Week 8 Game Theory

Note Title 5/16/2021

<u>Prisoner's dilemma</u> Two criminals are caught. During plea bargaining, the District Attorney urges both criminals to confess and plead guilty.

They are separately offered the same deal:

Jail time of criminal X

$$A = \begin{array}{cc} & \\ \begin{bmatrix} 5 & 0 \\ 10 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow X \text{ confesses} \\ \leftarrow X \text{ not confesses.} \end{array} \\ \begin{array}{cc} \nearrow \text{Y confesses} & \nwarrow \text{Y not confesses} \end{array} & \end{array}$$

Jail time of criminal Y

$$B = \begin{array}{cc} & \\ \begin{bmatrix} 5 & 10 \\ 0 & 1 \end{bmatrix} & \begin{array}{l} \leftarrow X \text{ confesses} \\ \leftarrow X \text{ not confesses} \end{array} \\ \begin{array}{cc} \nearrow \text{Y confesses} & \nwarrow \text{Y not confesses} \end{array} & \end{array}$$

If a (wise) Godfather advises his mob to never confess when caught, then there would be no dilemma, each would spend a year in jail.
Notice that it is a form of cooperation, and is a good compromise in this case.

What if these criminals are unorganized?

Not knowing what the other criminal has in mind, and each being rational, each try to optimize the worst scenario:

X solves : $\min_i \max_j A_{ij} = \min \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 = A_{11}$ so X chooses to confess

Y solves : $\min_j \max_i B_{ij} = \min [5, 10] = 5 = B_{11}$ Y chooses to confess also

A different (but subtly related) viewpoint :

$\begin{bmatrix} 5 & 0 \\ 10 & 1 \end{bmatrix}$ ← X chooses to confess, because he is better off regardless of Y's choice — (I)

$\begin{bmatrix} 5 < 10 \\ 0 < 1 \end{bmatrix}$ Y chooses to confess, because he is better off regardless of X's choice — (II)

Yet another viewpoint :

Y thinks X would confess because of (I), so Y confesses also.

X thinks Y would confess because of (II), so X confesses also.

$\boxed{\text{Conclusion : In a non-cooperative setting (and assuming each player is perfectly "rational"), each criminal chooses to confess.}}$

The prisoner's dilemma is an example of a non-zero sum game $(A+B \neq 0)$

In a non-zero sum game, the two players need not be hostile to each other. Cooperation may lead to a "win-win" situation, much better than that offered by a (non-cooperative) Nash equilibrium.

In a zero sum game $(A+B=0)$, there is no point to cooperate.

Examples of zero-sum games:

① A = payoff matrix for $X = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (A_{ij} = payoff to X if X applies strategy i and Y applies strategy j)

payoff matrix for $Y = -A$.

Notice $A = \begin{bmatrix} 2 \leq 2 \\ \vee \quad \wedge \\ 1 \leq 3 \end{bmatrix}$. Clearly Y would always choose strategy $j^*=1$ to minimize his loss. We call this a dominant strategy of Y. X, being perfectly rational, knows Y thinks this way, would always choose $i^*=1$ to maximize her gain. (And it would not matter that Y knows X thinks that way.)

The (trivial) Nash equilibrium is $(i^*, j^*) = (1,1)$.

② $A = \begin{bmatrix} 1 < 2 < 3 \\ 3 > 2 = 2 \end{bmatrix}$. There is neither a dominant strategy for X nor for Y.

Nonetheless, this payoff matrix satisfies the following "trivial Nash equilibrium property":

$$\max_i \left(\min_j A_{ij}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 = A_{22}$$

$$\min_j \left(\max_i A_{ij}\right) = [3, 2, 3] = 2 = A_{22}, \quad \text{with } (i^*, j^*) = (2,2)$$

And $\underbrace{A_{ij^*} \leq A_{i^*j^*} \leq A_{i^*j}}$

if X keeps wielding strategy i^*, Y is never better off using any strategy other than j^*
if Y keeps wielding strategy j^*, X is never better off using any strategy other than i^*.

