FALL 2024: MATH 790 EXAM 3

Throughout this exam, unless stated otherwise, all vector spaces will be defined over the field F. Each problem is worth 10 points. You will work in teams on this exam. You may use your notes, the Daily Summary, and any homework you have done (providing full details), but you may not consult any other sources, including, any algebra textbook, the internet, any graduate students not on your team, or any professor except your Math 790 instructor. You may not cite without proof any facts not covered in class or the homework. All members of each team should contribute to the team's effort. The solutions should be typeset in LaTex. Each team member should also participate in the typesetting effort. Each team should turn a hard copy of its solutions at the start of the final exam on Thursday, December 19 at 10:30am. **Good luck on the exam!**

1. Write V^* for the dual space of V, i.e., $\mathcal{L}(V, F)$. The elements of V^* are called *linear functionals* on V. Let $B := \{v_1, \ldots, v_n\}$ be a basis for V.

   (i) For each $1 \leq j \leq n$, define $v_j^* \in V^*$ by $v_j^*(v_i) = 1$, if $i = j$ and $v_j^*(v_i) = 0$, if $i \neq j$[1]. Show that $B^* := \{v_1^*, \ldots, v_n^*\}$ is a basis for V^*. This basis is called the *dual basis* to B.

   (ii) For $v \in V$, define $\hat{v} : V^* \to F$ by $\hat{v}(f) := f(v)$, for all $f \in V^*$. Prove that $\hat{v} \in (V^*)^*$ and $\hat{v_1}, \ldots, \hat{v_n}$ is a basis for $(V^*)^*$, the *double dual* of V.

   (iii) Show directly, without using bases, that the map from $\phi : V \to (V^*)^*$ given by $\phi(v) := \hat{v}$ is an isomorphism of vector spaces. Because this map is a natural one, we say that V and $(V^*)^*$ are *canonically* isomorphic. In other words, this maps is independent of the choice of basis, where as V and V^* are non-canonically isomorphic, as any isomorphism between them arises by identifying bases.

2. Maintaining the notation from the previous problem, suppose further that V is an inner product space over $F = \mathbb{R}$ or $\mathbb{C}$.

   (i) For $v_0 \in V$ fixed, show that $\phi : V \to F$ given by $\phi(v) := \langle v, v_0 \rangle$ belongs to V^*.

   (ii) Conversely, given any $f \in V^*$, prove that there exists a *unique* $v_0 \in V$ such that $f(v) = \langle v, v_0 \rangle$, for all $v \in V$. Hint: Work with an orthonormal basis for V.

3. Let $T \in \mathcal{L}(V, V)$. Suppose V is a finite dimensional inner product space over $\mathbb{C}$. This problem develops the standard definition of the adjoint of T. The goal is to construct a unique linear transformation $T^* \in \mathcal{L}(V, V)$ satisfying $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$, for all $v, w \in V$.

   (i) Fix $w \in V$. Define $\phi_w : V \to F$ by $\phi_w(v) = \langle T(v), w \rangle$. Show that $\phi_w \in V^*$.

   (ii) By The previous problem, there exists a unique $w_0 \in V$ such that $\phi_w(v) = \langle v, w_0 \rangle$, for all $v \in V$. Set $T^*(w) := w_0$. Then, by definition, $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$, for all v. Doing this for each w gives a function $T^* : V \to V$. Show that $T^* \in \mathcal{L}(V, V)$.

   (iii) Show that T^* is the unique element in $\mathcal{L}(V, V)$ satisfying $\langle T(v), w \rangle = \langle v, T^*(w) \rangle$, for all $v \in V$.

   (iv) Let B be an orthonormal basis for V. Show that $[T^*]_B^B = ([T]_B^B)^*$, so that the present definition of T^* agrees with the one given in class.

4. A sequence $\mathcal{V} : \quad 0 \xrightarrow{i} V_n \xrightarrow{T_n} V_{n-1} \xrightarrow{T_{n-1}} \cdots \xrightarrow{T_2} V_1 \xrightarrow{T_1} V_0 \xrightarrow{\pi} 0$ of vector spaces and linear transformations is a *complex* of vector spaces if the image of each T_{i+1} is contained in the kernel of T_i (including $i := T_{n+1}$ and $\pi := T_0$). The jth *homology* of the complex is the quotient space $\mathrm{H}_j(\mathcal{V}) := \ker(T_j)/\mathrm{im}(T_{j+1})$. Assume that each vector space has finite dimension. Show that $\Sigma_{j \geq 0}(-1)^{j+1}\dim(V_j) = \Sigma_{j \geq 0}(-1)^{j+1}\dim(\mathrm{H}_j(\mathcal{V}))$. Hint: Experiment with the $n = 2$ case.

5. Set $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{M}_n(\mathbb{R})$.

   (i) For $p \geq 1$, find p distinct pth roots of A

   (ii) Find the solution to the system of first order linear differential equations given by the vector equation $\mathbf{X}'(t) = A \cdot \mathbf{X}(t)$, with initial condition $\mathbf{X}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Here $\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$.

