S_2-IFICATIONS AND A UNIFORM BRIANÇON-SKODA THEOREM FOR NON-REDUCED RINGS

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ABSTRACT. We adapt the Uniform Briançon-Skoda Theorem to non-reduced rings via S_2-ifications. Notably, it is proven that if R is an excellent locally unmixed ring of finite Krull dimension, then there exists a nonzero divisor $c \in R$ so that for all ideals $I \subseteq R$, if $R[It, t^{-1}]$ is the extended Rees algebra of I and $R[It, t^{-1}]^{(S_2)}$ its S_2-ification, then $ct^{-B}R[It, t^{-1}]^{(S_2)} \subseteq R[It, t^{-1}]$.

1. Introduction

The purpose of this note is to establish a uniform containment property enjoyed by all excellent locally unmixed Noetherian rings of finite Krull dimension. Throughout, we use "unmixed" in the sense of Nagata: a local ring is unmixed if its $\mathfrak{m}$-adic completion is equidimensional and contains no embedded associated primes of zero. In [?], Ratliff proved that if R is a reduced unmixed local ring, then for each ideal $I \subseteq R$ there exists an integer $k > 0$, depending on I, such that

$$(I^{n+k})_1 \subseteq I^n \qquad \text{for all } n \geq 1.$$

For $j \geq 1$, the ideal

$$(I^j)_1 = t^{-j}\, R[It, t^{-1}]^{(1)} \cap R$$

is defined using the extended Rees algebra $R[It, t^{-1}]$. Here, for any ring A, the ring $A^{(1)}$ consists of elements in the total quotient ring of A whose conductor to A has height at least two. When R is excellent and locally unmixed, the ring $R[It, t^{-1}]^{(1)}$ agrees with the S_2-ification of the extended Rees algebra.

Ratliff's theorem was motivated by the classical result of Rees: if R is an analytically unramified local ring, then for every ideal $I \subseteq R$ there exists an integer $k > 0$, depending on I, such that

$$\overline{I^{n+k}} \subseteq I^n \qquad \text{for all } n \geq 1.$$

Briançon-Skoda type theorems are deep theorems of commutative algebra and algebraic geometry and show that for many classes of reduced rings, the constant k in Rees' theorem may be chosen independently of the ideal I, see [?, ?, ?, ?, ?, ?].

A reduced Noetherian ring R is said to enjoy the *Uniform Briançon–Skoda Property* if there exists an integer $B \in \mathbb{N}$ such that

$$\overline{I^{n+B}} \subseteq I^n \qquad \text{for all ideals } I \subseteq R \text{ and all } n \in \mathbb{N}.$$

Huneke conjectured all reduced excellent rings of finite Krull dimension enjoy the Uniform Briançon-Skoda Property, see [?, Conjecture 1.4]. In the same article, Huneke proved that if

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R is a reduced belonging to a large class of finite dimensional excellent rings[1], then R enjoys the Uniform Briançon–Skoda Property. Huneke's Uniform Briançon-Skoda conjecture and his related Uniform Artin-Rees conjecture, [**?**, Conjecture 1.3], were recently resolved in [**?**, Main Application].

The following theorem gives a formal statement of the Briançon-Skoda type containment results together with equivalent formulations via extended Rees algebras.

Theorem 1.1. *Let R be a commutative Noetherian ring with unity.*

(a) [**?**, **?**, Briançon–Skoda Theorem]. *Suppose R is a regular ring of finite Krull dimension d.*

i.) For all ideals $I \subseteq R$,

$$t^{-d} \in \operatorname{Ann}\Big(\overline{R[It, t^{-1}]}/R[It, t^{-1}]\Big).$$

ii.) Equivalently, for every $n \in \mathbb{N}$,

$$\overline{I^{n+d}} = t^{-n-d}\overline{R[It, t^{-1}]} \cap R \subseteq t^{-n}R[It, t^{-1}] \cap R = I^n.$$ [2]

(b) [**?**, Uniform Briançon–Skoda Theorem]. *Let R be an excellent reduced ring of finite Krull dimension.*

(i.) There exist a constant $B_1 \in \mathbb{N}$ and a nonzero divisor $c \in \operatorname{Ann}_R(\overline{R}/R)$ such that for all ideals $I \subseteq R$,

$$ct^{-B_1} \in \operatorname{Ann}\Big(\overline{R[It, t^{-1}]}/R[It, t^{-1}]\Big).$$

If R is normal, c may be chosen to be the unit element $1 \in R$.

(ii.) There exists a constant $B_2 \in \mathbb{N}$ such that for every ideal $I \subseteq R$,

$$\overline{I^{n+B_2}} = t^{-n-B_2}\overline{R[It, t^{-1}]} \cap R \subseteq t^{-n}R[It, t^{-1}] \cap R = I^n.$$

Remark 1.2. The statements of **??** (**??**) are equivalent to Huneke's $T(R)$-ideal, [**?**, Definition 2.8], not being contained in a minimal prime of the reduced ring R by the Uniform Artin-Rees Theorem, see [**?**, Proposition 2.10] and [**?**, Main Application].

Since the integral closure of a non-reduced ring containing non-zerodivisors is never finite, the normalization map $R[It, t^{-1}] \to \overline{R[It, t^{-1}]}$ is not finite when R is non-reduced. Consequently, the conclusions of the Uniform Briançon–Skoda Theorem cannot apply to non-reduced rings. When Huneke's theorem is applied to R_{red}, it only yields information

[1]Namely, R is either essentially of finite type over an excellent local ring, essentially of finite type over the integers, an F-finite ring of characteristic $p > 0$, or a homomorphic image of an excellent regular ring of finite Krull dimension such that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, the normalization of $R/\mathfrak{p}$ admits a resolution of singularities by blowing up an ideal.

[2]The conclusions of the Briançon–Skoda Theorem apply to broad classes of reduced rings that are not necessarily regular. Classically, [**?**] established that the same conclusions hold for pseudo-rational singularities. More recently, it has been shown that the conclusions of the Briançon–Skoda Theorem extend to F-pure rings and Cohen–Macaulay F-injective rings [**?**], as well as to further generalizations of these singularity classes, including Du Bois, lim-F-pure, and Cohen–Macaulay lim-perfectoid F-injective rings [**?**].

about ideals of R modulo nilpotents, i.e., up to their image in R_{red}.[3] Thus, a natural adaptation of the Uniform Briançon-Skoda Theorem to non-reduced rings, motivated by the principle of discovering uniformity in birational geometry, is to identify canonical finite birational extensions of the extended Rees algebras of R and a uniform conductor element of these extensions.

To this end, assume R is excellent and without embedded associated primes[4]. If

$$0 = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_t$$

is the minimal primary decomposition of the zero ideal, then each quotient $R/\mathfrak{q}_i$ is locally unmixed. If R is reduced (i.e., each $\mathfrak{q}_i$ is prime), then

$$\overline{R} \cong \prod_{i=1}^{t} \overline{R/\mathfrak{q}_i}.$$

If R is non-reduced, then at least one unmixed quotient $R/\mathfrak{q}_i$ fails to have finite normalization, i.e., it does not admit a finite birational extension that is both (S_2) and (R_1). Nevertheless, each unmixed quotient $R/\mathfrak{q}_i$ does admit an S_2-ification.

If R is a Noetherian ring without embedded associated primes, an *S_2-ification* of R is a finite birational extension $R \subseteq R^{(S_2)}$ such that $R^{(S_2)}$ is (S_2) as an R-module and $\mathrm{ht}(R :_R R^{(S_2)}) \geq 2$. When it exists, the S_2-ification of R is unique. For further details and justifications of these claims, see Section **??**.

Our main theorem provides an adaptation of the Uniform Briançon–Skoda Theorem to a suitably large class of non-reduced rings. When applied to a locally unmixed ring, it identifies a nonzerodivisor of $R[t^{-1}]$ that lies in the conductor of the S_2-ification of every extended Rees algebra of R, viewed as subalgebras of the Laurent polynomial ring $R[t, t^{-1}]$.

Main Theorem 1. *Let R be an excellent ring of finite Krull dimension and without embedded associated primes. For each ideal $I \subseteq R$ and $n \in \mathbb{N}$ let*

$$(I^n)_1 := \left(t^{-n} \prod_{i=1}^{\ell} \left(\frac{R[It, t^{-1}]}{Q_i} \right)^{(S_2)} \right) \bigcap R$$

where $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_\ell$ is the minimal primary decomposition of the 0-ideal in $R[It, t^{-1}]$.

*(a) (**??**) If R is locally unmixed, $I \subseteq R$ an ideal, and $n \in \mathbb{N}$, then*

$$(I^n)_1 = t^{-n} R[It, t^{-1}]^{(S_2)} \cap R.$$

*(b) (**??** and **??**) If R is locally unmixed, then there exists a nonzero divisor $c \in R$ and constant $B_1 \in \mathbb{N}$ so that for all ideals $I \subseteq R$,*

$$ct^{-B_1} \in \mathrm{Ann}\left(R[It, t^{-1}]^{(S_2)} / R[It, t^{-1}] \right).$$

If R is (S_2), then c can be chosen to be the unit element $1 \in R$.

*(c) (**??**) There exists a constant $B_2 \in \mathbb{N}$ so that for every ideal $I \subseteq R$ and $n \in \mathbb{N}$,*

$$(I^{n+B_2})_1 \subseteq I^n.$$

[3] Recently, Huneke's Uniform Briançon-Skoda Theorem was used to develop the theory of differential operators for algebras essentially of finite type over a field without embedded associated primes, see [**?**, **?**]. Their theorems provide insight on the behavior of the powers of ideals modulo the nilradical of the ring in relationship to a finite set of differential operators $\delta_i : R \longrightarrow R_{\mathrm{red}}$, $1 \leq i \leq \ell$, so that $\cap_{i=1}^{\ell} \ker(\delta_i) = 0$.

[4] This is essentially a requirement for $R^{(1)}$ to be finite. See **??** below.

Remark 1.3. Large classes of excellent reduced rings do not admit an S_2-ification, even though they admit finite normalization. The normalization of a reduced ring R is not necessarily (S_2) as an R-module. For example, if k is a field and

$$R = k[x, y, z]/\big((x) \cap (y, z)\big),$$

then R does not admit an S_2-ification (see **??** and **??**). The normalization is the product

$$\overline{R} = R/(x) \times R/(y, z) \cong k[y, z] \times k[x].$$

Note that, as an R-module, $\overline{R}$ is not (S_2) since localization at (x, y, z) yields a finitely generated $R_{(x,y,z)}$-module of depth 1 and dimension 2.

Remark 1.4. If $I \subseteq R$ is an ideal, then t^{-1} is a nonzero divisor of $R[It, t^{-1}]$. The extended Rees algebra of an ideal I can fail to enjoy the (S_2) property, even if R is a polynomial ring over a field. An illustrative example comes from [**?**, Section 5]. If k is a field and $R = k[x_1, x_2]$, then the extended Rees algebra of the $\mathfrak{m}$-primary ideal $I = (x^6, x^4y, xy^5, y^6)$ has the property $(t^{-1}, R[It, t^{-1}]_{>0})$ is an associated prime of $t^{-1}R[It, t^{-1}]$. See [**?**, Examples 6.1 E1] for additional details.

Moreover, if I is an $\mathfrak{m}$-primary ideal of an unmixed analytically unramified (S_2) local domain $(R, \mathfrak{m}, k)$, then $J := t^{-1}R[It, t^{-1}]^{(S_2)} \cap R$ is the unique largest ideal containing I so that the 0th and 1st Hilbert coefficients of J agree with those of I, see [**?**, Theorem 3.17] and [**?**, Theorem 2.5].

