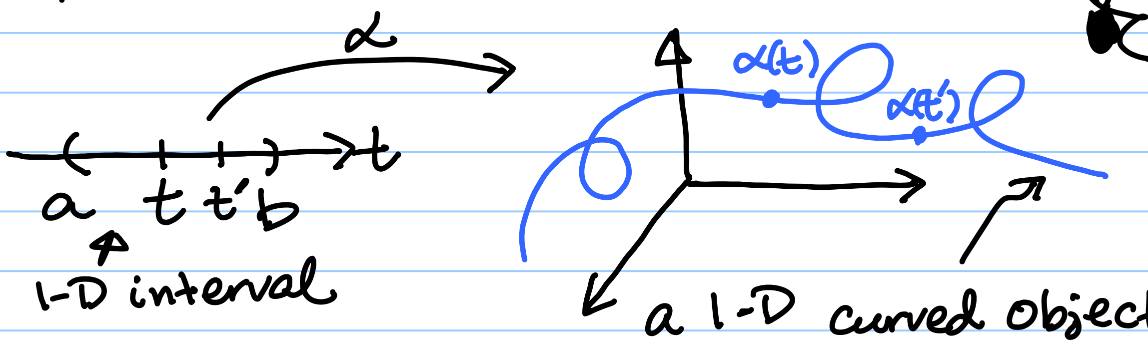
Lecture 1 : Local Theory of Curves

Note Title 12/29/2016

A map $\alpha : \overset{I}{(a,b)} \to \mathbb{R}^n$ is called a parameterized (or parametric) curve.

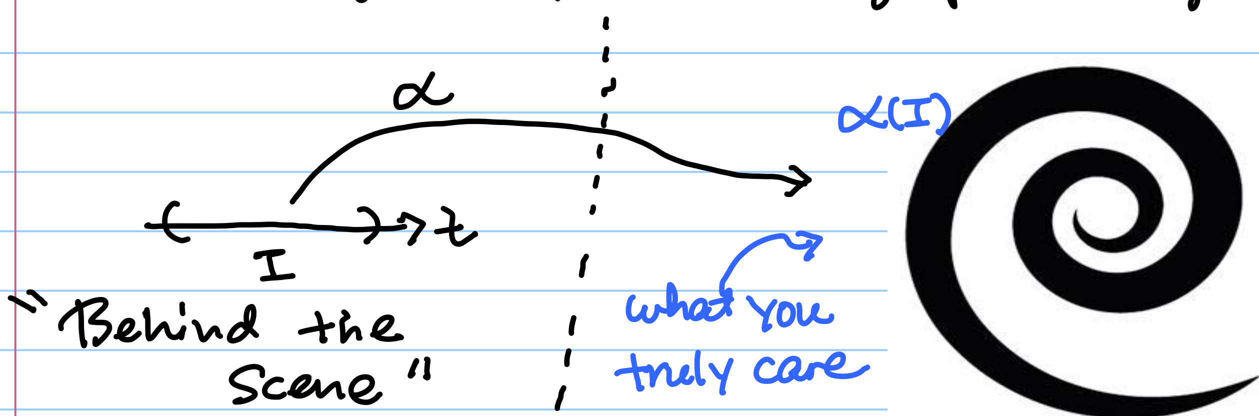


a 1-D curved object in a 3-D ambient space

Depending on the application, we may

① think of the map $\alpha : I \to \mathbb{R}^n$ as describing the motion of a point object, $\alpha'(t)$ is then the instantaneous velocity vector of the object at time t. (Time is a big deal, isn't it?)

② be only interested in the shape of $\alpha(I)$. E.g. if you are a graphics designer



In this case, what exactly the parameter 't' is isn't a big deal.

In particular, one is free to <u>reparameterize</u>.

From now on, we are interested in the latter case.

