Week 7

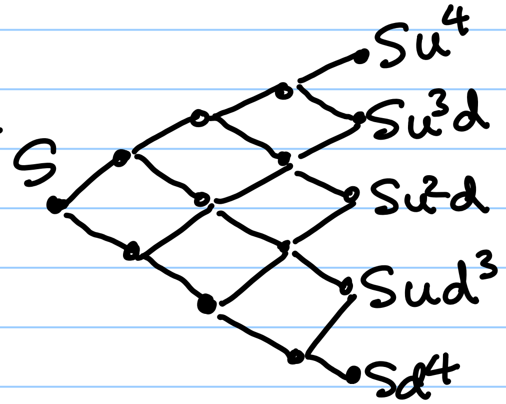
Note Title 5/11/2015

Derivations of Black-Scholes and the Greeks.

First, notice from the binomial lattice model:

e.g. $n=4$

$n\Delta t = T$



$\rightarrow C_{uuuu} = \max(Su^4 - K, 0)$

$\rightarrow C_{uuud} = \max(Su^3d - K, 0)$

$\rightarrow C_{uudd} = \max(Su^2d^2 - K, 0)$

$\rightarrow C_{uddd} = \max(Sud^3 - K, 0)$

$\rightarrow C_{dddd} = \max(Sd^4 - K, 0)$

$$R = e^{r\Delta t}, \quad u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}$$

$$\tilde{p} = \frac{R-d}{u-d} \qquad \text{write } \tilde{q} = 1 - \tilde{p}$$

By <u>linearity</u> of averaging,

$$C = R^{-4} \times (\text{a linear combination of } C_{uuuu}, C_{uuud}, C_{uudd}, C_{uddd}, C_{dddd})$$

OR

$$= R^{-4} \times \begin{bmatrix} \tilde{p} & \tilde{q} \end{bmatrix} \begin{bmatrix} \tilde{p} & \tilde{q} & 0 \\ 0 & \tilde{p} & \tilde{q} \end{bmatrix} \begin{bmatrix} \tilde{p} & \tilde{q} & 0 & 0 \\ 0 & \tilde{p} & \tilde{q} & 0 \\ 0 & 0 & \tilde{p} & \tilde{q} \end{bmatrix} \begin{bmatrix} \tilde{p} & \tilde{q} & 0 & 0 & 0 \\ 0 & \tilde{p} & \tilde{q} & 0 & 0 \\ 0 & 0 & \tilde{p} & \tilde{q} & 0 \\ 0 & 0 & 0 & \tilde{p} & \tilde{q} \end{bmatrix} \begin{bmatrix} C_{uuuu} \\ C_{uuud} \\ C_{uudd} \\ C_{uddd} \\ C_{dddd} \end{bmatrix}$$

