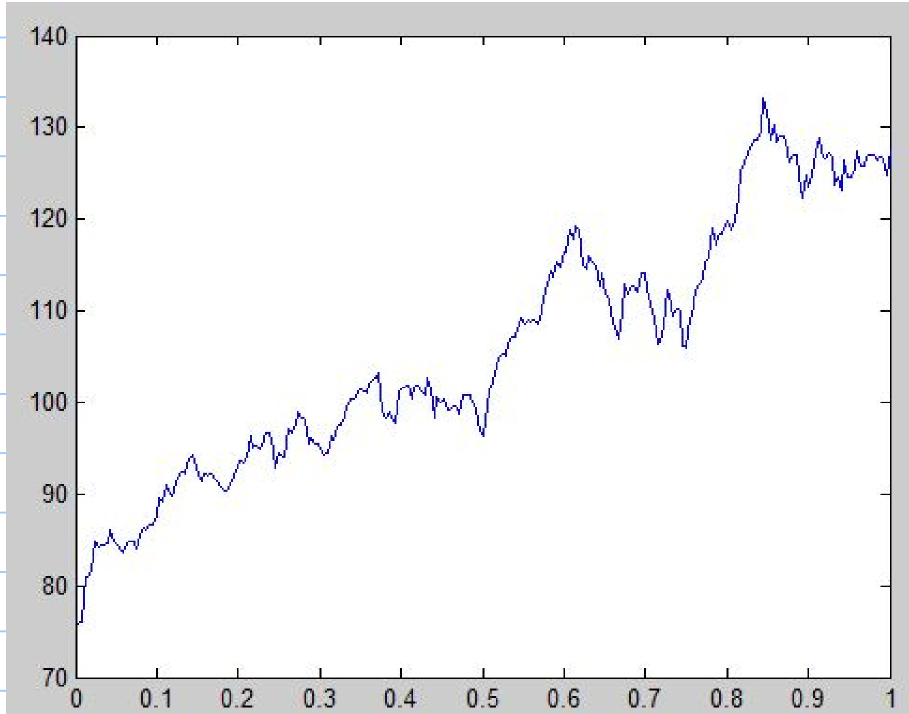
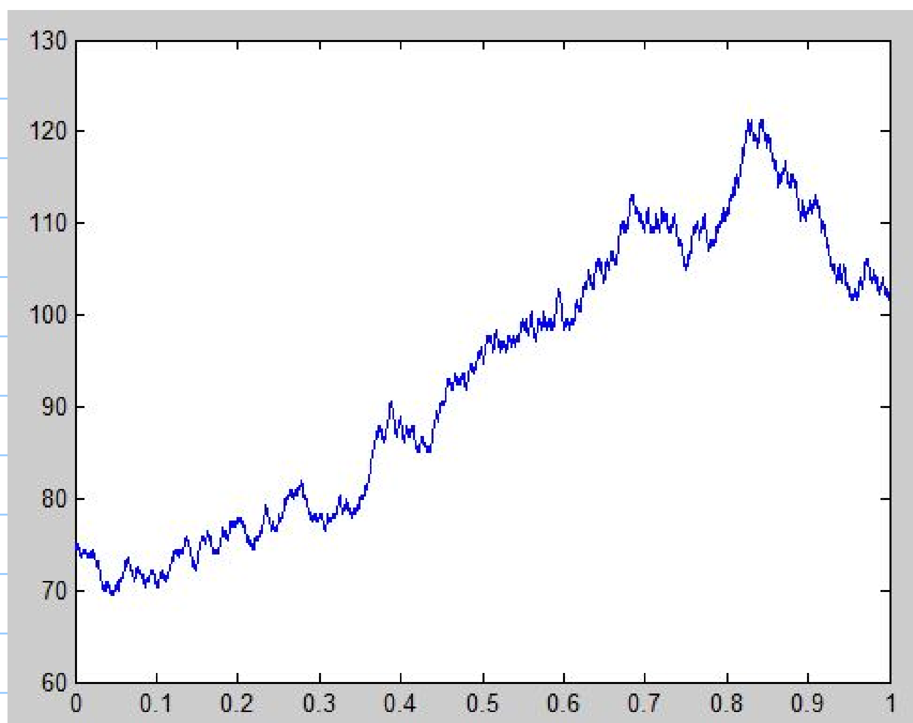
Week 5