A parameterized curve α is said to be <u>regular</u> if

- α is C^1
- $\alpha'(t) \neq 0 \quad \forall t \in (a,b)$

Recall the length of the curve is

$$\text{Arc lengt} = \int_a^b \sqrt{\alpha_1'(t)^2 + \cdots + \alpha_n'(t)^2}\, dt$$

(Why? Integration + Pythagorean theorem + differentiation)

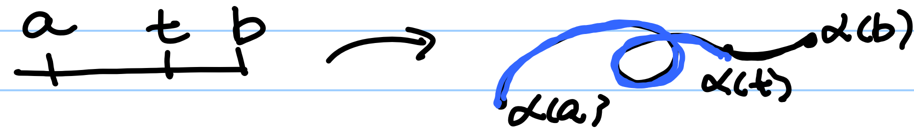
HW # 0

$$\text{or } \int_a^b \sqrt{\langle \alpha'(t), \alpha'(t) \rangle}\, dt$$

$$\text{or } \int_a^b \|\alpha'(t)\|\, dt$$

In particular,

$$s(t) := \text{Length of } \alpha([a,t)) = \int_a^t \|\alpha'(s)\|\, ds$$



α is regular $\Longrightarrow$ s is C^1, s^{-1} also C^1
(see HW # 1)

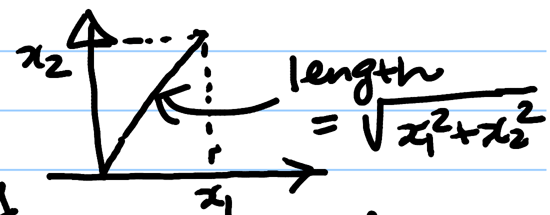
For latter purpose, we assume $\alpha(t)$ is C^3 smooth.

Recall the <u>inner product</u> of $\mathbb{R}^n$ is

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i \qquad x, y \in \mathbb{R}^n$$

$$= y^T x \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\|x\|^2 = \sum_{i=1}^{n} x_i^2 = \langle x, x \rangle$$

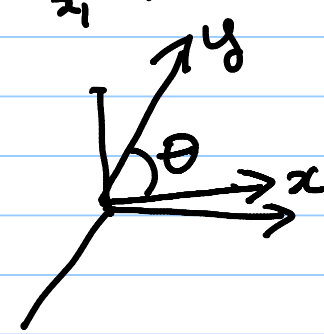


Cauchy-Schwartz inequality

$$\langle x, y \rangle \leq \|x\| \, \|y\|$$

$$|\langle x, y \rangle| = \|x\| \, \|y\| \cos\theta$$

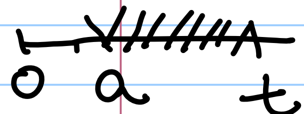
etc.



Note :

$$\|\alpha'(t)\| = 1 \quad \forall t$$

$$\Rightarrow \quad s(t) = \int_a^t \underbrace{\|\alpha'(\tau)\|}_{\equiv 1} d\tau = t - a$$



So the parameter value t is the arc length of α "measured from some point."

(easiest if you simply think $a = 0$) It's just a shift in the parameter line anyway.)

Conversely, if $t-a = \text{length}(\alpha[a,t])$ then

$$\int_a^t \|\alpha'(s)\|\,ds = t-a, \quad \forall t$$

$$\Big\downarrow \tfrac{d}{dt} \quad \text{(fundamental theorem of calculus)}$$

$$1 = \|\alpha'(t)\|, \quad \forall t.$$

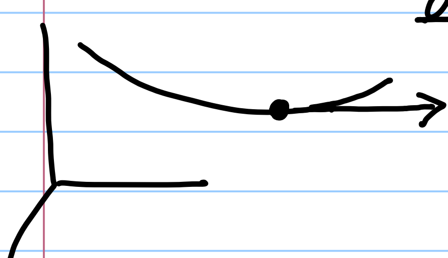
In this case, we say $\alpha : (a,b) \to \mathbb{R}$ is a curve <u>parametrized by arc length</u>

$\boxed{\text{Recap: if you only care about the shape of a curve, you may as well (re-) parameterize it by arclength.}}$

Why bother? Because it will help us to describe other geometrically meaningful quantities of the curve.

$\alpha'(s)$ = velocity vector of α at s

we don't happen to care about <u>velocity</u>, but just the <u>tangent direction</u>, so we may as well normalize this vector.

