CH 5 : The Mayer-Vietoris Sequence

Note Title 4/18/2017

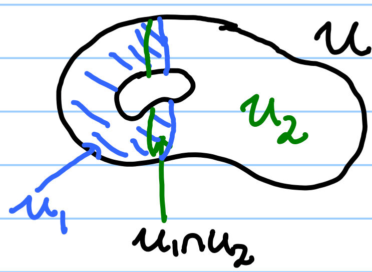
We introduce a technique for calculating

$$H^p(\mathcal{U}_1 \cup \mathcal{U}_2)$$

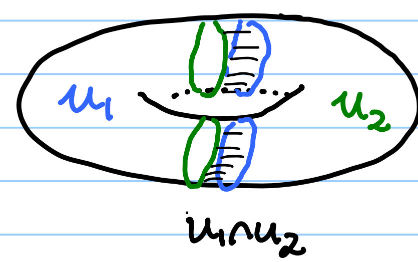
from

$$H^p(\mathcal{U}_1),\ H^p(\mathcal{U}_2),\ H^p(\mathcal{U}_1 \cap \mathcal{U}_2).$$

Combined with the Poincaré lemma, we have a tool for calculating $H^p(\mathcal{U})$ for quite general open sets in $\mathbb{R}^n$ (and for manifolds.)



Later: (manifold)



<u>Thm</u> (5.1)

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be open sets of $\mathbb{R}^n$

$$\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$$

Consider the inclusion maps:

$$i_\nu : \mathcal{U}_\nu \to \mathcal{U} \quad \text{and} \quad j_\nu : \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{U}_\nu,$$

$\nu = 1, 2.$

The following sequence is exact:

$$0 \to \Omega^p(\mathcal{U}) \xrightarrow{I^p} \Omega^p(\mathcal{U}_1) \oplus \Omega^p(\mathcal{U}_2) \xrightarrow{J^p} \Omega^p(\mathcal{U}_1 \cap \mathcal{U}_2) \to 0$$

where

$$I^p(\omega) := (i_1^*(\omega), i_2^*(\omega)), \quad J^p(\omega_1, \omega_2) = j_1^*(\omega_1) - j_2^*(\omega_2).$$

<u>Proof</u> ⓪ For a smooth map $\phi : V \to W$ and a p-form on W,

$$\omega = \sum_I f_I \, dx_I.$$

Its pullback can be written as

$$\phi^*\omega = \sum_I \underbrace{\phi^* f_I}_{= f_I \circ \phi} \underbrace{\phi^* dx_{i_1}}_{d\phi_{i_1}} \wedge \cdots \wedge \underbrace{\phi^* dx_{i_p}}_{d\phi_{i_p}}$$

When ϕ is an inclusion, i.e. $\phi_i(x) = x_i$

then $\quad d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_p} = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$.

Hence

$$\phi^*(\omega) = \sum_I (f_I \circ \phi)\, dx_I \,, \quad \text{if } \phi \text{ is an inclusion.}$$

① (Injectivity of I^p)

Assume $I^p(\omega) = 0$, then $i_1^*(\omega) = i_2^*(\omega) = 0$.

Write $\quad \omega = \sum_I f_I\, dx_I \quad , \quad f_I : \mathcal{U} \to \mathbb{R}$ smooth

$$0 = i_\nu^*(\omega) = \sum_I (\underbrace{f_I \circ i_\nu}_{f_I|_{\mathcal{U}_\nu}})\, dx_I \,, \quad \nu = 0, 1$$

The assumption means

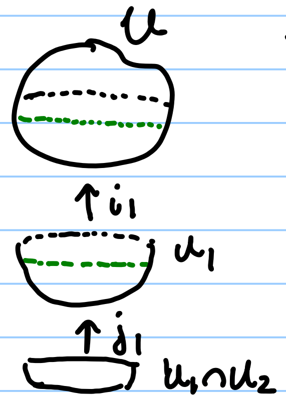
$$f_I|_{\mathcal{U}_1} \equiv 0 \,, \; f_I|_{\mathcal{U}_2} \equiv 0 \quad \forall I,$$

So $\quad f_I \equiv 0 \,, \; \forall I \quad$ since $\mathcal{U}_1 \cup \mathcal{U}_2 = \mathcal{U}$.

and $\quad \omega = 0$.

