CH 1 : Introduction / Motivation

Note Title 4/1/2017

Part I :

Cohomology of open sets in $\mathbb{R}^n$
[Ch 1-6 , M & T]

Applications : [Ch 7]

- Brouwer's fixed point theorem

$\Downarrow$

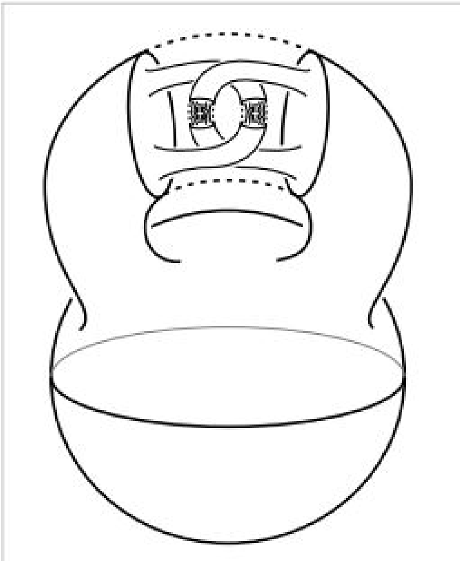
$\boxed{S^n \text{ has a continuous non-vanishing tangent vector field} \iff n \text{ is odd}}$

- Invariance of domain

$\Downarrow$

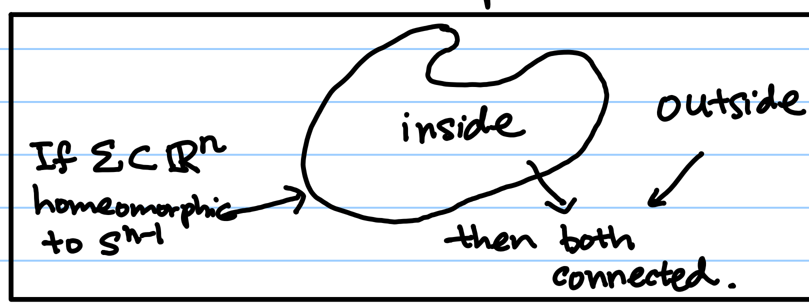
For $U \underset{\text{open}}{\subseteq} \mathbb{R}^n$, $V \underset{\text{open}}{\subseteq} \mathbb{R}^m$

U, V are homeomorphic $\Rightarrow m = n$



Alexander horned sphere shows how troublesome a set in R^3 homeomorphic to S^2 can be. It elucidates why the Jordan-Brouwer separation theorem is not obviously true. It is also used to show that a stronger version of the theorem, known as the Jordan-Schonflies theorem, does not in general hold in 3D.

- Jordan-Brouwer separation theorem:



[Both simply connected ?]

Part II :

Cohomology of smooth manifolds
[Ch 8, 9, 10, 11]

- Stoke's theorem [Ch 10]

Applications : [Ch 12]

- Poincaré - Hopf theorem
- Gauss - Bonnet theorem

Introduction

Given $f : (a,b) \to \mathbb{R}$, can always find $F : (a,b) \to \mathbb{R}$

st. $F' = f$.

What about multivariate functions?

Q: Given $f : \underset{\text{open}}{\mathcal{U} \subseteq} \mathbb{R}^2 \to \mathbb{R}^2$ smooth

is there a smooth function $F : \mathcal{U} \to \mathbb{R}^1$ st.

$$\underbrace{\frac{\partial F}{\partial x_1} = f_1 \ , \ \frac{\partial F}{\partial x_2} = f_2}_{\nabla F = f} \ , \ f = (f_1, f_2) \ ?$$

necessary condition:

$$\frac{\partial^2 F}{\partial x_1 \partial x_2} = \frac{\partial^2 F}{\partial x_2 \partial x_1}$$

$$\frac{\partial f_1}{\partial x_2} \qquad\qquad \frac{\partial f_2}{\partial x_1}$$

Is this condition also <u>sufficient</u>?

A: It depends on the domain $\mathcal{U}$.

