GAME THEORY*
Lecture Notes

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*These notes mainly follow the *Game Theory* textbook by Steven Tadelis.

Contents

Part I.

STATIC GAMES OF COMPLETE INFORMATION

1. Preliminaries

A static game is similar to the very simple decision problems in which a player makes a once-and-for-all decision, after which outcomes are realized. In a static game, a set of players independently choose once-and-for-all actions, which in turn cause the realization of an outcome. Thus a static game can be thought of as having two distinct steps:

Step 1: Each player simultaneously and independently chooses an action.

By simultaneously and independently, we mean that players must take their actions without observing what actions their counterparts take and without interacting with other players to coordinate their actions.

Step 2: Conditional on the players' choices of actions, payoffs are distributed to each player.

That is, once the players have all made their choices, these choices will result in a particular outcome, or probabilistic distribution over outcomes. The players have preferences over the outcomes of the game given by some payoff function over outcomes.

Steps 1 and 2 settle what we mean by *static*. What do we mean by *complete information*? The loose meaning is that all players understand the environment they are in–that is, the game they are playing–in every way.

Games of Complete Information. A game of complete information requires that the following four components be common knowledge among all the players of the game:

1. all the possible actions of all the players,
2. all the possible outcomes,
3. how each combination of actions of all players affects which outcome will materialize, and

4. the preferences of each and every player over outcomes.

Definition 1 *An event E is common knowledge if (1) everyone knows E, (2) everyone knows that everyone knows E, and so on ad infinitum.*

A event E is *mutual knowledge* among a set of agents if each agent knows E. Mutual knowledge by itself implies nothing about what, if any, knowledge anyone attributes to anyone else. Suppose each student arrives for a class meeting knowing that the instructor will be late. That the instructor will be late is mutual knowledge, but each student might think only she knows the instructor will be late. However, if one of the students says openly "Peter told me he will be late again," then each student knows that each student knows that the instructor will be late, each student knows that each student knows that each student knows that the instructor will be late, and so on, ad infinitum. The announcement made the mutually known fact common knowledge among the students.

The Barbecue Problem

N individuals enjoy a picnic supper together which includes barbecued spareribs. At the end of the meal, $k \geq 1$ of these diners have barbecue sauce on their faces. Since no one can see her own face, none of the messy diners knows whether he or she is messy. Then the cook who served the spareribs returns with a carton of ice cream. Amused by what he sees, the cook rings the dinner bell and makes the following announcement: "At least one of you has barbecue sauce on her face. I will ring the dinner bell over and over, until anyone who is messy has wiped her face. Then I will serve dessert." For the first $k-1$ rings, no one does anything. Then, at the kth ring, each of the messy individuals suddenly reaches for a napkin, and soon afterwards, the diners are all enjoying their ice cream.

How did the messy diners finally realize that their faces needed cleaning? The $k = 1$ case is easy, since in this case, the lone messy individual will realize he is messy immediately, since he sees that everyone else is clean. Consider the $k = 2$ case next. At the first ring, messy individual i_1 knows that one other person, i_2, is messy, but does not yet know about himself. At the second ring, i_1 realizes that he must be messy, since had i_2 been the only messy one, i_2 would have known this after the first ring when the cook made his announcement, and would have cleaned her face then. By a symmetric argument, messy diner i_2 also concludes that she is messy at the second ring, and both pick up a napkin at that time.

The general case follows by induction. Suppose that if $k = j$, then each of the j messy diners can determine that he is messy after j rings. Then if $k = j + 1$, then at the $j + 1$st ring, each of the $j + 1$ individuals will realize that he is messy. For if he were not messy, then the other j messy ones would have all realized their messiness at the jth ring and cleaned themselves then. Since no one cleaned herself after the jth ring, at the $j + 1$st ring each messy person will conclude that someone besides the other j messy people must also be messy, namely, himself.

By announcing a fact already known to every diner, the cook made this fact common knowledge among them, enabling each of them to eventually deduce the condition of his own face after sufficiently many rings of the bell.

Coordinated Attack

Consider a simple example of two allied armies situated on opposite hilltops waiting to attack their foe. Neither commander will attack unless he is sure that the other will attack at exactly the same time. The first commander sends a messenger to the other hilltop with the message "I plan to attack in the morning." The messenger's journey is perilous and he may die on the way to delivering the message. If he gets to the other hilltop and informs the other commander - can we be certain that both will attack in the morning? Note that both commanders now know the message, but the first cannot be sure that the second got the message. Thus, common knowledge implies not only that both know some piece of information, but can also be absolutely confident that the rest know it, and that the rest know that we know it, and so on.

Agreeing to Disagree

Aumann (1976) showed that people with the same priors cannot agree to disagree. If two people have the same priors, and their posteriors for a given event E are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on quite different information. The result is not true if we merely assume that the persons know each other's posteriors.

As an example, let $\Omega = \{\omega_1, \omega_1, \omega_3, \omega_4\}$, where each element of Ω occurs with equal (prior) probability. The information of player 1 is $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ and of player 2 is $\mathcal{F}_2 = \{\{\omega_1, \omega_2, \omega_3, \}, \{\omega_4\}\}$. These partitions are common knowledge. Suppose state ω_1 has actually occurred and the event is $E = \{\omega_1, \omega_4\}$. The posteriors of the two players are

$$\begin{aligned} q_1(E) &= \Pr(E|\{\omega_1, \omega_2\}) = \frac{1}{2} \\ q_2(E) &= \Pr(E|\{\omega_1, \omega_2,\, \omega_3\}) = \frac{1}{3}. \end{aligned}$$

Moreover, player 1 knows that player 2's information is the set $\{\omega_1, \omega_2,\, \omega_3\}$ so player 1 knows $q_2(E)$. Player 2 knows that player 1's information is either $\{\omega_1, \omega_2\}$ or $\{\omega_3, \omega_4\}$ and either way player 1's posterior probability of E is $\frac{1}{2}$, so player 2 knows $q_1(E)$. So, each player knows each other player's posteriors, yet the two posteriors differ. The explanation is that the posteriors are not common knowledge. This is because player 2 does not know what player 1 thinks player 2's posterior is. Given that state ω_3 is in player 2's information set, player 2 expects player 1 to believe that there is probability $\frac{1}{2}$ that $q_2(E) = \frac{1}{3}$ (if $\omega = \omega_1$) and probability $\frac{1}{2}$ that $q_2(E) = 1$ (if $\omega = \omega_4$).

Now suppose in round 1 both players announce their posteriors to each other, $\frac{1}{2}$ and $\frac{1}{3}$. This is not new information since players knew them already. In round 2 they do so again. Player 1 will announce again $\frac{1}{2}$. But this time this is new information for player 2. Since player 1 did not change his posterior upon learning the posterior of player 2 it must be that $\{\omega_3, \omega_4\}$ did not occur. Then player 2 updates his posterior to $\frac{1}{2}$.

Requiring common knowledge is not as innocuous as it may seem, but without this assumption it is quite impossible to analyze games within a structured framework. This difficulty arises because we are seeking to depict a situation in which players can engage in *strategic reasoning*. That is, I want to predict how you will make your choice, given my belief that you understand the game. Your understanding incorporates your belief about my understanding, and so on. Hence common knowledge will assist us dramatically in our ability to perform this kind of reasoning.

1.1. Normal-Form Games with Pure Strategies

A normal-form game consists of three features:

1. A set of players.
2. A set of actions for each player.
3. A set of payoff functions for each player that give a payoff value to each combination of the players' chosen actions.

We now introduce the commonly used concept of a *strategy*. A strategy is often defined as *a plan of action intended to accomplish a specific goal.*

Definition 2 *A pure strategy for player i is a deterministic plan of action. The set of all pure strategies for player i is denoted S_i. A profile of pure strategies $s = (s_1, s_2, ..., s_n)$, $s_i \in S_i$ for all $i = 1, 2, ..., n$, describes a particular combination of pure strategies chosen by all n players in the game.*

Later on we will introduce the concept of mixed strategies where a player chooses randomly among his pure strategies. We now formally define a normal-form game as follows.

Definition 3 *A normal-form game includes three components as follows:*

1. *A finite set of players, $N = \{1, 2, ..., n\}$.*
2. *A collection of sets of pure strategies, $\{S_1, S_2, ..., S_n\}$.*
3. *A set of payoff functions, $\{v_1, v_2, ..., v_n\}$, each assigning a payoff value to each combination of chosen strategies, that is, a set of functions $v_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$ for each $i \in N$.*

This representation is very general, and it will capture many situations in which each of the players $i \in N$ must simultaneously choose a possible strategy $s_i \in S_i$. Recall again that by simultaneous we mean the more general construct in which each player is choosing a strategy without knowing the choices of the other players. After strategies are selected, each player will realize his payoff, given by $v_i(s_1, s_2, ..., s_n) \in \mathbb{R}$, where $(s_1, s_2, ..., s_n)$ is the strategy profile that was selected by the agents. Thus from now on the normal-form game will be a triple of sets: $\langle N, \{S_i\}_{i=1}^n, \{v_i(\cdot)\}_{i=1}^n \rangle$.

1.1.1. Example: The Prisoner's Dilemma

The Prisoner's Dilemma is perhaps the best-known example in game theory, and it often serves as a parable for many different applications in economics and political science. It is a static game of complete information that represents a situation consisting of two individuals (the players) who are suspects in a serious crime, say, armed robbery. The police have evidence of only petty theft, and to nail the suspects for the armed robbery they need testimony from at least one of the suspects.

The police decide to be clever, separating the two suspects at the police station and questioning each in a different room. Each suspect is offered a deal that reduces the sentence he will get if he confesses, or "finks" (F), on his partner in crime. The alternative is for the suspect to say nothing to the investigators, or remain "mum" (M), so that they do not get the incriminating testimony from him. (As the Mafia would put it, the suspect follows the "omerta"–the code of silence.)

The payoff of each suspect is determined as follows: If both choose mum, then both get 2 years in prison because the evidence can support only the charge of petty theft. If, say, player 1 mums while player 2 finks, then player 1 gets 5 years in prison while player 2 gets only 1 year in prison for being the sole cooperator. The reverse outcome occurs if player 1 finks while player 2 mums. Finally, if both fink then both get only 4 years in prison. (There is some reduction of the 5-year sentence because each would blame the other for being the mastermind behind the robbery.) Because it is reasonable to assume that more time in prison is worse, we use the payoff representation that equates each year in prison with a value of -1. We can now represent this game in its normal form as follows:

Players: $N = \{1, 2\}$.

Strategies: $S_i = \{M, F\}$, for $i \in \{1, 2\}$.

Payoffs: Let $v_i(s_1, s_2)$ be the payoff to player i if player 1 chooses s_1 and player 2 chooses s_2. We can then write payoffs as

$$\begin{aligned} v_1(M,M) &= v_2(M,M) = -2 \\ v_1(F,F) &= v_2(F,F) = -4 \\ v_1(M,F) &= v_2(F,M) = -5 \\ v_1(F,M) &= v_2(M,F) = -1. \end{aligned}$$

This completes the normal-form representation of the Prisoner's Dilemma. We will soon analyze how rational players would behave if they were faced with this game.

1.1.2. Example: Cournot Duopoly

A variant of this example was first introduced by Augustin Cournot (1838). Two identical firms, players 1 and 2, produce some good. Assume that there are no fixed costs of production, and let the variable cost to each firm i of producing quantity $q_i \geq 0$ be given by the cost function, $c_i(q_i) = q_i^2$ for $i \in \{1, 2\}$. Demand is given by the function $q = 100 - p$, where $q = q_1 + q_2$. Cournot starts with the benchmark of firms that operate in a competitive environment in which each firm takes the market price, p, as given, and believes that its behavior cannot influence the market price. Under this assumption, as every economist knows, the solution will be the competitive equilibrium in which each firm produces at a point at which price equals marginal costs, so that the profits on the marginally produced unit are zero. In this particular case, each firm would produce $q_i = 25$, the price would be $p = 50$, and each firm would make 625 in profits.

Cournot then argues that this competitive equilibrium is naive because rational firms should understand that the price is not given, but rather determined by their actions. For example, if firm 1 realizes its effect on the market price, and produces $q_1 = 24$ instead of $q_1 = 25$, then the price will have to increase to $p(49) = 51$ for demand to equal supply because total supply will drop from 50 to 49. The profits of firm 1 will now be $v_1 = 51 \times 24 - 24^2 = 648 > 625$. Of course, if firm 1 realizes that it has such an effect on price, it should not just set $q_1 = 24$ but instead look for the best choice it can make. However, its best choice depends on the quantity that firm 2 will produce–what will that be? Clearly firm 2 should be as sophisticated, and thus we will have to find a solution that considers both the actions and the counteractions of these rational and sophisticated firms. For now, however, let's focus on the representation of the normal form of the game proposed by Cournot. The actions are choices of quantity, and the payoffs are the profits. Hence, the following represents the normal form:

Players: $N = \{1, 2\}$.

