Week 3

Note Title 4/10/2021

Standard form LP $\longrightarrow$ canonical form LP

$$\min_{x \in \mathbb{R}^n} \quad z = p^T x$$
$$\text{s.t.} \quad Ax \geq b$$
$$x \geq 0$$

$$\longrightarrow$$

$$\min_{\substack{x_N \in \mathbb{R}^n \\ x_B \in \mathbb{R}^m}} \quad z = p^T x_N + 0^T x_B$$
$$\text{s.t.} \quad x_B = A x_N - b \,, \quad x_B, x_N \geq 0$$

$$N = \{1, \cdots, n\}$$
$$B = \{n+1, \cdots, n+m\}$$

tableau representation :

$$\begin{array}{ccc|ccc|c}
 & & & \overbrace{x_1 \quad \cdots \quad x_n}^{\text{original variables}} & & & 1 \\
\hline
\text{slack variables} \left\{ \begin{array}{c} x_{n+1} \\ \vdots \\ x_{n+m} \end{array} \right. & \begin{array}{c} = \\ \\ = \end{array} & & \begin{array}{ccc} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{array} & & & \begin{array}{c} -b_1 \\ \vdots \\ -b_m \end{array} \\
\hline
z & = & & p_1 \quad \cdots \quad p_n & & & 0
\end{array}$$

or

$$\begin{array}{r|c|c|}
 & x_N & 1 \\
\hline
x_B = & A & -b \\
\hline
z = & p^T & 0 \\
\hline
\end{array}$$

An easy, but important, observation:

(1) The $n+m$ linear inequality constraints are

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_n &\geq 0 \\ x_{n+1} &\geq 0 \\ &\vdots \\ x_{n+m} &\geq 0 \end{aligned}$$

(written in this way, we almost do not distinguish the origin variables with the slack variables)

If $x_1 = x_2 = \cdots = x_n = 0$, then (i) the first n constraints are obviously satisfied, (ii) $z = 0$, and (iii)

$$\begin{aligned} x_{n+1} &= -b_1 \\ x_{n+2} &= -b_2 \\ &\vdots \\ x_{n+m} &= -b_m. \end{aligned}$$

$$x_{n+k} \geq 0 \text{ is satisfied} \iff -b_k \geq 0 \iff b_k \leq 0$$

In fact, in this case $x_1 = \cdots = x_n = 0$ is a <u>vertex</u> of the feasible region.

To conclude:

$b \leq 0$ (ie. all entries of b are non-positive) $\iff$ the origin $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ satisfies all $m+n$ constraints (and is a vertex of the feasible region)

E.g. $n=2$, $m=1$

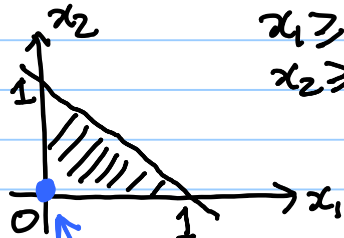
$A = [-1 \ \ -1]$

$b = [-1] \leq 0$

$-x_1 - x_2 \geq -1 \iff x_1 + x_2 \leq 1$

$x_1 \geq 0$

$x_2 \geq 0$



belongs to the feasible region

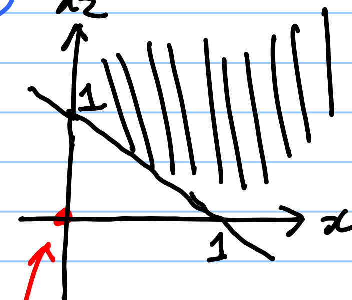
$A = [1, 1]$

$b = [1] \not\leq 0$

$x_1 + x_2 \geq 1$

$x_1 \geq 0$

$x_2 \geq 0$



does not belong to the feasible region

Now, imagine that we perform a series of Jordan exchanges :

$$\begin{array}{r|c|c|} & x_N & 1 \\ \hline x_B = & A & -b \\ \hline z = & p^T & 0 \\ \hline \end{array} \longrightarrow \cdots\cdots \longrightarrow \begin{array}{r|c|c|} & x_{N'} & 1 \\ \hline x_{B'} = & H & h \\ \hline z = & c^T & \alpha \\ \hline \end{array}$$

$N = \{1, \cdots, n\}$ $\qquad$ $N' \subset \{1, \cdots, m+n\}$, $|N'| = n$

$B = \{n+1, \cdots, n+m\}$ $\qquad$ $B' = \{1, \cdots, m+n\} \setminus N'$, so $|B'| = m$.

