Math 305 Introduction to Optimization **Midterm Examination**

Your name:

1. (10 points) Given a LP in the **standard form** with $n = 3$ and $m = 0$. (So A and b are empty. But note that this does not mean there are no constraints on the variables.)

   What is the minimum value of the LP if (i) $p = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$? (ii) if $p = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$? Explain your answers.

2. (5 points) What is a **vertex** of a feasible region $S := \{x \in \mathbb{R}^n : Ax \geqslant b, x \geqslant 0\}$? Give a rigorous mathematical definition.

   (5 points) What is the fundamental theorem of linear program?

   (5 points) Given a LP in the standard form with $m = 30$ and $n = 20$. A student tries to solve it by first computing all the vertices, followed by checking which vertex gives the lowest objective value.

   Assume that he has access to the state of the art NVDIA GPU cluster which is powerful enough to solve 100 million 20×20 linear systems in 1 second. Can he solve the linear program is this naive way in one week? What about $m = 35$ and $n = 20$?

3. (15 points) Given the following tableau for a LP in the standard form:

```
              x3          x2          1
       ----------------------------------
x1 = |      1.0000     -1.0000      1.0000
x4 = |     -1.0000      2.0000      3.0000
x5 = |     -1.0000      1.0000      2.0000
x6 = |     -1.0000      0.0000      4.0000
x7 = |      0.0000     -1.0000      3.0000
x8 = |      1.0000     -2.0000      3.0000
       ----------------------------------
z  = |     -1.0000     2.0000       1.0000
```

What is the maximum amount you can increase one of the (original or slack) variables in one simplex step (starting from the vertex represented by the tableau above)? What is the maximum amount you can reduce the objective value in the step? Explain your answer.

4. (15 points) The following tableau is resulted in solving a LP:

```
             x1          x5          x3          x6          1
        -------------------------------------------------------
 x2 = |    -0.4000      0.6000     -1.4000     -0.2000      2.4000
 x4 = |    -0.2000     -0.2000     -0.2000      0.4000      0.2000
        -------------------------------------------------------
 z  = |     0.0000      1.0000      1.0000      1.0000      8.0000
```

What is the minimum value of the LP?

List two vertex solutions of the LP.

Give 10 different non-vertex solutions of the LP.

5. (15 points) (i) The following tableau is resulted in solving a LP:

	x1	x5	x3	x4	1
x2 =	1.0000	-1.0000	1.0000	-0.5000	2.0000
x6 =	4.7500	-2.2500	2.7500	3.1250	8.5000
x7 =	0.0000	-1.0000	1.0000	-2.5000	5.0000
z =	1.0000	2.0000	0.0000	6.0000	-4.0000

What is the minimum value of the LP?

Determine the solution set (i.e. the set of minimizer(s)) of this LP.

(ii) The following tableau is resulted in solving a LP:

	x1	x5	x3	x4	1
x2 =	1.0000	-1.0000	1.0000	-0.5000	2.0000
x6 =	4.7500	-2.2500	2.7500	3.1250	8.5000
x7 =	0.0000	-1.0000	1.0000	-2.5000	5.0000
z =	1.0000	2.0000	-1.0000	6.0000	-4.0000

What is the minimum value of the LP?

Determine the solution set of this LP.

6. (15pts) Consider an LP in the standard form with $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ -1 & 0 \\ -1 & -1 \\ 0 & -1 \\ 1 & -1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ -2 \\ -3 \\ -5 \\ -3 \\ -2 \end{bmatrix}$, and $p = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

(a) Solve the LP **graphically**.

(b) If you use the simplex method to solve the LP, what will be the non-basic variables upon termination?

(c) Write down the Phase I LP of the feasible region. What is its minimum value? Also, list two different minimizers of this Phase I LP.

7. (15 pts) Consider the feasible region $S = \{[x_1, x_2]^T : x_1 + 2x_2 \geqslant 2, x_1 + 2x_2 \leqslant 1, x_1, x_2 \geqslant 0\}$. Clearly, S is an empty set.

Write down the Phase I LP of S.

Determine the minimum value of this Phase I LP.