CH 11

Degree

Let $f : N^n \to M^n$ smooth

connected, compact, oriented, same dimension

We have the commutative diagram

$$\begin{tikzcd} H^n(M) \arrow[r, "H^n(f)"] \arrow[d, "\cong"'] & H^n(N) \arrow[d, "\cong"'] \\ \mathbb{R} \arrow[r, "\deg(f)"] & \mathbb{R} \end{tikzcd},$$

i.e.

$$\int_N f^*(\omega) = \deg(f) \int_M \omega \quad , \quad \omega \in \Omega^n(M).$$

This can be generalized to the case where N is not connected:

Write N as a disjoint union of connected components $N_1, \cdots, N_k$. Write $f_j := f|_{N_j}$, so $\deg(f_j)$ is defined as above. Then define

$$\deg(f) = \sum_{j=1}^{k} \deg(f_j).$$

As such,

$$\int_N f^*(\omega) = \sum_{j=1}^{k} \int_{N_j} f_j^*(\omega) = \sum_j \deg(f_j) \int_M \omega = \deg(f) \int_M \omega;$$

so the same formula holds.

Fact: Since $H^n(f)$ depends only only on the homotopy class of f, so is $\deg(f)$.

Suppose $N^n \xrightarrow{f} M^n \xrightarrow{g} P^n$.

$\uparrow \quad \uparrow \quad \nearrow$

compact and oriented

M, P are connected

For $\omega \in \Omega^n(P)$,

$$\begin{array}{ccc} \int_N (g\circ f)^*\omega & = & \deg(g\circ f)\int_P \omega \\ \parallel & & \\ \int_N f^*(g^*\omega) & = & \deg(f)\int_M g^*\omega \\ & & \parallel \\ & & \deg(f)\cdot\deg(g)\int_P \omega \end{array}$$

so

$$\boxed{\deg(g\circ f) = \deg(f)\deg(g)}$$

Remark: If $f: M^n \to M^n$ is a smooth map

$\uparrow$ compact, connected, oriented $\quad \nwarrow$ oriented the same way

then $\deg(f)$ is independent of the choice of the orientation. (Because both $\int_M \omega$, $\int_M f^*\omega$ change sign when orientation is reversed.)

We'll show that $\deg(f)$ is always an integer, the proof brings out an important geometric interpretation of $\deg(f)$.

And it requires the concept of regular value:

<u>Def</u>: For a smooth map $f: N^n \to M^m$, $p \in M$ is called a regular value if

$$D_q f : T_q N \to T_p M$$

is surjective for all $q \in f^{-1}(p)$.

E.g. $id : S^n \to S^n$, $id(x) = x$
every $p \in S^n$ is a regular value.

$f : S^n \to S^n$, $f(x) = p$ (constant map)

p is not a regular value
but
every point in $S^n - \{p\}$ is, as

$$p' \neq p \Rightarrow f^{-1}(p') = \phi$$

Remark: If $n \geq m$, a "generic" linear map $D_q f : T_q N \to T_p M$ is surjective

BUT

If $n < m$, $D_q f : T_q N \to T_p M$ is <u>never</u> surjective.

Nonetheless, the following theorem holds for any n and m:

<u>Theorem</u> (Brown-Sard) For every smooth map $f : N^n \to M^m$ the set of regular values is dense in M.

<u>Theorem</u> (Sard, 1942) Let $f : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map.

Let $S = \{x \in \mathcal{U} : \operatorname{rank} D_x f < m\}$. Then $f(S)$ has (Lebesgue) measure zero in $\mathbb{R}^m$.

Ex: 1. Prove that the first theorem follows from the second.

2. Given the remark above, why isn't the theorem

obviously wrong when $n < m$?

It is a well-known and important result but the proof is quite long and the idea of the proof is more related to analysis than to geometry and topology.

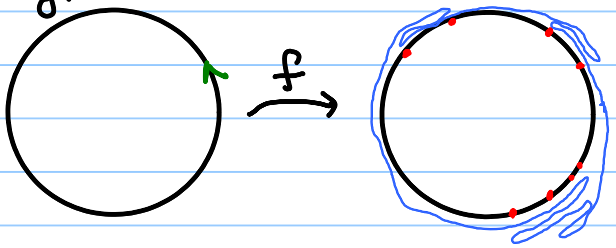
See M&T pg 99–100 / John Lee's manifold book.

Consider a smooth map $f: N^n \to M^n$ ← connected,
compact
oriented

for a regular value
$p \in M$ (there are lots of them by Sard's thm),
and $q \in f^{-1}(p)$,
define the local index:

$$\mathrm{Ind}(f,p) := \begin{cases} 1 & \text{if } D_qf : T_qN \to T_pM \text{ preserves} \\ -1 & \text{otherwise.} \end{cases}$$

E.g.