(I) consider $f : \overbrace{\mathbb{R}^2 \setminus \{0\}}^{\mathcal{U}} \to \mathbb{R}^2$

$$f(x_1, x_2) = \Big(\overbrace{\frac{-x_2}{x_1^2 + x_2^2}}^{= f_1} , \overbrace{\frac{x_1}{x_1^2 + x_2^2}}^{= f_2} \Big).$$

f satisfies the necessary condition $\frac{\partial f_1}{\partial x_2} = \frac{\partial f_2}{\partial x_1}$

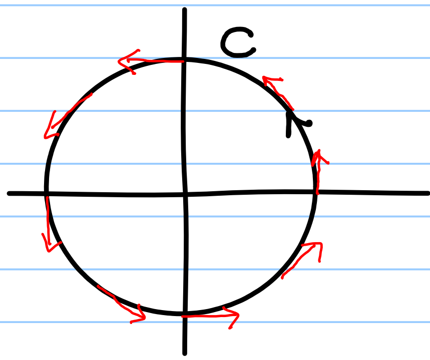
Claim: $\nexists \ F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ st $\nabla F = f$.

Proof (by contradiction) : Assume there were, then

$$\int_0^{2\pi} \frac{d}{d\theta} F(\cos\theta, \sin\theta)\, d\theta = F(1,0) - F(1,0) = 0$$

But

$$\frac{d}{d\theta} F(\cos\theta, \sin\theta) \underset{\substack{\uparrow \\ \text{chain} \\ \text{rule}}}{=} \left.\frac{\partial F}{\partial x}\right|_{(\cos\theta, \sin\theta)} \cdot (-\sin\theta) + \left.\frac{\partial F}{\partial y}\right|_{(\cos\theta, \sin\theta)} \cos\theta$$



$$= -f_1(\cos\theta, \sin\theta)\sin\theta + f_2(\cos\theta, \sin\theta)\cos\theta$$

$$= 1 \qquad \Rightarrow\!\Leftarrow \qquad \square$$

Note : $\int_0^{2\pi} f(\alpha(\theta)) \cdot \alpha'(\theta)\, d\theta$, $\alpha(\theta) = (\cos\theta, \sin\theta)$

$$= \int_0^{2\pi} \left[f_1(\cos\theta, \sin\theta)(-\sin\theta) + f_2(\cos\theta, \sin\theta)\cos\theta \right] d\theta$$

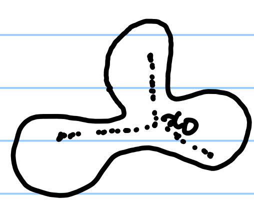
is the line integral of the vector field f along the circle C, usually denoted by

$\oint_C \vec{f} \cdot d\vec{x}$ in vector calculus.

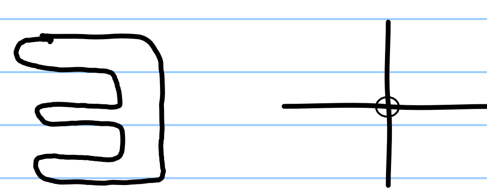
It has the physical interpretation of "work done".

(II) A subset $X \subset \mathbb{R}^n$ is said to be <u>star-shaped</u> if $\exists\, x_0 \in X$ s.t. the line segment

$$\{ t x_0 + (1-t)x : t \in [0,1] \} \subset X, \quad \forall x \in X.$$



star-shaped



not star-shaped

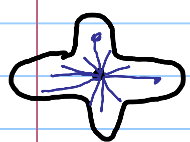
Theorem: If $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^2$ is star-shaped, then <u>yes</u> to the original question.

<u>Proof</u>:

WLOG, assume $x_0 = 0 \in \mathbb{R}^2$.

Consider

$$F(x_1,x_2) := \int_0^1 x_1 f_1(tx_1,tx_2) + x_2 f_2(tx_1,tx_2)\,dt$$



Then:

$$\partial F/\partial x_1 = \int_0^1 \Big[\underline{f_1(tx_1,tx_2)} + \underline{x_1 \cdot \frac{\partial f_1}{\partial x_1}(tx_1,tx_2)\cdot t} + x_2 \frac{\partial f_2}{\partial x_1}(tx_1,tx_2)\cdot t\Big]\,dt$$

Also,

$$\frac{d}{dt}\, t f_1(tx_1,tx_2) = \underline{f_1(tx_1,tx_2)} + \underline{t \frac{\partial f_1}{\partial x_1}(tx_1,tx_2)\, x_1} + t \frac{\partial f_1}{\partial x_2}(tx_1,tx_2)\, x_2$$

$$\Downarrow$$

$$\frac{\partial F}{\partial x_1} = \int_0^1 \frac{d}{dt}\, t f_1(tx_1,tx_2) + t x_2 \underbrace{\Big[\frac{\partial f_2}{\partial x_1}(tx_1,tx_2) - \frac{\partial f_1}{\partial x_2}(tx_1,tx_2)\Big]}_{=0}\,dt$$

$$= \Big[t f_1(tx_1,tx_2)\Big]_{t=0}^{1}$$

$$= f_1(x_1,x_2)$$

Similarly, $\partial F/\partial x_2 = f_2(x_1,x_2)$. $\square$

Let $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^2$,

$$C^\infty(\mathcal{U}, \mathbb{R}^k) = \{\phi : \mathcal{U} \to \mathbb{R}^k,\ C^\infty \text{ smooth}\}.$$

It is a vector space over $\mathbb{R}$.

