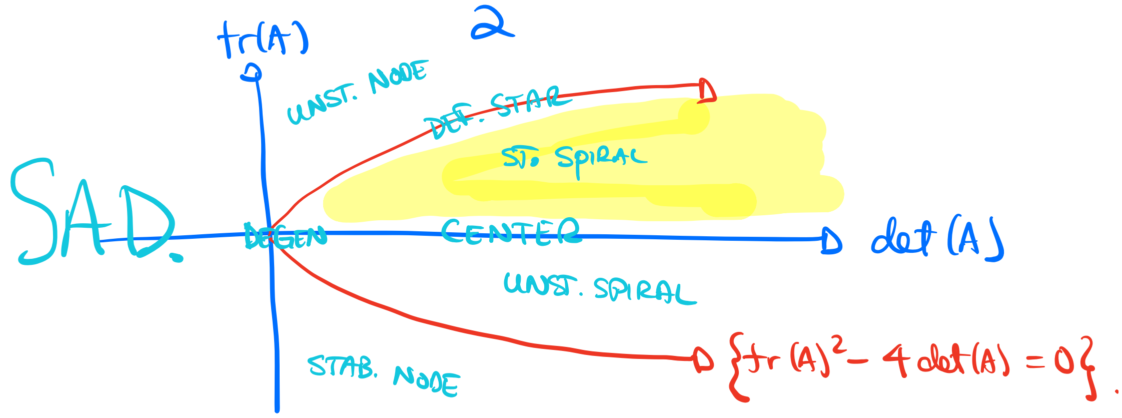
$$A = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}, \quad tr(A) = \alpha + \gamma, \quad det(A) = \alpha\gamma - \delta\beta.$$

$$\begin{aligned} det(A - \lambda I) &= (\alpha - \lambda)(\gamma - \lambda) - \delta\beta \\ &= \alpha\gamma - (\alpha+\gamma)\lambda + \lambda^2 - \delta\beta \\ &= \lambda^2 - tr(A)\lambda + det(A) \end{aligned}$$

$$\lambda_{\pm} = \frac{tr(A) \pm \sqrt{tr(A)^2 - 4\,det(A)}}{2} \qquad \Delta = tr(A)^2 - 4\,det(A)$$



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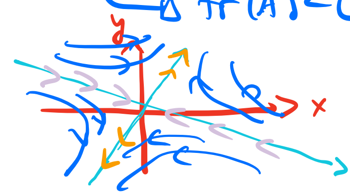
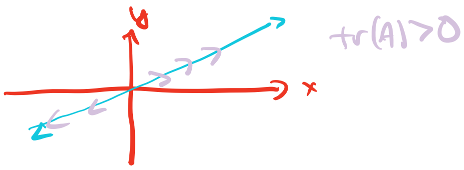
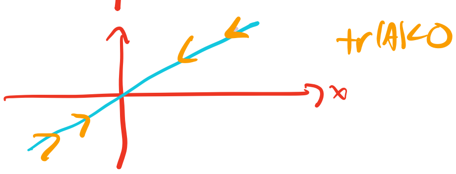
① $D > 0 \iff \lambda_+, \lambda_-$ have same sign.
  - $\to tr(A) > 0$
  - $\to tr(A) < 0$

② $D < 0 \iff \lambda_+, \lambda_-$ opposite sign.



③ $D = 0 \iff \lambda_+ = 0$ or $\lambda_- = 0$





Saddle, center, stable vs. unstable node

degen, star. stable vs. unstable spiral

Want to generate RANDOM SAMPLES of STAB. SP.

This will produce CONSTRAINTS on the PERTURBATIONS to the matrix.

$\bar{A}$ perturbation of A

$$\bar{A} = A + \underbrace{\begin{pmatrix} \tilde{\alpha} & \tilde{\beta} \\ \tilde{\delta} & \tilde{\gamma} \end{pmatrix}}_{\tilde{A}} \quad , \quad \tilde{\alpha}, \tilde{\beta}, \tilde{\delta}, \tilde{\gamma} \in \mathbb{R} .$$

$$\bar{A} = \begin{pmatrix} \alpha+\tilde{\alpha} & \beta+\tilde{\beta} \\ \delta+\tilde{\delta} & \gamma+\tilde{\gamma} \end{pmatrix} \qquad \lambda^2 - tr(A)\lambda + det(A)$$