In general,

Proved below

(i) $\max_i \left(\min_j A_{ij}\right) \leq \min_j \left(\max_i A_{ij}\right)$ (Proof: $\max_i \left(\min_j A_{ij}\right) \leq \max_i A_{ij} \ \forall j$)

(ii) $\underbrace{\max_i \left(\min_j A_{ij}\right) = \min_j \left(\max_i A_{ij}\right)}_{\text{the "max min = min max" property}} \iff \underbrace{\exists\, i^*, j^* \text{ s.t. } A_{ij^*} \leq A_{i^*j^*} \leq A_{i^*j}, \ \forall i, j}_{\text{a saddle point property (or what I called a "trivial Nash equilibrium")}}$

exercise for you

(iii) Existence of a dominant strategy for either player $\Longrightarrow$ / $\not\Longleftarrow$ the trivial Nash equilibrium property

Example ② above

i.e. $\exists\, i^*$ s.t. $\forall i,j$, $A_{i^*j} \geq A_{ij}$

OR

$\exists\, j^*$ s.t. $\forall i,j$, $A_{ij^*} \leq A_{ij}$

i.e.

$\exists\, i^*, j^*$ s.t.

$A_{ij^*} \leq A_{i^*j^*} \leq A_{i^*j}$, $\forall i,j$.

Note: Most payoff matrices do not satisfy the trivial Nash equilibrium property.

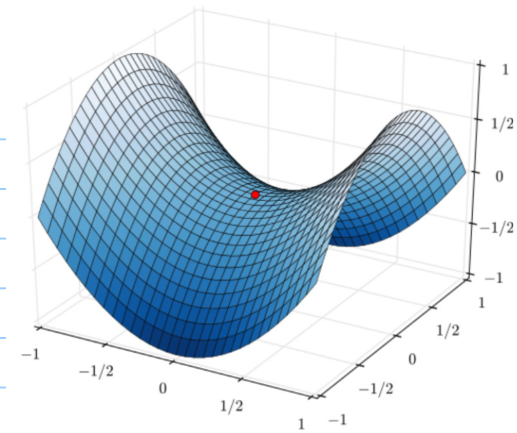
E.g. $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$ $\quad \max_i \min_j A_{ij} = \max \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2$, $\quad \min_j \max_i A_{ij} = \min [3 \;\; 1] = 1$

So there isn't a 'trivial' Nash equilibrium in this case.

Assume that you are alone, and you are lonely, a very attractive person invites you to be the X player (or the "row player").

An additional rule: If you agree to play, you have to play many games with this attractive person.

Can you resist?



A more general observation: (This proves (i) and (ii) above)
in particular

Let $f: A\times B \to \mathbb{R}$ (A, B can be any sets). Then:

(i) $\max\limits_{a\in A} \min\limits_{b\in B} f(a,b) \le \min\limits_{b\in B} \max\limits_{a\in A} f(a,b)$

(ii) $\max\limits_{a\in A} \min\limits_{b\in B} f(a,b) = \min\limits_{b\in B} \max\limits_{a\in A} f(a,b) \iff \exists (a^*, b^*) \in A\times B$ s.t. $f(a,b^*) \le f(a^*,b^*) \le f(a^*,b)$ $\forall a\in A, b\in B$.

Proof:

(i) $\max\limits_{a\in A} (\min\limits_{b\in B} f(a,b)) \le \max\limits_{a\in A} f(a,b)$ for any fixed $b\in B$

so $\max\limits_{a\in A} (\min\limits_{b\in B} f(a,b)) \le \min\limits_{b\in B} \max\limits_{a\in A} f(a,b)$.

(ii) ($\Rightarrow$) Assume a^* solves the 'maxmin', b^* solves the 'minmax'.

$$f(a^*,b^*) \le \max_{a\in A} f(a,b^*) = \min_{b\in B} \max_{a\in A} f(a,b) = \max_a \min_b f(a,b) = \min_b f(a^*,b) \le f(a^*,b)$$

Similarly, $f(a^*,b^*) \ge \min\limits_b f(a^*,b) = \max\limits_a \min\limits_b f(a,b) = \min\limits_b \max\limits_a f(a,b) = \max\limits_a f(a,b^*) \ge f(a,b^*)$

($\Leftarrow$) Assume that the saddle point property holds with (a^*, b^*).

We argue by contradiction that a^* solves the 'maxmin', b^* solves the 'minmax'.