This means : • we now express $x_{B'}$ as a $Hx_{N'} + h$

• z, originally defined as $p_1x_1 + \cdots + p_nx_n$, is now re-expressed as

$$z = \sum_{i \in N'} c_i x_i + \alpha$$

Recall from Week 2 what a vertex is. → • If $h \geq 0$, then $x_{N'} = 0$, which implies $x_{B'} = h$, corresponds to a vertex of the feasible region.

• When $x_{N'} = 0$, $z = \alpha$.

Terminologies

- If $h \geq 0$, we call the tableau

$$\begin{array}{r|c|c|} & x_{N'} & 1 \\ \hline x_{B'} = & H & h \\ \hline z = & c^T & \alpha \\ \hline \end{array}$$

feasible.

[Note : Even if some entries of h are negative,

$$x_{B'} = H x_{N'} + h$$ still correctly represents the linear relationships between $x_{B'}$ and $x_{N'}$. It is just that

$$x_{N'} = 0$$ does not correspond to a vertex.

(see example in class.)]

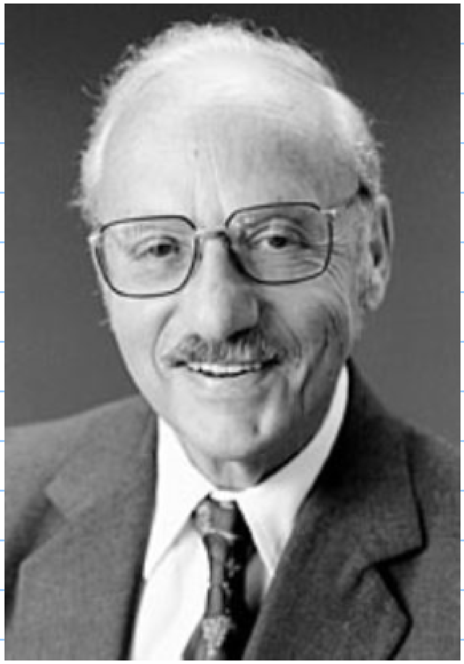
- The variables on top are referred to as "non-basic" (hence the 'N') those on the left are referred to as "basic" (hence the 'B').

Basic idea of the simplex method for solving LP :

At each iteration of the simplex method, we exchange one element between B and N, performing the corresponding Jordan exchange on the tableau representation, much as we did in Chapter 2 in solving systems of linear equations. We ensure that the tableau remains feasible at every iteration, and we try to choose the exchanged elements so that the objective function z decreases at every iteration. We continue in this fashion until either

1. a solution is found, or
2. we discover that the objective function is unbounded below on the feasible region, or
3. we determine that the feasible region is empty.

Geometrically, a cleverly chosen Jordan exchange corresponds to moving from one vertex to a neighboring vertex with a smaller objective value z.



Geroge B. Dantzig (1914-2005)

A simple example

Example 3-1-1.

$$
\begin{array}{lrcrcr}
\min\limits_{x_1,x_2} & 3x_1 - 6x_2 & & & & \\
\text{subject to} & x_1 & + & 2x_2 & \geq & -1, \\
 & 2x_1 & + & x_2 & \geq & 0, \\
 & x_1 & - & x_2 & \geq & -1, \\
 & x_1 & - & 4x_2 & \geq & -13, \\
 & -4x_1 & + & x_2 & \geq & -23, \\
 & & & x_1, x_2 & \geq & 0.
\end{array}
$$

all $b_i \leq 0$, so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a vertex

The slack variables are defined as:

$$
\begin{array}{rcrcrcr}
x_3 & = & x_1 & + & 2x_2 & + & 1, \\
x_4 & = & 2x_1 & + & x_2, & & \\
x_5 & = & x_1 & - & x_2 & + & 1, \\
x_6 & = & x_1 & - & 4x_2 & + & 13, \\
x_7 & = & -4x_1 & + & x_2 & + & 23.
\end{array}
$$