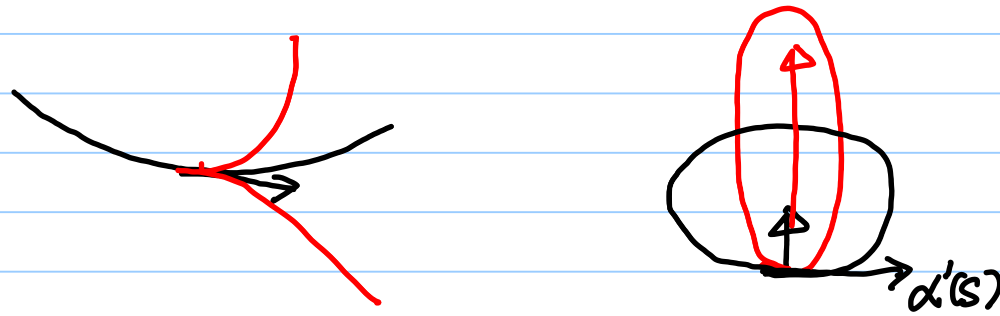


But this vector is already normalized.

NICE!

Next:

We would like to know how fast $\alpha'(s)$ is changing w.r.t arc length.



Consider the vector

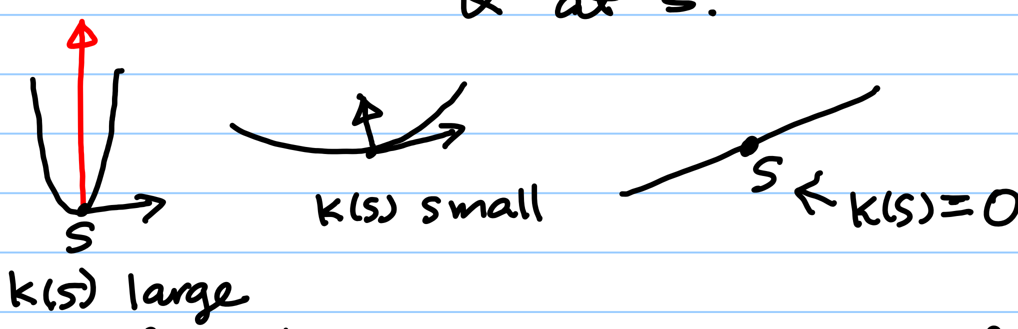
$$\frac{d}{ds}\alpha'(s) = \alpha''(s)$$

now a geometrically meaningful parameter

and the scalar

$$K(s) := \|\alpha''(s)\|$$

Def: $K(s)$ is called the curvature of α at s.



An interesting consequence of arc length parametrization

$$\langle \alpha'(s), \alpha'(s)\rangle \equiv 1 \Rightarrow \underbrace{\frac{d}{ds}\langle \alpha'(s), \alpha'(s)\rangle}_{} = 0$$

$$= \langle \alpha'(s), \alpha''(s)\rangle + \langle \alpha''(s), \alpha'(s)\rangle$$

$$= 2\langle \alpha'(s), \alpha''(s)\rangle$$

i.e. $\alpha'(s) \perp \alpha''(s) \quad \forall s$

On the one hand, $K(s)$ is meaningful already for planar curve, i.e. $\alpha: I \to \mathbb{R}^2$

(or a planar curve is disguise, i.e.
$\alpha: I \to \mathbb{R}^n \quad n \geq 3$, but
$\alpha(I) \subset$ a two-dimensional plane in $\mathbb{R}^n$

On the other hand, if a space (say) curve, i.e.

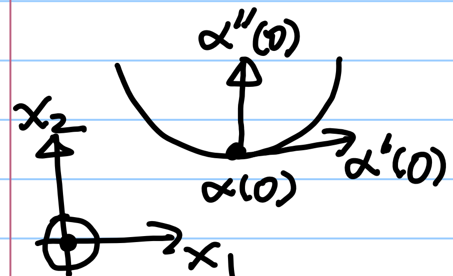
$$\alpha : I \to \mathbb{R}^{n=3}$$

is truly a space curve, then

$$\alpha'(s), \quad \alpha''(s), \quad K(s) = \|\alpha''(s)\|$$

do not completely describe the shape of the curve, at least not locally.