② ($\ker J^p = \operatorname{Im} I^p$)

First note that :



$$J^p \circ I^p(\omega) = \underbrace{j_1^* i_1^*(\omega)}_{\substack{\| \\ (\underbrace{i_1 \circ j_1}_{:=j})^*}} - \underbrace{j_2^* i_2^*(\omega)}_{\substack{\| \\ (\underbrace{i_2 \circ j_2}_{= \text{same } j})^*}} = 0$$

$j : \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathcal{U}$ is the inclusion.

So $\operatorname{Im} I^p \subseteq \ker J^p$.

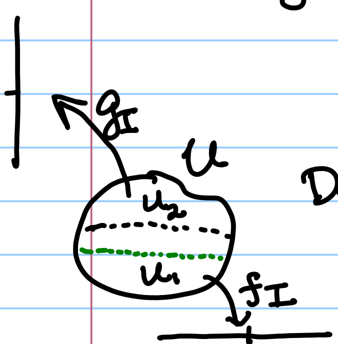
Let $\underset{\in \Omega^p(\mathcal{U}_1)}{\omega_1 = \sum f_I \, dx_I}$, $\underset{\in \Omega^p(\mathcal{U}_2)}{\omega_2 = \sum g_I \, dx_I}$

be such that $(\omega_1, \omega_2) \in \ker J^p$, i.e.

$$j_1^*(\omega_1) - j_2^*(\omega_2) = 0, \text{ or } j_1^* \omega_1 = j_2^* \omega_2$$

By ⓪, $\quad f_I \circ j_1 = g_I \circ j_2 \quad \forall I$

i.e. $\quad f_I|_{\mathcal{U}_1 \cap \mathcal{U}_2} = g_I|_{\mathcal{U}_1 \cap \mathcal{U}_2}$

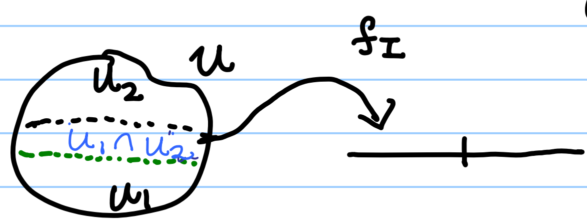


Define $h_I : \mathcal{U} \to \mathbb{R}$ by $h_I(x) = \begin{cases} f_I(x), & x \in \mathcal{U}_1 \\ g_I(x), & x \in \mathcal{U}_2 \end{cases}$

Then

$$I^p\left(\sum_I h_I \, dx_I\right) = (\omega_1, \omega_2).$$

③ (surjectivity of J^p – most technical step)



Given $\omega = \sum f_I \, dx_I \in \Omega^p(\mathcal{U}_1 \cap \mathcal{U}_2)$

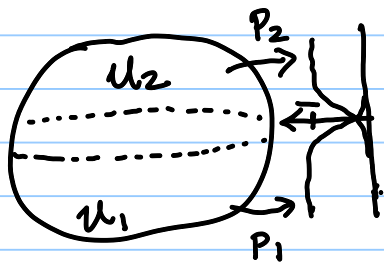
$f_I : \mathcal{U}_1 \cap \mathcal{U}_2 \to \mathbb{R}$

To construct a p-forms $\omega_\nu \in \Omega^p(\mathcal{U}_\nu)$, $\nu = 1, 2$
s.t.

$$\omega = J^p(\omega_1, \omega_2) = j_1^* \omega_1 - j_2^* \omega_2,$$

We need smooth functions $f_{I,\nu} : U_\nu \to \mathbb{R}$ s.t.

$$(\ast)\quad f_{I,1}(x) - f_{I,2}(x) = f_I(x), \quad x \in U_1 \cap U_2.$$



Choose a $\overset{\text{smooth}}{\text{partition of unity}}$ $\{p_1, p_2\}$ with support in $\{U_1, U_2\}$,

i.e. $p_\nu : U \to [0,1]$

for which $p_1(x) + p_2(x) = 1, \ \forall x \in U$
and
$\mathrm{supp}(p_\nu) = U_\nu, \quad \nu = 1, 2.$

(See Appendix A for proof of existence.)
(This is a standard technique in differential geometry and we will see it again later when we extend de Rham cohomology to manifolds)

Define

$$f_{I,1}(x) := \begin{cases} f_I(x)\, p_2(x), & x \in U_1 \cap U_2 \\ 0 & x \in U_1 \setminus \mathrm{supp}(p_2) \end{cases}$$

$$f_{I,2}(x) := \begin{cases} -f_I(x)\, p_1(x), & x \in U_1 \cap U_2 \\ 0 & x \in U_2 \setminus \mathrm{supp}(p_1) \end{cases}$$

These are smooth functions (why?) and satisfies $(\ast)$.