Strategies: $S_i = [0, \infty)$, for $i \in \{1, 2\}$ and firms choose quantities $s_i \in S_i$.

Payoffs: For $i, j \in \{1, 2\}$, $i \neq j$,

$$v_i(s_i, s_j) = \begin{cases} (100 - s_i - s_j)s_i - s_i^2, & \text{if } s_i + s_j < 100 \\ -s_i^2, & \text{if } s_i + s_j \geq 100. \end{cases}$$

1.2. Matrix Representation: Two-Player Finite Game

Definition 4 *A finite game is a game with a finite number of players, in which the number of strategies in S_i is finite for all players $i \in N$.*

As it turns out, any two-player finite game can be represented by a matrix that will capture all the relevant information of the normal-form game. This is done as follows:

Rows: Each row represents one of player 1's strategies. If there are k strategies in S_1 then the matrix will have k rows.

Columns: Each column represents one of player 2?s strategies. If there are m strategies in S_2 then the matrix will have m columns.

Matrix entries: Each entry in this matrix contains a two-element vector (v_1, v_2), where v_i is player i's payoff when the actions of both players correspond to the row and column of that entry.

1.2.1. Example: The Prisoner's Dilemma

Using the payoffs for the prisoner's dilemma given in the example above, the matrix representation of the Prisoner's Dilemma is

		Player 2	
		M	F
Player 1	M	$-2, -2$	$-5, -1$
	F	$-1, -5$	$-4, -4$

Figure 1.: Prisoner's dilemma game

1.2.2. Example: Rock-Paper-Scissors

Consider the famous child's game rock-paper-scissors. Recall that rock (R) beats scissors (S), scissors beats paper (P), and paper beats rock. Let the winner's payoff be 1 and the loser's be -1, and let the payoff for each player from a tie (i.e., they both choose the same action) be 0. This is a game with two players, $N = \{1, 2\}$, and three strategies for each player, $S_i = \{R, P, S\}$. Given the payoffs already described, we can write the matrix representation of this game as follows:

		Player 2		
		R	P	S
	R	$0, 0$	$-1, 1$	$1, -1$
Player 1	P	$1, -1$	$0, 0$	$-1, 1$
	S	$-1, 1$	$1, -1$	$0, 0$

Figure 2.: Rock-paper-scissors game

1.3. Solution Concepts

We have focused our attention on how to describe a game formally and fit it into a well-defined structure. This approach, of course, adds value only if we can use the structure to provide some analysis of what will or should happen in the game. Ideally we would like to be able to either advise players on how to play or try to predict how players will play. To accomplish this, we need some method to solve the game, and in this section we outline some criteria that will be helpful in evaluating potential methods to analyze and solve games.

In the prisoner's dilemma game, for example, each player is better off playing F regardless of his opponent's actions, but this leads the players to receive payoffs of -4 each, while if they could only agree to both choose M, then they would obtain -2 each. Left to their own devices the players should not be able to resist the temptation to choose F. Even if player 1 believes that player 2 will play M , he is better off choosing F (and vice versa).

Another classic game is the Battle of the Sexes, introduced by R. Duncan Luce and Howard Raiffa (1957) in their seminal book Games and Decisions. The story goes as follows. Alex and Chris are a couple, and they need to choose where to meet this evening. The catch is that the choice needs to be made while each is at work, and they have no means of communicating. (There were no cell phones or email in 1957, and even landline phones were not in abundance.) Both players prefer being together over not being together, but Alex prefers opera (O) to football (F), while Chris prefers the opposite. This implies that for each player being together at the venue of choice is better than being together at the other place, and this in turn is better than being alone. Using the payoffs of 2,1 and 0 to represent this order, the game is summarized in the following matrix:

		Chris	
		O	F
Alex	O	$2,1$	$0,0$
	F	$0,0$	$1,2$

Figure 3.: Battle of the sexes game.

What can you recommend to each player now? Unlike the situation in the Prisoner's Dilemma, the best action for Alex depends on what Chris will do and vice versa. If we want to predict or prescribe actions for this game, we need to make assumptions about the behavior and the beliefs of the players. We therefore need a solution concept that will result in predictions or prescriptions. A solution concept is a method of analyzing games with the objective of restricting the set of all possible outcomes to those that are more reasonable than others. That is, we will consider some reasonable and consistent assumptions about the behavior and beliefs of players that will divide the space of outcomes into "more likely" and "less likely." Furthermore, we would like our solution concept to apply to a large set of games so that it is widely applicable. Consider, for example, the solution concept that prescribes that each player choose the action that is always best, regardless of what his opponents will choose. As we saw earlier in the Prisoner's Dilemma, playing F is always better than playing M. Hence this solution concept will predict that in this game both players will choose F. For the Battle of the Sexes, however, there is no strategy that is always best: playing F is best if your opponent plays F, and playing O is best if your opponent plays O. Hence for the Battle of the Sexes, this simple solution concept is not useful and offers no guidance. We will use the term equilibrium for any one of the strategy profiles that emerges as one of the solution concept's predictions. We will often think of equilibria as the actual predictions of our theory. A more forgiving meaning would be that equilibria are the likely predictions, because our theory will often not account for all that is going on. Furthermore, in some cases we will see that more than one equilibrium prediction is possible for the same game. In fact, this will sometimes be a strength, and not a weakness, of the theory.

1.3.1. Assumptions and Set-up

To set up the background for equilibrium analysis, it is useful to revisit the assumptions that we will be making throughout:

1. Players are "rational": A rational player is one who chooses his action, $s_i \in S_i$, to maximize his payoff consistent with his beliefs about what is going on in the game.
2. Players are "intelligent": An intelligent player knows everything about the game: the actions, the outcomes, and the preferences of all the players.
3. Common knowledge: The fact that players are rational and intelligent is common knowledge among the players of the game.
4. Self-enforcement: Any prediction (or equilibrium) of a solution concept must be self-enforcing.

The requirement that any equilibrium must be self-enforcing is at the core of our analysis and at the heart of noncooperative game theory. We will assume throughout this book that the players engage in noncooperative behavior in the following sense: each player is in control of his own actions, and he will stick to an action only if he finds it to be in his best interest. That is, if a profile of strategies is to be an equilibrium, we will require each player to be happy with his own choice given how the others make their own choices. As you can probably figure out, the profile (F, F) is self-enforcing in the Prisoner's Dilemma game: each player is happy playing F. Indeed, we will see that this is a very robust outcome in terms of equilibrium analysis.

1.3.2. Evaluating Solution Concepts

In developing a theory that predicts the behavior of players in games, we must evaluate our theory by how well it does as a methodological tool. That is, for our theory to be widely useful, it must describe a method of analysis that applies to a rich set of games, which describe the strategic situations in which we are interested. We will introduce three criteria that will help us evaluate a variety of solution concepts: existence, uniqueness, and invariance.

Existence: How Often Does It Apply? A solution concept is valuable in so far as it applies to a wide variety of games, and not just to a small and select family of games. A solution concept should apply generally and should not be developed in an ad hoc way that is specific to a certain situation or game. That is, when we apply our solution concept to different games we require it to result in the existence of an equilibrium solution. For example, consider an ad hoc solution concept that offers the following prediction: "Players always choose the action that they think their opponent will choose." If this is our "theory" of behavior, then it will fail to apply to many–maybe most–strategic situations. In particular when players have different sets of actions (e.g., one chooses a software package and the other a hardware package) then this theory would be unable to predict which outcomes are more likely to emerge as equilibrium outcomes.

Any proposed theory for a solution concept that relies on very specific elements of a game will not be general and will be hard to adapt to a wide variety of strategic situations, making the proposed theory useless beyond the very special situations it was tailored to address. Thus one goal is to have a method that will be general enough to apply to many strategic situations; that is, it will prescribe a solution that will exist for most games we can think of.

Uniqueness: How Much Does It Restrict Behavior? Just as we require our solution concept to apply broadly, we require that it be meaningful in that it restricts the set of possible outcomes to a smaller set of reasonable outcomes. In fact one might argue that being able to pinpoint a single unique outcome as a prediction would be ideal. Uniqueness is then an important counterpart to existence. For example, if the proposed solution concept says "anything can happen," then it always exists: regardless of the game we apply this concept to, "anything can happen" will always say that the solution is one of the (sometimes infinite) possible outcomes. Clearly this solution concept is useless. A good solution concept is one that balances existence (so that it works for many games) with uniqueness (so that we can add some intelligent insight into what can possibly happen). It turns out that the nature of games makes the uniqueness requirement quite hard to meet. The reason, as we will learn to appreciate, lies in the nature of strategic interaction in a noncooperative environment. To foreshadow the reasons behind this observation, notice that a player's best action will often depend on what other players are doing. A consequence is that there will often be several combinations of strategies that will support each other in this way.

Invariance: How Sensitive Is It to Small Changes? Aside from existence and uniqueness, a third more subtle criterion is important in qualifying a solution concept as a reasonable one, namely that the solution concept be invariant to small changes in the game's structure. However, the term "small changes" needs to be qualified more precisely. Adding a player to a game, for instance, may not be a small change if that player has actions that can wildly change the outcomes of the game. Thus adding or removing a player cannot innocuously be considered a small change. Similarly if we add or delete strategies from the set of actions that are available to a player, we may hinder his ability to guarantee himself some outcomes, and therefore this too should not be considered a small change to the game. We are left with only one component to fiddle with: the payoff functions of the players. It is reasonable to argue that if the payoffs of a game are modified only slightly, then this is a small change to the game that should not affect the predictions of a "robust" solution concept. For example, consider the Prisoner's Dilemma. If instead of 5 years in prison, imposing a pain of -5 for the players, it imposed a pain of -5.01 for player 1 and -4.99 for player 2, we should be somewhat discouraged if our solution concept suddenly changed the prediction of what players will or ought to do. Thus invariance is a robustness property with which we require a solution concept to comply. In other words, if two games are "close," so that the action sets and players are the same yet the payoffs are only slightly different, then our solution concept should offer predictions that are not wildly different for the two games. Put formally, if for a small enough value $\varepsilon > 0$ we alter the payoffs of every outcome for every player by no more than ε, then the solution concept's prediction should not change.

1.3.3. Evaluating Outcomes

Once we subscribe to any particular solution concept, as social scientists we would like to evaluate the properties of the solutions, or predictions, that the solution concept will prescribe. This process will offer insights into what we expect the players of a game to achieve when they are left to their own devices. In turn, these insights can guide us toward possibly changing the environment of the game so as to improve the social outcomes of the players. We have to be precise about the meaning of "to improve the social outcomes." For example, many people may agree that it would be socially better for the government to take $10 away from the very rich Bill Gates and give that $10 to an orphan in Latin America. In fact even Gates himself might have approved of this transfer, especially if the money would have saved the child's life. However, Gates may or may not have liked the idea, especially if such government intervention would imply that over time most of his wealth would be dissipated through such transfers.

Economists use a particular criterion for evaluating whether an outcome is socially undesirable. An outcome is considered to be socially undesirable if there is a different outcome that would make some people better off without harming anyone else. As social scientists we wish to avoid outcomes that are socially undesirable, and we therefore turn to the criterion of Pareto optimality, which is in tune with the idea of efficiency or 'no waste.' That is, we would like all the possible value deriving from a given interaction to be distributed among the players. To put this formally:

Definition 5 *A strategy profile $s \in S$ Pareto dominates strategy profile $s' \in S$ if $v_i(s) \geq v_i(s')$ for all $i \in N$ and $v_i(s) > v_i(s')$ for at least one $i \in N$ (in which case, we will also say that s' is Pareto dominated by s). A strategy profile is Pareto optimal if it is not Pareto dominated by any other strategy profile.*

As social scientists, strategic advisers, or policy makers, we hope that players will act in accordance with the Pareto criterion and find ways to coordinate on Pareto-optimal outcomes, or avoid those that are Pareto dominated. However, as we will see time and time again, this result will not be achievable in many games. For example, in the Prisoner's Dilemma we made the case that (F, F) should be considered as a very likely outcome. In fact, as we will argue several times, it is the only likely outcome. One can see, however, that it is Pareto dominated by (M, M). (Notice that (M, M) is not the only Pareto-optimal outcome. (M, F) and (F, M) are also Pareto-optimal outcomes because no other profile dominates any of them. Don't confuse Pareto optimality with the best "symmetric" outcome that leaves all players "equally" happy.)

1.4. Summary

- A normal-form game includes a finite set of players, a set of pure strategies for each player, and a payoff function for each player that assigns a payoff value to each combination of chosen strategies.
- Any two-player finite game can be represented by a matrix. Each row represents one of player 1's strategies, each column represents one of player 2's strategies, and each cell in the matrix contains the payoffs for both players.
- A solution concept that proposes predictions of how games will be played should be

widely applicable, should restrict the set of possible outcomes to a small set of reasonable outcomes, and should not be too sensitive to small changes in the game.