Assume the contrary that a^* does not solve the maxmin, then

$$\max_a \min_b f(a,b) > \min_b f(a^*,b) = f(a^*,b^*) = \max_a f(a,b^*) \geq \min_b \max_a f(a,b),$$

which contradicts maxmin $\leq$ minmax.

So a^* solves the maxmin. Similarly, b^* solves the minmax. And we have

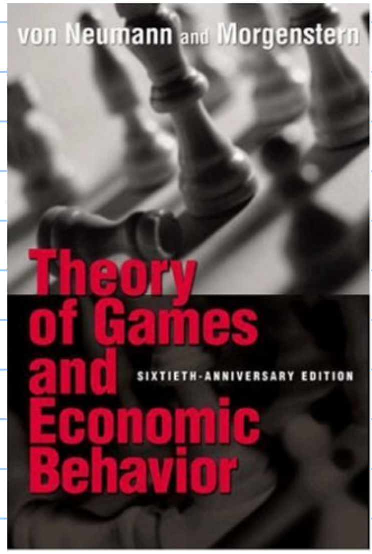
$$\max_a \min_b f(a,b) = f(a^*,b^*) = \min_b \max_a f(a,b)$$

Q.E.D.

If such a zero-sum game is to be played many times, and assuming each player is perfectly rational, then

- neither player wants to be predictable, meaning that the choice of strategy in each game should be made independent from any previous game.
- so the decision to be made for each player is how often he/she should employ each of his/her strategies.

Let x_i = frequency/probability that player X uses strategy i, $i=1,\cdots,m$
y_j = ________________________________ j, $j=1,\cdots,n$



Prob[X uses i <u>and</u> Y uses j] $= x_i y_j$
(in each game)

So, in the long run, player X's average payoff (= player Y's average loss) is:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}\, x_i y_j \quad \text{per game.}$$

$$= x^T A y \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

Player X wants to maximize $x^T A y$ by choosing x, but he has no control of y.
Player Y ———— minimize ———————— y, but she ———————— x.

However,
X expects that whatever x he chooses, Y would pick y to hurt him the most.
Likewise,
Y expects that whatever y she chooses, X would pick x to hurt her the most.

So, ideally, X would like to solve $\max_x \min_y x^T A y$

Y ———————————— $\min_y \max_x x^T A y$.

But if Y is busy hurting X, can she still afford to minimize her loss?
Same concern for X.

These problems look ill-defined, let alone having meaningful solutions!

Terminology: a choice by X of a probability vector $x \in \Delta_m = \{ x \in \mathbb{R}^m_+ : \sum x_i = 1 \}$ is called a mixed strategy of X.

Similarly, a $y \in \Delta_n$ is a mixed strategy of Y.

If $x = [0, \cdots, 1, \cdots 0]^T \in \Delta_m$, $y = [0, \cdots, 1, \cdots 0]^T \in \Delta_n$, that's called a **pure strategy**.

Applying our earlier result to $f : \Delta_m \times \Delta_n \to \mathbb{R}$, $(x, y) \mapsto x^T A y$, we have

- $\max\limits_{x} \min\limits_{y} x^T A y \leq \min\limits_{y} \max\limits_{x} x^T A y$
- Saying $\min\limits_{y} x^T A y = \min\limits_{y} \max\limits_{x} x^T A y$ is equivalent to $\exists\, x^* \in \Delta_m, y^* \in \Delta_n$ s.t.

$$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y \quad \forall x \in \Delta_m, y \in \Delta_n.$$

In the earlier (non-randomized, pure strategy) setting:

$$f : \{1, \cdots, m\} \times \{1, \cdots, n\} \to \mathbb{R}, \quad f(i,j) = A_{ij} = \begin{array}{c} \overset{i}{\downarrow} \\ [0 \cdots 1 \cdots 0] \end{array} A \begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \end{array} \end{array},$$

for most payoff matrix A, $\max \min < \min \max$.

