PHYS 4261 HW1

DLG

January 21, 2023

Problem 1

1a

I went through the table and made sure I understood how to extrapolate the state configurations however I'd rather not draw out the diagram here.

1b

We can start with the use of the raising operator on the state $|1,-1\rangle$:

$$F_+ |1,-1\rangle = \hbar\sqrt{1(1+1) - -1(-1+1)}\,|1,0\rangle = \sqrt{2}\hbar\,|1,0\rangle$$

$$F_+ |1,-1\rangle = (S_+ + I_+)\,|1,-1\rangle = (S_+ + I_+)\left(\frac{\left|-\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle}{2} - \frac{\sqrt{3}\left|-\frac{3}{2}\right\rangle\left|\frac{1}{2}\right\rangle}{2}\right)$$

$$= \frac{\hbar}{2}\left(2\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) - \frac{3\hbar}{2}\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle$$

$$= \hbar\left(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle - \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) \rightarrow |1,0\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle - \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right)$$

Another way to compute the state is to use the orthonormality criteria of states. We know that the state $|2,0\rangle$ has the form as given further in the HW. The state $|1,0\rangle$ should be orthonormal to it. Changing the sign of the $(\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle)$ on the state $|2,0\rangle$ satisfies this orthonormality constraint. If we apply the F^2 operator on the state, we get that the value is $2\hbar^2$, corresponding to $F = 1$

1c

finding the values of the uncoupled kets simply involve doing a little algebra along with combining the coupled kets:

$$\left|\frac{3}{2}\right\rangle\left|\frac{1}{2}\right\rangle = |2,2\rangle$$
$$\left|\frac{3}{2}\right\rangle\left|-\frac{1}{2}\right\rangle = \frac{(|2,1\rangle + \sqrt{3}\,|1,1\rangle)}{2}$$
$$\left|\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle = \frac{(\sqrt{3}\,|2,1\rangle - |1,1\rangle)}{2}$$
$$\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle = \frac{(|2,0\rangle + |1,0\rangle)}{\sqrt{2}}$$
$$\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle = \frac{(|2,0\rangle - |1,0\rangle)}{\sqrt{2}}$$
$$\left|-\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle = \frac{(|2,-1\rangle + \sqrt{3}\,|1,-1\rangle)}{2}$$
$$\left|-\frac{3}{2}\right\rangle\left|\frac{1}{2}\right\rangle = \frac{(|2,-1\rangle - \sqrt{3}\,|1,-1\rangle)}{2}$$
$$\left|-\frac{3}{2}\right\rangle\left|-\frac{1}{2}\right\rangle = |2,-2\rangle$$

Problem 2

It would be helpful to compute the action of $\mathbf{S}\cdot\mathbf{I}$ on a spin state so I'll start by doing that (this later proved to be redundant but I'm going to leave this in anyway):

$$\mathbf{S}\cdot\mathbf{I} = S_xI_x + S_yI_y + S_zI_z$$
$$= \frac{1}{4}(S_+ + S_-)(I_+ + I_-) + \frac{-1}{4}(S_+ - S_-)(I_+ - I_-) + S_zI_z$$
$$= \frac{1}{4}(S_+I_+ + S_+I_- + S_-I_+ + S_-I_- - (S_+I_+ - S_+I_- - S_-I_+ + S_-I_-)) + S_zI_z$$
$$= \frac{1}{4}(2S_+I_- + 2S_-I_+) + S_zI_z = \frac{1}{2}(S_+I_- + S_-I_+) + S_zI_z$$

$$H_{\text{hyp}} = \frac{\Delta}{\hbar}\mathbf{S}\cdot\mathbf{I} \qquad H_z = \Omega S_z \tag{1}$$

2a

Since $l = 0$ and $I = \frac{3}{2}$, we should have 8 states, $4\otimes 2$ 4 for each of the nuclear spin states and 2 for the two spin states of the electron. Since $\mathbf{S}\cdot\mathbf{I}$ can be defined in terms of the S^2, I^2, F^2 operators which all share the same eigenvectors as they commute with one another, we know that the eigenvectors given in 1a will also be eigenvectors of $\mathbf{S} \cdot \mathbf{I}$ which can be defined as $\frac{F^2 - S^2 - I^2}{2}$ we also know that these operators have the following behavior when acting on $|F, M_F\rangle$:

$$F^2 |I, S, F, M_F\rangle = F(F+1)\hbar^2 |I, S, F, M_F\rangle$$
$$S^2 |I, S, F, M_F\rangle = S(S+1)\hbar^2 |I, S, F, M_F\rangle$$
$$I^2 |I, S, F, M_F\rangle = I(I+1)\hbar^2 |I, S, F, M_F\rangle$$

