GUIDELINES AND PRACTICE PROBLEMS FOR THE FINAL EXAM

The final exam will cover all material presented in class during the semester, including whatever we covered since Exam 2. Questions on the exam will be of the following types: Stating definitions, propositions or theorems; short answer; true-false; and presentation of a proof of a theorem. I will try to keep time-consuming calculations to a minimum. You may have to do calculations for 2×2 or 3×3 matrices, but for questions involving larger matrices, you may be asked to sent things up or describe the process you would use to solve the problem.

Any definitions, propositions theorems, corollaries that you need to know how to state appear in the Daily Update, and all such are candidates for questions. You will need to be able to answer brief questions about these results as well as true-false statements about these results. Most of the definitions you need to know are also in the Daily Update, but it is best to check your notes for all definitions we have given during the semester.

You will also be responsible for working any type of problem that was previously assigned as homework.

On the Exam you will be required to state and provide a proof of **two** of the following Theorems. Note, you should be able to state the theorems in general (possibly as part of another section on the exam), even if you are asked to prove a special case of the theorem.

(i) The Exchange Theorem. You must state the theorem in its full generality and provide a proof that given a spanning set with four vectors, spanning the subspace W, and any independent set of two vectors in W, two of the vectors in the spanning set can be replaced by the given independent vectors and the new set still spans W.
(ii) If V is a finite dimensional vector space, then any linearly independent set in Vcan be extended to a basis for V.
(iii) Proof of the Singular Value Theorem, in the case that the matrix A is 2×2. (For this, just adapt the steps given in class with the justifications of the steps.)
(iv) Proof of the Graham-Schmidt process for a real inner product space, applied to the linearly independent set of vectors $\{w_1, w_2, w_2\}$.
(v) The Rank Plus Nullity Theorem.

Practice Problems

1. Verify the rank plus nullity theorem by calculating the rank and the dimension of the null space for the following matrices:

$$A = \begin{pmatrix} 2 & 0 & 10 & 4 \\ 4 & 0 & 2 & 2 \\ 6 & 0 & 12 & 6 \\ 12 & 0 & 24 & 12 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 & 0 & -7 & 9 \\ 2 & 1 & 0 & 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution. Recall that if D is an $m \times n$ matrix, then multiplication by D gives a linear transformation from $\mathbb{R}^n$ (column vectors) to $\mathbb{R}^m$ (column vectors). The null space of D is just the kernel of the transformation and the image is the column space of D.Thus, for A we have $A \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Thus, the rank is two and the null space has dimension 2, and 2+2 =4, the dimension of the domain.

$B \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \end{pmatrix}$, so B has rank 2 and the null space has a basis consisting of 3 elements (since the null space has three independent parameters), and 2+3 = 5.

$C \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so C has rank 3 and the null space has rank 0, and 3+0 = 3.

2. Suppose C in problem 1 is the matrix of $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the standard basis. Find $[T]_\alpha^\alpha$ for the basis $\alpha = \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}$ and $[T]_\beta^\beta$ for the basis $\beta = \{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}$. Find an invertible matrix P such that $[T]_\beta^\beta = P^{-1}[T]_\alpha^\alpha P$. Hint: For any $v \in \mathbb{R}^3$, $T(v) = Cv$.

Solution. Since C represents the matrix of T with respect to the standard basis, $T(v) = Cv$, for any column vector $v \in \mathbb{R}^3$. Let v_1, v_2, v_3 be the basis α and u_1, u_2, u_3 be the basis β. Thus:

$T(v_1) = Cv_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = 2 \cdot v_1 + 0 \cdot v_2 + 0 \cdot v_3$

$T(v_2) = Cv_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 1 \cdot v_1 + 0 \cdot v_2 + 1 \cdot v_3$

$T(v_3) = Cv_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 1 \cdot v_1 + -1 \cdot v_2 + 1 \cdot v_3$. Thus, $[T]_\alpha^\alpha = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

$T(u_1) = Cu_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 \cdot u_1 + 1 \cdot u_2 + 0 \cdot u_3$

$T(u_2) = Cu_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \cdot u_1 + 1 \cdot u_2 + 1 \cdot u_3$

$T(u_3) = Cu_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 1 \cdot u_1 + 0 \cdot u_2 + 1 \cdot u_3$. Thus, $[T]_\beta^\beta = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

