# Multiplicities and Rees valuations

Daniel Katz

*Department of Mathematics, University of Kansas, Lawrence, KS 66045*
E-mail: dlk@math.ku.edu

Javid Validashti

*Department of Mathematics, University of Kansas, Lawrence, KS 66045*
E-mail: jvalidas@math.ku.edu

*To Professor D. Rees, in honor of his nintieth birthday*

## Abstract

Let $(R, \mathfrak{m})$ be a local ring of Krull dimension $d$ and $I \subseteq R$ be an ideal with analytic spread $d$. We show that the $j$-multiplicity of $I$ is determined by the Rees valuations of $I$ centered on $\mathfrak{m}$. We also discuss a multiplicity that is the limsup of a sequence of lengths that grow at an $O(n^d)$ rate.

## 1. Introduction

Throughout, we let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $\mathsf{k}$. The purpose of this paper is to further elucidate the connections between the Rees valuations of an ideal $I \subseteq R$ and various multiplicities that can be associated to the ideal. Here we focus primarily on ideals that are *not* $\mathfrak{m}$-primary, since most of our results are attempts at establishing or using connections between valuations and multiplicities for non $\mathfrak{m}$-primary ideals that are already known in the $\mathfrak{m}$-primary case. For example, let $I$ be an $\mathfrak{m}$-primary ideal. In [14], Rees defines the degree function $d_I(x)$ as follows : For $x \in R$, $d_I(x)$ is the multiplicity of the image of ideal the $I$ in the ring $R/xR$. Here it is assumed that $\dim(R/xR) = d - 1$. The main theorem in [14]

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shows that there are a finite collection of discrete valuations $\{v_j\}$ and positive integers $d_j(I, v_j)$ so that for all appropriate $x$,

$$d_I(x) = \sum_j d_j(I, v_j) \cdot v_j(x). \tag{1.1}$$

In [15, Chapter 9], Rees revists this result (in a more general setting) and carefully identifies the valuations appearing in (1.1) as the set of Rees valuations of $I$ (*ideal valuations* in [15]) that are $\mathfrak{m}$-valuations. If one specializes $x$ in (1.1) to be a superficial element of $I$, one gets the following formula for the multiplicity of $I$,

$$e(I) = \sum_j d_j(I, v_j) \cdot v_j(I). \tag{1.2}$$

Now, let $I \subseteq R$ be an ideal which is not necessarily $\mathfrak{m}$-primary. Then, by definition, the $j$-multiplicity of $I$ is given by

$$j(I) = \lim_{n \to \infty} \frac{(d-1)!}{n^{d-1}} \cdot \lambda(\Gamma_{\mathfrak{m}}(I^n/I^{n+1})),$$

where $\lambda(C)$ denotes the length of the $R$-module $C$ and $\Gamma_{\mathfrak{m}}(D)$ is the zero-th local cohomology functor applied to $D$. Of course, when $I$ is an $\mathfrak{m}$-primary ideal, $j(I)$ is the usual multiplicity $e(I)$. One of our main results extends the multiplicity formula (1.2) to the case of ideals $I$ with maximal analytic spread. In particular we show that there exist positive integers $d_i(I, v_i)$ depending on $I$ such that

$$j(I) = \sum_i d_i(I, v_i) \cdot v_i(I), \tag{1.3}$$

where the $v_i$ are the Rees valuations of $I$ centered on $\mathfrak{m}$ with the property that each $v_i$ is an $\mathfrak{m}$-valuation. When $R$ is a quasi-unmixed local ring, this collection is precisely the set of Rees valuations of $I$ centered on $\mathfrak{m}$. The formula above is established in section three.

In section four, we consider a variant on the $j$-multiplicity of an ideal of maximal analytic spread, one which at first might seem more natural. In the $\mathfrak{m}$-primary case, one often writes

$$e(I) = \lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(R/I^n).$$

Thus, it might seem natural to consider the limit

$$\lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)).$$

However, it is not known whether this limit exists in general (though it is known that if this limit exists, it need not be rational, see [4]). In section four, we note that the corresponding limsup exists, which we denote by $\epsilon(I)$. We first show that this limsup is an invariant up to integral closure and that the corresponding limit exists for an ideal $I$ if and only if it the same limit exists for $\overline{I}$, the integral closure of $I$. Our main result in section four is that $\epsilon(I)$ is not zero when $I$ has maximal analytic spread. We then show that this limsup can be used to detect integral closure

in exactly the way that multiplicity detects integral closure in the Rees multiplicity theorem and $j$-multiplicity detects integral closure in the Flenner-Manaresi version of the Rees multiplicity theorem.

Finally, in a brief appendix, we show that there is a one-to-one correspondence between the Rees valuations of $I$ centered on $\mathfrak{m}$ and the Rees valuations of $I\widehat{R}$ centered on $\mathfrak{m}\widehat{R}$, where $\widehat{R}$ denotes the completion of $R$.

## 2. Preliminaries

In this section we establish some basic terminology and some relevant facts concerning Rees valuations and the various multiplicities at hand. Throughout $R$ is a local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $\mathsf{k}$. We will reserve $d$ for the Krull dimension of $R$ and assume throughout that $d > 0$. We set

$$\mathrm{G}_I(R) := \bigoplus_{n \geq 0} I^n/I^{n+1},$$

the associated graded ring of $R$ with respect to $I$. We will write $\mathcal{R}_I$ to denote the extended Rees algebra of $R$ with respect to $I$, i.e.,

$$\mathcal{R}_I := \bigoplus_{-\infty}^{\infty} I^n \cdot t^n,$$

$t$ an indeterminate. Thus, by definition, the $j$-multiplicity of $I$ is given by

$$j(I) := j(\mathrm{G}_I(R)) = j(\mathcal{R}_I/t^{-1}\mathcal{R}_I),$$

where $j(\mathrm{G}_I(R))$ is the normalized coefficient of the degree $d - 1$ term in the Hilbert polynomial of the $\mathrm{G}_I(R)$-module $\Gamma_{\mathfrak{m}}(\mathrm{G}_I(R))$) (see [1], [6, Section 6.1].) Since this polynomial has degree $d - 1$ if and only if $\Gamma_{\mathfrak{m}}(\mathrm{G}_I(R))$) has dimension $d$ if and only if the analytic spread of $I$ equals $d$, our attention will be focused primarily on ideals with this latter property. Recall that the analytic spread of $I$, denoted $\ell(I)$, is the Krull dimension of the ring $\mathrm{G}_I(R)/\mathfrak{m}\,\mathrm{G}_I(R)$. As is well known, when $\mathsf{k}$ is infinite, $\ell(I)$ is equal to the minimal number of generators of a minimal reduction of $I$.

Suppose for the moment that $R$ is an integral domain with quotient field $K$. The Rees valuation rings of $I$ may be obtained as follows. Take a set of generators $x_1, \ldots, x_s$ for $I$ (or equivalently, any reduction of $I$) and for each $1 \leq i \leq s$, let $S_i$ denote the subring of $K$ generated over $R$ by the fractions $x_1/x_i, \ldots, x_s/x_i$. Let $\overline{S_i}$ denote the integral closure of $S_i$. Each localization of $\overline{S_i}$ at a height one prime containing $x_i$ is a discrete valuation ring and the collection of all such discrete valuation rings (as $i$ varies) is the set of Rees valuation rings of $I$. (The text [17] is an excellent reference for Rees valuations and integral closure of ideals.) The *Rees valuations* of $I$ are the normalized valuations $v : K \to \mathbb{Z}$ associated with each Rees valuation ring $V$.

When $R$ is not an integral domain, the Rees valuation rings are just the Rees valuation rings of the image of $I$ in $R/q$ as $q$ ranges over the minimal prime ideals of $R$. If $I \subseteq q$ for some minimal prime $q$, we take the quotient field of $R/q$. Similarly, the Rees valuations of $I$ are the set of functions $v : R \to \mathbb{Z} \cup \{\infty\}$, where $v$ is the

composition of the canonical map from $R$ to $R/q$, for some minimal prime $q$, with a Rees valuation of $I$ extended to the ring $R/q$. In this case, $v$ takes the value $\infty$ on $q$ and the usual rules of arithmetic in $\mathbb{Z} \cup \{\infty\}$ apply.

Now suppose that $R$ is a local domain with finite residue field $\mathsf{k}$. Write $R(x)$ for the local ring $R[x]_{\mathfrak{m}R[x]}$, $x$ an indeterminate. Let $I \subseteq R$ be an ideal and suppose that $V_1, \ldots, V_r$ are the Rees valuation rings of $I$. Then, it is straightforward to check that $V_1(x), \ldots, V_r(x)$ are the Rees valuation rings of $IR(x)$, where for each $1 \le i \le r$, $V_i(x) := V_i[x]_{\mathfrak{m}_{V_i} V_i[x]}$. It follows that each Rees valuation of $IR(x)$ is the extension of the corresponding Rees valuation of $I$. Moreover, it also follows that

$$\mathrm{tr. \, deg.}_{\mathsf{k}} \, \mathsf{k}(\mathfrak{m}_{V_i}) = \mathrm{tr. \, deg.}_{\mathsf{k}(x)} \, \mathsf{k}(\mathfrak{m}_{V_i(x)}),$$

whenever $V_i$ is centered on $\mathfrak{m}$. Thus, when we change rings by extending the residue field in this manner, we will not disturb any of our statements involving the Rees valuations or Rees valuation rings of the ideal $I$. A similar statement applies when $R$ is not a domain.

We will also need the concept of $\mathfrak{m}$-*valuation*. Recall that a discrete valuation on the quotient field of a local domain $(R, \mathfrak{m})$ is said to be an '$\mathfrak{m}$-valuation' if its associated valuation ring $(V, \mathfrak{m}_V)$ satisfies the following conditions : (i) $\mathfrak{m}_V \cap R = \mathfrak{m}$ and (ii) $\mathrm{tr. \, deg.}_{\mathsf{k}} \, \mathsf{k}(\mathfrak{m}_V) = d - 1$ (see section two in [16] or [17, Chapter 9] where such valuations are called *divisorial valuations*). When $R$ is not an integral domain, then the $\mathfrak{m}$-valuations are the valuations from $R$ to $\mathbb{Z} \cup \infty$ obtained by following the canonical map $R \to R/q$ by an $\mathfrak{m} \cdot R/q$-valuation on $R/q$, where $q$ ranges over the minimal primes of $R$ with dimension $d$.

Finally, we will make frequent use of the following well known facts. Let $x_1, \ldots, x_s$ be elements that are analytically independent in $\mathfrak{m}$. For example, the $x_i$ could be generators for a minimal reduction of an ideal with analytic spread $s$. Assume that $x_1$ is not a zerodivisor in $R$ and set $S := R[x_2/x_1, \ldots, x_s/x_1]$. Then $\mathfrak{m}S$ is a prime ideal and $S/\mathfrak{m}S$ is isomorphic to a polynomial ring in $s - 1$ variables over $\mathsf{k}$. Since $\dim(S) \le d$, it follows that if $s = d$, then $\mathfrak{m}S$ is a height one prime ideal.

## 3. A formula for $j$-multiplicity

In this section we prove that the $j$-multiplicity of an ideal $I$ is determined by the Rees valuations of $I$ that are $\mathfrak{m}$-valuations. In order to do so, we need to establish some properties of Rees valuation rings of ideals with maximal analytic spread. The following observation plays a key role in our analysis. While it is likely to be known by experts on Rees valuations, we have not found it stated anywhere in the form below.

**Proposition 3.1**

Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal. Then $\ell(I) = d$ if and only if some Rees valuation of $I$ is an $\mathfrak{m}$-valuation.

