Journal of Pure and Applied Algebra 225 (2021) 106587

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Corrigendum

Corrigendum to "Uniform symbolic topologies and finite extensions" [J. Pure Appl. Algebra 219 (2015) 543–550]

Craig Huneke [a], Daniel Katz [b,*], Javid Validashti [c]

[a] Department of Mathematics, University of Virginia, Charlottesville, VA 22903, United States of America
[b] Department of Mathematics, University of Kansas, Lawrence, KS 66045, United States of America
[c] Department of Mathematics, DePaul University, Chicago IL 60614, United States of America

ARTICLE INFO

Article history:
Received 23 June 2020
Received in revised form 30 June 2020
Available online 19 November 2020
Communicated by S. Iyengar

MSC:
13A02; 13F20; 13H15

Keywords:
Symbolic powers

In this note we point out that Proposition 2.4 in [3] is not correct as stated. We thank Bernd Ulrich for making us aware that the proof of Proposition 2.4 is wrong.

In [2] and [3] we asked the following question: Let R be a complete local domain. Does there exist $b \geq 1$ such that $P^{(bn)} \subseteq P^n$, for all prime ideals $P \subseteq R$ and $n \geq 1$. Here $P^{(k)}$ denotes the kth symbolic power of the prime ideal P. Proposition 2.4 in [3] seeks to show that this question is well posed. In order for this question to be well posed, it must be the case that for every prime $P \subseteq R$, the adic and symbolic topologies of P are equivalent. Recall that for a prime ideal P in a Noetherian ring R, the adic and symbolic topologies of P are said to be equivalent, if for every $n \geq 1$, there exists $m \geq 1$ such that $P^{(m)} \subseteq P^n$. By [6], Theorem 1, the adic and symbolic topologies are equivalent for every prime ideal if R is locally analytically irreducible. This latter property holds if R is a complete, normal local domain. Thus the question above is well posed when R is a complete normal local domain. The error in the proof of Proposition 2.4 comes from attempting to descend the equivalence of adic and symbolic topologies from the integral closure of a

DOI of original article: https://doi.org/10.1016/j.jpaa.2014.05.012.
* Corresponding author.
E-mail addresses: huneke@virginia.edu (C. Huneke), dlk53@ku.edu (D. Katz), jvalids@depaul.edu (J. Validashti).

https://doi.org/10.1016/j.jpaa.2020.106587

complete local domain to the complete local domain itself. However, the proof of Proposition 2.4 in [3] does show that if the going down property holds between R and its integral closure, then the adic and symbolic topologies of all primes in R are equivalent. In the Proposition below, we use the work of McAdam [4], McAdam and Ratliff [5] and Schenzel [6] to give equivalent characterizations of when the adic and symbolic powers are equivalent for all primes in an excellent local domain R. First, we give an example, a standard example of the failure of the going down property, which can be modified to show that in a complete local domain, there can exist prime ideals whose adic and symbolic properties are not equivalent. The domain we give is even a hypersurface.

Example. Let k be a field and $S := k[[x, y, z]]$, the power series ring in three variables over k. Set $R := k[[x(z - x), yx, y, z]]$, so that R is a complete local domain, in fact, a hypersurface, and S is its integral closure. Set $Q := (z - x, y)S$ and $Q_0 := Q \cap R = (x(z - x), xy, y)R$. Set $P := xS \cap R = (x(z - x), xy)R$. Then Q is minimal over PS, and $Q \cap R = Q_0 \neq P$. Thus, the going down property fails for $P \subseteq Q_0$, which forces the P-adic and P-symbolic topologies to be not equivalent, by Proposition 3.3 in [5]. In fact, for this example, $P^{(n)}$ is not contained in P^2 for all $n \geq 1$. We can see this as follows: by induction on n we claim that $x^n y \in R$. For $n = 0, 1$ this is by definition of R. Suppose that $x^i y \in R$ for $i < n$. Then $x^n y = -(x(z - x))(x^{n-2}y) + (x^{n-1}y)z \in R$. On the other hand, $(x^n y)(y^{n-1}) = (xy)^n \in P^n$. Since $y \notin P$, this equation shows that $x^n y \in P^{(n)}$. However, $x^n y \notin P^2$, since it is clear that this is not true even after extension to $P^2 S$, as $x^n y \notin Q^2$.

While Schenzel's theorem [6] provides a sufficient condition for the equivalence of adic and symbolic topologies for every prime, the work of McAdam and Ratliff, together with a well known result about completions show that if a Noetherian domain is sufficiently nice, e.g., excellent, one has the following characterization. In the proposition below, we say that a prime ideal $P \subseteq R$ is *unibranched* in its integral closure R', if there exists just one prime ideal in R' lying over P.

Proposition. *Let R be an excellent Noetherian domain with integral closure R'. The following are equivalent.*

(i) *For every prime $P \subseteq R$, the adic and symbolic topologies of P are equivalent.*
(ii) *Going down holds between R and R'.*
(iii) *Every prime $P \subseteq R$ of height greater than one is unibranched in R'.*
(iv) *R_Q is analytically irreducible for every prime ideal Q having height greater than one.*

Proof. (i) implies (ii) follows from Proposition 3.3, in [5]. The equivalence of (ii) and (iii) is Theorem 2 in [4]. To see that (iii) implies (iv), suppose $Q \subseteq R$ is a prime ideal of height greater than one and the completion $\widehat{R_Q}$ of R_Q is not analytically irreducible. Since R is excellent, $\widehat{R_Q}$ is reduced, and therefore has more than one minimal prime. Thus, $(R_Q)'$ has more than one maximal ideal ([1], Theorem 6.5), and hence $Q \subseteq R$ is not unibranched in R'. That (iv) implies (i) follows from Schenzel's theorem [6].

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