2. Preliminary Results

2.1. **Terminology.** Let R be a Noetherian ring and M a finitely generated R-module.

- If $n \geq 1$ then M is *an (S_n)-module over its support*, or simply (S_n), if for all $\mathfrak{p} \in \operatorname{Spec}(R)$ so that $M_\mathfrak{p} \neq 0$,

$$\operatorname{depth}(M_\mathfrak{p}) \geq \min\{n, \dim(M_\mathfrak{p})\}.$$

- We say that a local ring $(R, \mathfrak{m}, k)$ is *unmixed* if for all $\mathfrak{q} \in \operatorname{Ass}(\widehat{R})$, $\dim(R) = \dim(\widehat{R}/\mathfrak{q}\widehat{R})$. We say that R is *locally unmixed* if $R_\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \operatorname{Spec}(R)$.

  Remark 2.1. If R is excellent, then the property of being locally unmixed is equivalent to the property for each $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{q} \in \operatorname{Ass}(R_\mathfrak{p})$, $\dim(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) = \dim(R_\mathfrak{p})$.

- The *total ring of fractions of R* is denoted by $Q(R)$ and is the localization of R at the complement of the union of its minimal primes.
- If R is without embedded associated primes, then a *birational extension of R* is an R-algebra extension $R \subseteq S \subseteq Q(R)$.

2.2. (S_2)**-ifications: Theorems of McAdam-Ratliff and Hochster-Huneke.** In this section we survey theory developed in [**?**] and [**?**] that pieced together prove every excellent locally unmixed ring admits a unique S_2-ification. The reader is directed to [**?**, **?**] for details on the local theory of S_2-ifications. We recall what is necessary to the article in our discussions.

Definition 2.2. Let R be a Noetherian ring without embedded associated primes and $Q(R)$ the total ring of fractions of R. An *S_2-ification of R* is a module-finite birational extension

$R \longrightarrow R^{(S_2)} \subseteq Q(R)$ so that $R^{(S_2)}$ satisfies Serre's (S_2)-condition and $(R :_R S)$ is an ideal of R of height at least 2.

Abstractly, an R-algebra S is an S_2-ification of R if S is isomorphic as an R-algebra to a module-finite birational extension of R satisfying Serre's (S_2)-condition as an R-module.

Let R be a Noetherian ring without embedded associated primes and $Q(R)$ the total ring of fractions of R. Let

$$R^{(1)} = \left\{ \frac{a}{b} \in Q(R) \mid \operatorname{ht}\left(R :_R \frac{a}{b}\right) \geq 2 \right\}.$$

If $\frac{a}{b} \in Q(R)$, then $\frac{a}{b} \in R^{(1)}$ if and only if $\frac{a}{b} \in R_{\mathfrak{p}}$ for all height 1 primes $\mathfrak{p} \in \operatorname{Spec}(R)$. Therefore

$$R^{(1)} = \bigcap_{\substack{\mathfrak{p} \in \operatorname{Spec}(R) \\ \operatorname{ht}(\mathfrak{p})=1}} R_{\mathfrak{p}}.$$

The map $R \longrightarrow R^{(1)}$ is a birational extension of R and is a natural candidate of an S_2-ification of R.

Theorem 2.3 ([**?**, Corollary 1.4(b)])**.** *Let R be a Noetherian ring without embedded associated primes. The following are equivalent:*

(a) $R \longrightarrow R^{(1)}$ is a finite ring homomorphism.

(b) R enjoys two properties:

(i) If $\mathfrak{p} \in \operatorname{Spec}(R)$ is a non-associated prime and there exists $\mathfrak{q} \in \operatorname{Ass}(\widehat{R_{\mathfrak{p}}})$ so that $\dim(\widehat{R_{\mathfrak{p}}}/\mathfrak{q}\widehat{R_{\mathfrak{p}}}) = 1$ then $\dim(R_{\mathfrak{p}}) = 1$.

(ii) There exists finitely many prime ideals $\mathfrak{q} \in \operatorname{Spec}(R)$ so that $\operatorname{depth}(R_{\mathfrak{q}}) = 1$ and $\dim(R_{\mathfrak{q}}) > 1$. Equivalently, there exists finitely many prime ideals $\mathfrak{q}$ of height at least 2 that is an embedded prime of a nonzero divisor of R.

Remark 2.4. The first property of Theorem **??** (**??**) is trivially enjoyed by every locally unmixed Noetherian ring. The second property of Theorem **??** (**??**) is enjoyed by every excellent Noetherian ring without an embedded associated prime by [**?**, (IV), 7.8.2]. Note that if R has an embedded prime divisor $\mathfrak{p}$ of zero such that $R/\mathfrak{p}$ has infinitely many height one primes, then $R^{(1)}$ will not be finite over R, since for any prime Q directly above $\mathfrak{p}$ and any non-zerodivisor $a \in Q$, Q is minimal over $(a, \mathfrak{p})$, and hence Q is an embedded prime divisor of aR. Since there are infinitely many such primes, $R^{(1)}$ is not finite over R, by the theorem above.

?? and [**?**, Proposition 2.4] is used in the following to show if R is an excellent and locally unmixed ring, then $R^{(1)}$ is the unique S_2-ification of R.

Theorem 2.5. *Let R be a Noetherian ring without embedded associated primes. If R admits an S_2-ification, then the S_2-ification of R is unique and $R^{(S_2)} = R^{(1)}$. If R is excellent and locally unmixed, then R admits an S_2-ification.*

Proof. Assume that R is locally unmixed and admits an S_2-ification $R \longrightarrow R^{(S_2)}$. Then $(R :_R R^{(S_2)})$ has height at least 2 implies $R^{(S_2)} \subseteq R^{(1)}$. For every prime $\mathfrak{p} \in \operatorname{Spec}(R)$, $(R^{(1)})_{\mathfrak{p}} = (R_{\mathfrak{p}})^{(1)}$. Uniqueness of the S_2-ification and the identification provided in the local case is the content of [**?**, Proposition 2.4].

Now assume R is excellent and locally unmixed. The birational extension $R \longrightarrow R^{(1)}$ is finite and birational, see Theorem **??** and Remark **??**. The property $R^{(1)}$ is (S_2) can be checked locally. Uniqueness of a finite birational extension of R can be verified locally. Indeed, if T_1, T_2 are finite birational extensions of R so that $(T_1)_{\mathfrak{p}} = (T_2)_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$, then for all $\frac{a}{b} \in T_1$, the cokernel of $T_2 \subseteq T_2\left[\frac{a}{b}\right]$ is not supported at any prime of $\operatorname{Spec}(R)$, implying $\frac{a}{b} \in T_2$ and $T_1 \subseteq T_2$. By symmetry, $T_2 \subseteq T_1$. The identification

$$R^{(S_2)} = \left\{ \frac{a}{b} \in Q(R) \mid \operatorname{ht}\left(\left(R :_R \frac{a}{b}\right)\right) \geq 2 \right\}$$

is also determined locally. The local case is the content of [?, Proposition 2.4]. $\square$

2.3. **Extended Rees Algebras and (S_2)-ifications.** Let $S = \bigoplus_{n \in \mathbb{Z}} S_n$ be a catenary $\mathbb{Z}$-graded ring. If M is a graded S-module and $n \in \mathbb{Z}$, then M_n denotes the degree n elements of M. We adopt the following notation concerning graded commutative algebra.

- If $P \subseteq S$ is a prime ideal then M_P denotes ordinary localization with respect to the complement of P.
- If $I \subseteq S$ is an ideal, then I^* is the homogeneous ideal of S generated by the homogeneous elements of I.
  - In particular, if $I = Q_1 \cap \cdots \cap Q_t$ is a primary decomposition of an ideal $I \subseteq S$, then $I^* = Q_1^* \cap \cdots \cap Q_t^*$ is a primary decomposition of I^* so that $\sqrt{Q_I^*} = \left(\sqrt{Q_i}\right)^*$.
  - If P is a non-homogeneous prime of S then
    - $\star$ $\dim(M_P) = \dim(M_{P^*}) + 1$ by [?, Lemma 1.5.6 and Theorem 1.5.8] and
    - $\star$ $\operatorname{depth}(M_P) = \operatorname{depth}(M_{P^*}) + 1$ by [?, Theorem 1.5.9].

Definition 2.6. Let R be a Noetherian ring, $I \subseteq R$ an ideal, and t an indeterminate of degree 1 over R. The *extended Rees algebra of* $I \subseteq R$ is the $\mathbb{Z}$-graded R-algebra $R[It, t^{-1}] \subseteq R[t, t^{-1}]$ whose graded pieces are defined as follows:

- If $n \geq 1$, $(R[It, t^{-1}])_n = I^n$;
- If $n \leq 0$, $R[It, t^{-1}]_n = R$.

If $n \leq 0$ then implicitly define $I^n = R$, allowing us to write $R[It, t^{-1}] = \bigoplus_{n \in \mathbb{Z}} I^n t^n$.

The theory of S_2-ifications of extended Rees algebras will take the place of normalization in our adaptation of the Uniform Briançon-Skoda Theorem to non-reduced rings described by **??**. To this end, we formally define our replacement for the integral closure of the powers of an ideal in rings that are not assumed to reduced in **??** and **??**.

Definition 2.7. Let R be a Noetherian ring and $I \subseteq R$ an ideal so that $R[It, t^{-1}]$ admits an S_2-ification $R[It, t^{-1}]^{(S_2)}$. For each $n \in \mathbb{N}$ we define

$$(I^n)_1 = t^{-n} R[It, t^{-1}]^{(S_2)} \cap R.$$

Suppose A is a Noetherian ring and $a \in A$ be a non-zerodivisor. Let C denote the intersection of the height one primary components of aA and S the non-zerodivisors in the complement of the height one primes containing aA. Finally, set $T := A_S \cap A[\frac{1}{a}]$. Then it is not hard to check that

$$aA^{(1)} \cap A = C = aA_S \cap A = aT \cap A.$$

This shows that for any ideal I in the locally unmixed local ring R, $n \geq 1$, and A the extended Rees ring of I, then $(I^n)_1 = I^{[n]}$, the notation used by Ratliff in [?].

?? below gives a natural way to extend the definition of $(I^n)_1$ to any ideal I belonging to an excellent Noetherian ring R without embedded associated primes, see **??**. We first record some well-known facts about extended Rees algebras.

Lemma 2.8. *Let R be a Noetherian ring and $I \subseteq R$ an ideal. There is a bijection of the associated primes of R and the associated primes of $R[It, t^{-1}]$ with the following properties:*

(a) If $P \subseteq R[It, t^{-1}]$ is an associated prime of $R[It, t^{-1}]$, then $P \cap R$ is a associated prime of R.

(b) If $\mathfrak{p} \subseteq R$ is an associated prime of R and $a \in R$ is so that $\operatorname{Ann}_R(a) = \mathfrak{p}$, then

$$P_I = \bigoplus_{n \in \mathbb{Z}} (I^n \cap \mathfrak{p})t^n = \operatorname{Ann}_{R[It,t^{-1}]}(a)$$

is an associated prime of $R[It, t^{-1}]$ and

$$aR[It, t^{-1}] \cong \frac{R}{\mathfrak{p}}\left[\frac{(I, \mathfrak{p})}{\mathfrak{p}}t, t^{-1}\right].$$

(c) If $0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t$ is a primary decomposition of the 0-ideal of R so that $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$, $Q_{I,i} = \bigoplus_{n \in \mathbb{Z}} I^n \cap \mathfrak{q}_i \subseteq R[It, t]$, then $0 = Q_{I,1} \cap \cdots \cap Q_{I,t}$ is a primary decomposition of the 0-ideal of $R[It, t^{-1}]$ so that $\sqrt{Q_{I,i}} = P_{I,i}$.

