180D Lab Report 1

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1 Introduction

The main objective of this experiment was to study shallow water waves and their dispsersive nature. Measurements of the damping of the waves were taken, and solitons were generated.

2 Theory

We begin with the same fundamental equations governing a fluid: conservation of mass (1) and Euler's equation (2), with p the pressure, ρ the density, and $\mathbf{u}$ the velocity field.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0. \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\frac{\boldsymbol{\nabla} p}{\rho} \tag{2}$$

Assuming an inviscid and irrotational flow,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = 0, \tag{3}$$

we can define the velocity potential ϕ using

$$\mathbf{u} = \boldsymbol{\nabla}\phi \tag{4}$$

Additionally, assuming an incompressible fluid, defined by

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{5}$$

we can write

$$\boldsymbol{\nabla}^2 \phi = 0. \tag{6}$$

Then, the second term in (2) is zero (left as an exercise to the reader). Plugging (4) into (2) and rearranging:

$$\boldsymbol{\nabla}(\rho \frac{\partial \phi}{\partial t} + p) = 0 \tag{7}$$

Using the hydrostatic balance equation

$$p = \rho g h \tag{8}$$

we can rearrange the equation into the form

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0. \tag{9}$$

Since this is a separable differential equation, we can assume a solution of the form

$$\phi = X(x)Z(z). \tag{10}$$

That solution is

$$\phi = (Ae^{ikx} + Be^{-ikx})(Ce^{kz} + De^{-kz})e^{\omega t}. \tag{11}$$

Using the boundary conditions

$$-\frac{\partial \phi}{\partial z}\bigg|_{z=0} = v_z(0) = 0, \quad \frac{\partial \phi}{\partial z}\bigg|_{z=\eta(x,t)} = v_z(\eta) = \frac{\partial \delta h}{\partial t} \tag{12}$$

it becomes clear $C = D$ and

$$\phi = e^{i\omega t}(Ae^{ikx} + Be^{-ikx})\cosh(kz) \tag{13}$$

This allows us to derive the dispersion relation for shallow water waves:

$$\omega^2 = gk\tanh(kh). \tag{14}$$

For higher frequencies with wavelengths similar to the capillary length of water, we also account for surface tension restoring forces, and the dispersion relation is

$$\omega^2 = (gk + \gamma k^3)\tanh(kh), \tag{15}$$

where γ is $0.5grh$, with g the acceleration due to gravity, r the radius of a pipette, and h the capillary height measured in said pipette. We also define the phase speed:

$$c_{phase} = \frac{\omega}{k}. \tag{16}$$

3 Experimental Setup

3.1 Materials

The experiment was performed using a glass channel with width 2.7±0.1 cm and length 34±0.5 cm. It was filled to a depth of 2.4±0.1 cm with a mixture of water, salt, and a solution called PhotoFlo. A function generator drives a low-frequency speaker, which in turn drives the channel that is mounted on a ball-bearing stage that moves back and forth. This generates surfaces waves in the channel, which are detected at the other end of the channel. The resistance between two wires in the water is detected by a Wheatstone bridge circuit and then measured by a lock-in amplifier which is passed through an AC to DC converter. The data was analyzed using LabView and Matlab.

3.2 Frequency Sweep

The first part of the experiment involved measuring the water surface variation produced by frequencies in the range of 0.5-10 Hz, split into 1999 second-long sweeps of 0.5-3.5 Hz, 3.5-7 Hz, and 7-10 Hz. This produced observations of the resonance modes.

3.3 Damping Coefficients

The next part of the experiment involved measuring damping coefficients for the first 6 resonances. This was accomplished by manually tuning to those frequencies (measured in the last part), and then turning the speaker off and recording the time decay.

3.4 Solitons

The final part of the experiment was a little different and started with laying the speaker flat underneath the channel to oscillate it in the vertical direction. Tuning to about 10 Hz and oscillating the water at the center of the channel produces what is known as a non-propagating soliton, detailed further in [2].

