# CH6 Homotopy invariance

Note Title 4/24/2017

Plan :
- Define what it means by two topological spaces $X$ and $Y$ being homotopy equivalent.
- For $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n$ (and later for manifolds) we show that

$$H^P(\mathcal{U})$$

depends only on the homotopy type of $\mathcal{U}$.
(We do not deal with cohomology/homology for general topological spaces in this course.)

**Def**: Two continuous maps $f_\nu : X \to Y$, $\nu = 0,1$ between topological spaces are said to be <u>homotopic</u> if

$$\exists \; F : X \times [0,1] \to Y \quad \text{s.t.}$$

$$F(x,\nu) = f_\nu(x) \;,\; \nu = 0,1 \;,\; x \in X .$$

In this case, we write $f_0 \simeq f_1$.

$F$ – "homotopy from $f_0$ to $f_1$"

Think of $F$ as a family of continuous maps

$$f_t = F(\cdot, t) : X \to Y \quad , \quad t \in [0,1]$$

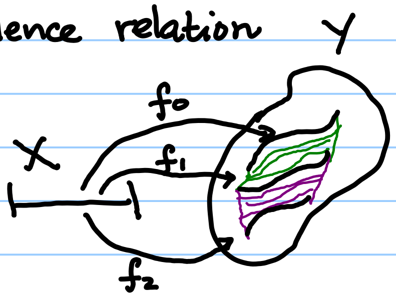
which deforms $f_0$ to $f_1$.

Easy to check : homotopy is an equivalence relation

i.e. $f_0 \simeq f_1 \Rightarrow f_1 \simeq f_0$

$f_0 \simeq f_0$

$f_0 \simeq f_1 \,,\, f_1 \simeq f_2 \Rightarrow f_0 \simeq f_2$



For transitivity, if $f_0 \simeq f_1$ via $F$, $f_1 \simeq f_2$ via $G$, then $f_0 \simeq f_2$ via

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 . \end{cases}$$

Easy to check:

$$X \xrightarrow[f_1]{f_0} Y \xrightarrow[g_1]{g_0} Z$$ all continuous maps between topological spaces

$$f_0 \simeq f_1 ,\ g_0 \simeq g_1 \Rightarrow g_0 \circ f_0 \simeq g_1 \circ f_1$$

**Def** A continuous map $f: X \to Y$ is called a

homotopy equivalence

if

$\exists\, g: Y \to X$ (called a homotopy inverse)

s.t.

$g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.

- Two topological spaces $X$ and $Y$ are called homotopy equivalent if $\exists$ a homotopy equivalence between them.

- A space $X$ is said to be contractible if $X$ is homotopy equivalent to a single-point space.

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & \{x\} \arrow[l, bend left, "g"] \end{tikzcd}$$

$g \circ f$ has to a constant map

and

$f \circ g = id_{\{x\}}$

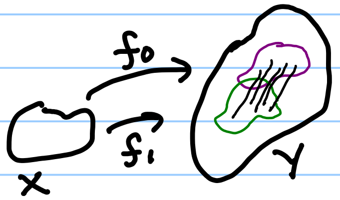
Hence, contractible

$\Leftrightarrow$ $id_X \simeq$ some constant map

$c: X \to \underset{\substack{\rotatebox{90}{$\in$} \\ x_0}}{X}$ $c(x) = x_0$ $\forall x \in X$

- We say two spaces are of the same **homotopy type** if they are homotopy equivalent.

E.g.



$$\underset{\substack{\uparrow \\ \text{any} \\ \text{space}}}{f_0, f_1 : X} \longrightarrow \underset{\substack{\uparrow \\ \text{need not} \\ \text{open}}}{Y \subseteq} \mathbb{R}^n$$

If $(1-t)f_0(x) + t f_1(x) \in Y$, $\forall t \in [0,1]$, $x \in X$

then $f_0 \simeq f_1$ via

$$F(x,t) := (1-t)f_0(x) + t f_1(x) \in Y$$

If $Y$ is star-shaped wrt. $y_0 \in Y$,

then $id_Y \simeq c_{y_0} : Y \to \{y_0\}$.

So every star-shaped set in $\mathbb{R}^n$ is contractible.

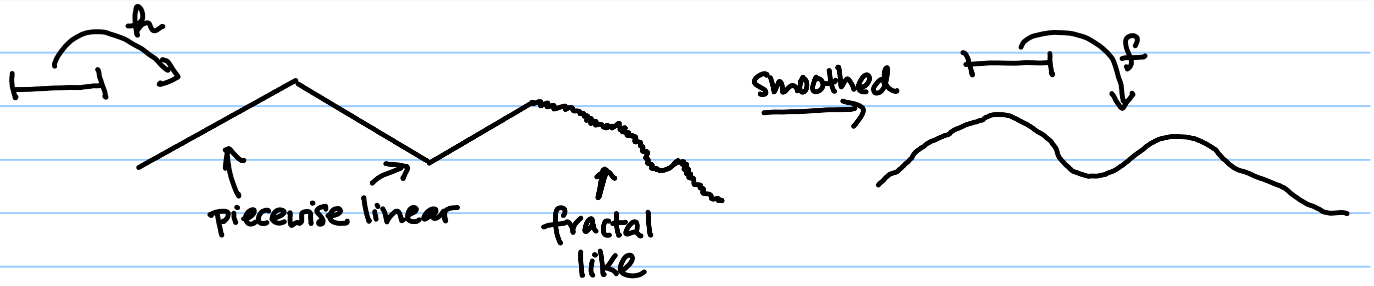
**Lemma** (6.6) If $U, V$ are open sets in Euclidean spaces, then

(i) Every continuous map $h: U \to V$ is homotopic to a smooth map $f: U \to V$

$$\underset{\substack{\uparrow \\ \text{only} \\ \text{continuous}}}{h} \simeq \underset{\substack{\nwarrow \\ \text{smooth}}}{f}$$

(ii) If two smooth maps $f_0, f_1 : U \to V$ are homotopic, then the homotopy can be made smooth, i.e.