E.g. $\alpha(s) = \begin{bmatrix} \alpha_1(s) \\ \alpha_2(s) \\ 0 \end{bmatrix}$ "planar curve in disguise"



$\left(\text{or } R \begin{bmatrix} \alpha_1(s) \\ \alpha_2(s) \\ 0 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix} \right.$ a rotation matrix

if you want to disguise it better)

consider

$$\tilde{\alpha}(s) = \alpha(s) + s^3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then $\left.\begin{aligned} \tilde{\alpha}(0) &= \alpha(0) \\ \tilde{\alpha}(0) &= \alpha'(0) \\ \tilde{\alpha}''(0) &= \alpha''(0) \\ \tilde{K}(0) &= K(0) \end{aligned}\right\}$ these "measurements" alone cannot tell $\tilde{\alpha}$ is (slowly) leaving the x_1-x_2 plane near $s=0$.

Ambitious Question:

How many more "measurement(s)" should we introduce in order to "characterize the shape of the curve"?

The answer appears to depend on the co-dimension (i.e. n)

For $n=3$, seems like it will be good enough to introduce just $3-2=1$ quantity.

For $n=3$ (space curve), this last quantity is called the <u>torsion</u>.

From now on, $n=3$.

Write:

$$\alpha'(s) =: t(s)$$ "unit tangent vector @ s"

"normal vector @ s" $$n(s) := \frac{\alpha''(s)}{\|\alpha''(s)\|}$$ $\leftarrow$ $\kappa(s)$ assumed $\neq 0$

or $$\alpha''(s) = \kappa(s) \cdot n(s)$$

$$\mathrm{Span}(t(s), n(s)) = \text{"Osculating plane @ } s \text{" of } \alpha$$

Define $$b(s) := t(s) \times n(s)$$

a unit vector normal to the osculating plane @ s

"binormal vector @ s"

How fast does the binormal vector change w.r.t. arc length?

$$b'(s) = ?$$

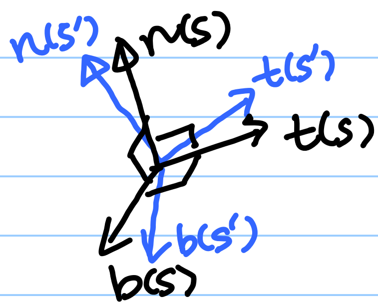
$$\frac{d}{ds} b(s) = \frac{d}{ds} t(s) \times n(s)$$

$$\overset{\text{why?}}{=} \underbrace{t'(s) \times n(s)}_{=0} + t(s) \times \underbrace{n'(s)}_{?}$$

This says $b' \perp t$

But $b' \perp b$
(same trick as before: $\langle b(s), b(s) \rangle \equiv 1$
$\Rightarrow \langle b'(s), b(s) \rangle = 0$)

Since there are only 3 dimensions,



$$b'(s) \parallel n(s)$$

so we may write

$$b'(s) = \underbrace{\tau(s)}_{\text{a scalar-valued function that we call the torsion @ s, well-defined when } \alpha''(s) \neq 0} n(s)$$

a scalar-valued function that we call the torsion @ s, well-defined when $\alpha''(s) \neq 0$

Recap:
$\alpha'(s) \neq 0$ means tangent vector exists

$\alpha''(s) \neq 0$ means osculating plane exists.

Note:

- if α is a "plane curve in disguise" (as defined earlier), then

$$\mathrm{span}(\alpha'(s), \alpha''(s)) = \text{the same plane for all } s.$$

Hence, $\tau(s) \equiv 0$

- conversely, if $\tau(s) \equiv 0$ (and $\kappa(s) \neq 0$) we have

$$b'(s) \equiv 0$$

$$\frac{d}{ds}\langle \alpha(s), b(0)\rangle = \langle \alpha'(s), \overset{\overset{b(s)}{\shortparallel}}{b(0)}\rangle = 0$$

so $\langle \alpha(s), \underbrace{b(0)}_{\text{a constant unit vector}} \rangle \equiv 0$

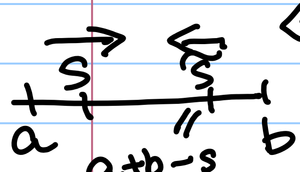
So $\alpha(s) \in$ the plane $\perp b(0)$.

We can reverse the orientation of a parameterized curve, and, in particular, that of an arc length parameterized curve.

