PHYS4261HW4

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Problem 1

The basic properties of the Laplace transform can be proved using basic identities from calculus.

1a

Proof.

$$L(\dot{f}(s)) = \int_0^\infty \dot{f}(t)e^{-st}\mathrm{d}t \tag{1}$$

$$\int_0^\infty f(t)e^{-st}\mathrm{d}t = -\frac{1}{s}f(t)e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^\infty \dot{f}(t)e^{-st}\mathrm{d}t \tag{2}$$

$$\rightarrow L(f(t)) = \frac{f(0)}{s} + \frac{L(\dot{f}(t))}{s} \rightarrow L(\dot{f}(t)) = sL(f(t)) - f(0) \tag{3}$$

□

0.1 1b

Proof.

$$L(e^{i\omega t}) = \int_0^\infty e^{-i\omega t}e^{-st}\mathrm{d}t = \int_0^\infty e^{-(s+i\omega)t}\mathrm{d}t \tag{4}$$

$$= \frac{-1}{s+i\omega}e^{-(s+i\omega)t}\Big|_0^\infty = \frac{1}{s+i\omega} \tag{5}$$

□

Problem 2

2a

We use the Laplace transform to reduce the differential equations:

$$\dot{c}_{e,vac}(t) = -i\omega_e c_{e,vac}(t) - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda} c_{g,\vec{k}\lambda}(t) \tag{6}$$

$$\dot{c}_{g,\vec{k},\lambda}(t) = -i(\omega_e + \Omega_k)c_{g,\vec{k}\lambda}(t) - ig*_{\vec{k},\lambda} c_{e,vac}(t) \tag{7}$$

$$\to L(\dot{c}_{e,vac}(t))(s) = -i\omega_e L(c_{e,vac}(t))(s) - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda} L(c_{g,\vec{k}\lambda}(t))(s) \tag{8}$$

$$\to s\hat{c}_{e,vac}(s) - c_{e,vac}(0) = -i\omega_e \hat{c}_{e,vac}(s) - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda}\hat{c}_{g,\vec{k}\lambda}(s) \tag{9}$$

$$\tag{10}$$

$$L(\dot{c}_{g,\vec{k},\lambda}(t))(s) = -i(\omega_e + \Omega_k)L(c_{g,\vec{k}\lambda}(t))(s) - ig*_{\vec{k},\lambda} L(c_{e,vac}(t))(s) \tag{11}$$

$$s\hat{c}_{g,\mathbf{k}\lambda}(s) - c_{g,\mathbf{k}\lambda}(0) = -i(\omega_e + \Omega_k)\hat{c}_{g,\mathbf{k}\lambda}(s) - ig^*_{\mathbf{k},\lambda}\hat{c}_{e,vac}(s) \tag{12}$$

I changed up the notation midway through the calculation but the result should still be obvious. We simply use the derivative property of the laplace transform and the initial value conditions.

2b

$$s\hat{c}_{e,vac}(s) - 1 = -i\omega_e\hat{c}_{e,vac}(s) - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda}\hat{c}_{g,\vec{k}\lambda}(s)$$

$$s\hat{c}_{g,\mathbf{k}\lambda}(s) = -i(\omega_e + \Omega_k)\hat{c}_{g,\mathbf{k}\lambda}(s) - ig^*_{\mathbf{k},\lambda}\hat{c}_{e,vac}(s)$$

$$\to \hat{c}_{g,\mathbf{k}\lambda}(s)(s + i(\omega_e + \Omega_k)) = -ig^*_{\mathbf{k},\lambda}\hat{c}_{e,vac}(s)$$

$$\to s\hat{c}_{e,vac}(s) - 1 = -i\omega_e\hat{c}_{e,vac}(s) - \sum_{\mathbf{k},\lambda} g_{\mathbf{k},\lambda} g^*_{\mathbf{k},\lambda}\frac{\hat{c}_{e,vac}(s)}{(s + i(\omega_e + \Omega_k))}$$

$$\to \hat{c}_{e,vac}(s)(s + i\omega_e + \sum_{\mathbf{k},\lambda}\frac{|g_{\mathbf{k},\lambda}|^2}{(s + i(\omega_e + \Omega_k))}) = 1$$

$$\to \hat{c}_{e,vac}(s) = \frac{1}{s + i\omega_e + i\Sigma(s)}$$

I think that there is a typo on the HW and that ω_e should be ω_g for the second differential equation given as that is what the definition of the self energy is. (I've just left it for now though)