$\exists$ smooth $F: U \times \underbrace{(-\varepsilon, 1+\varepsilon)}_{\text{may as well make it } \mathbb{R}} \to V$ s.t. $F(\cdot, 0) = f_0$, $F(\cdot, 1) = f_1$.



This lemma is instrumental for converting topological questions into questions pertaining to De Rham cohomology.
↑ requires smoothness

The proof of this result relies on a few results and techniques in math analysis. See Appendix A of M&T. Given the technical importance, it is not something I want to skip but there is simply not enough time.

Recall :

Any smooth map $f : \mathcal{U} \to V$ induces a (open sets in Euclidean spaces)

pullback map $\Omega^p(f)$, or $f^* : \Omega^p(V) \to \Omega^p(\mathcal{U})$

$$\boxed{\begin{tikzcd} f \arrow[d, squiggly] \\ \Omega^p(f) \arrow[d, squiggly] \\ H^p(f) \end{tikzcd}}$$

- In fact, not just one pullback map, but a whole <u>chain map</u> :

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^{p-1}(V) \arrow[r, "d_V^{p-1}"] \arrow[d, "\Omega^{p-1}(f)"] & \Omega^p(V) \arrow[r, "d_V^p"] \arrow[d, "\Omega^p(f)"] & \Omega^{p+1}(V) \arrow[r] \arrow[d, "\Omega^{p+1}(f)"] & \cdots \\ \cdots \arrow[r] & \Omega^{p-1}(\mathcal{U}) \arrow[r, "d_{\mathcal{U}}^{p-1}"'] & \Omega^p(\mathcal{U}) \arrow[r, "d_{\mathcal{U}}^p"'] & \Omega^{p+1}(\mathcal{U}) \arrow[r] & \cdots \end{tikzcd}$$

This is because pullback commutes with $d$

↳ = ↴

- Furthermore, this chain map induces (or "descends to") linear maps

$$H^p(f) : H^p(V) \to H^p(\mathcal{U}) \ , \ p = 0, 1, \cdots$$

The next result is vital, it says

$$\boxed{f \simeq g : \mathcal{U} \to V \implies H^p(f) = H^p(g) : H^p(V) \to H^p(\mathcal{U})}$$

In details, it goes as follows:

**Thm** If $f, g : \mathcal{U} \to V$ are smooth maps, $f \simeq g$, then the induced chain maps

$$f^*, g^* : \Omega^*(V) \to \Omega^*(\mathcal{U}) \text{ are chain homotopic}$$

i.e.

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega^{p-1}(V) \arrow[r, "d_V^{p-1}"] \arrow[d] & \Omega^p(V) \arrow[r, "d_V^p"] \arrow[d] \arrow[dl, "s^p"'] & \Omega^{p+1}(V) \arrow[r] \arrow[d] \arrow[dl, "s^{p+1}"'] & \cdots \\
\cdots \arrow[r] & \Omega^{p-1}(\mathcal{U}) \arrow[r, "d_{\mathcal{U}}^{p-1}"'] & \Omega^p(\mathcal{U}) \arrow[r, "d_{\mathcal{U}}^p"'] & \Omega^{p+1}(\mathcal{U}) \arrow[r] & \cdots
\end{tikzcd}$$

$\exists\ s : \Omega^p(V) \to \Omega^{p-1}(\mathcal{U})$ s.t.

$$\underset{\curvearrowright}{d_{\mathcal{U}}^{p-1} s^p} + \underset{\curvearrowleft}{s^{p+1} d_V^p} = \Omega^p(f) - \Omega^p(g)$$

$$[\text{ which } \underset{\substack{\uparrow \\ \text{Lemma 4.11 (easy, but important)}}}{\Longrightarrow} H^p(f) = H^p(g) : H^p(V) \to H^p(\mathcal{U}). ]$$

**Preparation:**

Recall that I complained about MBT's proof of the Poincaré lemma being unnecessarily indirect. Their construction of the map $S_p : \Omega^p(\underset{\substack{\uparrow\\ }}{\mathcal{U}}) \to \Omega^{p-1}(\underset{\substack{\nearrow\\ }}{\mathcal{U}})$ is via another map
star-shaped

$$\hat{S}_p : \Omega^p(\underset{\nwarrow}{\mathcal{U} \times \mathbb{R}}) \to \Omega^{p-1}(\underset{\nearrow}{\mathcal{U}})$$

need not be star-shaped

If $\omega \in \Omega^p(\mathcal{U}\times\mathbb{R})$ is expressed as

$$\omega = \sum_{\underset{\overset{\uparrow}{p\text{-tuple}}}{I}} f_I(x,t)\, dx_I \;+\; \sum_{\underset{\overset{\uparrow}{(p-1)\text{-tuple}}}{J}} g_J(x,t)\, dt\wedge dx_J$$

then

$$\hat{S}_p(\omega) := \sum_J \left[\int_0^1 g_J(x,t)\,dt\right] dx_J \,,$$

this family of maps satisfies

$$(*) \;-\; d\,\hat{S}_p(\omega) + \hat{S}_{p+1}\, d(\omega) = \sum_I f_I(x,1)\,dx_I - \sum_I f(x,0)\,dx_I .$$