Note Title 4/20/2015



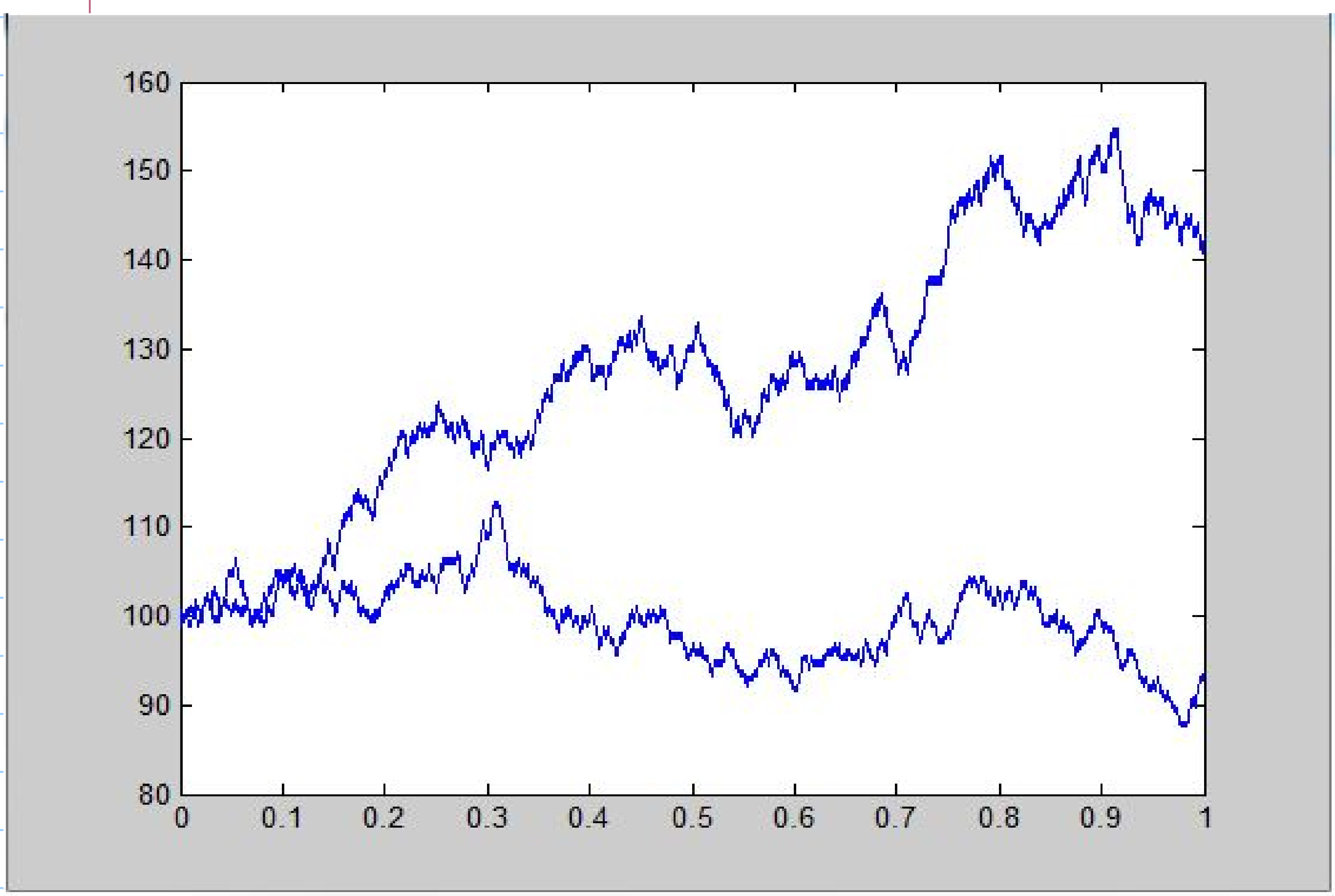
One of these is the daily closing prices of a real stock for one year, the other is a realization of a geometric brownian motion, can you tell which is which?



Apple stock price:

⟵ 1 year ⟶

Geometric Brownian Motion: ($T=1$, $\nu=0.1$, $\sigma=0.2$)



IBM 1 year :

discontinuity, not allowed by Geometric B.M.

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the results "point to the unprecedented pace of change in our industry."

"We are disappointed in our performance," said CEO Ginni Rometty

The fall in IBM (IBM, Tech30) shares took Dow futures, which had been positive earlier, into n territory.

IBM shares

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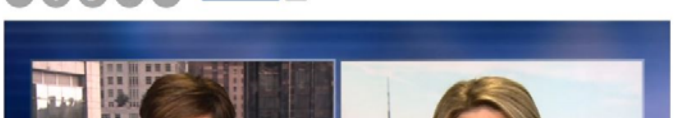
By Chris Isidore @CNNMoney



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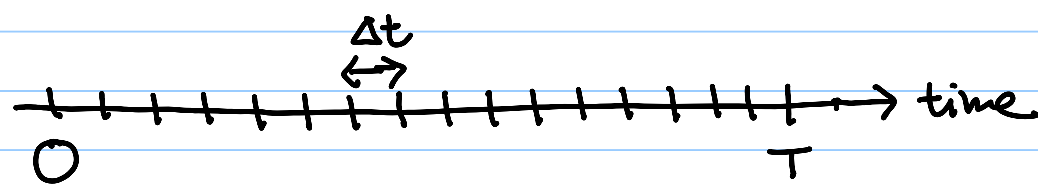
"All models are wrong, but some are useful."