• – not regular values

At any other point p,

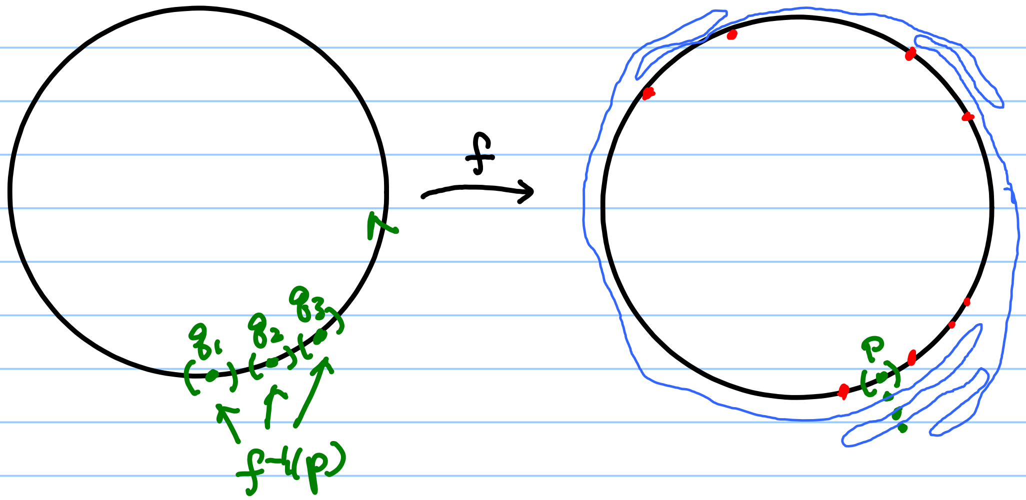
$$\sum_{q \in f^{-1}(p)} \mathrm{Ind}(f;q) = 1, \text{ or }$$
$$1 + (-1) + 1 \text{ or}$$
$$1 + (-1) + 1 + (-1) + 1$$
$$= 1$$

<u>Lemma</u> Let $p \in M^n$ be a regular value for the smooth map $f: N^n \to M^n$, N^n <u>compact</u>.

Then $f^{-1}(p)$ consists of a finitely many points $q_1, \dots, q_p$.

Moreover, $\exists$ disjoint open neighborhoods V_i of q_i

s.t. f maps V_i diffeomorphically onto $\mathcal{U}$
for $1 \le i \le k$.



Proof: For each $q \in F^{-1}(p)$, $D_q f : T_p N \to T_q \mathcal{M}$ is an isomorphism. By the inverse function theorem f is a local diffeomorphism around q. So points near q cannot be mapped to p by f, i.e. q is an isolated point in $f^{-1}(p)$.

Since N is compact, $f^{-1}(p)$ must be finite.
(An infinite subset of a compact set must have an accumulation point.)

Let $f^{-1}(p) = \{q_1, \cdots, q_k\}$. choose open nhbds W_i of q_i in N s.t. f maps W_i diffeomorphically onto an open nhbd $f(W_i)$ of p in $\mathcal{M}$.

It's not hard to see that

$$\mathcal{U} = \bigcap_{i=1}^{k} f(W_i) - f(N - \bigcup_{i=1}^{k} W_i) \quad \text{and}$$

$$V_i := W_i \cap f^{-1}(\mathcal{U})$$

prove the second part of the theorem. □

Thm Consider a smooth map $f: N^n \to M^n \leftarrow$ connected,

compact oriented

for any regular value $p \in M$
(there are lots of them),

$$\deg(f) = \sum_{q \in f^{-1}(p)} \operatorname{Ind}(f;q).$$

In particular $\deg(f)$ is an integer.

<u>Proof</u>: Let $f^{-1}(p) = \{q_1, \cdots, q_k\}$

$$f|_{V_i} : V_i \to \mathcal{U} \quad \text{diffeomorphisms}.$$

May assume $\mathcal{U}$ is connected. The diffeomorphisms are positively or negatively oriented, depending on whether $\operatorname{Ind}(f;q_i)$ is $+1$ or -1.

Let $\omega \in \Omega^n(M)$ be an n-form with

$$\operatorname{supp}_M(\omega) \subseteq \mathcal{U}, \ \int_M \omega = 1.$$

Then $\operatorname{supp}_N(f^*\omega) \subseteq f^{-1}(\mathcal{U}) = V_1 \cup \cdots \cup V_k$ and we can write

$$f^*\omega = \omega_1 + \cdots + \omega_k,$$

where

$$\omega_i \in \Omega^n(N) \ \text{ and } \ \operatorname{supp}(\omega_i) \subseteq V_i.$$

We can also write $\omega_i|_{V_i} = (f|_{V_i})^*(\omega|_{\mathcal{U}})$.

