MATH 147, FALL 2025: SOLUTIONS TO FINAL EXAM PRACTICE PROBLEMS

Below are problems to practice for the final exam. The problems below, together with the problems from the two midterm exams, are a good representation of what to expect on the final exam. There will also be a few short answer and true-false questions on the final exam. The final exam is comprehensive. The final exam is Thursday, December 11 from 4:30-7:00pm and will take place in **Snow Hall 302**.

1. Let $f(x,y) = \begin{cases} x^2 + y^2, \text{ if } x^2 + y^2 < 1 \\ 1, \text{ if } x^2 + y^2 \geq 1. \end{cases}$. Determine at which points $f(x,y)$ is continuous.

Solution. Note that $g(x,y) = x^2 + y^2$ and $h(x,y) = 1$ are both continuous on all of $\mathbb{R}^2$. Thus, if we let D denote the unit disk $0 \leq x^2 + y^2 \leq 1$, then $g(x,y)$ is continuous on the interior of D and $h(x,y)$ is continuous on $\mathbb{R}^2 \backslash D$, and thus $f(x,y)$ is continuous on both the interior of D and $\mathbb{R}^2 \backslash D$. For points (a,b) on the boundary of D, $a^2 + b^2 = 1$, and we can consider $\lim_{(x,y)\to(a,b)} f(x,y)$. Let $\epsilon > 0$. Since $g(x,y)$, as a function on $\mathbb{R}^2$, is continuous at (a,b) there exists $\delta > 0$ such that $||(x,y) - (a,b)|| < \delta$ implies $|g(x,y) - g(a,b)| = |g(x,y) - 1| < \epsilon$. Taking the same δ, if $||(x,y) - (a,b)|| < \delta$ and $(x,y) \in D$, then $g(x,y) = f(x,y)$, which gives $|f(x,y) - f(a,b)| = |g(x,y) - 1| < \epsilon$. If $(x,y) \notin D$, then $f(x,y) - f(a,b) = 1 - 1 = 0$, so $|f(x,y) - f(a,b)| < \epsilon$. Thus $f(x,y)$ is continuous at (a,b).

2. Show that the function $f(x,y) = \begin{cases} \frac{(2^x - 1)(\sin(y))}{xy}, \text{if } xy \neq 0 \\ \ln(2), \text{if } xy = 0 \end{cases}$ is continuous at (0,0).

Solution. Set $g(x) = \begin{cases} \frac{2^x - 1}{x} & \text{if } x \neq 0 \\ \ln(2) & \text{if } x = 0 \end{cases}$ and $h(y) = \begin{cases} \frac{\sin(y)}{y} & \text{if } y \neq 0 \\ 1 & \text{if } y = 0 \end{cases}$. Then $f(x,y) = g(x)h(y)$.

Thus, $\lim_{(x,y)\to(0,0)} f(x,y) = \{\lim_{x\to 0} g(x)\} \cdot \{\lim_{y\to 0} h(y)\}$. By L'Hospital's Rule, $\lim_{x\to 0} g(x) = \ln(2)$ and $\lim_{y\to 0} h(y) = 1$. Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = \ln(2)$.

3. Use the limit definition to show that $f(x,y) = 5x + 4y^2$ is differentiable at (2,1).

Solution. $\frac{\partial f}{\partial x}(2,1) = 5$, $\frac{\partial f}{\partial y}(2,1) = 8$, and $f(2,1) = 14$, so $L(x,y) = 5(x-2) + 8(y-1) + 14$. Therefore,

$$f(x,y) - L(x,y) = 5x + 4y^2 - (5x - 10 + 8y - 8 + 14)$$
$$= 4y^2 - 8y + 4$$
$$= 4(y-1)^2.$$

Thus, $\frac{f(x,y) - L(x,y)}{\sqrt{(x-2)^2 + (y-1)^2}} = \frac{4(y-1)^2}{\sqrt{(x-2)^2 + (y-1)^2}} \leq \frac{4(y-1)^2}{\sqrt{(y-1)^2}} = 4|y-1|$. Therefore,

$$\lim_{(x,y)\to(2,1)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-2)^2 + (y-1)^2}} \leq \lim_{y\to 1} 4|y-1| = 0,$$

which shows that $f(x,y)$ is differentiable at (2,1).

4. From class, we saw that if the first order partial derivatives of $f(x,y)$ are continuous in a neighborhood of (a,b), then $f(x,y)$ is differentiable at (a,b). This problem shows why those conditions are necessary. Let

$$g(x,y) = \begin{cases} \frac{2xy(x+y)}{x^2+y^2}, \text{if } (x,y) \neq (0,0) \\ 0, \text{if } (x,y) = (0,0) \end{cases} .$$

Show that:
   (i) $g(x,y)$ is continuous at $(0,0)$.
   (ii) Use the limit definitions to show that $g_x(0,0)$ and $g_y(0,0)$ exist and are equal to 0.
   (iii) Conclude that $L(x,y) = 0$.
   (iv) Show that $g(x,y)$ is not differentiable at (0,0).

