lecture 6 : Tangent Spaces and derivatives

Note Title 2/12/2017

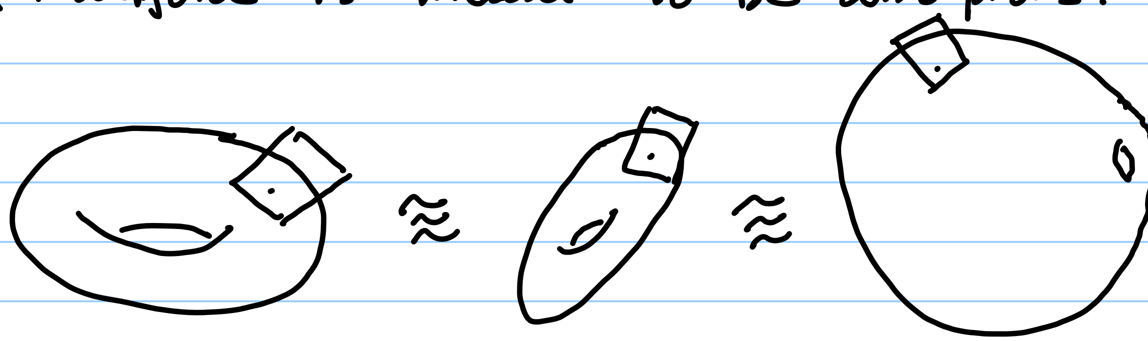
Without an ambient space, we can no longer define

$$T_pM := \{\alpha'(0) : \underset{\alpha(0)=p}{\alpha : (-\varepsilon, \varepsilon) \to M}\},$$

well, unless we have a new way to make sense out of a "velocity vector".

The concept of 'speed' (distance / time) makes no sense if there is no notion of distance on the manifold.

A manifold is meant to be amorphous.



But : there is an intimate relation between a surface with its tangent spaces. When we 'abstract away' the ambient space from a surface, maybe we can define tangent spaces to be some sort of abstract objects that encode that 'intimacy'.

Def : $C^\infty(M) = \{ f : M \to \mathbb{R} : f \text{ smooth} \}$.

Def : For a smooth $\alpha : I \to M$, $t_0 \in I$, $\alpha(t_0) = p$, we define

some open interval of $\mathbb{R}$

$$\alpha'(t_0) : C^\infty(M) \to \mathbb{R}$$

by

$$\alpha'(t_0) f = \frac{d}{dt} (\underbrace{f \circ \alpha}) (t_0).$$

Note : If $M = \mathbb{R}^n$, $\alpha'(0) = v$, then

$$\alpha'(0) f = \frac{d}{dt}(f \circ \alpha) = Df\big|_{\alpha(0)} \cdot \underbrace{\alpha'(0)}_{= v} = D_v f(p)$$

only makes sense in this case

is the directional derivative of f at p in the direction v.

$$D_v f(p) = \underset{1 \times n}{Df(p)} \cdot \underset{n \times 1}{v} = \underset{1 \times n}{v^T} \underset{n \times 1}{Df(p)^T}$$

Back to the abstract manifold setting,

Def : $T_p M = \{ \text{all } \alpha'(0) : \alpha(0) = p \}$ is called the tangent space to M at p.

