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On ideals with submaximal analytic spread

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Abstract – We construct examples of ideals I in a local ring R of dimension d with the property that, I has analytic spread $a \leq d-1$ and the polynomial giving the lengths of $\Gamma_{\mathfrak{m}}(R/I^n)$, for n large, has degree $t \leq a-1$.

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1. Introduction

Let $(R, \mathfrak{m})$ be a commutative local ring of Krull dimension d and $I \subseteq R$ an ideal such that $\mathfrak{m}$ is an associated prime of I^n, for all large n. In other words, $\Gamma_{\mathfrak{m}}(R/I^n) \neq 0$, for all large n. Here $\Gamma_{\mathfrak{m}}(-)$ denotes the zeroth local cohomology functor, so that for any R-module M, $\Gamma_{\mathfrak{m}}(M)$ is the set of elements in M annihilated by a power of $\mathfrak{m}$. We will write $(I^n : \mathfrak{m}^\infty)$ for $\bigcup_{k\geq 1}(I^n : \mathfrak{m}^k)$, so that $\Gamma_{\mathfrak{m}}(R/I^n) = (I^n : \mathfrak{m}^\infty)/I^n$, for all n. It is well known that under fairly mild circumstances (e.g., R is unmixed in the sense of Nagata) $\bigoplus(I^n : \mathfrak{m}^\infty)$ is a finitely generated module over the Rees ring $\mathcal{R}(I) := \bigoplus_{n\geq 0} I^n$ if and only if the analytic spread $a(I)$ of I is less than d, see for example [2], Theorem 4.1. This follows roughly because in this case, $\mathfrak{m}\mathcal{R}(I)$ has height greater than one, so that the ideal transform of $\mathfrak{m}\mathcal{R}(I)$ over $\mathcal{R}(I)$ is a finitely generated module over $\mathcal{R}(I)$, and clearly $\bigoplus_{n\geq 1}(I^n : \mathfrak{m}^\infty)$ is contained in this transform. It follows that when the analytic spread of I is less than d, and $\mathfrak{m}$ is an associated prime of I^n, for n large, the lengths of the modules $\Gamma_{\mathfrak{m}}(R/I^n)$ are given by a polynomial in n with rational coefficients. The situation when I has analytic spread d is much more

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complicated. In this case the lengths of the modules $\Gamma_{\mathfrak{m}}(R/I^n)$ need not be governed by a polynomial, but one can consider the growth of the function $\frac{d!}{n^d} \cdot \lambda(\Gamma_{\mathfrak{m}}(R/I^n))$, as n tends to infinity, where $\lambda(-)$ denotes length. Initially, the limsup of this sequence was called the ϵ multiplicity of I and it was known that the ϵ multiplicity of I was non-zero if and only if the analytic spread of I equals d (see [3] and [6]). In [1], this limit was ultimately shown to exist when R is an analytically unramified local ring, and this limit need not be rational.

As far as we know, little information has been given concerning the polynomial giving the lengths of $\Gamma_{\mathfrak{m}}(R/I^n)$ when I has analytic spread less than d, except for the observation that this polynomial must have degree $a(I) - 1$ or less. In this note we show that, in fact, such polynomials exist having any analytic spread $a \leq d - 1$ and degree $t \leq a - 1$. The ideals we construct are monomial ideals in a polynomial ring localized at its homogenous ideal, so our arguments come down to either counting, or estimating, the number of monomials in $(I^n : \mathfrak{m}^\infty)$, not in I^n, for various ideals I. We also note that unlike the case for traditional Hilbert polynomials, having the same integral closure does not guarantee having the same normalized leading coefficients.

2. The examples

We begin by recalling that for an ideal I contained in the local ring $(R, \mathfrak{m})$, the analytic spread $a(I)$ of I is the Krull dimension of the fiber ring $\bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$. When the residue field of R is infinite, $a(I)$ is the minimal number of generators in a minimal reduction of I. Recall that $J \subseteq I$ is a minimal reduction if J is minimal among ideals for which $JI^{t-1} = I^t$, for some t (see [5]).

From here on out, $(R, \mathfrak{m})$ will denote a local ring with Krull dimension d and infinite residue field k. Unless stated otherwise, when considering ideals $I \subseteq R$, we will assume $a(I) < d$. When $\Gamma_{\mathfrak{m}}(R/I^n) \neq 0$, for $n >> 0$, we will write $P_I(n)$ for the resulting polynomial.

We now turn to a useful construction. For this, let $(R, \mathfrak{m})$ be an unmixed local ring and suppose $J \subseteq I$ is a minimal reduction. Then $\bigoplus_{n\geq 0}(I^n : \mathfrak{m}^\infty)$ is also a finite module over the Rees ring of R with respect to J. Thus, there exists $b > 0$ such that for all $n > b$, $I^n = J^{n-b}I^b$ and $(I^n : \mathfrak{m}^\infty) = J^{n-b}(I^b : \mathfrak{m}^\infty)$. Set $a := a(I)$, choose generators $f_1, \ldots, f_a$ for J and indeterminates $U_1, \ldots, U_a$. Then we have the following exact sequence of graded $R[U_1, \ldots, U_a]$-modules:

$$(*) \qquad 0 \longrightarrow K \longrightarrow \frac{(I^b : \mathfrak{m}^\infty)}{I^b}[U_1, \ldots, U_a] \xrightarrow{\psi} \bigoplus_{n\geq 0} \frac{J^n(I^b : \mathfrak{m}^\infty)}{J^n I^b} \longrightarrow 0,$$

where $\psi(hU_1^{\alpha_1}\cdots U_a^{2\alpha_a})$ is the class of $hx_1^{2\alpha_1}\cdots x_a^{\alpha_a}$ in $\frac{J^n(I^b:\mathfrak{m}^\infty)}{J^nI^b}$, for $h\in\frac{(I^b:\mathfrak{m}^\infty)}{I^b}$, $U_1^{\alpha_1}\cdots U_a^{\alpha_a}$ a monomial of degree n and K is the kernel of ψ. Note that ψ is a graded homomorphism, so that in each degree we get a surjective R-module homomorphism

$$(**)\qquad \left(\frac{(I^b:\mathfrak{m}^\infty)}{I^b}[U_1,\ldots,U_a]\right)_n \longrightarrow \frac{J^n(I^b:\mathfrak{m}^\infty)}{J^nI^b}\longrightarrow 0.$$

The length of the module on the left in (**) is $\lambda\{(I^b:\mathfrak{m}^\infty)/I^b\}\cdot\binom{n+a-1}{a-1}$, which is a polynomial of degree $a(I)-1$. Our goal is to create examples where we can gain a good understanding of K.

