Lecture 5 : Differentiable Manifolds

Note Title 2/5/2017

A manifold is an object like a regular surface but it is <u>not</u> thought of as part of an ambient Euclidean space. <u>Locally</u>, it looks like a <u>k-dimensional</u> object, but is allowed to be <u>"nontrivial globally"</u>.

<u>Def</u>:

A k-dimensional differentiable (or smooth) manifold is a set M together with a family of injective maps (called 'charts')
$\varphi_\alpha : U_\alpha \subset M \to \mathbb{R}^k$
with
$\varphi_\alpha(U_\alpha)$ being <u>open</u> in $\mathbb{R}^k$, and
such that

1) $\bigcup_\alpha U_\alpha = M$

2) If $U_\alpha \cap U_\beta =: W \neq \emptyset$, then $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(W) \to \varphi_\beta(W)$ and its inverse $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(W) \to \varphi_\alpha(W)$ are smooth (C^∞).

"$(U_\alpha, \varphi_\alpha)$ (U_β, φ_β) are C^∞-compatible"

3) The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal relative to 1) and 2).

Not essential, we will clarify its purpose.

How are we supposed to define things like tangent planes without an ambient space?

and why bother?

One

Motivation of dispensing with ambient space:

Recall from vector calculus how we calculate length, area and volume of curved objects.

The formulas in standard textbooks look quite different in the three cases. The good news is that there is a way to <u>unify these formulas</u>.

This unification in turn helps understanding:

- why bother to think about "tangent space without an ambient space"
- what's the so-called "volume form" of "a Riemannian metric of a manifold"
- the meaning of "isometry"

and as a special case:

- what's so fundamental about "the first fundamental form of a surface"

From vector calculus:

- $\alpha : (a,b) \to \mathbb{R}^2$

or

$\alpha : (a,b) \to \mathbb{R}^n, \quad n \geq 2$

$$\text{length of } \alpha(a,b) = \int_a^b \sqrt{\dot{\alpha}_1(t)^2 + \cdots + \dot{\alpha}_n(t)^2}\, dt$$

- $\alpha : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^n, \quad n = 2 \text{ or } 3$

$$\text{area of } \alpha(\mathcal{U}) = \begin{cases} \displaystyle\iint_{\mathcal{U}} \left| \det \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} \end{bmatrix} \right| du\,dv & n=2 \\ \displaystyle\iint_{\mathcal{U}} \| \alpha_u \underset{\uparrow}{\times} \alpha_v \| \, du\,dv & n=3. \end{cases}$$

only defined in $\mathbb{R}^3$

- $\alpha : \mathcal{U} \subset \mathbb{R}^3 \to \mathbb{R}^3$

$$\text{volume of } \alpha(\mathcal{U}) = \iiint_{\mathcal{U}} \left| \det \begin{bmatrix} \frac{\partial x_1}{\partial u} & \frac{\partial x_1}{\partial v} & \frac{\partial x_1}{\partial w} \\ \frac{\partial x_2}{\partial u} & \frac{\partial x_2}{\partial v} & \frac{\partial x_2}{\partial w} \\ \frac{\partial x_3}{\partial u} & \frac{\partial x_3}{\partial v} & \frac{\partial x_3}{\partial w} \end{bmatrix} \right| du\,dv\,dw$$

Ex: ① Check $\| u \times v \| = \sqrt{\det \underbrace{[u\ v]^T}_{2\times 3} \underbrace{[u\ v]}_{3\times 2}}$

for $u, v \in \mathbb{R}^3$. $\quad = \sqrt{\det \begin{bmatrix} \langle u,u \rangle & \langle u,v \rangle \\ \langle v,u \rangle & \langle v,v \rangle \end{bmatrix}}$

② If $A \in \mathbb{R}^{n\times n}$, then

$$\sqrt{\underbrace{\det(A^T A)}_{\substack{\text{well defined} \\ \text{for a} \\ \text{rectangular} \\ \text{matrix}}}} = |\underbrace{\det A}_{\substack{\text{only makes sense for} \\ \text{a square matrix}}}|.$$

So we may unify these formulas as follows:

$$\underline{X} : \mathcal{U} \subset \mathbb{R}^k \to \mathbb{R}^n \qquad 1 \leq k \leq n \leq 3$$

The "k-dimensional volume" (recall Lecture 2) of $\underline{X}(\mathcal{U})$ is:

$$\mathrm{vol}\,(\underline{X}(\mathcal{U})) = \int_{\mathcal{U}}\!\cdots\!\int \sqrt{\det d\underline{X}(\underline{u})^T \underbrace{d\underline{X}(\underline{u})}_{\left[\underline{X}_{u_1} \cdots \underline{X}_{u_k}\right]}}\, d\underline{u} \quad \text{—} (*)$$

Is this unification just an accident, or is there a geometric interpretation behind it?

Does the same formula make sense (geometrically) for $k > 3$?

The relevant question is:

If $v_1, \ldots, v_k \in \mathbb{R}^n$, what is the k-dimensional volume of the k-dimensional parallelipiped

$$P = \left\{ \sum_{i=1}^{k} \alpha_i v_i \ : \ 0 \leq \alpha_i \leq 1 \right\}.$$

Note: In Lecture 2, we answered this question only in the case of $k = n$.