Then

$$\deg(f) = \deg(f)\int_M \omega = \int_N f^*\omega = \sum_{i=1}^{k} \int_N \omega_i$$

$$= \sum_{i=1}^{k} \int_{V_i} (f|_{V_i})^*(\omega|_{\mathcal{U}}) \underset{\substack{\uparrow \\ \text{Lemma} \\ 10.1}}{=} \sum_{i=1}^{k} \operatorname{Ind}(f;q_i) \underbrace{\int_{\mathcal{U}} \omega|_{\mathcal{U}}}_{=1}.$$

□

If $f^{-1}(p) = \emptyset$ for some $p \in M$ (ie. f is not surjective) then

$$f^*\omega = 0 \text{ in the proof above}$$
and
$$\deg(f) = 0.$$

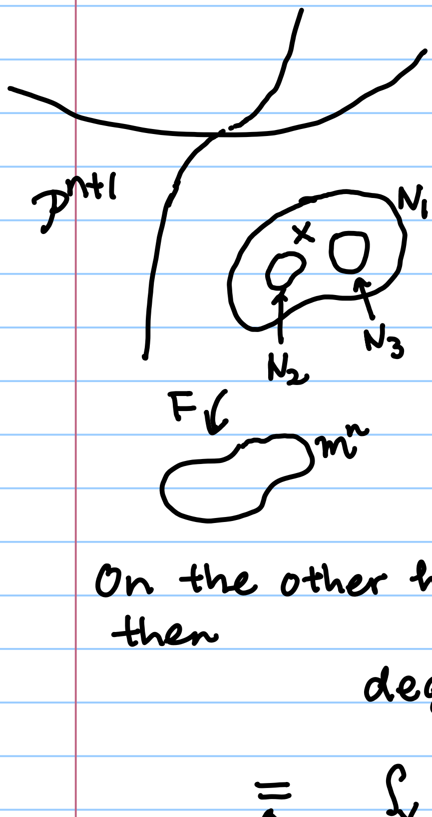
Thus we have:

Corollary: If $\deg(f) \neq 0$, then f is surjective.

<u>Proposition</u>: Suppose $\underset{\substack{\nearrow \\ \text{oriented}}}{P^{n+1}} \xrightarrow{F} \underset{\substack{\uparrow \\ \text{oriented, compact, connected}}}{M^n}$ is smooth.

Let $X \subseteq P$ be a compact domain with smooth boundary

$$N^n = \partial X = N_1^n \underset{\nwarrow}{\cup} \cdots \underset{\nearrow}{\cup} N_k^n.$$
$$\text{disjoint}$$



If $f_i = F|_{N_i}$, then
$$\sum_{i=1}^{k} \deg(f_i) = 0$$

<u>Proof</u> Let $f = F|_N$. So

$$\deg(f) = \sum_{i=1}^{k} \deg(f_i) \quad \text{(see pg 1)}.$$

On the other hand, let $\omega \in \Omega^n(M^n)$ with $\int_M \omega = 1$, then

$$\deg(f) = \deg(f)\int_M \omega = \int_N \underbrace{f^*\omega}_{= i^*(F^*\omega)} \qquad i : \overset{\text{“}N\text{”}}{\partial X} \to P$$

$$\underset{\substack{\uparrow \\ \text{Stokes'} \\ \text{theorem}}}{=} \int_X d(F^*\omega) = \int_X F^*(\underbrace{d\omega}_{\overset{\shortparallel}{0}}) = 0.$$

□

Two applications of degree :

- linking numbers
- indices of vector fields

Linking number

Def $\underbrace{J^d , K^\ell}_{\substack{\text{compact,}\\ \text{oriented,}\\ \text{connected}}} \overset{\text{submanifold}}{\subseteq} \mathbb{R}^{n+1}$, $k+\ell=n$

$J^d \cap K^\ell = \emptyset$.

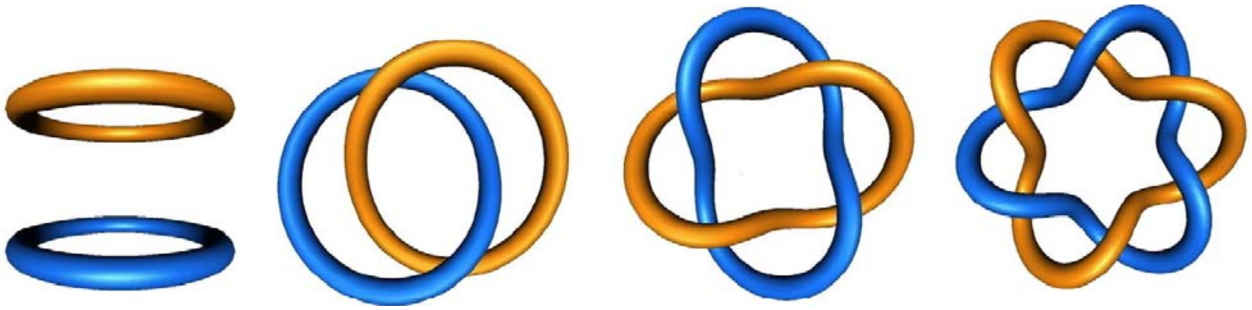
Consider

$$\boxed{\begin{array}{l} \Phi = \Phi_{J,K} : J\times K \to S^n , \quad \Phi(x,y) = y-x/\|y-x\| . \\ \qquad lk(J,K) := \deg(\Phi_{J,K}) \end{array}}$$

Orientation of $J\times K$ is the product orientation

S^n is oriented as the boundary of D^{n+1} with the standard orientation of $\mathbb{R}^{n+1}$.