- Outcomes should be evaluated using the Pareto criterion, yet self-enforcing behavior will dictate the set of reasonable outcomes.

2. Rationality and Common Knowledge

2.1. Dominance in Pure Strategies

It will be useful to begin by introducing some new notation. We denoted the payoff of a player i from a profile of strategies $s = (s_1, s_2, ..., s_{i-1}, s_i, s_{i+1}, ..., s_n)$ as $v_i(s)$. It will soon be very useful to refer specifically to the strategies of a player's opponents in a game. For example, the actions chosen by the players who are not player i are denoted by the profile

$$(s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_n) \in S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n.$$

To simplify we will hereafter use a common shorthand notation as follows: We define $S_{-i} \equiv S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$ as the set of all the strategy sets of all players who are not player i. We then define $s_{-i} \in S_{-i}$ as a particular possible profile of strategies for all players who are not i. Hence we can rewrite the payoff of player i from strategy s as $v_i(s_i, s_{-i})$, where $s = (s_i, s_{-i})$.

2.1.1. Dominated Strategies

The Prisoner's Dilemma was easy to analyze: each of the two players has an action that is best regardless of what his opponent chooses. Suggesting that each player will choose this action seems natural because it is consistent with the basic concept of rationality. If we assume that the players are rational, then we should expect them to choose whatever they deem to be best for them. If it turns out that a player's best strategy does not depend on the strategies of his opponents then we should be all the more confident that he will choose it.

It is not too often that we will find ourselves in situations in which we have a best action that does not depend on the actions of our opponents. We begin, therefore, with a less demanding concept that follows from rationality. In particular consider the strategy mum in the Prisoner's Dilemma:

As we argued earlier, playing M is worse than playing F for each player regardless of what the player's opponent does. What makes it unappealing is that there is another strategy, F, that is better than M regardless of what one's opponent chooses. We say that such a strategy is dominated. Formally we have

Definition 6 *Let $s_i \in S_i$ and $s_i' \in S_i$ be possible strategies for player i. We say that s_i' is strictly dominated by s_i if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, player i's payoff from s_i' is strictly less than that from s_i. That is,*

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

We will write $s_i \succ_i s_i'$ to denote that s_i' is strictly dominated by s_i.

Now that we have a precise definition for a dominated strategy, it is straightforward to draw an obvious conclusion:

Claim 7 *A rational player will never play a strictly dominated strategy.*

This claim is obvious. If a player plays a dominated strategy then he cannot be playing optimally because, by the definition of a dominated strategy, the player has another strategy that will yield him a higher payoff regardless of the strategies of his opponents. Hence knowledge of the game implies that a player should recognize dominated strategies, and rationality implies that these strategies will be avoided.

When we apply the notion of a dominated strategy to the Prisoner's Dilemma we argue that each of the two players has one dominated strategy that he should never use, and hence each player is left with one strategy that is not dominated. Therefore, for the Prisoner's Dilemma, rationality alone is enough to offer a prediction about which outcome will prevail: (F, F) is this outcome.

Many games, however, will not be as special as the Prisoner's Dilemma, and rationality alone will not suggest a clear-cut, unique prediction. As an example, consider Battle of the Sexes game.

2.1.2. Dominant Strategy Equilibrium

Because a strictly dominated strategy is one to avoid at all costs, there is a counterpart strategy, represented by F in the Prisoner's Dilemma, that would be desirable. This is a strategy that is always the best thing you can do, regardless of what your opponents choose. Formally we have

Definition 8 *$s_i \in S_i$ is a strictly dominant strategy for i if every other strategy of i is strictly dominated by it, that is,*

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i, s_i' \neq s_i, \text{ and all } s_{-i} \in S_{-i}.$$

If, as in the Prisoner's Dilemma, every player had such a wonderful dominant strategy, then it would be a very sensible predictor of behavior because it follows from rationality alone. We can introduce this simple idea as our first solution concept:

Definition 9 *The strategy profile $s^D \in S$ is a strict dominant strategy equilibrium if $s_i^D \in S_i$ is a strict dominant strategy for all $i \in N$.*

This gives a formal definition for the outcome "both players fink," or (F, F), in the Prisoner's Dilemma: it is a dominant strategy equilibrium. In this equilibrium the payoffs are $(-4, -4)$ for players 1 and 2, respectively.

Using this solution concept for any game is not that difficult. It basically requires that we identify a strict dominant strategy for each player and then use this profile of strategies to predict or prescribe behavior. If, as in the Prisoner's Dilemma, we are lucky enough to find a dominant strategy equilibrium for other games, then this solution concept has a very appealing property:

Proposition 10 *If the game* $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ *has a strictly dominant strategy equilibrium* s^D*, then* s^D *is the unique dominant strategy equilibrium.*

A related notion is that of **weak dominance**. We say that s_i' is weakly dominated by s_i if, for any possible combination of the other players' strategies, player i's payoff from s_i' is weakly less than that from s_i. That is,

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}.$$

This means that for some $s_{-i} \in S_{-i}$ this weak inequality may hold strictly, while for other $s'_{-i} \in S_{-i}$ it will hold with equality. We define a strategy to be weakly dominant in a similar way. This is still useful because if we can find a dominant strategy for a player, be it weak or strict, this seems like the most obvious thing to prescribe. An important difference between weak and strict dominance is that if a weakly dominant equilibrium exists, it need not be unique.

2.2. Iterated Elimination of Strictly Dominated Pure Strategies

As we saw in the previous chapter, our requirement that players be rational implied two important conclusions:

1. A rational player will never play a dominated strategy.
2. If a rational player has a strictly dominant strategy then he will play it.

We used this second conclusion to define the solution concept of strict dominance, which is very appealing because, when it exists, it requires only rationality as its driving force. A drawback of the dominant strategy solution concept is, however, that it will often fail to exist (as in the Battle of the Sexes game, for example). Hence if we wish to develop a predictive theory of behavior in games then we must consider alternative approaches that will apply to a wide variety of games.

2.2.1. Iterated Elimination and Common Knowledge of Rationality

We begin with the premise that players are rational, and we build on the first conclusion in the previous section, which claims that a rational player will never play a dominated strategy. This conclusion is by itself useful in that it rules out what players will not do. As a result, we conclude that rationality tells us which strategies will never be played.

Now turn to another important assumption introduced earlier: the structure of the game and the rationality of the players are common knowledge among the players. The introduction of common knowledge of rationality allows us to do much more than identify strategies that rational players will avoid. If indeed all the players know that each player will never play a strictly dominated strategy, they can effectively ignore those strictly dominated strategies that their opponents will never play, and their opponents can do the same thing. If the original game has some players with some strictly dominated strategies, then all the players know that effectively they are facing a "smaller" restricted game with fewer total strategies.

This logic can be taken further. Because it is common knowledge that all players are rational, then everyone knows that everyone knows that the game is effectively a smaller game. In this smaller restricted game, everyone knows that players will not play strictly dominated strategies. In fact we may indeed find additional strategies that are dominated in the restricted game that were not dominated in the original game. Because it is common knowledge that players will perform this kind of reasoning again, the process can continue until no more strategies can be eliminated in this way.

To see this idea more concretely, consider the following two-player finite game:

		Player 2		
		L	C	R
	U	$4,3$	$5,1$	$6,2$
Player 1	M	$2,1$	$8,4$	$3,6$
	D	$3,0$	$9,6$	$2,8$

A quick observation reveals that there is no strictly dominant strategy, neither for player 1 nor for player 2. Also note that there is no strictly dominated strategy for player 1. There is, however, a strictly dominated strategy for player 2: the strategy C is strictly dominated by R. Once we eliminate C for player 2, both M and D are strictly dominated by U for player 1, allowing us to perform a second round of eliminating strategies, this time for player 1. Eliminating these two strategies for player 2, we can conclude that the unique equilibrium is (U,L).

As the example demonstrates, this process of **iterated elimination of strictly dominated strategies** (IESDS) builds on the assumption of common knowledge of rationality. The first step of iterated elimination is a consequence of player 2's rationality; the second stage follows because players know that players are rational; the third stage follows because players know that players know that they are rational, and this ends in a unique prediction.

More generally we can apply this process to games in the following way. Let S_i^k denote the strategy set of player i that survives k rounds of IESDS. We begin the process by defining $S_i^0 = S_i$ for each i, the original strategy set of player i in the game.

Step 1: Define $S_i^0 = S_i$ for each i, the original strategy set of player i in the game, and set $k = 0$.

Step 2: Are there players for whom there are strategies $s_i \in S_i^k$ that are strictly dominated? If yes, go to step 3. If not, go to step 4.

Step 3: For all the players $i \in N$, remove any strategies $s_i \in S_i^k$ that are strictly dominated. Set $k = k + 1$, and define a new game with strategy sets S_i^k that do not include the strictly dominated strategies that have been removed. Go back to step 2.

Step 4: The remaining strategies in S_i^k are reasonable predictions for behavior.

In this chapter we refrain from giving a precise mathematical definition of the process because this requires us to consider richer behavior by the players, in particular, allowing them to choose randomly between their different pure strategies. We will revisit this approach briefly when such stochastic play, or mixed strategies, is introduced later. Using the process of IESDS we can define a new solution concept:

Definition 11 *We will call any strategy profile $s^{ES} = (s_1^{ES}, ..., s_n^{ES})$ that survives the process of IESDS an iterated-elimination equilibrium.*

Like the concept of a strictly dominant strategy equilibrium, the iterated- elimination equilibrium starts with the premise of rationality. However, in addition to rationality, IESDS requires a lot more: common knowledge of rationality. We will discuss the implications of this requirement later in this chapter.

2.2.2. Example: Cournot Duopoly

Recall the Cournot duopoly example we introduced in Section 1.1.2, but consider instead a simpler example of this problem in which the firms have linear rather than quadratic costs: the cost for each firm for producing quantity q_i is given by $c_i(q_i) = 10q_i$ for $i \in \{1, 2\}$. (Using economics jargon, this is a case of constant marginal cost equal to 10 and no fixed costs.) Let the demand be given by $p(q) = 100 - q$, where $q = q_1 + q_2$. Consider first the profit (payoff) function of firm 1:

$$v_1(q_1, q_2) = \overbrace{(100 - q_1 - q_2)q_1}^{\text{Revenue}} - \overbrace{10q_1}^{\text{Cost}} = 90q_1 - q_1^2 - q_1q_2.$$

What should firm 1 do? If it knew what quantity firm 2 will choose to produce, say some value of q_2, then the profits of firm 1 would be maximized when the first-order condition is satisfied, that is, when $90 - 2q_1 - q_2 = 0$. Thus, for any given value of q_2, firm 1 maximizes its profits when it sets its own quantity according to the function

$$q_1 = \frac{90 - q_2}{2}. \tag{2.2.1}$$

Though it is true that the choice of firm 1 depends on what it believes firm 2 is choosing, equation (2.2.1) implies that firm 1 will never choose to produce more than $q_1 = 45$. This follows from the simple observation that q_2 is never negative, in which case equation (2.2.1) implies that $q_1 \leq 45$. In fact, this is equivalent to showing that any quantity $q_1 > 45$ is strictly dominated by $q_1 = 45$.

It is easy to see that firm 2 faces exactly the same profit function, which implies that any $q_2 > 45$ is strictly dominated by $q_2 = 45$. This observation leads to our first round of iterated elimination: a rational firm produces no more than 45 units, implying that the effective strategy space that survives one round of elimination is $q_i \in [0, 45]$ for $i \in \{1, 2\}$.

We can now turn to the second round of elimination. Because $q_2 \leq 45$, equation (2.2.1) implies that firm 1 will choose a quantity no less than 22.5, and a symmetric argument applies to firm 2. Hence the second round of elimination implies that the surviving strategy sets are $q_i \in [22.5, 45]$ for $i \in \{1, 2\}$.

The next step of this process will reduce the strategy set to $q_i \in \left[22.5, 33\frac{3}{4}\right]$, and the process will continue on and on. Interestingly the set of strategies that survives this process converges to a single quantity choice of $q_i = 30$. To see this, notice how we moved from one surviving interval to the next. We started by noting that $q_2 \geq 0$, and using equation (2.2.1) we found that $q_1 \leq 45$, creating the first-round interval of $[0, 45]$. Then, by symmetry, it follows that $q_2 \leq 45$, and using equation (2.2.1) again we conclude that $q_1 \geq 22.5$, creating the second-round interval $[22.5, 45]$. If this were to converge to an interval and not to a single point, then by the symmetry between both firms, the resulting interval for each firm would be $[q_{min}, q_{max}]$ that simultaneously satisfy two equations with two unknowns: $q_{min} = \frac{90-q_{max}}{2}$ and $q_{max} = \frac{90-q_{min}}{2}$. However, the only solution to these two equations is $q_{min} = q_{max} = 30$. Hence using IESDS for the Cournot game results in a unique predictor of behavior where $q_1 = q_2 = 30$, and each firm earns a profit of $v_1 = v_2 = 900$.