Recall: when $k = n$, $\mathrm{vol}(P) = |\det \underbrace{[v_1 \cdots v_k]}_{\text{square matrix}}|$

I will guide you in the HW to prove:

Proposition: For $k \le n$, the k-dim. volume of P is:

$$\sqrt{\det\left[\langle v_i, v_j \rangle_{\mathbb{R}^n}\right]_{1 \le i,j \le k}}$$

Combining this linear algebra result with the technique of integration (or the "method of exhaustion"), the same formula ⊛ holds for any $k > 3$ and $n \ge k$.

Now, if the curve $(k=1)$, or surface $(k=2)$, or any higher dimensional analog belongs to a higher dimensional regular surface, ie.

$$X : \mathcal{U} \subset \mathbb{R}^k \to S$$

an n-dim. regular surface in $\mathbb{R}^N$ $(k \le n \le N)$

we can simply regard X as mapping into $\mathbb{R}^N$, the same formula ⊛ holds.

But in this case:

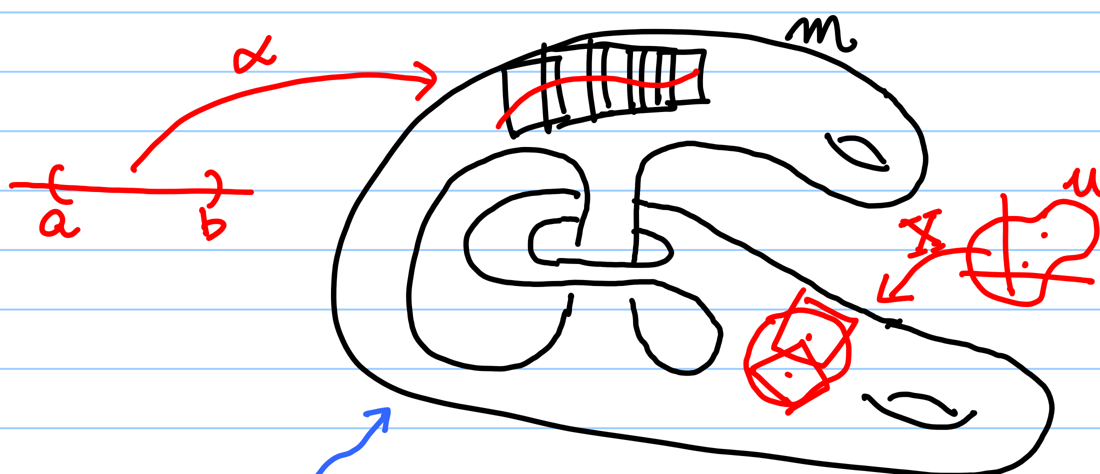
$X_{u_1}(\underset{\sim}{u}), \ldots, X_{u_k}(\underset{\sim}{u})$ are tangent vectors in the tangent space

$$T_{X(\underset{\sim}{u})} S \; ;$$

and ⊛ says if we only know how to calculate inner products of any pair of tangent vectors in any tangent space, then we have enough to calculate:

the k-dim. vol. of any k-dim. object in S, $\forall \; 1 \le k \le n$,

without ever refering to any sort of "measurements" in $\mathbb{R}^N$.



For the purpose of computing length and area, need not care where M sits, as long as $\langle \cdot , \cdot \rangle$ on T_pM can be computed $\forall p \in M$.

Imagine: length $\alpha(a,b)$

$$= \int_a^b \sqrt{\langle \underbrace{\alpha'(t)}, \underbrace{\alpha'(t)} \rangle}\, dt$$

$\in T_{\alpha(t)}M$, defined (abstractly) without refering to any ambient space of M.

called a "Riemannian metric on M" $\rightarrow$ $\boxed{\text{an (abstract) inner product defined on the (abstractly defined) tangent space } T_{\alpha(t)}M, \text{ again never referring to any ambient space of } M.}$

$$\text{Area}(X(U)) = \iint_U \left(\det \begin{bmatrix} \langle X_u, X_u \rangle, & \langle X_u, X_v \rangle \\ \langle X_v, X_u \rangle, & \langle X_v, X_v \rangle \end{bmatrix}\right)^{\frac{1}{2}} du\, dv$$

elements of $T_{X(u,v)}M$

Abstract vector spaces and inner-products are familiar objects in linear algebra, the most difficult new idea needed here is the concept of a

"tangent space of a manifold".

Note : For any inner product $\langle \cdot , \cdot \rangle$ on a (real) vector space V, knowing

$\langle v_i, v_j \rangle$ for any ordered basis

$\{v_1, \cdots, v_n\}$ of V is enough

to specify the whole inner-product,

$$\langle v, w \rangle = \langle \sum \alpha_i v_i, \sum \beta_j v_j \rangle$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \beta_j \langle v_i, v_j \rangle .$$
$$= [\alpha_1, \cdots, \alpha_n] \underbrace{\begin{bmatrix} \langle v_i, v_j \rangle \end{bmatrix}}_{n \times n} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} .$$

For a (2-dimensional) surface, knowing

$$E = \langle x_u, x_u \rangle, \quad F = \langle x_u, x_v \rangle, G = \langle x_v, x_v \rangle$$

is the same as knowing the Riemannian metric of the surface. This is why E, F, G are fundamental and

$\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ is called the

"first fundamental form" of the surface.