We are done by defining $\omega_\nu := \sum_I f_{I,\nu}\, dx_I$.

$\square$

$$0 \to \Omega^p(\mathcal{U}) \xrightarrow{I^p} \Omega^p(\mathcal{U}_1) \oplus \Omega^p(\mathcal{U}_2) \xrightarrow{J^p} \Omega^p(\mathcal{U}_1 \cap \mathcal{U}_2) \to 0$$

is not only a short exact sequence for every p, but also that

$$I : \Omega^*(\mathcal{U}) \to \Omega^*(\mathcal{U}_1) \oplus \Omega^*(\mathcal{U}_2)$$

$$J : \Omega^*(\mathcal{U}_1) \oplus \Omega^*(\mathcal{U}_2) \to \Omega^*(\mathcal{U}_1 \cap \mathcal{U}_2)$$

are chain maps, i.e. $\;\lrcorner\; = \;\urcorner$

$$\begin{tikzcd}
\Omega^p(\mathcal{U}) \arrow[r, "I^p"] \arrow[d, "d^p"] & \Omega^p(\mathcal{U}_1) \oplus \Omega^p(\mathcal{U}_2) \arrow[r, "J^p"] \arrow[d, "(d^p, d^p)"] & \Omega^p(\mathcal{U}_1 \cap \mathcal{U}_2) \arrow[d, "d^p"] \\
\Omega^{p+1}(\mathcal{U}) \arrow[r, "I^{p+1}"] & \Omega^{p+1}(\mathcal{U}_1) \oplus \Omega^{p+1}(\mathcal{U}_2) \arrow[r, "J^{p+1}"] & \Omega^{p+1}(\mathcal{U}_1 \cap \mathcal{U}_2)
\end{tikzcd}$$

(easy to check.)

Consequence by the "zig-zag lemma" :

Thm (Mayer - Vietoris)

$\exists$ exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H^p(\mathcal{U}) \arrow[r, "I^*"] & H^p(\mathcal{U}_1) \oplus H^p(\mathcal{U}_2) \arrow[r, "J^*"] & H^p(\mathcal{U}_1 \cap \mathcal{U}_2) \arrow[dll, "\partial^*"'] \\
& H^{p+1}(\mathcal{U}) \arrow[r, "I^*"] & H^{p+1}(\mathcal{U}_1) \oplus H^{p+1}(\mathcal{U}_2) \arrow[r, "J^*"] & H^{p+1}(\mathcal{U}_1 \cap \mathcal{U}_2) \arrow[r] & \cdots
\end{tikzcd}$$

Note :

$$H^p(\Omega^*(\mathcal{U}_1) \oplus \Omega^*(\mathcal{U}_2)) = H^p(\Omega^*(\mathcal{U}_1)) \oplus \Omega^*(\mathcal{U}_2)).$$

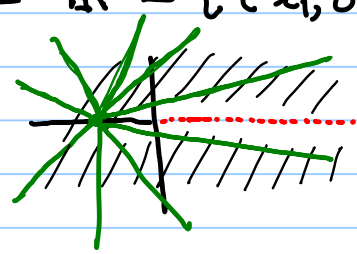
(see CH4.)

Our first example of using the M-V sequence is to show

$$H^p(\mathbb{R}^2 - \{0\}) = \begin{cases} \mathbb{R}, & p = 0, 1 \\ 0, & p \geq 2, \end{cases}$$

based on cutting $\mathbb{R}^2 - \{0\}$ into .

$$U_1 = \mathbb{R}^2 - \{(x_1, 0) : x_1 \geq 0\},$$



$$U_2 = \mathbb{R}^2 - \{(x_1, 0) : x_1 \leq 0\}$$

both star-shaped.

Note:
- both U_1 and U_2 are star-shaped
- $U_1 \cap U_2 = \mathbb{R}^2_+ \underset{\text{disjt}}{\cup} \mathbb{R}^2_-$ ← also star-shaped

Corollary of Theorem 5.1 and Lemma 4.13

If U_1 and U_2 are disjoint open sets in $\mathbb{R}^n$, then

$$I^*: H^p(U_1 \cup U_2) \to H^p(U_1) \oplus H^p(U_2)$$

is an isomorphism.