**Proof** :

Recall that $f \simeq g \Rightarrow \exists$ a smooth homotopy

$$F: \mathcal{U}\times\mathbb{R} \to V \;,\; F(x,0) = f(x),\; F(x,1) = g(x).$$

Consider the inclusion map

$$\phi_0, \phi_1 : \mathcal{U} \longrightarrow \mathcal{U}\times\mathbb{R} \;,\; \phi_0(x) = (x,0) \;,\; \phi_1(x) = (x,1).$$

Then $F\circ\phi_0 = f \;,\; F\circ\phi_1 = g$

$$\mathcal{U} \xrightarrow{\phi_\nu} \mathcal{U}\times\mathbb{R} \xrightarrow{F} V$$

Observe : For $\nu = 0$ or $1$, $\omega = \sum_I f_I\, dx_I + \sum_J g_J\, dt\wedge dx_J \,,$ $\in \Omega^p(\mathcal{U}\times\mathbb{R})$

$$\phi_\nu = \begin{bmatrix} (\phi_\nu)_1 = x_1 \\ \vdots \\ (\phi_\nu)_n = x_n \\ \nu \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow \text{constant } (0 \text{ or } 1) \end{matrix}$$

$$\phi_\nu^*\omega = \sum_I f_I\circ\phi_\nu\, \underset{\underset{dx_I}{\shortparallel}}{d(\phi_\nu)_I} + \sum_J g_J\circ\phi_\nu\, \underbrace{\phi_\nu^*(dt\wedge dx_J)}_{\substack{=0 \text{ because the} \\ \text{last comp.} \\ \text{(the } t\text{-comp.)} \\ \text{of } \phi_\nu \text{ is} \\ \text{constant}}}$$

$$\phi_\nu^*\omega = \sum_I f_I(x,\nu)\,dx_I \;,\; \nu = 0,1$$

And the property of the map $\hat{S}_p : \Omega^p(\mathcal{U}\times\mathbb{R}) \to \Omega^{p-1}(\mathcal{U})$ can be expressed as

$$(*') \text{ — } (d\hat{S}_p + \hat{S}_{p+1} d)\,\omega = \phi_1^*\omega - \phi_0^*\omega \quad (\text{see } (*))$$

Define $\quad S_p : \Omega^p(V) \to \Omega^p(\mathcal{U}) \quad$ by

$$S_p := \hat{S}_p \circ F^*$$

$$\begin{tikzcd} \Omega^{p-1}(\mathcal{U}) & \Omega^p(\mathcal{U}\times\mathbb{R}) \arrow[l, "\hat{S}_p"'] & \Omega^p(V) \arrow[l, "F^*"'] \end{tikzcd}$$

Claim : $d S_p + S_{p+1} d = g^* - f^*$ ( as desired)

check:

$$d\circ\hat{S}_p\circ F^*\omega + \hat{S}_{p+1}\circ \overbrace{F^*\circ d}^{= d\circ F^*}\omega$$

$$= (d\circ\hat{S}_p)(F^*\omega) + (\hat{S}_{p+1}\circ d)(F^*\omega) \qquad \text{( pullback commutes with } d)$$

$$= \underbrace{\phi_1^* F^*}_{\substack{\parallel \\ (F\circ\phi_1)^* \\ = g}}\omega - \underbrace{\phi_0^* F^*}_{\substack{\parallel \\ (F\circ\phi_0)^* \\ = f}}\omega = g^*\omega - f^*\omega .$$

$\square$

Note : If $\phi : \mathcal{U} \to V$ is merely continuous

we cannot define $\Omega^p(\phi)$ and $H^p(\phi)$.

However, $\phi \simeq f : \mathcal{U} \to V$ for some smooth $f$

and $H^p(f) : H^p(V) \to H^p(\mathcal{U})$ is well-defined

Moreover, by the last result, any such (smooth) $f$ gives the same $H^p(f)$.

Therefore, we can define

$H^p(\phi)$ (or $\phi^*$ when there is no source of confusion)

to be this unique linear map $H^p(V) \to H^p(\mathcal{U})$.

Conclusion:

$$\boxed{\begin{array}{ll} H^p(\phi) \overset{\text{def}}{:=} H^p(f) & \text{for any choice of} \\ \quad \| & \text{smooth } f \simeq \phi \text{, the} \\ \quad \phi^* & \text{choice doesn't matter.} \end{array}}$$

Using the previous lemma 6.6 (on "smooth homotopic approx.") to push all these a little further, we have

<u>Thm</u> For $p \in \mathbb{Z}$, open sets $\mathcal{U}, V, W$ in Euclidean spaces, we have

(i) If $\phi_0, \phi_1 : \mathcal{U} \to V$ are homotopic continuous maps, then

$$\phi_0^* = \phi_1^* : H^p(V) \to H^p(\mathcal{U}).$$

(ii) If $\phi : \mathcal{U} \to V$, $\psi : V \to W$ both continuous then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* : H^p(W) \to H^p(\mathcal{U})$$

(iii) If the continuous map $\phi : \mathcal{U} \to V$ is a homotopy equivalence, then

$$\phi^* : H^p(V) \to H^p(\mathcal{U}) \text{ is an isomorphism.}$$

[The proof should be a easy exercise.]

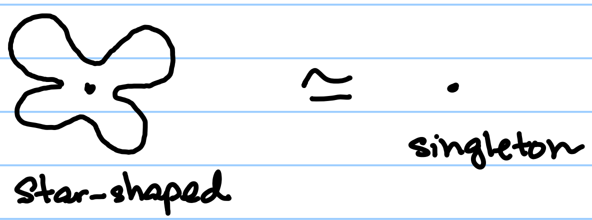
Recall: Two topological spaces are homotopy equivalent

$$\cancel{\Downarrow} \quad \Uparrow$$

They are homeomorphic

counterexamples:

(i)



but of course an open set in $\mathbb{R}^n$ cannot be homeomorphic to a singleton.