2.2.3. Evaluating IESDS

We turn to evaluate the IESDS solution concept using the criteria we introduced earlier. Start with existence and note that, unlike the concept of strict dominance, we can apply IESDS to any game by applying the algorithm just described. It does not require the existence of a strictly dominant strategy, nor does it require the existence of strictly dominated strategies. It is the latter characteristic, however, that gives this concept some bite: when strictly dominated strategies exist, the process of IESDS is able to say something about how common knowledge of rationality restricts behavior.

It is worth noting that this existence result is a consequence of assuming common knowledge of rationality. By doing so we are giving the players the ability to reason through the strategic implications of rationality, and to do so over and over again, while correctly anticipating that other players can perform the same kind of reasoning. Rationality alone does not provide this kind of reasoning.

It is indeed attractive that an IESDS solution always exists. This comes, however, at the cost of uniqueness. In the simple 3×3 matrix game described in 2.2.1 and the Cournot duopoly game, IESDS implied the survival of a unique strategy. Consider instead the Battle of the Sexes game. IESDS cannot restrict the set of strategies here for the simple reason that neither O nor F is a strictly dominated strategy for each player. This solution concept can be applied (it exists) to any game, but it will often fail to provide a unique solution. For the Battle of the Sexes game, IESDS can only conclude that "anything can happen."

After analyzing the efficiency of the outcomes that can be derived from strict dominance, you may have anticipated the possible efficiency of IESDS equilibria. An easy illustration can be provided by the Prisoner's Dilemma. IESDS leaves (F, F) as the unique survivor, or IESDS equilibrium, after only one round of elimination. The outcome from (F, F) is not Pareto optimal. Similarly, both previous examples (the 3×3 matrix game in 2.2.1 and the Cournot game) provide further evidence that Pareto optimality need not be achieved by IESDS: In the 3×3 matrix example, both strategy profiles (M, C) and (D, C) yield higher payoffs for both players–(8, 4) and (9, 6), respectively–than the unique IESDS equilibrium, which yields (4, 3). For the Cournot game, producing $q_1 = q_2 = 30$ yields profits of 900 for each firm. If instead they would both produce $q_1 = q_2 = 20$ then each would earn profits of 1000. Thus common knowledge of rationality does not mean that players can guarantee the best outcome for themselves when their own incentives dictate their behavior.

On a final note, it is interesting to observe that there is a simple and quite obvious relationship between the IESDS solution concept and the strict-dominance solution concept:

Proposition 12 *If for a game* $\Gamma = \langle N, \{S_i\}_{i=1}^n, \{v_i\}_{i=1}^n \rangle$ s^* *is a strict dominant strategy equilibrium, then* s^* *uniquely survives IESDS.*

Proof. If $s^* = (s_1^*, ..., s_n^*)$ is a strict dominant strategy equilibrium then, by definition, for every player i all other strategies s_i' are strictly dominated by s_i^*. This implies that after one stage of elimination we will be left with a single profile of strategies, which is exactly s ■

This simple proposition is both intuitive and straightforward. Because rationality is the only requirement needed in order to eliminate all strictly dominated strategies in one round, then if all strategies but one are strictly dominated for each and every player, both IESDS and strict dominance will result in the same outcome. This shows us that whenever strict dominance results in a unique outcome, then IESDS will result in the same unique outcome after one round. However, as we saw earlier, IESDS may offer a fine prediction when strict dominance does not apply. This is exactly what the extra assumption of common knowledge of rationality delivers: a more widely applicable solution concept. However, the assumption of common knowledge of rationality is far from innocuous. It requires the players to be, in some way, extremely intelligent and to possess unusual levels of foresight. For the most part, game theory relies on this strong assumption, and hence it must be applied to the real world with caution. Remember the rule about how assumptions drive conclusions: garbage in, garbage out.

2.3. Beliefs, Best Response, and Rationalizability

Both of the solution concepts we have seen so far, strict dominance and IESDS, are based on eliminating actions that players would never play. An alternative approach is to ask: what possible strategies might players choose to play and under what conditions? When we considered eliminating strategies that no rational player would choose to play, it was by finding some strategy that is always better or, as we said, that dominates the eliminated strategies. A strategy that cannot be eliminated, therefore, suggests that under some conditions this strategy is the one that the player may like to choose. When we qualify a strategy to be the best one a player can choose under some conditions, these conditions must be expressed in terms that are rigorous and are related to the game that is being played.

To set the stage, think about situations in which you were puzzled about the behavior of someone you knew. To consider his choice as irrational, or simply stupid, you would have to consider whether there is a way in which he could defend his action as a good choice. A natural way to determine whether this is the case is to simply ask him, “What were you thinking?" If the response lays out a plausible situation for which his choice was a good one, then you cannot question his rationality. (You may of course question his wisdom, or even his sanity, if his thoughts seem bizarre.)

This is a type of reasoning that we will formalize and discuss in this chapter. If a strategy s_i is not strictly dominated for player i then it must be that there are combinations of strategies of player i's opponents for which the strategy s_i is player i's best choice. This reasoning will allow us to justify or rationalize the choice of player i.

2.3.1. The Best Response

What makes a game different from a single-player decision problem is that once you understand the actions, outcomes, and preferences of a decision problem, then you can choose your best or optimal action. In a game, however, your optimal decision not only depends on the structure of the game, but it will often depend on what the other players are doing. Take the Battle of the Sexes as an example. The best choice of Alex depends on what Chris will do. If Chris goes to the opera then Alex would rather go to the opera instead of going to the football game. If, however, Chris goes to the football game then Alex's optimal action is switched around.

This simple example illustrates an important idea that will escort us throughout this book and that (one hopes) will escort you through your own decision making in strategic situations. In order for a player to be optimizing in a game, he has to choose a best strategy as a response to the strategies of his opponents. We therefore introduce the following formal definition:

Definition 13 *The strategy $s_i \in S_i$ is player i's best response to his opponents' strategies $s_{-i} \in S_{-i}$ if*

$$v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \qquad \forall s_i' \in S_i.$$

I can't emphasize enough how central this definition is to the concept of strategic behavior and rationality. In fact rationality implies that given any belief a player has about his opponents' behavior, he must choose an action that is best for him given his beliefs. That is,

Claim 14 *A rational player who believes that his opponents are playing some $s_{-i} \in S_{-i}$ will always choose a best response to s_{-i}.*

For instance, in the Battle of the Sexes, if Chris believes that Alex will go to the opera then Chris's best response is to go to the opera because $v_2(O,O) = 1 > 0 = v_2(O,F)$. Similarly if Chris believes that Alex will go to the football game then Chris's best response is to go to the game as well.

There are some appealing relationships between the concept of playing a best response and the concept of dominated strategies. First, if a strategy s_i is strictly dominated, it means that some other strategy s_i' is always better. This leads us to the observation that the strategy s_i could not be a best response to anything:

Proposition 15 *If s_i is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.*

A companion to this proposition would explore strictly dominant strategies, which are in some loose way the "opposite" of strictly dominated strategies. You should easily be able to convince yourself that if a strategy s_i^D is a strictly dominant strategy then it must be a best response to anything i's opponents can do. This immediately implies the next proposition, which is slightly broader than the simple intuition just provided and requires a bit more work to prove formally:

Proposition 16 *If in a finite normal-form game s^* is a strict dominant strategy equilibrium, or if it uniquely survives IESDS, then s_i^* is a best response to s_{-i}^* $\forall i \in N$.*

Proof. If s^* is a dominant strategy equilibrium then it uniquely survives IESDS, so it is enough to prove the proposition for strategies that uniquely survive IESDS. Suppose s^* uniquely survives IESDS, and choose some $i \in N$. Suppose in negation to the proposition that s^* is not a best response to s^*. This implies that there exists an $s' \in S_i \backslash \{s^*\}$ (this is the set S_i without the strategy s^*) such that $v_i(s', s^*) > v_i(s_i^*, s_{-i}^*)$. Let $S' \subset S_i$ be the set of all such s' for which $v_i(s', s_{-i}^*) > v_i(s^*, s_{-i}^*)$. Because s^* was eliminated while s^* was not (recall that s^* uniquely survives IESDS), there must be some s'' such that $v_i(s_i'', s_{-i}^*) > v_i(s_i', s_{-i}^*) > v_i(s_i^*, s_{-i}^*)$,implying that $s_i'' \in S_i'$. Because the game is finite, an induction argument on S' then implies that there exists a strategy $s_i \in S_i$ that must survive IESDS. But this is a contradiction to s^* being the unique survivor of IESDS. ■

With the concept of a best response in hand, we need to think more seriously about the following question: to what profile of strategies should a player be playing a best response? Put differently, if my best response depends on what the other players are doing, then how should I choose between all the best responses I can possibly have? This is particularly pertinent because we are discussing static games, in which players choose their actions without knowing what their opponents are choosing. To tackle this important question, we need to give players the ability to form conjectures about what others are doing. We have alluded to the next step in the claim made earlier, which stated that "a rational player who believes that his opponents are playing some $s_{-i} \in S_{-i}$ will always choose a best response to s_{-i} ." Thus we have to be mindful of what a player believes in order to draw conclusions about whether or not the player is choosing a best response.

2.3.2. Beliefs and Best-Response Correspondences

Belief is a player's assessment about the strategies of the other players in the game.

Definition 17 *A belief of player i is a possible profile of his opponents' strategies, $s_{-i} \in S_{-i}$.*

Given that a player has a particular belief about his opponents' strategies, he will be able to formulate a best response to that belief. The best response of a player to a certain strategy of his opponents may or may not be unique.

Definition 18 *The best-response correspondence of player i selects for each $s_{-i} \in S_{-i}$ a subset $BR_i(s_{-i}) \subset S_i$ where each strategy $s_i \in BR_i(s_{-i})$ is a best response to s_{-i}.*

In the Cournot model, for example, if firm 1's belies is that $q_2 = 20$, then using (2.2.1) $BR_1(20) = 35$.

2.3.3. Rationalizibility

In what follows we introduce another way of reasoning that rules out irrational behavior with a similar iterated process that is, in many ways, the mirror image of IESDS. This next solution concept also builds on the assumption of common knowledge of rationality. However, instead of asking "What would a rational player not do?" our next concept asks "What might a rational player do?" A rational player will select only strategies that are a best response to some profile of his opponents. Thus we have

Definition 19 *A strategy $s_i \in S_i$ is never a best response if there are no beliefs $s_{-i} \in S_{-i}$ for player i for which $s_i \in BR_i(s_{-i})$.*

The next step, as in IESDS, is to use the common knowledge of rationality to build an iterative process that takes this reasoning to the limit. After employing this reasoning one time, we can eliminate all the strategies that are never a best response, resulting in a possibly "smaller" reduced game that includes only strategies that can be a best response in the original game. Then we can employ this reasoning again and again, in a similar way that we did for IESDS, in order to eliminate outcomes that should not be played by players who share a common knowledge of rationality.

The solution concept of rationalizability is defined precisely by iterating this thought process. The set of strategy profiles that survive this process is called the set of rationalizable strategies. (We postpone offering a definition of rationalizable strategies because the introduction of mixed strategies, in which players can play stochastic strategies, is essential for the complete definition.)

Rationalizability merely requires that the players' beliefs and behavior be consistent with common knowledge of rationality. It does not require that beliefs are correct.

2.3.4. The Cournot Duopoly Revisited

We use the best-response given by (2.2.1). Examining $BR_1(q_2)$ implies that firm 1 will choose to produce only quantities between 0 and 45. That is, there will be no beliefs about q_2 for which quantities above 45 are a best response. By symmetry the same is true for firm 2. Thus a first round of rationalizability implies that the only quantities that can be best-response quantities for both firms must lie in the interval $[0, 45]$. The next round of rationalizability for the game in which $S_i = [0, 45]$ for both firms shows that the best response of firm i is to choose any quantity $q_i \in [22.5, 45]$. Just as with IESDS, this process will continue on and on. The set of rationalizable strategies converges to a single quantity choice of $q_i = 30$ for both firms.

2.3.5. The "p-Beauty Contest"

Consider a game with n players, so that $N = \{1, ..., n\}$. Each player has to choose an integer between 0 and 20, so that $S_i = \{0, 1, 2, ..., 19, 20\}$. The winners are the players who choose an integer number that is closest to 3/4 of the average.