In linear algebra, a lot can be said about general vector spaces and linear maps without refering to any inner-product:

Concepts only dependent on vector space structure:	concepts dependent on $\langle\cdot,\cdot\rangle$:
rank	
nullity	length $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$
basis	orthogonal basis
Projection $P^2 = P$	ortho-projection
linear independence	
subspace	self-adjoint operator
invariant subspace	
Jordan canonical form	polar decomposition
⋮	etc.

The same situation holds for manifold theory, a lot can be said about manifolds and differential geometric objects on them without refering to any Riemannian metric.

Let's begin ....

Terminology : $(U_\alpha, \varphi_\alpha)$ — 'charts' or 'coordinate neighborhood'

$\{(U_\alpha, \varphi_\alpha)\}$ (maximal or not) — 'atlas'

$\{(U_\alpha, \varphi_\alpha)\}$ (maximal) — a differentiable structure

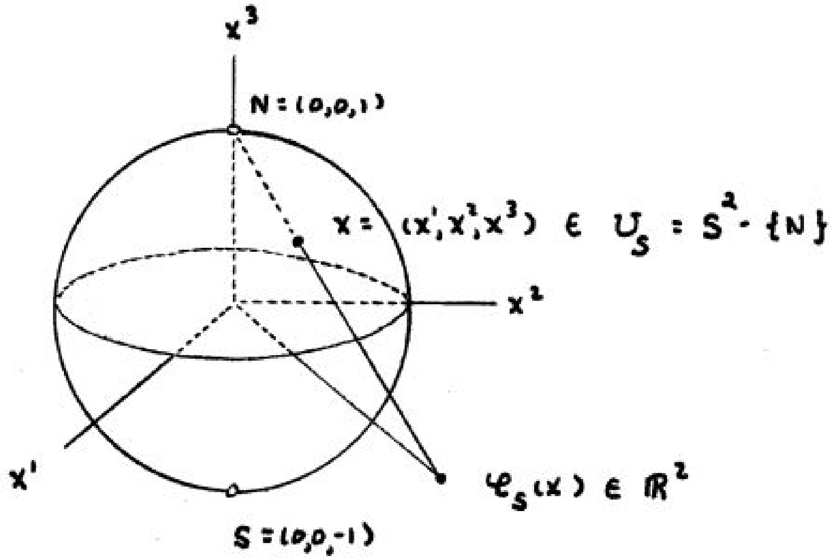
Examples :

1. Every regular surface with intrinsic dimension k is a k-dim. differentiable manifold.

2. $S^{n-1} = \{x = (x^1, \cdots, x^n) : \|x\|^2 = (x^1)^2 + \cdots + (x^n)^2 = 1\}$ can easily be shown to be an $(n-1)$-dimensional regular surface in $\mathbb{R}^n$ (i.e. a hypersurface in $\mathbb{R}^n$)

   Here, we provide a particular interesting atlas. (It has the extra property of being conformal; but we won't dive into it for now.)

   We do it for $n=3$, the construction works for any $n \geq 2$.



Define $\quad U_S = S^2 - \{N\}$

$\varphi_S : U_S \to \mathbb{R}^2$ (stereographic projection from N.)

$\varphi_S(x^1, x^2, x^3)$ = intersection with $x^3 = 0$ of the straightline through $N = (0,0,1)$ and $x = (x^1, x^2, x^3)$

$$= \left(\frac{x^1}{1-x^3} \, , \, \frac{x^2}{1-x^3} \right)$$

Ex: prove this formula, and generalize it n-dim.

φ_S is injective on U_S and maps <u>onto</u> $\mathbb{R}^2$ because its inverse is easily found (intersect the line through N and $y = (y^1, y^2) \in \mathbb{R}^2$ with S^2):

$$\varphi_S^{-1} : \mathbb{R}^2 \to U_S$$

$$\varphi_S^{-1}(y) = \frac{1}{1 + \|y\|^2} \left(2y^1, 2y^2, \|y\|^2 - 1 \right)$$

$\left[\begin{array}{l}\text{In fact, } \varphi_S, \varphi_S^{-1} \text{ are also continuous} \\ \text{with continuity defined based on the} \\ \text{relative topology of } S^2 \text{ as a} \\ \text{subspace of } \mathbb{R}^3.\end{array}\right]$

Similarly, one can define stereographic projection from S:

$$\varphi_N : \mathcal{U}_N = S^2 - \{S\} \to \mathbb{R}^2$$

$$\varphi_N(x) = \left(\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3} \right)$$

$$\varphi_N^{-1}(y) = \frac{1}{1+\|y\|^2} \left(2y^1, 2y^2, 1-\|y\|^2 \right)$$

Now, we check that the change of coordinates are smooth:

$$\varphi_S \circ \varphi_N^{-1} : \mathbb{R}^2 - \{(0,0)\} \to \mathbb{R}^2 - \{(0,0)\}$$

$$(y^1, y^2) \mapsto \left(y^1/(y^1)^2+(y^2)^2, \; y^2/(y^1)^2+(y^2)^2 \right)$$

$$\text{or} \quad y \mapsto y/\|y\|^2$$

(Aside: this map is called a "circle inversion")

Not hard to check that it is C^∞, i.e. mixed partial derivatives of any order exist and are continuous.

Similarly for $\varphi_N \circ \varphi_S^{-1}$.