4 Data/Analysis

4.1 Surface Tension

A measurement was made of the capillary height of the water using a pipette of volume 10 µL and length 12.8 cm, which means a radius of 5 mm. Using Appendix 1 of [1], we have

$$\gamma = \frac{1}{2}grh = \frac{1}{2}(9.81)(0.005)(0.0065) = 1.59 * 10^{-4}. \tag{17}$$

4.2 Resonant Frequencies

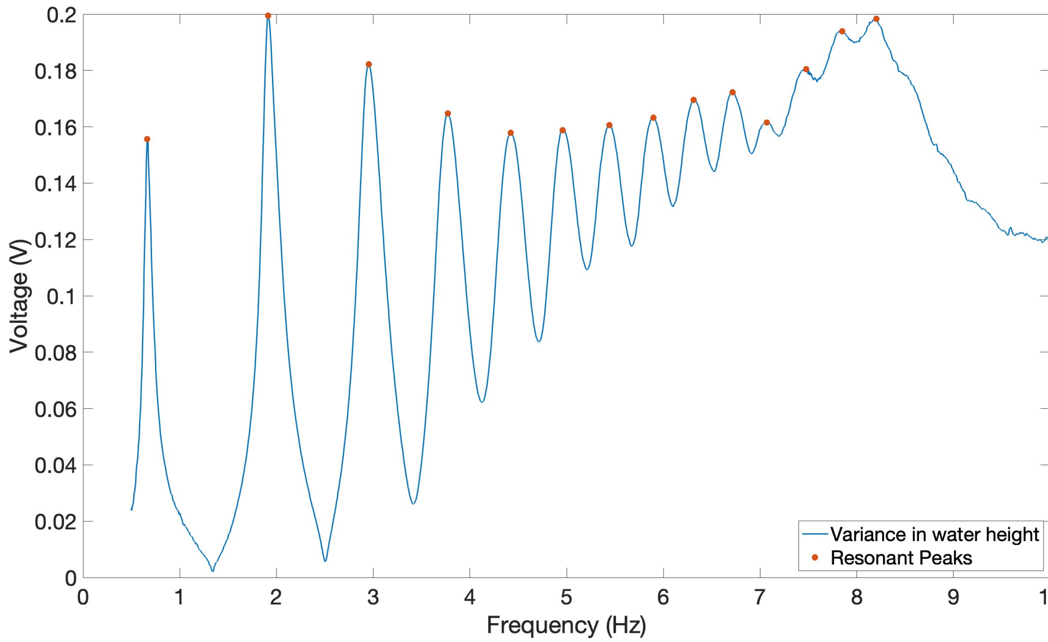

Figure 1: Recorded voltage from all three sweeps stitched together. Orange dots represent the resonance peaks.

Figure 1 shows the results from all three sweeps (0.5-3.5 Hz, 3.5-7 Hz, and 7-10 Hz) plotted together. The resonant modes are highlighted by orange dots and listed in Table 1. It is worth noting that past about 8 Hz, the results becomes mostly noise, which suggests that the waves break down past that point.

Listed in column 4 is the phase speed, calculated using equation [16] from the observed resonance frequencies. In column 5 is the theoretical phase speed calculated using equation [14], and in column 6 using equation [15] including the surface tension correction. Figure 2 represents these phase speeds plotted together with their uncertainties.

As expected, the observed phase theory tracks with the theoretical value well and underpredicts it for higher wavenumbers when we expect the surface tension to be non-negligible. The first few observed values are too small for unclear reasons. It suggests the values of the observed resonant frequencies in the data are slightly below what we would expect. The calculated values using the surface tension are far too big, which suggests the calculated value of γ is too large. This could be from a measurement error when finding the capillary height.

f (s^{-1})	ω (s^{-1})	k (m^{-1})	c $(\frac{m}{s})$	Theoretical c $(\frac{m}{s})$	with surface tension
0.663	4.165	9.24	0.451	0.481	0.482
1.914	12.026	27.72	0.434	0.454	0.457
2.954	18.563	46.2	0.402	0.413	0.42
3.771	23.695	64.68	0.366	0.372	0.385
4.421	27.778	83.16	0.334	0.337	0.356
4.958	31.151	101.64	0.306	0.308	0.333
5.442	34.192	120.12	0.285	0.285	0.317
5.898	37.057	138.6	0.267	0.266	0.304
6.312	39.662	157.08	0.252	0.25	0.296
6.716	42.195	175.56	0.24	0.236	0.29
7.069	44.416	194.04	0.229	0.225	0.285
7.476	46.974	212.52	0.221	0.215	0.283
7.848	49.31	230.999	0.213	0.206	0.282
8.202	51.535	249.479	0.207	0.198	0.281

