Lecture 9 : Gauss' Theorema Egregium

Note Title 3/2/2017

Gauss' applied work on geodesy must have motivated him to address the following question:

> Is it possible to map any (open) region of the sphere to the plane without distorting distance?

If you want to show that two topological spaces cannot be mapped continuously between each other (i.e. not homeomorphic), you would need to use topological invariants i.e. properties that are necessarily preserved by continuous maps

e.g. $\longleftrightarrow$ cannot be homeomorphic to a cross $\times$ in $\mathbb{R}^2$

use connectedness – a topological invariant

If f : (cross $\times$) $\to (a,b)$ is a homeomorphism then f : (cross $\setminus$ {center}) $\to (a,b) \setminus \{f(\text{center})\}$ is also a homeomorphism, which must preserve the number of connected components. This generates a contradiction as :

# of connected components in $\times$ = 4

# of connected components in $\longleftrightarrow$ = 2

e.g. $\longleftrightarrow$ cannot be homeomorphic to ∞ in $\mathbb{R}^2$

not compact — compact

use compactness – also a topological invariant

So to answer Gauss' question negatively, one needs an

"isometric invariant", i.e.

a property / quantity that is preserved by isometries.

Then show that any region of the sphere and any region of the plane would have different such properties / quantities.

Actually I don't know if Gauss was the first person to ask that question, but he was the first to answer it:

> (Theorema Egregium) The Gaussian curvature K of a regular surface in $\mathbb{R}^3$ is invariant by local isometries.

Part of the challenge of understanding this 'remarkable theorem' is to understand what local isometry means.

Def

Let $S, \overline{S}$ be two regular surfaces with the same intrinsic dimensions (but possibly different ambient dimensions.)

A smooth map $\varphi : V \subset S \to \overline{S}$ of a neighborhood V of $p \in S$ is a **local isometry** at p if

$$\varphi : V \to \varphi(V)$$

is a <u>diffeomorphism</u> satisfying:

$$\langle w_1, w_2 \rangle_q = \langle d\varphi_q(w_1), d\varphi_q(w_2) \rangle_{\varphi(q)}, \quad \forall q \in V.$$

Note: $\varphi : V \to \varphi(V)$ is a local isometry at $p \in S$
$\Leftrightarrow \varphi^{-1} : \varphi(V) \to V$ is a local isometry at $\varphi(p) \in \bar{S}$

<u>Def</u>
If there exists a local isometry into $\bar{S}$ at every $p \in S$, the surface S is said to be locally isometric to $\bar{S}$.

🚩 Note that it is not a symmetric relation
"S is locally isometric to $\bar{S}$"
$\not\Rightarrow$ "$\bar{S}$ is locally isometric to S"

e.g.

Every (small enough) open nhbd of $p \in \mathbb{R}^2$ can be mapped isometrically to an open set in the flat bottom of $\bar{S}$. So S is locally isometric to $\bar{S}$. But there is no reason to expect that $\bar{S}$ is locally isometric to S.
Therefore, we say:

<u>Def (continued)</u>
S and $\bar{S}$ are locally isometric if S is locally isometric to $\bar{S}$ and $\bar{S}$ is locally isometric to S.

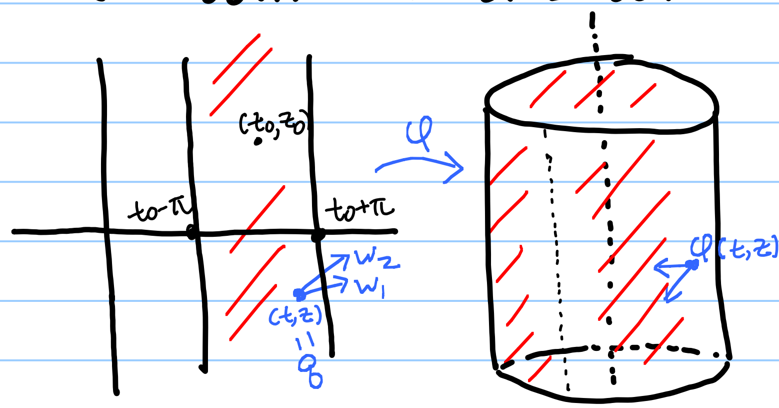
E.g. $\bar{S}$ = cylinder = $\{(x,y,z) : x^2+y^2=1\}$

S = plane = $\mathbb{R}^2$

Let $\varphi : S \to \bar{S}$, $[t,z]^T \mapsto [\cos t, \sin t, z]^T$

For any $(t_0, z_0) \in S = \mathbb{R}^2$, let $V := (t_0-\pi, t_0+\pi) \times \mathbb{R}$

$\varphi|_V : V \to \varphi(V)$ is a diffeomorphism
(smooth with a smooth inverse)



$$d\varphi = \begin{bmatrix} -\sin t & 0 \\ \cos t & 0 \\ 0 & 1 \end{bmatrix}$$

$$d\varphi^T d\varphi = \begin{bmatrix} -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin t & 0 \\ \cos t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sin^2 t + \cos^2 t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

so

$$\langle w_1, w_2 \rangle_q = \langle d\varphi_q w_1, d\varphi_q w_2 \rangle_{\varphi(q)}, \ \forall q.$$

Note that φ is surjective, for any $y \in \bar{S}$, choose any $p \in S$ s.t. $\varphi(p) = y$ and V around p s.t.

$$\varphi|_V : \underset{\underset{p}{\Psi}}{V} \to \underset{\underset{y}{\Psi}}{\varphi(V)} \text{ is a diffeomorphism.}$$

So $(\varphi|_V)^{-1} : \varphi(V) \to V$ is a local isometry at y.

The example above suggests that if we can find a function

$$\varphi : S \to \bar{S}$$

s.t.

① $\langle w_1, w_2 \rangle_p = \langle d\varphi_p w_1, d\varphi_p w_2 \rangle_{\varphi(p)}, \forall p \in S$

② $\varphi(S) = \bar{S}$, i.e. φ is surjective.

then

S and $\bar{S}$ are locally isometric.