(ii) $$\mathbb{R}^n - \{0\} \simeq S^{n-1}$$

via $$x \xmapsto{f} x/\|x\|$$

Corollary:

$$\begin{tikzcd}
U, V \text{ homeomorphic} \arrow[r, Rightarrow] \arrow[d, Rightarrow, shift right] & U, V \text{ homotopy equiv.} \arrow[dl, Rightarrow] \\
\text{Same de Rham cohomology} \arrow[u, Rightarrow, shift right, "/" marking] &
\end{tikzcd}$$

Note: This corollary cannot (yet) be applied to the two homotopy equivalent spaces above.

singleton and $S^{n-1}$ are not open sets in $\mathbb{R}^n$

But they are manifolds, and the corollary above will generalize after we extend de Rham cohomology to manifolds.

Another easy but important corollary :

If $\mathcal{U} \subseteq \mathbb{R}^n$ is an open contractible set (not necessarily star-shaped),

$$H^p(\mathcal{U}) = \begin{cases} \mathbb{R}^1 & p = 0 \\ 0 & p > 0 \end{cases}$$

(just as in the case $\mathcal{U}$ = star-shaped.)

Since $\mathcal{U}$ is star-shaped $\underset{\not\Leftarrow}{\Longrightarrow}$ $\mathcal{U}$ is contractible,

this result strengthens the Poincaré lemma.

Note : the <u>proof</u> of the Key result that leads to this corollary uses the same map $\hat{S}_a$ used in the proof of Poincaré lemma.

In the second half of this chapter, I want to

(I) Give a very coarse description of what simplicial / singular homology is about, so that

- You have some intuition of what

$$H_p(\mathbb{R}^n - \{0\}),\ H_p(\mathbb{R}^n - S^k) \text{ etc}$$

are supposed to mean.

- After we develop Stoke's theorem, you will have some intuition of why

Simplicial/Singular homology $H_p(M)$ $\overset{\text{duality}}{\cong}$ Simplicial/singular cohomology $H^p(M, \mathbb{R})$ $\cong$ deRham cohomology $H^p_{DR}(M)$

for manifolds.

de Rham theorem (not easy)

based on Stoke's theorem (much easier)

(II) Prove

$$H^p(\mathbb{R}^n - \{0\}) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

In particular,

$$H^p(\mathbb{R}^3 - \{0\}) = \begin{cases} \mathbb{R} & \text{if } p = 0, 2 \\ 0 & \text{if } p = 1, 3 \end{cases}$$

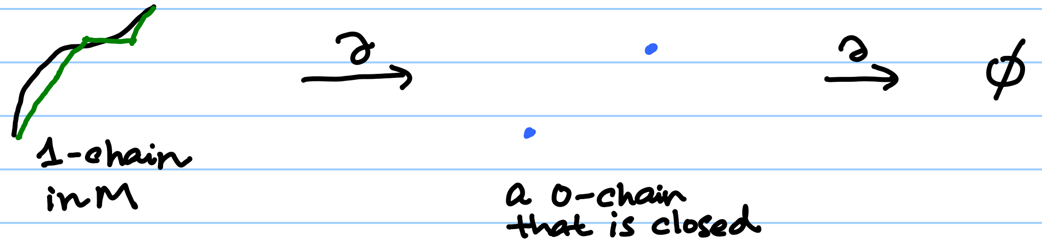
Compare

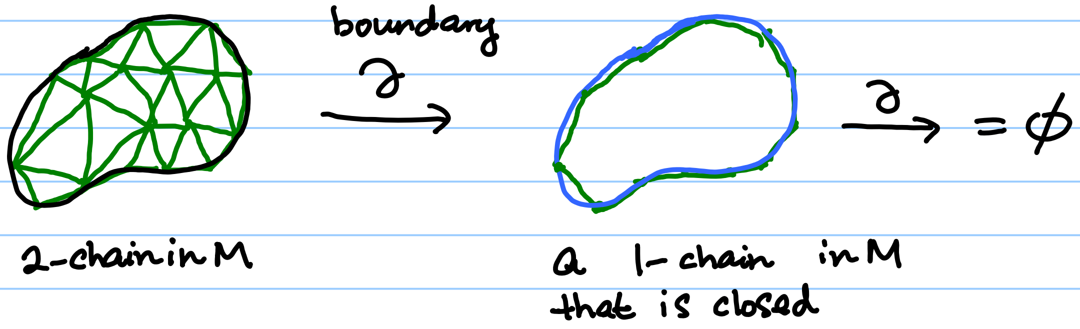
$$H^p(\mathbb{R}^3 - S^1) = \begin{cases} \mathbb{R} & \text{if } p = 0, 1, 2 \\ 0 & \text{if } p = 3 \end{cases}$$

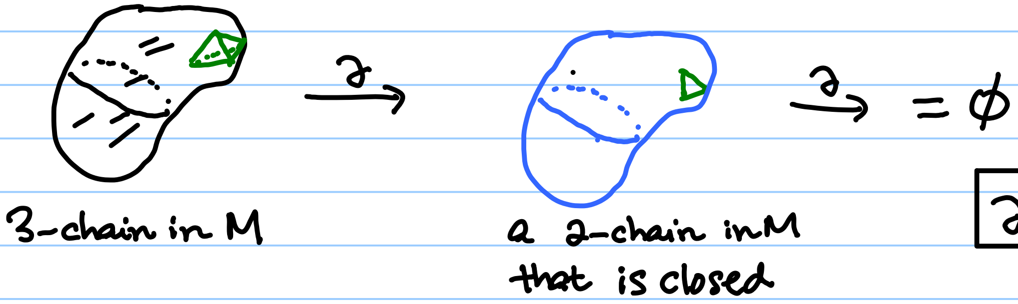
$$H^p(\mathbb{R}^n - S^k) = \begin{cases} \mathbb{R} & \text{if } p = 0, n-k-1, n-1 \\ 0 & \text{otherwise} \end{cases}$$

The first idea of homology:

"$\ell$-chain" — a $\ell$-dimensional object







$$\boxed{\partial \circ \partial = 0}$$

> **Central Question:** Is every $\ell$-chain in M the boundary of a $(\ell+1)$-chain?