Table 1: Observed Resonance Frequencies in the glass channel

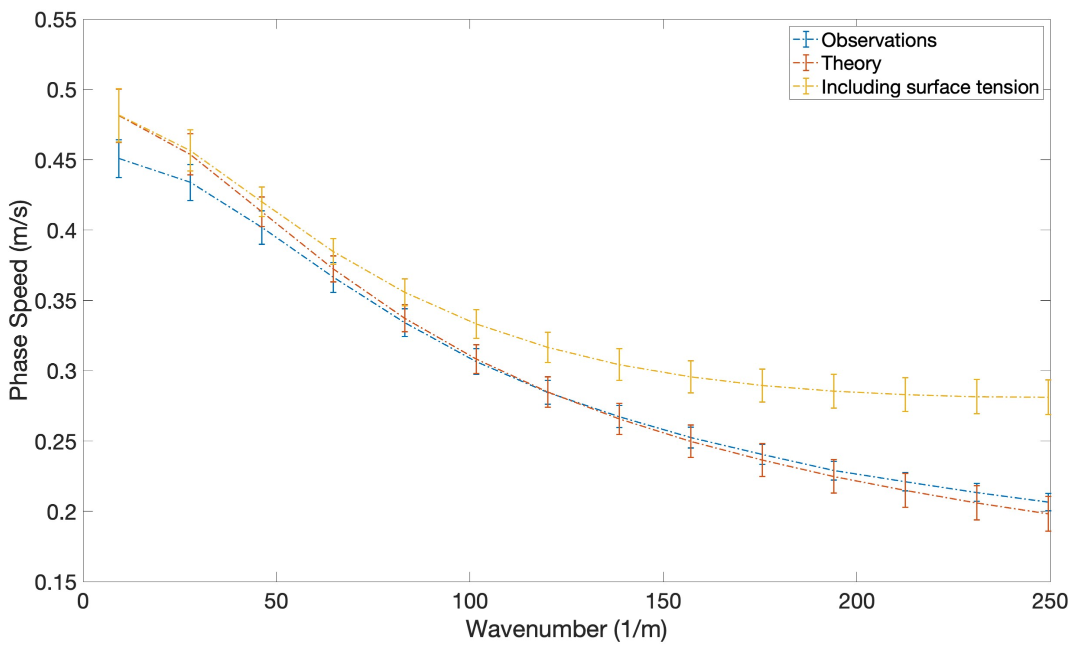

Figure 2: Phase Speed vs Wavenumber

Frequency (Hz)	α	Q factor
0.6629	0.1195	17.43
1.914	0.3127	19.23
2.954	0.5874	15.80
3.771	0.7848	15.10
4.421	0.8828	15.73
4.958	1.1112	15.74

Table 2: Damping Coefficients and Quality factors for first 6 resonance modes

4.3 Q factor

Based on the results of the previous section, measurements were made of the damping for each of the first six resonances, as shown in Figures 3-8. The crests and troughs were identified by finding one local maximum and one local minimum for each oscillation, using the Matlab functions `islocalmax` and `islocalmin`. By taking the inverse of the frequency in question, the period T was obtained, which gave the function the maximum spacing between maxima. An exponential decay of the form

$$A(t) = A_0 \exp(-\alpha t) \tag{18}$$

was fitted on each plot. This was accomplished by taking the logarithm of Equation 18 and then using an ordinary least-squares fit to find $\log(A_0)$ and α. Fitting was only done on the values after the decay started. Deciding when the decay started was based on when the maximum values were no longer at the peak amplitude, which is a potential source of uncertainty in the process. Fitting like this becomes difficult to do accurately as the decay becomes very fast for higher modes. Table 2 displays the measured damping coefficients for the first six resonances and the corresponding Q factors, calculated using

$$Q_{factor} = \frac{\omega}{2\alpha}. \tag{19}$$

On each plot, the crests and troughs of the water height are circled in red, and the fitted exponential is plotted as a line on top of the observations. The trend throughout the plots is that higher resonance frequencies mean faster damping, and this is reflected by higher damping coefficients. When calculating the Q factor however, we find very similar values.