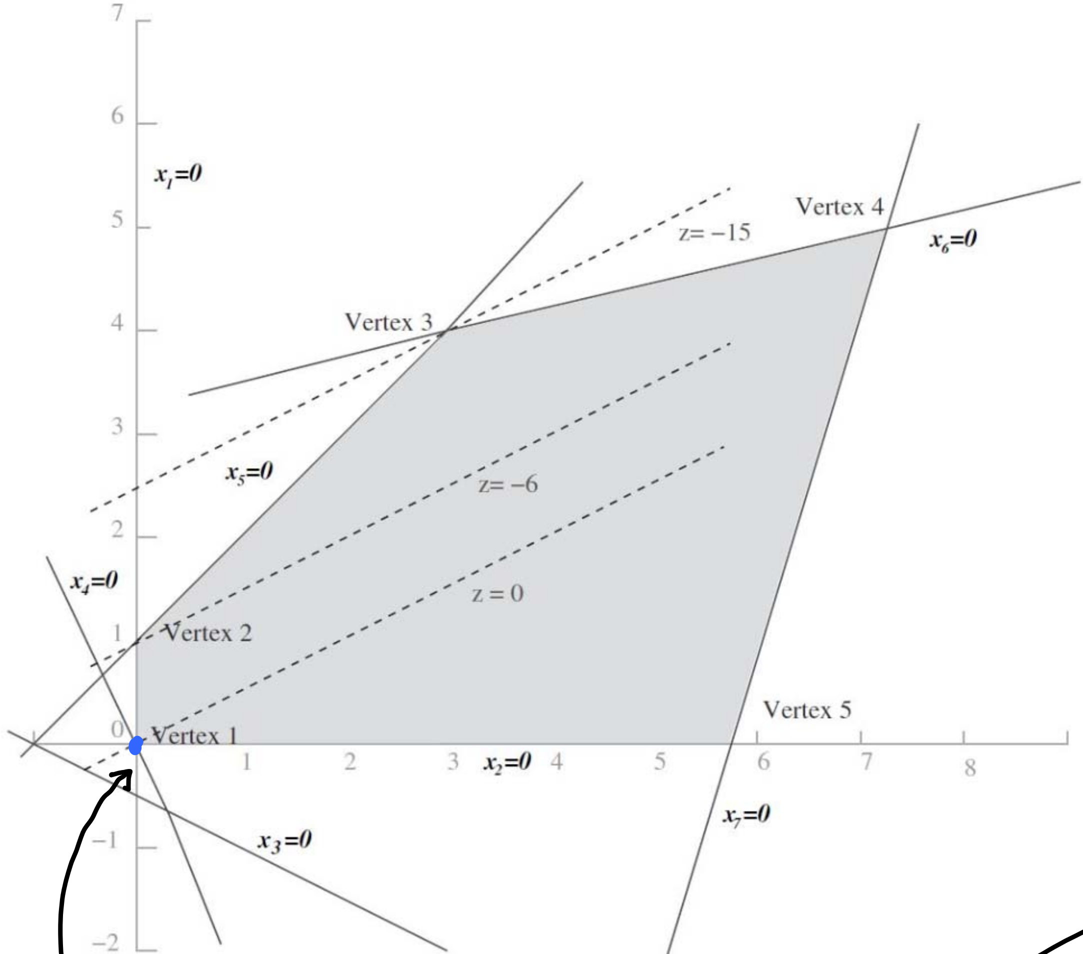


Tableau representation:

$$
\begin{array}{ccc|c}
 & & x_1 & x_2 & 1 \\
x_3 & = & 1 & 2 & 1 \\
x_4 & = & 2 & 1 & 0 \\
x_5 & = & 1 & -1 & 1 \\
x_6 & = & 1 & -4 & 13 \\
x_7 & = & -4 & 1 & 23 \\
\hline
z & = & 3 & -6 & 0
\end{array}
$$

And it is a feasible tableau.

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 13 \\ 23 \end{bmatrix} \geqslant 0
$$

and $z = 0$

We now seek a clever pivot — a Jordan exchange of a basic variable with a non-basic one — that

① yields a decrease in the objective z, and

② keeps the tableau feasible.

① is quite easy: choose any non-basic variable in a column with a negative value in the last row.

$$z = 3x_1 - 6x_2$$

if it becomes basic, x_1 increases from 0 to some positive value, this would increase z

if it becomes basic, x_2 increases from 0 to some positive value, this would decrease z

② is slightly trickier, but we just need to ask the following question:

Since we want to <u>minimize z</u>, we would like to increase x_2 (the "entering variable") as much as possible.

Q: How much can we increase x_2 without violating the constraints?