Note:

we don't need to check that φ provides local diffeomorphisms, it is implied by ①:

① $\Rightarrow d\varphi_p : T_pS \to T_{\varphi(p)}\bar{S}$ is a linear isometry

$\Rightarrow d\varphi_p : T_pS \to T_{\varphi(p)}\bar{S}$ is injective

$\Rightarrow d\varphi_p : T_pS \to T_{\varphi(p)}\bar{S}$ is bijective

same dimension

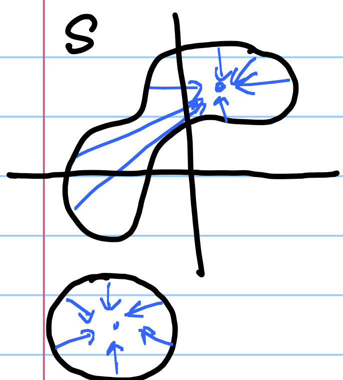
inverse function theorem $\Rightarrow \varphi$ is a local diffeomorphism

Note: This single, globally defined, map φ is not required to be, and typically is not (as in the previous example), injective.

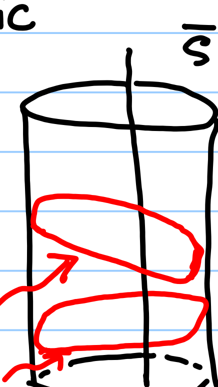
φ only furnishes a <u>local</u> <u>diffeomorphism</u> near each point $p \in S$, φ itself does not need be a (global) diffeomorphism.

Typically, S and $\bar{S}$ are not even homeomorphic, again as in the previous example.

This is almost getting off topic (if all that you care is Gauss' remarkable theorem), but let's quickly see why the cylinder and the plane are not homeomorphic, this time the topological invariant we use is called: Simple connectivity.



Every loop in $\mathbb{R}^2$ is homotopic to a trivial loop, but a loop that goes around a cylinder can be shown to be <u>not</u> homotopic to a trivial loop.



provably not contractible to a point.

$\mathbb{R}^2$ is simply connected but cylinder is not, so they cannot be homeomorphic.

Eg. $S = \mathbb{R}^2$, $\bar{S} = \{ [\cos u, \sin u, \cos v, \sin v] : u, v \in [0, 2\pi] \} \subset \mathbb{R}^4$

$$\varphi : S \to \bar{S}$$
$$[u,v]^T \mapsto [\cos u, \sin u, \cos v, \sin v]^T.$$

is clearly surjective. It is also a local isometry:

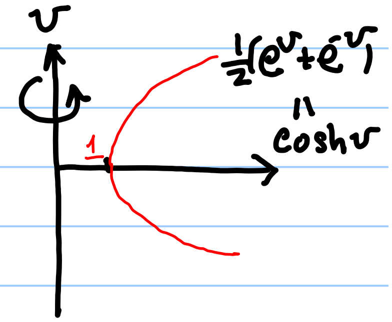
$$d\varphi_{(u,v)} = \begin{bmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & -\sin v \\ 0 & \cos v \end{bmatrix}, \quad d\varphi^T d\varphi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

But note that S, $\bar{S}$ are not homeomorphic ($\bar{S}$ is compact but S is not.) We conclude that

S and $\bar{S}$ are locally isometric. $\bar{S}$ is called a **flat torus.**

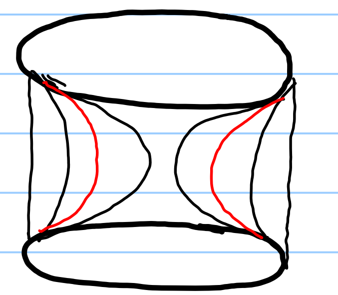
Eg. S = catenoid

$$= \left\{ \begin{bmatrix} \cosh v \cos u \\ \cosh v \sin u \\ v \end{bmatrix} : \begin{matrix} u \in [0, 2\pi] \\ v \in \mathbb{R} \end{matrix} \right\}$$



$\bar{S}$ = helicoid

$$= \left\{ \begin{bmatrix} \rho \cos\theta \\ \rho \sin\theta \\ \theta \end{bmatrix} : \begin{matrix} \rho \in \mathbb{R} \\ \theta \in \mathbb{R} \end{matrix} \right\}$$



catenoid minimizes area

Digression:

These are examples of minimal surfaces, with the property they have zero mean curvature, i.e. $H = 0$, everywhere.

What does minimal area have anything to with the $H \equiv 0$ condition?

It turns out that if a surface S is perturbed in the normal direction:

$$\underbrace{x(u,v,t)}_{\substack{\text{local} \\ \text{para.} \\ \text{of } S_t}} := \underbrace{x(u,v)}_{\substack{\text{a local} \\ \text{para. of} \\ S}} + \underbrace{t}_{\substack{\text{a small} \\ \text{parameter}}} \underbrace{h(u,v)}_{\substack{\text{some scalar field} \\ \text{on } S}} N(u,v)$$

Then $\frac{d}{dt} \text{Area}[S_t] \Big|_{t=0} = -\int 2hH \, dA$

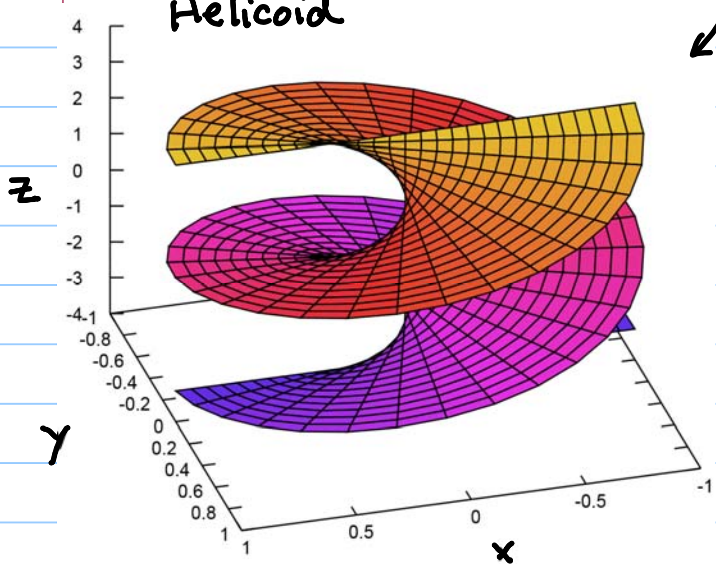
From this you see that $H \equiv 0$ is a necessary condition for area minimization.

