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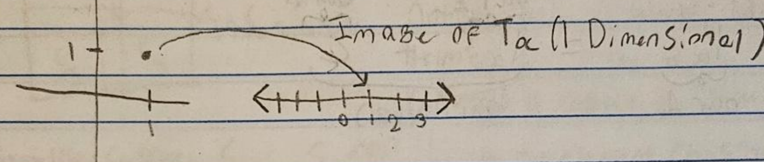
Math305
Homework#2

Question #1

a) rank is the Dimension of a linear maps Image, So the rank of $T_a : \mathbb{R}^n \to \mathbb{R}^1$, $rank(T_a) = 1$

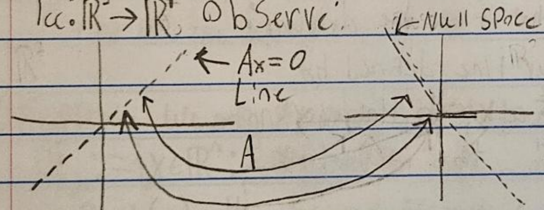
Nullity is the Dimension of the null space of our linear map T_a, So $nullity(a) = n - rank(T_a) = n-1$

For example Say $T_a : \mathbb{R}^2 \to \mathbb{R}^1$



Since we know $rank(T_a) + Nullity(T_a) = n$, and we know $n=2$ in our example, then $nullity(T_a) = 1$, for our made up example.

b) Noting that null space $= \{x \in \mathbb{R}^n : Ax = 0\}$, Supposing $T_a : \mathbb{R}^2 \to \mathbb{R}^1$ observe:



Imagine compressing the whole line down into a single point, the origin ($nullity(A) = 1$, and $rank(A) = 1$)

c) Assume that $P_{a,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i = b\}$ $\forall b \in \mathbb{R}$, Suppose $c \in \mathbb{R}^n$, for any $P_{a,b}$, in the language of linear algebra we have $P_{a,b} = \{x \in \mathbb{R}^n : a^T x = b\}$, thus we can say that for any b, $P : \mathbb{R}^n \to \mathbb{R}^1$. Furthermore, we can observe that $P_{a,b}$, where $b=0$, is equivalent to $P_{a,0} = \{x \in \mathbb{R}^n : a^T x = 0\}$ which is precisely the definition of the null space. Therefore, since $c \in \mathbb{R}^n$, by the following property of a vector in the null space ($Ax = A(x+z) = b$, for any vector in the null space) the we can say $P_{a,b} = P_{a,0} + c$, for any $c \in \mathbb{R}^n$, since $P_{a,0} \in$ nullspace.

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Question #1 - Continued

c) Since my explanation was verbose let me restate the reasoning:

For any linear map, call it A, any vector in null(A), call it z, has the following property $Ax = b \Rightarrow A(x+z) = b$.

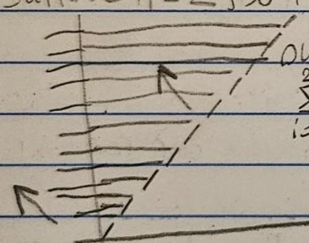
Using that reasoning:

$$\underbrace{P_{\alpha,b}}_{\text{our } A_x} = \underbrace{\overbrace{P_{\alpha,0}}^{\text{our "z"}} + \overbrace{C}^{\text{our "x"}}}_{\text{our } A(x+z)}$$

Note then that since $P_{\alpha,0} \in$ null space, if $P_{\alpha,b}$ such that $b \neq 0$, then C must also $C \neq 0$, our desired result.

d) Supposing $H_{\alpha,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} \alpha_i x_i \geq b\}$, observe the following

Suppose $n=2$, so that it can be drawn:



our line, defined by $\sum_{i=1}^{2} \alpha_i x_i = b$, However note the inequality.