A: If x_2 increases from 0 to $\lambda > 0$ (and x_1 stays zero) then

$$\begin{aligned} x_3 &= 2\lambda + 1 \\ x_4 &= \lambda \\ x_5 &= -\lambda + 1 \\ x_6 &= -4\lambda + 13 \\ x_7 &= \lambda + 23. \end{aligned} \quad \text{And} \quad \begin{aligned} x_3 &= 2\lambda + 1 \geq 0 &\Rightarrow \lambda \geq -1/2 \\ x_4 &= \lambda \geq 0 &\Rightarrow \lambda \geq 0 \\ x_5 &= -\lambda + 1 \geq 0 &\Rightarrow \lambda \leq 1 \\ x_6 &= -4\lambda + 13 \geq 0 &\Rightarrow \lambda \leq 13/4 \\ x_7 &= \lambda + 23 \geq 0 &\Rightarrow \lambda \geq -23 \end{aligned}$$

i.e. The largest value we can increase x_2 to is $\lambda = 1$.

And this corresponds to $x_5 = 0$

the "blocking variable"

		x_1	x_2	1
x_3	=	1	2	1
x_4	=	2	1	0
x_5	=	1	−1	1
x_6	=	1	−4	13
x_7	=	−4	1	23
z	=	3	−6	0

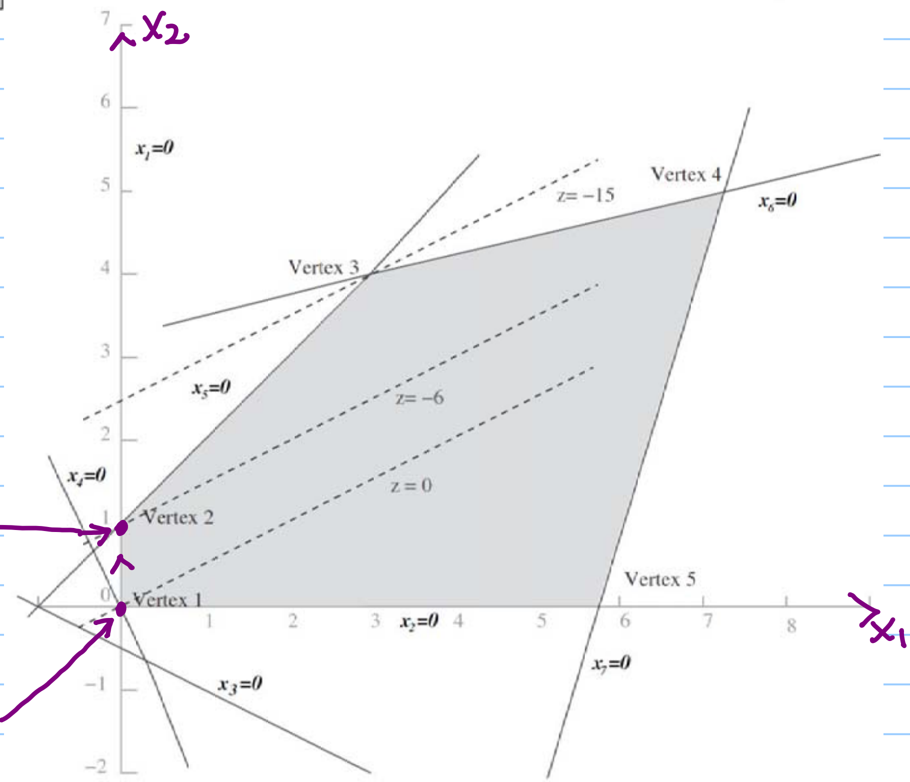
```
>> T = ljx(T,3,2);
```

		x_1	x_5	1
x_3	=	3	−2	3
x_4	=	3	−1	1
x_2	=	1	−1	1
x_6	=	−3	4	9
x_7	=	−3	−1	24
z	=	−3	6	−6

$x_1=0$
$x_2=1$

$\Uparrow$

$x_1=0$
$x_5=0$



$x_1=0$
$x_2=0$

This is the first step of the simplex method. By moving from $\begin{bmatrix}0\\0\end{bmatrix}$ to $\begin{bmatrix}0\\1\end{bmatrix}$, z decreases from 0 to -6.

Given a feasible tableau

$$\begin{array}{r|c|c|} & x_N & 1 \\ \hline x_B = & H & h \\ \hline z = & c^T & \alpha \\ \hline \end{array},$$

a step of the simplex method is a Jordan exchange between a basic and non-basic variable according to the following pivot selection rules:

1. *Pricing* (selection of pivot column s): The pivot column is a column s with a negative element in the bottom row. These elements are called *reduced costs*.