(v) Show that $g_x(x, y)$ is not continuous at (0,0).

Solution. (i) Taking limits, we have.

$$\lim_{(x,y)\to(0,0)} g(x, y) = \lim_{r\to 0} \frac{2r^2\cos(\theta)\sin(\theta)(r\cos(\theta) + r\sin(\theta))}{r^2}$$
$$= \lim_{r\to 0} r \cdot \{2\cos(\theta)\sin(\theta)(\cos(\theta) + \sin(\theta))\}$$
$$= 0$$
$$= g(0, 0),$$

so $g(x, y)$ is continuous at (0,0).

(ii) $\frac{\partial g}{\partial x}(0, 0) = \lim_{h\to 0} \frac{g(0+h,0)-g(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$ $\frac{\partial g}{\partial y}(0, 0) = \lim_{h\to 0} \frac{g(0,0+h)-g(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$

(iii) $L(x, y) = 0(x - 0) + 0(y - 0) = 0.$

(iv) Thus, $g(x, y) - L(x, y) = g(x, y)$. If $g(x, y)$ were differentiable at (0,0), then the limit (when $(x, y) \neq (0, 0)$)

$$\lim_{(x,y)\to(0,0)} \frac{g(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y)\to(0,0)} \frac{2xy(x + y)}{(x^2 + y^2)^{\frac{3}{2}}}$$

should equal zero. If we take the limit along the line $y = x$, then $\frac{g(x,y)}{\sqrt{x^2+y^2}} = \frac{4x^3}{2\sqrt{2x^3}} = \sqrt{2}$, so the limit along the line $y = x$ as $x \to 0$ is not 0. Thus, $g(x, y)$ is not differentiable at (0,0).

(v) Differentiating the non-zero part of $g(x, y)$ gives $g_x(, y) = \frac{-2x^2y^2+4xy^3+2y^4}{(x^2+y^2)^2}$. If we take the limit along the line $y = 0$, then $\lim_{(x,y)\to(0,0)} g_x(x, y) = 0$, while if we take the limit along the line $x = 0$, the limit becomes 2. Thus, $\lim_{(x,y)\to(0,0)} g_x(x, y)$ does not exist, so that $g_x(x, y)$ is not continuous at (0,0).

5. Find and classify the critical points for $f(x, y) = x^4 - 4xy + 2y^2$.

Solution. To find critical points we solve

$$f_x = 4x^3 - 4y = 0$$
$$f_y = -4x + 4y = 0.$$

Form the second equation, we get $x = y$. Using this in the first equation gives $4x^3 - 4x = 0$, so that $x = 0, -1, 1$. Thus, the critical points are (0,0), (-1,-1), and (1,1). For the discriminant, we have

$$D = f_{xx}f_{yy} - f_{xy}^2 = (12x^2)4 - (-4)^2 = 48x^2 - 16.$$

For (0,0): $D(0, 0) = -16$, so that $f(x, y)$ has a saddle point at (0,0).

For (-1,-1): $D(-1, -1) = 32 > 0$ and $f_{xx}(-1, -1) = 12 > 0$. Thus, $f(x, y)$ has a relative minimum value (of -3) at (-1,-1).

For (1,1): $D(1, 1) = 32 > 0$ and $f_{xx}(1, 1) = 12 > 0$. Thus, $f(x, y)$ has a relative minimum value (of -3) at (1,1).

6. Find the absolute maximum and absolute minimum values of $f(x, y) = x^2y$ on the closed and bounded set $D : 0 \leq 4x^2 + 9y^2 \leq 36$.

Solution. Solving

$$f_x = 2xy = 0$$
$$f_y = x^2 = 0$$

we see that $x = 0$, and y can be any real number. Thus, critical points in the interior of D are of the form $(0, y)$ with $0 \leq 9y^2 \leq 36$. However, $f(0, y) = 0$ for all such points. On the boundary of D, we must

2

maximize and minimize $f(x, y)$ subject to the constraint $4x^2 + 9y^2 = 36$. Calling this equation $g(x, y)$, we set $\nabla f = \lambda \nabla g$ and solve the resulting system of equations

$$2xy = \lambda 8x$$
$$x^2 = \lambda 18y$$
$$4x^2 + 9y^2 = 36.$$

Notice that if x or y equal 0, then $f(x, y) = 0$. So we can assume neither x nor y is zero. Dividing the first equation by $2x$ give $y = 4\lambda$. Using this in the second equation gives $x^2 = 72\lambda^2$. Putting both of these into the constraint equations gives $4(72\lambda^2) + 9(4\lambda)^2 = 36$, so that

$$3\lambda^2 = \frac{36}{144},$$

so that $\lambda = \pm \frac{1}{\sqrt{12}}$. Thus, $y = \pm \frac{4}{\sqrt{12}}$ and $x = \pm \sqrt{6}$. Thus, substituting these into $x^2 y$ gives $\pm \frac{24}{\sqrt{12}} = \pm 4\sqrt{3}$. Thus, on the domain D, the maximum value of $f(x, y)$ is $4\sqrt{3}$ and the minimum value is $-4\sqrt{3}$. Note that the critical points $(0, y)$ do note determine a minimum or maximum value of $f(x, y)$ on D.