$\Leftrightarrow$ $f(i,j)$ has no saddle point

But in the randomized, mixed strategy setting,

$$f : \underset{\text{convex}}{\underset{\uparrow}{\Delta_m}} \times \underset{\text{convex}}{\underset{\uparrow}{\Delta_n}} \to \mathbb{R}, \quad (x,y) \mapsto x^T A y, \quad \leftarrow \begin{array}{l}\text{linear (hence concave) in } x \text{ for fixed } y \\ \text{linear (hence convex) in } y \text{ for fixed } x\end{array}$$

So, from this point of view, the landscape of this function resembles that of

$$y^2 - x^2 \quad \leftarrow \text{concave in } x, \text{ convex in } y, \text{ saddle point at } (0,0).$$

And perhaps f always has a saddle point.

And this is exactly what J. von Neumann discovered in 1928:

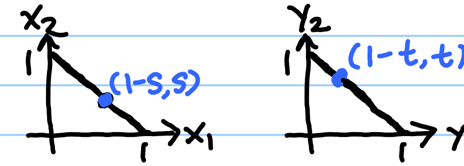
> For any $A \in \mathbb{R}^{m \times n}$, $\exists\, x^* \in \Delta_m$, $y^* \in \Delta_n$ s.t.
>
> $$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y \quad \forall\, x \in \Delta_m,\ y \in \Delta_n$$
>
> $$x^{*T} A y^* = \max_x \min_y x^T A y = \min_y \max_x x^T A y$$

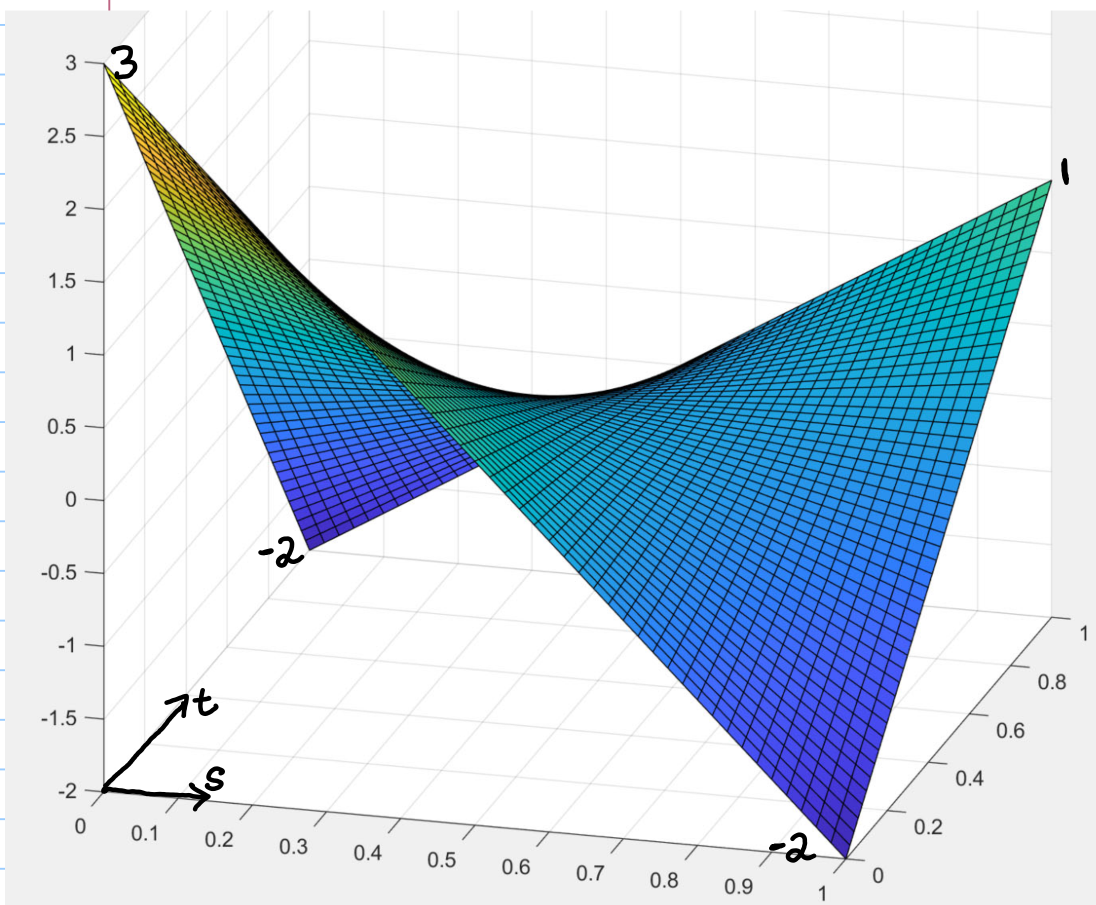
called the "value of the game"

(The equilibrium mixed strategies x^* and y^* need not be unique, but the value of the game is unique.)