On top of being a minimal surface, the helicoid is also a ruled surface, i.e. at every point, there is a straightline that goes through it and lies on the surface. Note:

$$\begin{bmatrix} \rho\cos\theta \\ \rho\sin\theta \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \theta \end{bmatrix} + \rho \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}$$

can you relate?

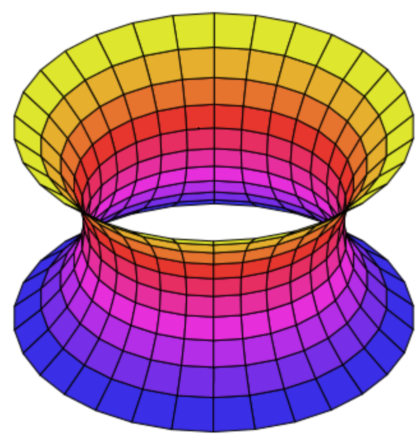
Helicoid



locally isometric?

shown: $-\pi \leq \theta \leq \pi$

$-1 \leq \rho \leq 1$



catenoid

More can be said about these two minimal surfaces; leading in particular to an explaination of why they should be locally isometric. But this will take us too far.

Instead, I will show you how to reparameterize the two surfaces to reveal a local isometry:

Helicoid: $\theta \leftrightarrow u$, $\rho \leftrightarrow \sinh v = \frac{1}{2}(e^{v} - e^{-v})$

$$(u,v) \overset{X}{\mapsto} (\sinh v \cos u, \sinh v \sin u, u)$$

Catenoid: use the original parameterization

$$(u,v) \underset{\bar{x}}{\mapsto} (\cosh v \cos u, \cosh v \sin u, v)$$

Let's compute

$$[x_u \; x_v]^T [x_u \; x_v] = \begin{bmatrix} \overset{=E}{\langle x_u, x_u \rangle} & \overset{=F}{\langle x_u, x_v \rangle} \\ \underset{=F}{\langle x_v, x_u \rangle} & \underset{=G}{\langle x_v, x_v \rangle} \end{bmatrix}$$

$$[\bar{x}_u \; \bar{x}_v]^T [\bar{x}_u \; \bar{x}_v] = \begin{bmatrix} \overset{=\bar{E}}{\langle \bar{x}_u, \bar{x}_u \rangle} & \overset{=\bar{F}}{\langle \bar{x}_u, \bar{x}_v \rangle} \\ \underset{=\bar{F}}{\langle \bar{x}_v, \bar{x}_u \rangle} & \underset{=\bar{G}}{\langle \bar{x}_v, \bar{x}_v \rangle} \end{bmatrix}$$

Easy to check:

$$E = \bar{E} = \cosh^2 v, \quad F = \bar{F} = 0, \quad G = \bar{G} = \cos^2 v$$

It follows that $\varphi = \bar{x} \circ x^{-1}$ must be a local isometry. (why? see below.)

Also:

$$\lambda_{1,2} = \bar{\lambda}_{1,2} = \pm 1/\cosh^2 v$$

$H = \bar{H} = 0$, (both surfaces are minimal surfaces)

$K = \bar{K} = -1/\cosh^4 v$ (exemplifies Gauss' theorem)

This example is perhaps the least obvious among the three, but from the point of view of exemplifying Gauss' theorem it is a bit boring because it turns out that not just

$$K = \bar{K}$$

but also $H = \bar{H} = 0$ (minimal surfaces), which also means the principle curvatures must be the same ($\lambda_1 \lambda_2 = \bar{\lambda}_1 \bar{\lambda}_2$, $\lambda_1 + \lambda_2 = \bar{\lambda}_1 + \bar{\lambda}_2 \Rightarrow \lambda_{1,2} = \bar{\lambda}_{1,2}$)

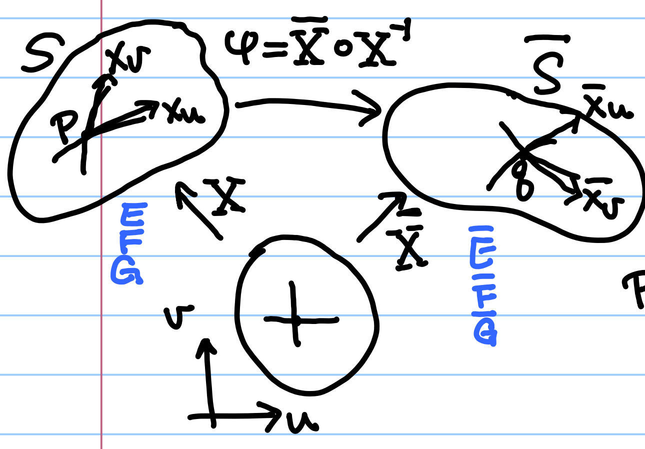
The situation of the first example is different :

$$\left.\begin{array}{l} S = \mathbb{R}^2 \quad , \quad \lambda_1 = \lambda_2 = 0 \\ \bar{S} = \text{cyclinder}, \ \bar{\lambda}_1 = 0 , \ \lambda_2 = 1 \end{array}\right\} \text{everywhere}$$

(assuming inward normal)

$$H = 0 \neq \bar{H} = 1/2 \quad \text{but} \quad K = \bar{K} = 0.$$

We cannot quite use the 2nd example to exemplify Gauss' theorem, because at this point the Gauss curvature is only defined for two-dimensional regular surface <u>in $\mathbb{R}^3$</u>. The definition <u>appears</u> to depend on the ambient space (the normal vectors in particular) although Gauss' theorem <u>suggests</u> that an alternate definition for K, without referring to the ambient space, <u>may be possible</u>.