(d) The bijective correspondence of associated primes of R and $R[It, t^{-1}]$ restricts to a bijective correspondence between the minimal primes of R and $R[It, t^{-1}]$.

(e) If R is universally catenary and locally unmixed, then $R[It, t^{-1}]$ is locally unmixed.

Proof. The element t^{-1} is a nonzero divisor of $R[It, t^{-1}]$ and statements (**??**), (**??**), and (**??**) are consequences of the analogous claims in the Laurent extension $R[It, t^{-1}] \subseteq R[It, t^{-1}][t] = R[t, t^{-1}]$ upon contraction in $R[It, t^{-1}]$.

For (**??**), if $\mathfrak{p} = \operatorname{Ann}_R(a) \in \operatorname{Ass}(R)$, then $\operatorname{Ann}_{R[It,t^{-1}]}(a)$ is a homogeneous ideal of $R[It, t^{-1}]$ whose degree n piece is

$$\operatorname{Ann}_{R[It,t^{-1}]}(a)_n = \{r \in I^n \mid ra = 0\} = I^n \cap \operatorname{Ann}_R(a) = I^n \cap \mathfrak{p}.$$

Therefore $\operatorname{Ann}_{R[It,t^{-1}]}(a) = P_I$ and

$$aR[It, t^{-1}] \cong \frac{R[It, t^{-1}]}{\operatorname{Ann}_{R[It,t^{-1}]}(a)} \cong \bigoplus_{n \in \mathbb{Z}} \frac{I^n}{I^n \cap \mathfrak{p}} t^n \cong \bigoplus_{n \in \mathbb{Z}} \frac{(I^n, \mathfrak{p})}{\mathfrak{p}} t^n \cong \frac{R}{\mathfrak{p}}\left[\frac{(I, \mathfrak{p})}{\mathfrak{p}}t, t^{-1}\right].$$

If R is universally catenary and locally unmixed as in (**??**), then the bijection of minimal primes of an extended Rees algebra $R[It, t^{-1}]$ and the minimal primes of R described above, in addition with $R[It, t^{-1}]$ being catenary, implies the finitely generated algebra $R[It, t^{-1}]$ is locally unmixed, see [**?**, Lemma 4.2 and Corollary Page 61]. $\square$

Recall that if R is reduced and $I \subseteq R$ is an ideal so that the extended Rees algebra $R[It, t^{-1}]$ has finite normalization $\overline{R[It, t^{-1}]}$, then

- $\overline{R[It, t^{-1}]}$ is a graded subring of $\overline{R}[t, t^{-1}]$;
- $\overline{I^n} = t^{-n}\overline{R[It, t^{-1}]} \cap R$;
- $\overline{I^n} = t^{-1}\overline{R[I^n t, t^{-1}]} \cap R$.

Our next lemma points out if R is a locally unmixed ring with S_2-ification $R^{(S_2)}$, then for each ideal $I \subseteq R$, $R[It, t^{-1}]$ admits an S_2-ifcation, and $R[It, t^{-1}]$ admits a graded structure analogous to the graded structure of the normalization of $R[It, t^{-1}]$. In particular, $(I^n)_1$ is well-defined for all ideals $I \subseteq R$ and $n \in \mathbb{N}$. Similar observations on the extended Rees algebra of an unmixed local domain can be found in [**?**, Section 2]. Moreover, [**?**, Theorem 2.5] proves if R is an analytically unramified quasi-unmixed local domain, $I_1 = I_{\{1\}}$, where $I_{\{1\}}$ is the first coefficient ideal of I defined by Shah in [**?**] and further studied in [**?**]

Lemma 2.9. *Let R be a Noetherian locally unmixed ring with S_2-ification $R^{(S_2)}$ and $I \subseteq R$ an ideal. Then*

(a) $R[It, t^{-1}]$ has an S_2-ification.
(b) $R[It, t^{-1}]^{(S_2)}$ is a graded subring of $R^{(S_2)}[t, t^{-1}]$.
(c) If R is (S_2), then $\left(R[It, t^{-1}]^{(S_2)}\right)_n = (I^n)_1$.
(d) For every $n \in \mathbb{N}$, $(I^n)_1 = t^{-1}R[I^n t, t^{-1}]^{(S_2)} \cap R$.

Proof. For (**??**), $R[It, t^{-1}]$ is locally unmixed by **??** (**??**). Since R satisfies the condition (ii) in theorem of McAdam-Ratliff, $R[It, t^{-1}]$ satisfies the condition as well since inverting t^{-1} in $R[It, t^{-1}]$ produces the Laurent polynomial ring $R[t, t^{-1}]$, which satisfies condition (ii), and t^{-1} will have at most finitely may embedded prime divisors. Thus, $R[It, t^{-1}] \longrightarrow R[It, t^{-1}]^{(1)}$ is finite by **??** and $R[IT, T^{-1}]$ has an S_2-ification.

For (b), the S_2-ification of R is the unique finite birational extension $R \longrightarrow R^{(S_2)}$ that is (S_2) as an R-module with the property that ht $\left(\left(R :_R R^{(S_2)}\right)\right) \geq 2$. It follows that $R[t, t^{-1}]^{(S_2)} = R^{(S_2)}[t, t^{-1}]$ and inverting t^{-1} in $R[It, t^{-1}]^{(S_2)}$ results in the S_2-ification of $R[t, t^{-1}]$. Therefore $R[It, t^{-1}]^{(S_2)}$ is a subring of $R^{(S_2)}[t, t^{-1}]$. It remains to check $R[It, t^{-1}]^{(S_2)}$ is a graded subring of $R^{(S_2)}[t, t^{-1}]$ to complete the proof of (**??**).

Assume that $f = a_{\ell_1}t^{\ell_1} + a_{\ell_1+1}t^{\ell_1+1} + \cdots + a_{\ell_1+s}t^{\ell_1+s} \in R^{(S_2)}[t, t^{-1}]$ belongs to $R[It, t^{-1}]^{(S_2)}$. Then $\left(R[It, t^{-1}] :_{R[It,t^{-1}]} f\right)$ is an ideal of height at least 2. Moreover, if P is a non-homogeneous prime of height at least 2, then P^* is a homogeneous prime of height at least 1 and by [**?**, Theorem 1.5.9],

$$\operatorname{depth}(R[It, t^{-1}]_P) = \operatorname{depth}(R[It, t^{-1}]_{P^*}) + 1 \geq 2.$$

Therefore if P is a non-homogeneous prime of height exactly 2 then $R[It, t^{-1}]_P$ is (S_2),

$$f \in R[It, t^{-1}]_P, \quad \text{and} \quad \left(R[It, t^{-1}] :_{R[It,t^{-1}]} f\right) \not\subseteq P. \tag{2.9.1}$$

We aim to show $a_i t^i \in R[It, t^{-1}]^{(S_2)}$ for each $\ell_1 \leq i \leq \ell_2$. By induction, it suffices to show $a_{\ell_1}t^{\ell_1} \in R[It, t^{-1}]^{(S_2)}$. If dt^n is a homogeneous element of $(R[It, t^{-1}] :_{R[It,t^{-1}]]} f)$. Then $dt^n f = da_{\ell_1}t^{\ell_1+n} + (\text{higher degree terms}) \in R[It, t^{-1}]$. By degree considerations, $dt^n a_{\ell_1}t^{\ell_1} \in R[It, t^{-1}]$. It therefore suffices to check that $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ contains a homogeneous parameter sequence of length 2.

If $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ has height 3 or larger, then the ideal generated by homogeneous element of $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ has height at least 2. If $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ has height exactly two, then all minimal primes of $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ are homogeneous by (**??**) and $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)^*$ is a height 2 homogeneous ideal contained in $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$. Either scenario, $(R[It, t^{-1}] :_{R[It,t^{-1}]} f)$ contains a homogeneous parameter sequence of length 2, the homogeneous components of f belong in $R[It, t^{-1}]^{(S_2)}$, and $R[It, t^{-1}]^{(S_2)}$ is a graded subring of $R^{(S_2)}[t, t^{-1}]$ as claimed in (**??**).

Claim (**??**) is immediate by (**??**). Claim (**??**) is also an application of (**??**) as $R[I^n t, t^{-1}]^{(S_2)} \cong R[I^n t^n, t^{-n}]^{(S_2)}$ is the nth Veronese of $R[It, t^{-1}]^{(S_2)}$. $\square$

The next proposition gives an equivalent characterization of the ideals $(I^n)_1$ in an excellent and locally unmixed ring. The equivalent characterization provides a natural extension of the definition of $(I^n)_1$ to an ideal $I \subseteq R$ belonging to an excellent ring R without embedded associated primes.

Proposition 2.10. *Let R be an excellent and locally unmixed ring, $I \subseteq R$ an ideal, and $0 = Q_1 \cap Q_2 \cap \cdots \cap Q_\ell$ the minimal primary decomposition of the* 0*-ideal in $R[It, t^{-1}]$. Then*

$$t^{-n}R[It,t^{-1}]^{(S_2)} \cap R = \left(\prod_{i=1}^{\ell} t^{-n}\left(\frac{R[It,t^{-1}]}{Q_i}\right)^{(S_2)}\right)\bigcap R.$$

Proof. For each ideal $I \subseteq R$, the extended Rees algebra $R[It, t^{-1}]$ is locally unmixed by **??** (**??**). Moreover, $r \in (I^n)_1$ if and only if r belongs to every height 1 primary components of $t^{-n}R[It,t^{-1}]^{(S_2)}$ by **??** and the discussion following **??**. The birational extension

$$R[It,t^{-1}] \to \prod_{i=1}^{\ell} \frac{R[It,t^{-1}]}{Q_i}$$

is finite. Therefore the proposition is proven if for every $1 \leq i \leq \ell$ and every height 1 prime P of $R[It,t^{-1}]/Q_i$, the contraction of P in $R[It,t^{-1}]$ has height 1.

Given P a height 1 prime of t^{-1} in $R[It,t^{-1}]/Q_i$, let $\mathfrak{p} = P \cap R[It,t^{-1}]$. If $\mathfrak{p}$ had height 2, then there exists minimal component $Q_j \subseteq \mathfrak{p}$ of 0 so that $\dim(R[It,t^{-1}]_{\mathfrak{p}}/Q_j) > \dim(R[It,t^{-1}]_{\mathfrak{p}}/Q_i) = 1$. This would contradict $R[It,t^{-1}]$ being locally unmixed. ∎

Definition 2.11. Let R be an excellent ring without embedded associated primes. If $I \subseteq R$ is an ideal, $n \in \mathbb{N}$, and $0 = Q_1 \cap \cdots \cap Q_\ell$ is the minimal primary decomposition of the 0-ideal in $R[It,t^{-1}]$, then we let

$$(I^n)_1 := \left(\prod_{i=1}^{\ell} t^{-n}\left(\frac{R[It,t^{-1}]}{Q_i}\right)^{(S_2)}\right)\bigcap R.$$

Remark 2.12. If R is excellent and locally unmixed, $I \subseteq R$ an ideal, and $n \in \mathbb{N}$ then **??** implies the definitions of $(I^n)_1$ provided by **??** and **??** agree.