7. Let S be the surface parametrized by $G(u, v) = (2u \sin(\frac{v}{2}), 2u \cos(\frac{v}{2}), 3v)$, with $0 \le u \le 1$ and $0 \le v \le 2\pi$.

    (i) Find the tangent plane to S at the point $P = G(1, \frac{\pi}{3})$.
    (ii) Find the surface area of S.

Solution.

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} i & j & k \\ 2\sin(v/2) & 2\cos(v/2) & 0 \\ u\cos(v/2) & -u\sin(v/2) & 3 \end{vmatrix} = (6\cos(v/2), -6\sin(v/2), -2u).$$
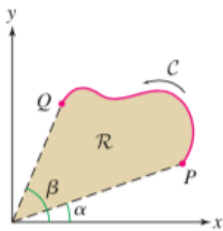
Therefore, $\mathbf{T}_u \times \mathbf{T}_v(1, \frac{\pi}{3}) = (3\sqrt{3}, -3, -2)$, since $G(1, \frac{\pi}{3}) = (1, \sqrt{3}, \pi)$, for the tangent plane we have:

$$3\sqrt{3}(x - 1) - 3(y - \sqrt{3}) - 2(z - \pi) = 0.$$

For the surface area, we have $||\mathbf{T}_u \times \mathbf{T}_v|| = \sqrt{36 + 4u^2} = 2\sqrt{9 + u^2}$.

$$\begin{aligned} \text{Surface area} &= \int \int_D ||\mathbf{T}_u \times \mathbf{T}_v|| \, dA \\ &= \int_0^{2\pi} \int_0^1 2\sqrt{9 + u^2} \, du dv \\ &= 4\pi \int_0^1 \sqrt{9 + u^2} \, du \\ &= 4\pi \{ \frac{u}{2}\sqrt{9 + u^2} + \frac{9}{2} \ln|u + \sqrt{9 + u^2}| \} \Big|_0^1 \quad \text{(using a table of integrals)} \\ &= 4\pi \{ \frac{\sqrt{10}}{2} + \frac{9}{2} \ln(1 + \sqrt{10}) - \frac{9}{2} \ln(3) \} \\ &= 4\pi \{ \frac{\sqrt{10}}{2} + \frac{9}{2} \ln(\frac{1 + \sqrt{10}}{3}) \}. \end{aligned}$$

8. Let C be a curve from the point P to the point Q in the xy-plane. Let $\mathcal{R}$ be the region enclosed by C and the two radial lines from the origin to P and Q. (See the figure below.) Use Green's Theorem to show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ gives the area of $\mathcal{R}$, for $\mathbf{F} = -\frac{y}{2}\vec{i} + \frac{x}{2}\vec{j}$.

Solution. Let D denote the oriented closed curve forming the boundary of $\mathbb{R}$, namely, the line segment from the origin to P, followed by C, followed by the line segment from Q to the origin. We take $\mathbf{F} = (\frac{-1}{2}y, \frac{1}{2}x)$, so that by Green's Theorem, $\int_D \mathbf{F} \cdot d\mathbf{r}$ equals the area enclosed by the closed curve D. Let C_1 denote the line segment from $(0,0)$ to $P = (a,b)$, C_2 denote the line segment from $Q = (c,d)$ to $(0,0)$, and C the curve given in the illustration. Let $D = C_1 \cup C \cup C_2$ so that

$$\text{area}(R) = \int_D \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

We have to show $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Take $\mathbf{r}_1(t) = (at, bt)$, $0 \le t \le 1$ for the parametrization of C_1. Then $\mathbf{F}(\mathbf{r}(t)) = (-\frac{1}{2}bt, \frac{1}{2}at)$ and $\mathbf{r}'(t) = (a,b)$. Therefore $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = -\frac{1}{2}abt + \frac{1}{2}abt = 0$. So $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$. The calculation showing $\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$ is similar.

9. Let C be the triangle with vertices $(1,0,0)$, $(0,2,0)$, $(0,0,1)$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, for the vector field $\mathbf{F} = (x^2 + yz, x + y, y - z^2)$.