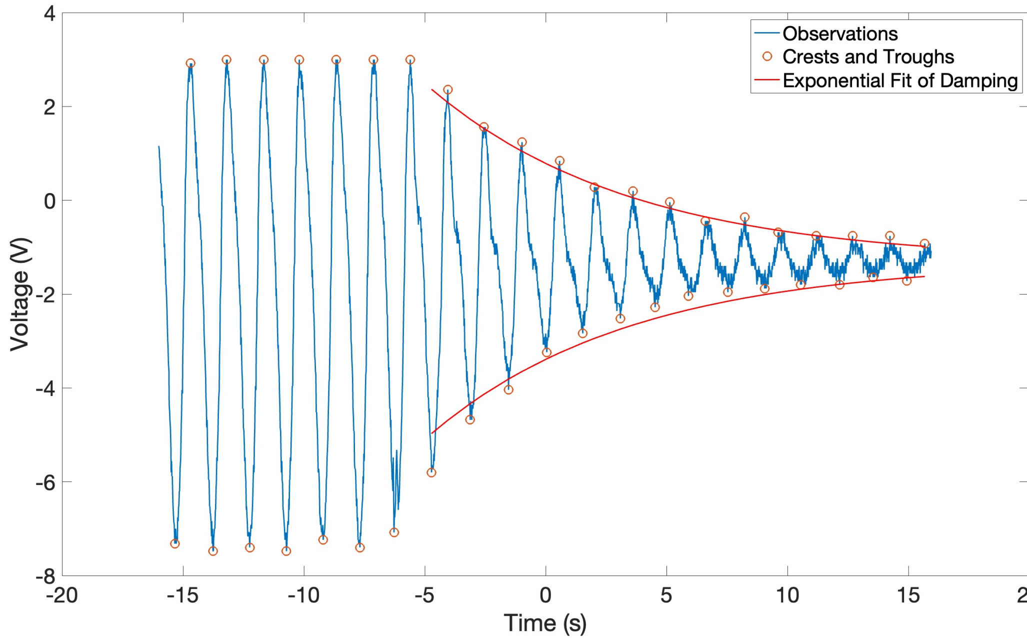

Figure 3: Variation in water height vs time, with f = 0.6629 Hz for the first resonant mode