The following remark will facilitate some of the calculations in the proof of the theorem below.

Remark. Assume $R=k[x_1,\ldots,x_d]$ is a polynomial ring over the field k and set $C:=(x_1,\ldots,x_a)$ and $J:=(x_1^2,\ldots,x_a^2)R$, for some $1\le a\le d$.

(i) Fix $p\ge 1$, and let μ be a monomial of degree $2p$ in the variables $x_1,\ldots,x_a$. Then all of the exponents in μ are even if and only if $\mu\in J^p$. More generally, t is the number of factors of μ of the form x_i^2, with $1\le i\le a$ if and only if $\mu\in J^tC^{2(p-t)}\backslash J^{t+1}C^{2(p-t-1)}$.

(ii) $JC^{2r}=C^{2r+2}$, for $r=\frac{a-1}{2}$, if a is odd and $r=\frac{a}{2}$, if a is even.

Proof. Part (i) follows from the definition of the ideal J. Part (ii) is more or less well known, but we include a proof. It suffices to show that the generators of C^{2r+2} belong to JC^{2r}. Suppose a is odd, $r=\frac{a-1}{2}$ and let $\mu=x_1^{e_1}\cdots x_a^{e_a}\in C^{2r+2}$ be a monomial of degree $2r+2=a+1$. We have $e_1+\cdots+e_a=a+1$, so some exponent $e_i\ge 2$. Therefore, $\mu=x_i^2\cdot\mu'$, for $\mu'\in C^{2r}$. Thus, $\mu\in JC^{2r}$, which is what we want. The case when a is even is similar. ∎

Before moving on to our general case, we consider the following example which shows a component of the general calculation. For an ideal $L\subseteq R$, we will write $\overline{L}$, for the integral closure of L. Recall that $x\in\overline{L}$ if and only if there exist $n\ge 1$ and $l_j\in L^j$ such that $x^n+l_1x^{n-1}+\cdots+l_n=0$.

Example. Let k be a field and set $R:=k[x_1,\ldots,x_5]_{(x_1,\ldots,x_5)}$, the polynomial ring in five variables over k localized at its homogenous maximal ideal. Set $\mathfrak{m}:=(x_1,\ldots,x_5)R$. In R, set $L:=(x_1^2,x_2^2,x_3^2)+(x_4)(x_1,\ldots,x_4)$ and $I:=L+\mathfrak{m}\overline{L}$. In order to understand K in (*) above, we need to understand which monomials in $(I^{n+b}:\mathfrak{m}^\infty)$ belong to I^{n+b}. For this example, we find the monomials in $(I^5:\mathfrak{m}^\infty)$ that belong to I^5. We begin by showing it suffices to determine the monomials in $(I^5:\mathfrak{m}^\infty)$ of degree ten that are in L^5. For this, set $C:=(x_1,\ldots,x_4)$ and $J:=(x_1^2,x_2^2,x_3^2,x_4^2)$, so that $\overline{J}=\overline{L}=C^2$. By the remark above, $JC^4=C^6$, so that $J^3C^4=C^{10}$. Therefore, $L^3C^4=C^{10}$. We now note that $L^3C^4 = L^4C^2$, and for this it suffices to see that $L^3C^4 \subseteq L^4C^2$. Let $\mu = \alpha\beta$ be a monomial generator of L^3C^4 such that $\alpha \in L^3$ and $\beta \in C^4$ are monomial generators. If the exponent of x_4 in β is zero, one of x_1, x_2, x_3, say x_1 has exponent at least two. Then $\mu = (\alpha x_1^2) \cdot \beta' \in L^4C^2$. If the exponent of x_4 in β is 1, then $\mu = (\alpha x_i x_4) \cdot \beta'$, for some $1 \le i \le 3$, belongs to L^4C^2. Finally, if the exponent of x_4 in β is two or more, $\mu = (\alpha x_4^2) \cdot \beta' \in L^4C^2$. Thus, $C^{10} = L^4C^2$. On the other hand, it follows from the proof of the theorem below that $(I^5 : \mathfrak{m}^\infty) = C^{10}$. We now have

$$\begin{aligned} I^5 &= L^5 + L^4\mathfrak{m}C^2 + \cdots + L\mathfrak{m}^4C^4 + \mathfrak{m}^5C^{10} \\ &= L^5 + \mathfrak{m}C^{10} + \cdots + L\mathfrak{m}^4C^8 + \mathfrak{m}^5C^{10} \\ &= L^5 + \mathfrak{m}C^{10}. \end{aligned}$$

Thus, any monomial of degree more than ten in the ideal C^{10} belongs to I^5 and the monomials of degree ten in I^5 are the same as the monomials of degree ten in L^5. So we must determine which monomials of degree ten in C^{10} are L^5. Note that a monomial of degree ten in C^{10} is just a monomial of degree ten in x_1, x_2, x_3, x_4.

Consider the monomial $\mu = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}x_4^{\alpha_4}$ of degree ten in $(I^5 : \mathfrak{m}^\infty) = C^{10}$. Let r denote the number of $\alpha_1, \alpha_2, \alpha_3$ that are odd. Suppose $\alpha_4 \ge r$. Then we can pair each x_i, with $i \neq 4$, whose exponent is odd, with an x_4. We then get r terms in L of the form x_ix_4 dividing μ. Let μ_0 denote the product of these r terms from L, so that can write $\mu = \mu_0\mu_1$. Writing $\mu_1 = x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}x_4^{\beta_4}$, $\beta_1, \beta_2, \beta_3$ are even. Moreover, since the degree of μ is even, β_4 is even. Thus, μ_1 is a product of squares from L and it follows that $\mu \in L^5 \subseteq I^5$.

