Lecture 7 : Submanifolds

Note Title 2/19/2017

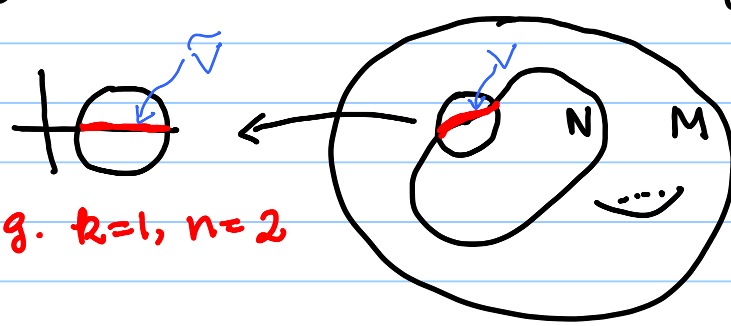
M - smooth manifold of dim. n

Def: $N \subset M$ is a submanifold (or embedded submanifold, or regular submanifold) of M if

$\forall p \in N$, $\exists$ chart (U, φ) of M s.t.

$p \in U$ and $U \cap N$ is a k-slice of U, i.e.

$$\varphi(q) = (x^1, \dots, x^k, 0, \dots, 0), \ \forall q \in U \cap N.$$



e.g. $k=1$, $n=2$

Let's check that a submanifold is a smooth manifold by itself.

N with relative topology from M is automatically Hausdorff and 2nd countable because M is.

Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$, $\pi(x^1, \dots, x^n) = (x^1, \dots, x^k)$.
$j : \mathbb{R}^k \to \mathbb{R}^n$, $j(x^1, \dots, x^k) = (x^1, \dots, x^k, 0, \dots, 0)$

For any "slice chart" (U, φ) in the definition above, write:

open in N $\to$ $V = U \cap N$, $\tilde{V} = \pi \circ \varphi(V)$ $\leftarrow$ open in $\mathbb{R}^k$

$\psi : \pi \circ \varphi|_V : V \to \tilde{V}$ is continuous

$\psi^{-1} = \varphi^{-1} \circ j|_{\tilde{V}}$ also continuous

Thus ψ is a homeomorphism.

By assumption, these charts cover N. We have shown that N is a topological k-manifold.

These charts are also smoothly compatible:

- If (U, φ) and (U', φ') are two slice charts,

$$V = U \cap N, \quad \psi = \pi \circ \varphi|_V$$
$$V' = U' \cap N, \quad \psi' = \pi \circ \varphi'|_{V'}$$

are the corresponding charts on N

$$\psi' \circ \psi^{-1} = (\pi \circ \varphi') \circ (\varphi^{-1} \circ j)$$
$$= \underbrace{\pi}_{\text{linear}} \circ \underbrace{(\varphi' \circ \varphi^{-1})}_{\text{smooth}} \circ \underbrace{j}_{\text{linear}} \quad \text{is smooth.}$$

Now, we see that $N \subset M$ is a smooth manifold by itself, with the topology being the subspace / relative topology induced by M.

The term "<u>regular</u> submanifold" has a lot to do with this last point.

To see what's important about "subspace topology", i.e.

$$\boxed{\begin{array}{c} V \subset N \text{ is open in the subspace} \\ \text{topology induced by } M \\ \text{iff} \\ V = \mathcal{U} \cap N \text{ for some open set in } M, \end{array}}$$

consider two examples of $N \subset M$ for which N is not a regular submanifold of M.