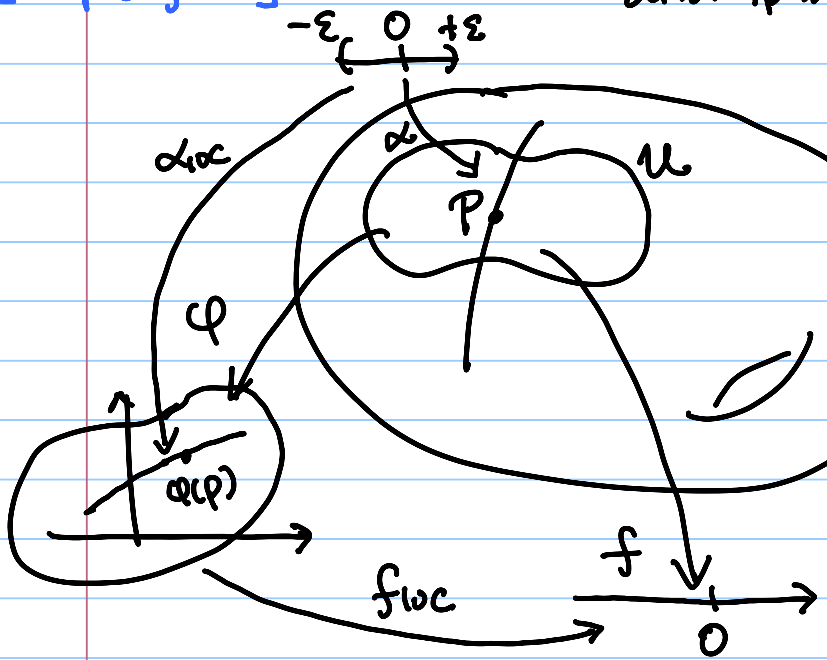
It would be fitting if the (abstract) tangent space of an (abstract) manifold is an (abstract) vector space.

Basis

Theorem: T_pM is a vector space over the reals, and

["coordinate proof"]

$$\dim T_pM = \dim M.$$



$$\alpha_{loc}(t) = \begin{bmatrix} \alpha_{loc}^1(t) \\ \vdots \\ \alpha_{loc}^n(t) \end{bmatrix} \in \mathbb{R}^n$$

$$f_{loc}(x^1, \dots, x^n) \in \mathbb{R}^1$$

Let $(\mathcal{U}, \varphi)$ be a chart around p.

$$\alpha'(0)f = \frac{d}{dt}[f \circ \alpha](0) \quad \text{(by definition)}$$

$$= \frac{d}{dt}[\underbrace{f \circ \varphi^{-1}}_{f_{loc}} \circ \underbrace{\varphi \circ \alpha}_{\alpha_{loc}}](0) \quad \text{(representing } f \text{ and } \alpha \text{ in local coordinates)}$$

$$= [d f_{loc}(\varphi(p))] \cdot [d\alpha_{loc}(0)] \quad \text{(chain rule)}$$

$$= \sum_{i=1}^{n} \frac{d\alpha_{loc}^i}{dt}(0) \cdot \frac{\partial f_{loc}}{\partial x^i}(\varphi(p)) \qquad (\star)$$

This (familar) calculation almost proves the theorem by establishing a basis for T_pM. To iron out the details, we need some notations.

Write:

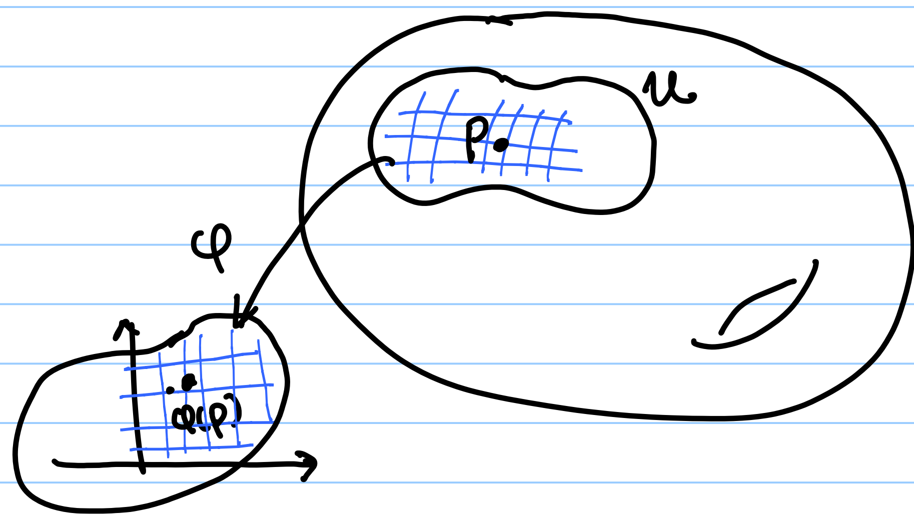
$x^i := \pi^i \circ \varphi$ (the ith coordinate function)

$\frac{\partial}{\partial x^i}\Big|_p : C^\infty(M) \to \mathbb{R}$ is defined by

$\frac{\partial}{\partial x^i}\Big|_p (f) :=$ the ordinary x^i-partial derivative of $f_{loc} = f \circ \varphi^{-1}$

Is this $\frac{\partial}{\partial x^i}\Big|_p$ an element of T_pM ?