Turning to rationalizable strategies, we must begin by finding strategies that are never a best response. Choosing $s_i = 20$ cannot be a best response. Even if all players choose 20 a player can deviate to 19 and become better off. This shows that only the numbers $S_i^1 = \{0, 1, ..., 19\}$ survive the first round of rationalizable behavior. Similarly after each additional round we will "lose" the highest number until we go through 19 rounds and are left with $S_i^{19} = \{0, 1\}$, meaning that after 19 rounds of dropping strategies that cannot be a best response, we are left with two strategies that survive: 1 and 0.

If $n > 2$, we cannot reduce this set further: if, for example, player i believes that all the other players are choosing 1 then choosing 1 is a best response for him. Similarly regardless of n, if he believes that everyone is choosing 0 then choosing 0 is his best response. Thus we are able to predict using rationalizability that players will not choose a number greater than 1, and if there are only two players then we will predict that both will choose 0.

Remark By now you must have concluded that IESDS and rationalizability are two sides of the same coin, and you might even think that they are one and the same. This is almost true, and for two-player games it turns out that these two processes indeed result in the same outcomes. We discuss this issue briefly in Section 6.3, after we introduce the concept of mixed strategies. A more complete treatment can be found in Chapter 2 of Fudenberg and Tirole (1991).

2.4. Summary

- Rational players will never play a dominated strategy and will always play a dominant strategy when it exists.
- When players share common knowledge of rationality, the only strategies that are sensible are those that survive IESDS.
- Rational players will always play a best response to their beliefs. Hence any strategy for which there are no beliefs that justify its choice will never be chosen.
- Outcomes that survive IESDS, rationalizability, or strict dominance need not be Pareto optimal, implying that players may not be able to achieve desirable outcomes if they are left to their own devices.

3. Pinning Down Beliefs: Nash Equilibrium

We have seen three solution concepts that offer some insights into predicting the behavior of rational players in strategic (normal-form) games. The first, strict dominance, relied only on rationality, and in some cases, like the Prisoner's Dilemma, it predicted a unique outcome, as it would in any game for which a dominant strategy equilibrium exists. However, it often fails to exist. The two sister concepts of IESDS and rationalizability relied on more than rationality by requiring common knowledge of rationality. In return a solution existed for every game, and for some games there was a unique prediction. Moreover, whenever there is a strict dominant equilibrium, it also uniquely survives IESDS and rationalizability. Even for some games for which the strict-dominance solution did not apply, like the Cournot duopoly, we obtained a unique prediction from IESDS and rationalizability. However, when we consider a game like the Battle of the Sexes, none of these concepts had any bite. Dominant strategy equilibrium did not apply, and both IESDS and rationalizability could not restrict the set of reasonable behavior.

For example, we cannot rule out the possibility that Alex goes to the opera while Chris goes to the football game, because Alex may behave optimally given his belief that Chris is going to the opera, and Chris may behave optimally given his belief that Alex is going to the football game. Yet there is something troubling about this outcome. If we think of this pair of actions not only as actions, but as a system of actions and beliefs, then there is something of a dissonance: indeed the players are playing best responses to their beliefs, but their beliefs are wrong! In this chapter we make a rather heroic leap that ties together beliefs and actions and results in the most central and best-known solution concept in game theory.

As already mentioned, for dominant strategy equilibrium we required only that players be rational, while for IESDS and rationalizability we required common knowledge of rationality. Now we introduce a much more demanding concept, that of the Nash equilibrium, first put forth by John Nash, who received the Nobel Prize in Economics for this achievement.

3.1. Nash Equilibrium in Pure Strategies

A Nash equilibrium is a system of beliefs and a profile of actions for which each player is playing a best response to his beliefs and, moreover, players have correct beliefs. Another common way of defining a Nash equilibrium, which does not refer to beliefs, is as a profile of strategies for which each player is choosing a best response to the strategies of all other players. Formally we have:

Definition 20 *The pure-strategy profile $s^* = (s_1^*, s_2^*, ..., s_n^*) \in S$ is a Nash equilibrium if s_i^* is a best response to s_{-i}^*, for all $i \in N$, that is,*

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i', s_{-i}^*) \text{ for all } s_i' \in S_i \text{ and all } i \in N.$$

There is a simple relationship between the concepts we previously explored and that of Nash equilibrium, as the following proposition clearly states:

Proposition 21 *Consider a strategy profile $s^* = (s_1^*, s_2^*, ..., s_n^*)$. If s^* is either*

1. *a strict dominant strategy equilibrium,*
2. *the unique survivor of IESDS, or*
3. *the unique rationalizable strategy profile,*

then s^* is the unique Nash equilibrium.

Let us emphasize the requirements for a Nash equilibrium:

1. Each player is playing a best response to his beliefs.
2. The beliefs of the players about their opponents are correct.

The first requirement is a direct consequence of rationality. It is the second requirement that is very demanding and is a tremendous leap beyond the requirements we have considered so far. It is one thing to ask people to behave rationally given their beliefs (play a best response), but it is a totally different thing to ask players to predict the behavior of their opponents correctly.

3.1.1. Evaluating the Nash Equilibria Solution

Considering our criteria for evaluating solution concepts, we can see from the Battle of the Sexes example that we may not have a unique Nash equilibrium. However, as alluded to in our earlier discussion of the Battle of the Sexes game, there is no reason to expect that we should. Indeed we may need to entertain other aspects of an environment in which players interact, such as social norms and historical beliefs, to make precise predictions about which of the possible Nash equilibria may result as the more likely outcome.

It turns out that for quite general conditions games will have at least one Nash equilibrium. Section 4.4 discusses some conditions that guarantee the existence of a Nash equilibrium, which was a central part of Nash's Ph.D. dissertation. This fact gives the Nash solution concept its power–like IESDS and rationalizability, the solution concept of Nash is widely applicable. It will, however, usually lead to more refined predictions than those of IESDS and rationalizability, as implied by proposition 21.

As with the previous solution concepts, we can easily see that Nash equilibrium does not guarantee Pareto optimality. The theme should be obvious by now: left to their own devices, people in many situations will do what is best for them, at the expense of social efficiency. This point was made quite convincingly and intuitively in Hardin's (1968) "tragedy of the commons" argument, which we explore in Section 3.2.4. This is where our focus on self-enforcing outcomes has its bite: our solution concepts took the game as given, and they imposed rationality and common knowledge of rationality to try to see what players would choose to do. If they each seek to maximize their individual well-being then the players may hinder their ability to achieve socially optimal outcomes.

3.2. Nash Equilibrium: Some Classic Applications

The previous section introduced the central pillar of modern noncooperative game theory, the Nash equilibrium solution concept. It has been applied widely in economics, political science, legal studies, and even biology. In what follows we demonstrate some of the best-known applications of the concept.

3.2.1. Two Kinds of Societies

The French philosopher Jean-Jacques Rousseau presented the following situation that describes a trade-off between playing it safe and relying on others to achieve a larger gain. Two hunters, players 1 and 2, can each choose to hunt a stag (S), which provides a rather large and tasty meal, or hunt a hare (H)–also tasty, but much less filling. Hunting stags is challenging and requires mutual cooperation. If either hunts a stag alone, the chance of success is negligible, while hunting hares is an individualistic enterprise that is not done in pairs. Hence hunting stags is most beneficial for society but requires "trust" between the hunters in that each believes that the other is joining forces with him. The game, often referred to as the Stag Hunt game, can be described by the following matrix:

		Player 2	
		S	H
Player 1	S	$5,5$	$0,3$
	H	$3,0$	$3,3$

Figure 4.: The Stag Hunt Game

It is easy to see that the game has two pure-strategy equilibria: (S, S) and (H, H). However, the payoff from (S, S) Pareto dominates that from (H, H). Why then would (H, H) ever be a reasonable prediction? This is precisely the strength of the Nash equilibrium concept. If each player anticipates that the other will not join forces, then he knows that going out to hunt the stag alone is not likely to be a successful enterprise and that going after the hare will be better. This belief would result in a society of individualists who do not cooperate to achieve a better outcome. In contrast, if the players expect each other to be cooperative in going after the stag, then this anticipation is self-fulfilling and results in what can be considered a cooperative society. In the real world, societies that may look very similar in their endowments, access to technology, and physical environments have very different achievements, all because of self-fulfilling beliefs or, as they are often called, norms of behavior.

3.2.2. Cournot Duopoly

The inverse demand function is $p = 1 - q_1 - q_2$ and the cost is zero. Firms sell homogeneous products. The profit functions are $\pi_1 = pq_1$ and $\pi_2 = pq_2$. We differentiate π_1 with respect to q_1 and π_2 with respect to q_2 to derive the best response functions: $q_1 = \frac{1-q_2}{2}$ and $q_2 = \frac{1-q_1}{2}$. We solve the best responses simultaneously to derive the Nash equilibrium.

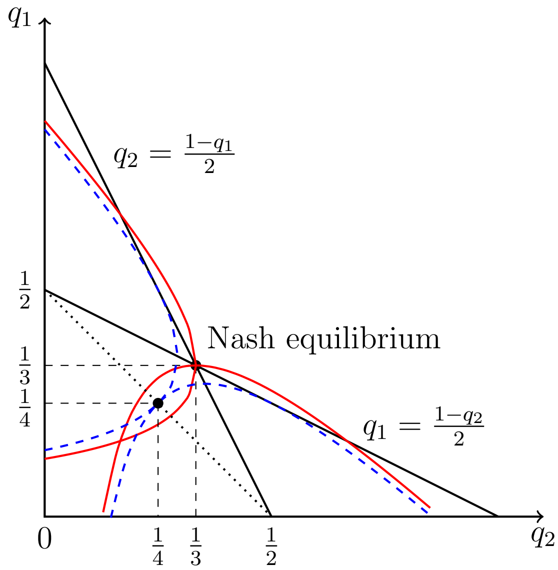

Figure 5.: Cournot equilibrium

Figure 5 depicts the Nash equilibrium, $q_1 = q_2 = \frac{1}{3}$. The solid (red) iso-profits depict the (q_1, q_2) locus where each firm's profits are fixed at the Nash equilibrium profits $\frac{1}{9}$, that is $(1 - q_1 - q_2)q_1 = \frac{1}{9}$ and $(1 - q_1 - q_2)q_2 = \frac{1}{9}$. Since the isoprofits are not tangent the outcome is not Pareto optimal (from the firms' perspective). If firms collude they maximize joint profits given by

$$\Pi = \pi_1 + \pi_2 = (1 - q_1 - q_2)q_1 + (1 - q_1 - q_2)q_2 = (1 - Q)Q,$$

where $Q = q_1 + q_2$. The joint profit maximizing quantity is $Q = \frac{1}{2}$. When each firm produces half of the monopoly output, $q_1 = q_2 = \frac{1}{4}$, the outcome is Pareto optimal. The dashed (blue) isoprofits in this case are tangent, and so there is no other allocation of outputs that can make one firm better off without hurting the other. Actually, any point on the black (dotted) line, $q_1 + q_2 = \frac{1}{2}$, is Pareto optimal since it yields the monopoly profits (which may not be distributed equally between the two firms).

The intuition behind why the Nash equilibrium is not Pareto optimal and why firms overproduce at the Nash equilibrium is as follows. Each firm when it decides whether it should increase its output exerts a negative externality on the other firm, because the market price p goes down. This negative externality is not internalized by either firm and so they overproduce in equilibrium. A cartel, which maximizes joint profits, does internalize this externality.

The first order condition of the joint profits with respect to q_1 is

$$\frac{\partial \Pi}{\partial q_1} = 1 - 2q_1 - q_2 - \underbrace{q_2}_{\text{negative externality}} = 0.$$

When a firm cares only about its own profits, as in the Nash equilibrium, the last term in the above equation is ignored.

3.2.3. Bertrand Duopoly

The Cournot model assumed that the firms choose quantities and the market price adjusts to clear the demand. However, one can argue that firms often set prices and let consumers choose from where they will purchase, rather than setting quantities and waiting for the market price to equilibrate demand. We now consider the game in which each firm posts a price for otherwise identical goods. This was the situation modeled and analyzed by Joseph Bertrand (1883).

As before the inverse demand is given by $p = 1 - Q$, so $Q = 1 - p$, where $p = \min\{p_1, p_2\}$. Costs are zero. Firms compete in prices. Since firms are selling homogeneous products, all consumers buy from the cheaper firm. If firms set the same price they split the demand evenly. The profit function of firm i is (j is the rival firm)

$$\pi_i(p_i, p_j) = \begin{cases} (1-p_i)p_i, & \text{if } p_i < p_j \\ \frac{(1-p_i)p_i}{2}, & \text{if } p_i = p_j \\ 0, & \text{if } p_i > p_j. \end{cases} \tag{3.2.1}$$

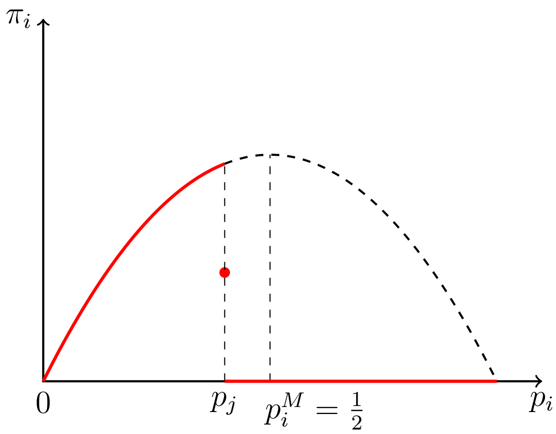

Figure 6.: Bertrand profit function (solid, red)

Figure 6 depicts (3.2.1). Note that for the Bertrand model the profit function is discontinuous at p_j.