2.4. **Canonical maps and higher Ext-modules.** When $(R, \mathfrak{m}, k)$ is local of Krull dimension d, then a canonical module of R is a finitely generated R-module ω_R so that $\widehat{\omega_R}$ is the Matlis dual of $H^d_{\mathfrak{m}}(R)$. If R is non-local, then a canonical module of R is a finitely generated R-module ω_R so that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, $(\omega_R)_{\mathfrak{p}}$ is a canonical module of $R_{\mathfrak{p}}$. When $(R, \mathfrak{m}, k)$ is a local equidimensional ring without embedded primes and has a canonical module ω_R, then R admits an S_2-ification and $R \longrightarrow \operatorname{End}_R(\omega_R)$ is isomorphic to the S_2-ification $R \longrightarrow R^{(S_2)}$ by [**?**, Proposition 2.7].

Let S be a Gorenstein ring, let $\mathfrak{a} \subseteq S$ be an ideal such that all components of $\mathfrak{a}$ have height h, and set $R = S/\mathfrak{a}$. Then $\operatorname{Ext}^h_S(R,S)$ is a canonical module of R and the natural map $R \longrightarrow \operatorname{End}_R(\operatorname{Ext}^h_S(R,S))$ serves as the S_2-ification of R locally, therefore globally, by the discussion above. The following proposition identifies a canonical map $\Phi : R \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(R,S),S)$, which coincides with the S_2-ification map $R \longrightarrow \operatorname{End}_R(\operatorname{Ext}^h_S(R,S))$. Explicit details of S_2-ifications provided by an analysis of the canonical map Φ are central to our main arguments. The theory of S_2-ifications through the canonical map Φ has proven useful in other contexts, notably in recent developments of test ideal theory in rings of prime characteristic, see [**?**, Section 4.2].

Proposition 2.13. *Let S be a Noetherian Gorenstein ring and M a finitely generated S-module so that $\mathrm{Ann}_R(M)$ has height at least h.*

(a) There is a canonical map $\Phi_M : M \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$ so that if $M \longrightarrow N$ is a homomorphism of S-modules and $\mathrm{Ann}_S(M), \mathrm{Ann}_S(N)$ both have height at least h, then Φ_M, Φ_N make the following diagram commutative:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\Phi_M"] & N \arrow[d, "\Phi_N"] \\
\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S) \arrow[r] & \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(N,S),S)
\end{tikzcd}$$

(b) $\mathrm{Ext}^h_S(M,S)$ and $\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$ are (S_2)-modules with common support. If all minimal components of the support of M have height h, then

$$\mathrm{Supp}(M) = \mathrm{Supp}(\mathrm{Ext}^h_S(M,S),S) = \mathrm{Supp}(\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)).$$

(c) There exists element $c \in S$ avoiding all associated primes of $\mathrm{Ann}_S(M)$ of height h so that

$$c \in \mathrm{Ann}(\mathrm{coker}(\Phi_M)).$$

(d) If $\mathfrak{a} \subseteq S$ an ideal whose associated primes all have height h and $R = S/\mathfrak{a}$, then the canonical map $\Phi_R : R \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(R,S),S)$ viewed as a map of R-modules is isomorphic to the S_2-ification of R.

Proof. Let $(F_\bullet, \partial_\bullet)$ and $(G_\bullet, \delta_\bullet)$ be free S-resolutions of M and $\mathrm{Ext}^h_S(M,S)$ respectively:

$$(F_\bullet, \partial_\bullet) : \cdots \xrightarrow{\partial_{h+1}} F_h \xrightarrow{\partial_h} F_{h-1} \xrightarrow{\partial_{h-1}} \cdots \xrightarrow{\partial_1} F_0 \longrightarrow M \longrightarrow 0$$

and

$$(G_\bullet, \delta_\bullet) : \cdots \xrightarrow{\delta_{h+1}} G_h \xrightarrow{\delta_h} G_{h-1} \xrightarrow{\delta_{h-1}} \cdots \xrightarrow{\delta_1} G_0 \longrightarrow \mathrm{Ext}^h_S(M,S) \longrightarrow 0.$$

For each $i \in \mathbb{N}$ let $(-)^* = \mathrm{Hom}_S(-,S)$ so that $\partial^*_i : F^*_{i-1} \longrightarrow F^*_i$ and $\delta^*_i : G^*_{i-1} \longrightarrow G^*_i$ are the maps of the cocomplex obtained by applying $\mathrm{Hom}_S(-,S)$ to $(F_\bullet, \partial_\bullet)$ and $(G_\bullet, \delta_\bullet)$ respectively.

Consider the truncated complex

$$(F^*_\bullet, \partial^*_\bullet)_{tr} : 0 \longrightarrow F^*_0 \xrightarrow{\partial^*_1} \cdots \xrightarrow{\partial^*_{h-1}} F^*_{h-1} \xrightarrow{\partial^*_h} F^*_h \longrightarrow 0.$$

We are assuming $\mathrm{ht}(\mathrm{Ann}_R(S)) \geq h$. By [?, Proposition 1.2.10 (e)], if $0 \leq i \leq h-1$,

$$H^i((F^*_\bullet, \partial^*_\bullet)_{tr}) \cong \mathrm{Ext}^i_S(M,S) = 0$$

and $(F^*_\bullet, \partial^*_\bullet)_{tr}$ is a free S-resolution of $\mathrm{coker}(\partial^*_h)$. There is a containment of modules

$$\mathrm{Ext}^h_S(M,S) \cong \frac{\ker(\partial^*_{h+1})}{\mathrm{im}(\partial^*_h)} \subseteq \mathrm{coker}(\partial^*_h).$$

The containment can therefore be extended to a map of complexes:

$$\begin{tikzcd}
 & 0 \arrow[r] & F^*_0 \arrow[r, "\partial^*_1"] & F^*_1 \arrow[r, "\partial^*_2"] & \cdots \arrow[r, "\partial^*_{h-1}"] & F^*_{h-1} \arrow[r, "\partial^*_h"] & F^*_h \arrow[r] & \mathrm{coker}(\partial^*_h) \\
\cdots \arrow[r, "\delta_{h+2}"] & G_{h+1} \arrow[r, "\delta_{h+1}"] \arrow[u] & G_h \arrow[r, "\delta_h"] \arrow[u] & G_{h-1} \arrow[r, "\delta_{h-1}"] \arrow[u] & \cdots \arrow[r, "\delta_2"] & G_1 \arrow[r, "\delta_1"] \arrow[u] & G_0 \arrow[r] \arrow[u] & \mathrm{Ext}^h_S(M,S) \arrow[u, "\subseteq"']
\end{tikzcd}$$

Applying $\mathrm{Hom}_S(-,S)$ to the truncated cocomplex $(F^*_\bullet, \partial^*_\bullet)_{tr}$ produces a cocomplex so that $H^h((F^*_\bullet, \partial^*_\bullet)^*_{tr}) \cong M$. The hth cohomology of the dualized complex $(G^*_\bullet, \delta^*_\bullet)$ is $H^h((G^*_\bullet, \delta^*_\bullet)) \cong \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$. Therefore the lift of $\mathrm{Ext}^h_S(M,S) \subseteq \mathrm{coker}(\partial^*_h)$ gives a map $\Phi_M : M \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$.

Any choice of free resolution of M as an S-module localizes to a trivial extension of the minimal free $S_\mathfrak{p}$-resolution of $M_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}(S)$. Therefore the map $\Phi_M : M \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$ is unique up to isomorphism for any choice of free S-resolution of M.

Let $M \longrightarrow N$ be a homomorphism of S-modules so that $\mathrm{Ann}_S(M)$ and $\mathrm{Ann}_S(N)$ both have height at least h. Then standard methods of homological algebra imply the canonical maps Φ_M and Φ_N constructed above make the following commutative:

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "\Phi_M"] & N \arrow[d, "\Phi_N"] \\
\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S) \arrow[r] & \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(N,S),S)
\end{tikzcd}$$

We continue using notation as before in the proof of (**??**). There is a short exact sequence

$$0 \longrightarrow \mathrm{Ext}^h_S(M,S) \longrightarrow \mathrm{coker}(\partial^*_h) \longrightarrow \mathrm{im}(\partial_{h+1}) \longrightarrow 0$$

and $\mathrm{im}(\partial_{h+1})$ is contained in a free S-module, hence an (S_1) S-module. By the Auslander-Buchsbaum formula, $\mathrm{coker}(\partial^*_h)_\mathfrak{p}$ has depth at least $\mathrm{ht}(\mathfrak{p})-h$ for all $\mathfrak{p} \in \mathrm{Spec}(S)$ in its support. Therefore $\mathrm{Ext}^h_S(M,S)$ is an (S_2)-module over its support. By symmetry, $\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)$ is an (S_2)-module over its support.

If $\mathrm{ht}(\mathrm{Ann}_R(M)) \geq h+1$, then $\mathrm{Ext}^h_S(M,S) = \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S) = 0$. Suppose that $\mathrm{ht}(\mathrm{Ann}_S(M)) = h$ and let $0 = M_1 \cap M_2 \subseteq M$ be a decomposition of $0 \subseteq M$ so that all associated primes of M/M_1 have height h and all associated primes of M/M_2 have height $h+1$ or larger. Then M_1 is not supported at a height h prime of S, $\mathrm{Ext}^h_S(M_1,S) = 0$, and

$$\mathrm{Ext}^h_S(M,S) \cong \mathrm{Ext}^h_S(M/M_1,S) \quad \text{and} \quad \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S) \cong \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M/M_1,S),S).$$

For an S-module N, $\mathrm{Supp}(\mathrm{Ext}^h_S(N,S)) \subseteq \mathrm{Supp}(N)$. So to complete the proof of (**??**) we can replace M by M/M_1, assume all minimal primes of M have height h, and show

$$\mathrm{Supp}(M) = \mathrm{Supp}(\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)).$$

To this end we only need to show if $M_\mathfrak{p} \neq 0$, then $\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(M,S),S)_\mathfrak{p} \neq 0$. This can be checked at the minimal primes of M.

If $\mathfrak{p}$ is a minimal prime in the support of M, then $\dim(M_\mathfrak{p}) = \mathrm{depth}(M_\mathfrak{p}) = 0$ and the height of $\mathfrak{p}S_\mathfrak{p}$ in $S_\mathfrak{p}$ is h. The Matlis dual of $M_\mathfrak{p} = H^0_{\mathfrak{p}S_\mathfrak{p}}(M_\mathfrak{p})$ as an $S_\mathfrak{p}$-module is nonzero and isomorphic to $\mathrm{Ext}^h_{S_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p}) \otimes_{S_\mathfrak{p}} \widehat{S}_\mathfrak{p} \cong \mathrm{Ext}^h_{S_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p})$. By symmetry, the Matlis dual of $\mathrm{Ext}^h_{S_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p})$ is $\mathrm{Ext}^h_{S_\mathfrak{p}}(\mathrm{Ext}^h_{S_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p}), S_\mathfrak{p}) \cong M_\mathfrak{p}$ and therefore is nonzero.