Note:

$\alpha : (a,b) \to \mathbb{R}^n$ satisfies $\|\alpha'(s)\| \equiv 1$

$\iff$ $\tilde{\alpha} : (a,b) \to \mathbb{R}^n$, $\tilde{\alpha}(\tilde{s}) := \alpha(a+b-\tilde{s})$ satisfies $\|\tilde{\alpha}'(\tilde{s})\| \equiv 1$)



When we reverse orientation:

- $$\tilde{t}(\tilde{s}) = -t(s)$$

  (check $\tilde{\alpha}(\tilde{s}) = \alpha(\underbrace{a+b-\tilde{s}}_{s})$

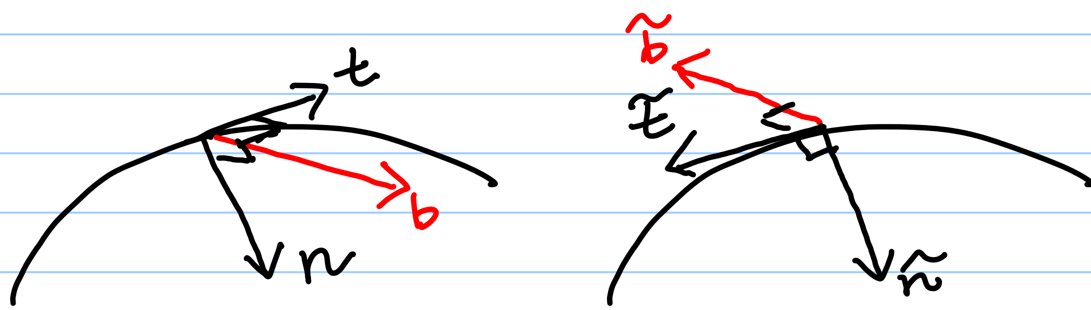
  $\Rightarrow \tilde{t}(\tilde{s}) = \tilde{\alpha}'(\tilde{s}) = -\alpha'(a+b-\tilde{s}) = -t(s)$)

- $$\tilde{n}(\tilde{s}) = n(s)$$ (check the same way)

- $$\tilde{b}(\tilde{s}) = -b(s)$$

  (check:
  $\tilde{b}(\tilde{s}) = \tilde{t}(\tilde{s}) \times n(\tilde{s}) = -t(s) \times n(s) = -b(s)$)

It follows $\tilde{b}'(\tilde{s}) = b(s)$
and $\tilde{\tau}(\tilde{s}) = \tau(s)$.



Note:

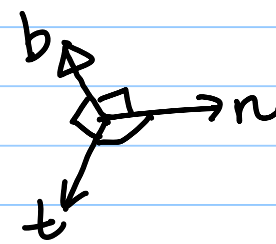
- curvature κ is always non-negative
- torsion τ has a sign (which has a geometric meaning)
- Both κ, τ remain invariant under a change of orientation.

So far, we have

$$t'(s) = \kappa(s)\, n(s)$$
$$b'(s) = \tau(s)\, n(s)$$

} by definition of curvature and torsion

Tempting to ask $n'(s) = ?$



$$b = t \times n \Rightarrow n = b \times t$$

$$n'(s) = b'(s) \times t(s) + b(s) \times t'(s)$$
$$= \tau(s) \underbrace{n(s) \times t(s)}_{-b(s)} + \underbrace{b(s) \times \kappa(s)\, n(s)}_{-\kappa(s)\, \underbrace{b(s) \times n(s)}_{-t(s)}}$$
$$= -\kappa(s)\, t(s) - \tau(s)\, b(s)$$

Now, write

$$\begin{bmatrix} | & | & | \\ t(s) & n(s) & b(s) \\ | & | & | \end{bmatrix}' = \begin{bmatrix} t(s) & n(s) & b(s) \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \quad (F)$$

3×3

These are called the Frenet formula.
It can be viewed as a system of ODE:

- first order
- linear, but
- variable coefficient

Basic ODE theory would tell us

that $\kappa(s), \tau(s)$ together with initial data

$$[t(0), n(0), b(0)]$$

would determine the curve uniquely.

BUT:

If we solve the system of ODEs (F), of which a solution is guaranteed by standard ODE theory by merely assuming

$\kappa(s)$, $\tau(s)$ are <u>continuous</u>,

would the solution necessarily satisfy:

$$\begin{bmatrix} t(s) & n(s) & b(s) \end{bmatrix}^T \begin{bmatrix} t(s) & n(s) & b(s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ?$$

If so, when we try to recover the curve by

$$\alpha(s) = \alpha(a) + \int_a^s t(s')\,ds' \quad (\Longleftarrow \alpha'(s) = t(s))$$

would this $\alpha(s)$ correspond to an arc-length parameterized curve with the prescribed curvature and torsion functions $\kappa(s)$ and $\tau(s)$?