Problem 3

3a

$$i\Sigma(s) \equiv \sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \left(s + i(\omega_g + \Omega_k)\right)^{-1} \tag{13}$$

We then substitute $s \to -i\omega_e + \epsilon$:

$$\Sigma_0 = \Sigma(-i\omega_e + \epsilon) = \sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \left(i(-i\omega_e + \epsilon) + -(\omega_g + \Omega_k)\right)^{-1} \tag{14}$$

$$= \sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \frac{1}{\omega_e + i\epsilon - \omega_g - \Omega_k} = \sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \frac{1}{\omega_0 + i\epsilon - \Omega_k} \tag{15}$$

3b

To find the imaginary component of Σ_0, we need to convert it into the form $z = a + bi$. We can use the relation $\frac{1}{z} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$:

$$\sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \frac{1}{\omega_0 + i\epsilon - \Omega_k} = \sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \left(\frac{(\omega_0 - \Omega_k)^2}{\epsilon^2 + (\omega_0 - \Omega_k)^2} - \frac{i\epsilon}{\epsilon^2 + (\omega_0 - \Omega_k)^2}\right)$$

$$\to \mathrm{Im}(\Sigma_0) = -\sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \frac{\epsilon}{\epsilon^2 + (\omega_0 - \Omega_k)^2} = -\sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \pi \frac{1}{\pi\epsilon(1 + \frac{(\omega_0 - \Omega_k)^2}{\epsilon^2})}$$

$$\lim_{\epsilon \to 0} \frac{1}{\pi\epsilon(1 + \frac{(\omega_0 - \Omega_k)^2}{\epsilon^2})} = \delta(\omega_0 - \Omega_k) \to \mathrm{Im}(\Sigma_0) = -\sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \pi \delta(\omega_0 - \Omega_k)$$

Thus using the property that the lorentzian approaches the dirac delta function in the appropriate limit, we find an expression for the self energy imaginary component.

3c

We can use the following relations for the sum over polarization and k vectors as well as the expression for the coupling coefficient:

$$\sum_{\mathbf{k}\lambda} = \int_0^\infty \frac{V}{8\pi^3} k^2 \mathrm{d}k \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \sum_{\lambda} \tag{16}$$

These identities, we can then evaluate $\mathrm{Im}(\Sigma_0)$. I'm going to assume that $\Omega_k = c|k|$

$$
\begin{aligned}
-\sum_{\mathbf{k}\lambda} |g_{\mathbf{k},\lambda}|^2 \pi\delta(\omega_0 - \Omega_k) = -\int_0^\infty \frac{V}{8\pi^3} k^2 \mathrm{d}k \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \sum_\lambda |g_{\mathbf{k},\lambda}|^2 \pi\delta(\omega_0 - \Omega_k) \\
= -\frac{c}{16\pi^2\epsilon_0\hbar} \int_0^\infty k^3 \mathrm{d}k \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} \sum_\lambda (\langle f,2| \mathbf{D} |i,3\rangle \cdot \mathbf{e}^*_{\mathbf{k},\lambda} \mathbf{e}_{\mathbf{k},\lambda} \cdot \langle i,3| \mathbf{D} |f,2\rangle) \frac{\delta(\frac{\omega_0}{c} - |k|)}{c} \\
= -\frac{c}{16\pi^2\epsilon_0\hbar} \frac{1}{c} (\frac{\omega_0}{c})^3 \int \mathrm{d}\Omega_{\hat{\mathbf{k}}} (\langle f,2| \mathbf{D} |i,3\rangle \cdot \sum_\lambda \mathbf{e}^*_{\mathbf{k},\lambda} \mathbf{e}_{\mathbf{k},\lambda} \cdot \langle i,3| \mathbf{D} |f,2\rangle) \\
= -\frac{c}{16\pi^2\epsilon_0\hbar} \frac{1}{c} (\frac{\omega_0}{c})^3 \frac{8\pi}{3} \langle f,2| \mathbf{D} |i,3\rangle \langle i,3| \mathbf{D} |f,2\rangle \\
= -\frac{2\pi(\frac{\omega_0}{c})^3}{3\pi[4\pi\epsilon_0]\hbar} \langle f,2| \mathbf{D} |i,3\rangle \langle i,3| \mathbf{D} |f,2\rangle = -\frac{A}{2}
\end{aligned}
$$

Also there may be a typo on the notes as there is an extra pi in the equation that I'm not sure where it comes from.