Special case : $d=\ell=1$, 2 closed curves in $\mathbb{R}^3$

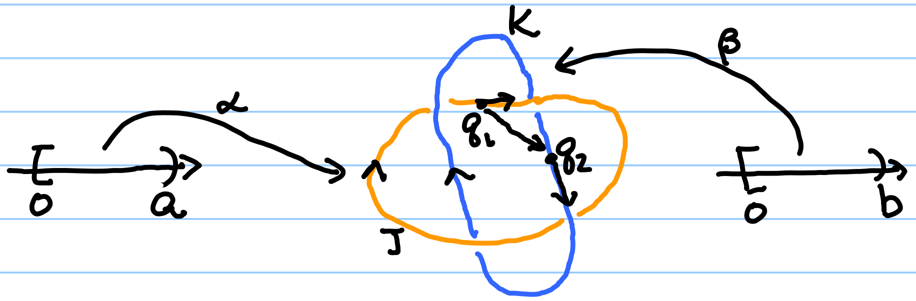


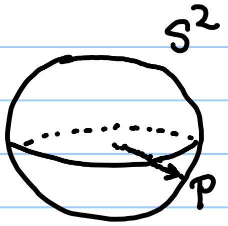
Both J, K diffeomorphic to S^1.

Choose regular parametrizations agreeing with the orientations of J and K :

$\alpha : \mathbb{R} \to J \subseteq \mathbb{R}^3$, $\beta : \mathbb{R} \to K \subseteq \mathbb{R}^3$

a-periodic b-periodic





$I(p) := \{(q_1, q_2) \in J \times K \mid q_2 - q_1 = \lambda p, \lambda > 0\}.$

$v(q_1) := \alpha'(\alpha^{-1}(q_1)) / \|\alpha'(\alpha^{-1}(q_1))\|$, $q_1 \in J$

$=$ positively oriented unit tangent vector to J @ q_1

$w(q_2) := \beta'(\beta^{-1}(q_2)) / \|\beta'(\beta^{-1}(q_2))\|$, $q_2 \in K$

Theorem: with the notation above we have:

(i) (Gauss)

$$\ell k(J, K) = \frac{1}{4\pi} \int_0^a \int_0^b \frac{\det(\alpha(u) - \beta(v), \alpha'(u), \beta'(v))}{\|\alpha(u) - \beta(v)\|^3} \, du \, dv$$

(ii) There exists a dense set of points $p \in S^2$ s.t.

$$\det(q_1 - q_2, v(q_1), w(q_2)) \neq 0 \quad \text{for} \quad (q_1, q_2) \in I(p).$$

(iii) For such points p,

$$\ell k(J, K) = \sum_{(q_1, q_2) \in I(p)} \delta(q_1, q_2)$$ where $\delta(q_1, q_2)$ is the sign of the determinant in (ii).

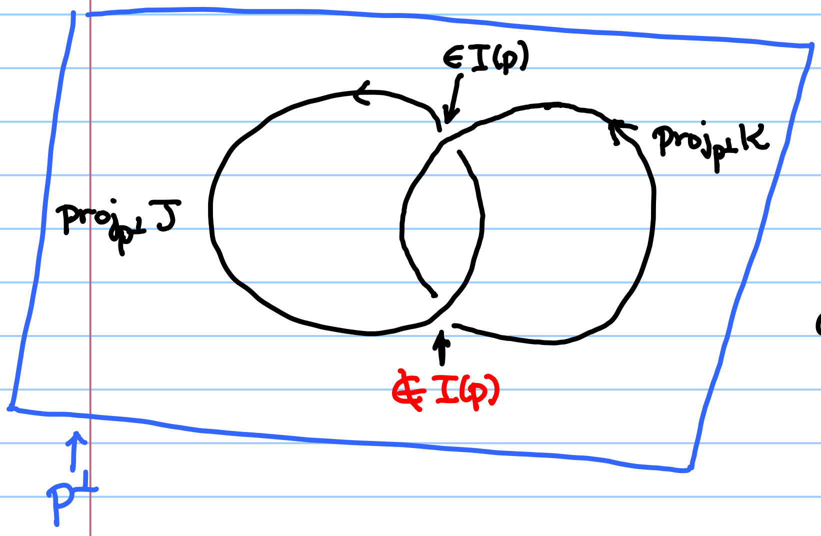
Comments: (ii) is related to Sard's theorem

The sum in (iii) is finite.

By a rotation, we can assume that a regular value in (ii) can be chosen to be

$$p = (0,0,1).$$

The projections of J and K on the x_1-x_2 plane may be drawn indicating over- and undercrossings and orientations, e.g.

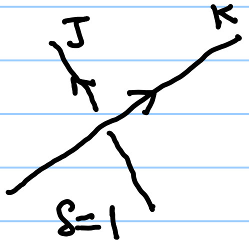


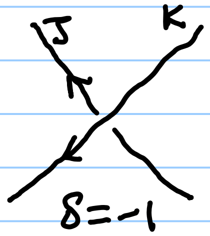
There is one element in I(p) for every place where K crosses over (and not under) J.

crosses over means :

$$q_1 - q_2 = \begin{bmatrix} <0 \\ 0 \\ 0 \end{bmatrix}$$

The corresponding sign δ is determined by the orientation of the curves and of the standard orientation of the plane as shown below :





Singularities of vector fields

Consider a vector field $F \in C^\infty(\mathcal{U}, \mathbb{R}^n)$ on $\overset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^n}$, $n \geq 2$.