[see my computer demos / examples]

- For instance, we can make up a $\kappa(s)$ with negative values, there is nothing from the view of standard ODE theory to deny a solution of (F) in $\mathbb{R}^9$. But sure enough, there is no curve with such a (negative-valued) curvature function.

- If we merely assume $\kappa(s)$, $\tau(s)$ are continuous, the resulted $t(s)$ may only be C^1, $\alpha(s) := \int_a^s t(\sigma)d\sigma$ only C^2.

With standard ODE theory + extra arguments (addressing the above issues):

Fundamental Theorem: Given C^1 functions $\kappa(s) > 0$, $\tau(s)$, $s \in I$, $\exists$ a C^3 regular parameterized curve $\alpha : I \to \mathbb{R}^3$ such that s = arc length, $\kappa(s)$ = curvature, $\tau(s)$ = torsion of α.

This α is unique up to rigid motion. (i.e. any other $\bar{\alpha}$ satisfying the same conditions is related to α by

$$\bar{\alpha} = A \circ \alpha + c \quad \text{for some } A \in SO(3),\ c \in \mathbb{R}^3.)$$

Def: $SO(3) := \{A \in \mathbb{R}^{3\times 3} : A^T A = I, \det(A) = 1\}$.

$$so(3) := \{A \in \mathbb{R}^{3\times 3} : A = -A^T\}$$

As we have seen, the Frenet frame ODE is not any ODE system in $\mathbb{R}^9$, it has additional structure.

As a hint:

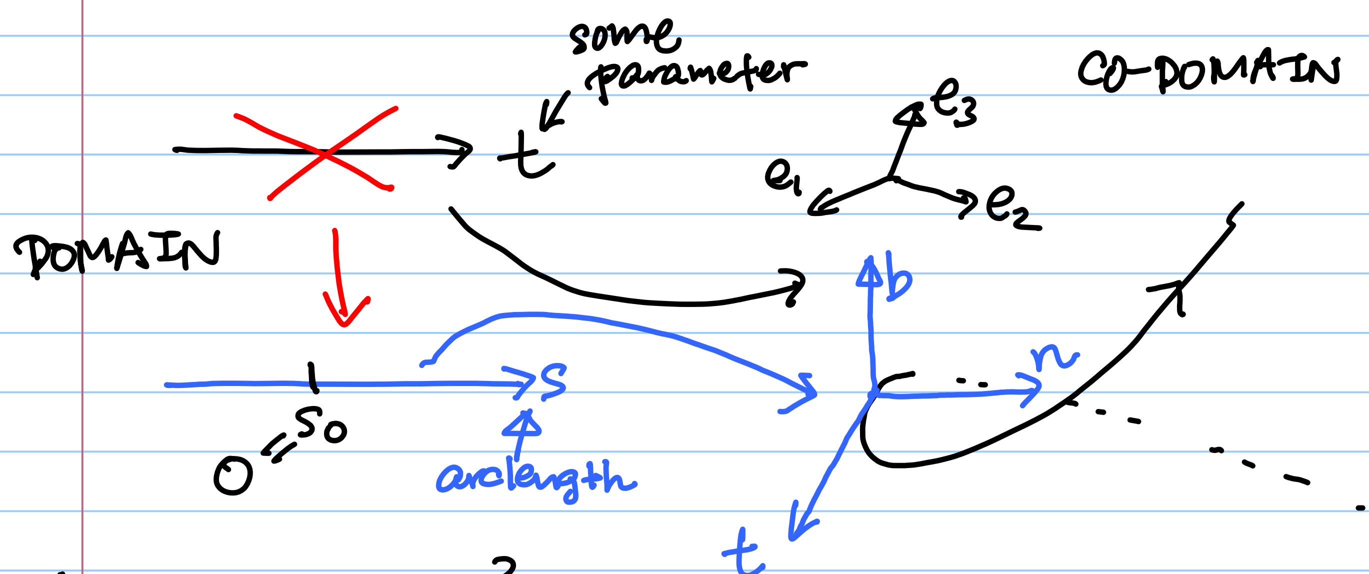
$$\begin{bmatrix} t(s) & n(s) & b(s) \end{bmatrix}' = \underbrace{\begin{bmatrix} t(s) & n(s) & b(s) \end{bmatrix}}_{\in SO(3)} \underbrace{\begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix}}_{\in so(3)}$$

We shall see that the appearance of a skew-symmetric matrix function is not an accident.