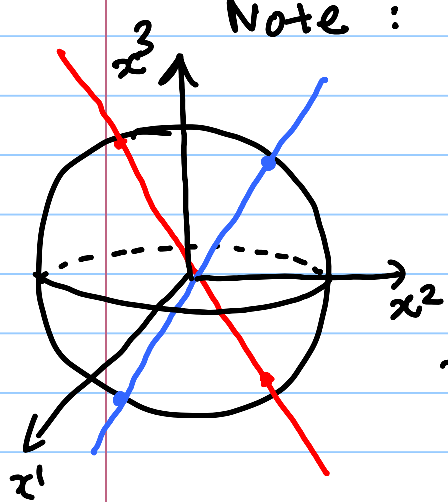
So, $\{(U_S, \varphi_S), (U_N, \varphi_N)\}$ is an example of atlas for S^2.

Of course, it is far from being maximal (As a matter of fact, it is minimal, it is impossible to cover S^2 with only one chart.)

3. $\mathbb{RP}^2$ (or $\mathbb{P}^2(\mathbb{R})$)

$= \{$ the set of all lines in $\mathbb{R}^3 \}$

$= G(3,1)$ (recall Lecture 4)



Note: its elements can be thought of as equivalence classes of $\mathbb{R}^3 - \{(0,0,0)\}$, with the equivalence relation $\sim$:

$$(x^1, x^2, x^3) \sim \lambda(x^1, x^2, x^3)$$
$$\lambda \in \mathbb{R},\ \lambda \neq 0$$

Denote the equivalent classes by $[x^1, x^2, x^3]$.

Let $V_i = \{ [x^1, x^2, x^3] : x^i \neq 0 \}$
$i = 1, 2, 3$

(V_1 = all the lines not lying on the x^2-x^3 plane, etc.)

$\varphi_1 : V_1 \to \mathbb{R}^2$, $[x^1, x^2, x^3] \mapsto (x^2/x^1, x^3/x^1)$
$\varphi_2 : V_2 \to \mathbb{R}^2$, $[x^1, x^2, x^3] \mapsto (x^1/x^2, x^3/x^2)$
$\varphi_3 : V_3 \to \mathbb{R}^2$, $[x^1, x^2, x^3] \mapsto (x^1/x^3, x^2/x^3)$

φ_1 is injective, if $\overset{\neq 0}{x^1}, x^2, x^3$ and $\overset{\neq 0}{\bar{x}^1}, \bar{x}^2, \bar{x}^3$ are such that

$$\left(x^2/x^1,\ x^3/x^1 \right) = \left(\bar{x}^2/\bar{x}^1,\ \bar{x}^3/\bar{x}^2 \right)$$

then

$$(x^1, x^2, x^3) = x^1/\bar{x}^1 \, (\bar{x}^1, \bar{x}^2, \bar{x}^3)$$

so

$$[x^1, x^2, x^3] = [\bar{x}^1, \bar{x}^2, \bar{x}^3].$$

Also $\varphi_1(V_1) = \mathbb{R}^2 \leftarrow$ open in $\mathbb{R}^2$, because for any

$(y^1, y^2) \in \mathbb{R}^2$, $\varphi_1([1, y^1, y^2]) = (y^1, y^2)$.

Similar for φ_2, φ_3.

<u>Ex</u> : Show that these 3 charts are C^∞-compatible.

Topological Issues :

- why condition 3) ?

It is hardly essential,

if $\{(U_\alpha, \varphi_\alpha)\}$ is not maximal, simply extend it by throwing in all the charts C^∞- compatible with the original ones.

- Topology induced by a differentiable structure :

$$\boxed{\text{Declare } A \subset M \text{ to be } \underline{\text{open}} \text{ if } \forall \alpha, \quad \varphi_\alpha(A \cap U_\alpha) \text{ is open in } \mathbb{R}^n.}$$

Ex : show that it is the (coarsest) topology for which $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \underset{\text{open}}{\subset} \mathbb{R}^n$ are homeomorphisms.

It's possible to define <u>different differentiable structures</u> on a set M with the <u>same underlying topology</u>.

The 'maximal' condition is introduced simply for the sake of comparing differentiable structures. We don't want to say two atlases are different simply because one has more charts in it than the other.

E.g. $M = \mathbb{R}$

1. $\mathcal{U} = M$, $\varphi : M \to \mathbb{R}$, $\varphi(x) = x$

   $\{(\mathcal{U}, \varphi)\}$ is an atlas on M, it gives the 'usual differentiable structure' of $\mathbb{R}$.

2. $V = M$, $\psi : M \to \mathbb{R}$, $\psi(x) = x^3$

   $\{(V, \psi)\}$ is also an atlas on M.

   But these two charts are not C^∞- compatible (in fact not even C^1- compatible.)