Yes, because it is the velocity vector of the i-th coordinate-curve.



Write $\varphi(p) = (x^1_*, \dots x^i_*, \dots, x^n_*)$

The i-th coordinate curve, denoted for now by γ_i, is

$$t \overset{\gamma_i}{\mapsto} \varphi^{-1}(x^1_*, \dots, x^i_* + t, \dots, x^n_*)$$

Easy to check : $\gamma_i'(0) = \frac{\partial}{\partial x^i}\Big|_p$.

so $\mathcal{B} := \{ \frac{\partial}{\partial x^1}|_p, \cdots, \frac{\partial}{\partial x^n}|_p \} \subset T_p M$.

The earlier calculation (*) shows that

$$\alpha'(0) = \sum_{i=1}^{n} (\alpha^i_{loc})'(0) \frac{\partial}{\partial x^i}|_p$$

i.e. every element of $T_p M$ is a linear combination of the elements in $\mathcal{B}$.

This shows $T_p M$ is a vector space over $\mathbb{R}$.

To show that the spanning set $\mathcal{B}$ is actually a basis, we just need to argue that $\mathcal{B}$ is a linearly independent set.

Note that $\frac{\partial}{\partial x^i}|_p (x^j) = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Then

$$\sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}|_p = 0$$

$$\Rightarrow \sum_{i=0}^{n} a^i \frac{\partial}{\partial x^i}|_p (x^j) = 0 \quad \forall j = 1, \cdots, n$$

$$\Rightarrow a^1 = \cdots = a^n = 0,$$

i.e. $\mathcal{B}$ is a linearly independent set.

And $\dim T_p M = n = \dim M$. ∎

Corollary of the proof: For two curves s.t. $\alpha(0) = \tilde{\alpha}(0) = p$, $\alpha'(0) = \tilde{\alpha}'(0) \Leftrightarrow \alpha_{loc}(0) = \tilde{\alpha}_{loc}(0)$.

Note : There is no canonical choice of basis for T_pM. But every chart around p gives rise to a basis for T_pM.

(We had seen the exact same statement in the study of regular surfaces.)

Change of basis matrix :

If (U, φ) and (V, ψ) are two charts around $p \in M$, and

$$x^1, \dots, x^n \quad , \quad y^1, \dots, y^n$$

are their coordinate functions, respectively.

(Formally, $x^i = \pi^i \circ \varphi$, $y^i = \pi^i \circ \psi$.)

Then $\mathcal{B}_1 = \{ \partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p \}$

$\mathcal{B}_2 = \{ \partial/\partial y^1|_p \dots, \partial/\partial y^n|_p \}$ are bases of T_pM.

If $$\frac{\partial}{\partial x^i}\Big|_p = \sum_{j=1}^{n} A_{ij} \frac{\partial}{\partial y^j}\Big|_p$$ The change of basis matrix

then $$\frac{\partial}{\partial x^i}\Big|_p (y^k) = \sum_{j=1}^{n} A_{ij} \underbrace{\frac{\partial}{\partial y^j}\Big|_p (y^k)}_{\delta_{jk}} = A_{ik}$$

So $$[A_{ik}]_{i,k} = \frac{\partial}{\partial x^i}\Big|_p (y^k) = d(\varphi \circ \psi^{-1})\Big|_{\psi(p)}$$

i.e. The differential of the (nonlinear) change of coordinate map gives the (linear) change of basis map.

An equivalent definition of T_pM.