Now suppose $r > \alpha_4$. If μ were an element of L^5, then μ would have to have α_4 factors of the form x_ix_4, with $1 \le i \le 3$. We can then write $\mu = \mu_0\mu_1$, where μ_0 is the product of α_4 factors of the form x_ix_4 with $1 \le i \le 3$ and μ_1 has degree $10 - 2\alpha_4$. (If $\alpha_4 = 0$, we just take $\mu = \mu_1$). Then $\mu_1 = x_1^{\beta_1}x_2^{\beta_2}x_3^{\beta_3}$ and at least one β_i is odd (since $r > \alpha_4$). But μ_1 is not divisible by x_4, so $\mu_1 \in (x_1^2, x_2^2, x_3^2)^{5-2\alpha_4}$. However this cannot happen if some x_i in μ_1 has odd exponent. Thus, $\mu \notin I^5$.

Putting this all together, we have that if $\mu \in (I^5 : \mathfrak{m}^\infty)$ then $\mu \in I^5$ if and only if μ has degree greater than ten or μ has degree ten and $r \le \alpha_4$. ■

We note that in the theorem below, our construction takes place in an integrally closed local domain R of Krull dimension greater than or equal to two. Thus, we do not consider the case where $a(I) = 1$, since such an ideal is principal and $(I^n : \mathfrak{m}^\infty) = I^n$, for all n.

Theorem. *Let k be a field and $(R, \mathfrak{m})$ the polynomial ring $k[x_1, \ldots, x_d]$ localized at its homogeneous maximal ideal. Then for any $2 \le a \le d-1$ and $0 \le t \le a-1$, there exists an ideal $I \subseteq R$ satisfying $\Gamma_{\mathfrak{m}}(R/I^n) \neq 0$, for all large n, $a(I) = a$ and $\deg(P_I(n)) = t$.*

Proof. Our ideal I will be a monomial ideal in R. We seek to understand K in the exact sequence $(*)$, so our goal is to calculate which monomials in $(I^p : \mathfrak{m}^\infty)$ belong to I^p, for any $p \ge 1$. In Cases 1 and 2 below, we set $J := (x_1^2, \ldots, x_a^2)R$, while in Case 3, we set $J := (x_1^3, \ldots, x_a^3)$, so that in each case, J is generated by a regular sequence, and thus has analytic spread a. If we set $C := (x_1, \ldots, x_a)$, then $\overline{J} = C^2$ in the first two cases and $\overline{J} = C^3$ in the third case since C is a normal ideal, i.e., $\overline{C^n} = C^n$, for all $n \ge 1$. In particular, $\overline{J^p} = (\overline{J})^p$, for all $p \ge 1$. Each I we construct will satisfy $J \subseteq I \subseteq \overline{J}$, so that J is a reduction of I, and hence $a(I) = a$.

We now show that $(I^p : \mathfrak{m}^\infty) = C^{2p}$ in Case 1 and Case 2. In Case 3, we have that $(I^p : \mathfrak{m}^\infty) = C^{3p}$, but the proof is identical to the previous two cases so we omit it. Note that for each I we construct and $p \ge 1$,

$$(I^p : \mathfrak{m}^\infty) \subseteq (\overline{I^p} : \mathfrak{m}^\infty) = (\overline{J^p} : \mathfrak{m}^\infty) = \overline{J^p} = (\overline{J})^p = C^{2p},$$

where the second equality follows since R satisfies the altitude formula and $a(I) < \dim(R)$ (see [4]). On the other hand, in Cases 1 and 2, each ideal we construct will be such that $I = L + \mathfrak{m}\overline{L}$, for an ideal L satisfying $J \subseteq L \subseteq \overline{J} = C^2$, so $I = L + \mathfrak{m}C^2$. Thus, $I^p = L^p + L^{p-1}\mathfrak{m}C^2 + \cdots + \mathfrak{m}^p C^{2p}$, so that $C^{2p}\mathfrak{m}^p \subseteq I^p$, showing that $C^{2p} \subseteq (I^p : \mathfrak{m}^\infty)$, and hence $C^{2p} = (I^p : \mathfrak{m}^\infty)$.

We proceed to investigate the kernel of the map ψ in (*) above. Note that by definition, $\psi(rU_i)$ is the class of rx_i^2, for $r \in (I^b : \mathfrak{m}^\infty)/I^b$. Since $J^n I^b$ is a monomial ideal, the kernel of ψ will be a "monomial" submodule in the variables $U_1, \ldots, U_a$. Thus, a homogeneous form $f(U_1, \ldots, U_a)$ belongs to K if and only if each term $rU_1^{q_1} \cdots U_a^{q_a}$ of $f(U_1, \ldots, U_a)$ belongs to K. In each of the cases below, we will provide a numerical criterion on the exponents of a monomial $\mu \in (I^p : \mathfrak{m}^\infty)$ which will determine whether or not $\mu \in I^p$. Since this criterion will behave nicely with respect to multiplication by x_i^2, for $1 \le i \le a$, this will give information about K.

