PHYS4261HW5

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1 Problem 1

We have the following state vector of an atom and quantized electromagnetic field and we want to confirm it's density matrix properties.

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \left(|g\rangle |n+1\rangle c_{g,n+1}(t) + |e\rangle |n\rangle c_{e,n}(t)\right) + c_{g,0}(t) |g\rangle |vac\rangle \tag{1}$$

1.1

We can confirm that $\rho = |\Psi(t)\rangle \langle\Psi(t)|$ is a density matrix. We can take advantage of the fact that both the trace and partial trace will only take the diagonal values of the matrix rep. of an operator. Thus, any outer products that aren't an outer product of a state with itself will result in the total trace components being 0. I assume this implicitly when calculating the traces.

$$|\Psi(t)\rangle \langle\Psi(t)| = \left(\sum_{n=0}^{\infty} \left(|g\rangle |n+1\rangle c_{g,n+1}(t) + |e\rangle |n\rangle c_{e,n}(t)\right) + c_{g,0}(t) |g\rangle |vac\rangle\right)$$

$$\left(\sum_{n=0}^{\infty} \left(\langle g| \langle n+1| c^*_{g,n+1}(t) + \langle e| \langle n| c^*_{e,n}(t)\right) + c^*_{g,0}(t) \langle g| \langle vac|\right)$$

$$Tr_A Tr_F(\rho) = Tr_A Tr_F \left(\sum_{n=0}^{\infty} \left(|c_{g,n+1}(t)|^2 |g\rangle \langle g| \otimes |n+1\rangle \langle n+1|\right)\right) \tag{2}$$

$$+Tr_A Tr_F \left(\sum_{n=0}^{\infty} |c_{e,n}(t)|^2 |e\rangle \langle e| \otimes |n\rangle \langle n|\right) + Tr_A Tr_F(|c_{g,0}(t)|^2 |g\rangle \langle g| \otimes |vac\rangle \langle vac|)$$

$$= \sum_{n=0}^{\infty} (|c_{g,n+1}(t)|^2 + c_{e,n}(t)|^2) + |c_{g,0}(t)|^2 = 1 \tag{3}$$

We can see on line 3 that all the possible probabilities for the states are represented here. Thus they sum to 1. It is also easy to see that $W(t)$ is hermitian

$$W^\dagger(t) = (|\Psi(t)\rangle \langle\Psi(t)|)^\dagger = |\Psi(t)\rangle \langle\Psi(t)| = W(t) \tag{4}$$

We know that each element on the diagonal of W will be a probability which will always be greater than or equal to 0. Since the definition of positivity is: $\langle\psi| W(t) |\psi\rangle \geq 0$

1.2

To find an expression for the atomic density operator, we can go back to the density operator calculation earlier except we must include terms with off diagonal atomic state outer products (e.g $|e\rangle \langle g|$):

$$
\begin{aligned}
&|\Psi(t)\rangle \langle\Psi(t)| = \left(\sum_{n=0}^{\infty} (|g\rangle |n+1\rangle c_{g,n+1}(t) + |e\rangle |n\rangle c_{e,n}(t)) + c_{g,0}(t) |g\rangle |vac\rangle\right) \\
&\left(\sum_{n=0}^{\infty} \left(\langle g| \langle n+1| c^*_{g,n+1}(t) + \langle e| \langle n| c^*_{e,n}(t)\right) + c^*_{g,0}(t) \langle g| \langle vac|\right) \\
&\rho_A = \mathrm{Tr}_F(\rho) = \mathrm{Tr}_F(|c_{g,0}(t)|^2 |g\rangle \langle g| \otimes |vac\rangle \langle vac|) \qquad (5) \\
&+ \mathrm{Tr}_F \left(\sum_{n,n'=0}^{\infty,\infty} (|g\rangle |n+1\rangle c_{g,n+1}(t) + |e\rangle |n\rangle c_{e,n}(t)) \left(\langle g| \langle n'+1| c^*_{g,n'+1}(t) + \langle e| \langle n'| c^*_{e,n'}(t)\right)\right) \\
&+ \mathrm{Tr}_F \left(c_{e,0}(t)c^*_{g,0}(t) |e\rangle |vac\rangle \langle g| \langle vac| + c_{g,0}(t)c^*_{e,0}(t) |g\rangle |vac\rangle \langle e| \langle vac|\right) \\
&= |c_{g,0}(t)|^2 |g\rangle \langle g| + c_{e,0}(t)c^*_{g,0}(t) |e\rangle \langle g| + c_{g,0}(t)c^*_{e,0}(t) |g\rangle \langle e| \qquad (6) \\
&+ \mathrm{Tr}_F \left(\sum_{n=0}^{\infty} |c_{g,n+1}(t)|^2 |g\rangle \langle g| \otimes |n+1\rangle \langle n+1| + |c_{e,n}(t)|^2 |e\rangle \langle e| \otimes |n\rangle \langle n|\right) \qquad (7) \\
&= |c_{g,0}(t)|^2 |g\rangle \langle g| + c_{e,0}(t)c^*_{g,0}(t) |e\rangle \langle g| + c_{g,0}(t)c^*_{e,0}(t) |g\rangle \langle e| \qquad (8) \\
&+ \sum_{n=0}^{\infty} |c_{g,n+1}(t)|^2 |g\rangle \langle g| + |c_{e,n}(t)|^2 |e\rangle \langle e| \qquad (9)
\end{aligned}
$$

This calculation hinges on the fact that we can only have diagonal elements of the field be non zero when traced out. We can see that this density matrix is not pure as we have off diagonal terms.