[$k=1$: "scalar fields" , $k=2$: "vector fields"]

Define the following linear operators :

$\text{grad} : C^\infty(\mathcal{U}, \mathbb{R}) \to C^\infty(\mathcal{U}, \mathbb{R}^2)$

$$\text{grad}\,\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}\right)$$

$\text{rot} : C^\infty(\mathcal{U}, \mathbb{R}^2) \to C^\infty(\mathcal{U}, \mathbb{R})$

$$\text{rot}(\phi_1, \phi_2) = \frac{\partial \phi_1}{\partial x_2} - \frac{\partial \phi_2}{\partial x_1}$$

$$C^\infty(\mathcal{U}, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\mathcal{U}, \mathbb{R}^2) \xrightarrow{\text{rot}} C^\infty(\mathcal{U}, \mathbb{R})$$

Note:

$$\text{rot} \circ \text{grad} = 0 \quad \text{— } (*)$$

And since these are linear operators ,

$$\ker(\text{rot}) = \{\phi : \mathcal{U} \to \mathbb{R}^2 : \text{rot}\,\phi = 0\}$$

$$\text{Im}(\text{grad}) = \{\text{grad}\,\phi : \phi : \mathcal{U} \to \mathbb{R}\}$$

are both linear subspaces of $C^\infty(\mathcal{U}, \mathbb{R}^2)$,

Moreover, by $(*)$,

Im(grad) is a subspace of ker(rot).

We can also talk about the quotient vector space:

$$H^1(\mathcal{U}) := \ker(\mathrm{rot}) / \mathrm{Im}(\mathrm{grad})$$

$$= \{ [\alpha]_\sim : \alpha \in \ker(\mathrm{rot}) \}$$

$$\alpha \sim \beta \iff \alpha - \beta \in \mathrm{Im}(\mathrm{grad}).$$

Note:

$\mathrm{Im}(\mathrm{grad}) = \mathrm{grad}(\underbrace{C^\infty(\mathcal{U}, \mathbb{R})}_{\infty\text{-dimensional}})$ is ∞-dim.

Ex: Prove this claim.

So $\ker(\mathrm{rot})$ ($\supset \mathrm{Im}(\mathrm{grad})$) must also be ∞-dim.

But usually the quotient space $H^1(\mathcal{U})$ is finite-dim.

The "star-shape theorem" (actually a special case of a later result called the <u>Poincare lemma</u>) can be reformulated as:

$$\boxed{H^1(\mathcal{U}) = 0 \quad \text{whenever } \mathcal{U} \subseteq \mathbb{R}^2 \text{ is star-shaped.}}$$

What about $H^1(\mathbb{R}^2 - \{0\})$?

Earlier example shows $\exists\, \phi \in C^\infty(\mathbb{R}^2 - \{0\}, \mathbb{R}^2)$

$$\phi \in \ker(\mathrm{rot})$$

$$\phi \notin \mathrm{Im}(\mathrm{grad})$$

so $H^1(\mathbb{R}^2 - \{0\}) \neq 0$

We shall prove:

$$H^1(\mathbb{R}^2 - \{0\}) \cong \mathbb{R}^1$$

$$H^1(\mathbb{R}^2 - \{x_1, \dots, x_k\}) \cong \mathbb{R}^k.$$

This suggests: (in a sense to be made precise)

$$\dim H^1(U) = \#\text{ of "Holes" in } U.$$

$$\{0\} \xrightarrow{id} C^\infty(U, \mathbb{R}^1) \xrightarrow{grad} C^\infty(U, \mathbb{R}^2)$$

the zero function

Trivial: $grad \circ id = 0$

$\ker(grad) \supset \operatorname{Im}(id)$

Define $H^0(U) := \ker(grad) / \operatorname{Im}(id)$

$= \ker(grad)$

This definition works for open sets U of $\mathbb{R}^k$ for any $k \geq 1$, when we define

$$grad(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_k}\right).$$

Moreover, it has a meaning:

Theorem: (i) $U \overset{\text{open}}{\subseteq} \mathbb{R}^k$ is connected $\Leftrightarrow H^0(U) = \mathbb{R}^1$.