The local canonical form.

High level remark:
it is often effective to find a coordinate system / representation adpated to the problem.

In this case, to understand the behavior of a curve locally, it is best to represent the curve in "t-b-n" coordinates.



Assuming $\alpha \in C^3$

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + \underbrace{R(s)}_{o(s^3)}$$

where $\lim_{s \to 0} \frac{R(s)}{s^3} = 0$

$\alpha'(0) = t$

$\alpha''(0) = \kappa n, \qquad \alpha''(s) = \kappa(s)\, n(s)$

$\alpha'''(0) = \kappa'(0)\, n(0) + \kappa(0) \underbrace{n'(0)}_{-\kappa t - \tau b}$

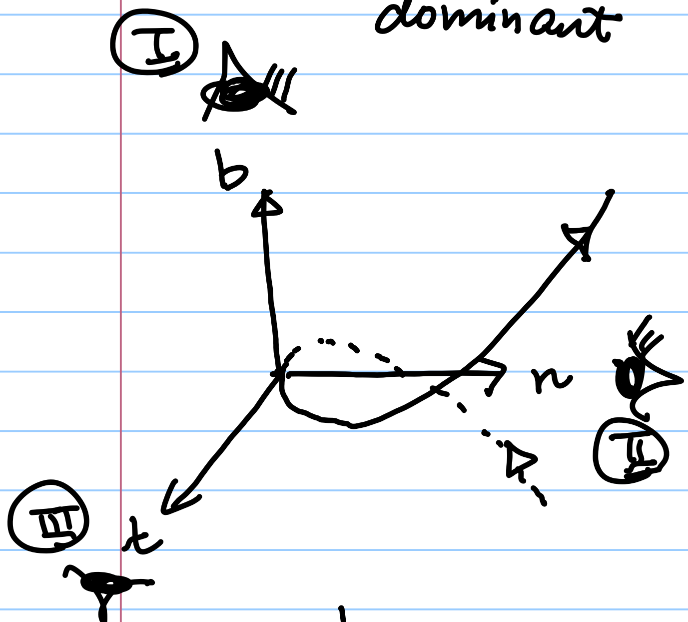
So

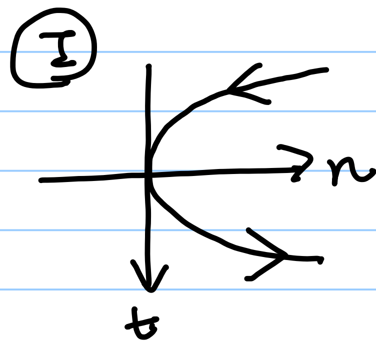
$$\alpha(s)-\alpha(0) = \underbrace{\left(s-\frac{\kappa^2 s^3}{3!}\right)t}_{O(s)} + \underbrace{\left(\frac{s^2\kappa}{2}+\frac{s^3\kappa'}{3!}\right)n}_{O(s^2)} - \underbrace{\frac{s^3}{3!}\kappa\tau\, b}_{O(s^3)} + \underbrace{R(s)}_{o(s^3)}$$

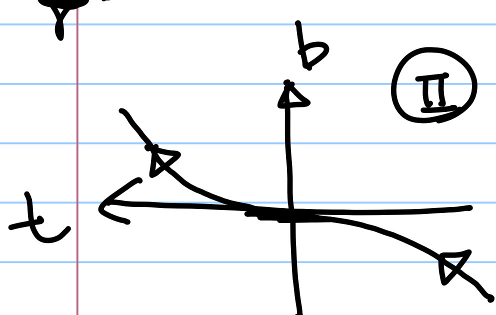
dominant

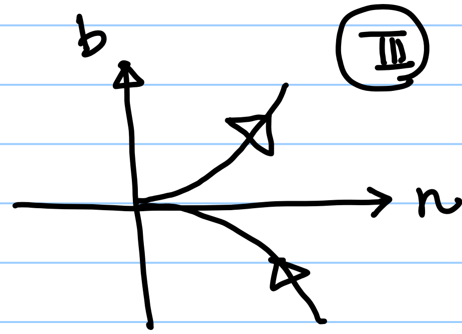
"2nd"

"3rd"









Of course, we know that the tangent direction provides the (DORMINANT!) local linear approximation,

(★) tells you what happen in the two remaining (subdominant) directions.