Assume $0 \in \mathcal{U}$ is an <u>isolated zero</u> of F. a.k.a. singularity

i.e. $F(0) = \vec{0}$, and $\exists\, \rho > 0$ s.t.

$$F(x) \neq 0 \quad \forall x \in \rho D^n = \{x \in \mathbb{R}^n : \|x\| < \rho\}$$

and $\rho D^n \subseteq \mathcal{U}$.

Define

$$F_\rho : S^{n-1} \to S^{n-1} \quad \text{by} \quad F_\rho(x) = F(\rho x)/\|F(\rho x)\|$$

The homotopy class of F_ρ is independent of the choice of ρ, so

$$\deg F_\rho \in \mathbb{Z} \text{ is independent of } \rho.$$

Def $\iota(F;0) := \deg F_\rho$ is called the local index of F at 0.

Lemma: suppose $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has the origin as its only zero. Then

$$F : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$$

descends to

$$H^{n-1}(F) : \underset{\mathbb{R}}{\underset{\cong}{H^{n-1}(\mathbb{R}^n - \{0\})}} \to \underset{\mathbb{R}}{\underset{\cong}{H^{n-1}(\mathbb{R}^n - \{0\})}}$$

as multiplication by $\iota(F;0)$ on $\mathbb{R}$.

Proof: There is little to prove; recall

$$\underset{\text{inclusion}}{i : S^{n-1} \to \mathbb{R}^n - \{0\}}, \qquad \underset{\text{retraction}}{r : \mathbb{R}^{n-1} - \{0\} \to S^{n-1}} \quad x \mapsto x/\|x\|$$

$$\iota(F;0) = \deg F_1, \quad F_1 = r \circ F \circ i : S^{n-1} \to S^{n-1}.$$

$$\begin{tikzcd}
H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[r, "H^{n-1}(F)"] \arrow[d, shift left, "H^{n-1}(i)"] & H^{n-1}(\mathbb{R}^n - \{0\}) \arrow[d, shift left, "H^{n-1}(i)"] \\
H^{n-1}(S^{n-1}) \arrow[r, "H^{n-1}(F_1)"] \arrow[u, shift left, "H^{n-1}(r)"] \arrow[d, "\int"] & H^{n-1}(S^{n-1}) \arrow[u, shift left, "H^{n-1}(r)"] \arrow[d, "\int"] \\
\mathbb{R} \arrow[r, "\cdot\, \iota(F;0) = \deg F_1"] & \mathbb{R}
\end{tikzcd}$$

linear isomorphisms

□

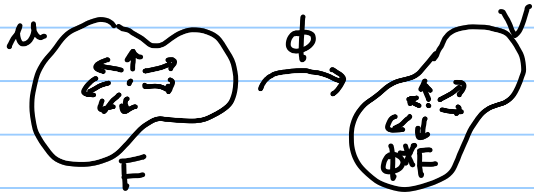
If $\phi : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow V \overset{\text{open}}{\subseteq} \mathbb{R}^n$ is a diffeomorphism

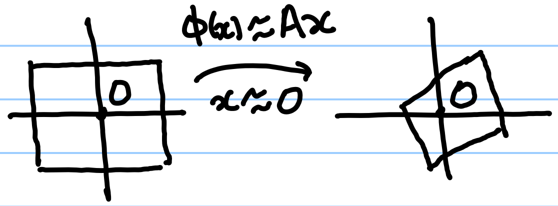
For a vector field on $\mathcal{U}$, $F \in C^\infty(\mathcal{U}, \mathbb{R}^n)$, we can use ϕ to push it forward to a vector field on V as follows:

$$\phi_* F(q) := D_p \phi\, (F(p)), \quad p = \phi^{-1}(q).$$

Lemma In the above, if $\phi(0) = 0$ and $F \in C^\infty(\mathcal{U}, \mathbb{R}^n)$ has 0 as an isolated singularity, then

$$\iota(\phi_* F; 0) = \iota(F; 0).$$





Idea:

The proof is not very short, and there is an interesting trick in it that I hate to skip. But the main idea of the proof is that the local index is about the local behavior vector field, and the local index of $\phi_* F$ is captured by the local behavior of the map ϕ at 0, which is governed by the matrix

$$A := D_0 \phi \quad \text{(the differential of } \phi \text{ at } 0\text{)}.$$

Let me prove the special case of the theorem when

$\mathcal{U} = V = \mathbb{R}^n$, $\phi = A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism.

Write $X = F : \mathbb{R}^n \to \mathbb{R}^n$ with 0 as its only zero.

$Y = A_* X : \mathbb{R}^n \to \mathbb{R}^n$ also has 0 as its only zero.