$C_\ell = \{\ell\text{-chains in } M\}$, $C_\ell = \{0\}$ if $\ell < 0$ or $\ell > \dim M$

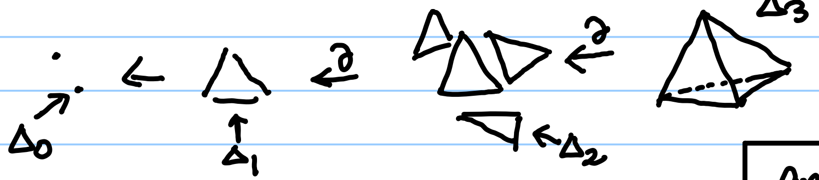
$$0 \leftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} \cdots C_{n-1} \xleftarrow{\partial} C_n \leftarrow 0$$

$$H_\ell(M) := \ker(\partial : C_\ell \to C_{\ell-1}) / \operatorname{Im}(\partial : C_{\ell+1} \to C_\ell)$$

Saying $H_\ell(M) = 0$ is the same as saying that

"every closed $\ell$-chain in M is the boundary of a $(\ell+1)$-chain in M"

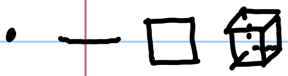
2nd idea: get help from simplices

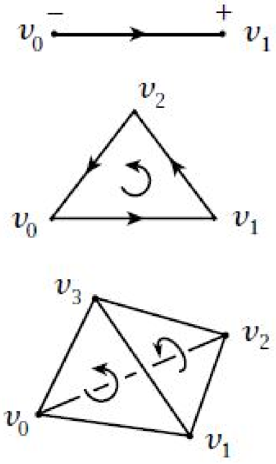


an $\ell$-chain
**def** a bunch of $\ell$-simplices

- easy to define in any dimension
- easy to define orientation ← why bother?
- easy to define boundary

can also use cubes





$\partial[v_0,v_1] = [v_1] - [v_0]$ ← 0-chain

$$\partial[v_0,v_1,v_2] = \overbrace{[v_1,v_2] - [v_0,v_2] + [v_0,v_1]}^{\text{1-chain}}$$

$$\partial[v_0,v_1,v_2,v_3] = \overbrace{[v_1,v_2,v_3] - [v_0,v_2,v_3]}^{\text{2-chain}}$$
$$+ [v_0,v_1,v_3] - [v_0,v_1,v_2]$$

Anything beyond these two ideas will take quite a lot of effort to spell out. A few things to pay attention:

- exactly how is $C_\ell$ defined?
  - Simplicial homology of X : think: X is "triangulated", the $\ell$-chains are only those in the triangulation

    Pros: - easier to understand
    - $\dim C_\ell < \infty$

    Cons: - Doesn't work for any topological space
    - showing that every manifold can be triangulated is a very difficult topic.

  - Singular homology : think of all the singular $\ell$-chains in X

$C_\ell(X)$ = the free abelian group generated by all continuous maps $\Delta_\ell \to X$

↑ not necessarily injective

dim $C_\ell(X) = \infty$

- how to define $\partial : C_\ell \to C_{\ell-1}$ as a linear map?
  - this is where the notion of orientation is relevant
- what is the effect of quotienting out $\operatorname{Im}(\partial : C_{\ell+1} \to C_\ell)$ from $\operatorname{Ker}(\partial : C_\ell \to C_{\ell-1})$?

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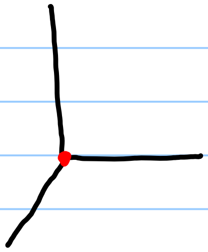
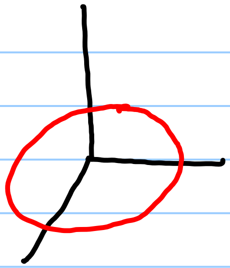
Assuming $H_p = H^p$, let's see what we should expect for $H^p(\mathbb{R}^3 - \{0\})$ and $H^p(\mathbb{R}^3 - S^1)$ just from the "first idea".





| | | | | | |
|---|---|---|---|---|---|
| 1 | $\underset{\text{a bunch of pts}}{\underset{\shortparallel}{\text{closed 0-chain}}} \overset{?}{\Rightarrow}$ boundary of a curve | Yes | 1 | $\underset{\text{a bunch of pts}}{\underset{\shortparallel}{\text{closed 0-chain}}} \overset{?}{\Rightarrow}$ boundary of a curve | Yes |
| 0 | closed curve $\overset{?}{\Rightarrow}$ boundary of a surface | Yes | 1 | closed curve $\overset{?}{\Rightarrow}$ boundary of a surface | NO |
| 1 | closed surface $\overset{?}{\Rightarrow}$ boundary of a solid | NO | 1 | closed surface $\overset{?}{\Rightarrow}$ boundary of a solid | NO |
| 0 | $\underset{\{0\}}{\underset{\shortparallel}{\left\{\begin{matrix}\text{closed}\\ \text{solid}\end{matrix}\right\}}} \underset{?}{/} \underset{\{0\}}{\underset{\shortparallel}{\left\{\begin{matrix}\text{boundary}\\ \text{of 4-chain}\end{matrix}\right\}}} = \{0\}$ | NO | 0 | | NO |