Proposition :

φ is a local isometry
$\iff$ $E = \bar{E}$, $F = \bar{F}$, $G = \bar{G}$

Proof : Note that

$$\left\{\begin{array}{l} d\varphi_p(X_u) = \bar{X}_u \\ d\varphi_p(X_v) = \bar{X}_v \end{array}\right. \quad (*)$$

$$\varphi \text{ is a local isometry}$$

$$\iff$$

$$d\varphi_p : (T_pS, \langle\ \rangle_p) \to (T_{\varphi(p)}\bar{S}, \langle\ \rangle_{\varphi(p)})$$

is a (linear) isometry $\forall p \in S$

i.e. $\langle d\varphi_p w_1, d\varphi_p w_2 \rangle_{\varphi(p)} = \langle w_1, w_2 \rangle$, $\forall p$, $\forall w_1, w_2$

$\iff$ (polarization identity)

$$\langle d\varphi_p w, d\varphi_p w \rangle_{\varphi(p)} = \langle w, w \rangle_p, \ \forall p, \forall w \in T_pS$$

But

$\{x_u, x_v\}$ is a basis of T_pS

$\{\bar{x}_u, \bar{x}_v\}$ is a basis of $T_{\varphi(p)}\bar{S}$

$$w = a x_u + b x_v$$

$$\Rightarrow \quad \langle w, w \rangle = [a, b] \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

and

$$\langle d\varphi_p w, d\varphi_p w \rangle \underset{\substack{\uparrow \\ \text{use } (*)}}{=} [a, b] \begin{bmatrix} \bar{E} & \bar{F} \\ \bar{F} & \bar{G} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

So:

φ is a local isometry $\iff E = \bar{E}, \ F = \bar{F}, \ G = \bar{G}$.

Note: knowing E, F, G is just the same as knowing $\langle \, , \, \rangle_p$ on T_pS, $\forall p \in$ (coord. nhbd)

Also: In manifold theory, $\langle \cdot, \cdot \rangle_p$ is a specific type of tensor field on S, and E, F, G are the component functions under the coordinate system.

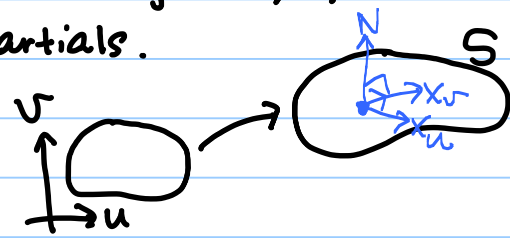
Knowing $E, F, G : (\text{coord. nhbd}) \subset S \to \mathbb{R}$ also means we know their derivatives

$E_u, E_v, E_{uu}, E_{uv}, E_{vv}, F_u, F_v, \cdots$ etc.

This will be exploited in the proof of Gauss' theorem below.

Proof of Gauss' Theorema Egregium:

The idea is to show that K can be expressed purely in terms of E, F, G and their (1st and 2nd) partials.



Recall

$$\langle N, N\rangle \equiv 1 \Rightarrow \langle N_u, N\rangle = 0 \text{ and } \langle N_v, N\rangle = 0$$
$$\Rightarrow N_u, N_v \in T_pS$$

so
$$N_u = a_{11}x_u + a_{21}x_v$$
$$N_v = a_{12}x_u + a_{22}x_v$$

$$[N_u\ N_v] = [x_u\ x_v]\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}}_{=A}$$

$$\underbrace{[x_u\ x_v]^T[N_u\ N_v]}_{= -\begin{bmatrix} e & f \\ f & g\end{bmatrix}} = \underbrace{[x_u\ x_v]^T[x_u\ x_v]}_{= \begin{bmatrix} E & F \\ F & G\end{bmatrix}} A$$

$$A = -\begin{bmatrix} E & F \\ F & G\end{bmatrix}^{-1}\begin{bmatrix} e & f \\ f & g\end{bmatrix}$$

Recall

$$\boxed{K = \det(A) = \frac{eg - f^2}{EG - F^2}}$$

Recall

$$\langle N, x_u\rangle \equiv 0 \Rightarrow \langle N, x_{uu}\rangle = -\langle N_u, x_u\rangle =: e$$
$$\Rightarrow \langle N, x_{uv}\rangle = -\langle N_v, x_u\rangle =: f$$
$$\langle N, x_v\rangle \equiv 0 \Rightarrow \langle N, \overset{\shortparallel}{x_{vu}}\rangle = -\langle N_u, x_v\rangle \ /\!/$$
$$\Rightarrow \langle N, x_{vv}\rangle = -\langle N_v, x_v\rangle =: g$$

Think of $\{x_u, x_v, N\}$ as a "moving frame" (as in the local theory of curves), see how it moves:

$$\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\{x_u, x_v, N\} = ?$$

$$x_{uu} = \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + L_1 N \quad — ①$$
$$x_{uv} = \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + L_2 N \quad — ②$$
$$\overset{\shortparallel}{x_{vu}} = \overset{\shortparallel}{\Gamma_{21}^1} x_u + \overset{\shortparallel}{\Gamma_{21}^2} x_v + \overset{\shortparallel}{L_2} N \quad — ③$$
$$x_{vv} = \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + L_3 N \quad — ④$$
$$N_u = a_{11} x_u + a_{21} x_v$$
$$N_v = a_{12} x_u + a_{22} x_v$$

Γ_{ij}^k $i,j,k=1,2$ are called Christoffel symbols, they are symmetric relative to the lower indices :

$$\Gamma_{ij}^k = \Gamma_{ji}^k \quad , \; i,j \in \{1,2\}.$$

$$① \Rightarrow \langle N, x_{uu} \rangle = 0 + 0 + L_1 \langle N, N \rangle$$
$$\Rightarrow L_1 = e$$

Similarly, $L_2 = f$, $L_3 = g$.

Now a key observation : the Christoffel symbols depend solely on the first fundemental form.