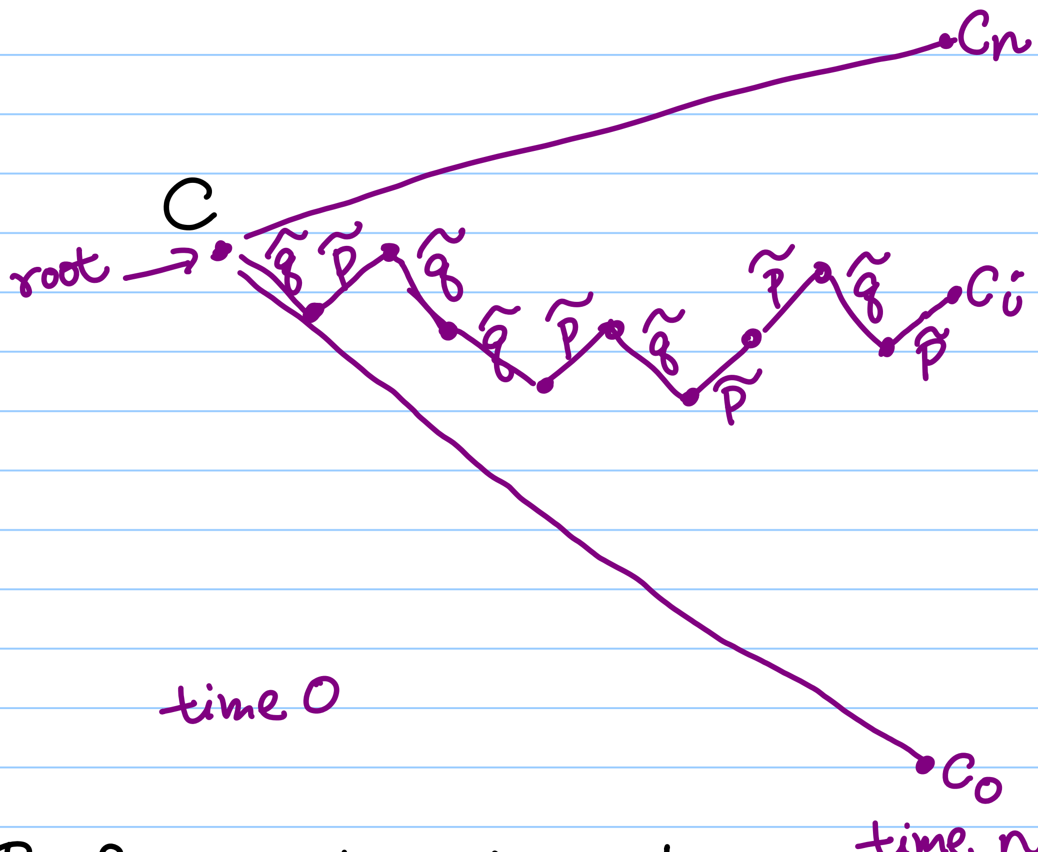
$$= R^{-4} \sum_{i=0}^{4} \left(\; ? \; \right) \max(Su^{4-i}d^i - K, 0)$$

$$R^{-4} = \left(e^{r\frac{T}{4}}\right)^{-4} = e^{-rT}$$

$\binom{4}{i}\tilde{p}^{4-i}\tilde{q}^{i}$ why?

For a general n,

$$C = e^{-rT} \sum_{i=0}^{n} \binom{n}{i} \tilde{p}^{i} \tilde{q}^{n-i} \overbrace{\max(Su^{i}d^{n-i} - K, 0)}^{\text{call it } C_i \text{ for now (think: } i \text{ up's } n-i \text{ down's)}} \qquad (\star)$$



Proof: In the pricing algorithm, the term C_i contributes exactly one term to the final value of C for each distinct path from the root to the leave associated with C_i.
The value of any such term is:

$$\tilde{p}^{i} \tilde{q}^{n-i}$$

and there are $\binom{n}{i}$ such paths.

Hence the formula $(\star)$. $\square$

Another interpretation of the formula (✡):

$$C = e^{-rT} E_{RN}(\max(S-K, 0))$$

the 'risk-neutral' probability measure is a binomial distribution $bin(n, \tilde{p})$ on the 'up-down' space

$$P[\, i \text{ up's and } (n-i) \text{ down's}] = \binom{n}{i} \tilde{p}^{i} (1-\tilde{p})^{n-i}.$$

($\tilde{p}$ is the one period risk-neutral probability.)

or $\boxed{P[S_T = S_0 u^i d^{n-i}] = \binom{n}{i} \tilde{p}^{i} (1-\tilde{p})^{n-i}}$

(not the "true" distribution of the stock, but the "risk-neutral distribution")

Derivation of Black-Scholes.

Hard part!

As $n \to \infty$, the corresponding risk-neutral distribution is:

$$S_T = S_0 e^{(r-q-\sigma^2/2)T + \sigma\sqrt{T}\underset{N(0,1)}{Z}} \quad \text{— (✡)}$$

(This is the geometric Brownian motion in Week 5, with 'ν' replaced by '$r - q - \frac{1}{2}\sigma^2$'.)