Note that any region above the line, our at it, defines $H_{\alpha,b}$

Expanding to higher dimensions, notice what we are doing more generally is seperating $\mathbb{R}^n$ into two parts. One part at or above the hyper plane defined by $H_{\alpha,b}$, and the part strictly below the hyper plane.

e) Suppose $H_{\alpha,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \geq b\}$. For any two points chosen in $H_{\alpha,b}$, say $\vec{L}$ and $\vec{K}$, if $H_{\alpha,b}$ is convex then $\theta \vec{L} + (1-\theta)\vec{K} \in H_{\alpha,b}$ $\forall \theta \in [0,1]$. Then,
$\theta \sum_{i=1}^{n} \alpha_i L_i \geq \theta b$ and $(1-\theta)\sum_{i=1}^{n} \alpha_i L_i \geq (1-\theta) b$, then we can observe

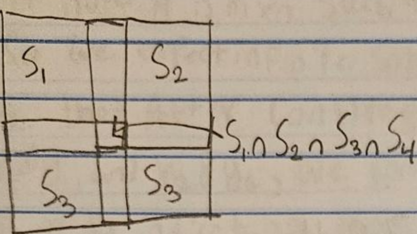
$$\theta \sum_{i=1}^{n} \alpha_i L_i + (1-\theta) \sum_{i=1}^{n} \alpha_i K_i = \sum_{i=1}^{n} \alpha_i (\theta L_i + (1-\theta) K_i) \geq \theta b + (1-\theta) b = b$$

Thus, we have the desired result. ■

Question #1 - Continued

f) Suppose $S_1, S_2, \ldots, S_m \in \mathbb{R}^n$ are Convex. First consider the Geometric reasoning behind $S_1 \cap S_2 \cap \ldots \cap S_m$ being Convex.

Consider 4 boxes (A convex Shape)



Notice how no matter We 'trim' a set, if the portion removed is convex, the result is convex

Formally, Suppose $S_1, \ldots, S_m \in \mathbb{R}^n$. Using an element chasing sort of approach, Consider $\vec{x}, \vec{y} \in S_1$, Since S_1 is convex $\theta\vec{x} + (1-\theta)\vec{y} = \vec{z} \in S_1 \ \forall \theta [0,1]$ by definition. Thus, if $\vec{z} \in S_2$, and S_2 is Convex then $\vec{z} \in S_1$ and $\vec{z} \in S_2$, Which implies $\vec{z} \in S_1 \cap S_2$. We can extend this reasoning to any $S_1 \cap S_2 \cap \ldots \cap S_m, \forall m \in \mathbb{N}$ Giving us the desired result. ∎

g) Suppose We are given a LP With constraints of the form $H^j_{a,b} := \{x \in \mathbb{R}^n : \sum_{i=1}^{n} a_i x_i \geq b\}$ For $j = \{1, \ldots, m\}$, by our reasoning in Part (e), $H^j_{a,b}$ are all Convex sets, and by our reasoning in Part (f), if All $H^j_{a,b}$ Sets are convex, then $H^1_{a,b} \cap H^2_{a,b} \cap \ldots \cap H^m_{a,b}$ is a Convex set. Since the Feasible region of our LP is the set of all Points Satisfy All Constraints, We can call our Feasible region $H^1_{a,b} \cap \ldots \cap H^m_{a,b}$. Since that Set is Convex, the feasible region of any LP With Constraints of the form of $H_{a,b}$, must be Convex. ∎

Question #2 (Exercise 2-2-3)

$$A=\begin{bmatrix}1 & 2 & 3 & 4\\ 3 & 1 & 3 & 0\\ 1 & 3 & -3 & -8\end{bmatrix}$$

$$T=\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4\\ \hline y_1= & 1 & 2 & 3 & 4\\ y_2= & 3 & 1 & 3 & 0\\ y_3= & 1 & 3 & -3 & -8\end{array}$$

Jordan exchange as many y_i to the top, the number of y_i that can be exchange is the number of linearly independent rows.

We can Jordan Exchange x_1 & y_1, x_2 & y_2, x_3 & y_3, thus A has 3 linearly independent rows.

Question #3 (Exercise 2-4-2)

We can find the solution $x=(1,-1,1)$ by first setting up A Tableau

$$T=\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & 1\\ \hline y_1 & 1 & 1 & 1 & -1\\ y_2 & 1 & -1 & -1 & -1\\ y_3 & 1 & -1 & 1 & -3\end{array} \xrightarrow{\substack{\text{ljx}(T,1,1)\\ \text{ljx}(T,2,2)\\ \text{ljx}(T,3,3)}} \begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & 1\\ \hline x_1 & 1/2 & 1/2 & 0 & 1\\ x_2 & 1/2 & 0 & -1/2 & -1\\ x_3 & 0 & -1/2 & 1/2 & 1\end{array} \begin{array}{l} \\ \leftrightarrow x_1=1\\ \leftrightarrow x_2=-1\\ \leftrightarrow x_3=.1\end{array}$$

We then Jordan exchange All y_i to the top and All x_i to the side. We can then read off the 4^{th} Column as the associated x_i Solution. Therefore, the solution $x=(1,-1,1)$, what was given in the beginning of the exercise.