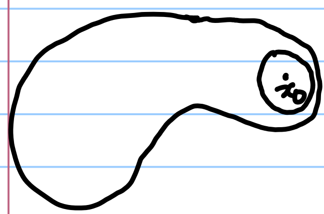
(ii) Moreover, $\dim H^0(U) = \#$ of connected components of U.

Proof of (i):

Assume $grad(f) = 0$.

For each $x_0 \in \mathcal{U}$, there is a ball $B(x_0, r) \subset \mathcal{U}$

Claim: f is constant on this ball



$\mathcal{U}$ - open + connected

$\overset{\bullet}{x_0} \to \hat{u}$ For any unit vector $\hat{u}$, consider

$$g_{\hat{u}}(t) = f(x_0 + t r \hat{u}) \quad t \in (-1,1)$$

$$g'_{\hat{u}}(t) = \underbrace{df(x_0 + t r \hat{u})}_{\substack{\mathrm{grad} f(x_0 + t r \hat{u}) \\ = 0}} \cdot r\hat{u} = 0$$

$$\text{So } g_{\hat{u}}(t) = \int_0^t g'_{\hat{u}}(s)\, ds + g_{\hat{u}}(0) = g_{\hat{u}}(0) = f(x_0), \quad \forall t, \forall \hat{u}$$

This claim is proved and we establish that f is locally constant.

Of course, f should be constant on the whole $\mathcal{U}$, to argue this, consider

$$X := \{x \in \mathcal{U} : f(x) = f(x_0)\} = f^{-1}(\underbrace{\{f(x_0)\}}_{\text{closed in } \mathbb{R}})$$

X is closed as f is continuous.

X is open as f is locally constant.

Since $\mathcal{U}$ is connected, $X = \mathcal{U}$.

This means every $f \in \ker(\mathrm{grad})$ is a constant function, with some constant value in $\mathbb{R}$, i.e.

$$H^0(\mathcal{U}) = \mathbb{R}.$$

Conversely, if $\mathcal{U}$ is not connected, then (by def.)

there exist $A, B \overset{\text{open}}{\subseteq} \mathcal{U}$ ($\Leftrightarrow$ $A, B \overset{\text{open}}{\subset} \mathbb{R}^k$, since $\mathcal{U}$ is open in $\mathbb{R}^k$)

s.t.

$$\mathcal{U} = A \cup B \quad , \quad A \cap B = \phi \quad , \quad A, B \neq \phi$$

so we can define

$$f : \mathcal{U} \to \mathbb{R} \quad , \quad f(x) = \begin{cases} a & , x \in A \\ b & x \in B \end{cases}$$

for some $a, b \in \mathbb{R}$, $a \neq b$.

This function is C^∞, is locally constant, so $\text{grad}(f) = 0$.

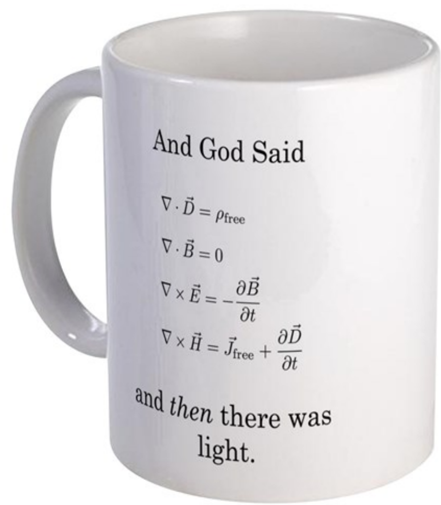
This shows $\dim H^0(\mathcal{U}) \geqslant 2$.

Ex: Extend this argument to prove part (ii) of the theorem. Be careful about the argument as the number of connected components can be infinite.

Next, consider the trivariate case.

Let $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^3$.