The no-arbitrage prices for an European call and put option are:

$$C(0) = e^{-rT} E_{RN}[\max(S(T)-K, 0)] \quad \text{— (✡✡)}$$

$$P(0) = e^{-rT} E_{RN}[\max(-S(T)+K, 0)]$$

where the expected value E_{RN} is based on the risk-neutral distribution.

Easier part (a basic probability exercise):

Let's calculate $C(0)$ based on (*) and (**):

$$S(T) \geq K \overset{(*)}{\Longleftrightarrow} S_0 e^{(r-q-\sigma^2/2)T + \sigma\sqrt{T} Z} \geq K$$

$$\Longleftrightarrow Z \geq \underbrace{\left[\ln\left(\tfrac{K}{S_0}\right) - (r-q-\sigma^2/2)\right]/\sigma\sqrt{T}}_{-d_2 \text{ (from week 6 with } t=0)}$$

So, $$C(0) = e^{-rT} \int_{-d_2}^{\infty} \left(S_0 e^{(r-q-\sigma^2/2)T + \sigma\sqrt{T}x} - K\right) \underbrace{\frac{e^{-x^2/2}}{\sqrt{2\pi}}}_{\text{pdf of } N(0,1)} dx$$

$$= \underbrace{\frac{S_0 e^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{(r-q-\frac{\sigma^2}{2})T + \sigma\sqrt{T}x - \frac{x^2}{2}} dx}_{} - \underbrace{\frac{Ke^{-rT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-x^2/2} dx}_{= Ke^{-rT}(1-N(-d_2)) = Ke^{-rT}N(d_2)}$$

$$\| \quad \frac{S_0 e^{-qT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(x^2 - 2\sigma\sqrt{T}x + \sigma^2 T)} dx$$

$$\| \quad \frac{S_0 e^{-qT}}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{1}{2}(x-\sigma\sqrt{T})^2} dx$$

$$\| \; y = x - \sigma\sqrt{T}$$

$$\frac{S_0 e^{-qT}}{\sqrt{2\pi}} \int_{-d_2-\sigma\sqrt{T}}^{\infty} e^{-\frac{1}{2}y^2} dy$$

$$\| \quad S_0 e^{-qT}(1 - N(-d_2 - \sigma\sqrt{T})) \; \| \; S_0 e^{-qT} N(\underbrace{d_2 + \sigma\sqrt{T}}_{:= d_1})$$

This verifies the B-S formula (with $t=0$)

(For any other $t \in [0,T]$, simply replace T by $\underbrace{T-t}_{\text{means time-to-expiry}}$.)

Greeks derivation:

$$\Delta(C) = \frac{\partial C}{\partial S} = e^{-q(T-t)} N(d_1) + S e^{-q(T-t)} \frac{\partial}{\partial S} N(d_1) - K e^{-r(T-t)} \frac{\partial}{\partial S} N(d_2)$$

(Note: d_1, d_2 depend on S)

By chain rule:

$$\frac{\partial}{\partial S} N(d_1) = N'(d_1) \frac{\partial d_1}{\partial S}$$

$$\frac{\partial}{\partial S} N(d_2) = N'(d_2) \frac{\partial d_2}{\partial S}$$

Note

$$N'(z) = \frac{d}{dz} \int_{-\infty}^{z} \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx = e^{-z^2/2} / \sqrt{2\pi}$$

It can be verified that

$$S e^{-q(T-t)} N'(d_1) = K e^{-r(T-t)} N'(d_2)$$

So,

$$\Delta(C) = e^{-q(T-t)} N(d_1) + S e^{-q(T-t)} N'(d_1) \frac{\partial d_1}{\partial S} - K e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial S}$$

$$= e^{-q(T-t)} N(d_1) + S e^{-q(T-t)} N'(d_1) \underbrace{\left(\frac{\partial d_1}{\partial S} - \frac{\partial d_2}{\partial S}\right)}_{= 0}$$

· But $\dfrac{\partial d_1}{\partial S} = \dfrac{\partial d_2}{\partial S} \quad \left(= \dfrac{1}{\sigma S \sqrt{T-t}}\right)$

So

$$\boxed{\Delta(C) = e^{-q(T-t)} N(d_1)}$$

Other Greeks can be derived similarly

Implied Volatility

Note: the only parameter needed in the BS formulas not directly observable in the markets is the volatility σ of the underlying asset.