So

$$Y q = A X(A^{-1} q)$$

$$Y = A \circ X \circ A^{-1} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$$

By the previous lemma, to show $\iota(X;0) = \iota(Y;0)$ it suffices to show

$$H^{n-1}(X) = H^{n-1}(Y) : H^{n-1}(\mathbb{R}^n - \{0\}) \to H^{n-1}(\mathbb{R}^n - \{0\}).$$

But $$H^{n-1}(Y) = \underbrace{H^{n-1}(A^{-1})}_{\cdot \frac{\det A^{-1}}{|\det A^{-1}|}} \circ H^{n-1}(X) \circ \underbrace{H^{n-1}(A)}_{\cdot \frac{\det A}{|\det A|}} \quad \text{Lemma 6.14}$$

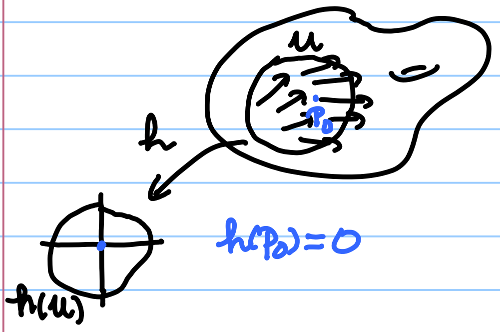
$$= H^{n-1}(X).$$

Note: It doesn't matter whether A is orientation preserving or reversing.

Def Let X be a smooth tangent vector field on the manifold M^n, $n \geq 2$ with $p_0 \in M$ as an isolated zero.

The local index $\iota(X;p_0) \in \mathbb{Z}$ of X is defined by

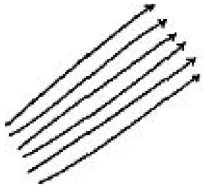
$$\iota(X;p_0) = \iota(h_* X|_u ; 0) \quad \text{where}$$

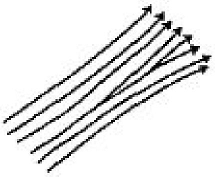


(u,h) is any chart around p_0 with $h(p_0)=0$.

The previous lemma says the local index does not depend on the choice (u,h).

E.g.


index 0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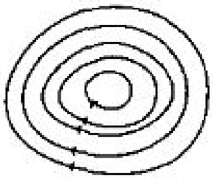
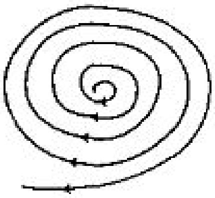
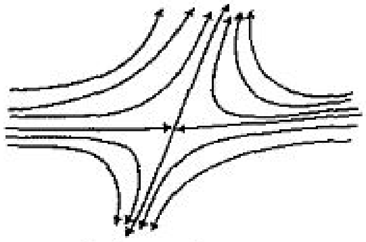
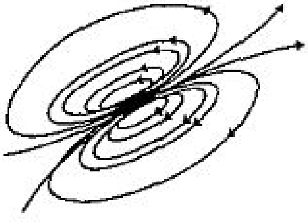
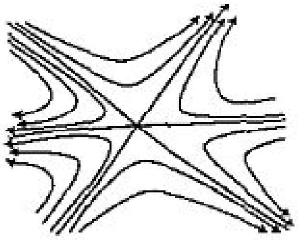

index 1

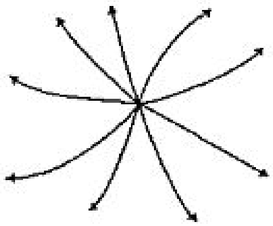

index 1

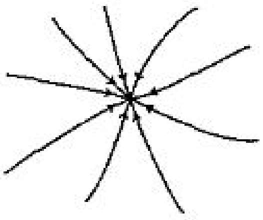

index -1


index 2


index -2

curves are integral curves (or 'flow') of the vector fields


index 1 in $\mathbb{R}^n$

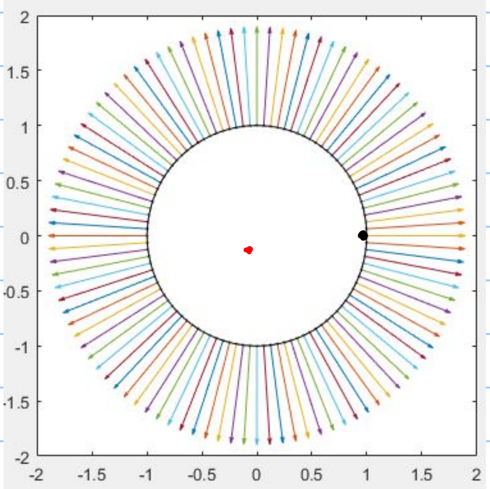

index $(-1)^n$ in $\mathbb{R}^n$

ODE of integral curves : $\dot{x}(t) = F(x(t))$

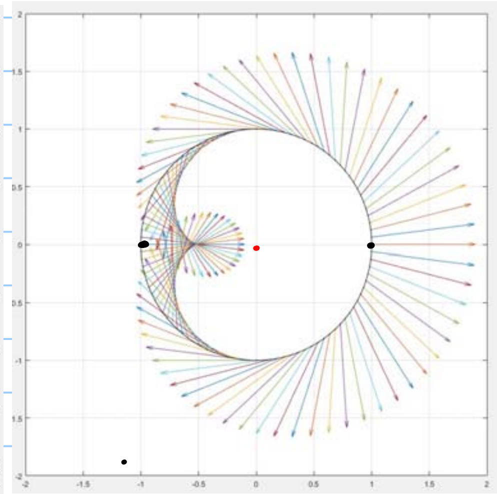
E.g. In dimension $n=2$, if we identify $\mathbb{R}^2$ with $\mathbb{C}$, the vector fields