$$\begin{aligned} det(\bar{A}-\tilde{\lambda}I) &= (\alpha+\tilde{\alpha}-\tilde{\lambda})(\gamma+\tilde{\gamma}-\tilde{\lambda}) - (\delta+\tilde{\delta})(\beta+\tilde{\beta}) \\ &= \tilde{\lambda}^2 - (\alpha+\tilde{\alpha}+\gamma+\tilde{\gamma})\lambda + \left[(\alpha+\tilde{\alpha})(\gamma+\tilde{\gamma}) - (\delta+\tilde{\delta})(\beta+\tilde{\beta})\right] \\ &= \tilde{\lambda}^2 - (tr(A)+tr(\tilde{A}))\lambda + \left[\alpha\gamma + \alpha\tilde{\gamma} + \tilde{\alpha}\gamma + \tilde{\alpha}\tilde{\gamma} - \delta\beta - \delta\tilde{\beta} - \tilde{\delta}\beta - \tilde{\delta}\tilde{\beta}\right] \\ &= \tilde{\lambda}^2 - \underbrace{\left[tr(A)+tr(\tilde{A})\right]}_{tr(\bar{A})}\tilde{\lambda} + \underbrace{\left[det(A) + det(\tilde{A}) + \overbrace{\boxed{\alpha\tilde{\gamma} + \tilde{\alpha}\gamma - (\delta\tilde{\beta} - \tilde{\delta}\beta)}}^{x}\right]}_{det(\bar{A})} \end{aligned}$$

STABLE SPIRAL. ($\Delta > 0$, $tr(A) > 0$, $det(A) > 0$)

Also WANT $\quad \bar{\Delta} > 0$, $tr(\bar{A}) > 0$, $det(\bar{A}) > 0$

Problem. Choose $\tilde{A}$ so that $\underbrace{\bar{\Delta} > 0, \ tr(\bar{A}) > 0, \ det(\bar{A}) > 0}_{\text{will produce constraints to } \tilde{A}}$

$$0 < \bar{\Delta} = tr(\bar{A})^2 - 4 \det(\bar{A})$$

$$= (tr(A) + tr(\tilde{A}))^2 - 4(\det A + \det \tilde{A} + x)$$

$$= tr(A)^2 + 2 tr(A) tr(\tilde{A}) + tr(\tilde{A})^2$$

$$- 4 \det A - 4 \det(\tilde{A}) - 4x$$

$$= \Delta + \tilde{\Delta} + 2(tr(A) tr(\tilde{A}) - 2x)$$

Only allow $\tilde{A}$ so that:

$$\tilde{\Delta} + 2 tr(A) tr(\tilde{A}) - 4x > -\Delta$$

Notice if only want $\tilde{A}$ as diagonal matrix,

then may choose $\tilde{\beta} = \tilde{\delta} = 0$

$$\Longrightarrow x = 0$$

$$\boxed{\tilde{\Delta} + 2 tr(A) tr(\tilde{A}) + \Delta > 0}$$

$$\tilde{A} = \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix}$$

SURFACE ?

$$\tilde{\Delta} = tr(\tilde{A})^2 - 4\det(\tilde{A}) \qquad , \; tr(A) = \alpha + \gamma$$
$$= \tilde{\alpha} + \tilde{\gamma} - 4\tilde{\alpha}\tilde{\gamma} \qquad tr(\tilde{A}) = \tilde{\alpha} + \tilde{\gamma}$$
$$\Delta = \alpha + \gamma - 4(\alpha\gamma - \beta\delta)$$

$$\to \underbrace{\tilde{\alpha} + \tilde{\gamma} - 4\tilde{\alpha}\tilde{\gamma} + 2(\alpha+\gamma)(\tilde{\alpha}+\tilde{\gamma}) + \alpha+\gamma - 4(\alpha\gamma-\beta\delta)}_{f(\tilde{\alpha},\tilde{\gamma})} > 0$$

Want $f_A(\tilde{\alpha},\tilde{\gamma}) > 0$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} \tilde{\alpha} & 0 \\ 0 & \tilde{\gamma} \end{pmatrix}$

$$f_A(\tilde{\alpha},\tilde{\gamma}) = \tilde{\alpha} + \tilde{\gamma} - 4\tilde{\alpha}\tilde{\gamma} + 6(\tilde{\alpha}+\tilde{\gamma}) + 3 + 8$$
$$\Longrightarrow \frac{7(\tilde{\alpha}+\tilde{\gamma})}{4} - \tilde{\alpha}\tilde{\gamma} + \frac{11}{4} > 0$$

You can make $\tilde{A}$ as simple as the constraint allows.

$$\to f_A(\tilde{\alpha},1) = \tilde{\alpha} - 4\tilde{\alpha} + 6(\tilde{\alpha}+1) + 12$$
$$= 3\tilde{\alpha} + 18$$

Want. $f_A(\tilde{\alpha},1) > 0 \iff 3\tilde{\alpha} + 18 > 0$