Case 1. Suppose $t = a - 1$. We set $I := J + \mathfrak{m}\overline{J} = J + \mathfrak{m}C^2$ and take $\mu := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ to be a monomial in $(I^p : \mathfrak{m}^\infty) = C^{2p}$ so that $\sum_{i=1}^d \alpha_i \ge 2p$. We will track three quantities which will determine whether or not μ is also in I^p.

$$\delta_{p,a}(\mu) := (\sum_{i=1}^a \alpha_i) - 2p \qquad \delta_{p,d}(\mu) := (\sum_{i=1}^d \alpha_i) - 2p \qquad r(\mu) := \sum_{i=1}^a r_2(\alpha_i),$$

where $r_2(c) \in \{0, 1\}$ denotes the residue class of c modulo 2. The last sum above merely counts the number of exponents in the first a variables that are odd. It follows that $\mu \in \mathfrak{m}^{\delta_{p,d}(\mu)}C^{2p}$. We claim that

$$(\star) \qquad \mu \in I^p \text{ if and only if } \delta_{p,d}(\mu) \geq \frac{r(\mu) - \delta_{p,a}(\mu)}{2}.$$

For this, we first identify the number of factors of μ of the form x_i^2, with $1 \leq i \leq a$. This is given by $\frac{(\sum_{i=1}^{a} \alpha_i) - r(\mu)}{2}$, which is an integer, since $r_2(\sum_{i=1}^{a} \alpha_i) \equiv \sum_{i=1}^{a} r_2(\alpha_i)$ mod 2. Rewriting this in terms of $\delta_{p,a}(\mu)$ gives

$$\frac{(\sum_{i=1}^{a} \alpha_i) - r(\mu)}{2} = \frac{\delta_{p,a}(\mu) + 2p - r(\mu)}{2} = p - \frac{r(\mu) - \delta_{p,a}(\mu)}{2}.$$

Setting $j := \frac{r(\mu) - \delta_{p,a}(\mu)}{2}$ it follows that μ is divisible by $p - j$ squares from J. Therefore $\mu \in J^{p-j}\mathfrak{m}^{\delta_{p,d}(\mu)}C^{2j}$. Thus, if $\delta_{p,d}(\mu) \geq j = \frac{r(\mu) - \delta_{p,a}(\mu)}{2}$, it follows that $\mu \in I^p$.

Conversely, suppose $\mu \in I^p$. Then $\mu \in \mathfrak{m}^g J^{p-g} C^{2g}$, for some $0 \leq g \leq p$. By part (i) of the Remark, $p - \frac{r(\mu) - \delta_{p,a}(\mu)}{2} \geq p - g$. Thus, $\delta_{p,d}(\mu) \geq g \geq \frac{r(\mu) - \delta_{p,a}(\mu)}{2}$, as required, so $(\star)$ holds.

We now show that, in the set-up of (*), the kernel of ψ is zero. Take $\mu \in (I^b : \mathfrak{m}^\infty)$ and suppose $\psi(\mu U_1^{e_1} \cdots U_a^{e_a}) = 0$. Set $E := \sum_{i=1}^{a} e_i$ and $\mu' := \mu x_1^{2e_1} \cdots x_a^{2e_a}$. Then $\mu' \in I^{b+E}$. We calculate $\delta_{b+E,a}(\mu')$, $\delta_{b+E,d}(u')$ and $r(\mu')$ from $\delta_{b,a}(\mu)$, $\delta_{b,d}(\mu)$ and $r(\mu)$. Note that

$$\begin{aligned}
\delta_{b,a}(\mu) &= (\sum_{i=1}^{a} \alpha_i) - 2b \\
&= (\sum_{i=1}^{a} \alpha_i) + 2E - 2(b + E) \\
&= \sum_{i=1}^{a} (\alpha_i + 2e_i) - 2(b + E) \\
&= \delta_{b+E,a}(\mu').
\end{aligned}$$

Similarly, one can see that $\delta_{b+E,d}(\mu') = \delta_{b,d}(\mu)$. Since all new factors are even powers of the x_i, we also have $r(\mu') = r(\mu)$.

Now, since $\mu' \in I^{b+E}$, by $(\star)$, we have $\delta_{b+E,d}(\mu') \geq \frac{r(\mu') - \delta_{b+E,a}(\mu')}{2}$ and thus, $\delta_{b,d}(\mu) \geq \frac{r(\mu) - \delta_{b,a}(\mu)}{2}$, so again by $(\star)$, $\mu \in I^b$, showing that the kernel of ψ is zero. Thus, for $n > b$,

$$P_I(n) = \lambda\left(\frac{J^{n-b}(I^b : \mathfrak{m}^\infty)}{J^{n-b}I^b}\right) = \lambda\left(\left(\frac{(I^b : \mathfrak{m}^\infty)[U_1, \ldots, U_a]}{I^b[U_1, \ldots, U_a]}\right)_{n-b}\right) = \lambda\left(\frac{(I^b : \mathfrak{m}^\infty)}{I^b}\right) \cdot \binom{n - b + a - 1}{a - 1},$$

which is a polynomial in n of degree $a-1$.

Case 2. Suppose $1 \leq t < a-1$. Set

$$L := (x_1^2, \dots, x_{t+1}^2) + (x_1, \dots, x_a)(x_{t+2}, \dots, x_a) = J + D,$$

where D is generated by the mixed terms of degree two in $x_1, \dots, x_a$ divisible by some x_j with $t+2 \leq j \leq a$. Note that, $\overline{L} = C^2 = \overline{J}$, so that if we set $I := L + \mathfrak{m}\overline{L} = L + \mathfrak{m}C^2$, we have $J \subsetneq L \subsetneq I \subsetneq \overline{J}$. Thus, J is a reduction of I and L, and hence these ideals have analytic spread a. We may therefore use the exact sequence in (*), for an appropriate choice of b.

Take $\mu := x_1^{\alpha_1} \cdots x_d^{\alpha_d} \in (I^p : \mathfrak{m}^\infty) = C^{2p}$ so that $\sum_i^d \alpha_i \geq 2p$. In particular, $\mu \in \mathfrak{m}^{\delta_{p,d}(\mu)} C^{2p}$.

Let $\delta_{p,a}$ and $\delta_{p,d}$ be as in Case 1. We redefine $r(\mu)$ as follows: $r(\mu) := \sum_{i=1}^{t+1} r_2(\alpha_i)$, so that $r(\mu)$ is the number of odd exponents among the first $t+1$ variables. Additionally, we define $\sigma_{t,a}(\mu) := \sum_{i=t+2}^{a} \alpha_i$.