In calculus, for a fixed vector $v \in \mathbb{R}^n$, the directional derivative operator

$$D_v$$

is a linear map on the vector space of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

$$D_v(af + bg) = aD_v(f) + bD_v(g).$$

Moreover, D_v satisfies the Leibniz product rule:

$$D_v(f \cdot g) = \left[\frac{\partial f \cdot g}{\partial x_1} \cdots \frac{\partial f g}{\partial x_n}\right] \cdot v$$

$$= \cdots$$

(usual product rule)

$$= f(p) D_v g(p) + D_v f(p) \cdot g(p).$$

(I) Proposition: The abstract definition of $\alpha'(t_0)$ for manifolds also satisfies linearity and the Leibniz product rule.

Conversely (and less obviously):

(II) Proposition: Every map from $C^\infty(M) \to \mathbb{R}$ that satisfies linearity and the Leibniz product rule is a $\alpha'(t_0)$ for some curve α.

Many authors like to first define T_pM as the set of all linear and Leibnizian functions $C^\infty(M) \to \mathbb{R}$, partly because under this definition it has an obvious real vector space structure defined by:

$$(a v_p + b w_p)(f) = a v_p(f) + b w_p(f). \quad \text{—} \circledast$$

Also, this alternate characterization of tangent vectors is slightly easier to work with (e.g. in setting up the concept of derivatives and the Lie bracket of vector fields.)

Ex: Prove that {all linear and Leibnizian functions $C^\infty \to \mathbb{R}$} is closed under the linear combination defined by $\circledast$.

Proof of (I):

(Linearity) $\alpha'(t_0)(af + bg)$

$$\stackrel{\text{def}}{=} [(af + bg) \circ \alpha]'(t_0)$$

$$= \frac{d}{dt}\left[a f(\alpha(t)) + b g(\alpha(t))\right]\Big|_{t=t_0}$$

$$\overset{\text{linearity of differentiation}}{=} a(f\circ\alpha)'(t_0) + b(g\circ\alpha)'(t_0)$$

$$= a\cdot\alpha'(t_0)(f) + b\cdot\alpha'(t_0)(g)$$

(Leibniz) $\alpha'(t_0)(fg)$

$$= ((fg)\circ\alpha)'(t_0)$$

$$= \frac{d}{dt}\left[f(\alpha(t))\cdot g(\alpha(t))\right]\Big|_{t=t_0}$$

$$\overset{\text{usual product rule}}{=} f(\alpha(t_0))\frac{d}{dt}g(\alpha(t))\Big|_{t=t_0} + \frac{d}{dt}f(\alpha(t))\Big|_{t=t_0}\cdot g(\alpha(t_0))$$

$$= f(p)\cdot\alpha'(t_0)(g) + \alpha'(t_0)(f)\cdot g(p).$$

$\blacksquare$

The proof of (II) is kind of long and for the interest of time we skip it.

From now on, we agree (with some regret) that:

$$T_pM = \{\alpha'(0) : C^\infty(M)\to\mathbb{R} \mid \alpha:(-\varepsilon,\varepsilon)\to M,\ \alpha \text{ smooth},\ \alpha(0)=p\}$$

$$= \{v_p : C^\infty(M)\to\mathbb{R} \mid v_p \text{ is linear and Leibnizian @ } p\}$$

more intuitive

allows for nicer coordinate-free proofs in some cases.

Def: A vector field on a manifold M is a map V that assigns to each $p \in M$ a tangent vector $V(p) = V_p \in T_p(M)$.

If $(\mathcal{U}, \varphi)$ is a chart with coordinate functions $x^1, \ldots, x^n$, then by the basis theorem

$$V(p) = \sum_i \underbrace{V_p(x^i)}_{\text{think of this as a function of } p} \partial/\partial x^i\big|_p$$

The real-valued functions

$$V^i : \mathcal{U} \to \mathbb{R}, \quad V^i(p) := V_p(x^i), \quad i = 1, \cdots, n$$

are the components of V relative to $(\mathcal{U}, \varphi)$.

The vector field is called smooth (resp. continuous) if its components are smooth (resp. continuous) for all charts in some atlas for M.