Solution. We have to integrate along each side of the triangle. $C_1 : \mathbf{r}(t) = (1 - t, 2t, 0)$., with $0 \le t \le 1$. $\mathbf{r}'(t) = (-1, 2, 0)$. $\mathbf{F}(\mathbf{r}(t)) = ((1-t)^2, 1 + t, 2t)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^1 -(1 - t)^2 + 2 + 2t \, dt$$

$$= \int_0^1 -1 + 2t - t^2 + 2 + 2t \, dt$$

$$= \int_0^1 1 + 4t - t^2 \, dt$$

$$= 1 + 2 - \frac{1}{3} = \frac{8}{3}.$$

$C_2 : \mathbf{r}(t) = (0, 2 - 2t, t)$, $0 \le t \le 1$, $\mathbf{r}'(t) = (0, -2, 1)$, $\mathbf{F}(\mathbf{r}(t)) = (2t - 2t^2, 2 - 2t, 2 - 2t - t^2)$.

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^1 -4 + 4t + 2 - 2t - t^2 \, dt$$

$$= \int_0^1 -2 + 2t - t^2 \, dt$$

$$= -2 + 1 - \frac{1}{3} = -\frac{4}{3}.$$

4

$C_3 : \mathbf{r}(t) = (t, 0, 1 - t)$, $\mathbf{r}'(t) = (1, 0, -1)$, $\mathbf{F}(\mathbf{r}(t)) = (t^2, t, -(1 - t)^2)$.

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

$$= \int_0^1 t^2 + (1 - t)^2 \, dt$$

$$= \{\frac{t^3}{3} - \frac{(1 - t)^3}{3}\}\Big|_0^1$$

$$= \frac{1}{3} - (-\frac{1}{3}) = \frac{2}{3}.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_C \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \frac{8}{3} - \frac{4}{3} + \frac{2}{3} = 2.$$

10. Let $f(x, y) = \sqrt{|xy|}$. Write out details showing:

 (a) $\frac{\partial f}{\partial x}(0, 0)$ and $\frac{\partial f}{\partial y}(0, 0)$ exist.
 (b) $f(x, y)$ is not differentiable at (0,0).
 (c) Part (b) does not contradict part (a).

Solution.

2.(a) $\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$. The calculation for $\frac{\partial f}{\partial y}(0, 0)$ is similar.

(b) From (a), the linear approximation to $f(x, y)$ at (0,0) is $L(x, y) = 0$. Therefore, $f(x, y) - L(x, y) = f(x, y)$. In order for $f(x, y)$ to be differentiable at (0,0), the limit

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y) - L(x, y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}}$$

should be 0. Taking the limit along the line $y = x$, we have

$$\lim_{(x,y) \to (0,0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = \lim_{x \to 0} \frac{\sqrt{|x^2|}}{\sqrt{2}\sqrt{x^2}} = \frac{1}{\sqrt{2}} \neq 0.$$

Therefore, $f(x, y)$ is not differentiable at (0,0).

(c) Part (b) does not contradict part (a) because the first order partial derivatives of $f(x, y)$ are not continuous at (0,0). In fact, the partial derivatives of $f(x, y)$ are not defined at all points (x, y) near (0,0), so we cannot evaluate the required limit to test continuity. To see this, let's try to calculate $\frac{\partial f}{\partial x}$ along the line $x = 0$, with $y \neq 0$.

$$\frac{\partial f}{\partial x}(0, y) = \lim_{h \to 0} \frac{f(0 + h, y) - f(0, y)}{h} = \lim_{h \to 0} \frac{|hy| - 0}{h} = |y| \lim_{h \to 0} \frac{|h|}{h},$$

which does not exist.

11. Evaluate $\int\int_S \text{Curl}\mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = (-y + z\sin(x), x, z^3)$ and S the surface defined by the equation $x^2 + \frac{y^2}{4} + z^2 + z^4 x^2 = 1$, with $z \geq 0$.

Solution. We use Stoke's Theorem. The surface lies above the xy-plane, and intersects the xy-plane along the ellipse $x^2 + \frac{y^2}{4} = 1$. By Stokes Theorem, $\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$, for $C : (\cos(t), 2\sin(t), 0)$, with $0 \leq t \leq 2\pi$. Note that the orientation of C is consistent with what is required by Stoke's Theorem.

$\mathbf{r}'(t) = (-\sin(t), 2\cos(t), 0)$ and $\mathbf{F}(\mathbf{r}(t)) = (-2\sin(t), \cos(t), 0)$. Therefore,

$$\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$
$$= \int_0^{2\pi} 2 \, dt$$
$$= 4\pi.$$

12. Verify the Divergence Theorem for $\mathbf{F} = (-x^2, y^2, -z^2)$ and S rectangular box $[0, 3] \times [-1, 2] \times [1, 2]$.

