Math 305
Homework #3

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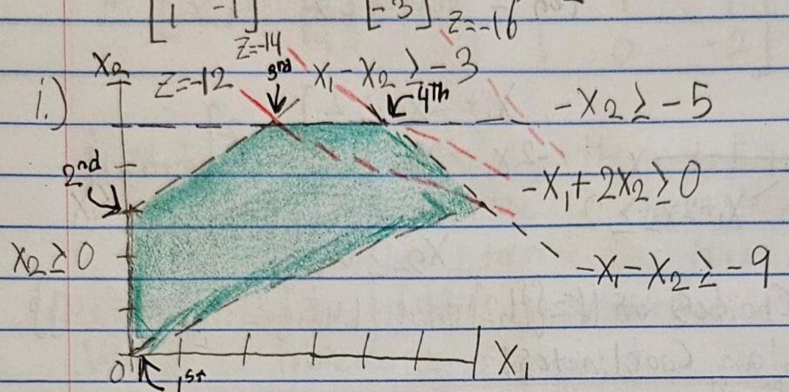
Question #1 (Exercise 3-1-2)

$\min_x p^T x$ s.t. $Ax \geq b,\ x \geq 0$

$$A=\begin{bmatrix}0 & -1\\ -1 & -1\\ -1 & 2\\ 1 & -1\end{bmatrix}\quad b=\begin{bmatrix}-5\\ -9\\ 0\\ -3\end{bmatrix},\quad p=\begin{bmatrix}-1\\ -2\end{bmatrix} \Rightarrow z=-x_1-2x_2$$

i.)



Green = Feasible region

Red = Contours at $z=-12,-14,-16$

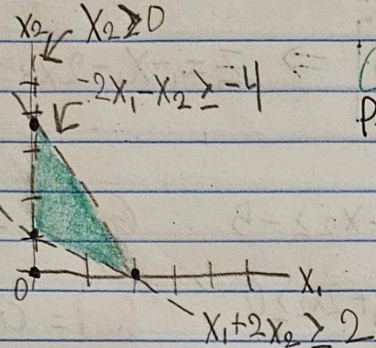
Pen = Simplex path

ii.) Observe Contours in Part i, Solution is at $x=\begin{bmatrix}4\\5\end{bmatrix}$. Can be observed graphically that is the largest $z=p^Tx$ can be while still having a point in the feasible region. Thus, Minimum Value is $z=-14$, Minimizer $= x=\begin{bmatrix}4\\5\end{bmatrix}$

iii.) Observe Arrows in Part i. Using MatLab determined Solution to be $x=\begin{bmatrix}4\\5\end{bmatrix}$

Question #2 (Exercise 3-2-1)

1.)



Green = feasible region
Pen = vertices

$$\left.\begin{array}{l} x_1 + 2x_2 \geq 2 \\ -2x_1 - x_2 \geq -4 \end{array}\right\} m=2$$
$$\left.\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \end{array}\right\} n=2$$
$m+n = 4$

All Possible Choices of $N = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$

In Cartesian Coordinates:

$(0,4)$: $N = \{\{2,4\}\}$

$(0,2)$: $N = \{\{1,4\}\}$

$(0,0)$: $N = \{\{3,4\}\}$

$(2,0)$: $N = \{\{1,2\},\{1,3\},\{2,3\}\}$

2.) For vertex $x = (0,4)$ using $N = \{2,4\}$

We get $B = \{1,3\}$

$$H = \begin{bmatrix} -1/2 & -1/2 \\ 3/2 & -1/2 \end{bmatrix} \qquad h = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} x_N & 1 \\ x_B = & \boxed{H \;|\; h} \end{array}$$

Question #3 (Exercise 3-3-2)

$\min_{x} p^T x$ S.t. $Ax \geq b, x \geq 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad p = \begin{bmatrix} 1 \\ -2 \\ -4 \\ 4 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & -2 & -1 \\ 2 & -1 & -1 & 4 \\ -1 & 1 & 0 & -2 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ -5 \\ -3 \end{bmatrix}$$

After Exchanging x_5 & x_3, then x_6 & x_2, we see using Pricing rule we have Pivots, but using ratio test we determine we have no Pivot rows. Therefore, we have an Unbounded Problem.

Using the tableau, we find

$$x_3 = \tfrac{2}{3}\lambda_1 + \lambda_2 + 3 \qquad x_2 = \tfrac{4}{3}\lambda_1 + 3\lambda_2 + 2$$

$$x_7 = \tfrac{1}{3}\lambda_1 + \lambda_2 + 5$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 2 \\ 3 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 4/3 \\ 2/3 \\ 0 \end{bmatrix}\lambda_1 + \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}\lambda_2$$

Wrong Picked multiple variables to be unbounded

Corrected ↓

Choose $x_4 = \lambda, \lambda \geq 0$, then $x_1 = 0$, thus:

$$x_3 = \lambda + 3 \quad x_2 = 3\lambda + 2 \quad x_7 = \lambda + 5$$

Therefore, our Solution set is:

$$x = \begin{bmatrix} 0 \\ \lambda+3 \\ 3\lambda+2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \end{bmatrix}$$

Therefore, if we want $z = -415$, from $z = p^T x$, we get:

$$-415 = -2(\lambda+3) - 4(3\lambda+2) + 4\lambda = 10\lambda - 14 \Rightarrow \lambda = 40.1$$

Therefore, we get $x^T = (0, 43.1, 122.3, 40.1)$

Question #4 (Exercise 3-3-5)