S_0 = initial stock price at time 0

S_T = stock price at time T (unit: year)

Growth factor : $R := S_T / S_0$

- model R as a random variable



- $T = n\Delta t$, n large

$$\frac{S_T}{S_0} = \left(\frac{S_{\Delta t}}{S_0}\right)\left(\frac{S_{2\Delta t}}{S_{\Delta t}}\right) \cdots\cdots \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right) \cdots\cdots \left(\frac{S_{n\Delta t}}{S_{(n-1)\Delta t}}\right)$$

Think, e.g, T = 1 year, Δt = 1 day

The yearly growth is the compound effect of daily growth.

$$R_i := \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}$$

$$R = R_1 R_2 \cdots\cdots R_n$$

Now the ingenious assumption:

$R_1, R_2, \dots, R_n$ are independent and identically distributed (i.i.d.)

[This assumption is definity wrong, but seems ok if T is not big and under "normal market conditions".]

Then:

$$\ln(R) = \ln(R_1) + \cdots + \ln(R_n)$$

also i.i.d.

But then:

$\ln(R)$ is approximately normally distributed in virtue of the central limit theorem.

(Probably a lot to digest here.....)

Review of probability results:

(I) Let $X_1, \cdots, X_n$ be rv's with means $\mu_1, \cdots, \mu_n$ and variances $\sigma_1^2, \cdots, \sigma_n^2$, respectively.

(a) whether or not $X_1, \cdots, X_n$ are independent
For any (deterministic) constants $a_1, \cdots, a_n$.

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$

$$= a^T \mu, \quad a^T = [a_1, \cdots, a_n]$$
$$\mu^T = [\mu_1, \cdots, \mu_n]$$

$$VAR\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \, COV(X_i, X_j)$$

$$= a^T C a, \quad C = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{bmatrix}$$

$$c_{ij} = COV(X_i, X_j)$$

(b) if $X_1, \cdots, X_n$ are independent then

$$COV(X_i, X_j) = 0 \quad \forall\, i,j \quad i \neq j$$

and

$$VAR\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \sigma_i^2$$

II) Let $X_1, \dots, X_n$ be independent normal r.v.'s :

$$X_i \sim N(\mu_i, \sigma_i^2), \quad i = 1, \dots, n.$$

Then

$$\sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \ \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

[Recall : a normal / Gaussian r.v., $N(\mu, \sigma^2)$, has a p.d.f.

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$]

III) (Central Limit theorem)

Let $X_1, \dots, X_n$ be i.i.d. with mean μ and variance σ^2 (need <u>not</u> be normal).

Then as $n \to \infty$

$\overline{X} := \frac{1}{n}(X_1 + \dots + X_n)$ has approximately a normal distribution with mean μ and standard deviation $\sigma/\sqrt{n}$.

more precisely, $\forall z \in \mathbb{R}$,

$$\lim_{n \to \infty} P\left[\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} < z\right] = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

OR : " $\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$ "

[see class demos]

Comment about technical difficulties:

I < II < III

I — easiest
need:
Concepts of joint distribution, expected value, variance, covariance, independence

II — harder
need: special structure of normal distribution, technique of moment generating function

III — most technical
Notice the "universality": the underlying distribution can be <u>any</u> distribution with a finite mean and variance.

need: all concepts from I and II
+
some approximations based on math analysis.

Back to :

$$(\star) \quad \ln(R) = \ln(R_1) + \cdots + \ln(R_n)$$

i.i.d.

If $E[\ln(R)] = \nu \cdot T$ annualized $\nu =$ "growth"

and

$VAR[\ln(R)] = \sigma^2 \cdot T$ $\sigma =$ "volatility"

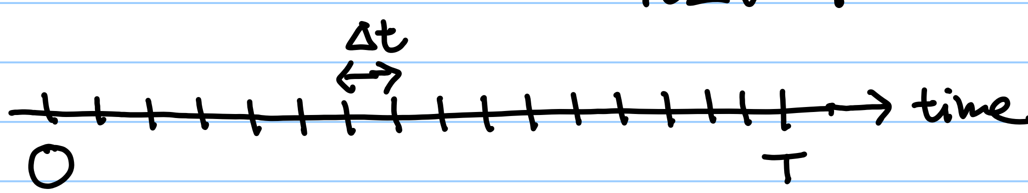
then

CLT $\Rightarrow$ $\ln(R)$ is approximately normal

and

$$(\star\star) \quad \begin{cases} E[\ln(R_i)] = \nu T/n = \nu \Delta t \\ VAR[\ln(R_i)] = \sigma^2 T/n = \sigma^2 \Delta t \end{cases}$$

$$n \Delta t = T$$



CLT also says : When n is large (equivalently Δt small), it does not quite matter what the exact distribution of $\ln(R_i)$ is, as long as its expected value and variance scale like $(\star\star)$.