manifolds

① $\gamma : (-\frac{\pi}{2}, \frac{3\pi}{2}) \to \mathbb{R}^2$

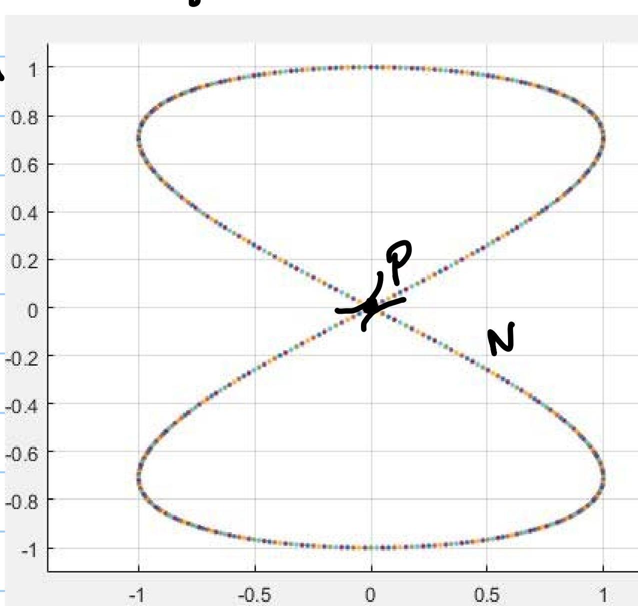
$$\gamma(t) = (\sin 2t, \cos t)$$

$\left.\begin{array}{l} \gamma \text{ is injective} \\ \gamma'(t) \neq \vec{0} \text{ for all } t \end{array}\right\}$ γ is both globally and locally injective

But

$N := \gamma(-\frac{\pi}{2}, \frac{3\pi}{2})$ is not a regular submanifold of $\mathbb{R}^2$.

$\mathbb{R}^2 = M$



Note: For $P = (0,0) \in N$ any open nbhd of P in $\mathbb{R}^2$ intersects with N at a cross 'X', which does not look homeomorphic to an open set in $\mathbb{R}^1$. (Some topological properties are needed to make this precise.)

But you can of course endow N with a manifold structure so that it is diffeomorphic to $\mathbb{R}^1$.

(2) $\gamma : \mathbb{R} \to \mathbb{T}^2 = S^1 \times S^1 \subset \underset{\underset{\mathbb{C}}{\wr\wr}}{\mathbb{R}^2} \times \underset{\underset{\mathbb{C}}{\wr\wr}}{\mathbb{R}^2}$

$c = \sqrt{2}$ or any irrational number

$$\gamma(t) = (e^{2\pi i t}, e^{2\pi i c t})$$

γ is injective :

$$\gamma(t_1) = \gamma(t_2) \iff e^{2\pi i (t_1 - t_2)} = 1 = e^{2\pi i c (t_1 - t_2)}$$

$$\iff t_1 - t_2 \in \mathbb{Z}, \quad c(t_1 - t_2) \in \mathbb{Z}$$

use $c \notin \mathbb{Q}$ $\iff t_1 - t_2 = 0$

$$\gamma'(t) = ((2\pi i) e^{2\pi i t}, (2\pi i c) e^{2\pi i c t}) \neq \vec{0}$$

(*) [I want to use this calculation to infer that

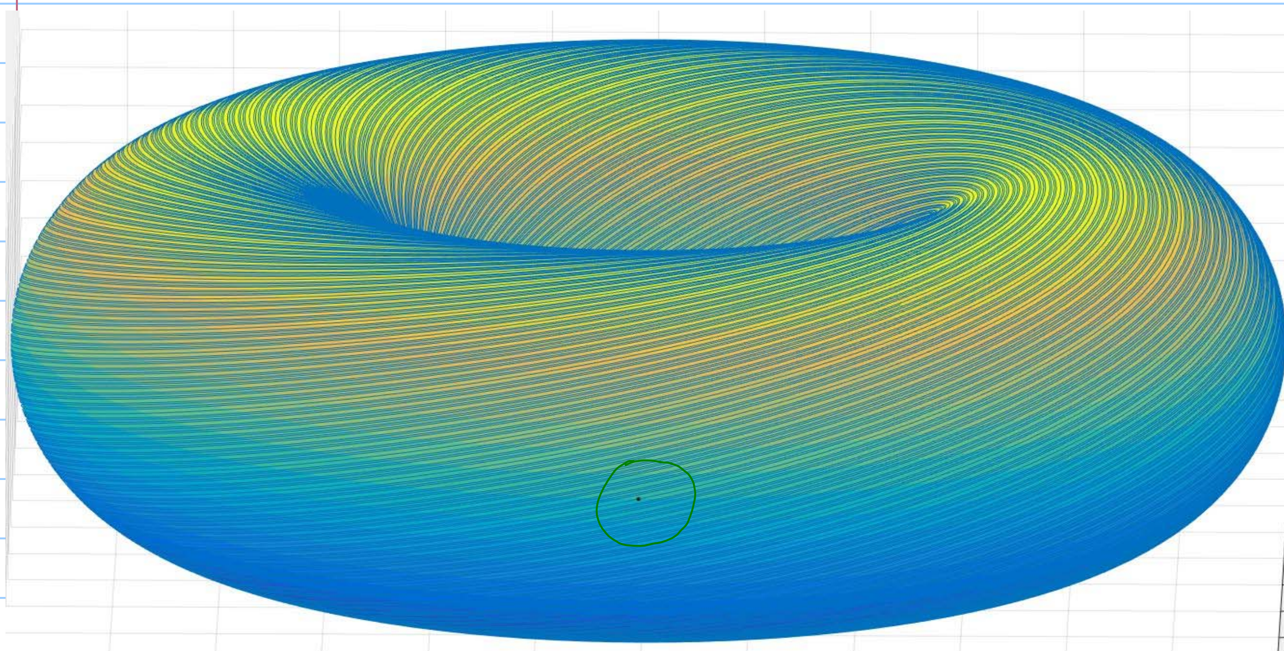
$$\gamma'(t) : T_t \mathbb{R} \to T_{\gamma(t)}(\mathbb{T}^2)$$