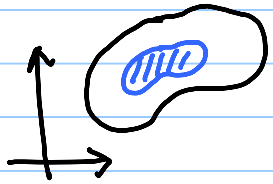
# Top dimensional homology space

what do you expect from $H_n(\mathcal{U})$ $(\cong H^n_{DR}(\mathcal{U}))$

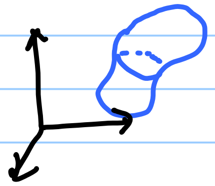
if $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n$ or $\mathcal{U}$ is an $n$-dim. manifold ?

$$H_n(\mathcal{U}) = \underset{\substack{\parallel \\ n\text{-chains with} \\ \text{an empty boundary}}}{\mathrm{Ker}(\partial : C_n \to C_{n-1})} \Big/ \underset{\substack{\parallel \\ 0}}{\mathrm{Im}(\partial : C_{n+1} \to C_n)}^{=0}$$

$$= \begin{cases} 0 & \text{if } \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \\ \mathbb{R} & \text{if } \mathcal{U} \text{ is a connected compact manifold} \end{cases}$$



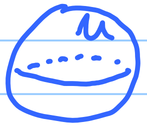
or



$\partial \mathcal{U} \neq \emptyset$

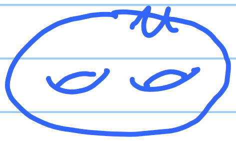
not possible if $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^n$, unless the chain is empty

but possible if $\mathcal{U}$ is a connected compact manifold (without boundary), just take the $n$-chain to be the whole space

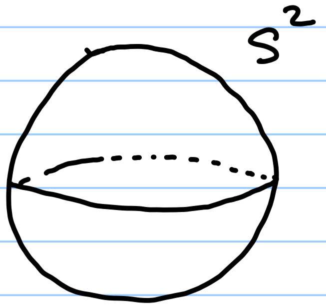


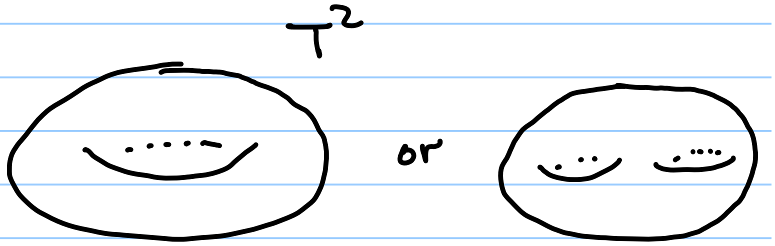
$\partial\mathcal{U}=\emptyset$

$\partial\mathcal{U}=\emptyset$



$\partial\mathcal{U}=\emptyset$





| | |
|---|---|
| closed 0-chain $\overset{?}{\Rightarrow}$ boundary of a curve<br>$\shortparallel$<br>a bunch of pts<br>Yes | closed 0-chain $\overset{?}{\Rightarrow}$ boundary of a curve<br>a bunch of pts<br>Yes |
| closed curve $\overset{?}{\Rightarrow}$ boundary of a surface<br>Yes | closed curve $\overset{?}{\Rightarrow}$ boundary of a surface<br>NO |
| closed surface $\overset{?}{\Rightarrow}$ boundary of a solid<br>NO | closed surface $\overset{?}{\Rightarrow}$ boundary of a solid<br>NO |

To distinguish the torus and the two-hole torus, we can use the fact that

$$\underset{\underset{2}{\shortparallel}}{\dim H^1(T^2)} \neq \underset{\underset{4}{\shortparallel}}{\dim H^1\left(\substack{\text{2-hole}\\ \text{torus}}\right)}$$

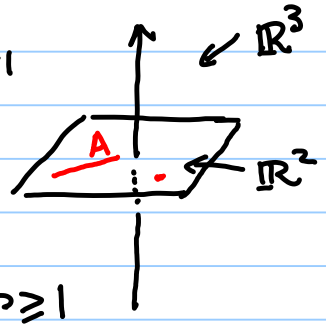
It turns out:

$$\begin{array}{c} V-E+F \\ \shortparallel \\ \text{Euler characteristics} \\ \text{of g-hole torus } T \\ \shortparallel \\ 2(1-g) \end{array} = \underset{\underset{1}{\shortparallel}}{\dim H^0(T)} - \dim H^1(T) + \underset{\underset{1}{\shortparallel}}{\dim H^2(T)}$$

$$\boxed{\dim H^1(T) = 2g}$$

[end of the very coarse exposition for homology]

Identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$



Proposition : For $A \overset{\text{closed}}{\subsetneq} \mathbb{R}^n$,

$$H^{p+1}(\mathbb{R}^{n+1}-A) \cong H^p(\mathbb{R}^n-A), \quad p \geq 1$$

$$H^1(\mathbb{R}^{n+1}-A) \cong H^0(\mathbb{R}^n-A)/\underbrace{\mathbb{R}\cdot 1}_{\text{"constant functions}}$$

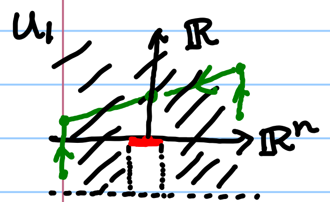
$$H^0(\mathbb{R}^{n+1}-A) \cong \mathbb{R}.$$

