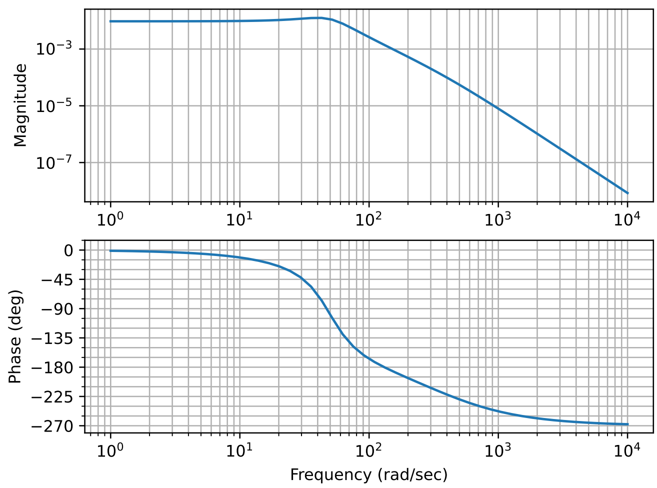

FIGURE 5. Graphs showing both low-frequency and high-frequency asymptotes of the magnitude and phase lag of the transfer function of the linearised system.

Problem B5.
The desired characteristics of a controller:
BIBO stability is a critical benchmark for our system to reach. To be exact, bounded changes in the set point should only cause bounded changes in the output
We should want the system to converge on a point quickly but with as little oscillation as possible. To do this the system would need to penalise large lateral velocities in the system. It would be preferable for the ball to move slower and reach the point with low oscillation than for it to come in contact with the physical boundaries of the system.
Evidence it is a good controller:
As George Box once said: "All models are wrong". What we aim for in a model may not be seen

in practise. The evidence we would have to show it is a good controller would be a very similar measurement from our laser proximity sensor as what our model would estimate. The *offset* should be as low as possible. The model also has to take into account the physical boundaries of the system that we explore in Problem B1.

Problem B6.
To control the system a Proportional Integral Differential (PID) controller was designed. The integral part of the PID controller will adjust for physical boundaries of the system and account for systematic errors and hopefully reduce and remove offset. This helps achieve the desired characteristic set out in Problem B5. The controller was written in Python and the implementaion can be found here. The step response from the simulated controlled system was recorded and is shown below in Figure 6.

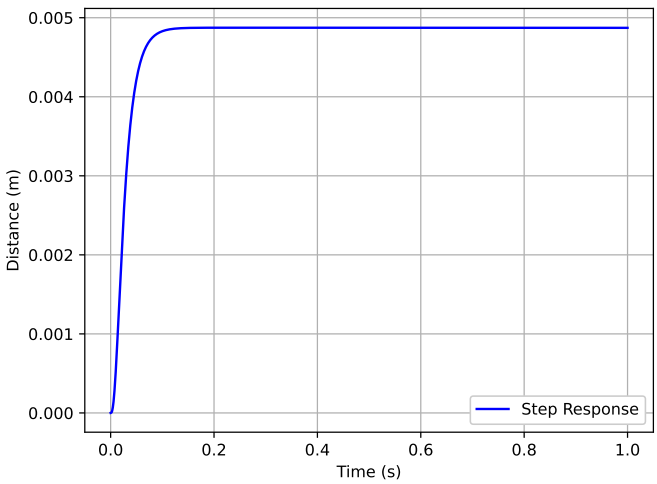

FIGURE 6. Graph to show the step response of the system with a PID controller implemented.

As Figure 6 shows the system's impulse response settles very quickly.

PART C: BONUS QUESTIONS

Problem C1. Determine the Laplace transform of $\mathrm{f(t)} = \log\left(t^3\right)$, $t > 0$.

The solution is:

$$\left(\frac{\sqrt{3}\left(-G_{5,2}^{2,3}\left(\begin{matrix}\frac{2}{3},\frac{1}{3},0 & 1,1\\ 0,0 & \end{matrix}\middle|\frac{27}{s^3}\right)+G_{5,2}^{0,5}\left(\begin{matrix}\frac{2}{3},\frac{1}{3},0,1,1 & \\ & 0,0\end{matrix}\middle|\frac{27}{s^3}\right)\right)}{2\pi s},\ 0,\ \text{True}\right) \tag{43}$$

Problem C2. Determine the Laplace transform of $\mathrm{f(t)} = |\cos(\omega)|$), $t \geq 0$, $\omega > 0$.

The solution is:

$$\left(\frac{|\cos(\omega)|}{\omega},\ 0,\ \text{True}\right) \tag{44}$$

Part D: Planning, Organisation & Collaboration

D1.
We used a GitHub repository to collaborate, sometimes with writing code but mainly with the writing of the report. We used the git issue tracker to keep track of the tasks. This issue tracker was critical as it allowed us to see how the tasks were assigned and how those tasks were progressing. We distributed the tasks on an initial call where we took problems that we felt confident with. These changed as we went on and might have found some problems tougher than we expected. Regular communication was the key to ensuring we were all aware of the progress on projects so a group chat was set up on social media as soon as the control groups were chosen. The link to the issue tracker can be found here.

D2.

Communication. We communicated clearly and regularly. This meant we were all aware of the progress of the coursework and what remained to be done.

Organisation. We began early having had our first meeting on the 16th November. This was important as work from other modules began to pile up nearing the end of Semester 1. Having an issue tracker was key to the organisation behind the project.

Education. When one person was struggling with a part of the coursework, or didn't understand the method behind a solution, we were able to explain our understandings of it. This definitely helps with our understanding of the theory from this module going forward.

D3.
The current global situation was a massive challenge to the group. Not only did it effect how the group met and communicated in which was the first group project started during the pandemic for all of us, but this period has also been a significant struggle and has effected academic performance for some of the members in the group.

The fact that we were physically apart meant communication was limited to calls, texts and emails. Because these forms of communication are inherently more limited than face-to-face conversing it was important that we we communicate, we clearly define the goals of it. Things like the issue tracker in GitHub was key to ensuring we were all on the same page when it came to progress and outstanding tasks. In hindsight weekly voice or video meetings should've been implemented as although the group had constant contact through social media it was difficult to stay motivated for some members in the group especially in the aforementioned global situation.

The coursework assignment was structured more linearly than expected which didn't inherently enable a simultaneous workflow. This meant that even though we started earlier than most groups we were still pushed for time as we found that tasks had to be completed in a set order and the way tasks were distributed in the beginning meant that group members were sometimes left waiting for someone to finish a task. This could've been helped by encouraging more people to engage with every task rather than splitting tasks up as you would in a traditional workload.

Finally, towards the submission dealing David started showing symptoms of COVID-19. Thankfully he tested negative in the following days however this was added stress to an already stressful situation and coursework due to the amount of work needed to complete it. However, Ben and Cerys managed to rise up to the challenge and get a version of this coursework submitted in time in case David's exceptional circumstances form was rejected.