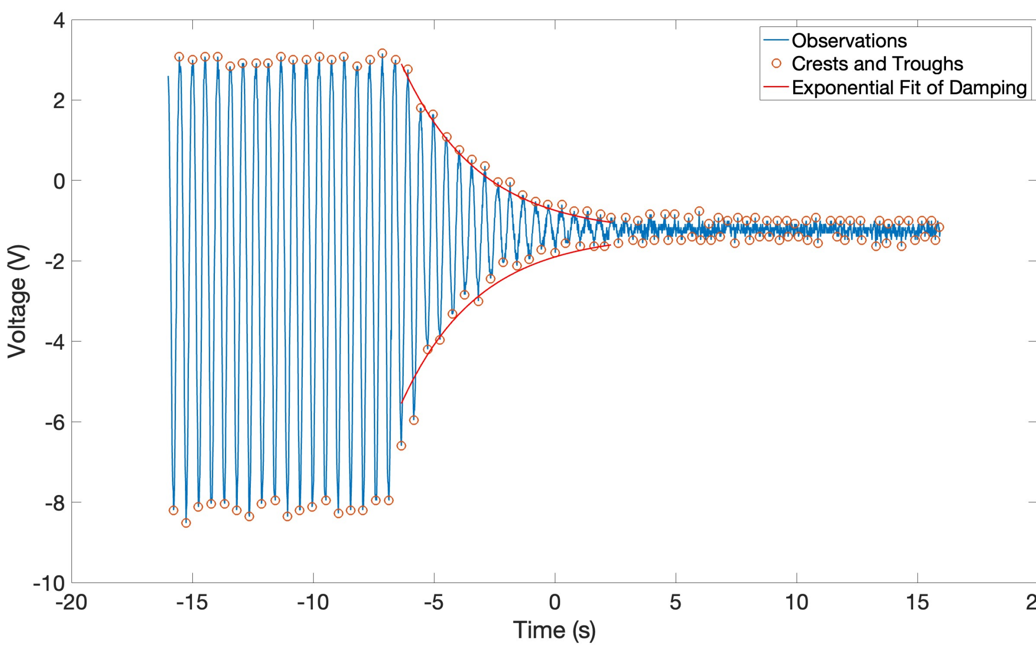

Figure 4: f = 1.914 Hz

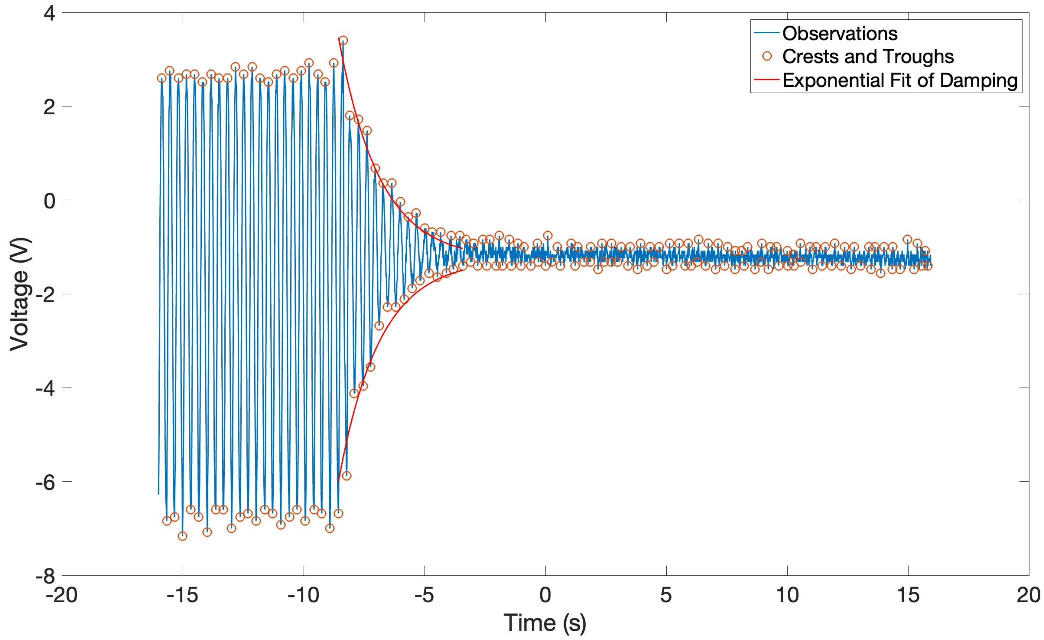

Figure 5: As in Figure 3 with f = 2.954 Hz

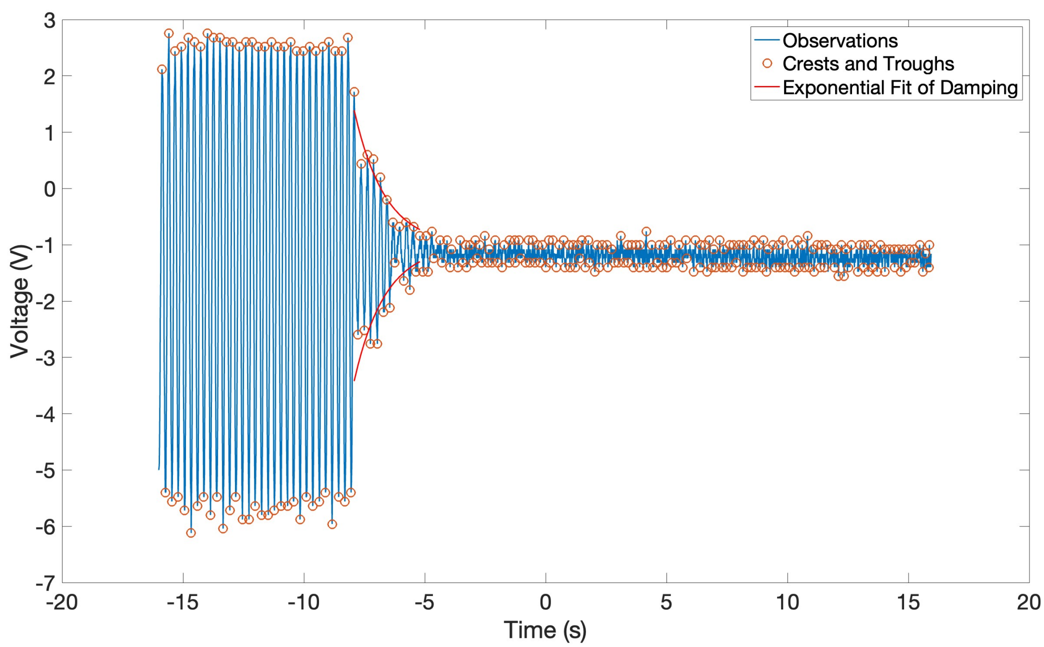

Figure 6: As in Figure 3 with f = 3.771 Hz

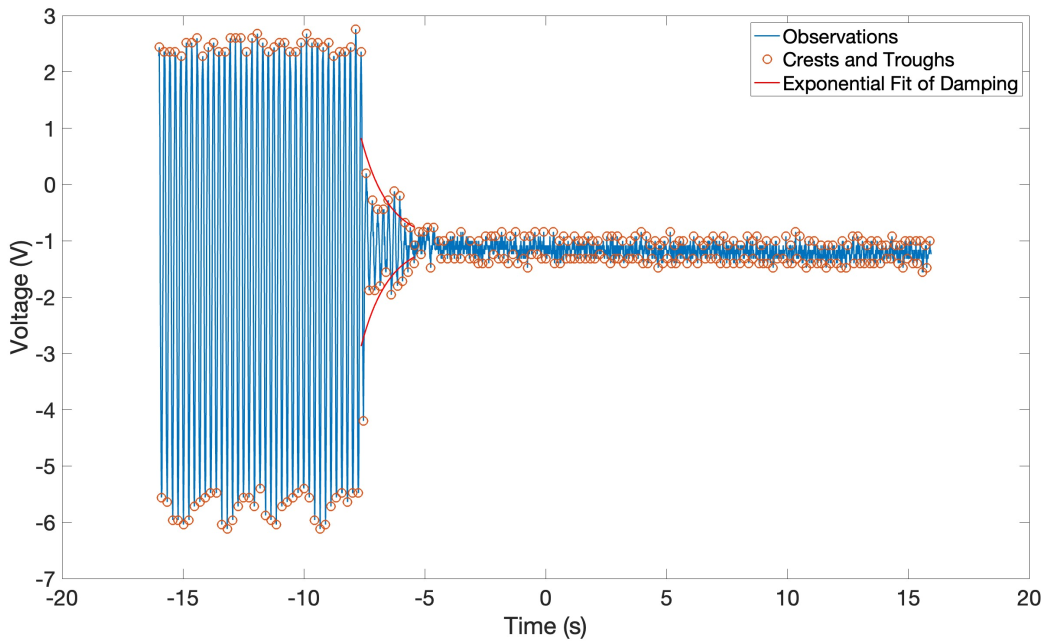

Figure 7: As in Figure 3 with f = 4.421 Hz

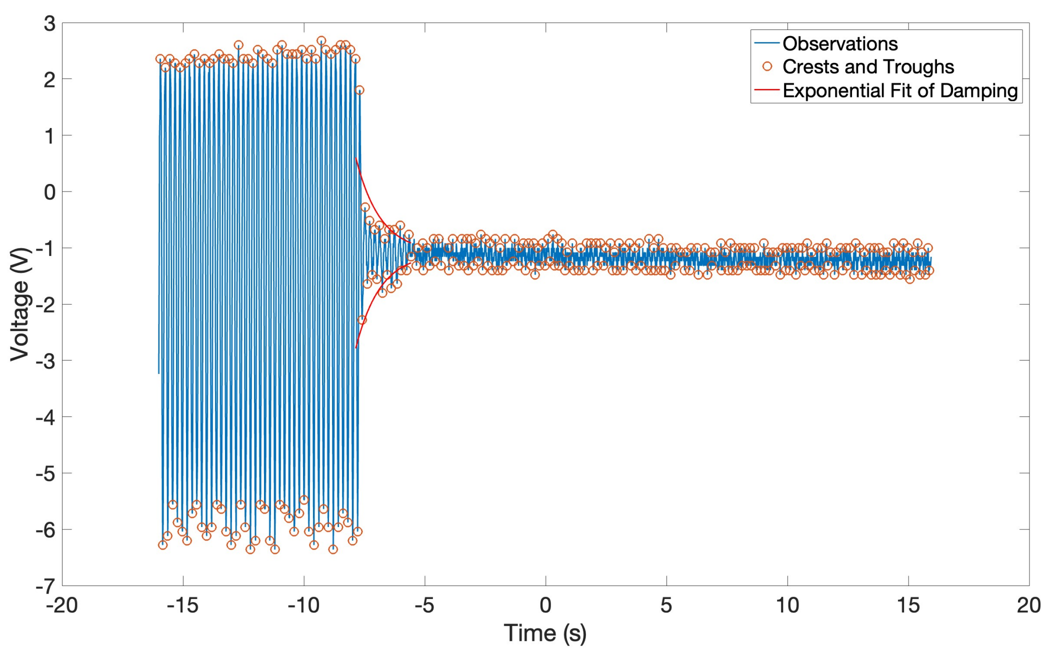

Figure 8: As in Figure 3 with f = 4.958 Hz

4.4 Solitons

As explained in Section 3.4, oscillating the channel in the y-direction (taking x along the length of the channel, y the width, and z the height) allowed the creation of non-propagating solitons. Figure 9 is an image of the result. The solitons do not move in the x-direction. They oscillate in the y-direction between the edges of the channel. Multiple solitons can be created at once along the channel; they will either merge or not interact with each other. Through trial and error, we found a range of frequencies that could support the solitons, between 10.1 and 11.4 Hz, and a range of voltages, between 0.5 and 1.05 V.
Creating a soliton in this way is at the first resonance mode in the y-direction. Therefore, given wavenumber