The implied volatility σ_{imp} is the value so that

$$C_{BS}(S, K, T, \sigma_{imp}, r, q) = C$$

the market price of the call option

or $P_{BS}(S, K, T, \sigma_{imp}, r, q) = P$

market price of the put

Questions:

(2) How to solve such a equation?

(1) Does a solution always exist? unique?

(4) Same solution for all parameters irrelevant to the underlying asset?

[Note: In the theory, σ depends only on the underlying asset.]

(3) Same solution for call and put with all other parameters fixed?

B-S formulas :

$$C = Se^{-q(T-t)} N(d_1) - Ke^{-r(T-t)} N(d_2)$$

$$P = Ke^{-r(T-t)} N(-d_2) - Se^{-q(T-t)} N(-d_1)$$

where

$$d_1 = \left[\ln(S/K) + (r-q+\sigma^2/2)(T-t)\right] / \sigma\sqrt{T-t}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$= \left[\ln(S/K) + (r-q-\sigma^2/2)(T-t)\right] / \sigma\sqrt{T-t}$$

Answers:

① $\frac{\partial C}{\partial \sigma} = \frac{\partial P}{\partial \sigma} = Se^{-qT}\sqrt{T}\,\frac{1}{\sqrt{2\pi}}\,e^{-d_1^2/2} > 0$

for any S, K, T, r, q and σ.

with all other factors fixed,
C and P $\Uparrow$ as $\sigma \Uparrow$

So if the solution exists, it must be unique

Existence is more subtle (see HW #5)

(Essentially, if the market price C does not produce any arbitrage opportunity, σ_{imp} exists.)

② a nonlinear equation in a single variable (σ), no closed-form expression but any standard numerical method would solve it easily.

(3) Yes if the market prices for P and C satisfy the put-call parity:

$$P + Se^{-qT} - C = Ke^{-rT}$$

Then

(1) — $$P_{BS}(\sigma_{imp,p}) + Se^{-qT} - C_{BS}(\sigma_{imp,c}) = Ke^{-rT}$$

But the B-S formulas themselves also satisfy the put-call parity:

(2) — $$P_{BS}(\sigma) + Se^{-qT} - C_{BS}(\sigma) = Ke^{-rT} \quad \forall \sigma > 0$$

In particular,

$$P_{BS}(\sigma_{imp,c}) \overset{(2)}{=} Ke^{-rT} - Se^{-qT} + C_{BS}(\sigma_{imp,c})$$

$$\overset{(1)}{=} P_{BS}(\sigma_{imp,p})$$

$$\underset{\Uparrow}{\Longrightarrow} \boxed{\sigma_{imp,c} = \sigma_{imp,p}}$$

since $P_{BS}(\sigma)$ is a strictly increasing function in σ

④ Same implied volatility for all strike price K and expiry T and time t ?

NO :

(i) For any time t, implied volatility changes with the strike price K and expiry date T :

t

K

"volatility smile" : implied volitility for ATM options is much lower than the deep OTM and deep ITM options.

(ii) Implied volatility changes with time t for any fixed expiry date T:

implied vol ↑ before earnings annoucement (for the near-term options)