What is the Nash equilibrium for this game? Any $p_i = p_j > 0$ (0 is the marginal cost) cannot be an equilibrium. A firm can lower its price by $\varepsilon > 0$ and capture the entire market. Any $1 \geq p_i > p_j > 0$ cannot be an equilibrium either. Firm i will set $p_i = p_j - \varepsilon$ and make positive profits. Finally, Any $1 \geq p_i > p_j = 0$ cannot be an equilibrium, because firm j will raise its price slightly and become better off.

We are left with $p_i = p_j = 0$, which is the Nash equilibrium. Firms earn zero profits but they have no incentive to unilaterally deviate, that is, no unilateral deviation will give a firm strictly positive profits.

Remark It is worth pointing out an interesting difference between the Cournot game and the Bertrand game. In the Cournot game the best-response function of each player is downward sloping. That is, the more player j produces, the lower is the best-response quantity of player i. In the Bertrand game, however, for prices between marginal costs (equal to 0 in the example) and the monopoly price (equal to $\frac{1}{2}$), the higher the price set by player j, the higher is the best-response price of player i. These differences have received some attention in the literature. Games for which the best response of one player decreases in the choice of the other, like the Cournot game, are called games with **strategic substitutes**. In contrast, games for which the best response of one player increases in the choice of the other, like the Bertrand game, are called games with **strategic complements**.

3.2.3.1. Comparison Between the Cournot and the Bertrand Models

The demand and cost functions in the two models are the same. So, why are the equilibrium outcomes so different? Recall that the equilibrium prices in the Cournot model are above marginal cost while in the Bertrand model they are equal to marginal cost. Consequently, profits are positive in the Cournot model and zero in the Bertrand model.

The implied equilibrium price in the Cournot model is $\frac{1}{3}$. Why can't firm i offer to sell its product at a price slightly lower than $\frac{1}{3}$ and raise its profits, as it would do in the Bertrand model. The residual demand of firm i is $q_i = 1 - p_i - q_j$. Firm i's profits using its residual demand are: $\pi_i = p_i(1 - p_i - q_j)$. When $q_j = \frac{1}{3}$, as it is the case in the Cournot equilibriuim, the profit maximizing price is $p_i = \frac{1}{3}$. So, a Cournot firm has no incentive to undercut the rival's price when $p_i = p_j = \frac{1}{3}$. In the Cournot model, when a firm deviates by lowering its price or raising its quantity, assumes that the rival will do whatever it takes to sell the quantity it has already produced, or has made irreversible plans to produce. Therefore, it takes $q_j = \frac{1}{3}$ as given.

In the Bertrand model, on the other hand, the residual demand of firm i is $q_i = 1 - p_i$, if $p_i < p_j$, $q_i = (1 - p_i)/2$, if $p_i = p_j$ and $q_i = 0$ if $p_i > p_j$. When firm i deviates by lowering its price it steals all the demand from the rival firm. For example, when $p_i = p_j = \frac{1}{3}$, the firms sell $q_i = q_j = \frac{1}{3}$. But if firm i set $p_i = \frac{1}{3} - \varepsilon$, q_j goes to zero. Thus, unilateral deviations are very profitable, but this implies low equilibrium prices.

The two models apply to different markets. The Cournot model is a better fit for markets where production is physical, production plans take time and are irreversible and it is costly to hold inventories, e.g., automobile market. The Bertrand model is a better fit for markets where production can adjust very quickly, e.g., digital products.

3.2.4. The Tragedy of the Commons

The tragedy of the commons refers to the conflict over scarce resources that results from the tension between individual selfish interests and the common good; the concept was popularized by Hardin (1968). The central idea has proven useful for understanding how we have come to be on the brink of several environmental catastrophes.

Hardin introduces the hypothetical example of a pasture shared by local herders. Each herder wants to maximize his yield, increasing his herd size whenever possible. Each additional animal has a positive effect for its herder, but the cost of that extra animal, namely degradation of the overall quality of the pasture, is shared by all the other herders. As a consequence the individual incentive for each herder is to grow his own herd, and in the end this scenario causes tremendous losses for everyone. To those trained in economics, it is yet another example of the distortion that results from the "free-rider" problem. It should also remind you of the Prisoner's Dilemma, in which individuals driven by selfish incentives cause pain to the group.

Imagine that there are n players, say firms, in the world, each choosing how much to produce. Their production activity in turn consumes some of the clean air that surrounds our planet. There is a total amount of clean air equal to K, and any consumption of clean air comes out of this common resource. Each player i chooses his own consumption of clean air for production, $k_i \geq 0$, and the amount of clean air left is therefore $K - \sum_{i=1}^{n} k_i$. The benefit of consuming an amount k_i gives player i a benefit equal to $\ln(k_i)$, and no other player benefits from i's choice. Each player also enjoys consuming the remainder of the clean air, giving each a benefit $\ln(K - \sum_{i=1}^{n} k_i)$. Hence the payoff for player i from the choice

$k = (k_1, k_2, ..., k_n)$ is equal to

$$v_i(k_i, k_{-i}) = \ln(k_i) + \ln\left(K - \sum_{i=1}^{n} k_i\right). \tag{3.2.2}$$

This means that if we derive all n best-response correspondences, and it turns out that they are functions (unique best responses), then we have a system of n equations, one for each player's best-response function, with n unknowns, the choices of each player. Solving this system will yield a Nash equilibrium. To get player i's best-response function (and we will verify that it is a function), we write down the first-order condition of his payoff function:

$$\frac{\partial v_i(k_i, k_{-i})}{\partial k_i} = \frac{1}{k_i} - \frac{1}{K - \sum_{j=1}^{n} k_j} = 0,$$

and this gives us player i's best response function

$$BR_i(k_{-i}) = \frac{K - \sum_{j \neq i} k_j}{2}.$$

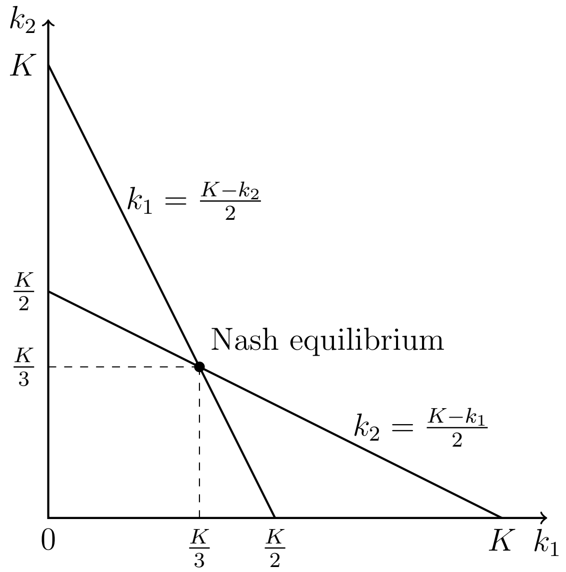

Figure 7.: Best-response functions: two-player tragedy of the commons

Suppose there are two players, $n = 2$. If we solve the two best-response functions simultaneously, we find the unique Nash equilibrium, which has both players playing $k_1 = k_2 = \frac{K}{3}$, as shown in Figure 7.

Is the Nash equilibrium Pareto optimal? We can answer this question by maximizing the sum of the players' payoffs $v_1 + v_2$. We differentiate the sum with respect to k_1 and k_2 and we solve the system to obtain $k_1 = k_2 = \frac{K}{4}$. Clearly, the Nash equilibrium is not Pareto optimal.

Thus, as Hardin puts it, giving people the freedom to make choices may make them all worse off than if those choices were somehow regulated. Of course the counterargument is whether we can trust a regulator to keep things under control; if not, the question remains which is the better of the two evils–an answer that game theory cannot offer!

3.2.5. Political Ideology and Electoral Competition

Given a population of citizens who vote for political candidates, how should candidates position themselves along the political spectrum? One view of the world is that a politician cares only about representing his true beliefs, and that drives the campaign. Another more cynical view is that politicians care only about getting elected and hence will choose a platform that maximizes their chances. This is precisely the view taken in the seminal model introduced by Hotelling (1929).

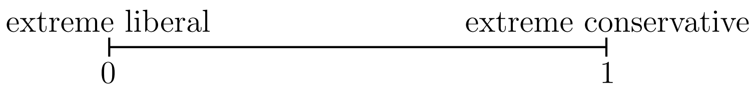

Figure 8.: The ideology spectrum

There are two candidates and voters are distributed across the ideology spectrum $[0, 1]$, see Figure 8, according to the distribution function $F(x)$. Candidates, simultaneously and independently, select their policy locations s_i and s_j. Then voting takes place. Each citizen votes for the candidate who is located closer to the citizen's ideal point. The winner is the candidate with the majority of votes, that is greater than 50%.

What is the Nash equilibrium? $0 \leq s_i < s_j \leq 1$ cannot be an equilibrium. One of the two candidates receives less than 50% of the votes and by moving closer to the other candidate will increase his votes and win. Hence, we are left with $s_i = s_j = x$. Can x be anywhere in $[0, 1]$? The answer is no. If $F(x) \neq 50\%$, then one of the two candidates will move his position and win for sure. Thus, we are left with $F(x) = 50\%$.

This simple example is related to a powerful result known as the **median voter theorem**. It states that if voters are different from one another along a single-dimensional "preference" line, as in Hotelling's model, and if each prefers his own political location, with other platforms being less and less attractive the farther away they fall to either side of that location, then the political platform located at the median voter will defeat any other platform in a simple majority vote. The theorem was first articulated by Black (1948), and it received prominence in Downs's famous 1957 book. Nevertheless one can see how the seed of the idea had been planted as far back as Hotelling's formalization of spatial competition.

3.3. Summary

- Any strategy profile for which players are playing mutual best responses is a Nash equilibrium, making this equilibrium concept self-enforcing.
- If a profile of strategies is the unique survivor of IESDS or is the unique rationalizable profile of strategies then it is a Nash equilibrium.
- If a profile of strategies is a Nash equilibrium then it must survive IESDS and it must be rationalizable, but not every strategy that survives IESDS or that is rationalizable is a Nash equilibrium.
- Nash equilibrium analysis can shed light on phenomena such as the tragedy of the commons and the nature of competition in markets and in politics.

4. Mixed Strategies

Consider the following classic zero-sum game called Matching Pennies.[1] Players 1 and 2 each put a penny on a table simultaneously. If the two pennies come up the same side (heads or tails) then player 1 gets both; otherwise player 2 does. We can represent this in the following matrix:

		Player 2	
		H	T
Player 1	H	$1, -1$	$-1, 1$
	T	$-1, 1$	$1, -1$

Figure 9.: Matching pennies game

It is easy to verify that the above game does not have a Nash equilibrium in pure strategies (the same is true for the Rock-paper-scissors game, Figure 2). Does this mean that a Nash equilibrium fails to exist? We will soon see that a Nash equilibrium will indeed exist if we allow players to choose random strategies, and there will be an intuitive appeal to the proposed equilibrium.

4.1. Strategies, Beliefs, and Expected Payoffs

We now introduce the possibility that players choose stochastic strategies, such as flipping a coin or rolling a die to determine what they will choose to do. This approach will turn out to offer us several important advances over that followed so far. Aside from giving the

[1] A zero-sum game is one in which the gains of one player are the losses of another, hence their payoffs always sum to zero. The class of zero-sum games was the main subject of analysis before Nash introduced his solution concept in the 1950s. These games have some very nice mathematical properties and were a central object of analysis in von Neumann and Morgenstern's (1944) seminal book.

players a richer set of actions from which to choose, it will more importantly give them a richer set of possible beliefs that capture an uncertain world. If player i can believe that his opponents are choosing stochastic strategies, then this puts player i in the same kind of situation as a decision maker who faces a decision problem with probabilistic uncertainty.