From the change of basis formula, we have $[T]_\beta^\beta = [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha$, so $P = [I]_\beta^\alpha$, i.e., we need to express the β basis in terms of the alpha basis. For this we have

$$\begin{aligned} u_1 &= 1 \cdot v_1 + 0 \cdot v_2 + -1 \cdot v_3 \\ u_2 &= 0 \cdot v_1 + 1 \cdot v_2 + 0 \cdot v_3 \\ u_3 &= 1 \cdot v_1 + (-1) \cdot v_2 + 1 \cdot v_3 \end{aligned}$$

Thus the change of basis matrix $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$. We then have

$$[T]_\alpha^\alpha \cdot P = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = P \cdot [T]_\beta^\beta,$$

which gives $[T]_\beta^\beta = [I]_\alpha^\beta \cdot [T]_\alpha^\alpha \cdot [I]_\beta^\alpha$.

3. For the matrix B in problem 1, find a basis for the solution space to the homogeneous system of equations $B \cdot (x, y, z, w, v)^t = \vec{0}$.

Solution. $B \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 & 3 & -3 \\ 0 & 1 & 0 & -5 & 6 \end{pmatrix}$. Thus, $x = -3w + 3v, y = 5w - 6v, z = z, w = w, v = v$. This yields basis $\begin{pmatrix} -3 \\ 5 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.

4. Find orthonormal basis for the following spaces by applying the Gram-Schmidt process to the given spanning set:

(i) $\text{Span}\{1, 1+x, x+x^2\}$, with $\langle f, g\rangle := \int_0^1 \, fg \; dx$.

(iii) $\text{Span}\{(-i,0,1,0),(0,2i,-1,0),(i,i,i,0)\}$ in $\mathbb{C}^4$, with the usual inner product over $\mathbb{C}$.

Solution. For (i), let v_1, v_2, v_3 denote the given vectors. We seek an orthogonal set w_1, w_2, w_3, using the Gram-Schmidt process. We take $w_1 = v_1 = 1$. For w_2, we need to calculate:

$$\langle v_2, w_1\rangle = \int_0^1 (1+x)\cdot 1 \; dx = \frac{3}{2} \qquad \text{and} \qquad \langle w_1, w_1\rangle = \int_0^1 1\cdot 1 \; dx = 1.$$

Thus, $w_2 = (1+x) - \frac{3}{2}\cdot 1 = -\frac{1}{2} + x$.

For w_3 we need to calculate $\langle w_1, w_1\rangle = 1$ and

$$\langle v_3, w_1\rangle = \int_0^1 (x+x^2)\cdot 1 \; dx = \frac{5}{6}, \langle v_3, w_2\rangle = \int_0^1 (x+x^2)\cdot(-\frac{1}{2}+x) \; dx = \frac{1}{6}, \langle w_2, w_2\rangle = \int_0^1 (-\frac{1}{2}+x)^2 \; dx = \frac{1}{12}.$$

Thus, $w_3 = (x+x^2) - \frac{5}{6}\cdot 1 - \frac{\frac{1}{6}}{\frac{1}{12}}(-\frac{1}{2}+x) = x^2 - x + \frac{1}{6}$.

Part (ii) proceeds in a similar fashion, using the standard inner product on $\mathbb{C}^4$.

5. For each of the matrices below, determine if the matrix is diagonalizable or orthogonally diagonalizable, and if so, find the diagonalizing matrix. If the matrix is orthogonally diagonalizable, find the corresponding orthogonal (or Herrmetian) diagonalizing matrix.

$$A = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 & i \\ 0 & 2 & 0 \\ -i & 0 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Solution. For A, we have $p_A(x) = (x-2)^2(x-1)$. E_2 = null space of $\begin{pmatrix} 0 & 2 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, E_2 has basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, which shows A is diagonalizable, since $\dim E_2 = 2$. E_1 = null space of $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, E_1 has basis $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Note that this vector is not orthogonal to any vector in E_2, so that A is not orthogonally diagonalizable. If we take $P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$, then $P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. For B, note that $B = B^*$ so that B is self adjoint, and therefore orthogonally diagonalizable.