For (**??**) let $\Phi_M : M \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^S_h(M,S),S)$ be the canonical map described above. If $\mathrm{Ann}_S(M)$ has height $h+1$ or larger, then $\mathrm{Ext}^h_S(\mathrm{Ext}^S_h(M,S),S) = 0$ and there is nothing to show. Else, if $\mathfrak{p}$ is a height h associated prime of M, then $\mathfrak{p}$ is a minimal prime of $\mathrm{Ann}_R(M)$, $M_\mathfrak{p}$ is 0-dimensional, and $(\Phi)_\mathfrak{p} : M_\mathfrak{p} \longrightarrow \mathrm{Ext}^h_{S_\mathfrak{p}}\left(\mathrm{Ext}^h_{S_\mathfrak{p}}(M_\mathfrak{p}, S_\mathfrak{p}), S_\mathfrak{p}\right)$ is an isomorphism as in the proof of (**??**). In particular, $\mathrm{coker}((\Phi_M)_\mathfrak{p}) = 0$. Therefore there exists $c \in S$ avoiding all associated primes of $\mathrm{Ann}_S(M)$ of height h so that

$$c \in \mathrm{Ann}(\mathrm{coker}(\Phi_M)).$$

For the claim of (**??**) we assume $R = S/\mathfrak{a}$ and $\mathfrak{a}$ is an ideal whose associated primes have common height h. Both $R^{(S_2)}$ and $\mathrm{Ext}^h_S(\mathrm{Ext}^h_S(R,S),S)$ are (S_2) modules whose support is that of R, see (**??**). Consequently, to show that $\Phi_R : R \longrightarrow \mathrm{Ext}^h_S(\mathrm{Ext}^h_S(R,S),S)$ agrees with $R \longrightarrow R^{(S_2)}$ as a map of R-modules, it suffices to show the map agrees with $R \longrightarrow R^{(S_2)}$ in codimension 1. Codimension 1 primes of $\mathrm{Spec}(R)$ belong to the Cohen-Macaulay locus of $\mathrm{Spec}(R)$. Let $y_1, \ldots, y_h \in \mathfrak{a}$ be a parameter sequence of height h, $\overline{S} = S/(y_1, y_2, \ldots, y_h)$, and $\overline{\mathfrak{a}} = \mathfrak{a}\overline{S}$. Then $\overline{S}$ is a Gorenstein ring mapping onto R so that $R \cong \overline{S}/\overline{\mathfrak{a}}$ and $\dim(\overline{S}_{\mathfrak{p}}) = \dim(R_{\mathfrak{p}})$. By [**?**, Lemma 3.1.16],

$$\mathrm{Ext}^h_{S_{\mathfrak{p}}}(R_{\mathfrak{p}}, S_{\mathfrak{p}}) \cong \mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}(R_{\mathfrak{p}}, \overline{S}_{\mathfrak{p}}) \cong (0 :_{\overline{S}_{\mathfrak{p}}} \overline{\mathfrak{a}}\overline{S}_{\mathfrak{p}}) \cong \omega_{R_{\mathfrak{p}}}.$$

Then $R \longrightarrow \mathrm{Ext}^h_{S_{\mathfrak{p}}}(\mathrm{Ext}^h_{S_{\mathfrak{p}}}(R_{\mathfrak{p}}, S_{\mathfrak{p}}), S_{\mathfrak{p}})$ is isomorphic to a canonical map

$$R_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}(\mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}(R_{\mathfrak{p}}, \overline{S}_{\mathfrak{p}}), \overline{S}_{\mathfrak{p}})$$

and

$$\mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}(\mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}(R_{\mathfrak{p}}, \overline{S}_{\mathfrak{p}}), \overline{S}_{\mathfrak{p}}) \cong \mathrm{Hom}_{\overline{S}_{\mathfrak{p}}}((0 :_{\overline{S}_{\mathfrak{p}}} \overline{\mathfrak{a}}\overline{S}_{\mathfrak{p}}), (0 :_{\overline{S}_{\mathfrak{p}}} \overline{\mathfrak{a}}\overline{S}_{\mathfrak{p}})) \cong \mathrm{Hom}_{R_{\mathfrak{p}}}(\omega_{R_{\mathfrak{p}}}, \omega_{R_{\mathfrak{p}}}).$$

By [**?**, Remark 2.2 (h)], $R_{\mathfrak{p}} \longrightarrow \mathrm{Hom}_{\overline{R}_{\mathfrak{p}}}(\omega_{R_{\mathfrak{p}}}, \omega_{R_{\mathfrak{p}}})$ is an isomorphism. □

3. The Uniform Briançon-Skoda Theorem for S_2-ifications

Let R be an excellent ring of finite Krull dimension. When R is locally unmixed, Main Theorem **??** (**??**) asserts the existence of a nonzerodivisor $c \in R$ and an integer $B \in \mathbb{N}$ such that for every ideal $I \subseteq R$,

$$ct^{-B} \in \mathrm{Ann}\big(R[It, t^{-1}]^{(S_2)}/R[It, t^{-1}]\big).$$

To establish such a result we utilize a theorem of Lyu, [**?**, Theorem 6.5]. If R is an excellent ring of finite Krull dimension, there exists a faithfully flat extension $R \longrightarrow R'$ so that R' is excellent and the homomorphic image of an excellent Gorenstein ring of finite Krull dimension. Lyu's theorem and the next lemma allow us to assume our rings are the homomorphic images of a Gorenstein ring.

Lemma 3.1. *Let $R \longrightarrow S$ be a faithfully flat extension of excellent rings. If R is locally unmixed, then S is locally unmixed, and for all ideals $I \subseteq R$ and $n \in \mathbb{N}$, $(I^n)_1 = (I^nS)_1 \cap R$.*

Proof. We are assuming R and S are excellent. To conclude S is locally unmixed, we only need to show for each $\mathfrak{p} \in \mathrm{Spec}(S)$ and $\mathfrak{q} \in \mathrm{Ass}(S_{\mathfrak{p}})$, that $\dim(S_{\mathfrak{p}}/\mathfrak{q}S_{\mathfrak{p}}) = \dim(S_{\mathfrak{p}})$. We can replace R by $R_{\mathfrak{p}\cap R}$ and S by $S_{\mathfrak{p}}$ and show for each $\mathfrak{q} \in \mathrm{Ass}(S)$ that $\dim(S) = \dim(S/\mathfrak{q})$. By flatness, the contraction $\mathfrak{q} \cap R$ is an associated prime of R. Then R locally unmixed implies implies $\dim(R) = \dim(R/\mathfrak{q} \cap R)$. By the going-down property, if $\mathfrak{m}$ is the maximal ideal of

R,

$$\dim\left(\frac{S}{\mathfrak{q}}\right) = \dim\left(\frac{R}{\mathfrak{q}\cap R}\right) + \dim\left(\frac{S}{\mathfrak{m}S}\right)$$
$$= \dim(R) + \dim\left(\frac{S}{\mathfrak{m}S}\right)$$
$$= \dim(S).$$

Given an ideal $I \subseteq R$ is an ideal of R, then there is a commutative diagram whose rows are faithfully flat:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] & S \arrow[d] \\
R[It, t^{-1}] \arrow[r] & S[(IS)t, t^{-1}].
\end{tikzcd}$$

If J is the intersection of the embedded associated primes of $t^{-1}R[It, t^{-1}]$, then

$$(I^n)_1 = \big(t^{-n}R[It : t^{-1}] :_{R[It,t^{-1}]}: J^\infty\big)\cap R \quad \text{and} \quad (I^nS)_1 = \big(t^{-n}S[(IS)t : t^{-1}] :_{S[(IS)t,t^{-1}]}: J^\infty\big).$$

The extension $R[It, t^{-1}] \longrightarrow S[(IS)t, t^{-1}]$ is faithfully flat implies

$$\big(t^{-n}R[It : t^{-1}] :_{R[It,t^{-1}]}: J^\infty\big) = \big(t^{-n}S[(IS)t : t^{-1}] :_{S[(IS)t,t^{-1}]}: (JS)^\infty\big) \cap R[It, t^{-1}].$$

A standard diagram chase of the commutative square above implies $(I^n)_1 = (I^nS)_1 \cap S$. $\square$

Assume S is an excellent Gorenstein ring of finite Krull dimension, $R \cong S/\mathfrak{a}$, and all associated primes of $\mathfrak{a}$ have a common height h. Given an ideal $I = (y_1, y_2, \ldots, y_n) \subseteq R$, we consider the polynomial extension

$$S_I := S[x_1, x_2, \ldots, x_n, t^{-1}],$$

and the surjective ring homomorphism

$$S_I \longrightarrow R[It, t^{-1}]$$

defined by the map $S \longrightarrow R$, $x_i \mapsto y_it$, and $t^{-1} \mapsto t^{-1}$. Then

$$R[It, t^{-1}] \cong S_I/\mathfrak{b},$$

and all primary components of $\mathfrak{b}$ have a common height $h + n$.

By **??** (**??**), there is a canonical map

$$\Phi_I \colon R[It, t^{-1}] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\big(\operatorname{Ext}^{h+n}_{S_I}(R[It, t^{-1}], S_I), S_I\big)$$

that is isomorphic to the S_2-ification map

$$R[It, t^{-1}] \to R[It, t^{-1}]^{(S_2)}.$$

Thus, finding a nonzerodivisor $c \in R$ and $B \in \mathbb{N}$ such that for all ideals $I \subseteq R$

$$ct^{-B} \in \operatorname{Ann}\big(R[It, t^{-1}]^{(S_2)}/R[It, t^{-1}]\big)$$

is equivalent to finding a nonzerodivisor $c \in R$ and $B \in \mathbb{N}$ such that

$$ct^{-B} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_I)\big).$$

Our first step to identifying such an element is a direct application of the Uniform Briançon-Skoda Theorem, **??** (**??**).

Proposition 3.2. *Let S be an excellent Gorenstein ring of finite Krull dimension. Let $\mathfrak{p} \in \operatorname{Spec}(S)$ be a prime of height h and $R = S/\mathfrak{p}$. Given an ideal $I \subseteq R$ and choice of generators $I = (y_1, y_2, \ldots, y_n)$, let $x_1, \ldots, x_n, t^{-1}$ be indeterminates, $S_I = S[x_1, \ldots, x_n, t^{-1}]$, and $S_I \longrightarrow R[It, t^{-1}]$ the onto ring homomorphism defined by $S \longrightarrow R$, $x_i \mapsto y_i t$, and $t^{-1} \mapsto t^{-1}$. Let*

$$\Phi_I : R[It, t^{-1}] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\big(\operatorname{Ext}^{h+n}_{S_I}(R[It, t^{-1}], S_I), S_I\big)$$

be the canonical map described by **??**.