2. *Ratio Test* (selection of pivot row r): The pivot row is a row r such that

$$-h_r / H_{rs} = \min_i \{-h_i / H_{is} \mid H_{is} < 0\}.$$

Back to the example, here is the next step:

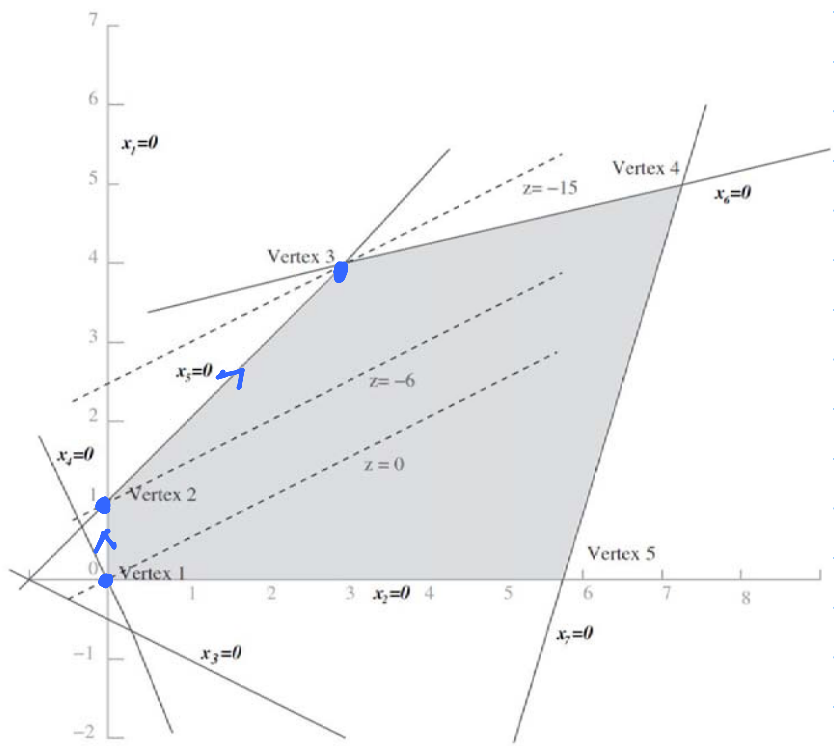
		x_1	x_5	1
x_3	=	3	−2	3
x_4	=	3	−1	1
x_2	=	1	−1	1
x_6	=	−3	4	9
x_7	=	−3	−1	24
z	=	−3	6	−6

$9/3$ ←

$24/3$

```
≫ T = ljx(T,4,1);
```

		x_6	x_5	1
x_3	=	−1	2	12
x_4	=	−1	3	10
x_2	=	−0.33	0.33	4
x_1	=	−0.33	1.33	3
x_7	=	1	−5	15
z	=	1	2	−15



By moving from $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$, z decreases from −6 to −15.

Note that there are no more negative entries in the last row.

$$z = x_6 + 2x_5 - 15.$$

Since every point in the feasible region satisfies

$$x_5 \geq 0, \quad x_6 \geq 0$$

The min value of is -15, and is attained when

$$x_5 = x_6 = 0 \quad (\Leftrightarrow x_1 = 3, x_2 = 4).$$

We say that the last tableau

		x_6	x_5	1
x_3	=	−1	2	12
x_4	=	−1	3	10
x_2	=	−0.33	0.33	4
x_1	=	−0.33	1.33	3
x_7	=	1	−5	15
z	=	1	2	−15

is optimal.

The simplex method solves the LP after 2 simplex steps.

Algorithm 3.1 (Simplex Method).

1. *Construct an initial tableau. If the problem is in standard form* (3.1), *this process amounts to simply adding slack variables.*

2. *If the tableau is not feasible, apply a Phase I procedure to generate a feasible tableau, if one exists (see Section* 3.4*). For now we shall assume the origin* $\underbrace{x_N = 0 \text{ is feasible}}_{\Leftrightarrow\ b \leq 0}$.

3. *Use the pricing rule to determine the pivot column* s. *If none exists,* **stop**; (a)*: tableau is optimal.*

4. *Use the ratio test to determine the pivot row* r. *If none exists,* **stop**; (b)*: tableau is unbounded.*

5. *Exchange* $x_{B(r)}$ *and* $x_{N(s)}$ *using a Jordan exchange on* H_{rs}.

