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Homework #1
Math 305

Problem #1 - 3 equations, 3 unknowns

$a = X + Y + Z$

$b = X - Y \Rightarrow Y = X - b$

$c = Y - Z \Rightarrow Y = c + Z \Rightarrow Z = X - b - c$

$a = X + X - b + X - b - c \Rightarrow a = 3X - 2b - c$

$$\boxed{\begin{aligned} X &= \tfrac{1}{3}a + \tfrac{2}{3}b + \tfrac{1}{3}c \\ Y &= \tfrac{1}{3}a - \tfrac{1}{3}b + \tfrac{1}{3}c \\ Z &= \tfrac{1}{3}a - \tfrac{1}{3}b - \tfrac{2}{3}c \end{aligned}}$$ Solution

Problem #2

$x + a = y$

$x + b = z \Rightarrow$ $x + a = x + b - c \Rightarrow a = b - c$

$y + c = z$

If $a = b - c$, then we can find $x, y, \& z$. You can uniquely determine $x, y,$ and z from just $a, b,$ and c.
If $a \neq b - c$, then you could not find $x, y,$ and z to satisfy the above system.
You require values for at least 2 of $a, b,$ and c in order for us to uniquely determine X, Y, Z.

Problem #3

Part I: Supposing we are restricted to only rectangle of a fixxed Perimeter, then:

$$\underset{A}{\text{Max}}\ A = h \cdot L \quad \text{S.t} \quad 2h + 2L = P_0, \ h \geq 0, \ L \geq 0$$

Here, h is the height of our rectangle, L is the Side length, and P_0 is our fixxed Perimeter.

Since we have a nice constraint and the Problem is Simple, we can make the following Sub.:

$$2h + 2L = P_0 \Rightarrow h = \tfrac{1}{2}P_0 - L$$

Thus, we Get:

$$A = (\tfrac{1}{2}P_0 - L) \cdot L = \tfrac{1}{2}LP_0 - L^2$$

Therefore:

$$\underset{A}{\text{Max}}\ A = \tfrac{1}{2}P_0 - L^2$$

Part 2: To find the triangle with the largest Area with a fixxed Perimeter of 2:

$$\underset{A}{\text{Max}}\ A = \sqrt{S(S-a)(S-b)(S-c)} \quad \text{S.t.} \quad a + b + c = 2, \ a, b, c \geq 0$$

Here, $S = \frac{1}{2}(a + b + c)$, and $a, b,$ and c are the three Sides of the triangle.

Continued →

Problem #3 - Continued

Part 2 Continued: Attempting our naive strategy from Part I;

$a+b+c=2 \Rightarrow a=2-b-c$

$s=\frac{1}{2}(a+b+c)=\frac{1}{2}(2-b+b-c+c)=1$

Thus,

$A=\sqrt{(1-a)(1-b)(1-c)}=\sqrt{(-1+b+c)(1-b)(1-c)}$.

However, our naive approach has come to an issue. We have more than one unknown, now what! Since we are stumped, consider some inequalities on b&c that must hold. Since, $a=2-b-c$ and:

$a+b>c \Rightarrow 2-c>c \Rightarrow c<1$

$b+c>a \Rightarrow b+c>2-b-c \Rightarrow b+c>1$

$a+c>b \Rightarrow 2-b>b \Rightarrow b<1$

Thus,

$c<1, b<1, b+c>1$

Clearly these are similar to a linear programs' constraints in the form $A_1X_1+A_2X_2\ldots>b$, as seen by $A_1=1, X_1=b, A_2=1, X_2=c, b=1$, then $A_1X_1+A_2X_2>b \Rightarrow b+c>1$, same as in a linear program.

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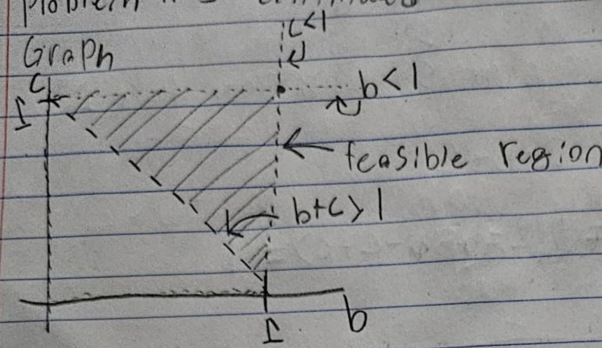
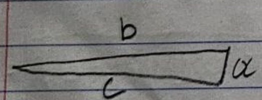
Problem #3 - Continued

Part 2 Continued: Graph



Part 3: I suggest, without rigorous proof, the the maximum for this function does not occur at a vertex. However, I believe it obvious that a triangle with any side length near zero is not a maximizer.