Let $B \in \mathbb{N}$ so that for every ideal $I \subseteq R$ and $n \in \mathbb{N}$, $\overline{I^{n+B}} \subseteq I^n$. Then for every conductor element $c \in \operatorname{Ann}\big(\overline{R}/R\big)$ and for every ideal $I = (y_1, \ldots, y_n) \subseteq R$,

$$ct^{-B} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_I)\big).$$

Proof. Let $c \in \operatorname{Ann}_S\big(\overline{R}/R\big)$ be an element of the conductor ideal of R. If $I \subseteq R$ is an ideal, then there are inclusions of R-algebras

$$R[It, t^{-1}] \subseteq R[It, t^{-1}]^{(S_2)} \subseteq \overline{R[It, t^{-1}]}.$$

The degree $n + B$ component of $\overline{R[It, t^{-1}]}$ is

$$\overline{R[It, t^{-1}]}_{n+B} = \overline{I^{n+B}\overline{R}}.$$

Therefore

$$c\left(\overline{R[It, t^{-1}]}\right)_{n+B} \subseteq \left(\overline{I^{n+B}\overline{R}}\right) \cap R = \overline{I^{n+B}} \subseteq I^n.$$

Consequently,

$$ct^{-B}\left(R[It, t^{-1}]\right)^{(S_2)} \subseteq ct^{-B}\overline{R[It, t^{-1}]} \subseteq R[It, t^{-1}]. \tag{3.2.1}$$

By **??** (**??**), the canonical map

$$\Phi_I : R[It, t^{-1}] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\big(\operatorname{Ext}^{h+n}_{S_I}(R[It, t^{-1}], S_I), S_I\big)$$

is isomorphic to the S_2-ification map $R[It, t^{-1}] \longrightarrow R[It, t^{-1}]^{(S_2)}$. In particular,

$$\operatorname{coker}(\Phi_I) \cong \frac{R[It, t^{-1}]^{(S_2)}}{R[It, t^{-1}]} \subseteq \frac{\overline{R[It, t^{-1}]}}{R[It, t^{-1}]}.$$

By (**??**),

$$ct^{-B} \in \operatorname{Ann}(\operatorname{coker}(\Phi_I))$$

for every ideal $I = (y_1, \ldots, y_n) \subseteq R$. ☐

Resume the assumption S is an excellent Gorenstein ring of finite Krull dimension, $R \cong S/\mathfrak{a}$, and all components of $\mathfrak{a}$ have common height h. Then R admits a prime filtration $0 = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \mathfrak{a}_\ell = R$ so that $\mathfrak{a}_i/\mathfrak{a}_{i-1} \cong R/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \operatorname{Spec}(R)$. Given ideal $I = (y_1, \ldots, y_n) \subseteq R$ and $S_I \longrightarrow R[It, t^{-1}]$ as above, the prime filtration of R extends to a *not necessarily prime* filtration of $R[It, t^{-1}]$,

$$0 = \mathfrak{c}_{I,0} \subset \mathfrak{c}_{I,1} \subset \cdots \subset \mathfrak{c}_{I,\ell} = R[It, t^{-1}]$$

where

$$\mathfrak{c}_{I,i} := \mathfrak{a}_i R[It, t^{-1}].$$

There are short exact sequences of S_I-modules

$$0 \longrightarrow \frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}} \longrightarrow \frac{R[It,t^{-1}]}{\mathfrak{c}_{I,i-1}} \longrightarrow \frac{R[It,t^{-1}]}{\mathfrak{c}_{I,i}} \longrightarrow 0.$$

The observations above motivate the technical **??** below. The corollary that follows, **??**, describes our strategy to prove **??** (**??**). We begin with the following.

Lemma 3.3. *Let $(S,\mathfrak{m},k)$ be a Gorenstein local ring of dimension $h+1$ and M a finitely generated S-module of dimension at most 1. Let $\Phi_M : M \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M,S),S)$ be the canonical map defined in* **??** *(**??**). Then Φ_M is surjective and $\ker(\Phi_M) = H^0_{\mathfrak{m}}(M)$.*

Proof. There is a short exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(M) \longrightarrow M \longrightarrow M/H^0_{\mathfrak{m}}(M) \longrightarrow 0.$$

The S-module $H^0_{\mathfrak{m}}(M)$ is 0-dimensional and $\operatorname{depth}(M/H^0_{\mathfrak{m}}(M)) = \dim(M/H^0_{\mathfrak{m}}(M))$. Therefore $\operatorname{Ext}^h_S(M,S) \cong \operatorname{Ext}^h_S(M/H^0_{\mathfrak{m}}(M),S)$ and

$$M/H^0_{\mathfrak{m}}(M) \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M/H^0_{\mathfrak{m}}(M),S),S) \cong \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M,S),S)$$

is an isomorphism. □

Lemma 3.4. *Let S be an excellent Gorenstein ring, and*

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

a short exact sequence of finitely generated S-modules so that $\operatorname{Ann}_S(M_i) \geq h$ for each $i \in \{1,2,3\}$. Let

$$\Phi_i : M_i \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_i,S),S)$$

be the canonical map as defined in **??** *(**??**) for each $i \in \{1,2,3\}$. If $c_1, c_3, d \in S$ are chosen so that*

- $c_1 \in \operatorname{Ann}(\operatorname{coker}(\Phi_1))$,
- $c_3 \in \operatorname{Ann}(\operatorname{coker}(\Phi_3))$,
- $d \in \operatorname{Ann}(\ker(\Phi_3))$,

then

$$c_1c_3d \in \operatorname{Ann}(\operatorname{coker}(\Phi_2)).$$

Proof. Consider the long exact sequence of $\operatorname{Ext}^\bullet_S(-,S)$-modules

$$0 \longrightarrow \operatorname{Ext}^h_S(M_3,S) \longrightarrow \operatorname{Ext}^h_S(M_2,S) \longrightarrow \operatorname{Ext}^h_S(M_1,S) \longrightarrow \operatorname{Ext}^{h+1}_S(M_3,S) \longrightarrow \cdots.$$

Break the long exact sequence of $\operatorname{Ext}^\bullet_S(-,S)$ into smaller exact sequences:

$$0 \longrightarrow \operatorname{Ext}^h_S(M_3,S) \longrightarrow \operatorname{Ext}^h_S(M_2,S) \longrightarrow N_1 \longrightarrow 0,$$

$$0 \longrightarrow N_1 \longrightarrow \operatorname{Ext}^h_S(M_1,S) \longrightarrow N_2 \longrightarrow 0, \text{ and} \tag{3.4.1}$$

$$0 \longrightarrow N_2 \longrightarrow \operatorname{Ext}^{h+1}_S(M_3,S) \longrightarrow \operatorname{Ext}^{h+1}_S(M_2,S).$$

A simple diagram chase implies there exists $\Psi_1 : M_1 \longrightarrow \operatorname{Ext}^h_S(N_1, S)$ so that the following is a commutative diagram with exact rows:
(3.4.2)

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "\Psi_1"] & M_2 \arrow[r] \arrow[d, "\Phi_2"] & M_3 \arrow[r] \arrow[d, "\Phi_3"] & 0 \\
0 \arrow[r] & \operatorname{Ext}^h_S(N_1, S) \arrow[r] & \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_2, S), S) \arrow[r] & \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_3, S), S) \arrow[r] & \operatorname{Ext}^{h+1}_S(N_1, S)
\end{tikzcd}$$

Moreover, if $g : \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S) \longrightarrow \operatorname{Ext}^h_S(N_1, S)$ is the induced map of $\operatorname{Ext}^h_S(-, S)$-modules obtained from (**??**), then $\Psi_1 : M_1 \longrightarrow \operatorname{Ext}^h_S(N_1, S)$ factors as

$$\Psi_1 : M_1 \xrightarrow{\Phi_1} \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S) \xrightarrow{g} \operatorname{Ext}^h_S(N_1, S).$$

Claim 3.5. *$d \operatorname{Ext}^h_S(N_1, S) \subseteq \operatorname{im}(g)$.*

Proof of Claim. The module N_1 is a submodule of $\operatorname{Ext}^h_S(M_1, S)$. If the minimal components of $\operatorname{Supp}(M_1)$ have height $h+1$ or larger, then $\operatorname{Ext}^h_S(N_1, S) = \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S) = 0$ and the claim is trivial. We therefore may assume all minimal components of the support of M_1 have height h.

By **??** (**??**), $\operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S)$ is an (S_2) module whose support agrees with the support of M_1. By (**??**), $\ker(g) \cong \operatorname{Ext}^h_S(N_2, S)$ and $N_2 \subseteq \operatorname{Ext}^{h+1}_S(M_3, S)$ is not supported at a prime of height h. Therefore $\ker(g) = 0$ and $\operatorname{im}(g) \cong \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S)$. Therefore the claimed containment can be checked after localizing at prime of height $h+1$ in the support of M_1.

Let $\mathfrak{p}$ be a prime ideal of height $h+1$ in the support of M_1. We are assuming $\operatorname{Ann}_S(M_i)$ has height at least h for each i. Therefore

- ∘ $\dim((M_1)_\mathfrak{p}) = \operatorname{depth}((M_1)_\mathfrak{p}) = 1$;
- ∘ $(M_2)_\mathfrak{p}$ and $(M_3)_\mathfrak{p}$ both have dimension at most 1.

By (**??**) there is an exact sequence

$$0 \to \operatorname{Ext}^{h+1}_S(\operatorname{Ext}^h_S(M_1, S), S)_\mathfrak{p} \to \operatorname{Ext}^{h+1}_S(N_1, S)_\mathfrak{p} \to \operatorname{Ext}^{h+2}_S(N_2, S)_\mathfrak{p}.$$

Then $\operatorname{Ext}^{h+2}_S(N_2, S)_\mathfrak{p} = \operatorname{Ext}^{h+1}_S(\operatorname{Ext}^h_S(M_1, S), S)_\mathfrak{p} = 0$ since $S_\mathfrak{p}$ has dimension $h+1$ and $\operatorname{Ext}^h_S(M_1, S)$ is an (S_2)-module over its support. Therefore the commutative diagram (**??**) is a diagram of short exact sequences:
(3.5.1)

$$\begin{tikzcd}
0 \arrow[r] & (M_1)_\mathfrak{p} \arrow[r] \arrow[d, "(\Psi_1)_\mathfrak{p}"] & (M_2)_\mathfrak{p} \arrow[r] \arrow[d, "(\Phi_2)_\mathfrak{p}"] & (M_3)_\mathfrak{p} \arrow[r] \arrow[d, "(\Phi_3)_\mathfrak{p}"] & 0 \\
0 \arrow[r] & \operatorname{Ext}^h_S(N_1, S)_\mathfrak{p} \arrow[r] & \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_2, S), S)_\mathfrak{p} \arrow[r] & \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_3, S), S)_\mathfrak{p} \arrow[r] & 0
\end{tikzcd}$$

By **??**, the localized maps $(\Phi_i)_\mathfrak{p}$ are surjective. By assumption $d \ker(\Phi_3) = 0$. A standard diagram chase implies $d \operatorname{Ext}^h_S(M_1, S)_\mathfrak{p} \subseteq \operatorname{im}((\Psi_1)_\mathfrak{p})$.