$$F(z)=\begin{cases} z^k & \text{has a local index } k \text{ at } z=0 \\ \bar{z}^k & \text{has a local index } -k \text{ at } z=0. \end{cases}$$

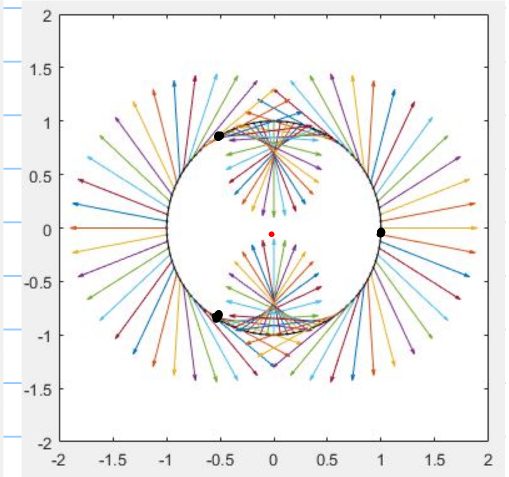
Shown below : $F(z)=z^k$ for $z\in S^1$, $k=1,2,3$



$\iota=1$



$\iota=2$



$\iota=3$

↑ non-degenerate zero at 0

↑ degenerate isolated zero at 0

Def Let X be a smooth vector field on $\mathcal{M}^n$ with
$$X_{p_0} = 0.$$
We say that p_0 is a non-degenerate singularity/zero if for any chart $(\mathcal{U}, h)$ with $h(p_0)=0$, the vector field
$$F = h_*(X|_{\mathcal{U}}) \in C^\infty(h(\mathcal{U}), \mathbb{R}^n)$$
has a non-singular derivative at 0.

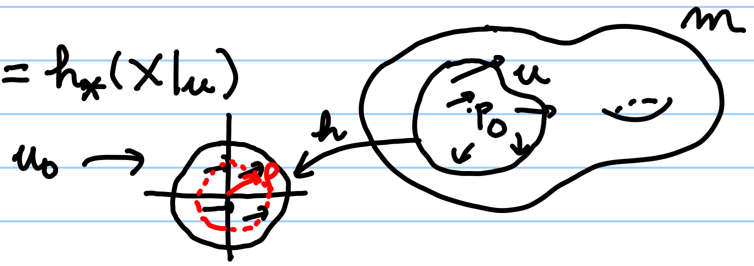
Note : (i) choice of chart is irrelevant

(ii) Non-degenerate zero $\overset{\not\Leftarrow}{\underset{\substack{\text{inverse}\\\text{function}\\\text{theorem}}}{\Longrightarrow}}$ isolated zero

Lemma If p_0 is a non-degenerate singularity, then
$$\iota(X, p_0) = \text{sign}(\det D_0F) \in \{+1, -1\}.$$

Sketch of proof

$F = h_*(X|_{\mathcal{U}})$



The same trick (I skipped) in the previous lemma :

Use $G : \mathcal{U}_0 \times [0,1] \to \mathbb{R}^n$; $G(x,t) := \begin{cases} \overset{=:A}{D_0F} & \text{if } t=0 \\ F(tx)/t & t \neq 0. \end{cases}$

to argue that
$$\iota(X, p_0) = \iota(F; 0) = \underset{\det A / |\det A| \in \{\pm 1\} \ [\text{Lemma } 6.14]}{\underset{\shortparallel}{\iota(\overset{\text{a linear vector field } \vec{V}(x) = Ax}{A}; 0)}}$$

Key point : If the singularity is degenerate, the local linear approximation of F @ 0 would not determine the local index.

Def Let X be a smooth vector field on M^n with only isolated singularities.