Solution. Div $(\mathbf{F}) = -2x + 2y - 2z$, therefore if B is the solid contained in the given rectangular box,

$$\int\int\int_B \text{Div } \mathbf{F} \, dV = \int_1^2 \int_{-1}^2 \int_0^3 -2x + 2y - 2z \, dxdydz$$
$$= \int_1^2 \int_{-1}^2 (-x^2 + 2xy - 2xz)\Big|_{x=0}^{x=3} \, dydz$$
$$= \int_1^2 \int_{-1}^2 -9 + 6y - 6z \, dydz$$
$$= \int_0^1 (-9y + 3y^2 - 6yz)\Big|_{y=-1}^{y=2} \, dz$$
$$= \int_1^2 -18 - 18z \, dz$$
$$= (-18z - 9z^2)\Big|_{z=1}^{z=2}$$
$$= (-36 - 36) - (-18 - 9)$$
$$= -45.$$

To calculate the surface integral, we must sum the integrals over each face of the given rectangular box.

Front Face, S_1: S_1 is given by $(3, y, z)$, with $-1 \leq y \leq 2$, $1 \leq z \leq 2$ and $\mathbf{n} = i$. We will see the bounds on y and z are just used to calculate the area of the front face. The same will hold for the other five faces. So: $\mathbf{F}$ on S_1 is given by $(-9, y^2, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = -9$. Therefore,

$$\int\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int\int_{S_1} -9 \, dS$$
$$= -9 \cdot \text{area}(S_1)$$
$$= -9 \cdot 3 = -27.$$

Back Face, S_2: S_1 is given by $(0, y, z)$, with $-1 \leq y \leq 2$, $1 \leq z \leq 2$ and $\mathbf{n} = -i$. $\mathbf{F}$ on S_2 is given by $(0, y^2, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore, $\int\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = 0$.

Left Face, S_3: S_3 is given by $(x, -1, z)$, with $0 \leq x \leq 3$, $1 \leq z \leq 2$ and $\mathbf{n} = -j$. $\mathbf{F}$ on S_3 is given by $(-x^2, 1, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = -1$. Therefore,

$$\int\int_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_3} \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int\int_{S_3} -1 \, dS$$
$$= -1 \cdot \text{area}(S_3)$$
$$= -1 \cdot 3 = -3.$$

Right Face, S_4: S_4 is given by $(x, 2, z)$, with $0 \leq x \leq 3$, $1 \leq z \leq 2$ and $\mathbf{n} = j$. $\mathbf{F}$ on S_4 is given by $(-x^2, 4, -z^2)$ and thus $\mathbf{F} \cdot \mathbf{n} = 4$. Therefore,

$$\int\int_{S_4} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_4} \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int\int_{S_4} 4 \, dS$$
$$= 4 \cdot \text{area}(S_3)$$
$$= 4 \cdot 3 = 12.$$

Top Face, S_5: S_5 is given by $(x, y, 2)$, with $0 \leq x \leq 3$, $-1 \leq y \leq 2$ and $\mathbf{n} = k$. $\mathbf{F}$ on S_5 is given by $(-x^2, y^2, -4)$ and thus $\mathbf{F} \cdot \mathbf{n} = -4$. Therefore,

$$\int\int_{S_5} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_5} \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int\int_{S_5} -4 \, dS$$
$$= -4 \cdot \text{area}(S_5)$$
$$= -4 \cdot 9 = -36.$$

Bottom Face, S_6: S_6 is given by $(x, y, 1)$, with $0 \leq x \leq 3$, $-1 \leq y \leq 2$ and $\mathbf{n} = -k$. $\mathbf{F}$ on S_6 is given by $(-x^2, y^2, -1)$ and thus $\mathbf{F} \cdot \mathbf{n} = 1$. Therefore,

$$\int\int_{S_6} \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_6} \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int\int_{S_6} 1 \, dS$$
$$= 1 \cdot \text{area}(S_6)$$
$$= 1 \cdot 9 = 9.$$

Putting these all together we have $\int\int_S \mathbf{F} \cdot d\mathbf{S} = -27 + 0 - 3 + 12 - 36 + 9 = -45$.

13. Let $\mathbf{F} = (z^2, x^2, -y^2)$. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the path traversing counterclockwise the square with sides of length s centered at $(x_0, y_0, 0)$. Then divide this number by the area of the square and take the limit as $s \to 0$. Compare this with $(\text{Curl } \mathbf{F})(x_0, y_0, 0) \cdot k$.

Solution. In this problem we are using the limit definition to calculate $(\text{Curl } \mathbf{F})(x_0, y_0, 0) \cdot k$. For this, we must compute a line integral of $\mathbf{F}$ over each side of the square C.