is injective for all t; I'll do it soon.]

Once again, γ is both globally and locally injective, and

$\boxed{\gamma(\mathbb{R}) \text{ fails (pretty badly!) to be a regular submanifold of } S^1 \times S^1.}$

It can be shown that $\gamma(\mathbb{R})$ is dense in $S^1 \times S^1$, so $\gamma(\mathbb{R})$ cannot be a regular submanifold of $S^1 \times S^1$.



$\gamma((-1000, 1000)) \subset \mathbb{T}^2$ (embedded into $\mathbb{R}^3$, just for the sake of an nice figure)

Consider $\gamma(0) = (1,1)$. If $\gamma(\mathbb{R})$ were a regular submanifold, there is an open set $\mathcal{U}$ in $\mathbb{T}^2$ s.t.

$\mathcal{U} \cap \mathbb{T}^2$ can be mapped homeomorphically to an open set in $\mathbb{R}$.

Claim: $\gamma(\mathbb{Z})$ has $\gamma(0)$ as a limit point (in the topology of $\mathbb{T}^2$)

To prove this, it suffices to show that:

$$\forall \varepsilon > 0, \ \exists \ k \in \mathbb{Z} \setminus \{0\} \ \text{s.t.} \ |\gamma(k) - \gamma(0)| < \varepsilon$$

$$\gamma(0) = (1,1), \quad \gamma(k) = (1, \underset{\overset{\Uparrow}{S^1}}{e^{2\pi i c k}})$$

But S^1 is compact, the infinite set $\{e^{2\pi i c \cdot k} : k \in \mathbb{Z}\}$ must have a limit point, say $z_0 \in S^1$.

Given $\varepsilon > 0$, choose n_1, n_2 $n_1 \neq n_2$ s.t.

$$|e^{2\pi i c n_1} - z_0| < \varepsilon/2, \quad |e^{2\pi i c n_2} - z_0| < \frac{\varepsilon}{2},$$

so

$$|e^{2\pi i c n_1} - e^{2\pi i c n_2}| < \varepsilon.$$

set

$$k = n_1 - n_2 \neq 0$$

$$|e^{2\pi i c k} - 1| = \overset{\shortparallel\, 1}{|e^{-2\pi i c n_2}|}\, |e^{2\pi i c n_1} - e^{2\pi i c n_2}| < \varepsilon.$$

And

$$|\gamma(k) - \gamma(0)| = |(1, e^{2\pi i c k}) - (1,1)| < \varepsilon.$$

From this, one can argue (i) γ does not map homeomorphically onto its image, and (ii) $N = \gamma(\mathbb{R})$ cannot be made a regular submanifold of $\mathbb{T}^2$.

Some work is needed to argue (ii) and I'll skip this; (i) is a easy consequence of the claim.

Just like the first example, N itself can be given a differentiable structure so that it is diffeomorphic to $\mathbb{R}^1$. It is just that N does not "look regular" in the torus.

Submersions, Immersions and Embeddings

Let $F: M \to N$ smooth, $\dim M = m$, $\dim N = n$.