Proof:
$$0 \to \Omega^p(\overbrace{U_1 \cup U_2}^{=U}) \xrightarrow{I^p} \Omega^p(U_1) \oplus \Omega^p(U_2) \xrightarrow{J^p} \Omega^p(\underbrace{U_1 \cap U_2}_{\text{empty in this case}}) \nearrow 0$$

is exact $\quad = 0$

so I^p is an isomorphism (both injective and surjective), $\forall p$

so the induced map on cohomology

$$I^*: H^p(\Omega^*(U_1\cap U_2)) \to H^p(\Omega^*(U_1)\oplus \Omega^*(U_2))$$

$$\| \text{ lemma 4.13}$$

$$H^p(\Omega^*(U_1))\oplus H^p(\Omega^*(U_2))$$

must also be an isomorphism. $\square$

Back to $H^p(\mathbb{R}^2-\{0\})$:

By Mayer-Vietoris and Poincaré,

$$\cdots \to \underbrace{H^p(U_1)\oplus H^p(U_2)}_{=\mathbb{R}\ (p=0),\ =0\ p\geq 1} \xrightarrow{J^*} \underbrace{H^p(U_1\cap U_2)}_{=H^p(\mathbb{R}^2_+)\oplus H^p(\mathbb{R}^2_-)}$$

$$\xrightarrow{\partial^*} \underset{?}{H^{p+1}(\mathbb{R}^2-(0))} \xrightarrow{I^*} \underset{=0}{H^{p+1}(U_1)} \oplus \underset{=0}{H^{p+1}(U_2)} \to \cdots \quad (p\geq 0)$$

For $p=1$,
$$0 \xrightarrow{J^*} \underbrace{H^1(\mathbb{R}^2_+)\oplus H^1(\mathbb{R}^2_-)}_{=0} \xrightarrow{\partial^*} \overset{?}{H^2(\mathbb{R}^2-(0))} \xrightarrow{I^*} 0$$

is (short) exact, so ∂^* is an isomorphism,

and $H^2(\mathbb{R}^2-\{0\}) \cong 0$.

For $p=0$,

$$\underset{=0}{H^{-1}(U_1\cap U_2)} \to \overset{\mathbb{R}}{\overset{\|}{H^0(\mathbb{R}^2-\{0\})}} \xrightarrow{I^0} \overset{\mathbb{R}}{\overset{\|}{H^0(U_1)}} \oplus \overset{\mathbb{R}}{\overset{\|}{H^0(U_2)}} \xrightarrow{J^0}$$

$$\mathbb{R}\oplus\mathbb{R} = H^0(U_1\cap U_2) \xrightarrow{\partial^*} \underset{?}{H^1(\mathbb{R}^2-\{0\})} \xrightarrow{I^1} \underset{0}{\underset{\|}{H^1(U_1)}} \oplus \underset{0}{\underset{\|}{H^1(U_2)}}$$

is exact.

$$0 \to \mathbb{R} \xrightarrow{\overset{\text{injective}}{I^0}} \mathbb{R} \oplus \mathbb{R} \xrightarrow{J^0} \mathbb{R} \oplus \mathbb{R} \xrightarrow{\overset{\text{surjective}}{\partial^*}} H^1(\mathbb{R}^2 - \{0\}) \to 0$$

is exact.

$\text{rank}(I^0) = 1 = \text{nullity}(J^0)$ by exactness

$$\begin{aligned} \text{rank}(J^0) &= 2 - \text{nullity}(J^0) \quad \text{by rank-nullity} \\ &= 2 - 1 \\ &= 1 \end{aligned}$$

$$\overset{\overset{1}{\shortparallel}}{\text{rank}(J^0)} = \text{nullity}(\partial^*) \quad \text{by exactness}$$

$$\begin{aligned} \dim H^1(\mathbb{R}^2 - \{0\}) &= \text{rank}(\partial^*) \quad \text{by exactness} \\ &= 2 - \text{nullity}(\partial^*) \quad \text{by rank-nullity} \\ &= 2 - 1 = 1 \end{aligned}$$

so $H^1(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}$.

We have proved $H^p(\mathbb{R}^2 - \{0\}) = \begin{cases} \mathbb{R}, & p = 0, 1 \\ 0, & p \geq 2. \end{cases}$

$\square$