Introduction to Optimization. HW#2. Due: Thursday, October 17, 2024

1. (20 points) The fundamental objects of this course are linear equations and linear inequalities. Let's try to understand what they mean – both *algebraically and geometrically.*

   Let $a \in \mathbb{R}^n$ be a non-zero vector. Think of $x \mapsto a^T x$ as a linear map from $\mathbb{R}^n$ to $\mathbb{R}^1$. Call this linear map T_a.

   (a) What are the rank and nullity of $T_a : \mathbb{R}^n \to \mathbb{R}$?

   (b) Note that the set of points $x \in \mathbb{R}^n$ that satisfies the linear equation $\sum_{i=1}^n a_i x_i = 0$ is exactly the null space of T_a. Describe how this set looks like geometrically.

   (c) For any scalar b, let $P_{a,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^n a_i x_i = b\}$. Prove that there is a vector $c \in \mathbb{R}^n$ such that $P_{a,b} = P_{a,0} + c$.[1] Verify that such a vector c is non-zero exactly when the scalar b is non-zero. (Note, however, that the choice of c is not unique.) Now, describe how $P_{a,b}$ looks like geometrically.

   (d) Describe how the set $H_{a,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^n a_i x_i \geqslant b\}$ looks like geometrically.

   (e) Prove that any $H_{a,b}$ is convex.

   (f) Prove: if $S_1, \ldots, S_m \in \mathbb{R}^n$ are convex, then so is $S_1 \cap \cdots \cap S_m$.

   (g) Using (e) and (f), give an **alternative proof** for the fact that the feasible region of a LP is convex. (This fact is proved in Week 2.)

2. (20 points) Page 27, Exercise 2-2-3. Recall that "the row rank of a matrix is the same as the column rank of a matrix": Given a $m \times n$ matrix A, the m rows of A span a certain subspace of $\mathbb{R}^n$, and the n columns of A spans a certain subspace of $\mathbb{R}^m$. There seems to be no palpable reason for why these two subspaces have anything to do with each other, but it turns out that these two subspaces, residing in two Euclidean spaces of different dimensions (assuming $m \neq n$), always have the same dimension.

3. (20 points) Page 35, Exercise 2-4-2.

4. (20 points) Page 37, Exercise 2-4-6.

5. (20 points) Page 38, Exercise 2-4-8. **Do not give just one example of A. Explain the type of matrices A with each property.**

[1]Note: If $S \subset \mathbb{R}^n$ and $c \in \mathbb{R}^n$, $S + c$ is the set $\{x + c : x \in S\}$. Geometrically, $S + c$ is a *translated* copy of S.