If F_{*p} is injective, i.e. $\operatorname{rank}(F_{*p}) = m$, $\forall p$
then F is called an immersion.

If F_{*p} is surjective, i.e. $\operatorname{rank}(F_{*p}) = n$, $\forall p$
then F is called a submersion.

F is called a <u>submersion at p</u> if $\operatorname{rank}(F_{*p}) = n$.

If F is an immersion and also a topological embedding, i.e.

$$F: M \to (F(M), \text{ subspace topology in } N)$$

is a homeomorphism,
then

F is called an embedding.

Note:

Immersion / embedding maps from <u>low to high dim</u>.

Submersion maps from <u>high to low dim.</u>

The two previous examples are immersions but not embeddings.

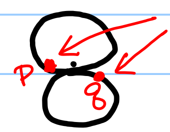
Example ① : the figure '8' is <u>compact</u> in $\mathbb{R}^2$

But a continuous function maps a compact space to a compact space.

Therefore $\gamma^{-1} : \underbrace{\gamma\left(\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)}_{\text{compact}}^{=N} \to \underbrace{\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)}_{\text{not compact}}$

is not continuous.

Alternatively, since N is a metric space, and so is $\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$, we can use a ε-δ argument to show that $\gamma^{-1} : N \to \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ is not continuous.



$\|p-q\|_{\mathbb{R}^2} < \varepsilon$ but

$|\gamma^{-1}(p) - \gamma^{-1}(q)| > (2\pi - 0.1)$

Similarly, it's easy to come up with an ε-δ argument to show that γ^{-1} in Example ② is also not continuous.

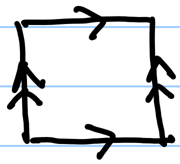
Apology:

In both examples, it is easy to argue that the immersion γ is not an embedding by arguing that γ^{-1} is not continuous. But it is harder to argue that the image of γ is not a regular submanifold of the range manifold.

Below, I give a few interesting examples of embedding without detailed proofs.

Examples of Embedding:

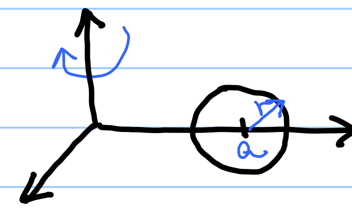
(A) Torus



$[0,2\pi]\times[0,2\pi]/\sim_{torus}$
+ quotient topology
+ differentiable structure (Ex: define it.)

Embedding into $\mathbb{R}^3$:

$$(u,v)\mapsto\begin{bmatrix}(r\cos v+a)\cos u\\(r\cos v+a)\sin u\\r\sin v\end{bmatrix}$$



Embedding into $\mathbb{C}\times\mathbb{C}$:

$$(u,v)\mapsto[e^{iu},e^{iv}]^T$$

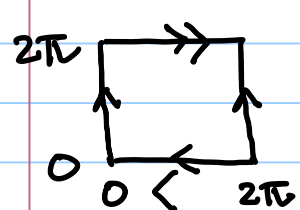
Embedding into $\mathbb{R}^4$ (essentially the same as above):

$$(u,v)\mapsto[\cos(u),\sin(u),\cos(v),\sin(v)]^T$$

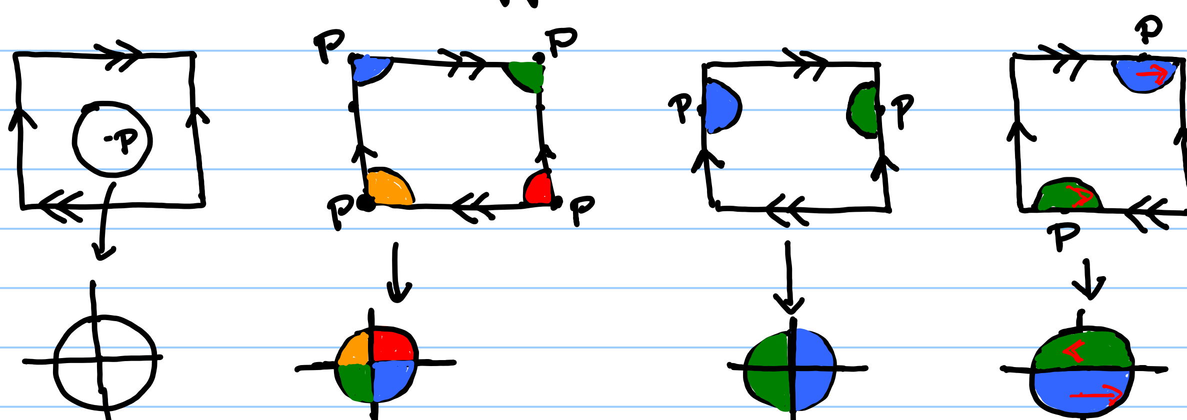
Embedding into S^3:

$$(u,v)\mapsto\frac{1}{\sqrt{2}}[\cos(u),\sin(u),\cos(v),\sin(v)]^T$$

(B) Klein Bottle



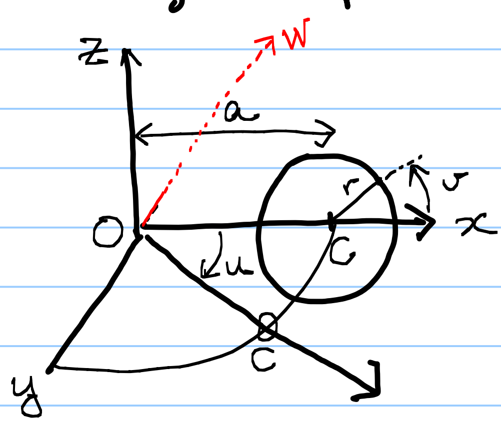
$K=[0,2\pi]\times[0,2\pi]/\sim_{klein}$
+ quotient topology
+ Differentiable structure



An embedding of the K to $\mathbb{R}^4$ is given by:

$$(u, v) \mapsto \begin{bmatrix} (r\cos v + a)\cos u \\ (r\cos v + a)\sin u \\ r\sin v\cos(u/2) \\ r\sin v\sin(u/2) \end{bmatrix} \begin{matrix} x \\ y \\ z \\ w \end{matrix} \quad (a > r).$$

Try to picture what this map does:



The circle is rotated about the z-axis in such a way that when the center C has described a rotation of an angle u in the $x-y$ plane, the plane of the circle has described a rotation of angle $u/2$ around the C-axis in the C-z-w space. (this is possible because we are in $\mathbb{R}^4$.)

<u>Ex</u>: Check that this is a well-defined map
Check that it is injective.

[More work is needed to verify that it is an embedding.]

(c) Projective plane $\mathbb{P}^2 = \{[x, y, z]_\sim : [x, y, z] \in S^2\}$

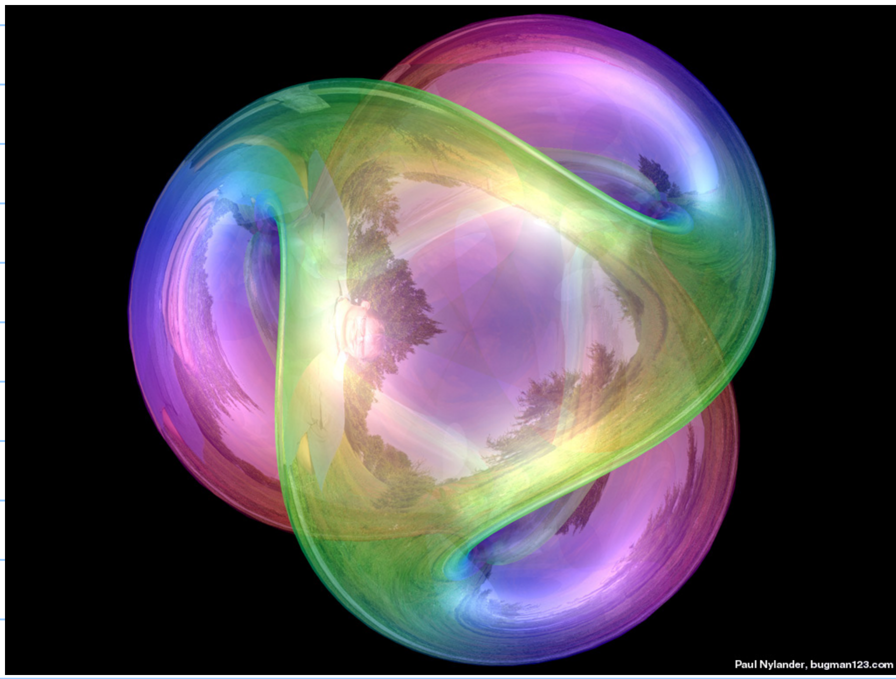
An embedding into $\mathbb{R}^4$:

$$[x, y, z]_\sim \mapsto [x^2 - y^2, xy, xz, yz]$$

David Hilbert once believed that $\nexists$ an immersion from

$$\mathbb{P}^2 \to \mathbb{R}^3,$$

and asked his student, Werner Boy, to look for a proof. Boy proved him wrong and constructed an immersion.



Boy's surface
(Immersion of $\mathbb{P}^2$ to $\mathbb{R}^3$)

Whitney Embedding Theorem:

Let X be a smooth manifold of dimension n. Then there exists a smooth embedding into $\mathbb{R}^{2n}$.

There is also a famous manifold embedding result by John Nash, on a more rigid kind of embedding called "isometric embedding". We will get to know what it means after we understand the meaning of a Riemannian metric.

Back to the basics :

Theorem: If X' is a submanifold of X then the inclusion map

$$i : X' \to X$$

($X' \subset X$, $i(p) = p$) is smooth. Then $\forall p \in X'$,

$$i_{*p} : T_p(X') \to T_{i(p)}X = T_pX$$

is well-defined and is linear isomorphism of $T_p(X')$ onto a subspace of T_pX.

Proof: Let $k = \dim X'$.

Let $p \in X'$, $(\mathcal{C}, \mathcal{U})$ be a slice chart. Denote its coordinate functions by

$$\underbrace{x^1, \cdots, x^k}_{\text{coordinate functions of a chart of } X' \text{ around } p}, \cdots, x^n.$$

So the coordinate representation of i is simply

$$(t^1, \cdots, t^k) \mapsto (t^1, \cdots, t^k, 0, \cdots, 0),$$

which is not only smooth, but linear.

This expression also shows that i_{*p} is represented by the matrix

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} \leftarrow I_{k \times k} \\ 0 \end{bmatrix},$$ which is clearly injective. $\blacksquare$

If M is a regular submanifold of $\mathbb{R}^n$, it should to be case that the (abstract) vectors in

$$T_pM$$

can somehow be naturally identified with what we usually call vectors in $\mathbb{R}^n$.

From the previous theorem, $\iota_{*p} : T_pM \to T_p\mathbb{R}^n$ is injective. So all we need is a way to identify $T_p\mathbb{R}^n$ with the "concrete vectors" in $\mathbb{R}^n$. But if you remember how those "abstract vectors" were invented, such an identification should be obvious.

For each $v \in \mathbb{R}^n$, let $v_p \in T_p(\mathbb{R}^n)$ be defined by

$$v_p = \alpha'(0) \ : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

where $\alpha : \mathbb{R} \to \mathbb{R}^n$

$$\alpha(t) = p + tv.$$

Then $v \to v_p$ is an isomorphism of $\mathbb{R}^n$ onto $T_p(\mathbb{R}^n)$ (Ex: check it.)

This says for any submanifold M of $\mathbb{R}^n$, every element of T_pM is the velocity vector of a smooth curve α in M, which can be regarded as a smooth curve in $\mathbb{R}^n$, whose velocity vector can be computed relative to standard coordinates and identified with an element in $\mathbb{R}^n$. Basically, we are back to exactly what we called "regular surface in $\mathbb{R}^n$."

Main theorems :

(I) [Stated in Lecture 4]
Let $F : X \to Y$ smooth, $\dim X = n$, $\dim Y = m$ $(n > m)$.

If $q \in F(X)$ and F is a submersion at each $p \in F^{-1}(q)$, then $F^{-1}(q)$ is a submanifold of X of dimension $n-m$.

(If $X = \mathbb{R}^n$, $F^{-1}(q)$ is a regular surface.)

(II) (i) Any immersion is locally an embedding.

(ii) If $F : X \to Y$ is an embedding, then $F(X)$ is a regular submanifold of Y.
Also, $F : X \to F(X)$ is a diffeomorphism.