We have $p_B(x) = (x-3)(x-2)(x-1)$. E_3 = null space of $\begin{pmatrix} -1 & 0 & i \\ 0 & -1 & 0 \\ -i & 0 & -1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, a basis for E_3 is $\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$. $E_2 =$ null space of $\begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, a basis for $E_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. E_1 is the null space of $\begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus a basis for E_1 is $\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}$. Note that the bases for the eigenspaces form an orthogonal set, using the complex inner product. Normalizing these vectors leads to the Hermetian matrix $Q = \begin{pmatrix} -\frac{i}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ satisfying $Q^{-1}BQ = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

For C, we have C is a symmetric real matrix and therefore orthogonally diagonalizable. $p_C(x) = x^3(x-4)$. $E_0 =$ null space of $C \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so E_0 has basis $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$, which is an orthogonal set. $E_4 =$ is one dimensional and $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is clearly an eigenvector. Note that this vector is orthogonal to the basis elements of E_0. All of the basis vectors found have length2. Thus, if we take the orthogonal matrix $P = \frac{1}{2} \cdot \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$, then $P^{-1}CP = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$

6. Verify that the matrix $A = \begin{pmatrix} 3i & i \\ i & 3i \end{pmatrix}$ is a normal matrix and find a unitary matrix Q such that $Q^{-1}AQ$ is diagonal.

Solution. $AA^* = \begin{pmatrix} 3i & i \\ i & 3i \end{pmatrix} \cdot \begin{pmatrix} 3i & -i \\ -i & 3i \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} 3i & -i \\ -i & 3i \end{pmatrix} \cdot \begin{pmatrix} 3i & i \\ i & 3i \end{pmatrix} = A^*A$, so A is normal. We have $p_A(x) = (x-4i)(x-2i)$. $E_{4i} =$ null space of $\begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$. Thus a basis for E_{4i} is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $E_{2i} =$ null space of $\begin{pmatrix} i & i \\ i & i \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Thus a basis for E_{2i} is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Note that the two basis elements are orthogonal and have length 2. Thus if we take $Q = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, Q is unitary (orthogonal in this case, since its entries are real numbers) and we have $Q^{-1}AQ = \begin{pmatrix} 4i & 0 \\ 0 & 2i \end{pmatrix}$.

7. In class we have seen that $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is not diagonalizable over $\mathbb{R}$, but is orthogonally diagonalizable over $\mathbb{C}$. Working over $\mathbb{R}$, find orthogonal matrices P, Q such that $Q^{-1}AP = \sum$, where $\sum$ is the diagonal matrix whose entries are the singular values of A.

Solution. Note: I have changed the notation in the statement of the problem to match the notation used in the lecture of April 3. $A^tA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so the singular values are just 1, 1 so that $\sum = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Also, since A^tA is already diagonal, $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ orthogonally diagonalizes A^tA. If we call the columns of P u_1, u_2 take $v_1 = \frac{1}{1}Au_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \frac{1}{1}Av_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$, we have that $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A$. Thus, $AP = A \cdot I_2 = A = Q \cdot I_2 = Q\sum$, so $Q^{-1}AP = \sum$.

8. Find the singular value decomposition of $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$, in two ways, first starting with A^tA and then starting with AA^t.

Solution. We do the AA^t case, since we have not done examples of that type in class. In this case we have $AA^t = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, with $p_{AA^t}(x) = (x-3)(x-1)x$, so the singular values of A are $\sqrt{3}, 1$. Thus, $\sum = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Each eigenspace of 3, 1, 0 has dimension one are it is easy to calculate that their bases are (respectively), $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. Thus, we take $P = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$. Denote the columns of P by u_1, u_2, u_3 respectively. Note that P is automatically an orthogonal matrix. We take

$$v_1 = \frac{1}{\sqrt{3}} A^t \cdot u_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad v_2 = \frac{1}{1} A^t \cdot u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

and set $Q := \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. This leads to

$$P\sum = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = AQ,$$

Thus, $A = P\sum Q$.