$E = \langle x_u, x_u \rangle$ $\quad F = \langle x_u, x_v \rangle$ $\quad G = \langle x_v, x_v \rangle$

$E_u = 2\langle x_u, x_{uu} \rangle$ $\quad F_u = \langle x_u, x_{uv} \rangle + \langle x_v, x_{uu} \rangle$ $\quad G_u = 2\langle x_v, x_{uv} \rangle$

$E_v = 2\langle x_u, x_{uv} \rangle$ $\quad F_v = \langle x_u, x_{vv} \rangle + \langle x_v, x_{uv} \rangle$ $\quad G_v = 2\langle x_v, x_{vv} \rangle$

So, we can solve for Γ_{ij}^k by taking inner-products of ①-④ with x_u and x_v (note : $\langle x_u, N \rangle = 0 = \langle x_v, N \rangle$) :

From ①, we have

$$\begin{cases} \Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle x_{uu}, x_u \rangle = \frac{1}{2} E_u \\ \Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle x_{uu}, x_v \rangle = F_u - \frac{1}{2} E_v \end{cases}$$

or
$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{bmatrix} \quad - (1')$$

Similarly, from (2) we have

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{12}^1 \\ \Gamma_{12}^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{bmatrix} \quad - (2')$$

and from (4),

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{22}^1 \\ \Gamma_{22}^2 \end{bmatrix} = \begin{bmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{bmatrix}. \quad - (3')$$

(1') − (3') show that $\underbrace{\Gamma_{ij}^k}_{\text{as functions}}$ depends only on $\overbrace{E, F, G}^{\text{as functions}}$, therefore:

All geometric concepts and properties expressed in terms of the Christofell symbols are invariant under isometries.

From $x_{uuv} = x_{uvu}$

$$x_{uuv} = \Gamma_{11,v}^1 x_u + \Gamma_{11}^1 \underbrace{x_{uv}}_{= \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + fN} + \Gamma_{11,v}^2 x_v + \Gamma_{11}^2 \underbrace{x_{vv}}_{= \Gamma_{22}^1 x_u + \Gamma_{22}^2 x_v + gN} + e_v N + e \overbrace{N_v}^{= a_{12} x_u + a_{22} x_v}$$

$$\|$$

$$x_{uvu} = \Gamma_{12,u}^1 x_u + \Gamma_{12}^1 \underbrace{x_{uu}}_{= \Gamma_{11}^1 x_u + \Gamma_{11}^2 x_v + eN} + \Gamma_{12,u}^2 x_v + \Gamma_{12}^2 \underbrace{x_{vu}}_{= \Gamma_{12}^1 x_u + \Gamma_{12}^2 x_v + fN} + f_u N + f \overbrace{N_u}^{= a_{11} x_u + a_{21} x_v}$$

$$\Downarrow$$

$$A x_u + B x_v + C N \equiv 0$$

$$\Downarrow$$

$$A, B, C \equiv 0$$

$\left(\{x_u(u,v),\ x_v(u,v),\ N(u,v)\} \right.$ linearly independent $\left. \forall u,v \right)$

$B \equiv 0$ means

$$\Gamma_{11}^{1}\Gamma_{12}^{2} + \Gamma_{11,v}^{2} + \Gamma_{11}^{2}\Gamma_{22}^{2} + e\,a_{22} \qquad a_{22} = \frac{fF - gE}{EG - F^2}$$

$$\|$$

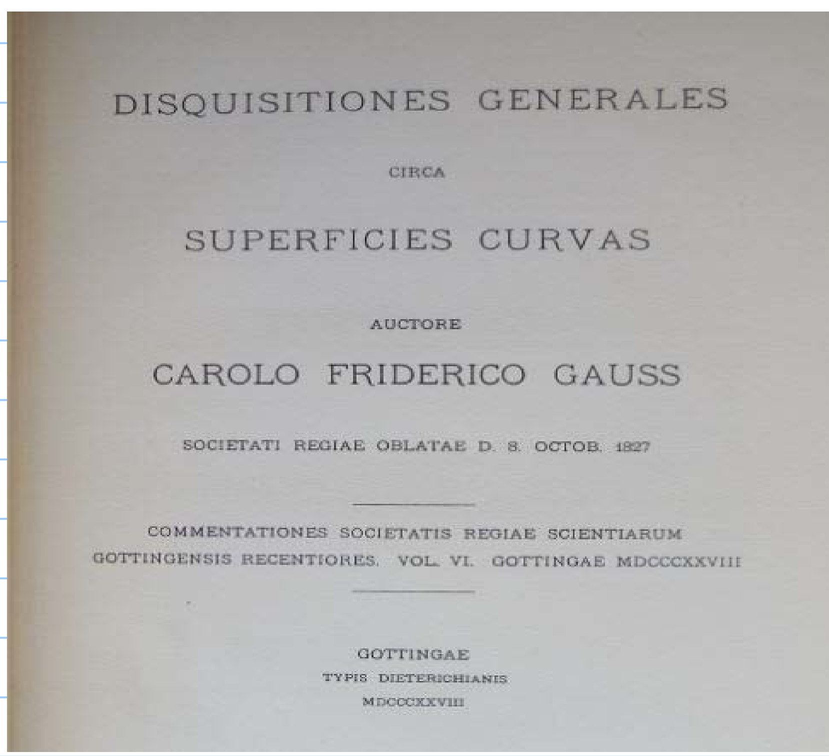
$$\Gamma_{12}^{1}\Gamma_{11}^{2} + \Gamma_{12,u}^{2} + \Gamma_{12}^{2}\Gamma_{12}^{2} + f\,\underbrace{a_{21}}_{=\frac{eF - fE}{EG - F^2}}$$

$$K = (eg - f^2)/(EG - F^2)$$

so

$$\Gamma_{11}^{1}\Gamma_{12}^{2} + \Gamma_{11,v}^{2} + \Gamma_{11}^{2}\Gamma_{22}^{2} - \Gamma_{12}^{1}\Gamma_{11}^{2} - \Gamma_{12,u}^{2} - \Gamma_{12}^{2}\Gamma_{12}^{2} = -EK$$

This shows K depends only on the first fundamental form, hence an isometric invariant. ∎



