A sequence of vector spaces and linear maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be <u>exact</u> when $\operatorname{Im}(f) = \ker(g)$.

A sequence

$$A^* = \{A^i, d^i\}$$

$$\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} A^{i+2} \to \cdots$$

of vector spaces and linear maps is called a chain complex provided

$$d^{i+1} \circ d^i = 0 \qquad (\Rightarrow \ker(d^{i+1}) \supset \operatorname{Im}(d^i))$$

for all i.

The chain complex is called exact if

$$\ker(d^{i+1}) = \operatorname{Im}(d^i) \quad \forall i.$$

An exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is called a short exact sequence. This is equivalent to requiring

- f is injective $(\Leftrightarrow \ker(f) = 0)$,
- $\operatorname{Im}(f) = \ker(g)$,
- g is surjective $(\Leftrightarrow \operatorname{Im}(g) = C)$.

Note : A linear map $g : B \to C$ always induces a linear map (also called g)

$$g : B/\ker(g) \to C$$

defined by $\bar{g}(b + \ker(g)) := g(b)$.

For a short exact sequence,

$\bar{g}: B/\underbrace{\ker(g)}_{= \operatorname{Im}(f)} \to C$ is an isomorphism

since:

$$\begin{aligned}\dim(B/\operatorname{Im}(f)) &= \dim(B/\ker(g)) \\ &= \dim(B) - \operatorname{nullity}(g) \\ &= \operatorname{rank}(g) \\ &= \dim(C).\end{aligned}$$

Another linear isomorphism:

<u>Lemma</u> For a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

$$\dim(A) < \infty,\ \dim(C) < \infty \implies$$

$$\dim(B) < \infty \text{ and } B \cong A \oplus C.$$

(prove it yourself or look up the proof.)

Recall a chain complex

$$A^* = \{\cdots \to A^{p-1} \xrightarrow{d^{p-1}} A^p \xrightarrow{d^p} A^{p+1} \to \cdots\}$$

$$d^p \circ d^{p-1} = 0 \quad \forall p.$$

So far we have only seen one kind of chain complex (namely de Rham's), but there are other kinds in order for the following general definition worthwhile:

Def pth cohomology vector space of A^* is

$$H^p(A^*) = \ker d^p / \operatorname{Im} d^{p-1}.$$

A chain map

$$f : A^* \longrightarrow B^*$$

chain complexes, e.g. de Rham's complexes of two open sets / manifolds

is a family of linear maps $f^p : A^p \to B^p$ s.t. the following diagram commutes :

$$\begin{tikzcd}
A^* = (\cdots \arrow[r] & A^{p-1} \arrow[r, "d^{p-1}"] \arrow[d, "f^{p-1}"] & A^p \arrow[r, "d^p"] \arrow[d, "f^p"] & A^{p+1} \arrow[r] \arrow[d, "f^{p+1}"] & \cdots) \\
B^* = (\cdots \arrow[r] & B^{p-1} \arrow[r, "d^{p-1}"] & B^p \arrow[r, "d^p"] & B^{p+1} \arrow[r] & \cdots)
\end{tikzcd}$$

i.e. $\quad d_B^p \circ f^p = f^{p+1} \circ d_A^p \quad , \forall p$

Next, a chain map induces a linear map
(Lemma 4.3 MBT)

$$H^p(A^*) \xrightarrow{\;H^*(f),\ \text{or}\ f^*\;} H^p(B^*)$$

For $a \in A^p$, $d^p a = 0$, define

$$f^*(\underset{a + \operatorname{Im}(d^{p-1})}{\underset{\shortparallel}{[a]}}) = [f^p(a)].$$

Check that it is a well-defined linear map.
(or look up the proof)

The true excitment happens when we have a

Short exact sequence of chain complexes

i.e.

$$0 \to A^* \xrightarrow{f} B^* \xrightarrow{g} C^* \to 0$$

① f, g are chain maps

② $0 \to A^p \xrightarrow{f^p} B^p \xrightarrow{g^p} C^p \to 0$ is exact $\forall p$.

So from the last lemma, f, g induce linear maps between the corresponding cohomology spaces

$$H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

moreover : This sequence is exact $\forall p$.