Since the values of I and S are fixed at $\frac{3}{2}$ and $\frac{1}{2}$ respectively, the action of S^2 and I^2 will be the same for all the eigenvectors, giving values of $\frac{3}{4}$ and $\frac{15}{4}$ respectively. Thus, we can now find the various energies of the states. Since there are only two possible F values (2,1), there will be two eigenenergies:

$$H_{\text{hyp}} |3/2, 1/2, F, M_F\rangle = \frac{\Delta\hbar}{2}(F(F+1) - \frac{3}{4} - \frac{15}{4}) |3/2, 1/2, F, M_F\rangle \quad (2)$$

$$\rightarrow E = \boxed{\frac{3\hbar\Delta}{4}, -\frac{5\hbar\Delta}{4}} \quad (3)$$

The diagram for the energies can be seen in Figure 1.

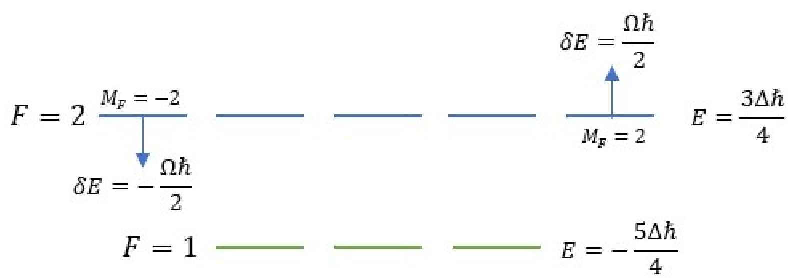

Figure 1: Energy Level Diagram for a joint spin 3/2-1/2 system. The energy level shifts for the $M_F = \pm 2$ are also shown

2b

(For this problem I'll write the nuclear spin states on the right) To find the energy shift we simply apply H_z to the two energy eigenvectors with $M_F = \pm 2$:

$$H_z |2, 2\rangle = \Omega S_z \left|\frac{1}{2}\right\rangle \left|\frac{3}{2}\right\rangle = \frac{\Omega\hbar}{2} \left|\frac{1}{2}\right\rangle \left|\frac{3}{2}\right\rangle$$
$$H_z |2, -2\rangle = \Omega S_z \left|-\frac{1}{2}\right\rangle \left|-\frac{3}{2}\right\rangle = -\frac{\Omega\hbar}{2} \left|-\frac{1}{2}\right\rangle \left|-\frac{3}{2}\right\rangle$$

Thus, the energy shift is linearly proportional to the applied magnetic field.

2c

We can start with the uncoupled rep of the two clock states

$$
\begin{aligned}
|2,0\rangle &= \frac{1}{\sqrt{2}}(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle + \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle) \\
|1,0\rangle &= \frac{1}{\sqrt{2}}(\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle - \left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle)
\end{aligned}
$$

To find the matrix of the net hamiltonian in the $M_F = 0$ subspace, we simply compute it's matrix elements on the vectors in the subspace. I'll start by first computing the values of the hamiltonian operating on each vector in the subspace (I'm also going to flip the order of the operators in the product to $\mathbf{I} \cdot \mathbf{S}$ as that is the order of the spin state product given in 1a and it doesn't change the calculation):

$$
\begin{aligned}
H|2,0\rangle &= \frac{\Delta}{\hbar}\mathbf{I}\cdot\mathbf{S}|2,0\rangle + \frac{\Omega\hbar}{2\sqrt{2}}\left(-\left|\frac{1}{2}\right\rangle\left|\frac{-1}{2}\right\rangle + \left|\frac{-1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) \\
\rightarrow \frac{\Delta}{\hbar}\mathbf{I}\cdot\mathbf{S}|2,0\rangle &= \frac{\Delta\hbar}{2}(2(2+1) - \frac{9}{2})|2,0\rangle = \frac{3\hbar\Delta}{4}|2,0\rangle \\
&\rightarrow \boxed{H|2,0\rangle = \frac{3\hbar\Delta}{4}|2,0\rangle + \frac{-\Omega\hbar}{2}|1,0\rangle} \\
H|1,0\rangle &= \frac{\Delta}{\hbar}\mathbf{I}\cdot\mathbf{S}|1,0\rangle + \frac{-\Omega\hbar}{2\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle\left|\frac{-1}{2}\right\rangle + \left|\frac{-1}{2}\right\rangle\left|\frac{1}{2}\right\rangle\right) \\
\frac{\Delta}{\hbar}\mathbf{I}\cdot\mathbf{S}|1,0\rangle &= \frac{\Delta\hbar}{2}(1(1+1) - \frac{9}{2})|1,0\rangle = \frac{-5\hbar\Delta}{4}|1,0\rangle \\
&\rightarrow \boxed{H|1,0\rangle = -\frac{5\hbar\Delta}{4}|1,0\rangle - \frac{\Omega\hbar}{2}|2,0\rangle}
\end{aligned}
$$