*Proof.* By our comments in Section 2, we may extend the residue field of $R$ in the usual way to assume that $\mathsf{k}$ is infinite. Now, since, $\ell(I) = d$ if and only if for some minimal

prime $q \subseteq R$, $\ell(I \cdot R/q) = d$, this together with the definitions of Rees valuation and $\mathfrak{m}$-valuation allows us to assume that $R$ is an integral domain. Suppose $\ell(I) = d$. Let $x_1, \ldots, x_d$ generate a minimal reduction of $I$. Then for

$$S_1 := R[x_2/x_1, \ldots, x_d/x_1],$$

$\mathfrak{m}S_1$ is a height one prime containing $I$. Thus, if $P \subseteq \overline{S_1}$ is a height one prime lying over $\mathfrak{m}S_1$, then $V := (\overline{S_1})_P$ is a Rees valuation ring of $I$. We have

$$\mathsf{k} \subseteq \mathsf{k}(\mathfrak{m}S_1) \subseteq \mathsf{k}(\mathfrak{m}_V).$$

But the images of the fractions $x_i/x_1$ in $\mathsf{k}(\mathfrak{m}S_1)$ are algebraically independent over $\mathsf{k}$ (since the $x_i$ are analytically independent) and the extension $\mathsf{k}(\mathfrak{m}S_1) \subseteq \mathsf{k}(\mathfrak{m}_V)$ is algebraic. It follows immediately from this that the valuation associated to $V$ is an $\mathfrak{m}$-valuation.

Conversely, suppose that $v$ is a Rees valuation of $I$ that is also an $\mathfrak{m}$-valuation, i.e., for $V$ the corresponding valuation ring, $\mathfrak{m}_V \cap R = \mathfrak{m}$ and $\operatorname{tr.deg.}_\mathsf{k} \mathsf{k}(\mathfrak{m}_V) = d - 1$. Let $x_1, \ldots, x_s$ be a minimal reduction of $I$, so that $s = \ell(I)$. Then, for some $1 \le i \le s$, $V$ is the localization of $\overline{S_i}$ at a height one prime containing $x_i$, where

$$S_i := R[x_1/x_i, \ldots, x_s/x_i].$$

Since $V$ is centered on $\mathfrak{m}$, $\mathsf{k} \subseteq \mathsf{k}(\mathfrak{m}_V \cap S_i) \subseteq \mathsf{k}(\mathfrak{m}_V)$. Moreover, $\mathsf{k}(\mathfrak{m}_V)$ is algebraic over $\mathsf{k}(\mathfrak{m}_V \cap S_i)$. Thus, the transcendence degree of this latter field over $\mathsf{k}$ is $d - 1$. But $S_i/\mathfrak{m}_V \cap S_i$ is a $\mathsf{k}$-algebra, so $\dim(S_i/\mathfrak{m}_V \cap S_i) = d - 1$. Since $\dim(S_i) \le d$, it follows that $\mathfrak{m}_V \cap S_i$ is a height one prime ideal, necessarily equal to $\mathfrak{m}S_i$, by the analytic independence of the $x_j$. Thus,

$$s - 1 = \operatorname{tr.deg.}_\mathsf{k} \mathsf{k}(\mathfrak{m}S_i) = \operatorname{tr.deg.}_\mathsf{k} \mathsf{k}(\mathfrak{m}_V \cap S_i) = d - 1,$$

so $s = d$, i.e., $\ell(I) = d$. $\qquad\square$

The foregoing proposition enables us to make the following definition.

DEFINITION 3.2 Let $I \subseteq R$ be an ideal with maximal analytic spread. We will write $\nu_\mathfrak{m}(I)$ to denote the Rees valuations of $I$ that are $\mathfrak{m}$-valuations. In a similar fashion, we use $\mathcal{V}_\mathfrak{m}(I)$ to denote the Rees valuation rings associated to the valuations in $\nu_\mathfrak{m}(I)$. Note that if $R$ is not an integral domain and $q_1, \ldots, q_c$ are the minimal prime ideals for which $\ell(I \cdot R/q_i) = d$, then

$$\mathcal{V}_\mathfrak{m}(I) = \mathcal{V}_\mathfrak{m}(I \cdot R/q_1) \cup \cdots \cup \mathcal{V}_\mathfrak{m}(I \cdot R/q_c).$$

A similar decomposition holds for $\nu_\mathfrak{m}(I)$.

**Corollary 3.3**

*Let $(R, \mathfrak{m})$ be a local domain with quotient field $K$ and $I \subseteq R$ an ideal with maximal analytic spread. Assume that $x_1, \ldots, x_d$ generate a minimal reduction of $I$ and set*

$$T := R[x_2/x_1, \ldots, x_d/x_1]_{\mathfrak{m}R[x_2/x_1, \ldots, x_d/x_1]}.$$

Then, $V \in \mathcal{V}_{\mathfrak{m}}(I)$ if and only if $V$ is an essential valuation ring of $T$, i.e., a discrete valuation ring obtained by localizing the integral closure of $T$ at one of its maximal ideals. Consequently, if $R$ is quasi-unmixed, then $\mathcal{V}_{\mathfrak{m}}(I)$ is just the set of Rees valuation rings of $I$ centered on $\mathfrak{m}$.

*Proof.* Retain the notation from the proof of Proposition 3.1. The proof of Proposition 3.1 shows that the elements of $\mathcal{V}_{\mathfrak{m}}(I)$ are exactly those discrete valuation rings $V$ with the property that there exists an $i$ such that $V = (\overline{S_i})_P$ for some height one prime ideal $P \subseteq \overline{S_i}$ with $P \cap S_i = \mathfrak{m}S_i$. Since $T := (S_i)_{\mathfrak{m}S_i}$ is independent of $i$, it follows that the valuation rings in $\mathcal{V}_{\mathfrak{m}}(I)$ are precisely the essential valuation rings of $T$.

The second statement is just a version of [17, Corollary 10.2.7], for ideals with maximal analytic spread. If $R$ is quasi-unmixed, let $V$ be a Rees valuation ring of $I$ centered on $\mathfrak{m}$. By definition, for some $i$, $V = (\overline{S_i})_P$ for a height one prime $P \subseteq \overline{S_i}$ with $P \cap R = \mathfrak{m}$. Thus $P \cap S_i = \mathfrak{m}S_i$, since by the quasi-unmixed hypothesis, $R$ satisfies the dimension formula (see [10]), so $\text{height}(S_i \cap P) = 1$. Thus $V$ corresponds to an essential valuation ring of $T$. $\square$

Our next proposition is closely related to the following results [17, Proposition 6.5.2, Proposition 6.5.4, Proposition 9.3.5 and Theorem 10.4.3]. This proposition plays a crucial role in our description of the valuations that determine $j(I)$. However, as we have not been able to parse the statements of these results to give the precise statement we have in the proposition below, we require a separate proposition. More importantly, this proposition shows that the known one-to-one correspondence between the Rees valuations of $I$ and $I\widehat{R}$ for $\mathfrak{m}$-primary ideals ([15, Theorem 6.23]) carries over to a similar correspondence between $\mathcal{V}_{\mathfrak{m}}(I)$ and $\mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ when $I$ has maximal analytic spread.

## Proposition 3.4

Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread. Let $\widehat{R}$ denote the completion of $R$.

(a) Take $V \in \mathcal{V}_{\mathfrak{m}}(I)$ and write $q$ for the corresponding minimal prime. Then there exists a minimal prime $z \subseteq \widehat{R}$ with $z \cap R = q$ such that, for $B := \widehat{R}/z$, $\ell(IB) = d$ and $V = W \cap \mathsf{k}(q)$ for some $W \in \mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ contained in $\mathsf{k}(z)$.

(b) Take $W \in \mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ and write $z$ for the corresponding minimal prime. Note that for $B := \widehat{R}/z$, $\ell(IB) = d$. Then $V := W \cap \mathsf{k}(z \cap R) \in \mathcal{V}_{\mathfrak{m}}(I)$.

Moreover, the correspondence from $\mathcal{V}_{\mathfrak{m}}(I)$ to $\mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ determined by (a) is a one-to-one, onto function of sets.

*Proof.* For (a), let $q \subseteq R$ be the minimal prime such that $V$ is a discrete valuation ring contained in $\mathsf{k}(q)$. Since a minimal prime in $\widehat{R}$ contracting to $q$ corresponds to a minimal prime in $\widehat{R/q}$ contracting to zero in $R/q$, we may pass to $R/q$. Note $\ell(I \cdot R/q) = d$. Changing notation, we begin again assuming that $R$ is a local domain with quotient field $K$. Let $V \in \mathcal{V}_{\mathfrak{m}}(I)$. By [17, Proposition 10.4.3], there exists a minimal prime $z \subseteq \widehat{R}$ such that $V = W \cap K$ for some Rees valuation ring $W$ of $I\widehat{R}$ contained in $\mathsf{k}(z)$. By [17, Proposition 6.5.2], $\mathsf{k}(\mathfrak{m}_V) = \mathsf{k}(\mathfrak{m}_W)$, so $\text{tr.deg.}_{\mathsf{k}} \mathsf{k}(\mathfrak{m}_W) = d - 1$, since $V \in \mathcal{V}_{\mathfrak{m}}(I)$. Since $W$ is clearly centered on $\mathfrak{m}\widehat{R}$, $W \in \mathcal{V}_{\mathfrak{m}}(I\widehat{R})$. Finally, if we set

$B := \widehat{R}/z$, then $W \in \mathcal{V}_{\mathfrak{m}}(IB)$, by definition, so $\ell(IB) = d$ either by Proposition 3.1 or [17, Theorem 10.4.2].

For part (b), let $V, W, B$ and $z$ be as in the statement of part (b) and set $q := z \cap R$. Note that since $B$ can be identified with $\widehat{R/q}$ modulo a minimal prime, it follows from the assumption $\ell(IB) = d$ that $\ell(I \cdot R/q) = d$. Thus, as in the proof of part (a), by replacing $R$ by $R/q$, we may further assume that $R$ is a domain with quotient field $K$ and that $V = W \cap K$. Note further that, by the hypothesis on $W$, $V$ is centered on $\mathfrak{m}$.

Next, we wish to take $d$ elements from $I$ generating a minimal reduction. If $\mathsf{k}$ is finite, rather than making a base change to $R(x)$, we will instead use the fact that the Rees valuation rings of $I$ and $I^n$ are the same for all $n > 0$. Thus, $\mathcal{V}_{\mathfrak{m}}(I) = \mathcal{V}_{\mathfrak{m}}(I^n)$ and $\mathcal{V}_{\mathfrak{m}}(I\widehat{R}) = \mathcal{V}_{\mathfrak{m}}(I^n\widehat{R})$ for all $n$. Since for all large $n$, a minimal reduction of $I^n$ can be generated by $d$ elements, we may use $I^n$ in place of $I$. But then changing notation, we can assume that there are $d$ elements generating a minimal reduction of $I$.

Now let $x_1, \ldots, x_d$ be a minimal generating set for a minimal reduction of $I$ and write $J$ for the ideal generated by the $x_i$. Since $JB$ is a reduction of $IB$ and $\ell(IB) = d$, the images $x_i'$ of the $x_i$ in $B$ form a minimal generating set for a minimal reduction of $IB$. It follows from Corollary 3.3 that $W$ is an essential valuation ring of

$$T_B := B[x_2'/x_1', \ldots, x_d'/x_1']_{\mathfrak{m}B[x_2'/x_1', \ldots, x_d'/x_1']}.$$

Thus the images of $x_2'/x_1', \ldots, x_d'/x_1'$ in $\mathsf{k}(\mathfrak{m}_W)$ form a transcendence basis over $\mathsf{k}$.

On the other hand, since $V = W \cap K$, $x_2/x_1, \ldots x_d/x_1 \in V$. Thus, in the notation of Proposition 3.1, $\overline{S_1} \subseteq V$. By [17, Proposition 6.5.2], $\mathsf{k}(\mathfrak{m}_V) = \mathsf{k}(\mathfrak{m}_W)$, so that the images of $x_2/x_1, \ldots, x_d/x_1$ in $\mathsf{k}(\mathfrak{m}_V)$ are algebraically independent over $\mathsf{k}$. Therefore, $\mathrm{tr.\,deg.}_{\mathsf{k}} \mathsf{k}(\mathfrak{m}_V \cap S_1) = d - 1$. As in the proof of Proposition 3.1, this forces $\mathfrak{m}_V \cap S_1 = \mathfrak{m}S_1$. Thus, $\mathfrak{m}_V \cap \overline{S_1}$ is a height one prime lying over $\mathfrak{m}S_1$. It follows immediately from this that $V = (\overline{S_1})_{\mathfrak{m}_V \cap \overline{S_1}}$, and thus $V \in \mathcal{V}_{\mathfrak{m}}(I)$ as required.