In analogy with Case 1, we claim that for $\mu \in C^{2p}$,

$$(***) \qquad \mu \in I^p \text{ if and only if } \delta_{p,d}(\mu) \geq \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu)).$$

Since mixed terms involving $x_{t+2}, \dots, x_a$ lie in L, we have more flexibility in identifying generators of L which are factors of μ. We again seek the number of such factors. We first consider the case $\sigma_{t,a}(\mu) \geq r(\mu)$. Note that the right hand side of the inequality in (***) is less than or equal to zero in this case, so the inequality automatically holds, i.e., the only if direction holds automatically. Continuing, we wish to show $\mu \in L^p$. Since $\mu \in C^{2p}$ we can write $\mu = \mu_0 x_{a+1}^{\alpha_{a+1}} \cdots x_d^{\alpha_d}$, where μ_0 is a monomial in $x_1, \dots, x_a$ of degree at least $2p$. Since the assumption $\sigma_{t,a}(\mu) \geq r(\mu)$ depends only on the variables $x_1, \dots, x_a$, it suffices to show $\mu_0 \in L^p$. Thus, after changing notation, we may assume that μ is a monomial in $x_1, \dots, x_a$ of degree at least $2p$ satisfying $\sigma_{t,a}(\mu) \geq r(\mu)$ and show $\mu \in L^p$.

Set $r(\mu) := r$ and assume for the moment that the degree of μ is $2p$. Without loss of generality we may further assume $\alpha_1, \dots, \alpha_r$ are odd. Then we may pair each of $x_1, \dots, x_r$ with the not necessarily distinct variables $x_{j_1}, \dots, x_{j_r}$, where $t+2 \leq j_1, \dots, j_r \leq a$. Thus, the product $x_i x_{j_i} \in L$, so $\mu_0 := (x_1 j_{j_1}) \cdots (x_r x_{j_r}) \in L^r$. We now have

$$(****) \qquad \mu = \mu_0 \cdot x_1^{\alpha_1 - 1} \cdots x_r^{\alpha_r - 1} x_{r+1}^{\alpha_{r+1}} \cdots x_{t+1}^{\alpha_{t+1}} x_{t+2}^{\beta_{t+2}} \cdots x_a^{\beta_a},$$

where the monomial $\frac{\mu}{\mu_0}$ has degree $2p - 2r$. Because the displayed exponents of $x_1, \dots, x_{t+1}$ are now even, we have $\beta_{t+1} + \cdots + \beta_a$ is even. Thus, $x_{t+2}^{\beta_{t+2}} \cdots x_a^{\beta_a}$ can be written as a product of generators of L (namely various squares of x_i, with $1 \le i \le a$), and similarly so can $x_1^{\alpha_1-1}\cdots x_r^{\alpha_r-1}x_{r+1}^{\alpha_{r+1}}\cdots x_{t+1}^{\alpha_{t+1}}$. We may therefore group the irreducible factors of μ in to monomials of degree two, each a generator of L, so that μ can be written as a product of p generators of L. Thus $\mu \in L^p \subseteq I^p$.

If μ has degree greater than $2p$. We can still choose r as in the previous paragraph and write μ as in (****). If the degree of μ is even, then $\sum_i \beta_i$ is even, and we may still pair the irreducible factors of μ as generators of L and there are at least p such pairs. If the degree of μ is odd, then $\sum_i \beta_i$ is odd, so some β_i is odd, say β_a. Then the $\frac{\mu}{x_a}$ has even degree at least $2p$ and the previous argument applies, showing that $\frac{\mu}{x_a}$, and hence μ belongs to $L^p \subseteq I^p$.

Now suppose $r(\mu) > \sigma_{t,a}(\mu)$. Set $r := r(\mu)$ and $s := \sigma_{t,a}(\mu)$. As before, we assume $\alpha_1, \dots, \alpha_r$ are odd. Then we can write

$$\mu = (x_1x_{j_1})\cdots(x_sx_{j_s})\cdot\{x_1^{\alpha_1-1}\cdots x_s^{\alpha_s-1}x_{s+1}^{\alpha_{s+1}}\cdots x_r^{\alpha_r}x_{r+1}^{\alpha_{r+1}}\cdots x_{t+1}^{\alpha_{t+1}}\}\cdot x_{a+1}^{\alpha_{a+1}}\cdots x_d^{\alpha_d},$$

where $t_2 \le j_1, \dots, j_s \le a$ and $(x_1x_{j_1})\cdots(x_sx_{j_s})$ is a product of $\sigma_{t,a}(\mu)$ mixed terms from L. In addition

$$x_1^{\alpha_1-1}\cdots x_s^{\alpha_s-1}x_{s+1}^{\alpha_{s+1}}\cdots x_r^{\alpha_r}x_{r+1}^{\alpha_{r+1}}\cdots x_{t+1}^{\alpha_{t+1}}$$

is divisible by $\frac{1}{2}\sum_{i=1}^{t+1}(\alpha_i - r(\mu)) + \sigma_{t,a}(\mu)$ squares x_i^2 from L. Thus, the product of these two latter expressions belongs to L raised to the

$$2\sigma_{t,a}(\mu) + \frac{1}{2}(\sum_{i=1}^{t+1}\alpha_i - r(\mu)) = \frac{1}{2}(\sum_{i=1}^{a}\alpha_i - r(\mu) + \sigma_{t,a}(\mu))$$

power. Simplifying, we have:

$$\begin{aligned}\frac{1}{2}(\sum_{i=1}^{a}\alpha_i - r(\mu) + \sigma_{t,a}(\mu)) &= \frac{1}{2}(2p + \delta_{p,a}(\mu) - r(\mu) + \sigma_{t,a}(\mu))\\ &= p - \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu)).\end{aligned}$$

Writing $j := \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu))$, it follows that $\mu \in L^{p-j}\mathfrak{m}^{\delta_{p,d}(\mu)}C^{2j}$. Therefore, $\mu \in I^p$ whenever $\delta_{p,d}(\mu) \ge j = \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu))$.