Two particularly appealing models:

① Assume $\ln(R_i) \sim N(\nu\Delta t, \sigma^2\Delta t)$

then $\ln(R)$ is <u>exactly</u> $N(\nu, \sigma^2)$

(This follows from I earlier. CLT is not needed.)

② Assume $\ln(R_i)$ is a (scaled) Bernoulli r.v.

$$\ln(R_i) = \begin{cases} \ln(u) = U & \text{with probability } p \\ \ln(\frac{1}{u}) = -U & \text{with probability } 1-p \end{cases}$$

$$= UB + (-U)(1-B) \qquad \text{with}$$
$$= 2UB - U \qquad B \sim \text{Bernoulli}(p)$$

Q: How to choose $U = \ln(u)$ and p to satisfy ⊛⊛ ?

$$E[2U\overset{R_i}{\overset{\shortparallel}{B}}-U] = 2Up - U \equiv \nu\Delta t$$
$$VAR[2UB-U] = 4U^2p(1-p) \equiv \sigma^2\Delta t$$

Solve for U and p in terms of ν, σ, T:

$$(2p-1)^2U^2 = \nu^2(\Delta t)^2$$

$$+ \qquad\qquad +$$

$$4U^2(p-p^2) \qquad \sigma^2\Delta t$$

$$(4p^2 - 4p + 1)u^2 + 4u^2(p - p^2) = \nu^2 \Delta t^2 + \sigma^2 \Delta t$$

$$" u^2$$

so $$u = \sqrt{\sigma^2 \Delta t + \nu^2 \Delta t^2}$$

$$p = (\nu \Delta t + u)/2u$$

$$= \frac{1}{2} + \frac{1}{2}\frac{\nu \Delta t}{u}$$

when Δt is small,

$$\ln(u) = \quad u \approx \sigma\sqrt{\Delta t}$$

$$p \approx \frac{1}{2} + \frac{1}{2}\frac{\nu \Delta t}{\sigma\sqrt{\Delta t}}$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t}$$

so the matching parameters can be chosen to be:

$$\boxed{u = e^{\sigma\sqrt{\Delta t}} \qquad (d = e^{-\sigma\sqrt{\Delta t}})}$$

and $$\boxed{p = \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t}}$$

Note: In the pricing method we don't care about p, only (u, d), and

u, d are only dependent on σ, not ν.

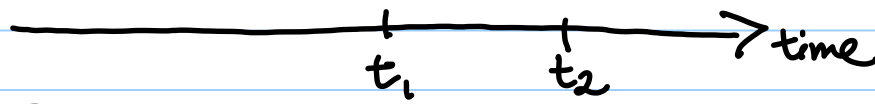
Finally, imagine what happen when $\Delta t \to 0$.

We get a special case of a continuous time stochastic process S_t.

This particular type of Stochastic process is called geometric Brownian motion.
parameters : ν, σ

Properties :

- S_t is a r.v. for any time t (on the continuous time axis)



$\ln\left(\frac{S_{t_2}}{S_{t_1}}\right)$ is a normal r.v. with mean $\nu(t_2-t_1)$ and variance $\sigma^2(t_2-t_1)$

- $\ln\left(\frac{S_{t_2'}}{S_{t_1'}}\right)$ and $\ln\left(\frac{S_{t_2}}{S_{t_1}}\right)$ are independent

if $[t_1', t_2']$ and $[t_1, t_2]$ do not overlap.

The famous Black-Scholes formula is based on this model.

[see again Example_Binomial_Lattice.m]

Using mean and variance of lognormal r.v., it can be shown that:

$$(*)\begin{cases} \ln\left(\frac{S_{t+\Delta t}}{S_t}\right) \sim \nu \Delta t + \sigma\sqrt{\Delta t}\, Z \\ \Longleftrightarrow \quad \frac{S_{t+\Delta t} - S_t}{S_t \Delta t} \sim \left(\nu + \frac{1}{2}\sigma^2\right) + \frac{\sigma}{\sqrt{\Delta t}} Z \end{cases}$$

Mean and Variance of lognormal r.v.:

If $Y = e^X$, $X \sim N(\mu, \sigma^2)$

then

$$E[Y] = e^{\mu + \sigma^2/2}$$
$$V[Y] = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)$$

[In the context of Stochastic differential equation, $(*)$ is related to a result known as the <u>Ito's lemma</u>.]