For the final statement, note that by part (b), the correspondence from $\mathcal{V}_{\mathfrak{m}}(I)$ to $\mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ given in (a) is surjective. The correspondence in question is clearly one-to-one. To see that the reverse correspondence given by (b) is one-to-one, we first note that by [17, Proposition 6.5.2], we cannot have a single minimal prime $z \subseteq \widehat{R}$ and distinct $W, W' \in \mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ contained in $\mathsf{k}(z)$ such that $W$ and $W'$ contract to the same valuation ring $V$ in $\mathsf{k}(z \cap R)$. Now suppose $W$ and $W'$ are defined over minimal primes $z \neq z'$ respectively such that $z \cap R = z' \cap R$ and $V = W \cap \mathsf{k}(z \cap R) = W' \cap \mathsf{k}(z' \cap R)$. If we show that $\widehat{W} = \widehat{V} = \widehat{W'}$, then this will provide the necessary contradiction from which the final statement in the proposition follows.

To see that $\widehat{V} = \widehat{W}$, we just modify the proof of [17, Proposition 6.5.2] showing $\mathsf{k}(\mathfrak{m}_V) = \mathsf{k}(\mathfrak{m}_W)$ to show that $V/\mathfrak{m}_V^n = W/\mathfrak{m}_W^n$, for all $n \geq 1$. To see this, we first note that the first paragraph in the proof of [17, Proposition 6.5.2], shows that the value groups of the valuations associated to $V$ and $W$ are equal, though, *a priori*, the value group associated to $V$ is a subgroup of the value group associated to $W$. It follows from this that $\mathfrak{m}_W^n \cap V = \mathfrak{m}_V^n$ and thus $V/\mathfrak{m}_V^n \subseteq W/\mathfrak{m}_W^n$, for all $n \geq 1$.

Let $x, y \in \widehat{R}/z$ with $x/y \in W$. Set $t := w(x)$ and $s := w(\mathfrak{m} \cdot \widehat{R}/z)$. Fix $n \geq 1$ and choose $r$ large enough so that $sr > nt$. Choose a non-zero $x' \in R$ such that $x - x' \in \mathfrak{m}^r\widehat{R}$. Then,

$$w(x - x') \geq sr > nt = nw(x) = nw(x') = nv(x').$$

Similarly, there exists a non-zero $y' \in R$ such that

$$w(y - y') > nw(y) = nw(y') = nv(y').$$

Note that $w(x/y) = v(x'/y')$, so $x'/y' \in V$. Moreover,

$$\frac{x}{y} - \frac{x'}{y'} = \frac{x - x'}{x} \cdot \frac{x}{y} + \frac{y' - y}{y} \cdot \frac{x'}{y'} \in \mathfrak{m}_W^n.$$

Since $x/y \in W$ was arbitrary, we have $V/\mathfrak{m}_V^n = W/\mathfrak{m}_W^n$. Taking inverse limits gives $\widehat{V} = \widehat{W}$. The proof that $\widehat{V} = \widehat{W'}$ is the same.                                  $\square$

*Remark 3.5* Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread. Then Proposition 3.4 shows that the elements of $\mathcal{V}_\mathfrak{m}(I)$ and the elements of $\mathcal{V}_\mathfrak{m}(I\widehat{R})$ are 'parameterized' by the minimal primes $z_1, \ldots, z_h$ in $\widehat{R}$ for which the extension of $I$ to $\widehat{R}/z_i$ has analytic spread $d$. In other words, we can label $\mathcal{V}_\mathfrak{m}(I) = \{V_{i,j}\}$ and $\mathcal{V}_\mathfrak{m}(I\widehat{R}) = \{W_{i,j}\}$ so that each $W_{i,j}$ belongs to the quotient field of $\widehat{R}/z_i$ and $V_{i,j} = W_{i,j} \cap \mathsf{k}(z_i \cap R)$. Pictorially, we have :
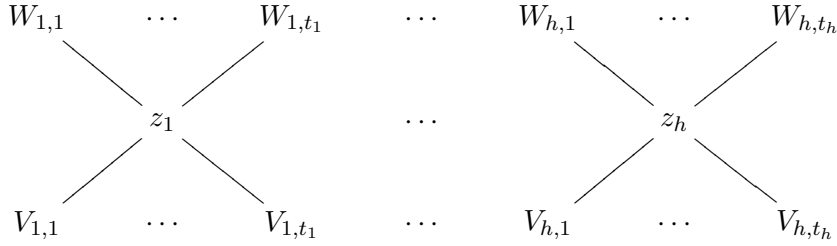


We are almost ready for the main result in this section. We first require an observation and a proposition.

**Observation 3.6** Let $I \subseteq R$ be an ideal and set $L := (0 : I^\infty)$. By the Artin-Rees lemma, $I^n \cap L = 0$ for all large $n$. It follows from this that $I^n/I^{n+1}$ is isomorphic to $(I^n + L)/(I^{n+1} + L)$ for $n$ large. Therefore, $\ell(I) = \ell((I + L)/L)$. In particular, if $I$ has maximal analytic spread, then so does the image of $I$ in the ring $R/L$. Moreover, it is also clear that $j(I) = j((I + L)/L)$.

The following proposition provides a $j$-multiplicity analogue of the main result in [8].

**Proposition 3.7**

Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread $d$. Set $L := (0 : I^\infty)$ and set $R' := R/L$. Suppose there exist $x_1, \ldots, x_d$ generating a minimal reduction of $I$ with the property that image $x_1'$ of $x_1$ in $R'$ is not a zerodivisor. Set

$$\tilde{T} := R'[x_2'/x_1', \ldots, x_d'/x_1']_{\mathfrak{m}R'[x_2'/x_1', \ldots, x_d'/x_1']}.$$

Then $j(I) = e(I\tilde{T})$.

*Proof.* Let $J$ denote the ideal generated by the $x_i$. One the one hand, $j(J) = j(I)$ (see [5]) and $L := (0 : J^\infty)$. On the other hand, by Observation 3.6, the image of $J$ in $R/L$ is still a minimal reduction of the image of $I$ and $j(J) = j((J + L)/L)$. Therefore, we may replace $I$ by $J$ and $R$ by $R'$. Changing notation, we just assume that $L = 0$, write $T$ instead of $\tilde{T}$ and assume that $I$ is generated by $d$ elements $x_1, \ldots, x_d$ analytically independent in $\mathfrak{m}$ such that $x_1$ is a non-zerodivisor.

Let $\mathcal{R}_I$ denote the Rees ring of $R$ with respect to $I$, so that $j(I) = j(\mathcal{R}_I/t^{-1}\mathcal{R}_I)$. Since the $x_i$ are analytically independent, $\mathcal{M} := (\mathfrak{m}, t^{-1})\mathcal{R}_I$ is a height one prime ideal and $\mathcal{R}_I/\mathcal{M}$ is a polynomial ring in $d$ variables over $\mathsf{k}$. Thus, $\mathcal{M}$ is the only prime in $\mathcal{R}_I$ of dimension $d$ containing $t^{-1}$ and $\mathfrak{m}$. Set $\mathcal{T} := (\mathcal{R}_I)_{\mathcal{M}}$. By the associativity formula for $j$-multiplicity (see [6, Proposition 6.1.3]), it follows that

$$j(\mathcal{R}_I/t^{-1}\mathcal{R}_I) = \lambda(\mathcal{T}/t^{-1} \cdot \mathcal{T}) \cdot e(\mathcal{R}_I/\mathcal{M}) = e(t^{-1} \cdot \mathcal{T}).$$

Thus $j(I) = e(t^{-1} \cdot \mathcal{T})$. To see that $e(t^{-1} \cdot \mathcal{T})$ is just $e(IT)$, we first note that $\mathcal{R}_I[(x_1 t)^{-1}] = S_1[x_1 t, (x_1 t)^{-1}]$ and the latter ring is a Laurent polynomial ring in the variable $x_1 t$ over $S_1$, where as before, $S_1 = R[x_2/x_1, \ldots, x_d/x_1]$. Note also that $x_1 t \notin \mathcal{M}$. Now,

$$\mathcal{M}\mathcal{R}_I[(x_1 t)^{-1}] = \mathfrak{m}\mathcal{R}_I[(x_1 t)^{-1}] = \mathfrak{m}S_1[x_1 t, (x_1 t)^{-1}],$$

and it follows from this that $\mathcal{T} = T[x_1 t]_{\mathfrak{m}T[x_1 t]}$, the latter ring being a polynomial ring in one variable over $T$ localized at the extension of $\mathfrak{m}T$. Thus,

$$e(IT) = e(IT[x_1 t]_{\mathfrak{m}T[x_1 t]}) = e(I \cdot \mathcal{T}) = e(t^{-1} \cdot \mathcal{T}).$$

It now follows that $j(I) = e(IT)$, as desired. $\qquad\square$

The following corollary of Proposition 3.7 gives a crucial special case of what we wish to prove.

**Corollary 3.8**

Let $(R, \mathfrak{m})$ be an analytically unramified local domain and let $I \subseteq R$ be an ideal with maximal analytic spread. Then for each $v \in \nu_{\mathfrak{m}}(I)$, there exists a positive integer $d(I, v)$ so that

$$j(I) = \sum_{v \in \nu_{\mathfrak{m}}(I)} d(I, v) \cdot v(I),$$

where $v(I)$ denotes the least value under $v$ of the elements from $I$.

*Proof.* Extending the residue field if necessary, let $x_1, \ldots, x_d$ generate a minimal reduction of $I$ and set

$$T := R[x_2/x_1, \ldots, x_d/x_1]_{\mathfrak{m}R[x_2/x_1, \ldots, x_d/x_1]}.$$

Proposition 3.7 yields $j(I) = e(IT)$. Let $V_1, \ldots, V_r$ denote the essential valuation rings of $T$. Since $R$ is analytically unramified, the integral closure of $T$ is a finite module over $T$, therefore, by the expansion formula for multiplicities in a finite extension,

$$e(IT) = \sum_{i=1}^{r} [\mathsf{k}(\mathfrak{m}_{V_i}) : \mathsf{k}(\mathfrak{m}T)] \cdot e(IV_i) = \sum_{i=1}^{r} [\mathsf{k}(\mathfrak{m}_{V_i}) : \mathsf{k}(\mathfrak{m}T)] \cdot v_i(I),$$

where $v_i$ is the valuation associated with $V_i$ and $[\mathsf{k}(\mathfrak{m}_{V_i}) : \mathsf{k}(\mathfrak{m}T)]$ is the degree of the residue field extension. On the other hand, $\mathcal{V}_\mathfrak{m}(I) = \{V_1, \ldots, V_r\}$, by Corollary 3.3. For $V = V_i$, we set $d(I,v) := [\mathsf{k}(\mathfrak{m}_{V_i}) : \mathsf{k}(\mathfrak{m}T)]$, where $v$ is the element in $\nu_\mathfrak{m}(I)$ corresponding to $V$. Putting this all together we get

$$j(I) = \sum_{v \in \nu_\mathfrak{m}(I)} d(I,v) \cdot v(I),$$

as desired.                                                                    $\square$

Here is the main result of this section.

