MATH 540: GUIDELINES AND PRACTICE PROBLEMS FOR THE FINAL EXAM

The final exam will be comprehensive, covering everything done this semester, including material covered since Exam 2. Questions on the exam will be of the following types: Stating definitions, propositions or theorems; short answer; true-false; and presentation of a proof of a theorem. Though the proofs you are responsible for are listed below, you will be responsible for verifying various theorems in class for particular examples. **You will be allowed to use a calculator on this exam.**

Any definitions, propositions theorems, corollaries that you need to know how to state appear in the Daily Update, and all such are candidates for questions. You will need to be able to answer brief questions about these results as well as true-false statements about these results. Most of the definitions you need to know are also in the Daily Update, but it is best to check your notes for all definitions we have given throughout the semester.

You will also be responsible for working any type of problem that was previously assigned as homework.

On the Exam you will be required to state and provide a proof of to of the following Theorems. You must state the full theorem, even if you are asked to prove a particular part of the theorem.

(i) If p is a prime number and $p|ab$, then $p|a$ or $p|b$.
(ii) Be able to state the Chinese Remainder Theorem in full generality, but provide a proof when just two congruences are given.
(iii) The existence of a division algorithm for the Gaussian integers
(iv) Fermat's Sum of Two Square's Theorem - showing only that primes congruent to 1 mod 4 are a sum of two square
(v) The part of the Pythagorean triples theorem that assumes x, y, z is a primitive Pythagorean triple.

Practice Problems

1. Prove the following statements, first using mathematical induction, and then using the well ordering principle.

(i) $2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$.
(ii) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$.

2. Let S be the set of $n \times n$ matrices over $\mathbb{R}$. For $A, B \in S$, define $A \sim B$ to mean there exists an invertible matrix $Q \in S$ such that $B = Q^{-1}AQ$. Prove that $\sim$ is an equivalence relation. Let C be an $n \times n$ scalar matrix, i.e., $C = \lambda \cdot I_n$, where I_n is the $n \times n$ identity matrix. Describe the equivalence class of C.

3. Calculate the GCD and LCM of 248 and 660 in two different ways. Write the GCD in terms of 248 and 660, as granted by Bezout's principle.

4. Calculate $\phi(2400)$ in two different ways. Verify Euler's formula for $n = 48$.

5. Solve the system of congruences $2x \equiv 7 \pmod{11}$, $3x \equiv 5 \pmod{4}$ and $x \equiv 4 \pmod{9}$ - and write your answer as the unique value less than $11 \cdot 4 \cdot 9 = 396$.

6. For a prime p, recall the Legendre symbol $(\frac{a}{p})$, with $p \nmid a$, equals 1 if a is a quadratic residue modulo p or -1 if a is a quadratic nonresidue modulo p.

(i) Calculate $(\frac{(240}{107})$.
(ii) Calculate $(\frac{12}{17})$ using Gauss's lemma, and then verify your answer using basic properties of the Legendre symbol.

7. Determine whether or not the polynomial $4x^2 + 8x - 8$ has any roots modulo 83. If the answer is yes, find two roots that are distinct modulo 83.

8. Find the GCD of the Gaussian integers $x = 8 + 6i$ and $y = 3 - 4i$ and express this GCD in terms of x and y, as granted by Bezout's principle.

9. Factor the following Gaussian integers into a product of Gaussian primes: 2400, $2 + 6i$, $3 + 5i$.

10. Solve the congruence $(2+4i)x \equiv 1 - 3i$ (modulo $2+5i$). Note, in the Gaussian integers G, we say $a \equiv b$ mod c if c divides $a - b$ in G.

11. Verify the following properties about Pythagorean triples x, y, z. These show that 3, 4, 5 are lurking around all primitive Pythagorean triples.

(i) Exactly one of x, y, z is divisible by 5.
(ii) If the triple is primitive, either x or y is divisible by 3.
(iii) At least one of x, y, z is divisible by 4.

12. Show that if a and b are positive integers, then $a^2 \mid b^2$ implies $a \mid b$.

13. Show that if a, b, and c are positive integers with $\gcd(a, b) = 1$ and $ab = c^n$, then there are positive integers d and e such that $a = d^n$ and $b = e^n$.

14. Calculate $\Phi_8(x), \Phi_{12}(x), \Phi_{35}(x)$.

15. Follow the proof of the theorem from the lecture of May 1 to find ten primes in the arithmetic progression $\{6t + 1\}_{t \geq 1}$. You may use a calculator or computer.

16. Find a positive integer that has at least three different representations as the sum of two squares, disregarding signs and the order of the summands.

17. Prove that of any four consecutive integers, at least one is not representable as a sum of two squares .

18. Any previous homework or quiz problem.