We have grad, curl, div that underlies the celebrated Maxwell's equations.



$$C^\infty(\mathcal{U}, \mathbb{R}) \xrightarrow{\text{grad}} C^\infty(\mathcal{U}, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\mathcal{U}, \mathbb{R}^3) \xrightarrow{\text{div}} C^\infty(\mathcal{U}, \mathbb{R})$$

$$\text{grad} f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$\operatorname{curl}(\overbrace{f_1, f_2, f_3}^{f}) = \nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\| $$

$$\left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$

$$\operatorname{div}(f_1, f_2, f_3) = \nabla \cdot f$$

these are just cute ways to help you remember the formulas

$$\| $$

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

And again we have

$$\operatorname{curl} \circ \operatorname{grad} = 0$$

$$\operatorname{div} \circ \operatorname{curl} = 0.$$

So we can define

$$H^0(U) = \ker(\operatorname{grad})$$
$$H^1(U) = \ker(\operatorname{curl}) / \operatorname{Im}(\operatorname{grad})$$
$$H^2(U) = \ker(\operatorname{div}) / \operatorname{Im}(\operatorname{curl})$$

These quotient vector spaces are vector spaces

<u>Theorem</u> (another special case of the Poincaré lemma)

For an open star-shaped set in $\mathbb{R}^3$, we have

$$H^0(U) = \mathbb{R},$$
$$H^1(U) = 0, \quad H^2(U) = 0.$$

<u>Proof</u>:

For $H^0(U)$ and $H^1(U)$, easy to adapt the proofs from the 2-D case. (Exercise.)

For $H^2(U)$, assume U is star-shaped wrt. O.

Consider $F \in \ker(\operatorname{div})$, i.e. $\operatorname{div} F = 0$

$F : U \to \mathbb{R}^3$

Let $\quad G(x) = \int_0^1 F(tx) \times tx \ dt$

$\uparrow$ cross product, only defined in $\mathbb{R}^3$

Check: $\quad \operatorname{curl}(F(tx) \times tx) = \frac{d}{dt}(t^2 F(tx))$.

$$F = (f_1, f_2, f_3)$$

$$F \times x = (f_2 x_3 - f_3 x_2,\ f_3 x_1 - f_1 x_3,\ f_1 x_2 - f_2 x_1)$$

$\operatorname{curl}(F(tx) \times tx)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ t\ \ f_2(tx)x_3 - f_3(tx)x_2 & f_3(tx)x_1 - f_1(tx)x_3 & f_1(tx)x_2 - f_2(tx)x_1 \end{vmatrix}$$

$$= t\Big[\hat{i}\Big(\frac{\partial f_1}{\partial x_2}(tx)\cdot t\, x_2 + f_1(tx) - \underline{\frac{\partial f_2}{\partial x_2}(tx)\cdot t\, x_1}$$
$$- \underline{\frac{\partial f_3}{\partial x_3}(tx)\cdot t\, x_1} + f_1(tx) + \frac{\partial f_1}{\partial x_3}(tx)\cdot t x_3\Big)$$
$$+$$
$$\hat{j}\Big(\frac{\partial f_2}{\partial x_3}(tx)\cdot t\, x_3 + f_2(tx) - \frac{\partial f_3}{\partial x_3}(tx)\cdot t\, x_2$$
$$\cdots \Big)$$
$$+$$
$$\hat{k}\Big(\frac{\partial f_3}{\partial x_1}(tx)\cdot t\cdot x_1 + f_3(tx) - \frac{\partial f_1}{\partial x_1}(tx)\cdot t\, x_3$$
$$\cdots \Big)\Big]$$

use $\operatorname{div} F = 0$

$$= t\Big[\hat{i}\Big(2 f_1(tx) + \underbrace{\underline{\frac{\partial f_1}{\partial x_1}(tx)\cdot t x_1} + \frac{\partial f_1}{\partial x_2}(tx)\cdot t x_2 + \frac{\partial f_1}{\partial x_3}(tx)\, t x_3}_{t\, df_1(tx)\cdot x}\Big)$$
$$+ j\Big(\ \cdots \Big)$$
$$+ k\Big(\ \cdots \Big)\Big]$$

$$= \begin{bmatrix} 2t\, f_1(tx) + t^2\, df_1(tx)\cdot x \\ 2t\, f_2(tx) + t^2\, df_2(tx)\cdot x \\ 2t\, f_3(tx) + t^2\, df_3(tx)\cdot x \end{bmatrix}$$

$$= 2t\,F(tx) + t^2 \overset{3\times 3}{[DF(tx)]}\overset{3\times 1}{x} \overset{\text{chain rule}}{=} \frac{d}{dt}\left(t^2 F(tx)\right).$$