9. Find the Jordan canonical form and the change of basis matrix for the following matrices:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix} \quad B = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -9 \\ 0 & 1 & 6 \end{pmatrix} \quad D = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 5 & -2 \\ 0 & 1 & 2 \end{pmatrix}.$$

For A: We first note that A is in block form, so that if we do things correctly, we can solve two 2×2 JCF problems instead of one 4×4 JCF problem. Note that the JCF of the upper left block is $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$, since this 2×2 has characteristic polynomial $(x+i)(x-i)$ and hence this matrix is diagonalizable over $\mathbb{C}$ with change of basis matrix $P_1 = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. The lower right block is already in JCF, so we can take for its change of basis matrix $P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Thus, the JCF for A is $J_A = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & i \end{pmatrix}$ and the change of basis matrix is $P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -i & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, where in each case, the 4×4 matrix is comprised of the blocks coming from the corresponding 2×2 matrices.

For B: $p_B(x) = x(x-1)^2$. For the eigenvalue 0, $E_0 =$ null space of $B \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (*). Thus, $v_1 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ is a basis for E_0.

$E_1 =$ null space of $\begin{pmatrix} -3 & 2 & 1 \\ -7 & 3 & 2 \\ 5 & 0 & -1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, E_1 has dimension 1, so there is a 2×2 Jordan box with one block corresponding to the eigenvalue 1. Thus the JCF J_B of B is $J_B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

We have $(B - 1 \cdot I_3)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 10 & -5 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We choose v_3 in the null space of this latter matrix, but not in E_1. Take $v_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. From (*) we have that E_1 is spanned by $w = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$, so v_3 is not in E_1. We take $v_2 = (B - 1 \cdot I_3)v_3 = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$. Therefore the change of basis matrix is $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 5 & 0 \end{pmatrix}$.

For C: $p_C(x) = (x-3)^3$, so there is one 3×3 Jordan box in J_C, the JCF of C. $E_3 =$ nul space of $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & -9 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, so E_3 is two dimensional. Thus, there are two Jordan blocks associated to C, and thus $J_C = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Note that $w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $w_2 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$ form a basis for E_3.

To find v_2, we take a vector not in E_3, say $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $v_1 = (C - 3 \cdot I_2)v_2 = \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix}$. For v_3 we may take $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, an eigenvector not a multiple of v_2. Thus, our change of basis matrix is $P = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

To find D: $p_D(x) = (x-3)^3$, so J_D, the JCF of D has one 3×3 Jordan box associated to 3. $E_3 =$ null space of $\begin{pmatrix} -1 & -1 & 1 \\ 1 & 2 & -2 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{\text{EROs}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$, so E_3 has basis $w = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and thus is one dimensional. Therefore, J_D has one Jordan block corresponding to 3, and hence $J_D = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

To find P, we first find v_3 not in the null space of $(D-3\cdot I)^2 =$ null space of $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{pmatrix}$, so we may take $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $v_2 = (D - 3I_3)v_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $v_1 = (D - 3\cdot I_3)v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Thus, for the change of basis matrix we have $P = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

10. Find all possible Jordan Canonical forms for 7×7 non-diagonalizable matrices A having characteristic polynomial $p_A(x) = (x+1)^4(x-2)^2(x+3)$. Do not count as different forms, two matrices A and B such that B is obtained from A by permuting Jordan blocks.

Solution. The eigenvalues of these matrices are: $\lambda = -1, 2, -3$. We the notation $J(\lambda s)$ to denote the $s\times s$ Jordan block with eigenvalue λ.

For $\lambda = -1$, the Jordan box is 4×4. Within that box, we have the following possible configurations:

(i) $J(-1,4)$
(ii) $J(-1,3), J(-1,1)$.
(iii) $J(-1,2), J(-1,2)$.
(iv) $J(-1,2), J(-1,1), J(-1,1)$.
(v) $J(-1,1), J(-1,1), J(-1,1), J(-1,1)$.

For $\lambda = 2$, we have a 2×2 Jordan box. Within that box, we have the following possible configurations:

(i) $J(2,2)$
(ii) $J(2,1), J(2,1)$.

For $\lambda = -3$ we have a 1×1 Jordan box consisting of the $1\times q$ Jordan block $J(-3,1)$.

Since any JCF consists of one box corresponding to -1, 2, -3, and we have 5 possible boxes associated to -1, two possible boxes associated to 2, and one box associated to -3. Putting these all together, shows that there are $5\cdot 2\cdot 1 = 10$ possible JCFs, each of which has three boxes, one associated to -1, one associated to 2, and one associated to -3.