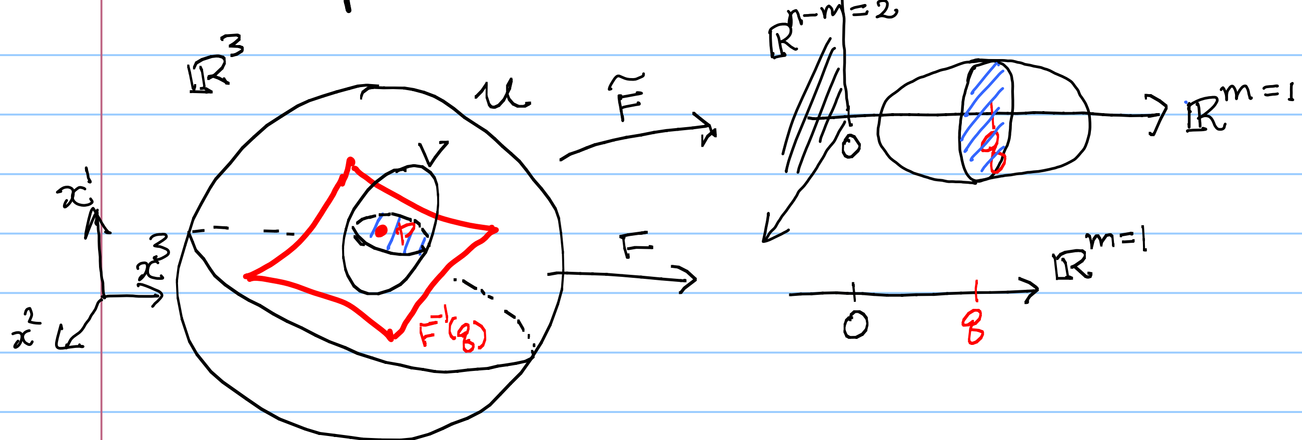
Proof of (I) :

We first prove the result in the special case in which
$X = \mathcal{U}$ is an open set in $\mathbb{R}^n$
$Y = \mathbb{R}^m$, $n > m$.

$F : \mathcal{U} \to \mathbb{R}^m$, $F(x^1, \dots, x^n) = (F^1(x^1, \dots, x^n), \dots, F^m(x^1, \dots, x^n))$

Fix a $q \in F(\mathcal{U})$ s.t. $\operatorname{rank} dF(p) = m$, $\forall p \in F^{-1}(q)$.

Fix a $p \in F^{-1}(q)$. We must find a slice chart around p.



$$dF(p) = \begin{bmatrix} \partial F^1/\partial x^1 & \cdots & \partial F^1/\partial x^{n-m} & \partial F^1/\partial x^{n-m+1} & \cdots & \partial F^1/\partial x^n \\ \vdots & & \vdots & \vdots & & \\ \partial F^m/\partial x^1 & \cdots & \partial F^m/\partial x^{n-m} & \partial F^m/\partial x^{n-m+1} & \cdots & \partial F^m/\partial x^n \end{bmatrix}$$

rank m

By renumbering the coordinates if necessary, we may assume this $m \times m$ submatrix is invertible.

Define $\tilde{F}: \mathcal{U} \to \mathbb{R}^{n-m} \times \mathbb{R}^m$ by ($\mathcal{U} \subset \mathbb{R}^n$)

$$\begin{aligned} \tilde{F}(x) &= \tilde{F}(x^1, \ldots, x^{n-m}, x^{n-m+1}, \ldots, x^n) \\ &= (x^1, \ldots, x^{n-m}, F^1(x), \ldots, F^m(x)) \end{aligned}$$

Then

$$d\tilde{F}(p) = \begin{bmatrix} I_{n-m} & \underbrace{0}_{(n-m)\times m} \\ \multicolumn{2}{c}{dF(p) \quad \underbrace{m \times m}_{\text{full rank}}} \end{bmatrix}$$

is an invertible $n \times n$ matrix,

so $\tilde{F}$ has a local inverse, i.e. $\exists$ an open nhbd V of $p = (p^1, \ldots, p^{n-m}, p^{n-m+1}, \ldots, p^n)$

and W of $\tilde{F}(p) = (p^1, \cdots, p^{n-m}, F^1(p), \cdots, F^m(p))$
$= (p^1, \cdots, p^{n-m}, q^1, \cdots, q^m)$
such that
$$\tilde{F}|_V : V \to W \text{ is a diffeomorphism.}$$