**Theorem 3.9**

   *Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread. Then for each $v \in \nu_\mathfrak{m}(I)$, there exists a positive integer $d(I,v)$ such that*

$$j(I) = \sum_{v \in \nu_\mathfrak{m}(I)} d(I,v) \cdot v(I),$$

*where $v(I)$ denotes the least value under $v$ of the elements from $I$.*

*Proof.* By the comments from section two, we may assume that $\mathsf{k}$ is infinite. We first reduce to the case that $R$ is a complete local domain using Proposition 3.4 and then apply Corollary 3.8. To begin, let $z_1, \ldots, z_h \subseteq \widehat{R}$ be the minimal primes in $\widehat{R}$ such that, upon setting $B_i := \widehat{R}/z_i$, we have $\ell(IB_i) = d$. Then, by faithful flatness and the associativity formula for $j$-multiplicity (see [6, Corollary 6.1.8]),

$$j(I) = j(I\widehat{R}) = \sum_{i=1}^{h} \lambda(\widehat{R}_{z_i}) \cdot j(IB_i).$$

Set $a_i := \lambda(\widehat{R}_{z_i})$ and suppose $\nu_\mathfrak{m}(IB_i) = \{w_{i,1}, \ldots, w_{i,t_i}\}$, for $1 \leq i \leq h$. By Corollary 3.8, the theorem holds for each $B_i$, thus there exist $d(IB_i, w_{i,j}) > 0$ such that

$$j(IB_i) = \sum_{j=1}^{t_i} d(IB_i, w_{i,j}) \cdot w_{i,j}(I),$$

for all $1 \leq i \leq h$. Therefore,

$$j(I) = j(I\widehat{R}) = \sum_{i=1}^{h} \sum_{j=1}^{t_i} a_i \cdot d(IB_i, w_{i,j}) \cdot w_{i,j}(I). \tag{3.1}$$

   Now, by definition, $\nu_\mathfrak{m}(I\widehat{R}) = \{w_{i,j} \mid 1 \leq i \leq h, 1 \leq j \leq t_i\}$. Thus, for each $i$ and $j$, if we set $B_w := B_i$ and $a_w := a_i$, for $w = w_{i,j}$, then we may rewrite (3.1) as

$$j(I) = \sum_{w \in \nu_\mathfrak{m}(I\widehat{R})} a_w \cdot d(IB_w, w) \cdot w(I). \tag{3.2}$$

On the other hand, by Proposition 3.4, there is a one-to-one correspondence between $\nu_\mathfrak{m}(I)$ and $\nu_\mathfrak{m}(I\widehat{R})$ such that if $v \in \nu_\mathfrak{m}(I)$ and $w \in \nu_\mathfrak{m}(I\widehat{R})$ correspond, then

we have $v(I) = w(I)$. Thus, if we set set $d(I, v) := a_w \cdot d(IB_w, w)$ and replace $w(I)$ by $v(I)$ in (3.2), then we may sum instead over $v \in \nu_{\mathfrak{m}}(I)$ to obtain the required expression for $j(I)$. □

We would like to refine our expression for the values of the $d(I, v)$ appearing in the proof of Theorem 3.9 and also show that they depend only on $R$, $I$ and $v$. The following proposition is relevant to the ensuing discussion and has independent interest.

**Proposition 3.10**

Let $(R, \mathfrak{m})$ be a local domain with infinite residue field and $I \subseteq R$ an ideal with maximal analytic spread. Fix $V \in \mathcal{V}_{\mathfrak{m}}(I)$. Let $x_1, \ldots, x_d$ be a minimal generating set for a minimal reduction of $I$ and set

$$T := R[x_2/x_1, \ldots, x_d/x_1]_{\mathfrak{m}R[x_2/x_1, \ldots, x_d/x_1]}.$$

Then $[\mathsf{k}(\mathfrak{m}_V) : \mathsf{k}(\mathfrak{m}T)]$ is independent of $T$.

*Proof.* We first reduce to the case that $R$ is a complete local domain. By Proposition 3.4, there exists a minimal prime $z \subseteq \widehat{R}$ such that if we set $B := \widehat{R}/z$, then $\ell(IB) = d$ and $V = W \cap K$ for some $W \in \mathcal{V}_{\mathfrak{m}}(I\widehat{R})$, where $K$ is the quotient field of $R$ and $W$ is contained in $\mathsf{k}(z)$. Since $\ell(IB) = d$, it follows that the images $x'_i$ of the $x_i$ in $B$ generate a minimal reduction of $IB$. Thus, we may form the one dimensional local domain $T_B$ derived from these elements. By [17, Proposition 6.5.2], $\mathsf{k}(\mathfrak{m}_W) = \mathsf{k}(\mathfrak{m}_V)$. It follows that the images in this field of the $x_i/x_1$ from $T/\mathfrak{m}T$ and the $x_i/x_1$ from $T_B/\mathfrak{m}T_B$ are equal, i.e., $T/\mathfrak{m}T = T_B/\mathfrak{m}T_B$. Thus,

$$[\mathsf{k}(\mathfrak{m}_W) : \mathsf{k}(\mathfrak{m}T_B)] = [\mathsf{k}(\mathfrak{m}_V) : \mathsf{k}(\mathfrak{m}T)]. \tag{3.3}$$

Therefore, since $W \in \mathcal{V}_{\mathfrak{m}}(IB)$ (by definition), it suffices to prove the lemma for $B$. In other words, we may begin again assuming that $R$ is a complete local domain.

Now, let $J$ denote the ideal generated $x_1, \ldots, x_d$. By definition of $\mathcal{V}_{\mathfrak{m}}(I)$ and Corollary 3.3, there exists a height one prime $P \subseteq \overline{S_1}$ with $V = (\overline{S_1})_P$ and $P \cap S_1 = \mathfrak{m}S_1$, where $S_1$ is as before (so that $(S_1)_{\mathfrak{m}S_1} = T$). Let $\mathcal{R}_J$ be the Rees ring of $J$. Set $\mathcal{S}_J := \overline{\mathcal{R}_J}$. Then $\mathcal{S}_J[(x_1t)^{-1}] = \overline{S_1}[x_1t, (x_1t)^{-1}]$. Let $\mathcal{P} \subseteq \mathcal{S}_J[(x_1t)^{-1}]$ be the prime corresponding to $P\overline{S_1}[x_1t, (x_1t)^{-1}]$. It follows that $V[x_1t]_{\mathfrak{m}_V V[x_1t]} = (\mathcal{S}_J)_{\mathcal{P}}$. Thus, the quotient field of $\mathcal{S}_J/\mathcal{P}$ is just the rational function field in one variable over $\mathsf{k}(\mathfrak{m}_V)$. Note that $\mathcal{S}_J/\mathcal{P}$ depends only on $V$. Set $\mathcal{M} := (t^{-1}, \mathfrak{m})\mathcal{R}_J$. Then, as in the proof of Proposition 3.7, $(\mathcal{R}_J)_{\mathcal{M}} = T[x_1t]_{\mathfrak{m}T[x_1t]}$, so that the quotient field of $\mathcal{R}_J/\mathcal{M}$ is the rational function field in one variable over $\mathsf{k}(\mathfrak{m}T)$. Thus,

$$[\mathsf{k}(\mathfrak{m}_V) : \mathsf{k}(\mathfrak{m}T)] = [\mathsf{k}(\mathcal{P}) : \mathsf{k}(\mathcal{M})].$$

Since $R$ is excellent, $\mathcal{S}_J/\mathcal{P}$ is a finitely generated graded module over $\mathcal{R}_J/\mathcal{M}$. Since the latter ring is a polynomial ring over $\mathsf{k}$, $[\mathsf{k}(\mathcal{P}) : \mathsf{k}(\mathcal{M})]$ is the multiplicity of $\mathcal{S}_J/\mathcal{P}$ when viewed as an $\mathcal{R}_J/\mathcal{M}$ module. On the other hand, $\mathcal{S}_J/\mathcal{P}$ is a finitely generated graded - not necessarily standard graded - $\mathsf{k}$-algebra, and as such, it has a Hilbert function given by a quasi-polynomial, which in this case is polynomial. Moreover, its

normalized leading coefficient depends only on $\mathcal{S}_J/\mathcal{P}$ and is equal to $[\mathsf{k}(\mathcal{P}) : \mathsf{k}(\mathcal{M})]$. Therefore, $[\mathsf{k}(\mathcal{P}) : \mathsf{k}(\mathcal{M})] = [\mathsf{k}(\mathfrak{m}_V) : \mathsf{k}(\mathfrak{m}T)]$ is independent of $T$. $\qquad\qquad\square$

**A description of** $d(I,v)$. For $I \subseteq R$ an ideal with maximal analytic spread, we would like to refine the description of $d(I, v)$ contained in the proof of Theorem 3.9, at least when $\mathsf{k}$ is infinite. We will maintain the notation from the proof of Theorem 3.9. From the last sentence of the proof of Theorem 3.9, we have

$$d(I,v) = a_w \cdot d(IB_w, w),$$

where $a_w$ and $d(IB_w, w)$ are as follows. By Proposition 3.4, there is a unique $w$ in $\nu_{\mathfrak{m}}(I\widehat{R})$ corresponding to $v$. If $z_w \subseteq \widehat{R}$ is the minimal prime such that $w$ is defined on $\mathsf{k}(z_w)$, then $a_w := \lambda(R_{z_w})$ and $B_w := \widehat{R}/z_w$. Note that from the proof of Corollary 3.8, $d(IB_w, w) := [\mathsf{k}(\mathfrak{m}_w) : \mathsf{k}(\mathfrak{m}T_{B_w})]$, where $T_{B_w}$ is the one dimensional local domain derived over $B_w$ from a minimal generating set of a minimal reduction of $IB_w$. At this point our description agrees with the one given for $\mathfrak{m}$-primary ideals in [15, Theorem 9.41].

Now, fix $x_1, \ldots, x_d$ a minimal generating set for a minimal reduction of $I$. Note that if $R$ maps to a local domain $D$ such that $\ell(ID) = d$, then the images of the $x_i$ in $D$ generate a minimal reduction of $ID$ and we can then form the one dimensional local domain over $D$ derived from these images. We denote this latter ring by $T_D$. Thus, in the preceding paragraph, all of the terms $d(IB_w, w) := [\mathsf{k}(\mathfrak{m}_w) : \mathsf{k}(\mathfrak{m}T_{B_w})]$ can be derived using $x_1, \ldots, x_d$ and these values depend only on the corresponding $w$, i.e., they are independent of the $x_i$, by Proposition 3.10. Now let $v \in \mathcal{V}_{\mathfrak{m}}(I)$. Then $v$ is defined on the quotient field of a unique minimal prime $q \subseteq R$. Set $C_v := R/q$. Since $\ell(IC_v) = d$, we may form the ring $T_{C_v}$. Let $w(v) \in \nu_{\mathfrak{m}}(I\widehat{R})$ be the unique element corresponding to $v$. By equality of residue fields and equation (3.3) we have

$$[\mathsf{k}(\mathfrak{m}_v) : \mathsf{k}(\mathfrak{m}T_{C_v})] = [\mathsf{k}(\mathfrak{m}_{w(v)}) : \mathsf{k}(\mathfrak{m}T_{B_{w(v)}})].$$

It now follows that if we set

$$d(I,v) := a_{w(v)} \cdot [\mathsf{k}(\mathfrak{m}_v) : \mathsf{k}(\mathfrak{m}T_{C_v})], \tag{3.4}$$

for all $v \in \nu_{\mathfrak{m}}(I)$, then

$$j(I) = \sum_{v \in \nu_{\mathfrak{m}}(I)} d(I,v) \cdot v(I). \tag{3.5}$$

Note that, even though the factor $[\mathsf{k}(\mathfrak{m}_v) : \mathsf{k}(\mathfrak{m}T_{C_v})]$ in (3.4) in does not depend on $\widehat{R}$, we cannot escape the inclusion of the term $a_{w(v)} = \lambda(\widehat{R}_{z_w})$, which does depend upon $\widehat{R}$. Of course, if $R$ is analytically unramified, then $a_w = 1$ and the expression for $d(I, v)$ in (3.4) agrees with the one derived from the proof of Corollary 3.8 (see Corollary 3.13 below). Finally, we note that by construction and Proposition 3.10, the integers $d(I, v)$ in (3.4) and (3.5) depend only on $R$, $I$, and $v$. $\qquad\square$

Our first corollary gives a quick proof showing that the $j$-multiplicity changes in the expected way when we replace $I$ by any ideal projectively equivalent to $I$.

**Corollary 3.11**

Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread. Then for all $q > 0$, $j(\overline{I^q}) = q^d \cdot j(I)$.