1.3

$$\mathbf{D} = \mathbf{d} |e\rangle \langle g| + \mathbf{d}^* |g\rangle \langle e| \qquad (10)$$

We know that the electric dipole operator only acts on the Hilbert space of the atom. Thus it will factor out of the partial trace over the field Hilbert space:

$$\langle\psi(t)| \mathbf{D} |\psi(t)\rangle = \mathrm{Tr}_A \mathrm{Tr}_F(W(t)\mathbf{D}) = \mathrm{Tr}_A(Tr_F(W(t))\mathbf{D}) = \mathrm{Tr}_A(\rho^A(t)\mathbf{D}) \qquad (11)$$

We can now insert the expression for $\rho^A(t)$. Once again we can take advantage of the partial trace properties to simplfy the expression. In this case, only the off diagonal elements will give nonzero results so we only need to take those components of $\rho^A(t)$ to compute the expectation value of the dipole operator:

$$\mathrm{Tr}_A(\rho^A(t)\mathbf{D}) = \mathrm{Tr}_A(\mathbf{D}\rho^A(t)) \tag{12}$$
$$= \mathrm{Tr}_A((\mathbf{d}\,|e\rangle\langle g| + \mathbf{d}^*\,|g\rangle\langle e|)(c_{e,0}(t)c^*_{g,0}(t)\,|e\rangle\langle g| + c_{g,0}(t)c^*_{e,0}(t)\,|g\rangle\langle e|)) \tag{13}$$
$$= Tr_A(\mathbf{d}c_{g,0}(t)c^*_{e,0}(t)\,|e\rangle\langle e| + \mathbf{d}^*c_{e,0}(t)c^*_{g,0}(t)\,|g\rangle\langle g|) \tag{14}$$
$$= \mathbf{d}c_{g,0}(t)c^*_{e,0}(t) + \mathbf{d}^*c_{e,0}(t)c^*_{g,0}(t) \tag{15}$$

2 Problem 2

2.1

$$S_+ \equiv |e\rangle\langle g| \qquad S_- \equiv |g\rangle\langle e| \qquad S_z \equiv \frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|) \tag{16}$$

We can compute the commutators of the operators:

$$[S_+, S_-] = |e\rangle\langle e| - |g\rangle\langle g| = 2S_z \tag{17}$$

It is also easy to see that $S_z^2 = \frac{I}{4}$, S_+ and S_- can be combined to create S_x and S_y operators which will also satisfy the relevant commutation relations.

2.2

The probability of the excited state can be derived from the trace properties. I'll assume in this case that $\rho^A(t) = \rho(t)$

$$\mathrm{Tr}\big(|e\rangle\langle e|\,\rho^A(t)\big) = \mathrm{Tr}_A(|e\rangle\langle e|\,\rho^A(t)) = \sum_{i=e,g}\langle i|e\rangle\langle e|\,\rho(t)\,|i\rangle \tag{18}$$
$$= \langle e|\,\rho(t)\,|e\rangle$$

By orthonormality, the $|e\rangle$ is the only non zero inner product, leaving the probability of the excited state.
We can now derive the equations for the time derivative of the excited state probabilities of the atom using the master equation. We can make some simplifications to start. We know that $S_+S_- = |e\rangle\langle e|$ and substitute this in:

$$\dot{\rho}(t) = \frac{-i}{\hbar}[H_A, \rho(t)] - \frac{\Gamma}{2}(|e\rangle\langle e|\,\rho(t) + \rho(t)\,|e\rangle\langle e| - 2\,|g\rangle\langle e|\,\rho(t)\,|e\rangle\langle g|) \tag{19}$$

Where $H_A = \hbar(\omega_e\,|e\rangle\langle e| + \omega_g\,|g\rangle\langle g|)$. We can now simplify. Note that the off diagonal terms will be zeroed out in the inner product:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle e|\,\rho(t)\,|e\rangle = \frac{-i}{\hbar}\langle e|\,[H_A, \rho(t)]\,|e\rangle - \frac{\Gamma}{2}(2\,\langle e|\,\rho(t)\,|e\rangle) \tag{20}$$
$$[H_A, \rho(t)] = \hbar(\omega_e\,|e\rangle\langle e| + \omega_g\,|g\rangle\langle g|)\rho(t) - \hbar\rho(t)(\omega_e\,|e\rangle\langle e| + \omega_g\,|g\rangle\langle g|) \tag{21}$$
$$\to \langle e|\,[H_A, \rho(t)]\,|e\rangle = \hbar\omega_e\,\langle e|\,\rho(t)\,|e\rangle - \hbar\omega_e\omega_e\,\langle e|\,\rho(t)\,|e\rangle = 0 \tag{22}$$
$$\to \frac{\mathrm{d}}{\mathrm{d}t}\langle e|\,\rho(t)\,|e\rangle = -\Gamma\,\langle e|\,\rho(t)\,|e\rangle \tag{23}$$