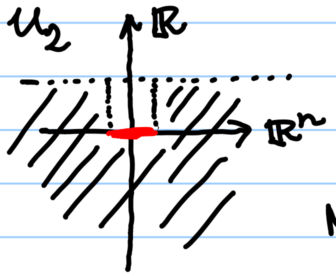
<u>Proof</u>

Use Mayer-Vietoris by choosing

$$\mathcal{U}_1 = [\mathbb{R}^n \times (0,\infty)] \cup [(\mathbb{R}^n-A)\times(-1,\infty)]$$

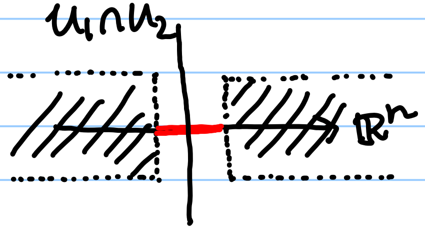
$$\mathcal{U}_2 = [\mathbb{R}^n \times (-\infty,0)] \cup [(\mathbb{R}^n-A)\times(-\infty,1)]$$





$$\mathcal{U}_1 \cup \mathcal{U}_2 = \mathbb{R}^{n+1} - A$$

$$\mathcal{U}_1 \cap \mathcal{U}_2 = (\mathbb{R}^n - A) \times (-1,1)$$



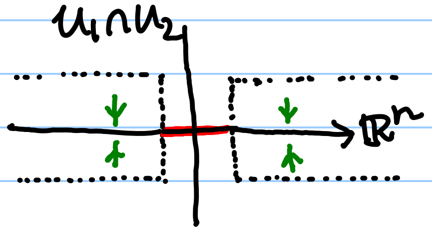
Note : $\mathcal{U}_1, \mathcal{U}_2$ may not (or never?) star-shaped, but they are contractible. (ex.)

Hint : Use the following "homotopy" in $\mathcal{U}_1$ :

$$\begin{tikzcd} (x_1,\dots,x_{n+1}+t) \arrow[r] & (0,\dots,0,1) \\ \mathcal{U}_1 \ni (x_1,\dots x_{n+1}) \arrow[u] & \end{tikzcd}$$

$\mathcal{U}_1 \cap \mathcal{U}_2$ and $\mathbb{R}^n - A$ are homotopy equivalent, so they have the same cohomology,

$$H^p(\mathcal{U}_1 \cap \mathcal{U}_2) \cong H^p(\mathbb{R}^n - A).$$



Check:

$pr : U_1 \cap U_2 \to \mathbb{R}^n - A$

$(x_1, \dots, x_n, x_{n+1}) \mapsto (x_1, \dots, x_n, 0)$

is a homotopy equivalence with homotopy inverse

$i : \mathbb{R}^n - A \to U_1 \cap U_2$

(inclusion)

$\Leftarrow$

$pr^* : H^p(\mathbb{R}^n - A) \to H^p(U_1 \cap U_2)$ is an isomorphism

Mayer-Vietoris: we have the following exact sequence

$$H^p(U) \xrightarrow{I^*} H^p(U_1) \oplus H^p(U_2) \xrightarrow{J^*} H^p(U_1 \cap U_2) \xrightarrow{\partial^*} H^{p+1}(U) \xrightarrow{I^*} H^{p+1}(U_1) \oplus H^{p+1}(U_2)$$

$p \geq 1$:

$$\left.\begin{array}{c} 0 \to \underset{\substack{\cong \\ H^p(\mathbb{R}^n - A)}}{H^p(U_1 \cap U_2)} \xrightarrow[\substack{\uparrow \\ \text{isomorphism}}]{\partial^*} \underset{?}{H^{p+1}(U)} \to 0 \end{array}\right\}$$ this proves (i)

connected in $\mathbb{R}^{n+1}$, since $A \neq \mathbb{R}^n \times \{0\}$

$p = 0$:

$$\underbrace{H^0(U_1) \oplus H^0(U_2)}_{\cong \mathbb{R}^2} \xrightarrow{J^*} \underbrace{H^0(U_1 \cap U_2)}_{\cong H^0(\mathbb{R}^n - A) \cong \mathbb{R}^C} \xrightarrow{\partial^*} \underset{?}{H^1(U)} \to 0$$

$C = \#$ of connected components of $\mathbb{R}^n - A$

If $C < \infty$, $\quad C = \underset{\substack{\| \\ \dim H^1(U)}}{\text{rank } \partial^*} + \underset{\substack{\| \\ \text{rank } J^*}}{\text{nullity } \partial^*}$

exactness

what we want

exactness

Fortunately, we know what $J^*$ is :

recall $\quad J(\omega_1, \omega_2) = j_1^*\omega_1 - j_2^*\omega_2$

$$\begin{tikzcd}
& \mathcal{U} & \\
\mathcal{U}_1 \arrow[ur, "i_1"] & & \mathcal{U}_2 \arrow[ul, "i_2"'] \\
& \mathcal{U}_1\cap\mathcal{U}_2 \arrow[ul, "j_1"] \arrow[ur, "j_2"'] &
\end{tikzcd}$$

$$J^*([\omega_1],[\omega_2]) = [j_1^*\omega_1] - [j_2^*\omega_2]$$

$p=0$, $[\omega_1], [\omega_2]$ just represent 2 constants $a_1, a_2$ and

$$[j_1^*\omega_1] - [j_2^*\omega_2]$$
$$= [a_1 - a_2]$$
$$\shortparallel$$

the constant function on $\mathcal{U}_1\cap\mathcal{U}_2$ with the value $a_1 - a_2$

i.e. $\operatorname{rank} J^* = 1$

In other words, $\dim H^1(\mathcal{U}) = C - 1$ if $C < \infty$

A better way to say it ( which takes care of $C=\infty$) would be

$$H^1(\mathcal{U}) \cong H^0(\mathcal{U}_1\cap\mathcal{U}_2) / \{\text{constant functions on } \mathcal{U}_1\cap\mathcal{U}_2\}$$

$$\cong H^0(\mathbb{R}^n - A) / \{\text{constant functions on } \mathbb{R}^n - A\}.$$

we have proved (ii)

(iii) follows from the assumption that $A \neq \mathbb{R}^n \times \{0\}$ So $\mathbb{R}^{n+1} - A$ is connected.