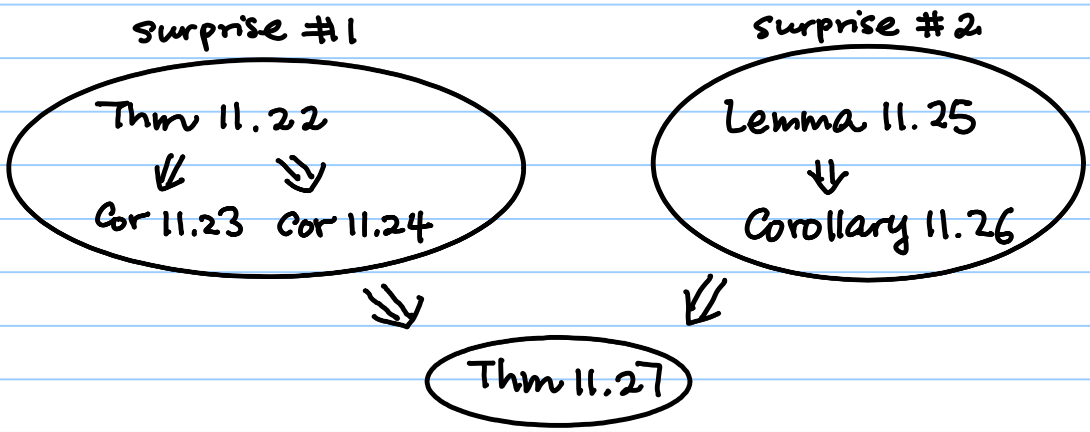
For a compact set $R \subseteq M$, define

$$\text{Index}(X;R) = \sum \iota(X;p),$$

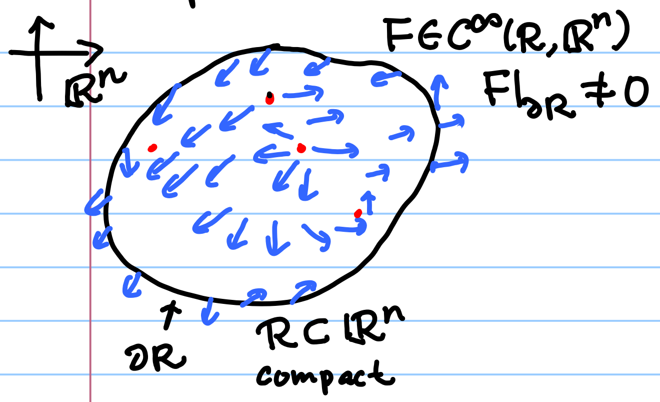
the summation runs over $p \in X^{-1}(\{0\})$ ← finite.

If M is compact, we write $\text{Index}(X)$ instead of $\text{Index}(X;M)$.

What come next are quite surprising.



Surprise #1

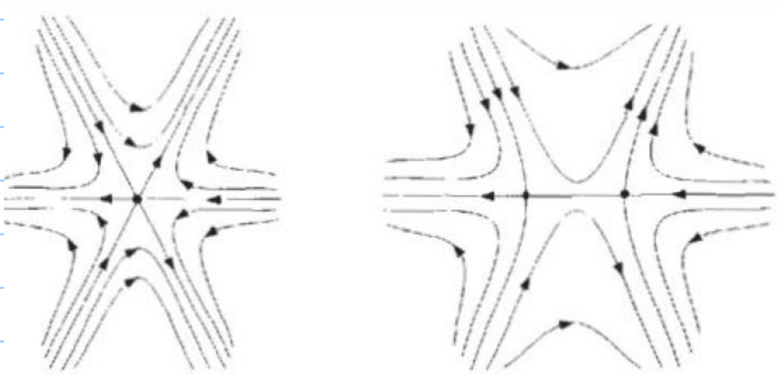


$\text{Index}(F,R) = \deg f$

$f : \partial R \to S^{n-1}$

$f(x) = F(x)/\|F(x)\|$

Surprise #2



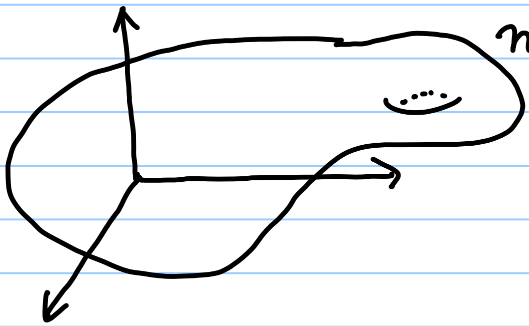
any degenerate singularity can be replaced by non-degenerate singularities

Surprise #1 + surprise #2 + tubular neighborhood thm

$\Downarrow$

Thm 11.27 Let $M^n \subseteq \mathbb{R}^{n+k}$ be a compact submanifold.

E.g.



$M^2 \hookrightarrow \mathbb{R}^3$

or

(Klein bottle) $\hookrightarrow \mathbb{R}^4$

N_ε = a tubular neighborhood of radius $\varepsilon > 0$ around M.

$g : \underbrace{\partial N_\varepsilon}_{} \to S^{n+k-1}$ outward pointing Gauss map

a compact hypersurface $\Rightarrow$ orientable

If X is an arbitrary smooth vector field on M^n with isolated singularities, then

$$\operatorname{Index}(X) = \deg g.$$

Surprise: Index(X) does not depend on X, it depends only on M

The Poincaré – Hopf theorem elucidates this in a more intrinsic way (i.e. not relying on any embedding of M):

$$\operatorname{Index}(X) = \chi(M^n) = \sum_{i=0}^{n} (-1)^i \underbrace{\dim_{\mathbb{R}} H^i(M^n)}_{b_i(M)}$$

ith Betti #