11. Consider $T : \mathbb{C}^2 \to \mathbb{C}^2$ defined by $T(x,y) = (2x-3y, 3x+2y)$.

(i) Verify that T is a linear transformation.
(ii) Find a formula for T^*,the adjoint of T
(iii) Show that T is normal, but not self adjoint.
(iv) Find an orthonormal basis α for $\mathbb{C}^2$ such that $[T]_\alpha^\alpha$ is a diagonal matrix.
(v) Is there a basis β of $\mathbb{C}^2$ such that $[T]_\beta^\beta$ is not a diagonal matrix? If so, find one.

(i) Set $v_1 = (x_1, y_1), v_2 = (x_2, y_2)$, then

$$\begin{aligned} T(v_1+v_2) &= T(x_1+x_2, y_1+y_2) \\ &= (2(x_1+x_2) - 3(y_1+y_2), 3(x_1+x_2) + 2(y_1+y_2)) \\ &= (2x_1 - 3y_1, 3x_1 + 2y_1) + (2x_2 - 3y_2, 3x_2 + 2y_2) \\ &= T(x_1,y_1) + T(x_2,y_2) \\ &= T(v_1) + T(v_2). \end{aligned}$$

Moreover, if $v = (x,y)$ and $a \in \mathbb{C}$,

$$T(av) = T(ax, ay)(2ax - 3ay, 3ax + 2ay) = a(2x-3y, 3x+2y) = aT(x,y) = aT(v),$$

so T is a linear transformation.

(ii) If we let α denote the standard basis of $\mathbb{C}^2$, then $[T]_\alpha^\alpha = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$, thus, $A^* = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} = [T^*]_\alpha^\alpha$. Thus, $T^*(1,0) = (2,-3)$ and $T^*(0,1) = (3,2)$. Therefore,

$$T^*(x,y) = xT^*(1,0) + yT^*(0,1) = x(2,-3) + y(3,2) = (2x+3y, -3x+2y).$$

(iii) On the one hand,

$$T^*T(a,b) = T^*(2a-3b, 3a+2b) = (2(2a-3b)+3(3a+2b), -3(2a-3b)+2(3a+2b)) = (13a, 13b),$$

while on the other hand,

$$TT^*(a,b) = T(2a+3b, -3a+2b) = (2(2a+3b)-3(-3a+2b), 3(2a+3b)+2(-3a+2b)) = (13a, 13b),$$

so $T^*T = TT^*$, showing that T is normal.

(iv) We have $A := \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} = [T]_\alpha^\alpha$, where α is the standard basis of $\mathbb{C}^2$. One easily checks that the eigenvalues of $[T]_\alpha^\alpha$ are $2-3i$ and $2+3i$ with corresponding eigenvectors $\begin{pmatrix} -i \\ 1 \end{pmatrix}$, $\begin{pmatrix} i \\ 1 \end{pmatrix}$, respectively. Taking $P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$, one has that $P^{-1}AP = \begin{pmatrix} 2-3i & 0 \\ 0 & 2+3i \end{pmatrix}$. It follows from the change of basis formula that $P = [I]_\alpha^\beta$, where β is the basis comprised of the columns of P. Therefore, $[T]_\beta^\beta = \begin{pmatrix} 2-3i & 0 \\ 0 & 2+3i \end{pmatrix}$.

(v) Yes. Just take α the standard basis, so that $[T]_\alpha^\alpha = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ is not diagonal.

12. Consider $T : \mathbb{C}^3 \to \mathbb{C}^3$ defined by $T(x,y,z) = (2x - y + z, x + 5y - 2z, y + 2z)$. Find a basis $\alpha \subseteq \mathbb{C}^3$ such that $[T]_\alpha^\alpha$ is in Jordan canonical form. Hint: You have done most of the work for this in problem 9D.

Solution. If we let E denote the standard basis for $\mathbb{C}^3$, then it is easy to see that $[T]_E^E = D$, where D is the matrix from problem 9. Taking P as in problem 9, we have $J_D = P^{-1}DP$. Let α denote the basis of $\mathbb{C}^3$ obtained by taking the columns of P, so that $[I]_\alpha^E = P$. By the change of basis theorem, we now have

$$[T]_\alpha^\alpha = [I]_E^\alpha \cdot [T]_E^E \cdot [I]_\alpha^E = P^{-1}DP = J_D.$$