4.1.1. Finite Strategy Sets

We start with the basic definition of random play when players have finite strategy sets S_i:

Definition 22 *Let $S_i = \{s_{i1}, s_{i2}, ..., s_{im}\}$ be player i's finite set of pure strategies. Define ΔS_i as the simplex of S_i, which is the set of all probability distributions over S_i. A mixed strategy for player i is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = \{\sigma_i(s_{i1}), \sigma_i(s_{i2}), ..., \sigma_i(s_{im})\}$ is a probability distribution over S_i, where $\sigma_i(s_i)$ is the probability that player i plays s_{ij}, $j = 1, ..., m$.*

Recall that any probability distribution $\sigma_i(\cdot)$ over a finite set of elements (a finite state space), in our case S_i, must satisfy two conditions:

1. $\sigma_i(s_i) \geq 0$ for all $s_i \in S_i$ and
2. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.

In the matching pennies game: $S_i = \{H, T\}$ and the simplex is

$$\Delta S_i = \{(\sigma_i(H), \sigma_i(T)) : \sigma_i(H) \geq 0, \sigma_i(T) \geq 0, \sigma_i(H) + \sigma_i(T) = 1\}.$$

In Figure 10 we depict the simplex for the matching pennies game.[2]

It will be useful to distinguish between pure strategies that are chosen with a positive probability and those that are not. We offer the following definition:

Definition 23 *Given a mixed strategy $\sigma_i(\cdot)$ for player i, we will say that a pure strategy $s_i \in S_i$ is in the support of $\sigma_i(\cdot)$ if and only if it occurs with positive probability, that is, $\sigma_i(s_i) > 0$.*

[2]In general the simplex of a strategy set with m pure strategies will be in an $(m-1)$-dimensional space, where each of the $m-1$ numbers is in $[0, 1]$, and will represent the probability of the first $m-1$ pure strategies. All sum to a number equal to or less than one so that the remainder is the probability of the mth pure strategy.

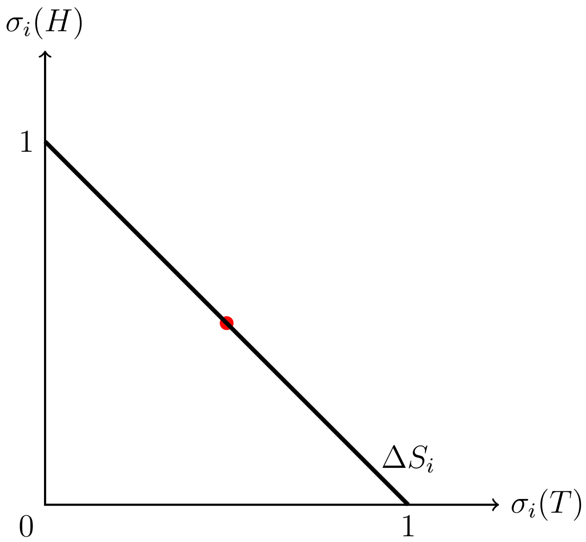

Figure 10.: Any point on the thick black line, like the red bullet, is a specific mixed strategy.

4.1.2. Continuous Strategy Sets

As we have seen with the Cournot and Bertrand duopoly examples, or the tragedy of the commons example, the pure-strategy sets that players have need not be finite. In the case in which the pure-strategy sets are well-defined intervals, a mixed strategy will be given by a cumulative distribution function:

Definition 24 *Let S_i be player i's pure-strategy set and assume that S_i is an interval. A mixed strategy for player i is a cumulative distribution function $F_i : S_i \to [0,1]$, where $F_i(x) = \Pr\{s_i \leq x\}$. If $F_i(\cdot)$ is differentiable with density $f_i(\cdot)$, then we say that $s_i \in S_i$ is in the support of $F_i(\cdot)$ if $f_i(s_i) > 0$.*

4.1.3. Beliefs and Mixed Strategies

As we discussed earlier, introducing probability distributions not only enriches the set of actions from which a player can choose but also allows us to enrich the beliefs that players can have. Consider, for example, player i, who plays against opponents $-i$. It may be that player i is uncertain about the behavior of his opponents for many reasons. For example, he may believe that his opponents are indeed choosing mixed strategies, which immediately implies that their behavior is not fixed but rather random. An alternative interpretation is the situation in which player i is playing a game against an opponent that he does not know, whose background will determine how he will play. This interpretation will be revisited later and it is a very appealing justification for beliefs that are random and behavior that is consistent with these beliefs. To introduce beliefs about mixed strategies formally we define them as follows:

Definition 25 *A belief for player i is given by a probability distribution $\pi_i \in \Delta S_{-i}$ over the strategies of his opponents. We denote by $\pi_i(s_{-i})$ the probability player i assigns to his opponents playing $s_{-i} \in S_{-i}$.*

4.1.4. Expected Payoffs

Consider the Matching Pennies game, and assume for the moment that player 2 chooses the mixed strategy $\sigma_2(H) = \frac{1}{3}$ and $\sigma_2(T) = \frac{2}{3}$. If player 1 plays H then he will win and get 1 with probability $\frac{1}{3}$ while he will lose and get -1 with probability $\frac{2}{3}$. Then, player 1's expected payoff from choosing H is $1 \times \frac{1}{3} - 1 \times \frac{2}{3} = -\frac{1}{3}$. Thus we define the expected payoff of a player as follows:

Definition 26 *The expected payoff of player i where he chooses the pure strategy $s_i \in S_i$ and his opponents play the mixed strategy $\sigma_i \in \Delta S_{-i}$ is*

$$v_i(s_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \sigma_{-i}(s_{-i}) v_i(s_i, s_{-i}).$$

4.2. Mixed-Strategy Nash Equilibrium

Now that we are equipped with a richer space for both strategies and beliefs, we are ready to restate the definition of a Nash equilibrium for this more general setup as follows:

Definition 27 *The mixed-strategy profile $\sigma^* = (\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ is a Nash equilibrium if for each player σ_i^* is a best response to σ_{-i}^*.*

This definition is the natural generalization of definition 20. We require that each player be choosing a strategy $\sigma_i^* \in \Delta S_i$ that is (one of) the best choice(s) he can make when his opponents are choosing some profile $\sigma_{-i}^* \in \Delta S_{-i}$.

Proposition 28 *If σ^* is a Nash equilibrium, and both s_i and s_i' are in the support of σ_i^*, then*

$$v_i(s_i, \sigma_{-i}^*) = v_i(s_i', \sigma_{-i}^*) = v_i(\sigma_i^*, \sigma_{-i}^*).$$

The above proposition says that in a mixed strategy equilibrium each player i is indifferent among his pure strategies in the support of σ_i^*.

4.2.1. Example: Matching Pennies

Let q and p be the probabilities with which players 1 and 2 plays H, respectively. The expected payoff of player 1 if he plays H is (where $\sigma_2 = \{p, 1-p\}$)

$$v_1(H, \sigma_2) = p - (1-p) = 2p - 1,$$

while his expected payoff if he plays T

$$v_1(T, \sigma_2) = -p + (1-p) = 1 - 2p,$$

Player 1 is indifferent between his two strategies if and only if $p = \frac{1}{2}$. So, player 2 mixes between H and T to keep player 1 indifferent between his two strategies. Similarly, player 1 will choose $q = \frac{1}{2}$ to keep player 2 indifferent. Therefore, the mixed strategy Nash equilibrium is

$$\sigma^* = \left\{ \left\{ \frac{1}{2}, \frac{1}{2} \right\}, \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right\}.$$

Verify that no player has an incentive to deviate unilaterally. Given σ_j^* player i's equilibrium expected payoff from σ_i^* is 0 which is equal to his payoff from playing either H or T. Given how player j mixes between his two pure strategies player i cannot increase his expected payoff from 0 by deviating to any other than the equilibrium mixed or pure strategy.

4.2.2. Example: Price Competition with Loyal Customers and Switchers

Two firms 1 and 2 produce homogeneous products and compete in prices, see Varian (1981) for a more complete description of this model. There is a continuum of consumers of total mass 1. Each consumer buys at most one unit of the good and has a reservation price of one. Consumers can be divided into two groups: loyal and comparison shoppers. The size of the loyal consumers to each firm is α and the size of the comparison shoppers is $1-2\alpha$. Loyal consumers buy from their favorite firm as long as the price is below one. Comparison shoppers compare prices and buy from the cheaper firm. The marginal cost of each firm is zero.

First, observe that a pure strategy Nash equilibrium does not exist. To see this suppose $p_1 = p_2 > 0$. Then, a firm has an incentive to undercut its rival in order to attract all the comparison shoppers. If $p_1 = p_2 = 0$, then profits are zero and a firm has an incentive to raise its price and earn revenue from its loyal customers and so on.

Let's look for a mixed strategy Nash equilibrium. We will search for a symmetric equilibrium where each firm chooses prices from a distribution $F(p)$ on the interval $\left[\underline{p}, \overline{p}\right]$.

The expected profits of, say, firm 1 when it chooses price p are

$$\pi(p) = \underbrace{(1-F(p))(1-\alpha)p}_{\text{The rival's price is higher}} + \underbrace{F(p)\alpha p}_{\text{The rival's price is lower}}.$$

In a mixed strategy equilibrium, firm 2 chooses prices from $F(p)$ to keep firm 1 indifferent among all prices in the support of the distribution. Therefore, $\pi(p)$ must be constant, i.e., $\pi(p) = k$

$$\begin{aligned}\pi(p) &= (1-F(p))(1-\alpha)p + F(p)\alpha p = k \\ \Rightarrow\ F(p) &= \frac{k - p(-\alpha+1)}{p\alpha - p(-\alpha+1)}.\end{aligned}$$

Now observe that the highest price in the support $\overline{p}$ must be 1. If not, then a firm's profit will be increasing in the interval $[\overline{p}, 1]$, because it will be the high priced firm with certainty selling only to its loyal customers. But then this is not an equilibrium.

If $\overline{p} = 1$, then $k = \alpha$. A firm can guarantee for itself a profit equal to α by setting $p = 1$ and selling only to its loyal consumers. Then, the distribution becomes

$$F(p) = \frac{\alpha - p(-\alpha + 1)}{p\alpha - p(-\alpha + 1)}.$$

Note that $F(1) = 1$, as it should be. Also, it must be that

$$F(\underline{p}) = 0 \Rightarrow \underline{p} = \frac{\alpha}{1 - \alpha}.$$

So, the mixed strategy Nash equilibrium is

$$F_1(p) = F_2(p) = \frac{\alpha - p(-\alpha + 1)}{p\alpha - p(-\alpha + 1)} \text{ on } \left[\frac{\alpha}{1 - \alpha}, 1\right].$$

No firm has a strict incentive to unilaterally deviate from $F(p)$. If firms follow the equilibrium, the expected profit of each firm is α. If a firm deviates to a price higher than 1 its profit will be zero, while if it deviates to a price less than $\alpha/(1-\alpha)$ its profit will be less than α. Finally, by construction, its profit will be α if it deviates to a different distribution than $F(p)$ on the interval $[\alpha/(1-\alpha), 1]$ and hence it has no strict incentive to do so.

The density is

$$f(p) = \frac{dF(p)}{dp} = \frac{\alpha}{(1 - 2\alpha)p^2}.$$

The density is decreasing in price, implying that low prices are more likely than high prices.

4.3. IESDS and Rationalizability Revisited

By introducing mixed strategies we offered two advancements: players can have richer beliefs, and players can choose a richer set of actions. This can be useful when we reconsider the concepts of IESDS and rationalizability, and in fact present them in their precise form using mixed strategies. In particular we can now state the following two definitions:

Definition 29 *Let $\sigma_i \in \Delta S_i$ and $s_i' \in S_i$ be possible strategies for player i. We say that s_i' is strictly dominated by σ_i if*

$$v_i(\sigma_i, s_{-i}) > v_i(s_i', s_{-i}) \quad \forall s_{-i} \in S_{-i}.$$

Definition 30 *A strategy $\sigma_i \in \Delta S_i$ is never a best response if there are no beliefs $\sigma_{-i} \in \Delta S_{-i}$ for player i for which $\sigma_i \in BR_i(\sigma_{-i})$.*

That is, to consider a strategy as strictly dominated, we no longer require that some other pure strategy dominate it, but allow for mixed strategies to dominate it as well. The same is true for strategies that are never a best response. It turns out that this approach allows both concepts to have more bite. For example, consider the following game:

Player 1 \ Player 2	L	C	R
U	$5,1$	$1,4$	$1,0$
M	$3,2$	$0,0$	$3,5$
D	$4,3$	$4,4$	$0,3$

Starting with IESDS, it is easy to see that no pure strategy is strictly dominated by another pure strategy for any player. Hence if we restrict attention to pure strategies then IESDS has no bite and suggests that anything can happen in this game. However, if we allow for mixed strategies, we can find that the strategy L for player 2 is strictly dominated by a strategy that mixes between the pure strategies C and R. That is, $(\sigma_2(L), \sigma_2(C), \sigma_2(R)) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ strictly dominates choosing L for sure because this mixed strategy gives player 2 an expected payoff of 2 if player 1 chooses U, of 2.5 if player 1 chooses M, and of 3.5 if player 1 chooses D.