$\square$

<u>Ex</u>: This proof relies on the fact that $\mathcal{U}_1, \mathcal{U}_2$ and $\mathbb{R}^{n+1} - A$ are connected / path connected, exactly why is it true?

Theorem: For $n \geq 2$,

$$H^p(\mathbb{R}^n - \{0\}) \cong \begin{cases} \mathbb{R} & \text{if } p = 0, n-1 \\ 0 & \text{otherwise} \end{cases} .$$

<u>Proof</u> The case $n=2$ was proved in the previous chapter (remember how?)

Assume the statement is true for a given dimension $n \geq 2$. Then

$$H^p(\mathbb{R}^{n+1} - \{0\}) \cong \begin{cases} H^{p-1}(\mathbb{R}^n - \{0\}) & p \geq 2 \\ H^0(\mathbb{R}^n - \{0\}) / \{\text{consts}\} & p = 1 \\ \mathbb{R} & p = 0 \end{cases}$$

$$\cong \begin{cases} \begin{cases} \mathbb{R} \\ 0 \end{cases} & \begin{matrix} p = n \\ p \geq 2, p \neq n \end{matrix} \\ 0 & p = 1 \\ \mathbb{R} & p = 0 \end{cases}$$

$$= \begin{cases} \mathbb{R} & p = 0, n \\ 0 & \text{otherwise} \end{cases}$$

The statement is proved by induction on $n$. ☐

For $n=1$, of course we have

$$H^p(\mathbb{R}^1 - \{0\}) \cong \begin{cases} \mathbb{R} \oplus \mathbb{R} & \text{if } p = 0 \\ 0 & \text{if } p \neq 0 . \end{cases}$$

As you see, De Rham cohomology is good enough to conclude that

(*) $\mathbb{R}^n - \{0\}$ is not homeomorphic to $\mathbb{R}^m - \{0\}$, $n \neq m$

as their cohomology are all different.

From this, we also have

**Thm** $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$, $n \neq m$.

**Proof**: Any possible homeomorphism $F: \mathbb{R}^n \to \mathbb{R}^m$ can be shifted to a homeomorphism s.t. $F(0) = 0$ which also gives a homeomorphism between $\mathbb{R}^n - \{0\}$ and $\mathbb{R}^m - \{0\}$. This contradicts $(*)$. $\square$

Recall: it is much easier to prove the same statement with 'homeomorphic' replaced by 'diffeomorphic', because

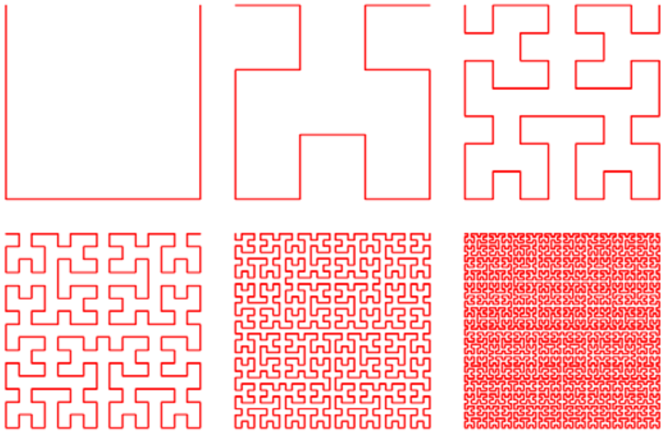
$F: \mathcal{U} \to V$ diffeomorphism

$\Rightarrow DF: \mathbb{R}^n \to \mathbb{R}^m$ is a linear isomorphism

$\Rightarrow \quad n = m$

On the other hand,

- $\exists$ continuous map $F: [0,1] \to [0,1]^2$ that is surjective (but not 1-1)



- $\exists$ bijection (but not continuous) $F: [0,1] \to [0,1]^2$.

(Given these bad boys, it was not clear why $\mathbb{R}^1$ cannot be homeomorphic to $\mathbb{R}^2$.)

$\pi_1$ $H^p$

$\pi_p$ $H_p$

## Final Remark: (homotopy group vs (co)homology group)

The first homotopy group (a.k.a. the fundamental group) can be used to prove

$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, $n > 2$.

The higher homotopy groups are needed for distinguishing $\mathbb{R}^m$ and $\mathbb{R}^n$ for arbitrary $m \neq n$.

They are never introduced in an introductory topology course (e.g. Munkres: Topology, a first course) because they are very hard to compute. Homology or cohomology happens to be easier and good enough for the problem at hand.

See Alan Hatcher's book on algebraic topology

Chapter 1 : $\pi_1$

Chapter 2 : $H_p$

Chapter 3 $H^p$ (no deRham)

Chapter 4 : $\pi_p$

$\cong$ ← De Rham's theorem (John Lee's manifold book, assumes you know $H_p / H^p$ )

$H^p_{D.R.}$ (this course, no de Rham theorem )
no $H_p / H^p$