Conversely, suppose $\mu \in I^p$. Maintaining the notation from the previous paragraph, by virtue of the assumption $r(\mu) > \sigma_{t,a}(\mu)$, we have $\mu \in J^{p-j}\mathfrak{m}^{\delta_{p,d}(\mu)}C^{2j}$. This forces $\delta_{p.d}(\mu) \ge j = \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu))$, which establishes (***)

We now wish to apply the criterion (***) to an analysis of the kernel of ψ. For ease of notation, we set $H := \frac{(I^b:\mathfrak{m}^\infty)}{I^b}$, $\underline{U} := \{U_1, \ldots, U_a\}$ and let $h_1, \ldots, h_q$ be a set of (non-zero) monomial generators of H as an R-module, so that each h_i is the image in H of a monomial in $(I^b : \mathfrak{m}^\infty)$, but not in I^b. Then $h_1, \ldots, h_q$ also generate $H[\underline{U}]$ as a module over $R[\underline{U}]$. Thus, a monomial in $H[\underline{U}]$ will be a monomial in $R[\underline{U}]$ times some h_i.

Now for any non-zero monomial $\mu \in H$, consider $\psi(\mu U_1^{e_1} \cdots U_a^{e_a}) := \mu'$. As in Case 1, with $E = \sum_{i=1}^a e_i$, we have

$$(\star\star) \qquad \delta_{b+E,d}(\mu') = \delta_{b,d}(\mu), \delta_{b+E,a}(\mu') = \delta_{b,a}(\mu) \text{ and } r(\mu') = r(\mu).$$

The only term in the new criterion (***) that might change is $\sigma_{t,a}$. If $e_{t+2} = \cdots = e_a = 0$, then we have $\sigma_{t,a}(\mu') = \sigma_{t,a}(\mu)$ and therefore $\mu' \notin I^{b+E}$. This shows that none of the monomials in $H[U_1, \ldots, U_{t+1}]$ belong to the kernel of ψ and since each of these monomial goes to a unique monomial in the image of ψ, we have

$$\lambda(H) \cdot \binom{n+t}{t} \leq \lambda\left(\frac{(I^{n+b} : \mathfrak{m}^\infty)}{I^{n+b}}\right) = P_I(n+b).$$

since there are $\binom{n+t}{t}$ monomials in $U_1, \ldots, U_{t+1}$ of degree n.

On the other hand, if $e_i \neq 0$, for some $t+1 < i$, then $\sigma_{t,a}(\mu) < \sigma_{t,a}(\mu')$, while all the other expressions in $(\star\star)$ remain the same. Suppose $\mu \in H$ is non-zero, so that $\delta_{b,d}(\mu) < \frac{1}{2}(r(\mu) - \delta_{p,a}(\mu) - \sigma_{t,a}(\mu))$. Write

$$\delta_{b,d}(\mu) + \epsilon = \frac{1}{2}(r(\mu) - \delta_{b,a}(\mu) - \sigma_{t,a}(\mu)),$$

for some $\epsilon > 0$. Set $E' := \sum_{i=t+2}^a e_i$, so $\sigma_{t,a}(\mu') = \sigma_{t,a}(\mu) + E'$. Then

$$\begin{aligned}
\delta_{b+E,d}(\mu') &= \delta_{b,d}(\mu) \\
&= \frac{1}{2}(r(\mu) - \delta_{b,a}(\mu) - \sigma_{t,a}(\mu)) - \epsilon \\
&= \frac{1}{2}(r(\mu') - \delta_{b+E,a}(\mu') - (\sigma_{t,a}(\mu') - E')) - \epsilon \\
&= \frac{1}{2}(r(\mu') - \delta_{b+E,a}(\mu') - \sigma_{t,a}(\mu')) + \frac{E'}{2} - \epsilon.
\end{aligned}$$

Thus, $\delta_{b+E}(\mu') \geq \frac{1}{2}(r(\mu') - \delta_{b+E,a}(\mu') - \sigma_{t,a}(\mu'))$ whenever $E' \geq 2\epsilon$. In particular, $\psi(\mu') = 0$. Let $\epsilon_1, \ldots, \epsilon_q$ be such that

$$\delta_{b,d}(h_i) + \epsilon_i = \frac{1}{2}(r(h_i) - \delta_{b,a}(h_i) - \sigma_{t,a}(h_i)),$$

for each $1 \le i \le q$. Let w denote the maximum of $\{2e_i\}_{i=1}^{i=q}$. Then it follows from (***) that $\psi(h_i M) = 0$, for all $1 \le i \le q$ and $M \in (U_{t+2}, \dots, U_a)^w R[\underline{U}]$. Therefore, $(U_{t+2}, \dots, U_a)^w H[\underline{U}]$ is contained in the kernel of ψ. Thus, $H[\underline{U}]/(U_{t+2}, \dots, U_a)^w H[\underline{U}]$ maps onto the image of ψ. However,

$$H[\underline{U}]/(U_{t+2}, \dots, U_a)^w H[\underline{U}] \cong (H[U_{t+2}, \dots, U_a]/(U_{t+2}, \dots, U_a)^w)\,[U_1, \dots, U_{t+1}].$$

Set $\Lambda := \lambda(H[U_{t+2}, \dots, U_a]/(U_{t+2}, \dots, U_a)^w)$. Then we have

$$P_I(n+b) = \lambda\left(\frac{(I^{n+b} : \mathfrak{m}^\infty)}{I^{n+b}}\right) \le \Lambda \cdot \binom{n+t}{t}.$$

Thus, $P_I(n+b)$ is bounded above and below by polynomials with positive leading coefficients and degrees equal to t. Therefore, $\deg(P_I(n)) = t$, as required.

Case 3. Suppose $t = 0$. Let L denote the ideal of R generated by all monomials of degree three in $x_1, \dots, x_a$, except the monomial $x_1^2 x_2$. Set $I := L + \mathfrak{m}\overline{L} = L + \mathfrak{m}C^3$. Thus, $\frac{C^3}{I}$ has dimension one, since every monomial of degree greater than three in C^3 belongs to I. Moreover, $J := (x_1^3, \dots, x_a^3)$ is a minimal reduction of L and $(I^n : \mathfrak{m}^\infty) = C^{3n}$.