DISQUISITIONES GENERALES

CIRCA

SUPERFICIES CURVAS

AUCTORE

CAROLO FRIDERICO GAUSS

SOCIETATI REGIAE OBLATAE D. 8. OCTOB. 1827

COMMENTATIONES SOCIETATIS REGIAE SCIENTIARUM
GOTTINGENSIS RECENTIORES. VOL. VI. GOTTINGAE MDCCCXXVIII

GOTTINGAE
TYPIS DIETERICHIANIS
MDCCCXXVIII

1827

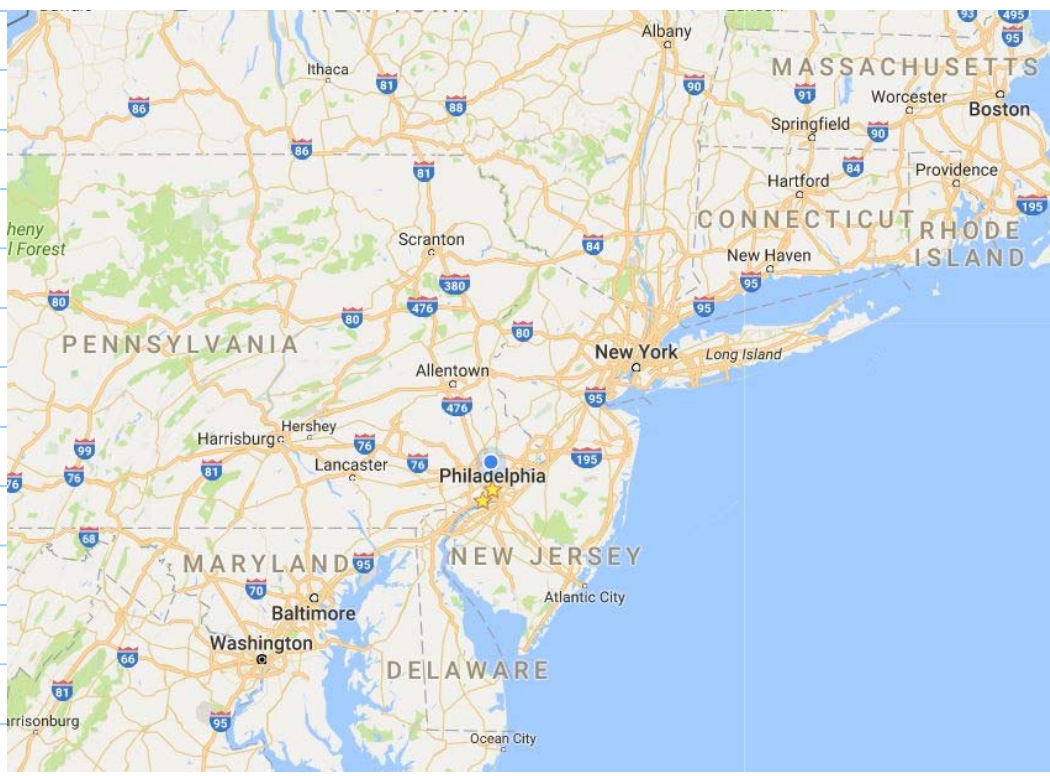
Implication :

$\leftarrow K = 1/R^2 \neq 0$

By Gauss' theorem, the sphere is not locally isometric to the plane

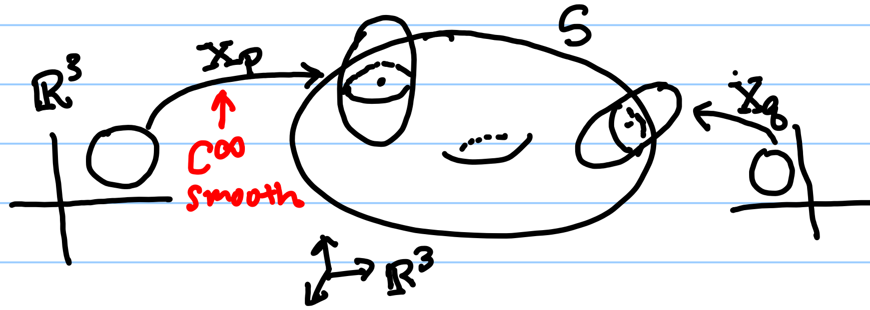
i.e.
it's impossible to build maps without distorting distance.

$K=0$



C^1 local isometry (optional)

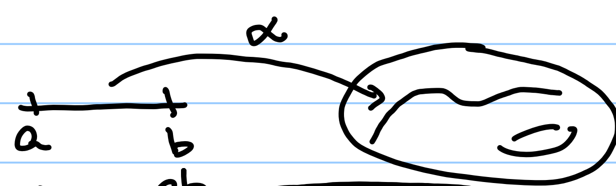
In this course, all maps are assumed to be C^∞ smooth. In particular, in our definition of regular surface, we assume that all the parameterizations are C^∞ smooth.



If we relax the smoothness requirement to C^k, we say S is a C^k regular surface.

Notice:

- the definition of curvatures only requires a C^2 regular surface
- the proof of Gauss' theorema egregium requires a C^3 regular surface
- the concept of length is, however, well-defined for any C^1 curve on a C^1 regular surface



$$\text{Length}(\alpha) = \underbrace{\int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle}\, dt}_{\text{$\alpha \in C^1$ is more than enough for this integral to be well-defined}}$$

Assume $\mathcal{M}$ is an abstract manifold. (We still assume that the change of coordinates maps are C^∞ smooth.)

Before (Lecture 7), we defined an embedding to be a C^∞ map from $\mathcal{M}$ to $\mathcal{N}$ to be an injective immersion so that $f: \mathcal{M} \to f(\mathcal{M})$ is also a homeomorphism according to the subspace topology of $f(\mathcal{M})$ in $\mathcal{N}$.

We can relax C^∞ to C^k $(k \geq 1)$ and call it a C^k embedding.

Ex: When a 2-manifold is C^k-embedded into $\mathbb{R}^n$, the resulted surface is a C^k regular surface.