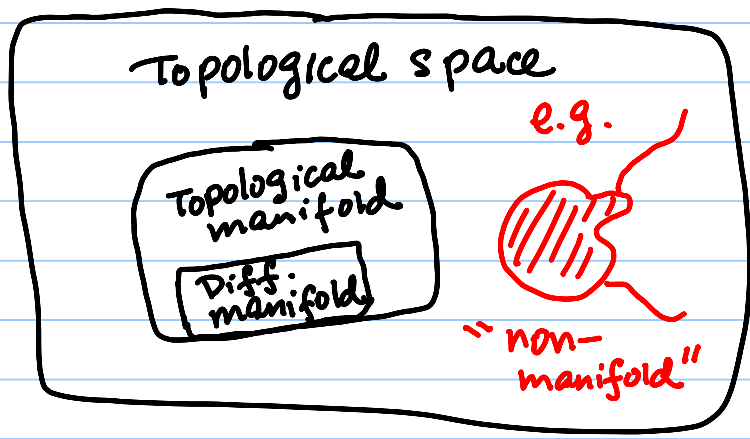
   $\psi \circ \varphi^{-1}(x) = x^3$ is C^∞ ,

   $\varphi \circ \psi^{-1}(x) = x^{1/3}$ is only continuous but not C^1.

The interaction between the topology and differentiable structure of a manifold is a very subtle issue I want to avoid for now.

Fortunately, we can get by with a few comments and a thoughtful result:

(I) Usually a differentiable manifold is defined by first assuming M is not an arbitrary set, but a <u>topological manifold</u>, followed by adding an additional differentiable structure.



Def: M is a topological manifold if it is a

- Hausdorff (T_2)
- second countable (C_{II})

topological space which is also

- locally Euclidean

[Try: John Lee's book or Greg Naber's notes for details]

Perhaps the most useful things to remember are that the Haudorff and second countable assumptions guarantee:

- uniqueness of limits
- existence of 'nice' partitions of unity

[I shall state the key theorem on the existence of partition of unity and bump functions, without providing a proof. Armed with this theorem, we can pretty much avoid dealing with basic point-set topology issues.]

(II) The relatively simple definition we use (on the 1st page) has the problem that the topology it induces may not be Hausdorff and/or 2nd countable.

The following result guarantees that we do not have a differentiable manifold with a bizarre underlying topology:

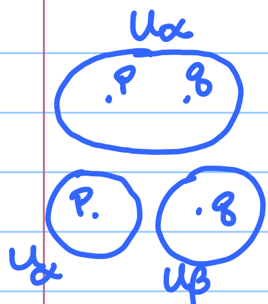
<u>Theorem</u> Let M be a differentiable manifold (according to our definition) with a not-necessarily maximal atlas

$$\{(U_\alpha, \phi_\alpha)\} = \mathcal{A}$$

If on top of the required properties on $\mathcal{A}$, we also check:

(i) $\forall \alpha, \beta$, $\phi_\alpha(U_\alpha \cap U_\beta)$, $\phi_\beta(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^k$.

(ii) Countably many of the sets U_α cover M



(iii) $\forall\, p, q \in M$, $p \neq q$, either:

$$\exists\, U_\alpha \text{ s.t. } p, q \in U_\alpha,$$

or

$$\exists\, U_\alpha, U_\beta \text{ s.t. } U_\alpha \cap U_\beta \neq 0,\quad p \in U_\alpha,\ q \in U_\beta.$$

Then

the topology induced by $\mathcal{A}$ is Hausdorff and 2nd countable.

The proof is pretty straightforward if you speak the language of point-set topology. See John Lee's Intro. to Smooth Manifold Ch. 1.

Eg. S^{n-1}. The two chart atlas

$$\{ (U_S, \varphi_S), (U_N, \varphi_N) \}$$

clearly satisfies (i) and (ii), but not (iii).

If $p = N$, $q = S$, none of the two charts contain both p and q,

$$S \in U_S = S^2 \setminus \{N\}, \quad N \in U_N = S^2 - \{S\}$$

But we can simply trim down U_S to the southern hemisphere U_S' and U_N to the northern hemisphere U_N', and throw the two charts to $\mathcal{A}$, then

$$\mathcal{A}' = \{ (U_S, \varphi_S), (U_N, \varphi_N), (U_S', \varphi_S), (U_N', \varphi_N) \}$$

satisfies all the properties in the theorem. Moreover it is easy to check that the topology induced by $\mathcal{A}'$ (or $\mathcal{A}$) coincides with the usual topology on S^{n-1} (as a subspace of $\mathbb{R}^n$.)

E.g. Grassmann manifolds $G(n,k)$

Let's build up some intuition, then turn the intuition into a proof.

If $G(n,k)$ is truly a manifold, how many degrees of freedom would it have?

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} | & | & \cdots & | \\ | & | & \cdots & | \end{bmatrix} \end{array}$$

a <u>basis</u> for a k-dim. subspace of $\mathbb{R}^n$ $\equiv$ a rank k $n\times k$ matrix

From linear algebra:

- $\binom{n}{k}$ ways to pick a $k\times k$ sub-matrix, at least one of them is non-singular.
- $\underset{n\times k}{\operatorname{span}(A) = \operatorname{span}(A')} \iff A = A'G$ for a $k\times k$ invertible matrix G

say, the first k rows give a non-singular square matrix G

$$\begin{array}{cc} & k \\ n & \begin{bmatrix} | & | & \cdots & | \\ | & | & \cdots & | \end{bmatrix} \end{array} \begin{bmatrix} G^{-1} \end{bmatrix} = \begin{bmatrix} I_{k\times k} \\ \hline \underbrace{\boxed{}}_{(n-k)\times k} \end{bmatrix}$$

Claim: these uniquely specify the subspace i.e. the map

$$\mathbb{R}^{(n-k)\times k} \ni u \longmapsto \operatorname{span}\left[\begin{array}{c} I_{k\times k} \\ \hdashline u \end{array}\right] \in G(n,k)$$

is <u>injective</u>, but <u>not surjective</u>.