For our earlier example,

$$x^T A y = [1-t,\ t]\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1-s \\ s \end{bmatrix}$$

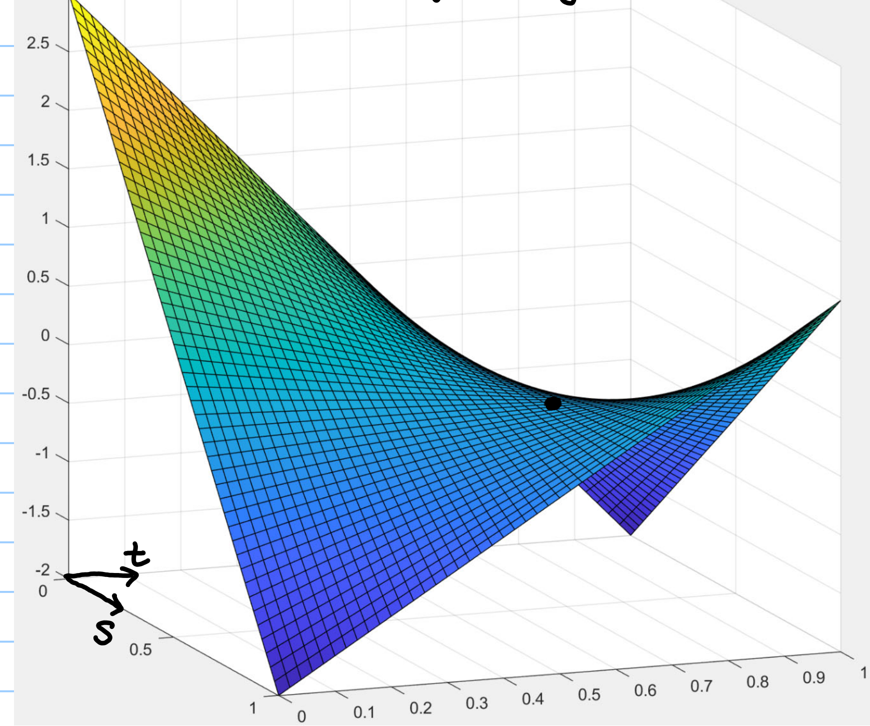




$\Delta_2 \cong [0,1]$

Where is the saddle point?

Value of the game?



When $m=n=2$, write $[x_1, x_2] = [1-s, s]$, $[y_1, y_2] = [1-t, t]$

$$p(s,t) = [x_1, x_2] A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [1-s, s] \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1-t \\ t \end{bmatrix} = A_{11} + (A_{21} - A_{11}) s + (A_{12} - A_{11}) t + (A_{11} - A_{12} - A_{21} + A_{22}) st$$

$$\| $$

Critical point : $s^* = (A_{11} - A_{12}) / (A_{11} - A_{12} - A_{21} + A_{22})$

$t^* = (A_{11} - A_{21}) / (A_{11} - A_{12} - A_{21} + A_{22})$

$$\text{const.} + \underbrace{\frac{1}{2} \begin{bmatrix} s-s^* \\ t-t^* \end{bmatrix}^T \begin{bmatrix} 0 & A_{11}-A_{12}-A_{21}+A_{22} \\ A_{11}-A_{12}-A_{21}+A_{22} & 0 \end{bmatrix} \begin{bmatrix} s-s^* \\ t-t^* \end{bmatrix}}_{= (s-s^*)(t-t^*)(A_{11}-A_{12}-A_{21}+A_{22})}$$

As long as $A_{11} - A_{12} - A_{21} + A_{22} \neq 0$, the Hessian is negative definite, and the critical point is a saddle point. In fact,

$$p(s, t^*) = p(s^*, t^*) = p(s^*, t), \quad \forall s, t.$$

This does not prove von Nemann's theorem even when $m=n=2$, because s^* or t^* may not be in $[0,1]$.