(Lemma 4.4)

Proof :

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r] \arrow[d, "d"] & B^{p-1} \ni b_1 \arrow[r, "g^{p-1}"] \arrow[d, "d"] & C^{p-1} \ni c \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^p \arrow[r, "f^p"] \arrow[d, "d"] & B^p \arrow[r, "g^p"] \arrow[d, "d"] & C^p \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r] \arrow[d] & B^{p+1} \arrow[r] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
\Rightarrow H^p(A^*) \xrightarrow{f^*} H^p(B^*) \xrightarrow{g^*} H^p(C^*)$$

$a' \qquad b'$

$$[a] \in H^p(A^*), \quad [b] \in H^p(B^*)$$

(i) Since $g^p \circ f^p = 0$, for any $[a] \in H^p(A^*)$,

$$g^* \circ f^*([a]) = g^*([f(a)]) = [g^p(f^p(a))] = 0.$$

so $\quad g^* \circ f^* = 0$

(ii) Assume $[b] \in H^p(B^*)$ s.t. $g^*([b]) = 0$, so

$g^p(b) \in \mathrm{Im}(d_C^{p-1})$, or $g^p(b) = d_C^{p-1}(c)$ for some $c \in C^{p-1}$.

But g^{p-1} is surjective,

so $g^{p-1}(b_1) = c$ for some $b_1 \in B^{p-1}$.

Then $g^p \circ d_B^{p-1}(b_1) = \underbrace{d_C^{p-1} \circ g^{p-1}(b_1)}_{g^p(b)}$ $\quad \downarrow\!\_ = \urcorner\!\downarrow$

or $g^p(\underbrace{d_B^{p-1} b_1 - b}_{\in \ker g^p \, = \, \mathrm{Im} f^p}) = 0$

Then

$$d_B^{p-1} b_1 - b = f^p(a) \text{ for some } a \in A^p.$$

We are done if we can show:

① $d_A^p(a) = 0$, so $[a]$ is a cohomology class in $H^p(A^*)$

② $f^*([a]) = [b]$.

① $d_A^p(a) = 0 \iff f^{p+1}(d_A^p(a)) = 0$, since f^{p+1} is injective.

But then

$$f^{p+1} \circ d_A^p(a) = d_B^p \circ \underbrace{f^p(a)}_{\shortparallel} \qquad \downarrow\!\_ = \urcorner\!\downarrow$$

$$= d_B^p(d_B^{p-1} b_1 - b)$$

$$= \underbrace{d_B^p \circ d_B^{p-1}}_{=0} b_1 - \underbrace{d_B^p b}_{=0} = 0$$

② $f^*[a] = [f^p(a)]$ (by def. of f^*)

$$= [\underbrace{d_B^{p-1} b_1}_{\in \, \mathrm{Im}\, d_B^{p-1}} - \underbrace{b}_{\in \, \ker d_B^p}] = [b]$$

$\square$

A key result (from algebraic topology) says

Thm: $\boxed{\text{a short exact sequence of chain complexes gives a long exact sequence of the corresponding cohomology spaces}}$.

i.e.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A^{p-1} \arrow[r, "f^{p-1}"] \arrow[d, "d"] & B^{p-1} \arrow[r, "g^{p-1}"] \arrow[d, "d"] & C^{p-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{p} \arrow[r, "f^{p}"] \arrow[d, "d"] & B^{p} \arrow[r, "g^{p}"] \arrow[d, "d_B^p"] & C^{p} \ni c \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A^{p+1} \arrow[r, "f^{p+1}"] \arrow[d] & B^{p+1} \arrow[r, "g^{p+1}"] \arrow[d] & C^{p+1} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd}
H^p(A^*) \arrow[r, "f^*"] & H^p(B^*) \arrow[r, "g^*"] & H^p(C^*) \arrow[dll, "\partial^*"'] \\
H^{p+1}(A^*) \arrow[r, "f^*"] & H^{p+1}(B^*) \arrow[r, "g^*"] & H^{p+1}(C^*) \arrow[dll] \\
\cdots\cdots & &
\end{tikzcd}$$

$$\partial^*([c]) := [(f^{p+1})^{-1}(d_B^p((g^p)^{-1}(c)))]$$

Direct sum of vector spaces A, B

$$A \oplus B := \{(a,b) : a \in A,\ b \in B\}$$

$$\lambda(a,b) = (\lambda a, \lambda b)$$

$$(a_1,b_1) + (a_2,b_2) = (a_1+a_2,\ b_1+b_2).$$

$$\dim(A \oplus B) = \dim A + \dim B.$$

If A^*, B^* are chain complexes

$$A^* \oplus B^* := \left(\cdots \to A^p \oplus B^p \xrightarrow[(d_A^p,\, d_B^p)]{} A^{p+1} \oplus B^{p+1} \to \cdots\right)$$

is also a chain complex.

Moreover: (Lemma 4.13) $\quad H^p(A^* \oplus B^*) = H^p(A^*) \oplus H^p(B^*).$

(Easy to check.)