Ex: Prove this claim.

For $J \subset \{1,2,\dots,n\}$, $A \in \mathbb{R}^{n\times k}$

Write: A_J = the $|J|\times k$ submatrix of A with rows indexed by J.

Consider:

$$V_J = \{\operatorname{span}(A) : A_J \text{ is non-singular}\}$$

$$\operatorname{span}(A) \xmapsto{\varphi_J} (A A_J^{-1})_{I\setminus J}$$

Then $\mathcal{A} := \{(V_J, \varphi_J) : J \subset \{1,\dots,n\}\}$

is an atlas of $G(n,k)$ (with $\binom{n}{k}$ charts.)

- Change of coordinate map:

$$W = V_{J_1} \cap V_{J_2}$$

$$\varphi_{J_2} \circ \varphi_{J_1}^{-1} : \varphi_{J_1}(W) \to \varphi_{J_2}(W)$$

$$M \xmapsto{\varphi_{J_1}^{-1}} \underset{A_{J_1} = M}{\operatorname{span}[A]} = \operatorname{span}[A]A_{J_2}^{-1} \xmapsto{\varphi_{J_2}} (AA_{J_2}^{-1})_{J_2^c} = N$$

By the Cramer's rule, the entries of N are rational functions of the entries of M, so

$$M \mapsto N \quad \text{is a } C^\infty \text{ function.}$$

(The assumption that $M \in \varphi_{J_1}(W)$ guarantees that there is no 'division by zero' in the expression of $N(M)$.)

Similar to the case of S^{n-1}, the atlas satisfies condition (i) and (ii) trivially. But requires throwing in some "trimmed charts" to satisfy condition (iii).

Ex: Fill in this final detail.

$\blacksquare$

If $\mathcal{U}$ is any open set in $\mathbb{R}^n$, then

$(\mathcal{U}, id)$ defines a differentiable structure.

(And the corresponding topology is just that of $\mathcal{U}$ as a subspace of $\mathbb{R}^n$, so is Hausdorff and 2nd countable.)

<u>E.g.</u> $Gl(n) = \{$ all non-singular $n \times n$ matrices $\}$

$$= \{ A : \mathbb{R}^{n \times n} : \det(A) \neq 0 \}$$

$$= \det^{-1}(\underbrace{\mathbb{R} \setminus \{0\}}_{\text{open in } \mathbb{R}})$$

Since $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is a continuous function, for which the preimage of an open set is an open set.

$Gl(n)$ is an open subset of $\mathbb{R}^{n\times n}$, hence is a differentiable manifold.

E.g. A generalization of $Gl(n)$.

$$\{ A \in \mathbb{R}^{m\times n} : A \text{ is full rank} \}$$

is also an open subset of $\mathbb{R}^{m\times n}$, hence a differentiable manifold.

The proof follows from:

A is full rank

$$\iff \quad 0 \neq \begin{cases} \det(A^T A), & m \geq n \\ \det(A A^T), & m < n. \end{cases}$$

- More generally, if $\mathcal{M}$ is any smooth n-manifold, and $\mathcal{O} \subset \mathcal{M}$ is an open subset, then $\mathcal{O}$ has the following atlas:

$$\mathcal{A}_{\mathcal{U}} := \{ (\mathcal{U} \cap \mathcal{O}, \varphi) : (\mathcal{U}, \varphi) \text{ is a chart for } \mathcal{M} \}.$$

$\mathcal{O}$ is called an <u>open submanifold</u> of $\mathcal{M}$.

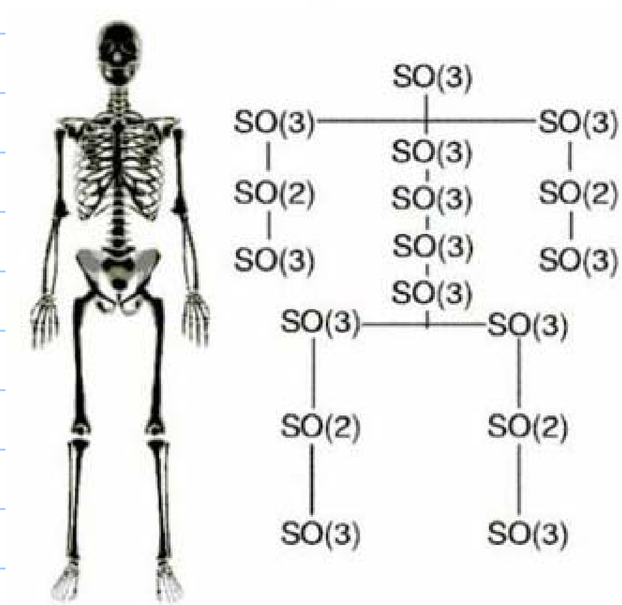
- Product manifolds

If $M_1, \ldots, M_k$ are smooth manifolds of dim. $n_1, \ldots, n_k$, respectively,

then $M_1 \times \cdots \times M_k$ has a manifold structure with dim. $n_1 + \cdots + n_k$.

