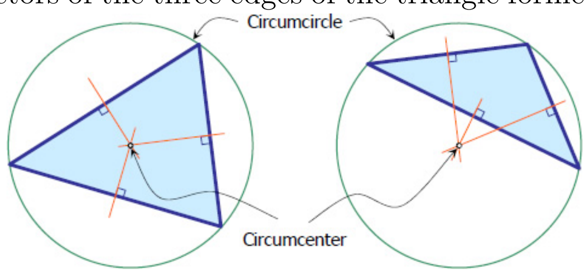
Math 538 Differential Geometry and Manifolds

$\boxed{\text{HW \#1}}$

Due: Friday October 2, 2020

While the primary goal of this first homework is to fill in some details in the local theory of curves, by doing it you will also begin to see the importance of *inner-product* as a tool for computing various geometric quantities. Later in this course, we shall see that inner-product is a key ingredient of a ***Riemannian manifold***.

1. In classical geometry, we learned that given three non-collinear points there is a unique circumcircle passing through the three points; the circumcenter is the intersection of the (concurrent) perpendicular bisectors of the three edges of the triangle formed by the three points.



We put this fact in a modern setting. Given two linearly independent vectors $v_1, v_2 \in \mathbb{R}^n$, $n \geqslant 2$, there is a unique circumcircle that passes through the three points $0, v_1, v_2$ and this circle lies on the plane spanned by v_1 and v_2. On the one hand, the dimension of the ambient space, namely n, is irrelevant to the geometry. Even if n is set to the 'intrinsic dimension', namely $n = 2$,[1] the actual coordinates are still rather meaningless from a geometric viewpoint. On the other hand, the vectors v_1, v_2 are given to you as (column) n-vectors, and ultimately we would like to know the coordinates of the circumcenter in $\mathbb{R}^n$.

How do we 'compute in $\mathbb{R}^n$' but yet 'focus on the geometry'? A moment of thought suggests that the only things that matter to the geometry are the lengths of v_i, $i = 1, 2$, and the angle between them. These quantities are recorded by the **inner-products**

$$E := \langle v_1, v_1 \rangle, \ F := \langle v_1, v_2 \rangle, \ G := \langle v_2, v_2 \rangle.$$

The circumcenter, denoted by c, must be in the plane spanned by v_1, v_2, so $c = \alpha v_1 + \beta v_2$ for some scalars α, β. It seems plausible that we can compute the 'intrinsic coordinates' (α, β) entirely based on E, F, G.

(i) Show that the circumcenter c is given by

$$c = \underbrace{\begin{bmatrix} v_1, \ v_2 \end{bmatrix}}_{n \times 2} \ \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} E \\ G \end{bmatrix} \cdot \frac{1}{2}}_{\substack{2\times 2 \\ \text{extrinsic coordinates forgotten!}}} \quad .$$

Hence the radius of the circumcircle is given by $R = \sqrt{\langle c, c \rangle}$.

(ii) Show that R can be expressed purely in terms of E, F and G. Derive such an expression.

[1]The problem is 'intrinsically' a 2-dimensional one, regardless of what n is. We may call n the 'extrinsic dimension'.

2. Let $\alpha : I \to \mathbb{R}^n$ ($n \geqslant 2$) be a curve parameterized by arc length s such that $\alpha''(s) \neq 0$, $s \in I$. This assumption means that the curvature at s, defined by $\kappa(s) := \|\alpha''(s)\|$ is non-zero; moreover, when h_1 and h_2 are different and small enough, the two vectors $\alpha(s+h_i) - \alpha(s)$, $i = 1, 2$, cannot be collinear. (Why?)

   Let $R(h_1, h_2)$ be the radius of the circumcircle of $\alpha(s)$, $\alpha(s+h_1)$, $\alpha(s+h_2)$. Using the Taylor expansion $\alpha(s+h) = \alpha(s) + \alpha'(s)h + \alpha''(s)h^2/2 + \alpha'''(s)h^3/6 + O(h^4)$, show that

$$\lim_{h_1, h_2 \to 0} R(h_1, h_2) = \kappa(s)^{-1}.$$

   This reproduces the classical fact (credited to Newton and Leibniz?) that the curvature at s is the reciprocal of the radius of the *osculating circle* at s.

   (Optional) Show that the *osculating plane*, defined by the plane through $\alpha(s)$ and spanned by $t = \alpha'(s)$ and $n = \alpha''(s)/||\alpha''(s)\|$, is the limit position of the plane passing through $\alpha(s)$, $\alpha(s+h_1)$, $\alpha(s+h_2)$ when $h_1, h_2 \to 0$. (To prove this statement we must first make precise of what we mean by 'a family of planes converging to a limit plane'.)

3. Let's work out some details for *arc length reparameterization*. From this, we can also derive curvature and torsion formulas for parametric curve *not* parameterized by arc length.

   Let $\alpha : (a, b) \to \mathbb{R}^n$ be a regular parameterized curve. Define

$$s(t) := \text{length of } \alpha((a, t]) = \int_a^t \|\alpha'(\sigma)\| d\sigma.$$

   (a) [2] Show that $s : (a, b) \to \mathbb{R}$ is C^1 smooth with a C^1 inverse. More precisely, if $L = s(b)$, then $s^{-1} : (0, L) \to (a, b)$ exists and is C^1. Moreover, prove that if α is C^k, then s and s^{-1} are also C^k. (Hint: inverse function theorem.)

   (b) Explain why $\overline{\alpha} := \alpha \circ s^{-1}$ is the right way to "reparameterize α by arc length."

4. With the theoretical foundation in the previous problem, derive the following formulas using the chain rule: For any C^3 regular parameterized curve,

$$\kappa(t) = \text{curvature of } \alpha \text{ at } t = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$$

$$\tau(t) = \text{torsion of } \alpha \text{ at } t = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2}.$$

[2] Optional if you are an undergraduate student or a non-math major.