6. *Go to Step* 3.

Interestingly, and perhaps confusing for you, the inner workings of Phase I depend on the Phase II procedure!

Stop (a) indicates optimality, as

$$z = \underset{\substack{\vee\!/ \\ 0}}{c^T}\, x_N + \alpha$$

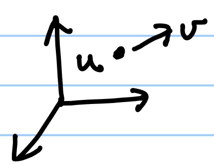
any point in the feasible region must satisfy $x_N \geq 0$

so $z \geq \alpha$, and $\min z = \alpha$, with $\underbrace{x_N = 0}$ as a minimizer.

the corresponding values of $x_1, \cdots, x_n$ (original variables) can be easily read off from the tableau.

Stop (b) means $\min z = -\infty$, and we can read off from the tableau a "ray"

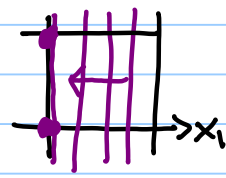
$$\underset{\substack{\| \\ u + \lambda v}}{x(\lambda)} \quad \text{so that} \quad \lim_{\lambda \to +\infty} z(x(\lambda)) = -\infty.$$



See Example 3-3-1.

Non-uniqueness of minimizer

Linear programs may have more than one solution.

E.g.  $\min x_1$ s.t. $0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 1$

$\min p^T x$ s.t. $Ax \geq b, x \geq 0$

Proposition: If $x^1, \cdots, x^K \in \mathbb{R}^n$ are solutions of a LP, any other point in the <u>convex hull</u> of these solutions, defined by

$$\left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^K \alpha_i x^i, \ \sum_{i=1}^K \alpha_i = 1, \ \alpha_i \geq 0, \ i = 1, \cdots, K \right\},$$

is also a solution.

Proof: ① Check that every point in the convex hull is feasible.

By assumption, $x^1, \cdots, x^K$ are feasible, i.e. $x^1, \cdots, x^K \geq 0$
$Ax^1, \cdots, Ax^K \geq b$

so $\sum_{i=1}^K \overset{\geq 0}{\alpha_i} \overset{\geq 0}{x^i} \geq 0$, and $A\left(\sum_{i=1}^K \alpha_i x^i\right) = \sum_{i=1}^K \alpha_i A x^i \geq \sum_{i=1}^K \alpha_i b = \Big(\overset{1}{\overset{\shortparallel}{\sum_{i=1}^K \alpha_i}}\Big) b = b.$

② Check that every point in the convex hull attains the same optimal objective value.

By assumption, $p^T x^i = z_{opt}$ = min value of the LP.

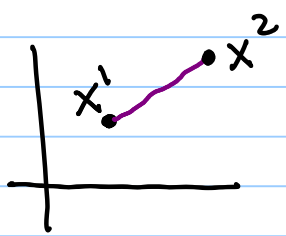
Then, $$p^T\left(\sum_{i=1}^{K}\alpha_i x^i\right) = \sum_{i=1}^{K}\alpha_i p^T x^i = \sum_{i=1}^{K}\alpha_i z_{opt} = \overbrace{\left(\sum_{i=1}^{K}\alpha_i\right)}^{=1} z_{opt} = z_{opt}.$$

Q.E.D.

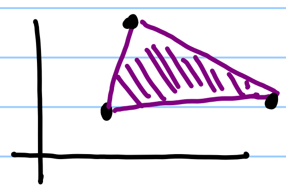
Convex hull of $x^1, \cdots, x^K$ = the smallest convex set containing $x^1, \cdots, x^K$

Detailed explaination omitted (with regret). Here are some examples:

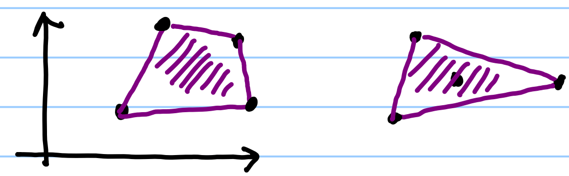
$K=2,\ n=2$



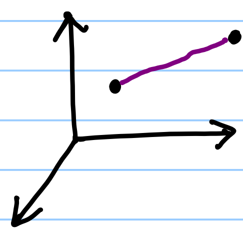
$K=3,\ n=2$



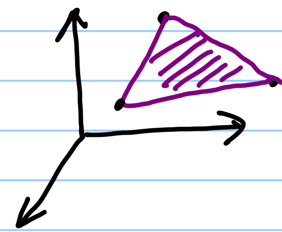
$K=4,\ n=2$



$K=2,\ n=3$

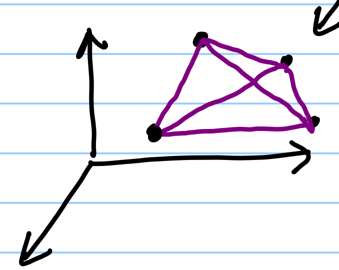


$K=3,\ n=2$

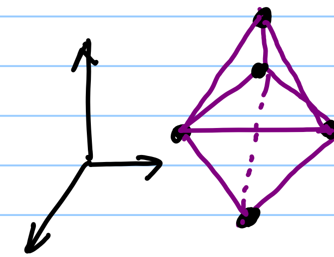


$K=4,\ n=3$

tetrahedron



$K=5,\ n=3$

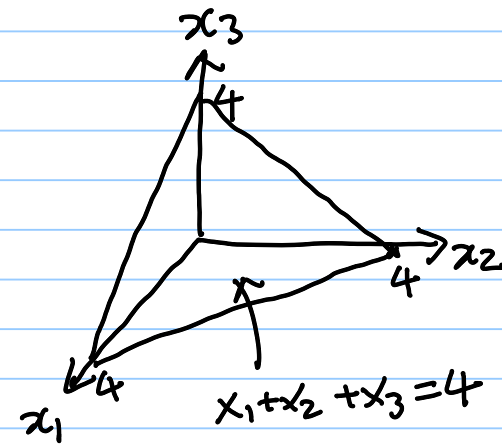


Example 3-3-3.

$$z = \min_{x_1,x_2,x_3} \; -x_1 - x_2 - x_3$$

$$\text{s.t.} \quad \begin{aligned} x_1 - x_2 + x_3 &\geq -2 \\ -x_1 + x_2 + x_3 &\geq -3 \\ x_1 + x_2 - x_3 &\geq -1 \\ -x_1 - x_2 - x_3 &\geq -4 \end{aligned}$$

$$x_1, x_2, x_3 \geq 0$$

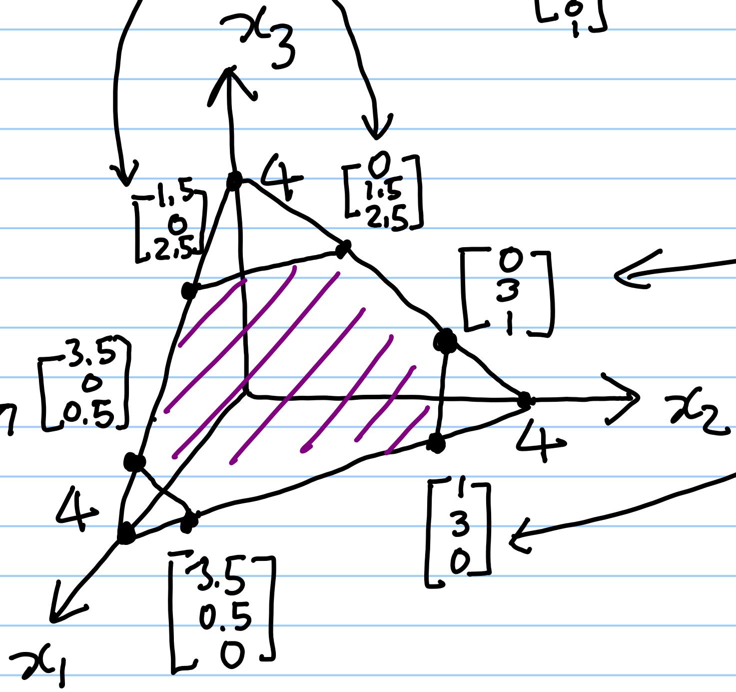


It's clear that

$$z \geq -4$$

visualizing the
other 3 constraints:

$x_1 + x_2 - x_3 = -1$ goes through
these 2 pts and
$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