There are more details to work out.

For example, in practice one may use specific functions (notably <u>splines</u>) to model curves parametrically. There is no reason that such a parameterization is an arc length parameterization. Would be useful to have formulas for curvature and torsion.

It can be shown:

If $\alpha : I \to \mathbb{R}^3$ is regular (but not necessarily by arc length)

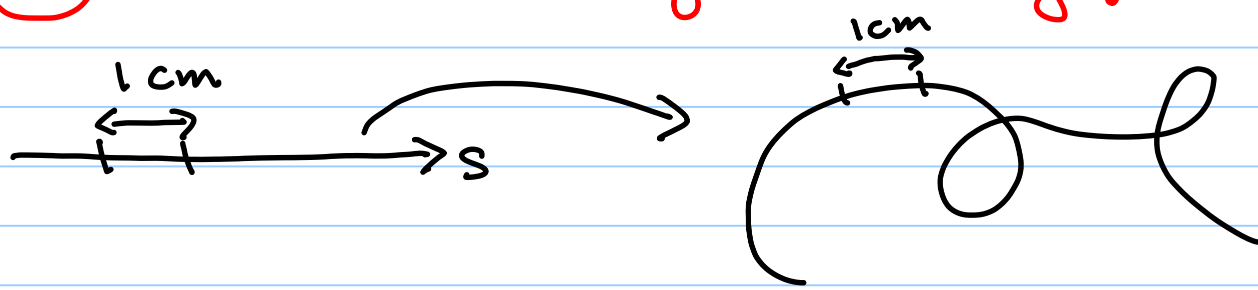
$$\kappa(t) = \|\alpha'(t) \times \alpha''(t)\| / \|\alpha'(t)\|^3$$

$$\tau(s) = -\frac{(\alpha'(t) \times \alpha''(t)) \cdot \alpha'''(t)}{\|\alpha'(t) \times \alpha''(t)\|^2}.$$

Final remark :

The fundamental theorem also means :

we can think of a curve in $\mathbb{R}^3$ as being obtained from a straight line by bending (curvature) and twisting (torsion), but without any stretching !



As we shall see (but not so soon), the same is not true for surfaces.

E.g. Impossible to map a part of the sphere to the plane without distorting distance / area.

Understanding this phenomenon is a big part of classical differential geometry (Gauss' Theorema Egregium)

It is also what stimulates the modern theory of Riemannian manifold.

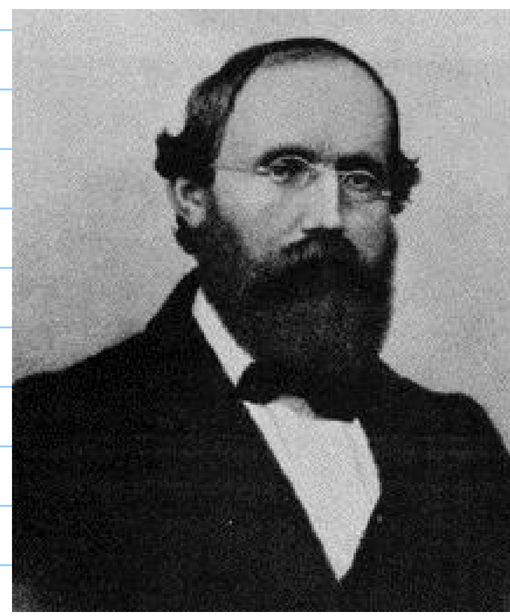
Now, I have written 18 pages or so telling you that "curves have curvatures".

Confusingly, because of $\star$, in the modern theory of Riemannian manifold:

" curves (= a 1-D Riemannian manifold) always have zero curvature"

But this latter 'curvature' refers to a kind of 'intrinsic curvature' that we can only begin to understand through a careful study of surfaces.

Gauss



Riemann