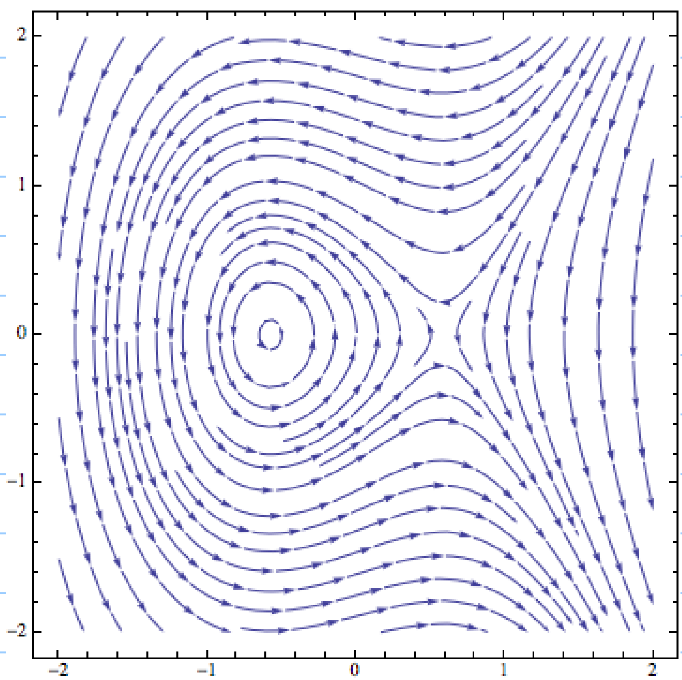
Obvious question:
if $(\mathcal{U}, \varphi)$, (V, ψ) are two charts with $\mathcal{U} \cap V \neq \phi$, can the component functions of V smooth in one chart but not in the other?

Ex: work out a formula for how the components of V change under a change of coordinates. Use it to answer the above question.

At this point, we can define (and prove the basic properties) of :

- flow of a vector field
- Lie bracket of vector fields

which are very fundamental objects in both physics and geometry. Before we delve into these topics, perphaps you should recall the basic concept of 'flow' from vector Calculus/ ODE :



Given velocity vector field

$$V : \underbrace{U \overset{\text{open}}{\subset} \mathbb{R}^n}_{\substack{\uparrow \\ \text{position}}} \to \underbrace{\mathbb{R}^n}_{\substack{\uparrow \\ \text{velocity} \\ \text{vector}}}$$

its 'flow' map

$$\phi(t, x)$$

satisfies

$$\frac{\partial}{\partial t} \phi(t, x) = V(\phi(t, x))$$

$$\phi(0, x) = x.$$

Next, it would be very helpful if you try to answer the following question :

Does it make sense to talk about "the flow of a vector field" in the abstract manifold setting? Yes

[Hint: comment of pg. 1]

Before we do these I want to first go through some basic materials related to <u>submanifolds</u> (not too connected to the vector field materials)

Regardless, we need to first extend the basic results of <u>derivatives</u> from calculus to manifolds. We do so in the rest of this lecture.

Suppose X, Y are smooth manifolds with dimensions n and m, respectively. Let $F: X \to Y$ a smooth map. , $p \in X$.

$F_{*p} : T_pX \to T_{F(p)}Y$ is defined by
(or dF_p)

$$F_{*p}(\alpha'(t_0)) = (F \circ \alpha)'(t_0).$$

Here we have the same well-definedness question as we had in the case of regular surfaces: If $\alpha, \tilde{\alpha}$ are two curves with $\alpha'(t_0) = \tilde{\alpha}'(t_0)$, is it always true that