*Proof.* We may assume that $\mathsf{k}$ is infinite and, using the associativity formula for $j$-multiplicity, reduce to the case that $R$ is a local domain. We first note that since $\nu_{\mathfrak{m}}(I) = \nu_{\mathfrak{m}}(\overline{I^q})$, by Theorem 3.9 we have

$$j(I) = \sum_{v \in \nu_{\mathfrak{m}}(I)} d(I, v) \cdot v(I)$$

and

$$j(\overline{I^q}) = \sum_{v \in \nu_{\mathfrak{m}}(I)} d(\overline{I^q}, v) \cdot v(\overline{I^q}).$$

Clearly, $v(\overline{I^q}) = q \cdot v(I)$, for all $v \in \mathcal{V}_{\mathfrak{m}}(I)$. If we show $d(\overline{I^q}, v) = q^{d-1} \cdot d(I, v)$, for all $v$, the proof will be complete. On the one hand, (3.4) gives a formula for each $d(v, I)$. On the other hand, maintaining the notation leading to (3.4), $x_1^q, \ldots, x_d^q$ generate a minimal reduction of $\overline{I^q}$. Let $T_q$ denote the one dimensional ring derived from these latter generators. Now applying (3.4) to $T_q$ and $\overline{I^q}$, we see that it suffices to show that $[\mathsf{k}(\mathfrak{m}T) : \mathsf{k}(\mathfrak{m}T_q)] = q^{d-1}$. (Note that in fact, $T_q \subseteq T$ is a finite extension, so $\mathsf{k}(\mathfrak{m}T_q) \subseteq \mathsf{k}(\mathfrak{m}T)$.) Write $u_i$ for the image of the fraction $x_i/x_1$ in the residue field $\mathsf{k}(\mathfrak{m}_V)$ of $V$. Then, since $\mathsf{k}(\mathfrak{m}T_q) \subseteq \mathsf{k}(\mathfrak{m}T)$, the images of the $x_i^q/x_1^q$ in $\mathsf{k}(\mathfrak{m}_V)$ are just $u_i^q$. Moreover, since $\mathsf{k}(\mathfrak{m}T)$ is the rational function field in the $u_i$ over $\mathsf{k}$, it follows that $[\mathsf{k}(\mathfrak{m}T) : \mathsf{k}(\mathfrak{m}T_q)] = q^{d-1}$, and this gives what we want. $\square$

Our second corollary yields the following interesting formula for the $j$-multiplicity of a principal ideal times an $\mathfrak{m}$-primary ideal.

**Corollary 3.12**

Let $(R, \mathfrak{m})$ be a local domain, $Q \subseteq R$ an $\mathfrak{m}$-primary ideal and $0 \neq a \in R$. Set $I := a \cdot Q$. Then $j(I) = e(Q) + e(Q + (a)/(a))$.

*Proof.* We first note that since $I^n/\mathfrak{m}I^n$ is isomorphic to $Q^n/\mathfrak{m}Q^n$ for all $n$, $I$ has maximal analytic spread. Moreover, if $x_1, \ldots, x_d$ is a minimal generating set of a reduction of $Q$, then $a \cdot x_1, \ldots, a \cdot x_d$ generate a minimal reduction of $I$. Now, since

$$T := R[x_2/x_1, \ldots, x_d/x_1]_{\mathfrak{m}R[x_2/x_1, \ldots, x_d/x_1]}$$
$$= R[a \cdot x_2/a \cdot x_1, \ldots, a \cdot x_d/a \cdot x_1]_{\mathfrak{m}R[a \cdot x_2/a \cdot x_1, \ldots, a \cdot x_d/a \cdot x_1]},$$

Corollary 3.3 implies that $\nu_{\mathfrak{m}}(I) = \nu_{\mathfrak{m}}(Q)$ and $\mathcal{V}_{\mathfrak{m}}(I) = \mathcal{V}_{\mathfrak{m}}(Q)$. It follows from equation (3.4) that for each $v \in \nu_{\mathfrak{m}}(I)$, $d(v, I) = d(v, Q)$. Thus, by Theorem 3.9

$$j(I) = \sum_{v \in \nu_{\mathfrak{m}}(I)} d(I, v) \cdot v(I) \tag{3.6}$$

$$= \sum_{v \in \nu_{\mathfrak{m}}(I)} d(I, v) \cdot (v(a) + v(Q)) \tag{3.7}$$

$$= \sum_{v \in \nu_{\mathfrak{m}}(Q)} d(Q, v) \cdot v(a) + \sum_{v \in \nu_{\mathfrak{m}}(Q)} d(Q, v) \cdot v(Q). \tag{3.8}$$

But for $x \in R$, the degree function $d_Q(x)$ is, by definition, $e((Q + (x))/(x))$. Thus by formula (1.1), the first term on the right in (3.8) is the multiplicity of the image of $Q$ in $R/(a)$, while, by [14, Theorem 3.9], the second term on the right is just $j(Q) = e(Q)$, which completes the proof. $\qquad \square$

Our final corollary is an amusing application of the description of $d(I, v)$ given in (3.4). For the sake of clarity, we just state it for local domains.

**Corollary 3.13**

Let $(R, \mathfrak{m})$ be a quasi-unmixed local domain. Then $R$ is analytically unramified if and only if for some (respectively, for every) system of parameters $x_1, \ldots, x_d$,

$$e(I) = \sum_{v \in \nu_{\mathfrak{m}}(I)} [\mathsf{k}(\mathfrak{m}_v) : \mathsf{k}(\mathfrak{m}T)] \cdot v(I),$$

where $I$ is the ideal generated by $x_1, \ldots, x_d$ and $T$ is the one dimensional local domain derived from the $x_i$.

*Proof.* Since $R$ is quasi-unmixed, $\ell(I \cdot \widehat{R}/z) = d$, for all minimal primes $z \subseteq \widehat{R}$. The result now follows from Theorem 3.9, equation (3.4), and the definiton of $a_{w(v)}$, since $R$ is analytically unramified if and only if $a_{w(v)} = 1$ for all $v$. $\qquad \square$

*Remark 3.14* As one might expect, $j$-multiplicity lends itself to geometric interpretation. Some of the associated geometric ideas can be found in the original work of Achilles and Manaresi (see [1, 2]).

## 4. A limit superior multiplicity

In this section we wish to discuss the limsup-derived multiplicity mentioned in the introduction and derive a few of its basic properties. Our main result uses Rees valuations to show that this multiplicity is non-zero whenever the ideal in question has maximal analytic spread. We also show that this multiplicity can be used to give a Rees-type result for arbitrary ideals in a locally quasi-unmixed Noetherian ring. This latter result is sort of a hybrid between the original result of Rees and the Flenner-Manaresi version given in [5] using $j$-multiplicity. We begin by observing that the limsup in question exists.

**Observation 4.1** Let $(R, \mathfrak{m})$ be a local ring of dimension $d$ and $I \subseteq R$ an ideal. Then

$$\limsup_n \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)),$$

exists and is a finite real number.

*Proof.* For each $n$ we have $R \supset I \supset I^2 \supset \cdots \supset I^n$. Since the zero-th local cohomology functor is left exact, if we apply $\Gamma_{\mathfrak{m}}$ to the quotients in this filtration, we only get

subadditivity of the corresponding lengths. Thus,

$$\lambda_R(\Gamma_{\mathfrak{m}}(R/I^n)) \leq \sum_{i=0}^{n-1} \lambda_R(\Gamma_{\mathfrak{m}}(I^i/I^{i+1})).$$

If we now think of the term on the right in the expression above as a function of $n$, then it is ultimately just the sum transform of the Hilbert polynomial giving the lengths of the graded components of $\Gamma_{\mathfrak{m}}(\mathrm{G}_I(R))$. In other words,

$$\lim_{n\to\infty} \frac{d!}{n^d} \cdot \sum_{i=0}^{n-1} \lambda_R(\Gamma_{\mathfrak{m}}(I^i/I^{i+1})) = j(I) < \infty.$$

It follows immediately from this that the terms in our sequence are bounded, so the required limsup exists and is finite. $\qquad\square$

It is an open question whether or not in an arbitrary local ring the limit of the sequence in Observation 4.1 exists. In [4, 3], and [7] this limit is shown to exist in many special cases. In particular, in [4] it is shown to exist for graded ideals in a finitely generated standard graded algebra of depth greater than one over a field of characteristic zero. Moreover, in that same paper it is shown that the limit need not be a rational number. Nevertheless, we make the following definition.

DEFINITION 4.2 Let $(R, \mathfrak{m})$ be a local ring of dimension $d$ and $I \subseteq R$ an ideal. We set

$$\epsilon(I) := \limsup_n \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)).$$

We call $\epsilon(I)$ the $\epsilon$-multiplicity of $I$.

*Remark 4.3* Suppose that $R$ is a local domain and $I := a \cdot Q$, for some $\mathfrak{m}$-primary ideal $Q$ and $0 \neq a \in R$. If we assume depth$(R) \geq 2$, then it is easy to see that $\Gamma_{\mathfrak{m}}(R/I^n)$ is isomorphic to $R/Q^n$, so that $\epsilon(I) = e(Q)$. Comparing this with the statement of Corollary 3.12, we see that the difference between $j(I)$ and $\epsilon(I)$ can be made arbitrarily large.

The first property of $\epsilon$-multiplicity we wish to verify is that it is an invariant up to integral closure. The next proposition accomplishes this.

**Proposition 4.4**

Let $(R, \mathfrak{m})$ be a local ring and $I, J \subseteq R$ two ideals with the same integral closure. Then $\epsilon(I) = \epsilon(J)$. Moreover, given the sequences

$$\left\{ \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)) \right\}_n \quad \text{and} \quad \left\{ \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/J^n)) \right\}_n,$$

the limit of the first sequence exists if and only if the limit of the second sequence exists.

*Proof.* Both statements in the lemma clearly reduce to the case that $J \subseteq I$ are ideals and $J$ is a reduction of $I$. Assuming this, take $h > 0$ so that $I^{n+h} = J^n I^h$ for all $n > 0$. From the exact sequence $0 \to I^n/J^n \to R/J^n \to R/I^n \to 0$ and subadditivity, we obtain

$$\lambda(\Gamma_{\mathfrak{m}}(R/J^n)) \leq \lambda(\Gamma_{\mathfrak{m}}(R/I^n)) + \lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)), \tag{4.1}$$

for all $n$. If we show that $\lim_n \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)) = 0$, then it follows immediately that $\epsilon(J) \leq \epsilon(I)$. To see this, note that $\lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)) \leq \lambda(\Gamma_{\mathfrak{m}}(J^{n-h}/J^n))$, for all $n \geq h$. But $\bigoplus_{n \geq h} \Gamma_{\mathfrak{m}}(J^{n-h}/J^n)$ is a finite graded $\mathcal{R}_J/t^{-h} \cdot \mathcal{R}_J$-module. Thus, its Hilbert polynomial has degree less than $d$, so

$$\lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)) = 0, \tag{4.2}$$

as required.

To see that $\epsilon(I) \leq \epsilon(J)$, we follow a similar path, starting with the exact sequences $0 \to J^{n-h}/I^n \to R/I^n \to R/J^{n-h} \to 0$. Applying $\Gamma_{\mathfrak{m}}(-)$ as before, we get

$$\lambda(\Gamma_{\mathfrak{m}}(R/I^n)) \leq \lambda(\Gamma_{\mathfrak{m}}(R/J^{n-h})) + \lambda(\Gamma_{\mathfrak{m}}(J^{n-h}/I^n)), \tag{4.3}$$

for all $n \geq h$. Since $\lambda(\Gamma_{\mathfrak{m}}(J^{n-h}/I^n)) \leq \lambda(\Gamma_{\mathfrak{m}}(I^{n-h}/I^n))$ and the dimension of the $\mathcal{R}_I/t^{-h} \cdot \mathcal{R}_I$-module $\bigoplus_{n \geq h} \Gamma_{\mathfrak{m}}(I^{n-h}/I^n)$ is at most $d$,

$$\lim_{n \to \infty} \frac{d!}{(n-h)^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(J^{n-h}/I^n)) = 0. \tag{4.4}$$

Moreover, since

$$\limsup_n \frac{d!}{(n-h)^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)) = \limsup_n \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n)),$$

it follows immediately from (4.3) that $\epsilon(I) \leq \epsilon(J)$, and the proof of the first statement is complete.