The map $(\Psi_1)_\mathfrak{p}$ factors as

$$(\Psi_1)_\mathfrak{p} M_\mathfrak{p} \xrightarrow{(\Phi_1)_\mathfrak{p}} : \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M, S), S)_\mathfrak{p} \xrightarrow{(g)_\mathfrak{p}} \operatorname{Ext}^h_S(N_1, S)_\mathfrak{p}.$$

The localized map $(\Phi_1)_\mathfrak{p}$ is an isomorphism, therefore $d \operatorname{Ext}^h_S(M_1, S)_\mathfrak{p} \subseteq \operatorname{im}((g)_\mathfrak{p})$ as needed to complete the proof of the claim. $\square$

We continue the proof of the lemma. The map Ψ_1 factors as

$$\Psi_1 : M_1 \xrightarrow{\Phi_1} \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(M_1, S), S) \xrightarrow{g} \operatorname{Ext}^h_S(N_2, S). \tag{3.5.2}$$

By Claim **??**, $d \operatorname{Ext}_S(N_2, S) \subseteq \operatorname{im}(g)$. By assumption, $c_1 \in \operatorname{Ann}(\operatorname{coker}(\Phi_1))$. Therefore $c_1 d \in \operatorname{Ann}(\operatorname{coker}(\Psi_1))$. By assumption, $c_3 \in \operatorname{Ann}(\operatorname{coker}(\Phi_3))$. A standard diagram chase of (**??**) implies $dc_1c_3 \in \operatorname{Ann}(\operatorname{coker}(\Phi_2))$. □

Corollary 3.6. *Let S be an excellent Gorenstein ring, $\mathfrak{a} \subseteq S$ an ideal whose associated primes have have a common height h, and $R = S/\mathfrak{a}$. Let $0 = \mathfrak{c}_0 \subset \mathfrak{c}_1 \subset \cdots \subset \mathfrak{c}_\ell = R$ be a filtration of R by ideals and for each $1 \leq i \leq \ell$ let*

$$\Phi_i : \mathfrak{c}_i \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(\mathfrak{c}_i, S), S) \quad \text{and} \quad \overline{\Phi}_i : \frac{\mathfrak{c}_i}{\mathfrak{c}_{i-1}} : \operatorname{Ext}^h_S\left(\operatorname{Ext}^h_S\left(\frac{\mathfrak{c}_i}{\mathfrak{c}_{i-1}}, S\right), S\right)$$

*be the canonical maps as defined in **??** (**??**). For each $1 \leq i \leq \ell$ let $c_i \in S$ be chosen so that*

$$c_i \in \operatorname{Ann}\big(\operatorname{coker}(\overline{\Phi}_i)\big).$$

For each $2 \leq i \leq \ell$ let $d_i \in S$ be chosen so that

$$d_i \ker(\overline{\Phi}_i) = 0.$$

*Let $c = \prod_{i=1}^{\ell} c_i$ and $d = \prod_{i=2}^{\ell} d_i$. If $\Phi : R \longrightarrow \operatorname{Ext}^h_S(\operatorname{Ext}^h_S(R, S), S)$ is the canonical map defined in **??** (**??**), then*

$$cd \in \operatorname{Ann}(\operatorname{coker}(\Phi)).$$

Proof. For each $1 \leq i \leq \ell$, $\mathfrak{c}_i \subseteq R$, therefore all associated primes of $\mathfrak{c}_i$, as an S-module, are associated primes of R, and therefore have the common height h. For each $1 \leq i \leq \ell$ there is a short exact sequence

$$0 \longrightarrow \mathfrak{c}_{i-1} \longrightarrow \mathfrak{c}_i \longrightarrow \frac{\mathfrak{c}_i}{\mathfrak{c}_{i-1}} \longrightarrow 0.$$

When $i = 2$, $\mathfrak{c}_{i-1} = \mathfrak{c}_1 \cong \frac{\mathfrak{c}_1}{\mathfrak{c}_0}$. By **??**,

$$c_1c_2d_2 \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_2)\big).$$

Inductively, assume that if $2 \leq i \leq \ell - 1$ that

$$c_1c_2 \cdots c_i d_2 d_3 \cdots d_i \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_i)\big).$$

Then there is a short exact sequence

$$0 \longrightarrow \mathfrak{c}_i \longrightarrow \mathfrak{c}_{i+1} \longrightarrow \frac{\mathfrak{c}_{i+1}}{\mathfrak{c}_i} \longrightarrow 0.$$

By **??**,

$$c_1c_2 \cdots c_i c_{i+1} d_2 d_3 \cdots d_i d_{i+1} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_{i+1})\big).$$

By induction, $cd \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_\ell)\big)$. □

Theorem 3.7. *Let R be an excellent ring of finite Krull dimension. If R is locally unmixed, then there exists $B \in \mathbb{N}$ and nonzero divisor $c \in R$ so that for every ideal $I \subseteq R$,*

$$ct^{-B} \in \operatorname{Ann}\big(R[It, t^{-1}]^{(S_2)}/R[It, t^{-1}]\big).$$

Proof. By [?, Theorem 6.5], there exists a faithfully flat extension $R \longrightarrow R'$ so that R' is excellent, of finite Krull dimension, and the homomorphic image of an excellent Gorenstein ring of finite Krull dimension. By **??**, we can pass the content of the theorem to R' and assume R is the homomorphic image of a finite dimensional excellent Gorenstein ring S. Assume $R \cong S/\mathfrak{a}$ and S. We are assuming R is locally unmixed. Therefore the minimal primes of $\mathfrak{a}$ agrees with the set of associated primes of $\mathfrak{a}$. Even further, if $\mathfrak{q}_1, \mathfrak{q}_2$ are primary components of $\mathfrak{a}$ of different height, then there does not exist $\mathfrak{p} \in \operatorname{Spec}(S)$ so that $\mathfrak{q}_1 \subseteq \mathfrak{p}$ and $\mathfrak{q}_2 \subseteq \mathfrak{p}$. Therefore R is isomorphic to a direct product $R \cong \prod \frac{S}{\mathfrak{a}_i}$ so that each $\mathfrak{a}_i$ is an ideal of S whose associated primes have a common height h. The content of the theorem is easily reduced to the quotient rings $S/\mathfrak{a}_i$. We therefore may assume $R \cong S/\mathfrak{a}$ and all primary components of $\mathfrak{a}$ have common height h.

Adopt the following notation when given an ideal $I \subseteq R$ and a choice of generators $y_1, y_2, \ldots, y_n$ of I:

- Let $x_1, x_2, \ldots, x_n, t^{-1}$ be indeterminates, $S_I = S[x_1, x_2, \ldots, x_n, t^{-1}]$, and
$$S_I \longrightarrow R[It, t^{-1}]$$
the onto A-algebra homomorphism defined by $x_i \mapsto y_i t$.

  Remark 3.8. The kernel of $S_I \longrightarrow R[It, t^{-1}]$ is an ideal of S_I whose components all have height $h + n$.

- Let $\Phi_I : R[It, t^{-1}] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}(\operatorname{Ext}^{h+n}_{S_I}(R[It, t^{-1}], S_I), S_I)$ the canonical map defined in **??** (**??**).

By **??** (**??**), $\Phi_I : R[It, t^{-1}] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}(\operatorname{Ext}^{h+n}_{S_I}(R[It, t^{-1}], S_I), S_I)$ is isomorphic as a map of $R[It, T^{-1}]$-modules $R[It, t^{-1}] \longrightarrow R[It, t^{-1}]^{(S_2)}$. The theorem is therefore equivalent to identifying $c \in S$ avoiding all associated primes of R and $B \in \mathbb{N}$ so that for all ideals $I = (y_1, y_2, \ldots, y_n) \subseteq R$,
$$ct^{-B} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_I)\big).$$

Begin by building a prime filtration of R:

$$0 = \mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_\ell = R \tag{3.8.1}$$

so that for each $1 \leq i \leq \ell$, $\mathfrak{p}_i \in \operatorname{Spec}(R)$ and
$$\frac{\mathfrak{a}_i}{\mathfrak{a}_{i-1}} \cong R/\mathfrak{p}_i.$$

Further adopt the following notation:

(a) For each ideal $I = (y_1, y_2, \ldots, y_n) \subseteq R$ let $\mathfrak{c}_{I,i} = \mathfrak{a}_i R[It, t^{-1}]$.
   - For each ideal $I = (y_1, y_2, \ldots, y_n) \subseteq R$, we have a (not necessarily prime) filtration of ideals $R[It, t^{-1}]$ of bounded length ℓ.
$$0 = \mathfrak{c}_{I,0} \subset \mathfrak{c}_{I,1} \subset \cdots \subset \mathfrak{c}_{I,\ell} = R[It, t^{-1}].$$

(b) For each ideal $I = (y_1, y_2, \ldots, y_n) \subseteq R$ and $1 \leq i \leq \ell$ let
$$\Phi_{\mathfrak{c}_{I,i}} : \mathfrak{c}_{I,i} \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\left(\operatorname{Ext}^{h+n}_{S_I}\left(\mathfrak{c}_{I,i}, S_I\right), S_I\right)$$
be the canonical map as defined in **??** (**??**).

(c) For each ideal $I = (y_1, y_2, \dots, y_n) \subseteq R$ and $1 \le i \le \ell$ let
$$\Phi_{\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}} : \frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}} \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\left(\operatorname{Ext}^{h+n}_{S_I}\left(\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}, S_I\right), S_I\right)$$
be the canonical map as defined in **??** (**??**).

Claim 3.9. *Assume that there are elements $c_i \in S$ and $B_i \in \mathbb{N}$ for each $1 \le i \le \ell$ and $A_i \in \mathbb{N}$ for each $1 \le i \le \ell - 1$ so that for each ideal $I = (y_1, y_2, \dots, y_n) \subseteq R$,*

- $c_i t^{-B_i} \in \operatorname{Ann}\left(\operatorname{coker}\left(\Phi_{\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}}\right)\right)$;
- $t^{-A_i} \ker\left(\Phi_{\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}}\right) = 0$.

Let $c = \prod_{i=1}^{\ell} c_i$ and $B = (\sum_{i=1}^{\ell} B_i) + (\sum_{i=1}^{\ell-1} A_{i-1})$. Then for each ideal $I = (y_1, y_2, \dots, y_n) \subseteq R$,
$$ct^{-B} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_I)\big).$$

Proof of Claim. The claim is a direct consequence of Corollary **??**. □

We continue with the proof of the theorem. We can identify constants B_i, A_i and elements $c_i \in S$ avoiding the associated primes of R satisfying the hypotheses of Claim **??**. The prime filtration (**??**) is built by identifying $0 \neq a_1 \in R$ so that $\operatorname{Ann}_R(x_1) = \mathfrak{p}_1$, $a_1 R = \mathfrak{a}_1$, built inductively by identifying elements $a_i \in R$ so that
$$\frac{\mathfrak{p}_i}{\mathfrak{a}_{i-1}} = \operatorname{Ann}_{R/(a_1,\dots a_{i-1})}\left(\frac{(a_1, a_2, \dots, a_{i-1}, a_i)}{(a_1, \dots a_{i-1})}\right),$$
and setting $\mathfrak{a}_i = (a_1, a_2, \dots, a_i)$.

Adopt the following notation:

(i) B_i is a Uniform Briançon-Bound of the integral domain $R/\mathfrak{p}_i$.

(ii) For each $1 \le i \le \ell$, let $\overline{R/\mathfrak{p}_i}$ be the normalization of the domain $R/\mathfrak{p}_i$ and $c_i \in \operatorname{Ann}_S\left(\overline{R/\mathfrak{p}_i}/R/\mathfrak{p}_i\right)$ a conductor element of $R/\mathfrak{p}_i$ chosen to avoid all associated primes of R. For each ideal $I \subseteq R$ and $1 \le i \le \ell$, let
$$\Phi_{\mathfrak{p}_i, I} : \frac{R}{\mathfrak{p}_i}\left[\frac{(I, \mathfrak{p}_i)}{\mathfrak{p}_i}t, t^{-1}\right] \longrightarrow \operatorname{Ext}^{h+n}_{S_I}\left(\operatorname{Ext}^{h+n}_{S_I}\left(\frac{R}{\mathfrak{p}_i}\left[\frac{(I, \mathfrak{p}_i)}{\mathfrak{p}_i}t, t^{-1}\right], S_I\right), S_I\right)$$
be the canonical map as defined in **??** (**??**).