But when $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$, $s^* = t^* = 5/8 \in [0,1]$.

The value of the game is $[3/8 \;\; 5/8] \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} = -1/8$.

Interpretation: if the Y player holds her hand up $3/8$ of the time and hand down $5/8$ of the time,
the X player losses at least \$ $1/8$ per game on average regardless of what he does.

$A^T \neq -A$, so the game is by definition not fair. And indeed it is not fair!
(i.e. its value $\neq 0$)

(It is easy to construct a game with value 0, but $A^T \neq -A$:

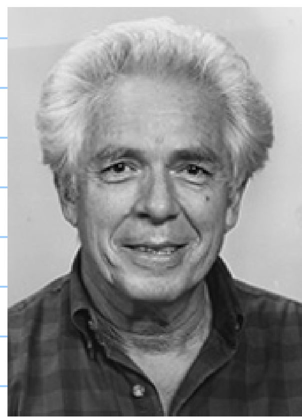
Take an arbitrary zero-sum game with payoff matrix A, subtract from every entry of A the value of the game. The resulted matrix $\tilde{A}$ must be the payoff matrix of a game with 0 value.)

Von Neumann's original proof (1928) is based on a non-constructive argument, using a fixed point theorem.

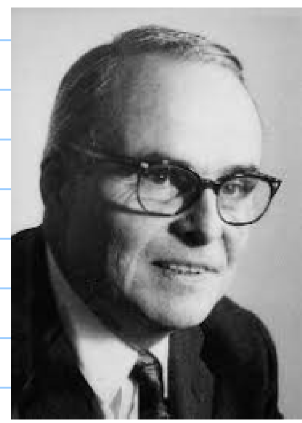
It inspired J. Nash to extend the result to multiple players, non-zero sum, non-cooperative games (1950), again using a non-constructive fixed point argument.

LP began before 1947, Gale, Kuhn, Tucker published the first proof of duality in 1951, again inspired by von Neumann's minimax theorem.

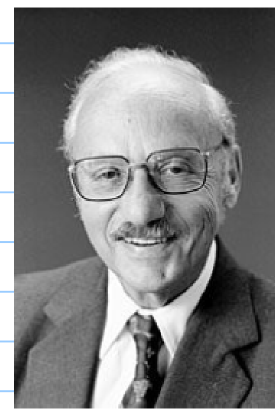
J. von Neumann



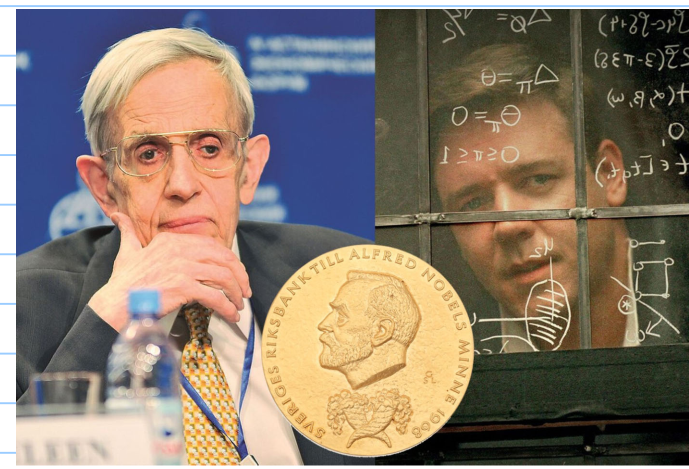
H. Kuhn



A. Tucker



G. Dantzig



J. Nash

KKT

We now reverse history by proving the minimax theorem — constructively — using LP duality.

Consider the primal-dual pair of LPs:

(P) $\min_{x \in \mathbb{R}^m} [1,\cdots,1]x$ st. $A^T x \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $x \geq 0$ (D) $\max_{y \in \mathbb{R}^n} [1,\cdots,1]y$ st $Ay \leq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $y \geq 0$

Note that (D) is always feasible, as $y = [0,\cdots,0]^T$ is a feasible point for (D).

(P), however, is not always feasible.

Ex (tricky): Prove that for a fair game (i.e. $A^T = -A$), (P) is never feasible.

Nonetheless, if we change the payoff matrix A to $A + \alpha \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ for a big enough $\alpha > 0$, the resulted (P) must be feasible. (why?)