For the second statement, assume that the sequence $\{\frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/J^n))\}_n$ has a limit. Combining the expressions (4.1) and (4.3) we get

$$
\begin{aligned}
\lambda(\Gamma_{\mathfrak{m}}(R/J^n)) &\leq \lambda(\Gamma_{\mathfrak{m}}(R/I^n)) + \lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)) \\
&\leq \lambda(\Gamma_{\mathfrak{m}}(R/J^{n-h})) + \lambda(\Gamma_{\mathfrak{m}}(J^{n-h}/I^n)) + \lambda(\Gamma_{\mathfrak{m}}(I^n/J^n)).
\end{aligned}
$$

If we now multiply by $d!/n^d$ and take the limit as $n$ goes to infinity, it follows from equations (4.2) and (4.4) that the limit

$$\lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n))$$

exists and agrees with $\lim_{n \to \infty} \frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/J^n))$. The reverse implication in the second statement follows along similar lines. $\square$

We now turn to showing that for an ideal $I \subseteq R$, the $\epsilon$-multiplicity of $I$ is non-zero if and only if $I$ has maximal analytic spread. To do this, we need to generalize a key lemma from [16]. To elaborate, in [16, Lemma 4.4], it is shown that if $(R, \mathfrak{m})$ is a

quasi-unmixed local ring of dimension two and $v$ is an $\mathfrak{m}$-valuation of $R$, then there exists a real (rational) number $\delta > 0$ such that $\lambda(R/\mathfrak{a}_n) \geq \delta \cdot n^2$, for all $n \geq 1$, where $\mathfrak{a}_n$ denotes the set of $x \in R$ with $v(x) \geq n$. It turns out that essentially the same proof works for arbitrary local rings. The lemma below applies to any $\mathfrak{m}$-valuation, since any $\mathfrak{m}$-valuation is a Rees valuation of an $\mathfrak{m}$-primary ideal. However, we have phrased the lemma in such a way that it serves our present purposes.

**Lemma 4.5** ([16, c.f. Lemma 4.4])

*Let $(R, \mathfrak{m})$ be a local ring with infinite residue field and $I \subseteq R$ an ideal with maximal analytic spread. Take $v \in \nu_{\mathfrak{m}}(I)$. For each $n \geq 1$, let $\mathfrak{a}_n$ denote the set of elements $x \in R$ such that $v(x) \geq n$ (so that $\mathfrak{a}_n$ is an $\mathfrak{m}$-primary ideal). Then there exists a real number $\delta > 0$ such that $\lambda(R/\mathfrak{a}_n) \geq \delta \cdot n^d$, for all $n \geq 1$.*

*Proof.* Let $q \subseteq R$ be the minimal prime such that $v$ is a valuation on the quotient field of $R/q$, and thus $\ell(I) = \ell(I \cdot R/q)$. Since $\mathfrak{a}_n$ is the inverse image in $R$ of the elements in $R/q$ with value greater than or equal to $n$ under $v$, we may clearly replace $R$ by $R/q$ and assume that $R$ is an integral domain. (Recall that $q$ is the set of elements in $R$ that receive the value $\infty$ under $v$.) Thus, if we let $x_1, \ldots, x_d$ be minimal generators for a minimal reduction of $I$ and $V$ be the Rees valuation ring corresponding to $v$, then $V$ is an essential valuation ring of $T := R[x_2/x_1, \ldots, x_d/x_1]_{\mathfrak{m}R[x_2/x_2, \ldots, x_d/x_1]}$.

To find $\delta$, note that $\mathfrak{m} \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$, so that $\mathfrak{a}_n/\mathfrak{a}_{n+1}$ is a vector space over $\mathsf{k}$ for all $n$. Set $v(x_1) := a$ and recall that the images of the $x_i/x_1$ in $\mathsf{k}(\mathfrak{m}_V)$ are algebraically independent over $\mathsf{k}$. We now claim that for all $r$, the images in $\mathfrak{a}_{ar}/\mathfrak{a}_{ar+1}$ of the monomials of degree $r$ in $x_1, \ldots, x_d$ are linearly independent over $\mathsf{k}$. To see this, suppose that $\tilde{F}$ is a homogeneous polynomial of degree $r$ with coefficients in $\mathsf{k}$ such that the image of $\tilde{F}$ evaluated at the $x_i$ in $\mathfrak{a}_{ar}/\mathfrak{a}_{ar+1}$ is zero. Then, writing $F$ for a pre-image of $\tilde{F}$ with coefficients in $R$, we have $F(\underline{x}) \in \mathfrak{m}_V^{ar+1}$. Thus, $x_1^r \cdot F(1, x_2/x_1, \ldots, x_d/x_1)$ belongs to $\mathfrak{m}_V^{ar+1}$, so $F(1, x_2/x_1, \ldots, x_d/x_1) \in \mathfrak{m}_V$. Thus $\tilde{F}(1, x_2/x_1, \ldots, x_d/x_1) = 0$ in $\mathsf{k}(\mathfrak{m}_V)$. Therefore, the coefficients of $\tilde{F}$ are zero, which is what we want. It follows that $\lambda(\mathfrak{a}_{ar}/\mathfrak{a}_{ar+1}) \geq \binom{r+d-1}{d-1}$. Now let $n$ be a positive integer. Then there is a greatest integer $r$ such that $ar + 1 \leq n$. Thus

$$\lambda(R/\mathfrak{a}_n) \geq \lambda(R/\mathfrak{a}_1) + \lambda(\mathfrak{a}_a/\mathfrak{a}_{a+1}) + \cdots + \lambda(\mathfrak{a}_{ar}/\mathfrak{a}_{ar+1})$$

$$\geq 1 + \binom{d}{1} + \cdots + \binom{d+r-1}{d-1} = \binom{d+r}{d}.$$

Hence,

$$\lambda(R/\mathfrak{a}_n) \geq \frac{1}{d!}(r+1)\cdots(r+d) \geq \frac{1}{d! \cdot a^d}(ar+a)\cdots(ar+ad) \geq \frac{1}{d! \cdot a^d} n^d.$$

So, we may take $\delta := (d! \cdot a^d)^{-1}$, and this completes the proof of the lemma. $\qquad \square$

The following proposition was inspired by the [16, proof of Theorem 4.5].

**Proposition 4.6**

*Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal with maximal analytic spread. Let $x \in R$ be such that there exists $V \in \mathcal{V}_{\mathfrak{m}}(I)$ with $x \notin IV$, yet $x \in \overline{I}_P$, for all prime ideals $P \neq \mathfrak{m}$. Then there exists a real number $\gamma > 0$ such that $\lambda(R/(\overline{I^n} : x^n)) \geq \gamma \cdot n^d$, for all $n > 0$.*

*Proof.* We first reduce to the case that $R$ is a local domain. Let $q \subseteq R$ be the minimal prime such that $V$ is a valuation ring in the quotient field of $R/q$. Then, for $B := R/q$, $\ell(IB) = d$ and $V \in \mathcal{V}_{\mathfrak{m}}(IB)$, by definition of $\mathcal{V}_{\mathfrak{m}}(I)$. Furthermore, $x \notin IV$ and $x \in \overline{IB}_P$, for all prime ideals $P$ in $B$ not equal to $\mathfrak{m}B$. Thus, the hypotheses of the proposition carry over to $B$. Moreover,

$$\lambda(R/(\overline{I^n} : x^n)) \geq \lambda(B/(\overline{I^n B} :_B x^n)),$$

for all $n > 0$, so that it suffices to prove the existence of a real number $\delta > 0$ with

$$\lambda(B/(\overline{I^n B} :_B x^n)) \geq \delta \cdot n^d,$$

for all $n > 0$. Thus we may replace $R$ by $B$, change notation, and begin again assuming that $R$ is a local domain. Invoking the usual base change, we may further assume that $\mathsf{k}$ is infinite.

Now let $v$ be the valuation on the quotient field of $R$ corresponding to $V$. Take integers $0 < b < a$, such that $v(x) = b$ and $v(I) = a$ and set $\tau := a - b$. Then for any $n > 0$ and $r \in (\overline{I^n} : x^n)$, $r \cdot x^n \in I^n V$, so $v(r) \geq \tau \cdot n$. If we now set $\mathfrak{a}_n$ to be the set of elements $y \in R$ with $v(y) \geq n$, then we have that $(\overline{I^n} : x^n) \subseteq \mathfrak{a}_{\tau \cdot n}$, for all $n$. Thus, $\lambda(R/(\overline{I^n} : x^n)) \geq \lambda(R/\mathfrak{a}_{\tau \cdot n})$, for all $n$. It follows from Lemma 4.5 that there exists a real number $\delta > 0$, so that $\lambda(R/\mathfrak{a}_n) \geq \delta \cdot n^d$, for all $n$. We then have that

$$\lambda(R/(\overline{I^n} : x^n)) \geq \delta \cdot (\tau n)^d,$$

and it follows from this that we may take $\gamma := \delta \cdot \tau^d$. In fact, from the proof of Lemma 4.5, $\gamma = \frac{(a-b)^d}{d! \cdot a^d}$.                                                            $\square$

We can now give a proof of the non-vanishing of $\epsilon$-multiplicity.

**Theorem 4.7**

*Let $(R, \mathfrak{m})$ be a local ring and $I \subseteq R$ an ideal. Then $\epsilon(I) \neq 0$ if and only if $I$ has maximal analytic spread.*

*Proof.* Without loss of generality, we may assume that $R$ has an infinite residue field. It follows from the proof of Observation 4.1 that $\epsilon(I) \leq j(I)$. Thus, if $\epsilon(I) \neq 0$, then $j(I) \neq 0$, so $\ell(I) = d$. Conversely, suppose that $\ell(I) = d$. Then, by Proposition 3.1, $\mathcal{V}_{\mathfrak{m}}(I)$ is not empty. Take $V \in \mathcal{V}_{\mathfrak{m}}(I)$. If $V$ is the only Rees valuation ring of $I$, then $I$ is an $\mathfrak{m}$-primary ideal and clearly $\epsilon(I) = e(I)$ is non-zero. Thus, we may assume that $V$ is not the only Rees valuation ring of $I$. Let $V, V_2, \ldots, V_r$ denote the Rees valuation rings of $I$. By the irredundancy of Rees valuations ([17, Chapter 10]), there exists $c \geq 1$ such that

$$\overline{I^c} \neq (I^c V_2 \cap R) \cap \cdots \cap (I^c V_r \cap R).$$

Now, on the one hand, the sequence $\{\frac{d!}{(cn)^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^{cn}))\}$ is a subsequence of the sequence $\{\frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n))\}$, so the limsup associated to the subsequence is less than or equal to the limsup of the entire sequence. If the former is not zero, then the latter is not zero. On the other hand, $\ell(I^c) = \ell(I)$ and $I^c$ and $I$ have the same Rees valuation rings. Thus, after replacing $I$ by $I^c$ and changing notation, we may assume that

$$\overline{I} \neq (IV_2 \cap R) \cap \cdots \cap (IV_r \cap R).$$

Now, take $x \in (IV_2 \cap R) \cap \cdots \cap (IV_r \cap R) \backslash (IV \cap R)$. Then, $x \in \overline{I}_P$, for all primes $P \neq \mathfrak{m}$. Since $\ell(I) = \ell(\overline{I}) = d$ and $\epsilon(I) = \epsilon(\overline{I})$ (by Proposition 4.4), we may replace $I$ by $\overline{I}$ and assume that $x \in I_P$, for all prime ideals $P \neq \mathfrak{m}$. Thus, for each $n$, the module $(x^n, I^n)/I^n$ has finite length and is contained in $\Gamma_{\mathfrak{m}}(R/I^n)$. Therefore it suffices to show that

$$\limsup_n \frac{d!}{n^d} \cdot \lambda((x^n, I^n)/I^n) > 0.$$

Let $q \subseteq R$ be the minimal prime ideal such that $V$ is contained in $\mathsf{k}(q)$. Then the hypotheses on $x$, $I$ and $R$ carry over to $B := R/q$. Since

$$\limsup_n \frac{d!}{n^d} \cdot \lambda((x^n, I^n)/I^n) \geq \limsup_n \frac{d!}{n^d} \cdot \lambda((x^n, I^n)B/I^n B),$$

we may reduce to the case that $R$ is a local domain. But now, since $x$ is a non-zerodivisor, $(x^n, I^n)/I^n$ is isomorphic to $R/(I^n : x^n)$, for all $n \geq 1$. By Proposition 4.6 there exists a real number $\gamma := \frac{(a-b)^d}{d! \cdot a^d} > 0$ such $\lambda(R/(I^n : x^n)) \geq \gamma \cdot n^d$, for all $n$, where $a := v(I)$ and $b := v(x)$. It follows that $\lambda((x^n, I^n)/I^n) \geq \gamma \cdot n^d$ for all $n$. Therefore,

$$\limsup_n \frac{d!}{n^d} \cdot \lambda((x^n, I^n)/I^n) \geq d! \cdot \gamma = \frac{(a-b)^d}{a^d} > 0,$$

and the proof of the theorem is complete. $\square$

We close this section with an $\epsilon$-multiplicity test for equality of integral closure that can be thought of as a hybrid of the Rees multiplicity theorem with the Flenner-Manaresi theorem for $j$-multiplicity (see [5]). As we shall see, our theorem is in some sense a formal consequence of [13, Theorem 2.1]. We would also like to point out that in the theorem below, we only need to check equality of $\epsilon$-multiplicity locally at finitely many prime ideals (c.f. [18]).