Charts: $\varphi_1 \times \cdots \times \varphi_k : U_1 \times \cdots \times U_k \longrightarrow \mathbb{R}^{n_1 + \cdots + n_k}$

Ex: check details.



<u>Informal discussion of quotient manifold</u>

In point set topology, given an equivalence relation $\sim$ on a topological space X, there is natural topology (called the quotient topology) on $X/\sim$.

What about manifold structure?

Eg. If $\mathbb{R}^{n\times k}_{*}$ = all rank k $n\times k$ matrices $(n \geq k)$

then

$$G(n,k) = \mathbb{R}^{n\times k}_{*} / \sim_{GL(k)}$$

where "$\sim_{GL(k)}$" is the equivalence relation

$$A \sim B \text{ if } A = BG \text{ for some } G \in GL(k).$$

In this case, it happens so all

$$G(n,k), \ \mathbb{R}^{n\times k}_{*}, \ GL(k) \text{ are manifolds.}$$

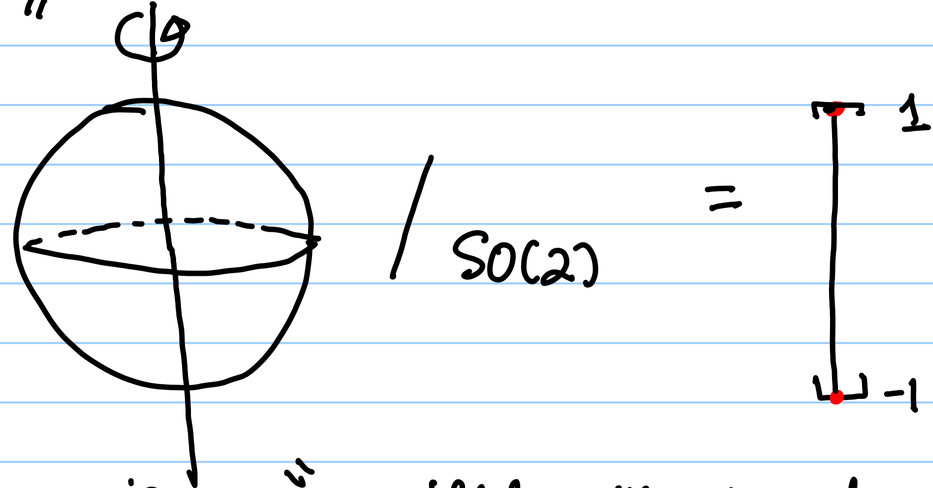
But it is not always true that

$M/\sim$ is a manifold, even if $\sim$ is somehow given by a manifold

E.g. $S^2/\sim$ where

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \sim \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \text{ if } \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

$\sim$ can be thought of as "SO(2) acting on S^2"



is a "<u>manifold with boundary</u>".

There is a systematic study of when such a quotient manifold structure exists when a <u>Lie group</u> acts on a manifold.

''

a manifold + group with group operations smooth according to manifold differentiable structure

Not sure if we have enough time for the "quotient manifold theorem", but we will definitely learn more about Lie groups.

Smooth Maps

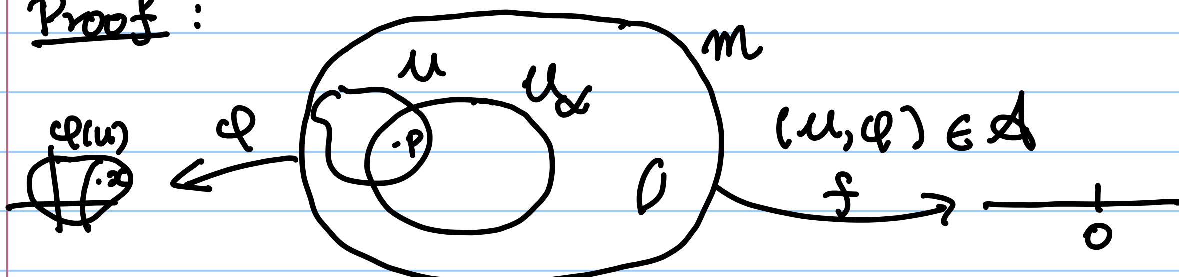
M - n-manifold with maximal atlas $\mathcal{A}$.

$f : M \to \mathbb{R}$ is called smooth if

$\underbrace{f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}}_{\text{called a "coordinate representation" of } f}$ is C^∞, $\forall (U, \varphi) \in \mathcal{A}$.

Proposition: Enough to check $f \circ \varphi^{-1}$ is smooth for an atlas.

Proof:



Is $f : \varphi(U) \to \mathbb{R}$ smooth if (U, φ) is <u>not</u> in the atlas?

Let $x \in \varphi(U) \subset \mathbb{R}^n$ be arbitrarily chosen. We want to show f is smooth near x.

$p := \varphi^{-1}(x)$ is contained in some chart U_α in the atlas.