Since the action of the hamiltonian on the 2 states gives an answer in terms of the coupled rep states, it is easy to see what the matrix elements will be:

$$
\begin{aligned}
\langle 2,0| H |2,0\rangle &= \frac{3\hbar\Delta}{4} \\
\langle 2,0| H |1,0\rangle &= -\frac{\Omega\hbar}{2} \\
\langle 1,0| H |1,0\rangle &= -\frac{5\hbar\Delta}{4} \\
\langle 1,0| H |2,0\rangle &= -\frac{\Omega\hbar}{2}
\end{aligned}
$$

We can find the eigenvalues of H in the standard way by finding the determinant of $H - \lambda I$, I'll give the polynomial and then the resulting eigenvalues to avoid including excess computation:

$$P(\lambda) = \left(\frac{3\Delta\hbar}{4} - \lambda\right)\left(-\frac{5\Delta\hbar}{4} - \lambda\right) - \frac{\Omega^2\hbar^2}{4} = 0 \to \lambda = -\frac{\Delta\hbar}{4} \pm \hbar\sqrt{\Delta^2 + \frac{\Omega^2}{4}}$$

$$\to \boxed{\lambda = -\frac{\Delta\hbar}{4} \pm \hbar\Delta\sqrt{1 + \frac{\Omega^2}{4\Delta^2}}}$$

Using the binomial expansion, we get that the energy eigenvalues are:

$$\lambda \approx -\frac{\Delta\hbar}{4} \pm \hbar\Delta(1 + \frac{1}{2}\frac{\Omega^2}{4\Delta^2}) \tag{4}$$

Thus, we can see the quadratic dependence on Ω and therefore B ($\Omega = \frac{2\mu_b B}{\hbar}$). This means that there is no linear dependence for the eigenenergies on B

Problem 3

Intial state $|\Psi\rangle$:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{3}{2}, -\frac{3}{2}\right\rangle \otimes \left|\frac{1}{2}, \frac{1}{2}\right\rangle - \left|\frac{3}{2}, \frac{1}{2}\right\rangle \otimes \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right) \tag{5}$$

3a

It will be helpful to redefine $|\Psi\rangle$ in terms of the coupled states of the form $|F, M_F\rangle$. I'm also going to drop the first quantum number in the kets for convenience (assume the nuclear angular quantum number is the leftmost one):

$$\left|-\frac{3}{2}\right\rangle\left|\frac{1}{2}\right\rangle = \frac{(|2,-1\rangle - \sqrt{3}\,|1,-1\rangle)}{2} \qquad \left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle = \frac{(|2,0\rangle + |1,0\rangle)}{\sqrt{2}}$$

$$\to |\Psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{(|2,-1\rangle - \sqrt{3}\,|1,-1\rangle)}{2} - \frac{1}{\sqrt{2}}(|2,0\rangle + |1,0\rangle)\right)$$

We can now calculate estimate the probabilities using the F^2 operator. (Note, it has the action of $F^2\,|F, M_F\rangle = F(F+1)\hbar^2\,|F, M_F\rangle$) Using the coupled representation of $|\Psi\rangle$, we can see that the states that would yield a $2\hbar^2, (F = 1)$ have a total probability of $\left|-\frac{1}{2}\right|^2 + \left|-\frac{\sqrt{3}}{2\sqrt{2}}\right|^2 = \frac{5}{8}$ while the states that correspond to a measurement being $6\hbar^2, (F = 2)$ being $\left|-\frac{1}{2}\right|^2 + \left|\frac{1}{2\sqrt{2}}\right|^2 = \frac{3}{8}$.

3b

A measurement of F_z resulting in $0\hbar$ corresponds to the quantum number $M_F = 0$. Thus, the states $|2,0\rangle, |1,0\rangle$ are the two possible states that would give this result. In the case of $|\Psi\rangle$ each of those states have probabilities of 1/4 . To determine the probabilities for respective spin states of the spin 3/2 and 1/2 systems given that the measurement of F_z has already occurred, we need to convert back to the uncoupled representation to see the respective probabilities for the each of the joint spin states. We can simply look at the coefficients for the two clock states to determine the probabilities for each of the spin states. Each of these states have a probability of 1/2. Thus, we have probabilities of 1/8 for the states. Since the clock states share the same states, we can add up the probabilities to get that the state $\left|\frac{1}{2}\right\rangle\left|-\frac{1}{2}\right\rangle$ has a probability of 1/4 and $\left|-\frac{1}{2}\right\rangle\left|\frac{1}{2}\right\rangle$ also has a probability of 1/4 meaning that the probability that $M_F = 0$ gets measured is 1/2