**Theorem 4.8**

*Let $R$ be a locally quasi-unmixed Noetherian ring and $J \subseteq I \subseteq R$ ideals. Then the following statements are equivalent.*

(i) *$J$ is a reduction of $I$.*
(ii) *$\epsilon(J_P) = \epsilon(I_P)$, for all prime ideals $P \subseteq R$.*
(iii) *$\epsilon(J_P) = \epsilon(I_P)$, for all prime ideals $P \in \mathrm{Ass}(R/\overline{J})$.*

*Proof.* If $J$ is a reduction of $I$, then by Proposition 4.4, $\epsilon(J_P) = \epsilon(I_P)$ holds for all $P$. Thus, (i) implies (ii). Clearly, (ii) implies (iii).

To see that (iii) implies (i), we first note that we may assume that $(R, \mathfrak{m})$ is a quasi-unmixed local ring. We now proceed by induction on $d = \dim(R)$. When $d = 0$, there is nothing to prove. Suppose $d > 0$. By induction, $J$ is a reduction of $I$ locally on the punctured spectrum of $R$. Now, if $\mathfrak{m} \notin \mathrm{Ass}(R/\overline{J})$, then in fact we have that $J$ is already a reduction of $I$. Thus, we may assume $\mathfrak{m} \in \mathrm{Ass}(R/\overline{J})$, and therefore $\epsilon(J) = \epsilon(I)$. Using Proposition 4.4, we may replace $J$ by $\overline{J} \cap I$ and further assume that $I$ is equal to $J$ locally on the punctured spectrum of $R$. Now consider the following short exact sequence,

$$0 \to I^n/J^n \to R/J^n \to R/I^n \to 0.$$

Since $I^n/J^n$ has finite length, the sequence above remains exact after applying the zero-th local cohomology functor. Thus, taking lengths, we obtain

$$\lambda_R(\Gamma_\mathfrak{m}(R/J^n)) = \lambda_R(\Gamma_\mathfrak{m}(R/I^n)) + \lambda_R(I^n/J^n).$$

If we multiply by $d!/n^d$ and take the limsup, then, since $\epsilon(I) = \epsilon(J)$, we see that the polynomial giving the lengths $\lambda_R(I^n/J^n)$ must have degree less than $d$. Thus, $I$ is integral over $J$, by [13, Theorem 2.1]. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

*Remark 4.9*  We make two comments concerning Theorem 4.8. The first is that since $R$ is locally quasi-unmixed, a prime ideal $P \subseteq R$ satisfies $\ell(J_P) = \dim(R_P)$ if and only if for some $n \geq 1$, $P \in \mathrm{Ass}(R/\overline{J^n})$ (see [9, Proposition 4.1]). Thus, in part (iii) of the theorem we are potentially using a smaller set of primes than those at which locally $J$ has maximal analytic spread.

Our second comment is that using Proposition 4.6, we can quickly recover that portion of [13, Theorem 2.1] used in the proof of Theorem 4.8. For simplicity, we just do the case that $R$ is an integral domain. In other words, we will show that if $J \subseteq I$ are ideals in a quasi-unmixed local domain with $\lambda(I/J) < \infty$ and $P(n)$ is the rational polynomial giving the lengths $\lambda(I^n/J^n)$ for $n$ large, then $J$ is a reduction of $I$ if $\deg(P(n)) < d$. To do this, suppose $J$ is not a reduction of $I$. Let $x$ be an element of $I$ that is not integral over $J$. Thus, $x \notin JV$, for some Rees valuation ring $V$ of $J$. Since $x \in J_P$ for all primes $P \neq \mathfrak{m}$, $V$ must be centered on $\mathfrak{m}$. But $R$ is quasi-unmixed, so $V \in \mathcal{V}_\mathfrak{m}(J)$, by Corollary 3.3. Thus, we may apply Proposition 4.6 to conclude that there exists a rational number $\gamma > 0$, such that $\lambda(R/(J^n : x^n)) \geq \gamma \cdot n^d$, for all $n \geq 1$. Since $R/(J^n : x^n)$ is isomorphic to a submodule of $I^n/J^n$, we have $P(n) \geq \gamma \cdot n^d$ for $n$ large. Thus, $\deg(P(n)) = d$.

# 5. Appendix

Let $(R, \mathfrak{m})$ be a local ring with completion $\widehat{R}$. The purpose of this appendix is to show that for *any* ideal $I \subseteq R$, there is a one-to-one correspondence between the Rees valuation rings of $I$ centered on $\mathfrak{m}$ and the Rees valuation rings of $I\widehat{R}$ centered on $\mathfrak{m}\widehat{R}$. This extends Proposition 3.4, in that we are no longer considering only those Rees valuation rings whose corresponding valuations are $\mathfrak{m}$-valuations. Of course, if $R$ is quasi-unmixed, then every Rees valuation centered on $\mathfrak{m}$ is an $\mathfrak{m}$-valuation, so that in the quasi-unmixed case Theorem 5.3 is equivalent to Proposition 3.4.

A key point in our analysis is the following fundamental relation discovered by Ratliff in [10], namely : If $A$ is a local domain, then there is a height one maximal ideal in the integral closure of $A$ if and only if there is a minimal prime of dimension one in the completion of $A$ (see [10, Proposition 3.5]). In fact, it is this relation that is the key to the proof of Ratliff's celebrated result that a local domain is quasi-unmixed if and only if it satisfies the dimension formula ([10, Theorem 3.1]).

We will need to refine [10, Proposition 3.5] as follows. If the local domain $A$ has the property that its completion has a minimal prime of dimension one, let $B$ denote the one dimensional local domain obtained by modding out this minimal prime. Then the integral closure of $B$ is a discrete valuation ring $W$. If $V$ denotes the contraction

of $W$ to the quotient field of $A$, the following lemma will show that $V$ is an essential valuation ring of $A$.

**Lemma 5.1**

Let $(A, \mathfrak{m}, \mathsf{k})$ be a local ring with completion $\widehat{A}$.

(a) *Suppose there exists a minimal prime $q \subseteq A$ such that for $C := A/q$, there exists a height one maximal ideal $P \subseteq \overline{C}$. Set $L := (\overline{C})_P$. Then there exists a minimal prime $z \subseteq \widehat{A}$ such that $z \cap A = q$ and writing $B := \widehat{A}/z$, $\dim(B) = 1$, and for $M := \overline{B}$, $M \cap \mathsf{k}(q) = L$.*

(b) *Suppose there exists a minimal prime $z \subseteq \widehat{A}$ such that writing $B := \widehat{A}/z$, $M := \overline{B}$ is a discrete valuation ring. Then for $L := M \cap \mathsf{k}(z \cap A)$ and $C := A/z \cap A$, $L = (\overline{C})_{\mathfrak{m}_W \cap \overline{C}}$. In particular, $L$ is an essential valuation ring of $C$.*

*Proof.* Since the minimal primes in $\widehat{A}$ contracting to $q$ correspond to the minimal primes of $\widehat{C}$ contracting to zero in $C$, for both (a) and (b) we may assume that $A = C$ is a local domain with quotient field $K$. To prove (a), let $L := (\overline{A})_P$, where $P \subseteq \overline{A}$ is a height one maximal ideal. Let $I$ be any non-zero principal ideal in $A$. Then $L$ is a Rees valuation ring of $I$. By [17, Proposition 10.4.3], there exists a minimal prime $z \subseteq \widehat{A}$ such that for $B := \widehat{A}/z$, $L = M \cap K$, where $M$ is a Rees valuation ring of $I\widehat{A}$ contained in $\mathsf{k}(z)$. Moreover, by the same proposition,

$$\operatorname{tr.deg.}_{\mathsf{k}} \mathsf{k}(\mathfrak{m}_L) = \dim(B) - 1.$$

Therefore, $\dim(B) = 1$. Since $B$ is a complete local domain, we must have $M = \overline{B}$, which is what we want.

To prove (b), first note that $\dim(B) = \dim(\widehat{A}/z) = 1$. Let $\tilde{K}$ denote the total quotient ring of $\widehat{A}$. By [10, Lemma 2.17] and the proof of Theorem 3.1, there exists $c/b \in \overline{A} \backslash A$ with the following property. For $D := A[c/b]$ and $\widehat{P_0} \subseteq \widehat{D} = \widehat{A}[c/b]$, the maximal ideal containing $z\tilde{K} \cap \widehat{D}$, $z\tilde{K} \cap \widehat{D}$ is the only minimal prime ideal contained in $\widehat{P_0}$. Thus, $\operatorname{height}(\widehat{P_o}) = 1$. As in the proof of [10, Theorem 3.1], $\widehat{P_0} = P_0 \cdot \widehat{D}$, for some height one maximal ideal $P_0 \subseteq D$.

Let $z = z_1, \dots, z_t$ be the minimal primes of $\widehat{A}$, set $B_i := \widehat{A}/z_i$ and write $E$ for $\widehat{A}$ modulo its nilradical, $N$. Then writing $\tilde{A}$ for the integral closure of $\widehat{A}$ we have,

$$\tilde{A}/J \subseteq \overline{E} = \overline{B_1} \oplus \cdots \oplus \overline{B_t} = M \oplus \overline{B_2} \oplus \cdots \oplus \overline{B_t},$$

for $J := N\tilde{K} \cap \tilde{A}$. Let $\mathfrak{m}' \subseteq \tilde{A}$ be the maximal ideal such that $\mathfrak{m}'/J$ is the contraction to $\tilde{A}/J$ of the maximal ideal in $\overline{E}$ corresponding to $\mathfrak{m}_M$. Thus, $\mathfrak{m}' \cap \widehat{D} = \widehat{P_0}$, since $\widehat{D}/z\tilde{K} \cap \widehat{D}$ is an integral extension of $B$ and is therefore local. It follows that

$$P_0 = \mathfrak{m}' \cap D = (\mathfrak{m}' \cap \overline{A}) \cap D.$$

Thus, $\mathfrak{m}' \cap \overline{A}$ lies over $P_0$, so that $\mathfrak{m}' \cap \overline{A}$ has height one. By construction, the maximal ideal of $\overline{B} = M$ lies over $\mathfrak{m}' \cap \overline{A}$. Therefore, $L := \overline{A}_{\mathfrak{m}' \cap \overline{A}} \subseteq M$. Since $L$ is a discrete valuation ring, $L = M \cap K$, which completes the proof. $\qquad\square$

*Remark 5.2* Before stating the main theorem of this section, we make a comment about minimal reductions. Suppose $R$ is a local ring and $I \subseteq R$ is an ideal such that

$\ell(I) = s \geq 1$. Suppose that for some minimal prime $z \subseteq \widehat{R}$ and $B := \widehat{R}/z$, $\ell(IB) = e \leq s$. Then there exists $n > 0$ and a minimal reduction of $I^n$ generated by $x_1, \ldots, x_s$ such that the images of any $e$ of the $x_i$ in $B$ generate a minimal reduction of $I^nB$. If $\mathsf{k}$ is infinite, then we can take $n = 1$. This follows since minimal reductions correspond to homogeneous systems of parameters in fiber rings (see [17, Proposition 8.3.8 and its proof]). In fact, if $\mathcal{F}$ denotes the fiber ring of $R$ with respect to $I$ and $\mathcal{F}_B$ denotes the fiber ring of $B$ with respect to $IB$, then $\mathcal{F}$ maps onto $\mathcal{F}_B$. Thus, by prime avoidance, one may choose a homogeneous system of parameters of length $s$ for $\mathcal{F}$ such that the images of any $e$ of these elements in $\mathcal{F}_B$ form a homogeneous system of parameters. One then adjusts the degrees of the elements so that they all have degree $n$ for some $n$. When $\mathsf{k}$ is infinite, one can take $n = 1$, as in the standard form of Noether Normalization.