$$k = \frac{\pi}{W} = \frac{\pi}{0.027} = 116.36 \text{ m}^{-1}, \tag{20}$$

the dispersion relation gives us

$$f = \frac{\omega}{2\pi} = \frac{\sqrt{gk\tanh(kh)}}{2\pi} = \frac{\sqrt{9.81 \cdot 116.36 \cdot \tanh(116.36 \cdot 0.024)}}{2\pi} = 5.357 \text{ Hz}. \tag{21}$$

Twice that number is about 10 Hz, as expected.

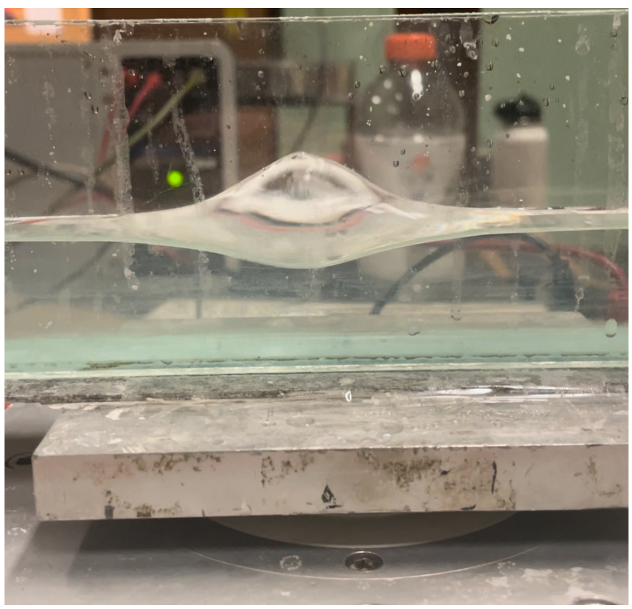

Figure 9: Picture of a non-propagating soliton

4.5 Error Analysis

The uncertainty in the calculations of frequency, wavenumber, and phase velocity are reflected in Figure 2. Accuracy throughout the experiment was limited by the ability of humans to accurately measure lengths. Beyond that, an unknown error in the calculation of the surface tension of water produced too strong of a deviation from the no-surface tension limit. The oscilloscope's data, used for the damping coefficient section, recorded only 4 significant digits, which made the reported time separations inaccurate for $|t| > 10$ seconds. The relevant damping occurred between -10 and 10 seconds in every instance but if working with the full record was necessary, the times would have needed to be corrected.

5 Conclusion

Resonant modes were identified for shallow dispersive water waves in a narrow glass channel. The frequencies of the modes were used to calculate phase speeds, which were compared to theoretical predictions of the phase speed using dispersion relations. These speeds agreed within uncertainty. However, when adding in a correction for surface tension, which should improve prediction accuracy, there was a clear disparity between prediction and observation. Quality factors were calculated for each of the first six resonance modes, which had the same order of magnitude. Given that they were greater than $\frac{1}{2}$, the system is underdamped in its resonant modes, which is expected. Finally, non-propagating solitons were produced in the channel, which are a unique self-sustaining oscillation that can be produced in a narrow chanenel.

References

[1] Physics 180D. *Experiment 2: Water Waves and Solitons.*

[2] Junru Wu, Robert Keolian, and Isadore Rudnick. Observation of a nonpropagating hydrodynamic soliton. *Phys. Rev. Lett.*, 52:1421–1424, Apr 1984.