6. Let $\{V_i\}_{i \in I}$ be a collection of (not necessarily finite dimensional) vector spaces. Prove the following properties of the direct sum $\bigoplus_{i \in I} V_i$.

   (a) For each $i_0 \in I$ there is a natural inclusion of vector spaces $j_{i_0} : V_{i_0} \to \bigoplus_{i \in I} V_i$.

   (b) The direct sum $\bigoplus_{i \in I} V_i$ satisfies the following universal property: Given a vector space U and a collection of linear transformations $f_i : V_i \to U$, for each $i \in I$, there exists a unique linear transformation $f : \bigoplus_{i \in I} V_i \to U$ satisfying $f \circ j_i = f_i$, for all $i \in I$.

   (c) For any vector space U, $(\bigoplus_{i \in I} V_i) \otimes U \cong \bigoplus_{i \in I}(V_i \otimes U)$.

7. Let V, W, U be vector spaces that need not be finite dimensional. Prove that $(V \otimes W) \otimes U \cong V \otimes (W \otimes U)$.

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[1]Recall that to define a linear transformation with domain V, it suffices to specify its value on a basis

8. The vector spaces below need not be finite dimensional.
   (i) Suppose that $T : L \to M$ and $S : M \to L$ are linear transformations of vector spaces such that ST is the identity on L and TS is the identity on M. Prove that T is an isomorphism with inverse S.
   (Ii) Suppose (P, f) is a tensor product of V and W. Suppose $\alpha : P \to P_1$ is an isomorphism of vector spaces. Set $f_1 := \alpha \circ f$. Show that (P_1, f_1) is a tensor product of V and W.

9. Let V be a vector space over F. Let L denote $\operatorname{Span}\{v \otimes v' - v' \otimes v \mid v, v' \in V\} \subseteq V \otimes V$. Let $v_1 * v_2$ denote the coset $v_1 \otimes v_2 + L$ in the quotient space $(V \otimes V)/L$. Set $S^2(V) := (V \otimes V)/L$, the *symmetric square* of V.
   (i) Show that the same bilinear properties holding in $V \otimes V$ hold with respect to the product $*$ in $S^2(V)$.
   (ii) Show that $v_1 * v_2 = v_2 * v_1$ in $S^2(V)$, for all $v_1, v_2 \in V$.
   (iii) Given a vector space U, a bilinear map $h : V \times V \to U$ is *symmetric* if $h(v_1, v_2) = h(v_2, v_1)$ for all $v_1, v_2 \in V$. Let $\hat{f} : V \times V \to S^2(V)$ be the natural map i.e., the usual bilinear map $f : V \times V \to V \otimes V$ followed by the quotient map from $V \otimes V \to S^2(V)$. Prove that $\hat{f}$ is a symmetric bilinear map, and given any vector space U and a symmetric bilinear map $g : V \times V \to U$, there exists a unique linear transformation $T : S^2(V) \to U$ such that $T \circ \hat{f} = g$
   (iv) Suppose $v_1, \ldots, v_n$ is a basis for V. Find a basis for $S^2(V)$.
   (v) If $\dim(V) = n$, what is $\dim(S^2(V))$?

10. Let V and W be vector spaces. Recall that V^* denotes the dual space of V.
   (i) Prove that there exists a unique linear transformation $T : V^* \otimes W \to \mathcal{L}(V, W)$ such that $T(f \otimes w)(v) = f(v)w$, for all $f \otimes w \in V^* \otimes W$ and $v \in V$.
   (ii) Prove that if V and W are finite dimensional, then T is an isomorphism.
   (iii) Let $V = \mathbb{R}^4$, $W = \mathbb{R}^3$, $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \in V^*$, $f_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} \in V^*$, $w_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $w_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Find the matrix of $T(f_1 \otimes w_1 + f_2 \otimes w_2)$ with respect to the standard bases of V and W.

Bonus Problems. Each problem below is worth 10 points. Solutions must be completely correct in order to receive any credit.
   (i) Let $\{f_n\}$ be the Fibonacci sequence $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, ..., $f_n = f_{n-1} + f_{n-2}$. Prove that for all $n \geq 1$,
$$f_n = \frac{1}{\sqrt{5}}\left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\}.$$
   Hint: Write $\begin{pmatrix} f_n \\ f_{n-1} \end{pmatrix} = A^{n-2} \cdot \begin{pmatrix} u \\ v \end{pmatrix}$, for some 2×2 matrix A and $u, v \in \mathbb{Z}$.
   (ii) Let A and B be $n \times n$ matrices and set $C := AB - BA$. If $AC = CA$, prove that C is a nilpotent matrix.
   (iii) Let V and W be vector spaces over $\mathbb{C}$ of dimensions n and m. Set $U := \mathcal{L}(V, W)$. Fix isomorphisms $\alpha \in \mathcal{L}(V, V)$ and $\beta \in \mathcal{L}(W, W)$. Define $\phi \in \mathcal{L}(U, U)$ by $\phi(T) = \beta^{-1} T \alpha$, for all $T \in U$. Find formulas for $\operatorname{trace}(\phi)$ and $\det(\phi)$ in terms of α and β.