Math 538 Differential Geometry and Manifolds

$\boxed{\text{HW \#2}}$

Due: Monday October 12, 2020

1. In Lecture 2, I suggest to use properties (1'), (2'), (3'), together with the 'normalization' condition $D(e_1, \ldots, e_n) = 1$, as the 'defining axioms' of 'the signed volume of the parallelepiped with sides $v_1, \ldots, v_n$'. I give a proof, modulo that of a lemma, for the fact that these four axioms completely specify the value of $D(v_1, \ldots, v_n)$ for any $v_1, \ldots, v_n \in \mathbb{R}^n$, and that value coincides with what everyone in mathematics call the determinant.

   This homework problem is about filling in the proof of that lemma, and at the same time connects the exposition in Lecture 2 to what is usually seen in a linear algebra textbook.

   Recall: I like axioms (1')-(3') because they are geometrically meaningful. However, it is the multi-linearity and alternating properties that allow us to finally derive the formula for determinant.

   The property in axiom (2') is also called **alternating**.

   It is because of the latter situation that when you read a textbook on linear algebra, the defining axioms of determinant are usually the following (see, e.g., Hoffman and Kunze):

   Definition: $D(v_1, \ldots, v_n)$ is a determinant function if (i) D is n-linear, (ii) alternating, and (iii) $D(e_1, \ldots, e_n) = 1$.

   Prove: conditions (i) and (ii) hold $\Longleftrightarrow$ axioms (1')-(3') hold.

   In other words, the three axioms in my notes are equivalent to the 'alternating n-linear' condition that seems to come out of the blue.

2. In contrast to 'alternating multi-linear functions', i.e. k-linear functions $T : \overbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}^{k \text{ copies}} \to \mathbb{R}$ that satisfy

$$T(v_1, \ldots v_i, \ldots, v_j, \ldots v_k) = -T(v_1, \ldots v_j, \ldots, v_i, \ldots v_k), \quad \forall i \neq j,\ i, j \in \{1, \ldots, k\},$$

   'symmetric k-linear functions' are k-linear functions that satisfy

$$T(v_1, \ldots v_i, \ldots, v_j, \ldots v_k) = T(v_1, \ldots v_j, \ldots, v_i, \ldots v_k), \quad \forall i, j \in \{1, \ldots, k\}.$$

   You have seen that the determinant function is of the former type, with $k = n$. Symmetric k-linear functions arise naturally from the multivariate Taylor's expansion: if $f : \mathbb{R}^n \to \mathbb{R}$ is C^k smooth, then[1]

$$f(\mathbf{x_0} + \mathbf{h}) \approx f(\mathbf{x_0}) + \frac{d}{dt} f(\mathbf{x_0} + t\mathbf{h})|_{t=0} + \frac{1}{2!} \frac{d^2}{dt^2} f(\mathbf{x_0} + t\mathbf{h})|_{t=0} + \cdots + \frac{1}{k!} \frac{d^k}{dt^k} f(\mathbf{x_0} + t\mathbf{h})|_{t=0}.$$

[1] There is a clever trick here for deriving the multivariate Taylor expansion from the 1-D Taylor expansion.

An application of the chain rule (with some details omitted) shows that

$$\frac{1}{k!}\frac{d^k}{dt^k}f(\mathbf{x_0}+t\mathbf{h})|_{t=0} = \frac{1}{k!}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{\partial^k f(\mathbf{x_0})}{\partial x_{i_1}\cdots\partial x_{i_k}}h_{i_1}\cdots h_{i_k}, \quad \mathbf{h}=[h_1,\ldots,h_n]^T. \tag{1}$$

(i) Explain why the right-hand side of (1) can be re-expressed as $T_k(\mathbf{h},\ldots,\mathbf{h})$ for a certain symmetric k-linear function.

T_1 here has many names: the 'derivative of f', 'differential of f', 'jacobian (matrix) of f',[2] 'the local linear approximation of f', whereas T_2 is usually called the 'Hessian', or, quite confusingly, the 'Hessian matrix' of f.

T_k is usually called the 'k-th derivative of f'; make sure you understand that when $n>1$, T_k is not a scalar, but a symmetric k-linear function.

(ii) You may wonder why one bothers to write (1) using a function with k arguments, when ultimately we are only going to make the k arguments the same? An answer to this is that sometimes in analysis the perturbation vector $\mathbf{h}$ can be decomposed into a sum of easier pieces, e.g. $\mathbf{h}=\mathbf{h}_1+\mathbf{h}_2+\mathbf{h}_3$, and then the multi-linearity of T_k is instrumental for splitting the error term $T_k(\mathbf{h},\ldots,\mathbf{h})$ into a sum of (3^k, in this case) easier pieces $T_k(\mathbf{h},\ldots,\mathbf{h}) = \sum_{i_1,\ldots,i_k} T_k(\mathbf{h}_{i_1},\ldots,\mathbf{h}_{i_k})$. Now the arguments are not always the same!

But there is yet another interesting answer. Surprisingly, knowing just the 'polarized version' of T_k is as good as knowing the whole T_k. This may suggest that if you care about $T_k(v,\ldots,v)$, you may as well also care about $T_k(v,w,\ldots)$.[3]

For $k=2$ (the most familiar case), prove the polarization identity:

$$T_2(u,v) = \frac{1}{2}(T_2(u+v,u+v) - T_2(u,u) - T_2(v,v)).$$

Next, figure out a polarization identity for $k=3$. Hint: perhaps we should mimic the $k=2$ case and try to express $T_3(u,v,w)$ in terms of $T_3(u+v+w,u+v+w,u+v+w)$, $T_3(u,u,u)$, $T_3(v,v,v)$ and $T_3(w,w,w)$? Doesn't seem to work... What to do?

(iii) Explain: While the polarization $T(v,\ldots,v)$ means everything to a symmetric k-linear map, it means nothing for an alternating k-linear maps T.

Hint: What is $T(v,\ldots,v)$ for a symmetry k-linear map?

[2]In the modern literature, the term 'jacobian' is often used to refer to the determinant of the derivative of a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, which measures locally how much the nonlinear map distorts volume, and would only make sense when the co-domain has the same dimension as the domain. See `https://upload.wikimedia.org/wikipedia/en/9/96/Jacobian_determinant_and_distortion.svg` for a nice illustration in the case of $n=2$.

[3]At the beginning of Lecture 2, we find a use of the $k=2$ polarization identity. The symmetric 2-linear function there is the familiar Euclidean inner-product, which seems to have nothing to do with any Taylor's expansion.