Bottom side, C_1: C_1 is given by $\mathbf{r}(t) = (x_0 - \frac{s}{2}, y_0 - \frac{s}{2}, 0) + t(s, 0, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (s, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 - \frac{s}{2})^2, -(y_0 - \frac{s}{2})^2)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$$
$$= \int_0^1 0 \; dt$$
$$= 0.$$

Top side, C_2: C_2 is given by $\mathbf{r}(t) = (x_0 + \frac{s}{2}, y_0 + \frac{s}{2}, 0) + t(-s, 0, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (-s, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 + \frac{s}{2})^2, -(y_0 + \frac{s}{2})^2)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$$
$$= \int_0^1 0 \; dt$$
$$= 0.$$

Right side, C_3: C_3 is given by $\mathbf{r}(t) = (x_0 + \frac{s}{2}, y_0 - \frac{s}{2}, 0) + t(0, s, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (0, s, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 + \frac{s}{2})^2, -(y_0 - \frac{s}{2})^2)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$$
$$= \int_0^1 (x_0 + \frac{s}{2})^2 s \; dt$$
$$= x_0^2 s + x_0 s^2 + \frac{s^3}{4}.$$

Left side, C_4: C_4 is given by $\mathbf{r}(t) = (x_0 - \frac{s}{2}, y_0 + \frac{s}{2}, 0) + t(0, -s, 0)$, with $0 \le t \le 1$. $\mathbf{r}'(t) = (0, -s, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, (x_0 - \frac{s}{2})^2, -(y_0 + \frac{s}{2})^2)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \; dt$$
$$= \int_0^1 -(x_0 - \frac{s}{2})^2 s \; dt$$
$$= -x_0^2 s + x_0 s^2 + -\frac{s^3}{4}.$$

We now have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + (x_0^2 + x_0 s^2 + \frac{s^3}{4}) + (-x_0^2 + x_0 s^2 + -\frac{s^3}{4}) = 2s^2 x_0.$$

Therefore,

$$\lim_{s \to 0} \frac{1}{\text{area}(S)} \int_C \mathbf{F} \cdot d\mathbf{r} = \lim_{s \to 0} \frac{1}{s^2} 2s^2 x_0 = 2x_0.$$

To Check:

$$\text{Curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & -y^2 \end{vmatrix} = (-2y, 2z, 2x).$$

Thus, Curl $\mathbf{F}(x_0, y_0, z_0) = (-2y_0, 2z_0, 2x_0)$ and Curl $\mathbf{F}(x_0, y_0, z_0) \cdot k = 2x_0$.

14. Let C be the curve obtained by intersecting the cylinder $x^2 + y^2 = 1$ with the plane $x + y + z = 1$, and $\mathbf{F} = -y^3 \vec{i} + x^3 \vec{j} + -z^3 \vec{k}$. Set up the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ as a single integral over an interval of the form $[a, b]$. Now evaluate this line integral by using Stoke's Theorem.

Solution. The curve C lies on the plane $z = 1-x-y$ but also on the cylinder $x^2+y^2 = 1$. So a parametrization and tangent of C are

$$\mathbf{r}(t) = (\cos(t), \sin(t), 1 - \cos(t) - \sin(t)) \text{ and } \mathbf{r}'(t) = (-\sin(t), \cos(t), \sin(t) - \cos(t)).$$

$$\mathbf{F}(\mathbf{r}(t)) = (-\sin^3(t), \cos^3(t), -(1 - \cos(t) - \sin(t))^3).$$

$$\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \sin^4(t) + \cos^4(t) - (1 - \cos(t) - \sin(t))^3 \cdot (\sin(t) - \cos(t)).$$

Integrating this last expression from 0 to 2π is doable .... but not much fun.

To apply Stoke's Theorem we will integrate $\nabla \times \mathbf{F}$ over S, that portion of the given plane lying above the disk $D : 0 \le x^2 + y^2 \le 1$ in the xy-plane.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & -z^3 \end{vmatrix} = (3x^2 + 3y^2)\vec{k} = (0, 0, 3x^2 + 3y^2).$$

S is given by $G(u, v) = (u, v, 1 - u - v)$, with $0 \le u^2 + v^2 \le 1$. Thus,

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \vec{i} + \vec{j} + \vec{k} = (1, 1, 1).$$
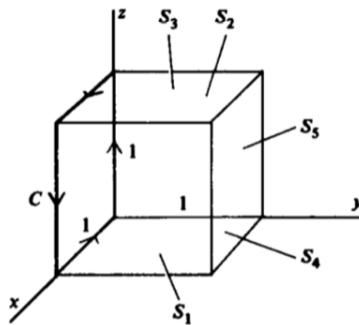
Moreover, $\nabla \times \mathbf{F}$ on S is $(0, 0, 3u^2 + 3v^2)$. Thus, on S,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{T}_u \times \mathbf{T}_v = (0, 0, 3u^2 + 3v^2) \cdot (1, 1, 1) = 3u^2 + 3v^2.$$

Therefore:

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_D 3u^2 + 3v^2 \; dA$$
$$= 3 \int_0^{2\pi} \int_0^1 r^2 \cdot r \; dr d\theta$$
$$= 6\pi \int_0^1 r^3 \; dr$$
$$= \frac{6\pi}{4} = \frac{3\pi}{2}.$$

15. Verify Stoke's Theorem for $\mathbf{F} = (z^2, -y^2, 0)$ and C the square of side 1 oriented as shown, lying in the xz-plane and S the open box with sides S_1, S_2, S_3, S_4, S_5. What happens, if instead, you take S to be the square enclosed by C?