$$F_{*p}(\alpha'(t_0)) = F_{*p}(\tilde{\alpha}'(t_0))?$$

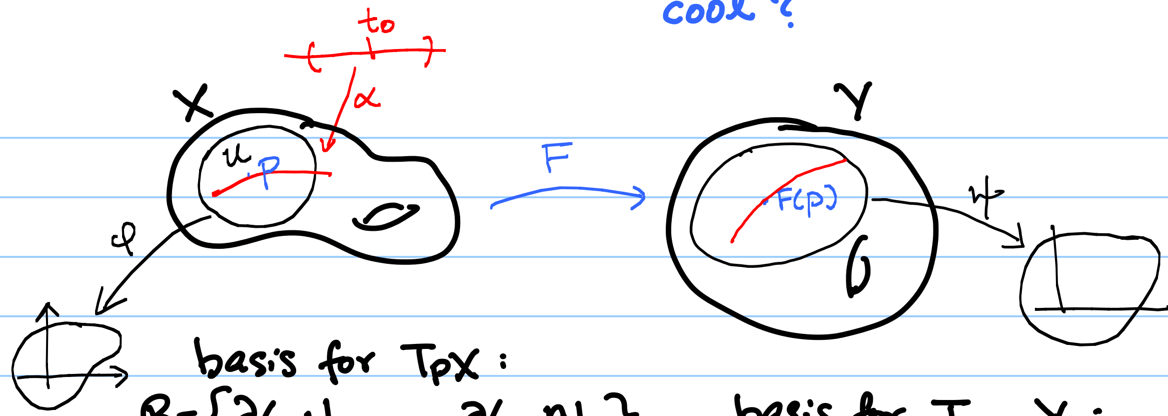
We use the basis theorem to "kill two birds with one stone":

Proposition: F_{*p} is well-defined and is a linear map.

<u>Proof</u>: choose charts $(\mathcal{U}, \varphi)$ around p in X, (V, ψ) around $F(p)$ in Y.

["coordinate proof"]

Q: Are coordinate proofs not cool?



basis for T_pX:
$\mathcal{B} = \{ \partial/\partial x^1|_p, \dots, \partial/\partial x^n|_p \}$

basis for $T_{F(p)}Y$:
$\mathcal{B}' = \{ \partial/\partial y^1|_{F(p)}, \dots, \partial/\partial y^m|_{F(p)} \}$

$(F \circ \alpha)'(t_0) \in T_{F(p)}(Y)$

$\{$ $(\psi \circ F \circ \alpha)'(t_0) \in \mathbb{R}^m$ is the vector of coefficients of $(\alpha \circ F)'(t_0)$ in the basis $\mathcal{B}'$.

$$\underbrace{(\psi \circ F \circ \alpha)}_{(F\circ\alpha)_{loc}} = \underbrace{(\psi \circ F \circ \varphi^{-1})}_{F_{loc}} \circ \underbrace{(\varphi \circ \alpha)}_{\alpha_{loc}}$$

so by the chain rule:

$$\underset{m \times 1}{(F\circ\alpha)'_{loc}(t_0)} = \underset{m \times n}{\left[dF_{loc}|_{\varphi(p)} \right]} \cdot \underset{n \times 1}{\alpha'_{loc}(t_0)}$$

This shows

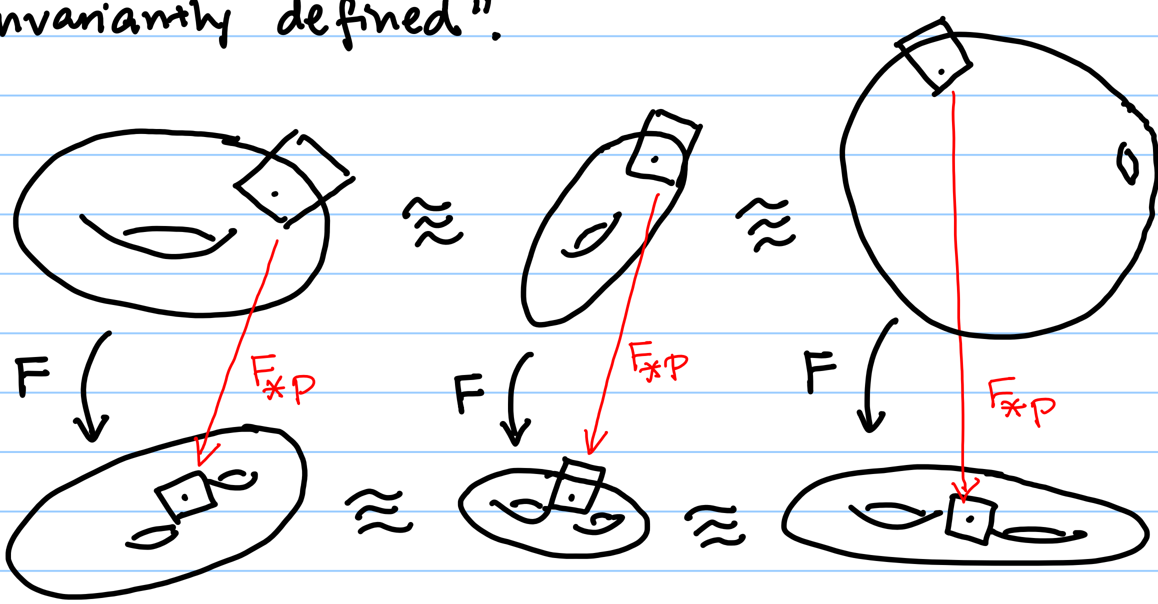
① $F_{*p}(\alpha'(t_0))$ depends only on the vector $\alpha'_{loc}(t_0)$ in $\mathbb{R}^n$.