We first note that $LC^3 = C^6$. To see this, we have $C^3 = (L, x_1^2 x_2)$, so that

$$C^6 = (L^2, x_1^2 x_2 L, x_1^4 x_2^2) = LC^3 + (x_1^4 x_2^2).$$

But $x_1^4 x_2^2 = x_1 x_2^2 \cdot x_1^3 \in LC^3$, which gives $C^6 = LC^3$. From this equation it follows that that

$$C^{3n} = LC^{3(n-1)} = L^{n-1}C^3,$$

for all $n \ge 2$.

We now want to show that $\lambda\left(\frac{(I^n : \mathfrak{m}^\infty)}{I^n}\right) = 1$, for all $n \ge 1$, which will give that the degree of $P_I(n)$ equals zero. This is clear for $n = 1$, by the first paragraph in Case 3 above. Now, for any $n \ge 1$, it is not hard to see that $x_1^{3n-1} x_2$ is a monomial of degree $3n$ which is not in L^n, and hence not in I^n, since the monomials of degree $3n$ in I^n are the monomials of degree $3n$ in L^n (see (†) below). In particular we have $x_1^{3n-1} x_2 \in (I^n : \mathfrak{m}^\infty) \setminus I^n$. However, any other monomial of total degree $3n$ in C^{3n} belongs to L^n. To see this, we have $C^{3n} = L^{n-1}C^3 = L^{n-1}(L, x_1^2 x_2) = L^n + L^{n-1} x_1^2 x_2$, for all $n \ge 2$. Suppose M is a monomial of degree $3n$ in $L^{n-1} x_1^2 x_2$ that is not equal to $x_1^{3n-1} x_2$. Then $M = l_1 \cdots l_{n-2} l_{n-1} x_1^2 x_2$, where each $l_i \in L$. Since $l_1 \cdots l_{n-1} \ne x_1^{3n-3}$, without loss of generality, we can write $l_{n-1} = x_i x_j x_k$, where we fix $x_i \ne x_1$. If $x_j x_k \ne x_1 x_2$, then we can write $M = l_1 \cdots l_{n-2} \cdot (x_1 x_j x_k) \cdot (x_1 x_i x_2) \in L^n$. If $x_j x_k = x_1 x_2$ and $x_i \ne x_2$, then $M = l_1 \dots l_{n-2} \cdot (x_i x_1 x_2) \cdot x_1^2 x_2 = {}_1 \cdots l_{n-2} \cdot (x_1 x_2^2) \cdot (x_1^2 x_i) \in L^n$. Finally, if $x_i = x_2$

and $x_j x_k = x_1x_2$, then $M = l_1 \cdots l_{n-2} \cdot x_1x_2^2 \cdot x_1^2x_2 = l_1 \cdots l_{n-2} \cdot x_1^3 \cdot x_2^3 \in L^n$. Thus, every monomial of degree $3n$ in C^{3n} except $x_1^{3n-1}x_2$ belongs to L^n, and hence I^n.

What about monomials of degree greater than $3n$ in C^{3n}? We have

$$(\dagger)\qquad I^n = (L+\mathfrak{m}C^3)^n = L^n + \mathfrak{m}L^{n-1}C^3 + \mathfrak{m}^2L^{n-2}C^6 + \cdots + \mathfrak{m}^nC^{3n} = L^n + \mathfrak{m}C^{3n},$$

since $C^{3n} = L^{n-1}C^3$. Thus the monomials in C^{3n} of degree greater than three belong to I^n. It follows that $\lambda\left(\frac{(I^n:\mathfrak{m}^\infty)}{I^n}\right) = 1$, for $n \geq 1$, so that

$$P_I(n) = \lambda\left(\frac{(I^n : \mathfrak{m}^\infty)}{I^n}\right) = 1,$$

which gives $\deg(P_I(n)) = 0$. ■

We close with the following remarks, the second of which is more or less known to experts.

Remarks. 1. Unlike the case where $I \subseteq R$ is an $\mathfrak{m}$ primary ideal, so $\lambda\left((I^n : \mathfrak{m}^\infty)/I^n\right) = \lambda\left(R/I^n\right)$ or the case of $\lambda\left(\Gamma_\mathfrak{m}(I^n/I^{n+1})\right)$, when $a(I) = d$, the degree and leading coefficients of the associated polynomials $P_I(n)$ above are not controlled by the integral closure of I. Indeed, Case 1 and Case 2 above show that the degree of $P_I(n)$ can change while the integral closure stays the same. On the other hand, the leading coefficients might change, even though the ideals in question have the same degree and the same integral closure. For example, when $n = 2$, and $\mathfrak{m} = (x, y)$, if we set $J_1 := (x^4, y^4)$ and $J_2 := (x^4, x^2y^2, y^4)$, and $I_1 = J_1 + \mathfrak{m}\overline{J_1}$ and $I_2 := J_2 + \mathfrak{m}\overline{J_2}$, then it is not difficult to show that $P_{I_1}(n) = 4n - 1$ and $P_{I_2}(n) = 2n$, so that even though $I_1$and I_2 have the same integral closure, and $\deg(P_{I_1}(n)) = \deg(P_{I_2}(n))$, $P_{I_1}(n)$ and $P_{I_2}(n)$ have different leading coefficients.

