Math T680 Topics in Geometry

$\boxed{\text{HW \#5}}$

Due: Wednesday, May 17, 2017

This Brouwer fixed point theorem was first proved by Brouwer around 1910. As explained in the notes, the $n = 1$ case can be proved using one of the most elementary topological properties, namely connectedness. The $n = 2$ case can be proved using the fundamental group π_1 of S^1. To prove the theorem for a general dimension $n \geqslant 2$, one could use either the homology group or the higher homotopy group π_n. In this course, we use the De Rham cohomology group (which contains the same information as the homology group.) Brouwer's original proof used neither homology nor homotopy groups, which had not been invented at the time. Instead, Brouwer developed the the notion of **degree** for maps $S^n \to S^n$. In Chapter 11, we first define 'degree' using De Rham cohomology, which Brouwer defined directly in more geometric terms.

In this homework, we reprove Brouwer's fixed point theorem and the hairy ball theorems using Brouwer's original approach, but we make this homework much easier by taking Brouwer's degree for granted:

Suppose that to every continuous map $h : S^n \to S^n$ we have assigned an integer, denoted by $\deg h$ and called the degree of h, such that:

(i) Homotopic maps have the same degree.

(ii) $\deg(h \circ k) = (\deg h) \cdot (\deg k)$.

(iii) The identity map has degree 1, the constant map has degree 0, and the reflection map $\rho(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_{n+1})$ has degree -1.

1. Offer some intuitive explanations of why such a notion is plausible. Use your best writing and drawings. Construct some examples; be creative.

   In the following, assume Brouwer's degree but do not assume any knowledge in De Rham cohomology, prove:

2. (a) M&T's Lemma 7.2 (There is no continuous map $g : D^n \to S^{n-1}$ with $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$. Such a map is called a retraction in topology.)
   Explain how Brouwer's fixed point theorem follows.

   (b) If $h : S^n \to S^n$ has degree different from $(-1)^{n+1}$, then h has a fixed point. (Hint: Show that if h has no fixed point, then h is homotopic to the antipodal map.)

   (c) If $h : S^n \to S^n$ has degree different from 1, then h maps some point x to $-x$.

   (d) (Hairy ball theorem.) If S^n has a non-vanishing tangent vector field, then n is odd. (Hint: Borrow a key trick from our proof in class.)

3. (a) A harder theorem says that the degree of any odd function $h : S^n \to S^n$ (i.e. $h(-x) = -h(x)$) must have an odd degree.
   Assuming this fact, prove that every continuous map $g : S^n \to \mathbb{R}^n$ must have a point $x \in S^n$ such that $g(x) = g(-x)$. This is called the Borsuk-Ulam theorem.

(b) Show that if a continuous homogenous (but possibly nonlinear) function $F : \mathbb{R}^n \to \mathbb{R}^n$ is injective, it must be surjective. F is homogeneous means $F(cx) = cF(x)$ for all $c \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

(c) The rank-nullity theorem tells us that for a linear map $F : \mathbb{R}^n \to \mathbb{R}^n$, F is injective $\iff$ F is surjective.

i. Explain why topological methods give a nonlinear generalization of the '$\Rightarrow$' part of this statement; use part (b).

ii. Explain why the '$\Leftarrow$' implication is not true for general nonlinear maps; adding the 'homogeneous' assumption to the map would not save this situation.

4. Try to construct two independent vector fields on S^5, you will fail according to Adams' theorem. Share your failing experience.