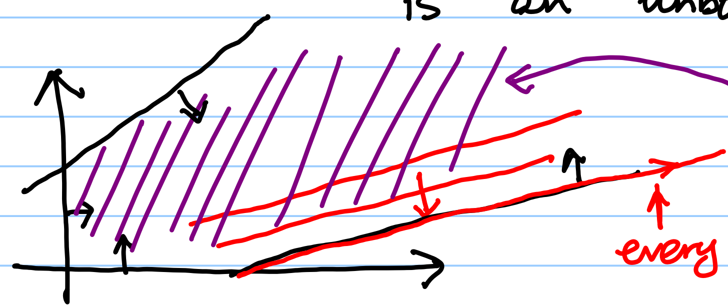
$x_1 - x_2 + x_3 = -2$
goes thr. these
2 pts and $\begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$

$-x_1 + x_2 + x_3$
$= -3$
goes thr.
these 2 pts and $\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$

Note that it is also possible to have a LP s.t.

- the feasible region is unbounded
- the min value is bounded
- but the solution set (i.e. the set of minimizers) is an unbounded set

E.g.



Contour lines of z parallel to one of the boundary edges of the feasible region.

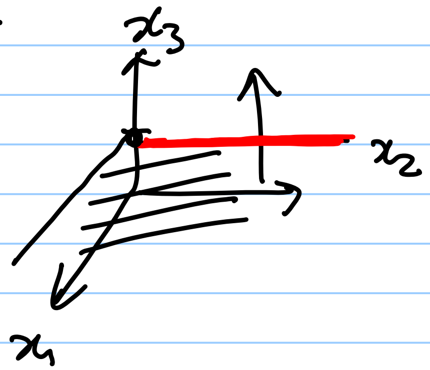
$-x_3 \geq -1$

$x_4 := -x_3 + 1$

Exercise 3-3-7

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 1$

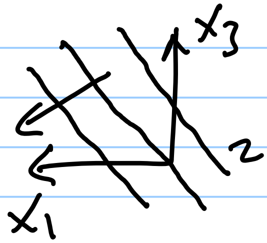
1.



2. $x_1 + x_2 + x_3$. $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ min $= 1$

3. $x_1 + x_3$ $\begin{bmatrix} 0 \\ \lambda \geq 0 \\ 1 \end{bmatrix}$

4. $-x_1 + x_3$ min $= -\infty$



In class: simplex method in action

Note that the origin is <u>not</u> part of the feasible region.

(But clearly $x_1 = 0,\ x_2 = 0,\ x_4 = 0$ is feasible.)

The Phase I Procedure (Sec 3.4)

Origin LP

$$\min\ p^T x \quad \text{s.t. } Ax \geq b,\ x \geq 0$$

$\Longleftrightarrow x_{n+i} := A_{i\cdot} x - b_i \geq 0$, $i = 1, \cdots, m$

$\longrightarrow$

Phase I LP

$$\min_{x_0, x}\ z_0 = x_0$$

$$\text{s.t. } x_{n+i} := \begin{cases} A_{i\cdot} x - b_i & \text{if } b_i \leq 0 \\ A_{i\cdot} x - b_i + x_0 & \text{if } b_i > 0 \end{cases}$$

$$x_0 \geq 0$$
$$x_1 \geq 0$$
$$\vdots$$
$$x_n \geq 0$$
$$\vdots$$
$$x_{m+n} \geq 0$$

$S = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$

when $b \not\leq 0$, how do we identify a vertex, or determine that S is 'empty'?

Note: the phase I LP concerns only the feasible region, it has nothing to do with the vector p.

(I) While the original LP may or may not be feasible, the Phase I LP is always feasible:

set $x_0 = \max\left(\max_{1 \le i \le m} b_i , 0\right)$ (> 0 when you need Phase I)

and $x_1 = \cdots = x_n = 0$

then the remaining constraints ($x_{n+i} \geq 0$, $i = 1, \cdots, m$) are obviously satisfied.

(II) $\bar{x}$ is feasible for the original LP $\Longleftrightarrow$ $(0, \bar{x})$ is feasible for the Phase I LP

$\Updownarrow$

$0 \leq A_{i\cdot}\bar{x} - b_i \ \forall i$

$\Updownarrow$ (because $z_0 \geq 0$)

$(0, \bar{x})$ is a solution for the Phase I LP

(Note: There are as many vertices for the original LP as there are solutions for the Phase I LP. So, the solution of the Phase I LP is typically far from unique.)

(III) (contrapositive of (II))

If (x_0^*, x^*) is a solution of the Phase I LP, but

$$x_0^* > 0,$$

then the original LP must be infeasible (i.e. $S = \phi$.)

Example 3-4-1.