Hence,

$$\operatorname{curl} Q(x) = \operatorname{curl} \int_0^1 (F(tx) \times tx)\, dt$$

$$= \int_0^1 \frac{d}{dt}\left(t^2 F(tx)\right) dt = F(x).$$

So, $F \in \ker(\operatorname{div})$ and $\mathcal{U}$ star-shaped

$$\Downarrow$$

$$F \in \operatorname{Im}(\operatorname{curl}) \qquad \square$$

Examples of nontrivial cohomology in 3-D

① $H^1(\mathbb{R}^3 - \{\text{a circle}\}) \neq 0$

② $H^2(\mathbb{R}^3 - \{\text{a point}\}) \neq 0$

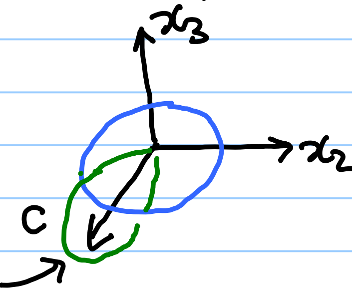
①: see Example 1.7 (P5, M&T)

This example is based on a vector field $\vec{f}$ on

$$\mathbb{R}^3 - \{\text{a circle}\}$$

with

$$\operatorname{curl} \vec{f} \equiv 0$$



and there is a closed loop

s.t.

$$\oint_C \vec{f} \cdot ds \neq 0$$

This means $\vec{f}$ cannot be the grad of a $\phi \in C^\infty(\mathbb{R}^3 - \{\text{circle}\}, \mathbb{R})$; see the next page.

The mathematics (or physics ?) underlying this and our earlier example of $H^1(\mathcal{U})$:

In general, for any smooth vector field in $\mathbb{R}^k$

$$\vec{f} : U \overset{\text{open}}{\subset} \mathbb{R}^k \to \mathbb{R}^k,$$

$$\gamma : [a,b] \to U$$

$$\int_\gamma \vec{f} \cdot ds := \int_a^b \vec{f}(\gamma(t)) \cdot \gamma'(t)\, dt \quad \text{"work done"}$$

If $\vec{f} = \nabla \phi$ ($\vec{f}$ is called a conservative vector field)

"potential (energy) function"

$$\vec{f}(\gamma(t)) \cdot \gamma'(t) = \nabla \phi(\gamma(t)) \cdot \gamma'(t)$$

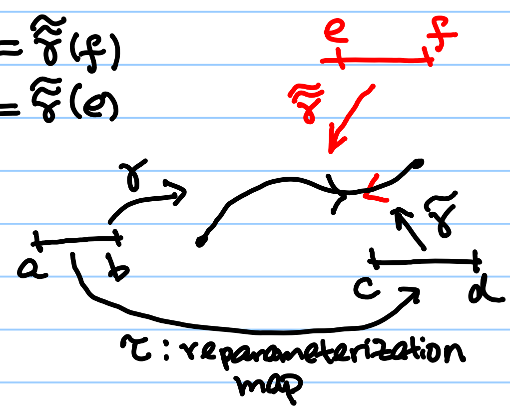
$$= \frac{d}{dt} \phi(\gamma(t)) \quad \text{(chain rule)}$$

and $$\int_\gamma \vec{f} \cdot ds = \underbrace{\phi(\gamma(b)) - \phi(\gamma(a))}_{\text{"potential difference"}}.$$ [Fundamental thm. of calculus]

Two things to remember:

1. $\int_\gamma \vec{f} \cdot ds$ is invariant under reparameterization

$\gamma(a) = \tilde{\gamma}(c) = \tilde{\tilde{\gamma}}(f)$
$\gamma(b) = \tilde{\gamma}(d) = \tilde{\tilde{\gamma}}(e)$



$$\int_\gamma \vec{f} \cdot ds = \int_{\tilde{\gamma}} \vec{f} \cdot d\tilde{s}$$

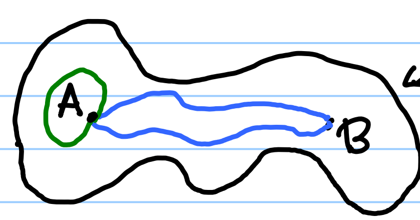
$$= -\int_{\tilde{\tilde{\gamma}}} \vec{f} \cdot d\tilde{\tilde{s}}$$

2. If $\vec{f}$ is conservative,

$\int_\gamma \vec{f} \cdot ds$ only depends on the endpoint of γ

In particular,

$$\boxed{\int_{\text{closed loop}} \vec{f} \cdot ds = 0}$$



open domain where $\vec{f} = \nabla \phi$

② $\vec{V}(x_1,x_2,x_3)$ = "inverse square law vector field"

$$= \frac{1}{r^3}(x_1,x_2,x_3), \quad r=(x_1^2+x_2^2+x_3^2)^{1/2}$$

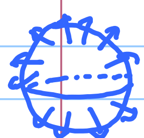
$$\in C^\infty(\mathbb{R}^3-\{0\},\mathbb{R}^3)$$

has the property that $\operatorname{div}\vec{V}=0$ everywhere

Is $\vec{V}=\operatorname{curl}\vec{F}$ for some $\vec{F}\in C^\infty(\mathbb{R}^3-\{0\},\mathbb{R}^3)$?