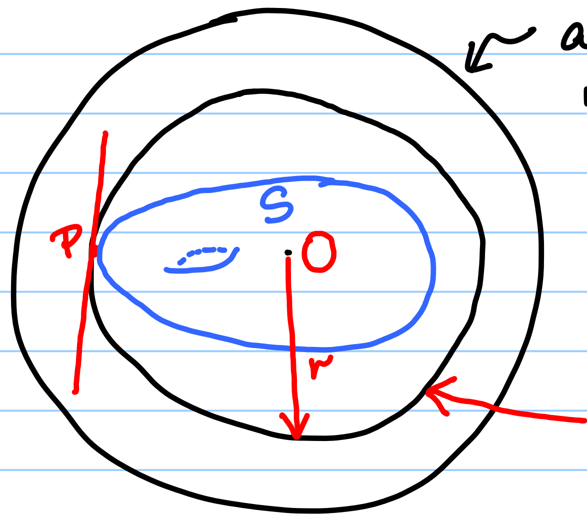
🚩 It is not possible to C^2 embed, say, the abstract sphere or torus into $\mathbb{R}^3$ so that the resulted C^2 regular surface has vanishing Gauss curvature everywhere.

Fact: A C^2 regular compact surface $S \subset \mathbb{R}^3$ has at least one elliptic point.

i.e. $K > 0$

<u>Sketch of proof</u>

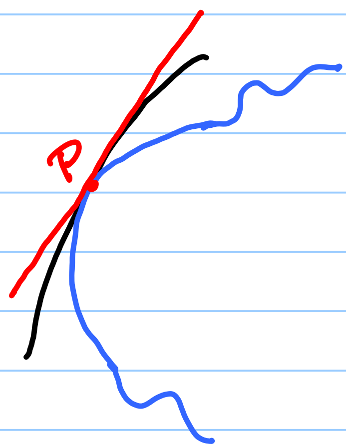
Compact $\Rightarrow$ bounded



any ball that contains S in its interior, centered at O, let

$r =$ infimum of the radii of all such balls

S must share a point with this "kissing ball" and S is on one side of the tangent plane at p



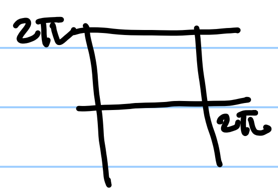
It is evident that

$$K_S(p) \geq K_{\substack{\text{kissing}\\ \text{sphere}}}(p) = \frac{1}{r^2} > 0$$

as wished. $\square$

(See Do Carmo Ch 5 for more details.)

So according to Gauss, it is impossible to construct a C^2 map $T : \mathbb{R}^2 \to \mathbb{R}^3$. s.t.



① $T(u+2\pi, v) = T(u)$ $\forall u, v$
$T(u, v+2\pi) = T(v)$,
and
Image (T) is C^2-regular torus

② T is a local isometry.

For if such a T exists, then according to Gauss' theorem, $S =$ Image (T) must have zero

Gauss curvature everywhere, which contradicts the fact we just established.

In the language of Riemannian manifold (Lecture 10), we say that it is impossible to C^2 embed the 'flat torus' into $\mathbb{R}^3$.

The Nash-Kuiper theorem, however, showed that if we relax C^2 to C^1, it is possible!

ANNALS OF MATHEMATICS
Vol. 60, No. 3, Nnvember, 1954
Printed in U.S.A.

1954

C^1 ISOMETRIC IMBEDDINGS

BY JOHN NASH

(Received February 26, 1954)
(Revised June 21, 1954)

Introduction

The question of whether or not in general a Riemannian manifold can be isometrically imbedded in Euclidean space has been open for some time. The local problem was discussed by Schlaefli [1] in 1873 and treated by Janet [2] and Cartan [3] in 1926 and 1927.

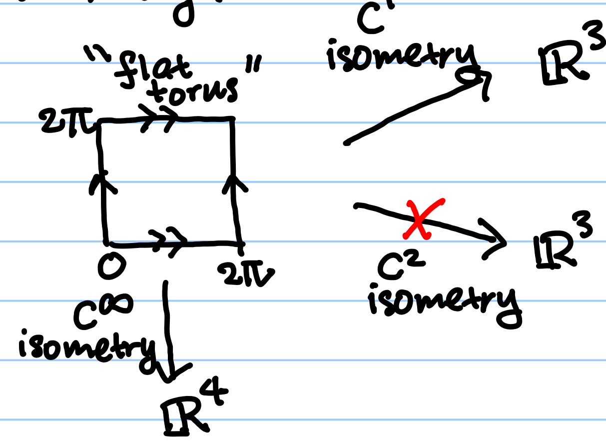
This question comes up in connection with the alternative extrinsic and intrinsic approaches to differential geometry. The historically older extrinsic attitude sees a manifold as imbedded in Euclidean space and its metric as derived from the metric of the surrounding space. The metric is considered to be given abstractly from the intrinsic viewpoint.

This intrinsic approach has seemed the more general, so long as there was no contravening evidence. Now it develops that the two attitudes are equally general, and any (positive) metric on a manifold can be realized by an appropriate imbedding in Euclidean space.

This paper is limited to the construction of C^1 isometric imbeddings. It turns out that the C^1 case is easier to treat and that surprisingly low dimensional Euclidean spaces can be used. A closed n-manifold always has C^1 isometric imbeddings in E^{2n}. But to get a C^3 imbedding of an n-manifold with C^3 metric I have (as of this writing) needed $1\frac{1}{2}n^2 + 5\frac{1}{2}n$ dimensions. One expects this number to be reduced, but it is clear that there will always be a sharp transition between the C^1 case and more differentiable imbeddings. At least $(n^2 + n)/2$ dimensions will be required beyond the C^1 case. This many dimensions were used in the analytic local theory.