Question #4 (Exercise 2-4-6)

1.) $Ax=a$ $A=\begin{bmatrix}2 & -1 & 1 & 1\\ -1 & 2 & -1 & -2\\ 4 & 1 & 1 & -1\end{bmatrix}$ $a=\begin{bmatrix}1\\1\\5\end{bmatrix}$

First note A is $m\times n$ such that $m=3, n=4$, so $m<n$,
So we expect infinite solutions.

Note that after constructing tableau and exchanging x_1 & y_1, and x_2 & y_2, we cannot exchange y_3 with either x_3 or x_4, indicating that y_3 is a linearly dependent row.

More particularly, noting the values in the tableau
$y_3=3y_1+2y_2$

Moreover, there are infinite solutions as expected, in the form
$x_1=-1/3x_3+1$
$x_2=1/3x_3+x_4+1$

Therefore,

$$x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}1\\1\\0\\0\end{bmatrix}+\begin{bmatrix}-1/3\\1/3\\1\\0\end{bmatrix}\lambda_1+\begin{bmatrix}0\\1\\0\\1\end{bmatrix}\lambda_2$$

Such that λ_1, λ_2 are arbitrary

2.) $Bx=b$ $B=\begin{bmatrix}1 & -1 & 1 & 2\\ 1 & 1 & 0 & -1\\ 1 & -3 & 2 & 5\end{bmatrix}$ $b=\begin{bmatrix}2\\1\\1\end{bmatrix}$

Again note B is 3×4, so we expect infinitely many solutions.

Note after constructing tableau and exchanging x_1 & y_1 and x_2 & y_2, we find we can't exchange y_3 with x_3 or x_4. Thus, y_3 is linearly dependent on y_1 and y_2; $y_3=2y_1-y_2$

Since we have a constant in the last column, we can see that we have no solution since we can't set $y_1=y_2=y_3=0$, because we have $y_3=2y_1-y_2+2$.

Question #4 - Continued (Exercise 2-4-6)

3.) $Cx = c$ $\quad C = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}$ $\quad c = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix}$

Note C is 4×3, So we expect no Solution, unless One or more rows of C are linearly dependent.

After Constructing the tableau and exchanging x_1 & y_1, x_2 & y_2, and x_3 & y_3, we find

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So we Check to insure a Solution

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1+1 \\ 2-1+1 \\ -1+1+2 \\ 1-1-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ -1 \end{bmatrix}$$

Thus we do have a Solution X.

Question #5 (Exercise 2-4-8)

1. In order for $\underbrace{A}_{m\times n}x=b$ to have a single Solution, regardless of b, We need two thing. First we need $\text{rank}(A)=m$ So that a Solution Exists for all b. Second, we need $\text{rank}(A)=n$ to insure the uniqueness of our Solution. Therefore, We need $\text{rank}(A)=m=n$ to insure one Solution regardless of b.

Ex:

$$A=\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 2 & 2 & 0\end{bmatrix}$$ A 3x3 matrix with $\text{rank}(A)=3$

2. In order for $\underbrace{A}_{m\times n}x=b$ to have Zero or infinitely many Solutions, depending on b, we want two things. First, We want atleast one "free" variable, One Whose Value does not effect the Solution. Second, we Want our not free variables to be linerly indepent. In the language of linear Algebra, we want $\text{rank}(A)<m$ and $\text{rank}(A)<n$.

Ex:

$$A=\begin{bmatrix}1 & 4 & 7\\ 3 & 12 & 21\end{bmatrix}$$

3. In order to insure $\underbrace{A}_{m\times n}x=b$ has either Zero or one Solution, depending on b, we need two things. We need to insure Matrix A has $\text{rank}(A)=n$ and $n<m$. $\text{rank}(A)=n$ is to insure uniqueness, and $n<m$ is to insure Zero Solutions at some Values of b.

Ex:

$$A=\begin{bmatrix}4 & 9\\ 1 & 3\\ 3 & 6\end{bmatrix}$$

Question #5 - Continued (Exercise 2-4-8)

4.) To gurantee infinitely many solutions to $Ax=b$, independent of b, we need two things. First, $rank(A)=m$ to insure existance of a solution. Second, $n > m$ to insure infinitely many solutions

Ex.
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & -1 & 2 \end{bmatrix}$$