For example, consider the following triangle:



It appears obvious we should make α bigger, by some amount.

Question #4

Part i) $\min_{a_0, a_1, a_2} \sum_{i=1}^{m} (a_0 + a_1 x_i + a_2 x_i^2 - y_i)^2$

$\min_{A} \|Ax-b\|^2$ such that;

$A = [a_0 \quad a_1 \quad a_2]$ $\quad b = [y_1, \dots, y_m]$

If we know the least squared solution follows the form $X = (A^T A)^{-1} (A^T b)$, thus:

$$X = (A^T A)(A^T b) = \begin{bmatrix} a_0^2 & a_0 a_1 & a_0 a_2 \\ a_0 a_1 & a_1^2 & a_1 a_2 \\ a_0 a_2 & a a_2 & a_2^2 \end{bmatrix}^{-1} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} [y_1, \dots, y_m]$$

Thus, it follows that this least squars problem comes down to solving the above 3x3 system of equation.

Continued →

Question #4 - Continued

Part ii) $\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^{m} |\alpha_0+\alpha_1 x_i+\alpha_2 x_i^2 - b_i|$, to fit lecture notes.

We can rewrite this problem in the form:

$$\min_{\alpha_0,\alpha_1,\alpha_2} y_1+\dots+y_m \quad \text{s.t.} \; -y_i \le \alpha_0+\alpha_1 x_i+\alpha_2 x_i^2 - b_i \le y_i$$
$$y_1,\dots,y_m \ge 0$$

We can then transform this non-standard form into standard form where $\alpha_0=\alpha_0^+-\alpha_0^-$, $\alpha_1=\alpha_1^+-\alpha_1^- \dots$ where $\alpha_0^+,\alpha_0^-,\alpha_i^+,\alpha_i^- \ge 0$.

$$\min_{\alpha_0,\alpha_1,\alpha_2} y_1+\dots+y_m \quad \text{s.t.} \; \alpha_0^+-\alpha_0^-+\alpha_1^+x_i-\alpha_1^-x_i+\alpha_2^+x_i^2-\alpha_2^-x_i^2+y_i \ge b_i$$
$$-\alpha_0^++\alpha_0^--\alpha_1^+x_i+\alpha_1^-x_i-\alpha_2^+x_i^2+\alpha_2^-x_i^2-y_i \ge -b_i$$

Require both constraints for $i=1,\dots,m$

and $\alpha_0^+,\alpha_0^-,\alpha_1^+,\alpha_1^-,\alpha_2^+,\alpha_2^-,y_1,\dots,y_m \ge 0$

Then, to get this into matrix form:

$\min_{\vec{\alpha}} \vec{p}^T\vec{\alpha}$ s.t. $\vec{A}\vec{x} \ge \vec{b}$

We have

$$\vec{\alpha}=\begin{bmatrix}\alpha_0^+\\ \alpha_0^-\\ \alpha_1^+\\ \alpha_1^-\\ \alpha_2^+\\ \alpha_2^-\\ y_1\\ \vdots\\ y_m\end{bmatrix} \quad p=\begin{bmatrix}0\\0\\0\\0\\0\\0\\1\\ \vdots\\ 1\end{bmatrix} \quad \vec{A}=\begin{bmatrix}1 & -1 & x_1 & -x_1 & x_1^2 & -x_1^2 & 1 & \cdots & 0\\ \vdots & & \vdots & & \vdots & & & \ddots & \vdots\\ 1 & -1 & x_m & -x_m & x_m^2 & -x_m^2 & 0 & \cdots & 1\\ -1 & 1 & -x_1 & x_1 & -x_m^2 & x_m^2 & -1 & \cdots & 0\\ \vdots & & \vdots & & \vdots & & & \ddots & \vdots\\ -1 & 1 & -x_m & x_m & -x_m^2 & x_m^2 & 0 & \cdots & -1\end{bmatrix}$$

$(m+6)\times 1 \qquad (m+6)\times 1$

$$b=\begin{bmatrix}b_1\\ \vdots\\ b_m\\ -b_1\\ \vdots\\ -b_m\end{bmatrix}$$