John Nash

Summary:



Followup work of:
Gromov (1970s, 1980s)
Borrelli, Jabrane, Lazarus, Thibert (2012):

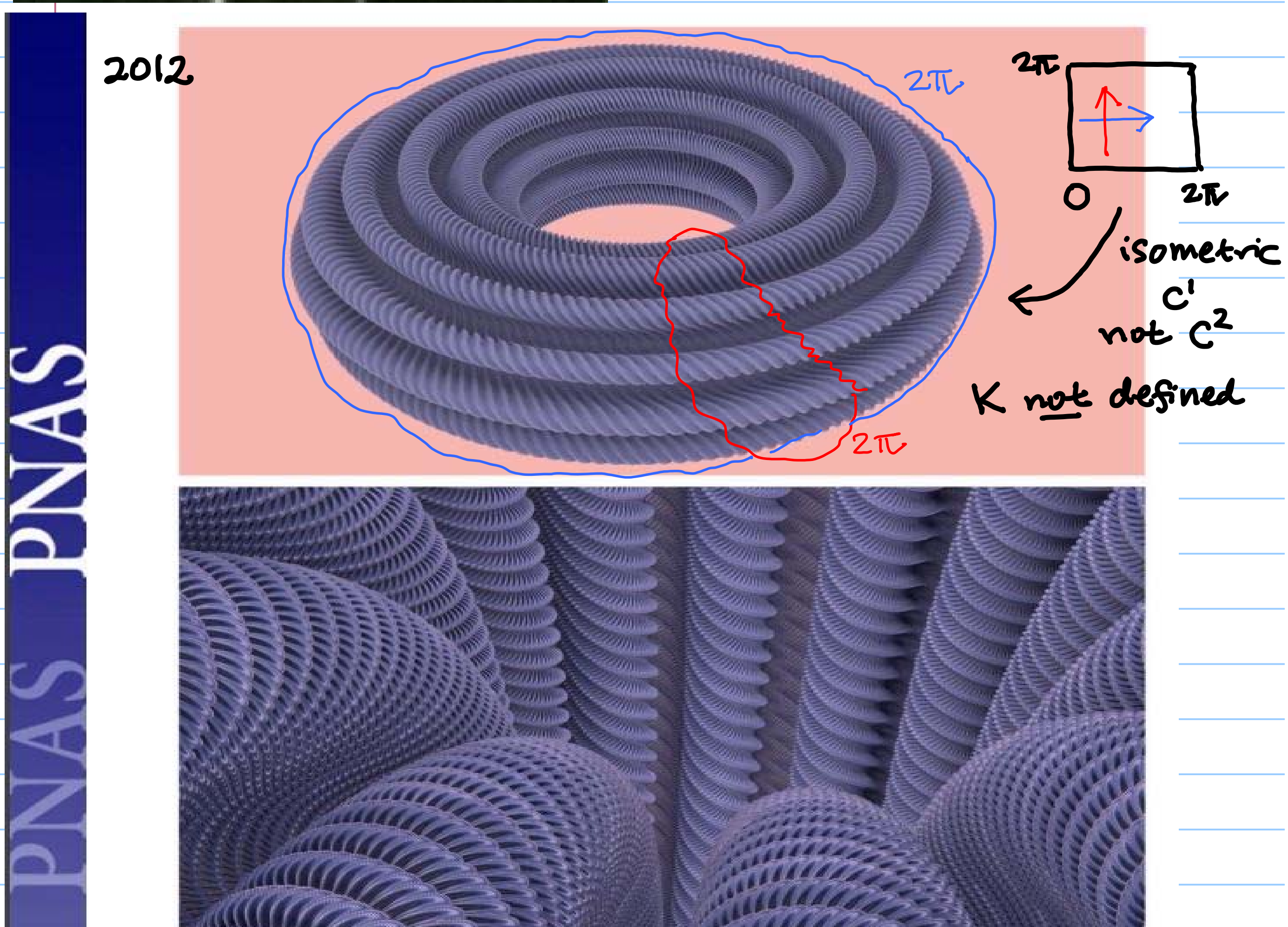


Fig. 3. The image of a square flat torus by a C^1 isometric map. Views are from the outside and from the inside.

The two big results in surface theory:

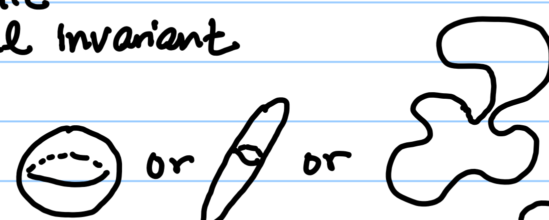
1. Perhaps you should ask: why do geometers care about this flat torus? Is there a "flat sphere", "flat 2-hole torus"?

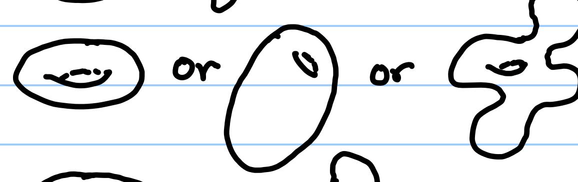
   It turns out there isn't, the reason is topological and the fact that the torus is the only topology (among all compact surfaces) with a flat metric has interesting consequences, accumulating into a big result called the

   uniformization theorem.

2. The Gauss-Bonnet theorem says

$$\int_M K\,dA = 2\pi \underbrace{\chi(M)}_{\substack{\text{Euler characteristic}\\ \text{of } M \text{, a topological invariant}}} \quad \text{for closed surfaces.}$$

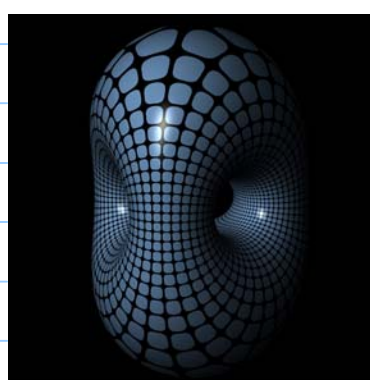
E.g. $\int_M K\,dA = 2$, $M =$ 

$\int_M K\,dA = 0$, $M =$ 

$\int_M K\,dA = -2$ $M =$ 

$$\boxed{\int K\,dA = 2-2g}$$

g: genus of an orientable surface



or