2. Let $(R, \mathfrak{m})$ be an arbitrary local ring of positive dimension and $I \subseteq R$ an ideal whose analytic spread is less that the dimension of R with the properties that $\Gamma_\mathfrak{m}(R/I^n) \neq 0$, for n large and $\bigoplus_{n\geq 0}(I^n : \mathfrak{m}^\infty)$ is a finite module over the Rees ring of R with respect to I, e.g., any ideal with submaximal analytic spread in an unmixed local ring. As before, let $P_I(n)$ denote the polynomial giving the lengths of $\frac{(I^n:\mathfrak{m}^\infty)}{I^n}$, for n sufficiently large. It is not hard to express the degree of $P_I(n)$ in terms of the associated graded ring of R with respect to I. One first notes that the polynomial $Q_I(n)$ giving the lengths of $\Gamma_\mathfrak{m}(I^n/I^{n+1})$ has the same degree as $P_I(n)$. This follows from the subadditivity of $\Gamma_\mathfrak{m}(-)$. To see this, our assumption on I implies that there exists $c > 0$ such that $(I^n : \mathfrak{m}^\infty) \subseteq I^{n-c}$, for $n > c$. Thus,

$$\lambda(\Gamma_\mathfrak{m}(I^{n-1}/I^n)) \leq \lambda(\Gamma_\mathfrak{m}(R/I^n)) \leq \lambda(\Gamma_\mathfrak{m}(I^{n-1}/I^{n-1})) + \cdots + \lambda(\Gamma_\mathfrak{m}(I^{n-1}/I^{n-c+1})) \leq \sum_{i=n-c+1}^{n} Q_I(i).$$

Since $Q_i(n)$ has a positive leading coefficient, $\sum_{i=n-c+1}^{n} Q_i(i) \le cQ_I(n)$, which shows that $P_I(n)$ and $Q_I(n)$ have the same degree. It now follows that

$$(\star\star\star) \quad \deg(P_I(n)) = \max\{\dim(\mathcal{G}(I)/Q \mid Q \in \mathrm{Ass}(\mathcal{G}(I)) \text{ and } \mathfrak{m}\mathcal{G}(I) \subseteq Q\} - 1,$$

where $\mathcal{G}(I) := \bigoplus_{n\ge 0} I^n/I^{n+1}$, the associated graded ring of I. This follows from above and the general fact that if A is a Noetherian ring and $J \subseteq A$ is an ideal, such that $\Gamma_J(A) \neq 0$, then the Krull dimension of the module $\Gamma_J(A)$ is $\max\{\dim(A/Q) \mid J \subseteq Q$ and $Q \in \mathrm{Ass}(A)\}$. To see this, note that Q is in the support of $\Gamma_J(A)$ if and only if $\Gamma_{J_Q}(A_Q) \neq 0$ if and only if $J \subseteq Q_0 \subseteq Q$, for some $Q_0 \in \mathrm{Ass}(A)$. Note further, that since we are assuming $a(I) < d$, the prime $Q \subseteq \mathcal{G}(I)$ with $\deg(P_I(n)) = \dim(\mathcal{G}(I)/Q) - 1$ must be an embedded prime of $\mathcal{G}(I)$. In the case of our Theorem, it was much easier to simply count (or estimate) the monomials in $(I^n : \mathfrak{m}^\infty)\backslash I^n$, than to calculate the associated graded ring of I and its various associated primes.

3. One can think of R/I^n as $\mathrm{Tor}_0^R(R/I^n, R)$, so that $\Gamma_\mathfrak{m}(R/I^n) = \Gamma_\mathfrak{m}(\mathrm{Tor}_0^R(R/I^n, R))$. Suppose M is a non-free finite R-module such that the lengths of $\Gamma_\mathfrak{m}(M/I^nM)$ do not have polynomial growth, i.e., $\Gamma_\mathfrak{m}(\mathrm{Tor}_0^R(R/I^n, M))$ does not have polynomial growth. In this case, the higher Tor modules need not vanish, so one can ask about the growth of the lengths of $\Gamma_\mathfrak{m}(\mathrm{Tor}_i^R(R/I^n, M))$, for $i > 0$. In these cases, the lengths of the higher Tor modules are eventually given by a polynomial. It is not difficult to see that $\mathrm{Tor}_i^R(R/I^n, M)$ is a module of the form $(I^nF_{i-1} \cap B_{i-1})/I^nB_{i-1}$, where F_{i-1} is the $(i-1)$st free module in a free resolution of M and B_{i-1} is the corresponding boundary module. However, $\mathcal{M} := \bigoplus_{n\ge 0}(I^nF_{i-1} \cap B_{i-1})/I^nB_{i-1}$ is a finite module over over the Rees ring of R with respect to I and hence, so is the module $\Gamma_\mathfrak{m}(\mathcal{M})$. It follows that the lengths of the higher Tor modules in question have polynomial growth.

To see an example with a non-zero Tor module in homological degree greater than 0, suppose $(R, \mathfrak{m})$ is a local ring and $I \subseteq R$ an ideal such that the lengths of $\Gamma_\mathfrak{m}(R/I^n)$ are not given by a polynomial for $n >> 0$. Take $M = I$. Then the exact sequence $0 \to I/I^{n+1} \to R/I^{n+1} \to R/I \to 0$ gives rise to the exact sequence

$$0 \to \Gamma_\mathfrak{m}(M/I^nM) \to \Gamma_\mathfrak{m}(R/I^{n+1}) \xrightarrow{f_n} \Gamma_\mathfrak{m}(R/I).$$

If $\Gamma_\mathfrak{m}(R/I) = 0$, then $\Gamma_\mathfrak{m}(M/I^nM) \cong \Gamma_\mathfrak{m}(R/I^{n+1})$, so the lengths of the former module are not polynomial. Suppose $\Gamma_\mathfrak{m}(R/I) \neq 0$. Then this is an Artinian R-module. If we let C_n denote the image of f_n, then $C_1 \supseteq C_2 \supseteq \cdots$ is a descending sequence of submodules in $\Gamma_\mathfrak{m}(R/I)$, and thus is stable at some C_{n_0}. Then, for $n \ge n_0$ we have a short exact sequence

$$0 \to \Gamma_\mathfrak{m}(M/I^nM) \to \Gamma_\mathfrak{m}(R/I^{n+1}) \xrightarrow{f_n} C_{n_0} \to 0.$$

for all $n \geq n_0$. It follows that the lengths of $\Gamma_{\mathfrak{m}}(M/I^n M) = \Gamma_{\mathfrak{m}}(\mathrm{Tor}_0^R(R/I^n, M))$ do not have polynomial growth. On the other hand, by the comments in the preceding paragraph, the lengths of $\Gamma_{\mathfrak{m}}(\mathrm{Tor}_i^R(R/I^n, M))$ do have polynomial growth, for values of $i > 0$.

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