$(V, \tilde{F}|_V)$ is pretty much the slice-chart we are looking for, except that it maps any
$$(x^1, \cdots, x^{n-m}, x^{n-m+1}, \cdots, x^n) \in V \cap F^{-1}(q)$$
to
$$(x^1, \cdots, x^{n-1}, \overbrace{q^1, \cdots, q^m}^{=q}) \text{ instead of } (x^1, \cdots, x^{n-1}, 0, \cdots, 0).$$
But this easy to fix, simply define
$$\psi : V \to \mathbb{R}^n, \quad \psi(x) = \tilde{F}(x) - (0, \cdots, 0, q^1, \cdots, q^m),$$
then
$$(V, \psi) \text{ is a desired slice chart.}$$

Note that this part of the proof takes care of the case of regular surfaces needed in Lecture 4. Now we use this special case to prove the theorem in general.

X n-manifold $\qquad$ Y - m-manifold
$n > m$ $\qquad$ $F : X \to Y$
$q \in Y$ $\qquad$ rank $dF(p) = m$, $\forall p \in F^{-1}(q)$.

Fix an arbitrary $p \in F^{-1}(q)$. Choose charts $(\mathcal{U}, \varphi)$ at p and (V, ψ) at q, and consider the coordinate representation of F

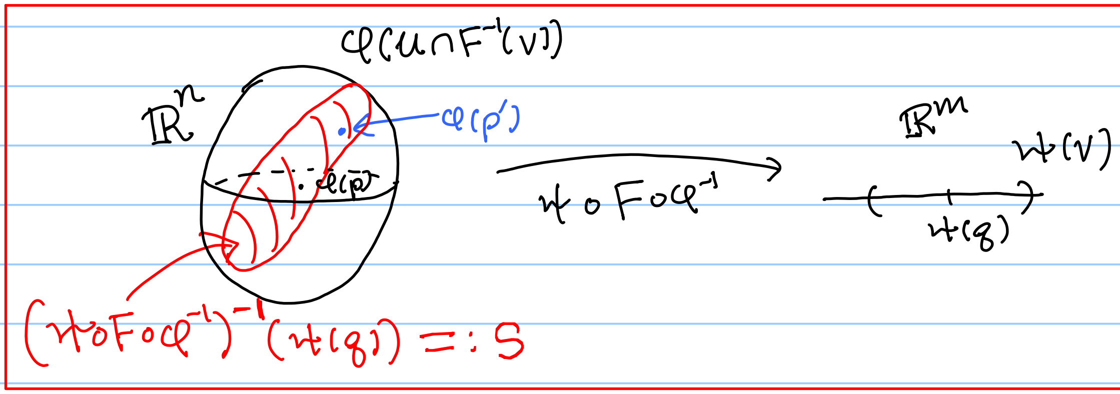
$$\psi \circ F \circ \varphi^{-1} : \varphi(\mathcal{U} \cap F^{-1}(V)) \to \psi(V)$$

We have

$$(\psi \circ F \circ \varphi^{-1})(\varphi(p)) = \psi(q)$$

and

$$(\psi \circ F \circ \varphi^{-1})_{*\varphi(p)} = \psi_{*q} \circ F_{*p} \circ \varphi^{-1}_{*\varphi(p)} .$$



All three derivatives on the right-hand side are full rank, so the composition is surjective. Same is true if $\varphi(p)$ is replaced by any

$$\varphi(p') \in (\psi \circ F \circ \varphi^{-1})^{-1}(\psi(q)) \quad , \quad q \in V$$

So we are back to the Euclidean setting, $\exists$

(W, ξ) for $\mathbb{R}^n$ around $\varphi(p)$ s.t.

$W \subset \varphi(\mathcal{U} \cap F^{-1}(V))$

and for every vector in $\xi(W \cap S)$, the last m coordinates are zero.

Now it is easy to check that $(\varphi^{-1}(W), \xi \circ \varphi|_{\varphi^{-1}(W)})$ is a desired slice chart. ■

The proof of (II) is quite similar to that of (I), we omit it.