Solution. Both terms in Stoke's Theorem require computing several integrals. We start by computing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $C = C_1 \cup C_2 \cup C_3 \cup C_4$ is the curve indicated in the diagram.

$C_1 : \mathbf{r}(t) = (1 - t, 0, 0)$, $0 \leq t \leq 1$, $\mathbf{r}'(t) = (-1, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (0, 0, 0)$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$
$$= \int_0^1 0 \, dt$$
$$= 0.$$

$C_2 : \mathbf{r}(t) = (0, 0, t)$, $0 \leq t \leq 1$, $\mathbf{r}'(t) = (0, 0, 1)$, $\mathbf{F}(\mathbf{r}(t)) = (t^2, 0, 0)$.

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$
$$= \int_0^1 0 dt$$
$$= 0.$$

$C_3 : \mathbf{r}(t) = (t, 0, 1)$, $0 \leq t \leq 1$, $\mathbf{r}'(t) = (1, 0, 0)$, $\mathbf{F}(\mathbf{r}(t)) = (1, 0, 0)$.

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$
$$= \int_0^1 1 \, dt$$
$$= 1.$$

$C_4 : \mathbf{r}(t) = (1, 0, 1 - t)$, $0 \leq t \leq 1$, $\mathbf{r}'(t) = (0, 0, -1)$, $\mathbf{F}(\mathbf{r}(t)) = ((1 - t)^2, 0, 0)$.

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$
$$= \int_0^1 0 \, dt$$
$$= 0.$$

We now have,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 1 + 0 = 1.$$

To calculate the curl of $\mathbf{F}$

$$\text{Curl } \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & -y^2 & 0 \end{vmatrix} = (0, 2z, 0).$$

For the surface integral $\int \int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S}$ is the sum of the surface integrals of the curl of $\mathbf{F}$ over the five faces indicated in the diagram.

Front Face, S_1: S_1 is given by $(1, y, z)$, with $0 \leq y \leq 1$, $0 \leq z \leq 1$ and $\mathbf{n} = i$. Curl $\mathbf{F}$ on S_1 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$\int \int_{S_1} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int \int_{S_1} \text{Curl } \mathbf{F} \cdot \mathbf{n} \, dS$$
$$= \int \int_{S_1} 0 \, dS$$
$$= 0.$$

10

Back Face, S_2: S_2 is given by $(0, y, z)$, with $0 \le y \le 1$, $0 \le z \le 1$ and $\mathbf{n} = -i$. Curl $\mathbf{F}$ on S_2 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$
\int\int_{S_2} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_2} \text{Curl } \mathbf{F} \cdot \mathbf{n} \, dS
$$
$$
= \int\int_{S_3} 0 \, dS
$$
$$
= 0.
$$

Top Face, S_3: S_3 is given by $(x, y, 1)$, with $0 \le x \le 1$, $0 \le y \le 1$ and $\mathbf{n} = k$. Curl $\mathbf{F}$ on S_3 is given by $(0, 2, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$
\int\int_{S_3} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_3} \text{Curl } \mathbf{F} \cdot \mathbf{n} \, dS
$$
$$
= \int\int_{S_3} 0 \, dS
$$
$$
= 0.
$$

Bottom Face, S_4: S_4 is given by $(x, y, 0)$, with $0 \le x \le 1$, $0 \le y \le 1$ and $\mathbf{n} = -k$. Curl $\mathbf{F}$ on S_4 is given by $(0, 0, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 0$. Therefore,

$$
\int\int_{S_4} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_4} \text{Curl } \mathbf{F} \cdot \mathbf{n} \, dS
$$
$$
= \int\int_{S_4} 0 \, dS
$$
$$
= 0.
$$

Right Face, S_5: S_5 is given by $(x, 1, z)$, with $0 \le x \le 1$, $0 \le z \le 1$ and $\mathbf{n} = j$. Curl $\mathbf{F}$ on S_5 is given by $(0, 2z, 0)$ and thus $\mathbf{F} \cdot \mathbf{n} = 2z$. Therefore,

$$
\int\int_{S_5} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_5} \text{Curl } \mathbf{F} \cdot \mathbf{n} \, dS
$$
$$
= \int\int_{S_5} 2z \, dS
$$
$$
= \int_0^1 \int_0^1 2z \, dz \, dx
$$
$$
= \int_0^1 2z \, dz
$$
$$
= 1.
$$

Putting these all together we have $\int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S} = 0 + 0 + 0 + 0 + 1 = 1$, as expected, thereby confirming Stoke's Theorem.