We can now derive the spontaneous emission rate of the atom, again taking advantage of the inner product properties of the orthonormal states:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle g| \rho(t) |e\rangle = \frac{-i}{\hbar} \langle g| [H_A, \rho(t)] |e\rangle - \frac{\Gamma}{2}(\langle g| \rho(t) |e\rangle) \tag{24}$$

$$[H_A, \rho(t)] = \hbar(\omega_e |e\rangle \langle e| + \omega_g |g\rangle \langle g|)\rho(t) - \hbar\rho(t)(\omega_e |e\rangle \langle e| + \omega_g |g\rangle \langle g|) \tag{25}$$

$$\rightarrow \langle g| [H_A, \rho(t)] |e\rangle = \hbar\omega_g \langle g| \rho(t) |e\rangle - \hbar\omega_e \langle g| \rho(t) |e\rangle \tag{26}$$

$$\rightarrow \frac{\mathrm{d}}{\mathrm{d}t} \langle g| \rho(t) |e\rangle = -\frac{\Gamma}{2}(\langle g| \rho(t) |e\rangle) + i(\omega_e - \omega_g)(\langle g| \rho(t) |e\rangle) \tag{27}$$

I'll go ahead and solve this differential equation as it will be useful in 2.3. It's solution is a simple exponential function:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle g| \rho(t) |e\rangle = -(\frac{\Gamma}{2} - i(\omega_e - \omega_g))(\langle g| \rho(t) |e\rangle) \tag{28}$$

$$\rightarrow \langle g| \rho(t) |e\rangle = \langle g| \rho(0) |e\rangle e^{-(\frac{\Gamma}{2} - i(\omega_e - \omega_g))t} \tag{29}$$

$$\langle e| \rho(t) |g\rangle = (\langle g| \rho(t) |e\rangle)^\dagger = \langle e| \rho(0) |g\rangle e^{-(\frac{\Gamma}{2} + i(\omega_e - \omega_g))t} \tag{30}$$

2.3

We can solve for the expectation value of the dipole operator. We can first take advantage of the linear properties of the trace:

$$\langle \mathbf{D}(t)\rangle = \mathrm{Tr}_A((\mathbf{d} |e\rangle \langle g| + \mathbf{d}^* |g\rangle \langle e|)\rho(t)) \tag{31}$$

$$= \mathbf{d}\, \mathrm{Tr}_A(|e\rangle \langle g| \rho(t)) + \mathbf{d}^*\, \mathrm{Tr}_A(|g\rangle \langle e| \rho(t)) \tag{32}$$

$$= \mathbf{d} \langle g| \rho(t) |e\rangle + \mathbf{d}^* \langle e| \rho(t) |g\rangle \tag{33}$$

We can see that the two operator traces just correspond to the expectation value of the $S_\pm$ operators which we can calculate using the expressions derived in 2.2 We can then subsitute the expressions derived at the end of 2.2:

$$\langle \mathbf{D}(t)\rangle = \mathbf{d} \langle g| \rho(t) |e\rangle + \mathbf{d}^* \langle e| \rho(t) |g\rangle \tag{34}$$

$$= \mathbf{d} \langle g| \rho(0) |e\rangle e^{-(\frac{\Gamma}{2} - i(\omega_e - \omega_g))t} + \mathbf{d}^* \langle e| \rho(0) |g\rangle e^{-(\frac{\Gamma}{2} + i(\omega_e - \omega_g))t} \tag{35}$$

3 Problem 3

We have the following state vector for a two level atom and a field with one photon:

$$|\Psi(t)\rangle = \sum_s (c_{g,s}(t) |g\rangle |1_s\rangle) + c_{e,vac}(t) |e\rangle |vac\rangle \tag{36}$$

We can calculate the density matrix. I'll use the same tricks from problem 1:

$$|\Psi(t)\rangle \langle\Psi(t)| = |c_{e,vac}(t)|^2 |e\rangle |vac\rangle \langle e| \langle vac| + \sum_s \left(|c_{g,s}(t)|^2 |g\rangle |1_s\rangle \langle g| \langle 1_s|\right) \tag{37}$$

$$+ \sum_s (c_{g,s}(t) c^*_{e,vac}(t) |g\rangle |1_s\rangle \langle e| \langle vac| + c^*_{g,s}(t) c_{e,vac}(t) |e\rangle |vac\rangle \langle g| \langle 1_s|) \tag{38}$$

We can now trace over the field:

$$\mathrm{Tr}_F(|\Psi(t)\rangle\langle\Psi(t)|) = |c_{e,vac}(t)|^2 |e\rangle\langle e| + \sum_s |c_{g,s}(t)|^2 |g\rangle\langle g| \tag{39}$$

Note that the off diagonal mixed terms trace out to zero because the terms only contain field states which are orthonormal to one another. Thus the expectation value of the electric dipole moment will be 0 because there are no off diagonal terms.