We are now ready for a version of Proposition 3.4 for Rees valuations centered on $\mathfrak{m}$ that are not necessarily $\mathfrak{m}$-valuations. We state it in global form.

**Theorem 5.3**

Let $R$ be a Noetherian ring an $I \subseteq R$ an ideal. For a prime ideal $P$ containing $I$, let $\tilde{\mathcal{V}}_P(I)$ denote the Rees valuation rings of $I$ centered on $P$ and $\tilde{\mathcal{V}}_P(I\widehat{R}_P)$ denote the Rees valuation rings of $I\widehat{R}_P$ centered on $P\widehat{R}_P$.

(a) Take $V \in \tilde{\mathcal{V}}_P(I)$ and write $q$ for the corresponding minimal prime. Then there exists a minimal prime $z \subseteq \widehat{R}_P$ with $z \cap R = q$ such that $V = W \cap \mathsf{k}(q)$ for some $W \in \tilde{\mathcal{V}}_P(I\widehat{R}_P)$ contained in $\mathsf{k}(z)$.

(b) Take $W \in \tilde{\mathcal{V}}_P(I\widehat{R}_P)$ and write $z$ for the corresponding minimal prime. Then $V := W \cap \mathsf{k}(z \cap R) \in \tilde{\mathcal{V}}_P(I)$.

Moreover, the correspondence from $\tilde{\mathcal{V}}_P(I)$ to $\tilde{\mathcal{V}}_P(I\widehat{R}_P)$ determined by (a) is a one-to-one, onto function of sets.

*Proof.* We may clearly assume that $R$ is local and $P = \mathfrak{m}$. Assuming that (a) and (b) hold, the proof that the correspondence from $\tilde{\mathcal{V}}_{\mathfrak{m}}(I)$ to $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R})$ determined by (a) is a one-to-one onto function is exactly the same as the one given in the proof of Proposition 3.4. Therefore, we only need to prove statements (a) and (b). However, since (a) follows immediately from [17, Theorem 10.4.3], we only need to prove statement (b).

To prove (b), take $W \in \tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R})$ and assume that $W$ is contained in $\mathsf{k}(z)$, for some minimal prime $z \subseteq \widehat{R}$. As before, we may replace $R$ by $R/z \cap R$, change notation, and assume that $R$ is a local domain with quotient field $K$. Write $B := \widehat{R}/z$ and set $s := \ell(I)$ and $e := \ell(IB)$. Since for all $n \geq 1$, $\tilde{\mathcal{V}}_{\mathfrak{m}}(I) = \tilde{\mathcal{V}}_{\mathfrak{m}}(I^n)$ and $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R}) = \tilde{\mathcal{V}}_{\mathfrak{m}}(I^n\widehat{R})$, we may, by Remark 5.2, replace $I$ by $I^n$, for a suitable $n$, change notation, and assume that there exist $x_1, \ldots, x_s$ in $I$ generating a minimal reduction of $I$ such that the images of any $e$ of these elements in $B$ generate a minimal reduction of $IB$.

We first note that since $B$ is quasi-unmixed and $W$ is a Rees valuation ring of $IB$ centered on $\mathfrak{m}$, then $e = \ell(IB) = \dim(B)$ (see for example, [17, Theorem 10.4.2]). Thus, $IB$ is an ideal with maximal analytic spread. Temporarily take any subset $x_{i_1}, \ldots, x_{i_e}$ of the elements $x_1, \ldots, x_s$. By Corollary 3.3, $W$ is an essential valuation ring of $T_B$, where $T_B$ is our standard one dimensional local domain derived from the

images of $x_{i_1}, \ldots, x_{i_e}$ in $B$. In particular, if $w$ is the Rees valuation associated to $W$, then since $x_{i_j} T_B = I T_B$, $w(x_{i_j}) = w(I)$, for all $1 \leq j \leq e$. Therefore $w(x_i) = w(I)$, for all $1 \leq i \leq s$.

Now fix $x_1, \ldots, x_e$ and let $T_B$ denote the one dimensional ring derived from the images of these elements in $B$. Set $U := R[x_2/x_1, \ldots, x_e/x_1]_{\mathfrak{m}R[x_2/x_1, \ldots, x_e/x_1]}$. Note that $U$ is a local domain, but $U$ does not have dimension one unless $e = d$. Let $U_{\widehat{R}}$ be the local ring derived in the same manner over $\widehat{R}$ from $x_1, \ldots, x_e$. Let $\tilde{K}$ denote the total quotient ring of $\widehat{R}$. Then, since the images of the $x_i$ in $B$ are analytically independent in $\mathfrak{m}B$, $T_B = U_{\widehat{R}}/z\tilde{K} \cap U_{\widehat{R}}$. Thus, $z\tilde{K} \cap U_{\widehat{R}}$ is a minimal prime of dimension one in $U_{\widehat{R}}$. Therefore, $\widehat{U_{\widehat{R}}}$ has a minimal prime of dimension one. Note that since $\widehat{U_{\widehat{R}}} = \widehat{U}$ (see [11, Lemma 3.2]), it follows that the local domain $U$ has a minimal prime of dimension one in its completion.

Now, by part (a) of Lemma 5.1 with $q := z\tilde{K} \cap U_{\widehat{R}}$, $A := U_{\widehat{R}}$ and $L := W$, there exists a minimal prime $z_1 \subseteq \widehat{U_{\widehat{R}}} = \widehat{U}$ with $\dim(\widehat{U}/z_1) = 1$ such that

$$z_1 \cap U_{\widehat{R}} = z\tilde{K} \cap U_{\widehat{R}} \quad \text{and} \quad M \cap \mathsf{k}(z\tilde{K} \cap U_{\widehat{R}}) = M \cap \mathsf{k}(z) = W,$$

where $M$ denotes the integral closure of $\widehat{U}/z_1$. By part (b) of Lemma 5.1,

$$V := M \cap \mathsf{k}(z_1 \cap U) = M \cap K = W \cap K$$

is an essential valuation ring of $U$. In particular, $V = (\overline{U})_Q$, for $Q = \mathfrak{m}_W \cap \overline{U}$.
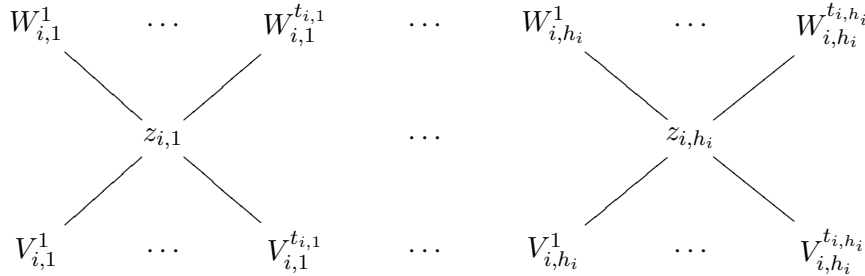
Finally, from the last sentence in the third paragraph of this proof, we have that $w(x_i) = w(I)$ for all $1 \leq i \leq s$, so that $x_2/x_1, \ldots, x_s/x_1 \in W \cap K = V$. Let $S$ be the $R$ algebra generated by these fractions. Then $V$ is a localization of the integral closure of a finitely generated $S$-algebra. If we show that $\mathrm{tr.\,deg.}_{\mathsf{k}(\mathfrak{m}_V \cap S)} \mathsf{k}(\mathfrak{m}_V) = 0$, then by [9, Lemma 3.1], $\mathrm{height}(\mathfrak{m}_V \cap \overline{S}) = 1$. It follows that $V = (\overline{S})_{\mathfrak{m}_V \cap \overline{S}}$ and therefore $V$ is a Rees valuation ring of $I$. Since $V$ is centered on $\mathfrak{m}$, $V \in \tilde{\mathcal{V}}_{\mathfrak{m}}(I)$, which is what we want.

To see that $\mathrm{tr.\,deg.}_{\mathsf{k}(\mathfrak{m}_V \cap S)} \mathsf{k}(\mathfrak{m}_V) = 0$, note that we have $U \subseteq S_{\mathfrak{m}_V \cap S} \subseteq V$, since $V$ is an essential valuation ring of $U$. Moreover, $\mathrm{tr.\,deg.}_{\mathsf{k}(\mathfrak{m}U)} \mathsf{k}(\mathfrak{m}_V) = 0$ and therefore $\mathrm{tr.\,deg.}_{\mathsf{k}(\mathfrak{m}_V \cap S)} \mathsf{k}(\mathfrak{m}_V) = 0$. This completes the proof of the theorem. $\qquad\square$

*Remark 5.4* In Remark 3.5 we noted that the one-to-one correspondence between the elements of $\mathcal{V}_{\mathfrak{m}}(I)$ and $\mathcal{V}_{\mathfrak{m}}(I\widehat{R})$ are 'parameterized' by the minimal primes $z \subseteq \widehat{R}$ for which $\ell(I \cdot \widehat{R}/z) = d$. The proof of Theorem 5.3 shows that the correspondence between $\tilde{\mathcal{V}}_{\mathfrak{m}}(I)$ and $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R})$ is parameterized in the same way by the minimal primes $z \subseteq \widehat{R}$ with $\ell(I \cdot \widehat{R}/z) = \dim(\widehat{R}/z)$. To elaborate, let $1 \leq d_1 < \cdots < d_r$ be the dimensions of the rings of the form $\widehat{R}/z$, where $z \subseteq \widehat{R}$ is a minimal prime with $\ell(I \cdot \widehat{R}/z) = \dim(\widehat{R}/z)$. List these minimal primes as $\{z_{i,j}\}$ with $1 \leq i \leq r$ and $1 \leq j \leq h_i$, where $d_i := \dim(\widehat{R}/z_{i,j})$ for all $i$ and $j$. Set $B_{i,j} := \widehat{R}/z_{i,j}$ for all $i$ and $j$. Then we may write the elements in $\tilde{\mathcal{V}}_{\mathfrak{m}}(I)$ and $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R})$ in the form $\tilde{\mathcal{V}}_{\mathfrak{m}}(I) = \{V_{i,j}^l\}$ and $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R}) = \{W_{i,j}^l\}$ with $1 \leq l \leq t_{i,j}$ so that each $W_{i,j}^l$ is contained in $\mathsf{k}(z_{i,j})$ and $V_{i,j}^l = W_{i,j}^l \cap \mathsf{k}(z_{i,j} \cap R)$, for all $i, j$ and $l$. Note that for fixed $i, j, l$ we have

$$d_i = \dim(B_{i,j}) = \ell(IB_{i,j}) = \mathrm{tr.\,deg.}_{\mathsf{k}} \mathsf{k}(\mathfrak{m}_{W_{i,j}^l}) = \mathrm{tr.\,deg.}_{\mathsf{k}} \mathsf{k}(\mathfrak{m}_{V_{i,j}^l}).$$

Thus, the correspondence between $\tilde{\mathcal{V}}_{\mathfrak{m}}(I)$ and $\tilde{\mathcal{V}}_{\mathfrak{m}}(I\widehat{R})$ for $i$ fixed can be visualized as

$$
\begin{array}{ccccccccc}
W_{i,1}^1 & \cdots & W_{i,1}^{t_{i,1}} & \cdots & W_{i,h_i}^1 & \cdots & W_{i,h_i}^{t_{i,h_i}} \\
& \diagdown \quad \diagup & & & & \diagdown \quad \diagup & \\
& z_{i,1} & & \cdots & & z_{i,h_i} & \\
& \diagup \quad \diagdown & & & & \diagup \quad \diagdown & \\
V_{i,1}^1 & \cdots & V_{i,1}^{t_{i,1}} & \cdots & V_{i,h_i}^1 & \cdots & V_{i,h_i}^{t_{i,h_i}}
\end{array}
$$

where $z_{i,1}, \ldots, z_{i,h_i}$ are the minimal primes for which $\dim(\widehat{R}/z_{i,j})) = d_i$. Moreover, there is one such diagram for each $1 \le i \le r$.

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