Finally, an important consequence of Stoke's theorem is that the surface integrals of Curl $\mathbf{F}$ over two surfaces sharing a common oriented boundary are the same. Let S_0 be the left face of the square in the diagram, so that S and S_0 share the same oriented boundary.

S_0 is given by $(x, 0, z)$, with $0 \le x \le 1$, $0 \le z \le 1$ and $\mathbf{n} = j$. Note that j is the correct normal when considering S_0 as an open, oriented surface with boundary C. If we were considering S_0 as the 6th side of the cube, we would take $-j$ as the unit normal. Curl $\mathbf{F}$ on S_0 is given by $(0, 2z, 0)$ and thus Curl $\mathbf{F} \cdot \mathbf{n} = 2z$.

Therefore,

$$\int\int_{S_0} \text{Curl } \mathbf{F} \cdot d\mathbf{S} = \int\int_{S_0} \mathbf{F} \cdot \mathbf{n} \, dS$$

$$= \int\int_{S_0} 2z \, dS$$

$$= \int_0^1 \int_0^1 2z \, dz \, dx$$

$$= \int_0^1 2z \, dz$$

$$= 1$$

$$= \int\int_S \text{Curl } \mathbf{F} \cdot d\mathbf{S}.$$

16. Calculate, without using Stoke's Theorem, $\int\int_{S_1} \nabla \times \mathbf{F} \cdot d\mathbf{S}$, for $\mathbf{F} = (3y^2 + 2y)\vec{i} + 3z^2\vec{j} + 3x^2\vec{k}$ and S_1 the inverted cone $z = 1 - \sqrt{x^2 + y^2}$, with vertex $(0,0,1)$, and $z \geq 0$. Then calculate directly $\int\int_{S_2} \nabla \times \mathbf{F} \cdot d\mathbf{S}$, for S_2 the unit disk in the xy-plane. The answers you get should be the same. This shows the consequence of Stoke's Theorem, that surfaces integrals of the curl of a vector field over surfaces sharing the same boundary are independent of the surface.

Solution. $\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y^2 + 2y & 3z^2 & 3x^2 \end{vmatrix} = (-6z, -6x, -6y - 2).$

If we integrate directly over the cone, we use the parametrization

$$G(u,v) = (v\cos(u), v\sin(u), 1 - v),$$

with $0 \leq u \leq 2\pi$, $0 \leq v \leq 1$. It follows that

$$\nabla \times \mathbf{F}(G(u,v)) = (-6(1-v), -6v\cos(u), -6v\sin(u) - 2).$$

Moreover,

$$\mathbf{T}_u \times \mathbf{T}_v = (v\cos(u), v\sin(u), -v).$$

Note that this vector, has a negative z-component, and thus is *inside* of the inverted cone. If we flatten the cone, this would point downward, and contradict the right hand thumb rule. Thus, we need to use the vector $-(\mathbf{T}_u \times \mathbf{T}_v)$ when integrating $\nabla \mathbf{F}$. (An important point: We could also parametrize the inverted cone using $H(u,v) = (u, v, 1 - \sqrt{u^2 + v^2})$, with $0 \leq u^2 + v^2 \leq 1$, and in this case $\mathbf{T}_u \times \mathbf{T}_v$ gives the correct normal vector.)

We now have

$$\{\nabla \times \mathbf{F}(G(u,v))\} \cdot -(\mathbf{T}_u \times \mathbf{T}_v) = 6v(1-v)\cos(u) + 6v^2\sin(u)\cos(u) - 6v^2\sin(u) - 2v.$$

When we calculate $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, first integrating with respect to u, the trig terms integrated from 0 to 2π all become 0. We are left with,

$$\int_0^1 \int_0^{2\pi} -2v \, dudv = 2\pi \int_0^1 -2v \, dv$$

$$= -4\pi \cdot \frac{v^2}{2}\Big|_0^1$$

$$= -2\pi.$$

To integrate over the disk we have $G(u,v) = (u,v,0)$ with $0 \leq u^2 + v^2 \leq 1$ and $\mathbf{T}_u \times \mathbf{T}_v = (0,0,1)$. $\nabla \times \mathbf{F}(G(u,v)) = (0, -6u, -6v - 2)$, so $\nabla \times \mathbf{F}(G(u,v)) \cdot \mathbf{T}_u \times \mathbf{T}_v = -6v - 2$. Thus,

$$\int\int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int\int_{S_2} -6v - 2 \ dA$$

$$= \int_0^{2\pi} \int_0^1 (-6r\sin(\theta) - 2) \ r dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 -6r^2 \sin(\theta) - 2r \ dr \ d\theta$$

$$= \int_0^{2\pi} -2\sin(\theta) - 1 \ dr$$

$$= -2\pi,$$

which is what we want.