- If $\mathfrak{p}_i$ is not a minimal prime of R, then the kernel of the surjective composition
$$S_I \longrightarrow R[It, t^{-1}] \longrightarrow \frac{R}{\mathfrak{p}_i}\left[\frac{(I, \mathfrak{p}_i)}{\mathfrak{p}_i}t, t^{-1}\right]$$
is a prime ideal of height at least $h + n + 1$. In particular,
$$\operatorname{Ext}^{h+n}_{S_I}\left(\operatorname{Ext}^{h+n}_{S_I}\left(\frac{R}{\mathfrak{p}_i}\left[\frac{(I, \mathfrak{p}_i)}{\mathfrak{p}_i}t, t^{-1}\right], S_I\right), S_I\right) = 0.$$
- If $\mathfrak{p}_i$ is a minimal prime of R, then the kernel of the surjective composition
$$S_I \longrightarrow R[It, t^{-1}] \longrightarrow \frac{R}{\mathfrak{p}_i}\left[\frac{(I, \mathfrak{p}_i)}{\mathfrak{p}_i}t, t^{-1}\right]$$

is a prime ideal of S_I of height $h+n$. By **??**,

$$c_i t^{-B_i} \in \operatorname{Ann}\big(\operatorname{coker}(\Phi_{\mathfrak{p}_i,I})\big).$$

(iii) By the Uniform Artin-Rees Theorem, [**?**, Main Application], there exists constants $A_i \in \mathbb{N}$ so that for all ideals $I \subseteq R$ and $n \in \mathbb{N}$,

$$\mathfrak{a}_i \cap I^n \subseteq \mathfrak{a}_i I^{n-A_i}.$$

If $1 \le i \le \ell$ then there is an onto homomorphism $\pi_{I,i}$ of $R[It,t^{-1}]$-modules

$$\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}} = \frac{(a_1,a_2,\dots,a_{i-1},a_i)R[It,t^{-1}]}{(a_1,a_2,\dots,a_{i-1})R[It,t^{-1}]} \xrightarrow{\pi_{I,i}} a_i\left(\frac{R}{\mathfrak{a}_{i-1}}\left[\frac{(I,\mathfrak{a}_{i-1})}{\mathfrak{a}_{i-1}}t,t^{-1}\right]\right)$$

by sending a_i to its residue class in $\frac{R}{\mathfrak{a}_{i-1}}\left[\frac{(I,\mathfrak{a}_{i-1})}{\mathfrak{a}_{i-1}}t,t^{-1}\right]$. The element $a_i \in R$ has the property that

$$\frac{\mathfrak{p}_i}{\mathfrak{a}_{i-1}} = \operatorname{Ann}_{R/(a_1,\dots a_{i-1})}\left(\frac{(a_1,a_2,\dots,a_{i-1},a_i)}{(a_1,\dots a_{i-1})}\right).$$

By **??** (**??**),

$$a_i\left(\frac{R}{\mathfrak{a}_{i-1}}\left[\frac{(I,\mathfrak{a}_{i-1})}{\mathfrak{a}_{i-1}}t,t^{-1}\right]\right) \cong \frac{R}{\mathfrak{p}_i}\left[\frac{(I,\mathfrak{p}_i)}{\mathfrak{p}_i}t,t^{-1}\right].$$

After inverting t^{-1}, $\pi_{I,i}$ is an isomorphism. Therefore $\operatorname{Ann}_{S_I}(\ker(\pi_{I,i}))$ has height at least $h+n+1$, implying

$$\begin{aligned}\operatorname{Ext}^{h+n}_{S_I}\left(\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}},S_I\right) &\cong \operatorname{Ext}^{h+n}_{S_I}\left(a_i\left(\frac{R}{\mathfrak{a}_{i-1}}\left[\frac{(I,\mathfrak{a}_{i-1})}{\mathfrak{a}_{i-1}}t,t^{-1}\right]\right),S_I\right)\\ &\cong \operatorname{Ext}^{h+n}_{S_I}\left(\frac{R}{\mathfrak{p}_i}\left[\frac{(I,\mathfrak{p}_i)}{\mathfrak{p}_i}t,t^{-1}\right],S_I\right).\end{aligned}$$

Consequently, if $d \in S_I$ then

$$d \in \operatorname{Ann}\left(\operatorname{coker}\left(\Phi_{\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}}\right)\right) \quad \text{if and only if} \quad d \in \operatorname{Ann}\left(\operatorname{coker}\left(\Phi_{\mathfrak{p}_i,I}\right)\right).$$

By (**??**),

$$c_i t^{-B_i} \in \operatorname{Ann}\left(\operatorname{coker}\left(\Phi_{\frac{\mathfrak{c}_{I,i}}{\mathfrak{c}_{I,i-1}}}\right)\right).$$

The degree n piece of $\ker(\pi_{I,i})$ is an element of $\frac{\mathfrak{a}_i I^n}{\mathfrak{a}_{i-1}I^n}$ mapped to 0 in $a_i\left(\frac{(I^n,\mathfrak{a}_{i-1})}{\mathfrak{a}_{i-1}}\right)$. Therefore if $f \in R$ is a lift of an element belonging to $\ker(\pi_{I,i})_n$ then

$$f \in \mathfrak{a}_i I^n \cap \mathfrak{a}_{i-1} \subseteq I^n \cap \mathfrak{a}_{i-1}.$$

By (**??**), $I^n \cap \mathfrak{a}_{i-1} \subseteq I^{n-A_{i-1}}\mathfrak{a}_{i-1}$. It follows that

$$t^{-A_{i-1}}\ker(\pi_{I,i}) = 0.$$

Let $c = \prod_{i=1}^{\ell} c_i$ and $B = (\sum_{i=1}^{\ell} B_i) + (\sum_{i=2}^{\ell} A_{i-1})$. The element c avoids all associated primes of R by construction and the constant B is independent of the ideal I. By Claim **??**, for each ideal $I \subseteq R$,

$$ct^{-B} \in \operatorname{Ann}(\operatorname{coker}(\Phi_I)). \qquad \square$$

Before completing the proof of **??**, we remind the reader of a well-known application of the Uniform Artin-Rees Property.

Lemma 3.10 (Application of the Uniform Artin-Rees Property)**.** *Let R be an excellent ring of finite Krull dimension and $c \in R$ a nonzero divisor. There exists a constant $A \in \mathbb{N}$ so that for all ideals $I \subseteq R$ and $n \in \mathbb{N}$, $(I^{n+A} :_R c) \subseteq I^n$.*

Proof. By the Uniform Artin-Rees Theorem, [**?**, Main Application], there exists a constant $A \in \mathbb{N}$ so that for every ideal $I \subseteq R$ and $n \in \mathbb{N}$,

$$(c) \cap I^{n+A} = c(I^{n+A} :_R c) \subseteq cI^n.$$

Canceling the nonzero divisor c implies the claimed containment. □

Corollary 3.11. *Let R be an excellent ring of finite Krull dimension. If R is locally unmixed and (S_2), then there exists $B \in \mathbb{N}$ so that for all ideals $I \subseteq R$,*

$$t^{-B}R[It, t^{-1}]^{(S_2)} \subseteq R[It, t^{-1}].$$

Proof. By **??**, there exists nonzero divisor $c \in R$ and $B \in \mathbb{N}$ so that for all ideals $I \subseteq R$,

$$ct^{-B}R[It, t^{-1}]^{(S_2)} \subseteq R[It, t^{-1}].$$

Let A be as in **??** with respect to the nonzerodivisor c. Multiplying the above containment t^{-n-A} and contracting back in R implies

$$(I^{n+A+B})_1 \subseteq (I^{n+A} :_R c) \subseteq I^n.$$

We are assuming R is (S_2). Therefore the nth graded piece of $R[It, t^{-1}]^{(S_2)}$ is $(I^n)_1$ by **??** (**??**). Therefore $t^{-A-B}R[It, t^{-1}]^{(S_2)} \subseteq R[It, t^{-1}]$. The constant $A + B$ is independent of the ideal $I \subseteq R$. □

Theorem 3.12. *Let R be an excellent ring of finite Krull dimension and without embedded associated primes. There exists a constant $B \in \mathbb{N}$ so that for every ideal $I \subseteq R$ and $n \in \mathbb{N}$,*

$$(I^{n+B})_1 \subseteq I^n.$$

Proof. If $0 = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_\ell$ is the minimal primary decomposition of the 0-ideal of R, $Q_i = \bigoplus_{n\in\mathbb{Z}} I^n \cap \mathfrak{q}_i$, then $0 = Q_1 \cap \cdots \cap Q_\ell$ is the unique minimal primary decomposition of the 0-ideal of $R[It, t^{-1}]$. We can apply **??** to the locally unmixed rings $R/\mathfrak{q}_i$. There exist nonzerodivisors c_i of $R/\mathfrak{q}_i$ so that for every ideal $I \subseteq R$ and $1 \le i \le \ell$,

$$c_i t^{-B} \left(\frac{R[It, t^{-1}]}{Q_i}\right)^{(S_2)} \subseteq \frac{R[It, t^{-1}]}{Q_i}. \tag{3.12.1}$$

After multiplication by t^{-n}, contracting in $R/\mathfrak{q}_i$,

$$c_i \left(\frac{I^{n+B} + \mathfrak{q}_i}{\mathfrak{q}_i}\right)_1 \subseteq \frac{I^n + \mathfrak{q}_i}{\mathfrak{q}_i}.$$

By **??**, we can choose $A \in \mathbb{N}$ so that for all ideals $I \subseteq R$, $n \in \mathbb{N}$, and $1 \le i \le \ell$,

$$\left(\frac{I^{n+B+A} + \mathfrak{q}_i}{\mathfrak{q}_i}\right)_1 \subseteq \frac{I^n + \mathfrak{q}_i}{\mathfrak{q}_i}.$$

Contraction in R with respect to the inclusion $R \subseteq \prod_{i=1}^{\ell} \frac{R}{\mathfrak{q}_i}$ implies

$$(I^{n+B+A})_1 = \left(\prod_{i=1}^{\ell} \left(\frac{I^{n+B+A} + \mathfrak{q}_i}{\mathfrak{q}_i}\right)_1\right) \bigcap R \subseteq \bigcap_{i=1}^{\ell} (I^n + \mathfrak{q}_i).$$

The inclusion $R \subseteq \prod_{i=1}^{\ell} \frac{R}{\mathfrak{q}_i}$ is generically an isomorphism. The cokernel of C of the inclusion is then annihilated by a nonzero divisor $d \in R$. The kernel of

$$\frac{R}{I^n} \to \prod_{i=1}^{\ell} \frac{R}{(I^n, \mathfrak{q}_i)}$$

is

$$\frac{\bigcap_{i=1}^{\ell}(I^n + \mathfrak{q}_i)}{I^n} \cong \operatorname{Tor}_1^R(R/I^n, C).$$

The module C is annihilated by the nonzero divisor d, therefore

$$d \bigcap_{i=1}^{\ell}(I^n + \mathfrak{q}_i) \subseteq I^n.$$

Consequently, for every ideal $I \subseteq R$, for every $n \in \mathbb{N}$,

$$(I^{n+B+A})_1 \subseteq \bigcap_{i=1}^{\ell}(I^n + \mathfrak{q}_i) \subseteq (I^n :_R d).$$

The constants A, B are independent of the ideal $I \subseteq R$ and $n \in \mathbb{N}$. Let D be as in **??** with respect to the nonzerodivisor d. Then for every ideal $I \subseteq R$ and every $n \in \mathbb{N}$,

$$(I^{n+B+A+D})_1 \subseteq (I^{n+D} :_R d) \subseteq I^n.$$

□

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