IBM 1st Quarter earnings announcement on April 20, 2015 at 4:30 pm

implied vol ↓ after

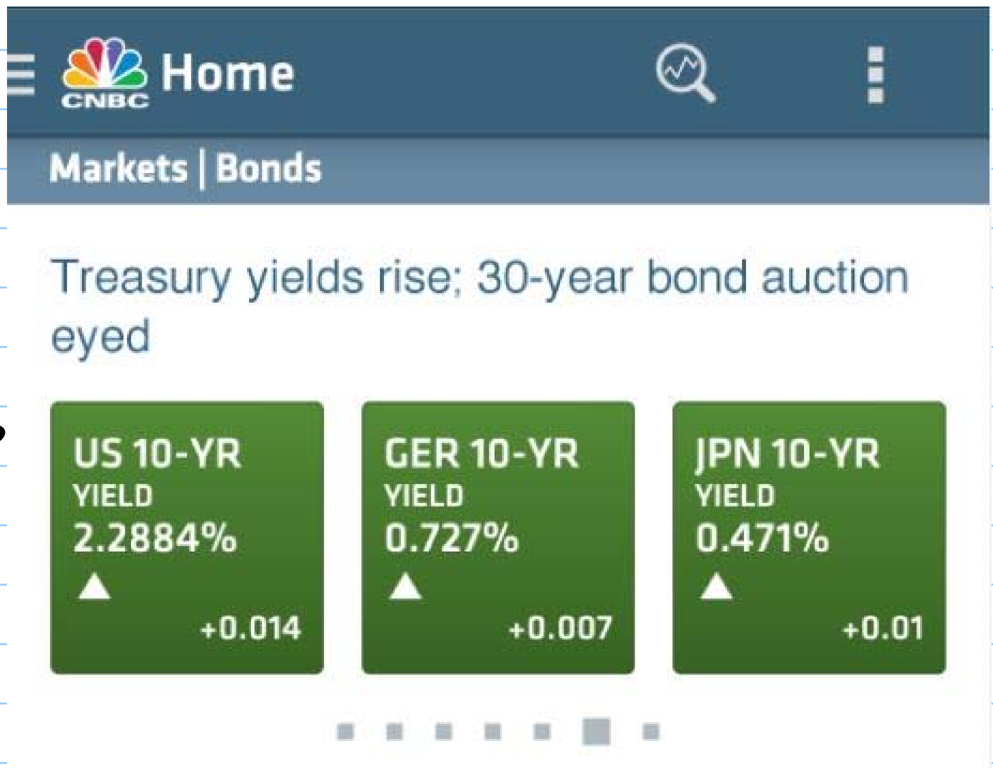
From ibm.com

Recent events

IBM 1Q 2015 Earnings Announcement	20 Apr 2015 16:30 EDT

We shall make a detour to talk about numerical methods for solving nonlinear equations. The implied vol. computation is one application of such methods.

Another financial application is the computation of a bond's yield (a.k.a. the yield to maturity (YTM)), which we now discuss.



Bond quotes on May 13, 2015

Recall from week 1:

$\$F$ – face value

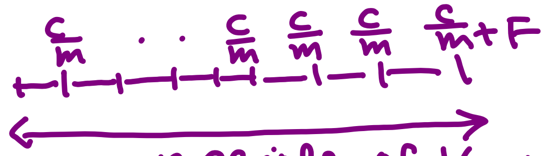
m coupon payments of $\$C/m$ each year

n periods remaining

$\$P$ – current market price of the bond

Q: What does it mean to you if you purchase this bond?

(*) cash flow



n periods of $1/m$ year per period

Assume $F + n\frac{c}{m} > P$.

A: If you hold the bond to maturity, it is like you deposit \$P in a bank, the bank pays an annualized interest rate of $100\lambda\%$ (λ to be calculated). compounded m times a year. But you must withdraw the money from the bank according to the predetermined cash flow pattern (*).

If you think this way, the interest rate λ satisfies:

$$P = \frac{F}{[1+\lambda/m]^n} + \underbrace{\sum_{k=1}^{n} \frac{c/m}{[1+\frac{\lambda}{m}]^k}}_{\text{geometric series}}$$

$$\boxed{P = \frac{F}{[1+\frac{\lambda}{m}]^n} + \frac{c}{\lambda}\left\{1 - \frac{1}{[1+\frac{\lambda}{m}]^n}\right\}}$$

Bond price formula

Note: Given P, F, m, n, C, it is a <u>nonlinear</u> equation in λ.

When P changes in real-time, one needs to solve this equation for λ also in real-time.