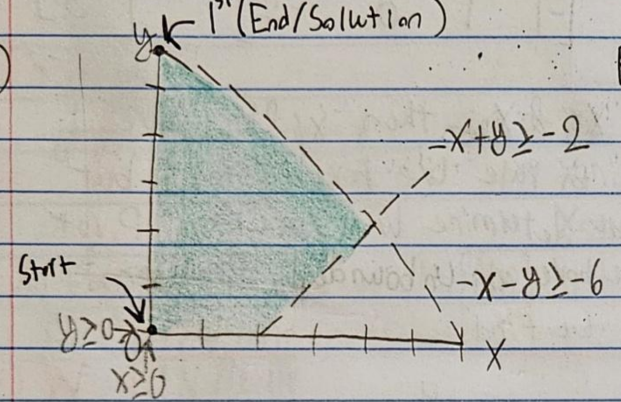
1.) $\min_{x,y} z = x - y$ S.t. $-x + y \geq -2$ $\quad x \geq 0$

$-x - y \geq -6$ $\quad y \geq 0$

Using a single exchange of Row 2 & Column 2 as Pivots

Minimum Value $= -6$, minimizer $x = 0, y = 6$

2.)



Pen = SimPlex Path

3.) $\min_{x,y} z = -x + y$ S.t. $-x + y \geq -2$ $\quad x \geq 0$

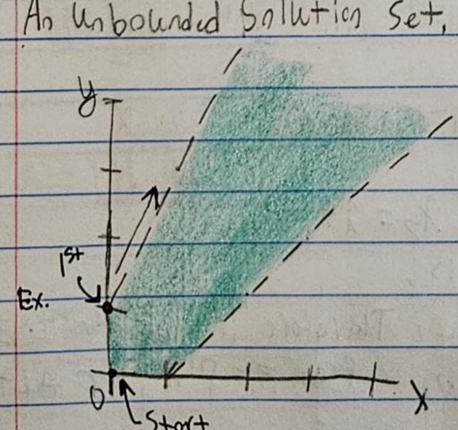
$-x - y \geq -6$ $\quad y \geq 0$

Solution Set $= (1-\theta)\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \theta \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ For any $0 \leq \theta \leq 1$

4.) $\min_{x,y} z = -x - y$ S.t. $2x - y \geq -1$ $\quad x \geq 0$

$-x + y \geq -1$ $\quad y \geq 0$

An Unbounded Solution Set,



Solution Set $W = \begin{bmatrix} x \\ y \end{bmatrix}$,

$W = \begin{bmatrix} \lambda \\ 2\lambda + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Pen = SimPlex Path

Question #5 (Exercise 3-3-6)

$$\min_{X} Z = P^T x \quad \text{s.t. } Ax \geq b, \; x \geq 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad P = \begin{bmatrix} 3 \\ -3 \\ -2 \\ 5 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -1 & -1 & -1/2 \\ 5/2 & 9/4 & -1/2 & 17/4 \\ -1 & 1 & 0 & -2 \end{bmatrix} \quad b = \begin{bmatrix} -2 \\ -4 \\ -3 \end{bmatrix}$$

Before starting notice A's size, A is 3x4, thus $m < n$ meaning we expect infinite solutions.

As expected, we find Infinite Solution. Note that the feasible region is unbounded, also note the solution set is unbounded.

Let $x_1 = \lambda$, and $\lambda \geq 0$, then our solution set is as follows:

$$X = \begin{bmatrix} \lambda \\ \lambda + 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \text{ thus } U = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \; V = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Question #6 (Exercise 3-4-2)

1. $\min_{X} z = -3x_1 + x_2$ S.t.: $-x_1 - x_2 \geq -2$

$$2x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

To show this LP is infeasible, consider first that, using the Phase I procedure, if the original LP is feasible, we can find a $X = (0, \bar{X})$. Therefore if the minimum value of $x_0 \neq 0$, the LP is not feasible.

We find our final tableau:

	x_3	x_2	x_4	I
$x_1 =$	-1	-1	0	2
$x_0 =$	2	0	1	6
$z =$	3	4	0	-6
$z_0 =$	2	0	1	6

Observe that $z_0 = x_0$ is minimized, however $x_0 \neq 0$. Therefore this LP is infeasible.

2.) $\min_{X} z = -x_1 + x_2$ S.t. $2x_1 - x_2 \geq 1$

$$x_1 + 2x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

First, since the origin is not within the feasible region we must use the Phase I procedure to find a starting vertex.

Using Phase I procedure we find $x^T = (0.8, 0.6)$ as a starting vertex. Proceeding with Phase 2, we find a pivot column but no pivot row, meaning this LP is unbounded.

Question #7 (Exercise 3-4-4)

$\min_{x} Z = P^T X$ s.t. $Ax \geq b$, $X \geq 0$

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad P = \begin{bmatrix} 2 \\ 3 \\ 6 \\ 4 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Noticing that the origin is not in the feasible region thus, using the Phase I procedure to find a vertex. We find a vertex at $X^T = (0, 0, 5/3, 0)$. Doing a Jordan Exchange of X_6 & X_4, then X_3 & X_2, We find an infinite Solution Set:

$$X = (1-\theta)\begin{bmatrix} 0 \\ 2.4 \\ 0 \\ 0.2 \end{bmatrix} + \theta \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \quad \text{for any } 0 \leq \theta \leq 1$$

Minimum Value = 8