② In the bases $\mathcal{B}$ and $\mathcal{B}'$, the map
$$F_{*p} : T_pX \to T_{F(p)}Y$$
is represented by the matrix $dF_{loc}|_{\varphi(p)}$.

Of course ② also shows the map is linear.

Note: the derivative of F at p is just the standard derivative of F when represented in local coordinates.

But I hope you try to convince yourself (at a higher level) why the derivative F_{*p} can be "invariantly defined".



Theorem (Chain rule in manifold setting)

Let $F: X \to Y$, $G: Y \to Z$ be smooth maps of diff. manifolds. Let $p \in X$.

Then $G \circ F: X \to Z$ is smooth and

$$(G \circ F)_{*p} = G_{*F(p)} \circ F_{*p}.$$

$$\left[\text{or } d(G \circ F)|_p = dG|_{F(p)} \circ dF|_p\right]$$

Proof: Let (U, φ), (V, ψ), (W, ϕ) be charts around $p \in X$, $F(p) \in Y$, $G(F(p)) \in Z$, respectively.

["coordinate proof"]

Denote by $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ the corresponding induced bases for T_pX, $T_{F(p)}Y$, $T_{G(F(p))}Z$.

Then

$$[(G\circ F)_{*p}]_{\mathcal{B}_1,\mathcal{B}_3} = d(\phi\circ G\circ\psi^{-1}\circ\psi\circ F\circ\varphi^{-1})|_{\varphi(p)}$$

$$= d(\phi\circ G\circ\psi^{-1})|_{\psi(F(p))}\cdot d(\psi\circ F\circ\varphi^{-1})|_{\varphi(p)}$$

$$= [G_{*F(p)}]_{\mathcal{B}_2,\mathcal{B}_3}\cdot[F_{*p}]_{\mathcal{B}_1,\mathcal{B}_2}$$

$$= [G_{*F(p)}\cdot F_{*p}]_{\mathcal{B}_1,\mathcal{B}_3}$$

Since $(G\circ F)_{*p}$ and $G_{*F(p)}\circ F_{*p}$ have the same matrix representation in some bases, they must be the same linear map from T_pX to $T_{F(G(p))}Z$. ∎

<u>Note</u>: A coordinate-free proof is possible, but it requires us to use an equivalent definition of F_{*p} based on the alternate (equivalent) definition for tangent spaces:

Recall: $T_pM = \{v_p : C^\infty(M)\to\mathbb{R} : \text{linear + Leibnizian at } p\}$

$F: X\to Y$, $F_{*p}(v_p)$ can be equivalently re-defined as:

$$F_{*p}(v_p)(g) := v_p(g\circ F).$$

Ex.: check that this definition indeed defines a linear and Leibnizian function on $C^\infty(Y)$.

check that this defintion of F_{*p} is equivalent to the earlier one.

Explain why this definition of F_{*p} allows for a coordinate-free proof of the chain rule in the manifold setting.