If so, $\iint_{S^2} \underbrace{\operatorname{curl}\vec{F}}_{\vec{V}}\cdot dS = \int_{\partial S^2=\phi} \vec{F}\cdot ds = 0$ (Stoke's thm.)

Flux of $\vec{V}$ across S^2



But $\iint_{S^2}\vec{V}\cdot dS \neq 0$, so $\vec{F}$ does not exist and

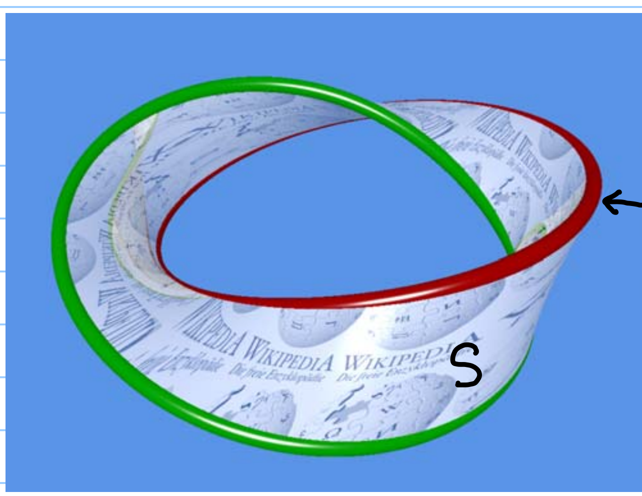
$$H^2(\mathbb{R}^3-\{0\})\neq 0.$$

Ex: check $\iint_{S^2}\vec{V}\cdot dS\neq 0$.

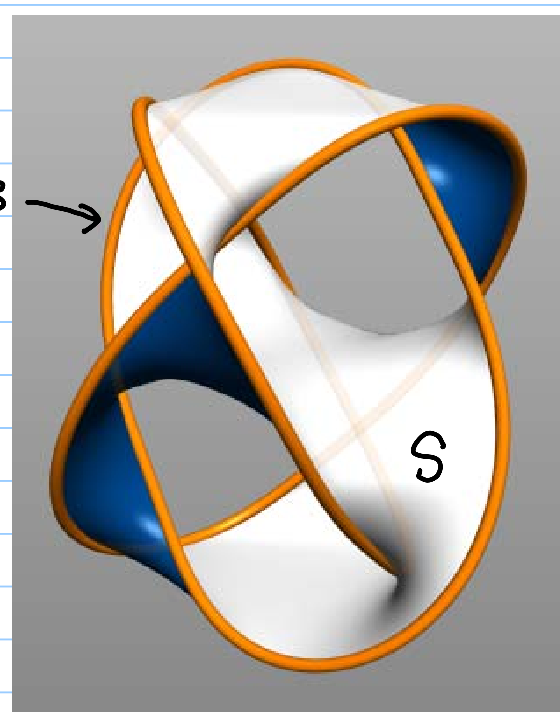
Vector calculus underlying these examples:

$H^1(\mathcal{U})$	$H^2(\mathcal{U})$
line integral over an oriented curve C	flux integral over an oriented (orientable) surface S ($\mathbb{R}^3$ only at this stage)
independent of parametrization of C, sign dependent on orientation	independent of parametrization of S, sign dependent on orientation
Fundamental thm of Calculus $\Downarrow$	Stoke's theorem $\Downarrow$
$\vec{F} = \nabla\phi \Rightarrow \int_C \vec{F}\cdot ds$ only depends on ∂C	$\vec{F} = \mathrm{curl}\,\vec{V} \Rightarrow \iint_S \vec{F}\cdot dS$ only depends on ∂S; called "vector potential" in physics

Examples of orientable surfaces S with $\partial S \neq \phi$:



∂S



Note: not a Möbius band

Where is the divergence theorem?