But $f \circ \varphi^{-1} = \underbrace{(f \circ \varphi_\alpha^{-1})}_{\text{smooth}} \circ \underbrace{(\varphi_\alpha \circ \varphi^{-1})}_{\text{smooth}}$,

so $f \circ \varphi^{-1}$ is smooth around x. ∎

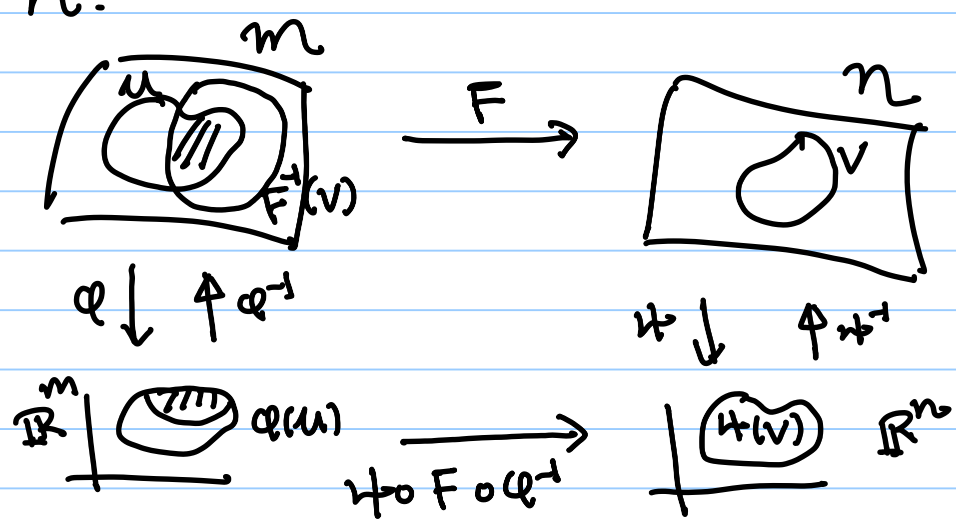
Similarly, if $\mathcal{M}, \mathcal{N}$ are smooth manifold of dim. m, n, resp.

$F : \mathcal{M} \to \mathcal{N}$ is smooth if

$$\underbrace{\psi \circ F \circ \varphi^{-1} : \varphi(\mathcal{U} \cap F^{-1}(V)) \to \psi(V)}_{\text{a coordinate representation of } F}$$

is smooth for all charts $(\mathcal{U}, \varphi)$ of $\mathcal{M}$ and (V, ψ) of $\mathcal{N}$.

Again, enough to check smoothness for all charts in an atlas of $\mathcal{M}$ and an atlas of $\mathcal{N}$.



E.g. (Group operations on $GL(n)$

$$\underbrace{(A,B)}_{\in \underbrace{GL(n)\times GL(n)}_{\text{product manifold structure}}} \longmapsto \underbrace{A\cdot B}_{\in GL(n)}$$

The coordinate representation (in the standard chart) have component functions that are quadratic polynomials, hence C^∞.

Similarly $\underset{\in GL(n)}{A} \mapsto \underset{\in GL(n)}{A^{-1}}$ is C^∞ (Cramer's rule)

$GL(n)$ is an example of a Lie group:

<u>Def</u> A Lie group is a group G that also has the structure of a smooth manifold for which the group operations

$$(a,b) \in G\times G \longmapsto ab \in G$$
$$a \in G \mapsto a^{-1} \in G$$

are C^∞.

We will see more examples, some of them are

$$\text{Subgroups} + \underbrace{\text{submanifolds}}_{\text{to be defined}}$$

of $GL(n)$.

Diffeomorphisms

If $\mathcal{M}$ and $\mathcal{N}$ are differentiable manifolds, $F: \mathcal{M} \to \mathcal{N}$ is a bijection that is smooth with a smooth inverse $F^{-1}: \mathcal{N} \to \mathcal{M}$, then

F is a diffeomorphism and $\mathcal{M}$ and $\mathcal{N}$ are said to be diffeomorphic.

(Analogue of "homeomorphism" for topological spaces, or "isomorphism" for vector spaces, groups, etc.)

Earlier, we explained that

$$\{(\mathbb{R}^1, id)\}, \{(\mathbb{R}^1, id^3)\} \text{ define}$$

two <u>different</u> differentiable structures on $\mathcal{M} = \mathbb{R}^1$. Here we see that these two differentiable manifolds are, in fact, diffeomorphic.

Let X be the $\mathbb{R}^1$ with the standard diff. structure, and X' be the "non-standard $\mathbb{R}^1$".

Claim : $id^3 : X' \to X$ $(x \mapsto x^3)$ is a diffeomorphism

$$\begin{tikzcd} X' \arrow[r, "id^3"] \arrow[d, shift left, "id^3"] & X \arrow[d, "id"] \\ \mathbb{R}^1 \arrow[u, shift left, "\sqrt[3]{id}"] \arrow[r, dashed, "id"'] & \mathbb{R}^1 \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[r, "\sqrt[3]{id}"] \arrow[d, shift right, "id"'] & X' \arrow[d, "id^3"] \\ \mathbb{R}^1 \arrow[u, shift right, "id"'] \arrow[r, dashed, "id"] & \mathbb{R}^1 \end{tikzcd}$$

smooth $\qquad\qquad$ smooth

Aside: a famous story that involves at least two Fields' Medalists:

- any differentiable structure on $(\mathbb{R}^1$, usual topology$)$ is diffeomorphic to the standard one.

- same is true for $\mathbb{R}^n$, $\forall n \neq 4$

- there is a whole continuum of non-diffeomorphic diff. structures on $(\mathbb{R}^4$, usual topology$)$.

[work of Donaldson and Freedman]
(on the
Yang-Mills
equations)