Also, adding a constant α to all entries of A only increases the expected payoff by the same α and cannot affect the optimal mixed strategies of either player, since

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j (A_{ij} + \alpha) = \sum_i \sum_j x_i y_j A_{ij} + \alpha \Big(\underbrace{\sum_i \sum_j x_i y_j}_{=1} \Big).$$

With such a fix, both (P) and (D) are feasible, and hence, by strong duality, have the same finite optimal value. In other words,

$$\exists\, \bar{x} \in \mathbb{R}^m, \bar{y} \in \mathbb{R}^n \text{ s.t. } \bar{x} \text{ solves (P)}, \bar{y} \text{ solves (D) and}$$

$$\sum \bar{x}_i = \sum \bar{y}_j \leftarrow \text{call this common value } \theta$$ ($\theta > 0$, as $x=0$ is infeasible for (P))

Let $x^* = \frac{1}{\theta}\bar{x} \in \Delta_m$, $y^* = \frac{1}{\theta}\bar{y} \in \Delta_n$. (These are then probability distributions.)

We now show that x^*, y^* have the desired saddle point property.

Since $A^T\bar{x} \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, for any $y \in \Delta_n$, $y^T A^T \bar{x} \geq y^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1$.

Similarly, since $A\bar{y} \leq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, for any $x \in \Delta_m$, $x^T A \bar{y} \leq x^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1$.

So, $\quad x^T A \bar{y} \leq 1 \leq \bar{x}^T A y$, or $x^T A y^* \leq \underset{\substack{\uparrow \\ \text{value of the game}}}{1/\theta} \leq x^{*T} A y$, $\forall x \in \Delta_m, y \in \Delta_n$

QED.

Example: $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (This is the # of days the Allies had in bombing the Japanese convoy in the Battle of the Bismarck Sea.)

Note that this matrix has the "trivial Nash equilibrium" property:

$$\max_i \min_j A_{ij} = \max\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = 2 = A_{11}$$

$$\min_j \max_i A_{ij} = \min[2, 3] = 2 = A_{11}$$

In fact, there is a dominant strategy for Y:

$$\begin{array}{cc} & \downarrow \\ \rightarrow & \begin{bmatrix} 2 \leq 2 \\ 1 \leq 3 \end{bmatrix} \end{array}$$

This means no matter how X chooses his pure strategy, Y is better off choosing his first pure strategy.

Knowing this, X chooses his first pure strategy to maximize his gain.

$x^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $y^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ value of the game $= A_{11} = 2$

Let's double check this using our LP approach.

Solve the primal-dual pair of LP:

(P) $\min\limits_{x} \; [1,1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t. $A^T x \geq \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x \geq 0$

(D) $\max\limits_{y} \; [1,1]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ s.t. $y^T A^T \leq [1,1], x \geq 0$

Note: using the dual simplex method avoids Phase I.

```
>> A = [2 2; 1 3]; % payoff matrix
>> b = [1;1]; p = [1;1];
>> T = totbl(A',b,p);
              x1         x2          1
        -----------------------------------
 x3 = |     2.0000     1.0000    -1.0000
 x4 = |     2.0000     3.0000    -1.0000
        -----------------------------------
 z  = |     1.0000     1.0000     0.0000
>>  T = dualbl(T);
                  u3 =       u4 =        w  =
                  x1         x2          1
            -----------------------------------
-u1   x3 = |     2.0000     1.0000    -1.0000
-u2   x4 = |     2.0000     3.0000    -1.0000
            -----------------------------------
 1    z  = |     1.0000     1.0000     0.0000
```

```
>> T = ljx(T,1,1);
                  u1 =       u4 =        w  =
                  x3         x2          1
            -----------------------------------
-u3   x1 = |     0.5000    -0.5000     0.5000
-u2   x4 = |     1.0000     2.0000     0.0000
            -----------------------------------
 1    z  = |     0.5000     0.5000     0.5000
>> T = ljx(T,2,2);
                  u1 =       u2 =        w  =
                  x3         x4          1
            -----------------------------------
-u3   x1 = |     0.7500    -0.2500     0.5000
-u4   x2 = |    -0.5000     0.5000    -0.0000
            -----------------------------------
 1    z  = |     0.2500     0.2500     0.5000
```