$$\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^3$$

$$0 \xrightarrow{id} C^\infty(\mathcal{U},\mathbb{R}) \xrightarrow{grad} C^\infty(\mathcal{U},\mathbb{R}^3) \xrightarrow{curl} C^\infty(\mathcal{U},\mathbb{R}^3) \xrightarrow{div} C^\infty(\mathcal{U},\mathbb{R}) \xrightarrow{\text{"}0\text{"}} 0$$

$$H^0(\mathcal{U}) = \ker(grad) / Im(id) = \mathbb{R}^{\#\text{ connected comps.}}$$
$$H^1(\mathcal{U}) = \ker(curl) / Im(grad) = ?$$
$$H^2(\mathcal{U}) = \ker(div) / Im(curl) = ?$$
$$H^3(\mathcal{U}) = \underbrace{\ker(\text{"}0\text{"})}_{C^\infty(\mathcal{U},\mathbb{R})} / Im(div) = ?$$

Divergence theorem:

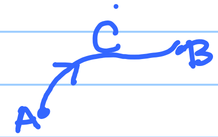
$$\underset{\substack{\Omega \\ \uparrow \\ \text{solid (3D)}}}{\iiint} div\,\vec{F}\, dV = \underset{\substack{\partial\Omega \\ \uparrow \\ \text{surface (2-D)}}}{\oiint} \vec{F}\cdot dS$$

Stoke's:

$$\underset{\substack{S \\ \uparrow \\ \text{surface (2D)}}}{\oiint} curl\,\vec{V}\cdot dS = \underset{\substack{\partial S \\ \uparrow \\ \text{curve (1-D)}}}{\int} \vec{V}\cdot ds$$

FTOC:

$$\underset{\substack{C \\ \uparrow \\ \text{curve (1D)}}}{\int} grad\,\phi \cdot ds = \text{"}\underset{\substack{\partial C = \{A,B\} \\ \uparrow \\ \text{points (0-D)}}}{\int} \phi\text{"} = \phi(B) - \phi(A)$$



What follows is a highly nontrivial generalizations of all these, first to $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n$ then to manifolds.

The first step of this extensive generalization is to notice that the objects that show up in

volume, flux and line integrals "act pointwise"

on the <u>tangent vectors</u> of the corresponding

solid, surface and curve (resp.)

in a

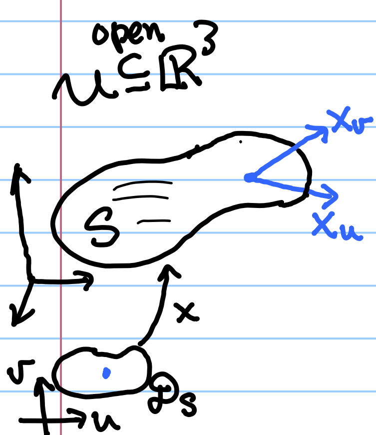
trilinear, bilinear and linear (resp.)

and

alternating manner.

- Flux integral :

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}_S} \underbrace{\vec{F}(u,v) \cdot (X_u \times X_v)}_{\text{think pointwise}} \, du\, dv$$



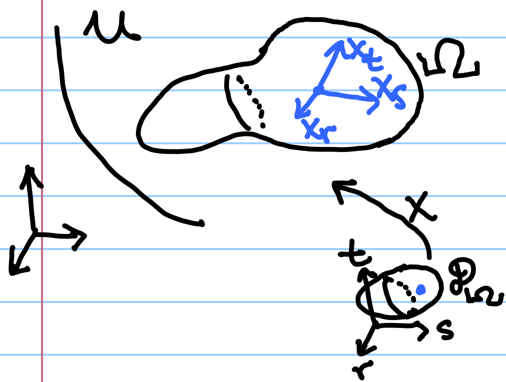
The 2-linear map

$$(X_u, X_v) \mapsto \vec{F}(u,v) \cdot (X_u \times X_v)$$

is <u>alternating</u>.

- Volume integral

$$\iiint_{\Omega} g \, dV = \iiint_{\mathcal{D}_{\Omega}} g(r,s,t) \cdot \det[x_r, x_s, x_t] \, dr \, ds \, dt$$

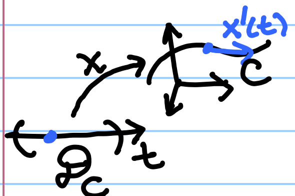


The 3-linear map

$(x_r, x_s, x_t) \mapsto g(r,s,t) \cdot \det[x_r, x_s, x_t]$

is <u>alternating</u>.

- Line integral $\displaystyle\int_C \vec{F} \cdot ds = \int_{\mathcal{D}_C} \vec{F}(t) \cdot x'(t) \, dt$



The linear map : $x'(t) \mapsto \vec{F}(t) \cdot x'(t)$

is alternating (but pointless to say it when there is only one input vector.)