Next: the manifold version chain rule

+

the calculus version of inverse fcn. thm

gives

the manifold version of inverse fcn. thm.

Thm (IFT, manifold version)

Let $F: X \to Y$ be a smooth map of differentiable manifolds, $p \in X$.

Then $F_{*p}: T_p(X) \to T_{F(p)}(Y)$ is a linear isomorphism iff $F: X \to Y$ is a local diffeomorphism near p (i.e. $\exists$ open nbhds $\mathcal{U}$ of p and V of $F(p)$ s.t. $F|_{\mathcal{U}}: \mathcal{U} \to V$ is a diffeomorphism.

($\Leftarrow$) use the manifold version chain rule

($\Rightarrow$) set up coordinates and use the calculus IFT:

<u>IFT (Calulus version)</u>:

Let

$$G : A \overset{\text{open}}{\subset} \mathbb{R}^n \to \mathbb{R}^n \text{ be a smooth map.}$$

Suppose $dG(a)$ is non-singular for $a \in A$.

Then

$\exists$ open sets B_a, $B_{G(a)} \subset \mathbb{R}^n$

s.t. $a \in B_a$, $G(a) \in B_{G(a)}$

and

$G|_{B_a} : B_a \to B_{G(a)}$ is a smooth bijection and has a smooth inverse

$$(G|_{B_a})^{-1} : B_{G(a)} \to B_a.$$

<u>Ex</u>: Fill in the details. In the ($\Leftarrow$) proof, you should establish

$$(F_{*p})^{-1} = (F^{-1})_{*F(p)}.$$

from the chain rule.

<u>Corollary</u>: If two <u>smooth</u> manifolds are diffeomorphic, then they have the same dimension.

🚩 The proof of this "obvious" result depends crucially on the differentiable structures. The corresponding statement for topological manifolds (homeomorphic $\Rightarrow$ same dimension) is true but much harder to prove.

[What tool is need to show $\nexists\ \mathbb{R}^m \underset{F^{-1}}{\overset{F}{\rightleftarrows}} \mathbb{R}^n$?
F, F^{-1} continuous]

To get ready for the study of submanifolds, we must recall that the concept of <u>rank</u> of a linear map

$$L : V \to W$$

'abstract' linear map

'abstract' vector spaces (over $\mathbb{R}$)

$$\text{rank}(L) = \dim(L(V))$$
$$\text{nullity}(L) = \dim(\text{null}(L))$$

can be "invariantly defined" in the sense that it has nothing to do with any choice of basis in V or W. Equivalently, the <u>matrix rank</u> of:

$[L]_{B_V, B_W}$ = matrix representation of L in the basis B_V of V and basis B_W of W

is invariant under change of bases.

So, for any smooth map $F : X \to Y$

smooth manifolds

we can talk about the <u>rank</u> (and <u>nullity</u>) of

$$F_{*p} : T_p X \to T_{F(p)} Y.$$

The maps F with <u>constant rank</u>
i.e.

$\operatorname{rank}(F_{*p})$ is the same for all $p \in X$,

are of interests in the study of submanifold.

Recall the map that we use to prove $SO(3)$ is a regular surface in Lecture 4:

$$\mathbb{R}^{3\times 3} \ni A \xrightarrow{F} \begin{bmatrix} \ddots & & \vdots \\ & A^TA & \vdots \\ & & \ddots \end{bmatrix} \in \mathbb{R}^6$$

$$F([A_1, A_2, A_3]) = [\langle A_i, A_j \rangle]_{1 \le i \le j \le 3}$$

It does not have constant rank on the whole $\mathbb{R}^{3\times 3}$:

e.g. $$dF|_0 = \overset{6\times 9}{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}$$ has rank 0

But we showed in Lecture 4 that:

$dF|_A$ has constant (full) rank 6,
$$\forall A \in SO(3) = F^{-1}\left(\begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}\right).$$

And this very fact is the key to establish that $SO(3)$ is a regular surface. Recall also that the complete proof relies on an unproved theorem which we will prove in earnest in the next lecture.