Solutions:

$\bar{x} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \bar{y} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

$\theta = 1/2$

or

$\bar{x} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}, \bar{y} = \begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$

or

$\bar{y} = (1-t)\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1/4 \\ 1/4 \end{bmatrix}$

$0 \leq t \leq 1$

$$x^* = \frac{1}{\theta}\begin{bmatrix}1/2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad y^* = \frac{1}{\theta}\begin{bmatrix}1/2\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}, \quad \text{value of the game} = 1/\theta = 2$$

OR

$$y^* = (1-t)\begin{bmatrix}1\\0\end{bmatrix} + t\begin{bmatrix}1/2\\1/2\end{bmatrix} = \begin{bmatrix}1-\frac{1}{2}t\\ \frac{1}{2}t\end{bmatrix} \quad \text{any } 0 \le t \le 1$$

this means Y uses his 1st pure strategy no less often than the 2nd.

Back to our original example $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$. ← your payoff matrix

Let's see the value of this game. (If it is non-negative, play it. Otherwise, don't play it for too long, unless you really cannot resist Y!)

```
>> A = [3 -2; -2 1]; % payoff matrix
>> b =[1; 1]; p=[1; 1];
>> T = totbl(A',b,p); T = dualbl(T);
               x1        x2          1
          ----------------------------------
 x3  = |    3.0000   -2.0000    -1.0000
 x4  = |   -2.0000    1.0000    -1.0000
          ----------------------------------
 z   = |    1.0000    1.0000     0.0000
                  u3 =      u4 =        w   =
                  x1        x2          1
              ----------------------------------
-u1  x3 = |    3.0000   -2.0000    -1.0000
-u2  x4 = |   -2.0000    1.0000    -1.0000
              ----------------------------------
 1   z  = |    1.0000    1.0000     0.0000
>> T = ljx(T,1,1);
                  u1 =      u4 =        w   =
                  x3        x2          1
              ----------------------------------
-u3  x1 = |    0.3333    0.6667     0.3333
-u2  x4 = |   -0.6667   -0.3333    -1.6667
              ----------------------------------
 1   z  = |    0.3333    1.6667     0.3333
>> T = ljx(T,2,?????);
```

This means the dual problem is <u>unbounded</u>.

(and the primal is infeasible.)

Trick: Add a big enough constant to A.

$\alpha = 2$

```
>> T = totbl(A'+2,b,p); T = dualbl(T);
                x1           x2           1
          ----------------------------------
 x3 = |       5.0000       0.0000      -1.0000
 x4 = |       0.0000       3.0000      -1.0000
          ----------------------------------
 z  = |       1.0000       1.0000       0.0000
                u3 =         u4 =          w  =
                x1           x2            1
          ----------------------------------
-u1   x3 = |    5.0000       0.0000      -1.0000
-u2   x4 = |    0.0000       3.0000      -1.0000
          ----------------------------------
 1    z  = |    1.0000       1.0000       0.0000
>> T = ljx(T,1,1);
                u1 =         u4 =          w  =
                x3           x2            1
          ----------------------------------
-u3   x1 = |    0.2000      -0.0000       0.2000
-u2   x4 = |    0.0000       3.0000      -1.0000
          ----------------------------------
 1    z  = |    0.2000       1.0000       0.2000
>> T = ljx(T,2,2);
                u1 =         u2 =          w  =
                x3           x4            1
          ----------------------------------
-u3   x1 = |    0.2000      -0.0000       0.2000
-u4   x2 = |   -0.0000       0.3333       0.3333
          ----------------------------------
 1    z  = |    0.2000       0.3333       0.5333
```

$$\bar{x} = \begin{bmatrix} 1/5 \\ 1/3 \end{bmatrix}, \quad \bar{y} = \begin{bmatrix} 1/5 \\ 1/3 \end{bmatrix}, \quad \theta = 8/15$$

$$x^* = y^* = \frac{15}{8}\begin{bmatrix} 1/5 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

Value of the shifted game $= \frac{15}{8}$

value of the original game $= \frac{15}{8} - \underbrace{2}_{=\alpha}$

$= -1/8$

Conclusion: In the long run, you lose on average \$1/8 per game.

The game is not as fair as it seems!