Once L is removed for player 2, strategy U for player 1 is strictly dominated by a strategy that mixes between the pure strategies M and D. That is $(\sigma_1(U), \sigma_1(M), \sigma_1(D)) = \left(0, \frac{1}{2}, \frac{1}{2}\right)$ strictly dominates U. So the game is reduced to the following 2×2 game

Player 1 \ Player 2	C	R
M	$0,0$	$3,5$
D	$4,4$	$0,3$

which cannot be reduced any further.

Turning to rationalizability, in Section 2.3.3 we introduced the concept that after eliminating all the strategies that are never a best response, and employing this reasoning again and again in a way similar to what we did for IESDS, the strategies that remain are called the set of rationalizable strategies. If we use this concept to analyze the game we just solved with IESDS, the result will be the same. Starting with player 2, there is no belief that he can have for which playing L will be a best response. This is easy to see because either C or R will be a best response to one of player 1's pure strategies, and hence, even if player 1 mixes then the best response of player 2 will either be to play C, to play R, or to mix with both. Then after reducing the game a similar argument will work to eliminate U from player 1's strategy set. As we mentioned briefly in Section 2.3.3, the concepts of IESDS and rationalizability are closely related. To see one obvious relation, the following fact is easy to prove:

Fact If a strategy σ_i is strictly dominated then it is never a best response.

This fact is useful, and it implies that the set of a player's rationalizable strategies is no larger than the set of a player's strategies that survive IESDS. This is true because if a strategy was eliminated using IESDS then it must have been eliminated through the process of rationalizability. Is the reverse true as well?

Proposition 31 *For any two-player game a strategy σ_i is strictly dominated if and only if it is never a best response.*

Hence for two-player games the set of strategies that survive IESDS is the same as the set of strategies that are rationalizable. Proving this is not that simple and is beyond the scope of this text. For games with more than two players the set of rationalizable strategies is smaller than the IESDS, unless we allow for *correlated beliefs*. For example, in a three player game, player i's belief about the strategies of the other two players is a two dimensional probability distribution. It makes a difference whether we assume independence or we allow for correlation. The latter enlarges the set of beliefs and consequently the set of rationalizable strategies.

To better understand this consider the following 3-person simultaneous-move game in which player 1 chooses the row, player 2 chooses the column and player 3 the matrix that will be played. The first number in each cell is the payoff to player 1, the second number is the payoff to player 2, and the third number the payoff to player 3.

1/2	L	R
T	$(-,-,5)$	$(-,-,2)$
B	$(-,-,2)$	$(-,-,1)$

3:A

1/2	L	R
T	$(-,-,4)$	$(-,-,0)$
B	$(-,-,0)$	$(-,-,4)$

3:B

1/2	L	R
T	$(-,-,1)$	$(-,-,2)$
B	$(-,-,2)$	$(-,-,5)$

3:C

We will show that there exists a strategy that is not strictly dominated but it is never a best response if the beliefs are independent. This demonstrates that in games with more than two players and with independent beliefs, the IESDS set is larger than the set of rationalizable strategies. Strategy B for player 3 is not strictly dominated, since a dominating mixture of A and C would need to put at least probability 3/4 on both A (in case 1 and 2 play (T, L)) and C (in case 1 and 2 play (B, R)). If player 3's conjectures about player 1's and 2's choices are independent, B is not a best response. Independence implies that for some $t, l \in [0,1]$, we can write $\mu_3(T,L) = tl$, $\mu_3(T,R) = t(1-l)$, $\mu_3(B,L) = (1-t)l$, and $\mu_3(B,R) = (1-t)(1-l)$. Then,

$$u_3(B,\mu_3) > u_3(C,\mu_3) \Leftrightarrow l > \frac{1+t}{6t-1}.$$

Similarly,

$$u_3(B,\mu_3) > u_3(A,\mu_3) \Leftrightarrow l < \frac{5t-3}{6t-5}.$$

It can be verified that these two inequalities are never satisfied simultaneously, so B is never a best response. But B is a best response to the correlated conjecture $\mu_3(T,L) = \mu_3(B,R) = \frac{1}{2}$.

The eager and interested reader is encouraged to read Chapter 2 of Fudenberg and Tirole (1991), and the daring reader can refer to the original research papers by Bernheim (1984) and Pearce (1984), which simultaneously introduced the concept of rationalizability.

4.4. Nash's Existence Theorem

The Nash solution concept usually leads to more refined predictions than those of IESDS and rationalizability, yet the reverse is never true. In his seminal Ph.D. dissertation, Nash defined the solution concept that now bears his name and showed some very general conditions under which the solution concept will exist. We first state Nash's theorem:

Theorem 32 *Any n-player normal-form game with finite strategy sets S_i for all players has a (Nash) equilibrium in mixed strategies.*

The central idea of Nash's proof builds on what is known in mathematics as a fixed-point theorem. The most basic of these theorems is known as Brouwer's fixed-point theorem:

Theorem 33 (Brouwer's fixed point theorem) *If $f(x)$ is a continuous function from the domain $[0,1]$ to itself then there exists at least one value $x^* \in [0,1]$ for which $f(x^*) = x^*$.*

Figure 11 depicts a continuous function from $[0,1]$ to $[0,1]$ and the fixed point x^*. We need continuity and the domain and range of the function to be compact (closed and bounded) and convex sets. If, for example, the function is discontinuous then there may not exist a fixed point, i.e., when the jump occurs around the 45° line. Also, if the domain and range are not compact then again a fixed point may not exist. For instance, consider $f(x) = \sqrt{x}$ from $(0,1)$ to $(0,1)$. The domain and range are not compact because they are not closed. There does not exist an $x^* \in (0,1)$ such $x^* = f(x^*)$. Finally, consider the domain $\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$ which is not convex and the function $f(x) = \frac{3}{4}$ on $\left[0, \frac{1}{3}\right]$ and $f(x) = \frac{1}{4}$ on $\left[\frac{2}{3}, 1\right]$. The function is continuous, but a fixed point does not exist.

To gain intuition about the existence proof, consider the game in Figure 12.

Player 1 chooses M with probability p and player 2 chooses C with probability q. The player's best response correspondences are

$$BR_1(q) = \begin{cases} p = 1 & \text{if } q < \frac{3}{7} \\ p \in [0,1], & \text{if } q = \frac{3}{7} \\ p = 0, & \text{if } q > \frac{3}{7} \end{cases}$$

and

$$BR_2(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{6} \\ q \in [0,1], & \text{if } p = \frac{1}{6} \\ q = 0, & \text{if } p > \frac{1}{6}. \end{cases}$$

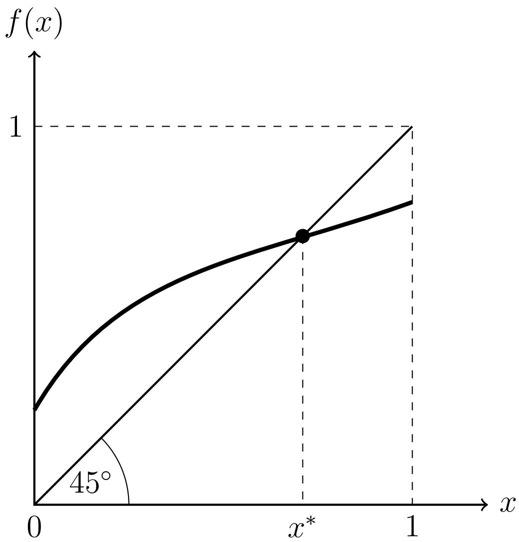

Figure 11.: Brouwer's fixed point theorem

		Player 2	
		C	R
Player 1	M	$0, 0$	$3, 5$
	D	$4, 4$	$0, 3$

Figure 12.

Figure 13 depicts the two best responses and the three equilibria, two in pure strategies and one in mixed. Note that the best response need not be a function, it can be a set, so we have best response correspondences. For example, when $q = 3/7$ the best response of player 1 is any p in $[0, 1]$.

An extension of Brouwer's fixed point theorem to correspondences is the Kakutani's fixed point theorem that we state next.

Theorem 34 (Kakutani's fixed point theorem) *A correspondence $C : X \rightrightarrows X$ has a fixed point $x \in C(x)$ if four conditions are satisfied: (1) X is a non-empty, compact, and convex subset of $\mathbb{R}^n$; (2) $C(x)$ is non-empty for all x; (3) $C(x)$ is convex for all x; and (4) C has a closed graph.*

Take now the collection of the two best response correspondences $BR \equiv BR_1 \times BR_2$ that maps from $[0, 1]^2$ to $[0, 1]^2$. If, for example, you pick a point in $[0, 1]^2$ say $\left(\frac{1}{2}, \frac{1}{2}\right)$, BR will map

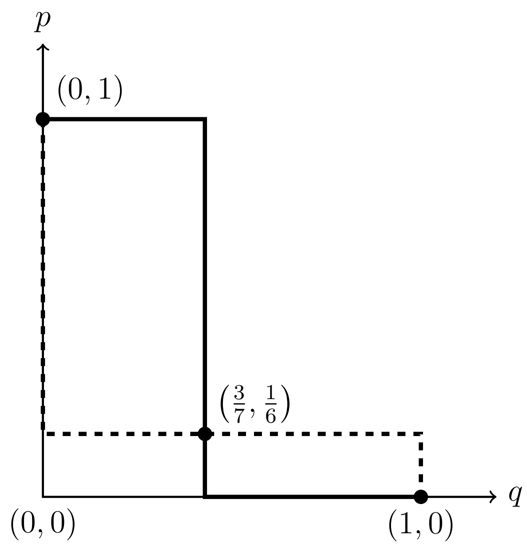

Figure 13.: The solid lines are the best responses of player 1 and the dashed of player 2.

it to $(0,0)$. This is not a fixed point, or in other words $\left(\frac{1}{2}, \frac{1}{2}\right)$ is not a Nash equilibrium. But a Nash equilibrium exists. If you take $\left(\frac{3}{7}, \frac{1}{6}\right)$, the BR will map it to $[0,1]^2$, which contains $\left(\frac{3}{7}, \frac{1}{6}\right)$, so it is a Nash equilibrium.

So in this example $X = [0,1]^2$ which is non-empty, compact and convex subset or $\mathbb{R}^2$ and the best-response correspondences are convex-valued and have a closed graph. Closed graph is almost equivalent to a notion of continuity for correspondences called *upper hemi-continuity*, which follows from the fact that the expected payoffs are continuous in the probabilities. So a small change in q will not result in a 'jump' of v_1 and so the best-response of player 1 is in that sense continuous in q.

4.5. Summary

- Allowing for mixed strategies enriches both what players can choose and what they can believe about the choices of other players.
- In games for which players have opposing interests, like the Matching Pennies game, there will be no pure strategy equilibrium but a mixed-strategy equilibrium will exist.
- Allowing for mixed strategies enhances the power of IESDS and of rationalizability.

- Nash proved that for finite games there will always be at least one Nash equilibrium.

Part II.

DYNAMIC GAMES OF COMPLETE INFORMATION

5. Preliminaries

One of the drawbacks of the normal form representation is its inability to capture games that unfold over time. That is, there is a sense of how players' strategy sets correspond to what they can do, and of how the combination of their actions affects each other's payoffs, but there is no way to represent situations in which the order of moves might be important. As an example, consider the familiar Battle of the Sexes game but with a slight modification. Imagine that Alex finishes work at 3:00 p.m. while Chris finishes work at 5:00 p.m. This gives Alex ample time to get to either the football game or the opera and then to call Chris at 4:45 p.m. and announce "I am here." Chris then has to make a choice of where to go. Where should Chris go? If the choice is to the venue where Alex is waiting then Chris will get some payoff. (It would be 1 if Alex is at the opera and 2 if Alex is at the football game.) If Chris's choice is to go to the other venue, then he will get 0. Hence a rational Chris should go to the same venue that Alex did. Anticipating this, a rational Alex ought to choose the opera, because then Alex gets 2 instead of 1 from football.

As you can see, there is a fundamental difference between this example and the simultaneous-move Battle of the Sexes game. Here Chris makes a move after learning what Alex did, hence the difference is in what Chris knows when Chris makes a move. In other words, in the simultaneous-move Battle of the Sexes game neither player knows what the other is choosing, so each player conjectures a belief and plays a best response to this belief. Here, in contrast, when it is his turn to make a move, Chris knows what Alex has done, and as a result the notion of conjecturing beliefs is moot. Furthermore Alex knows, by common knowledge of rationality, that Chris will choose to follow Alex because it is Chris's best response to do so. As a result Alex can get what Alex wants.

5.1. The Extensive-Form Game

In this section we derive the most common representation for games that unfold over time and in which some players move after they learn the actions of other players. The innovation over the normal-form representation is to allow the knowledge of some players, when it is their turn to move, to depend on the previously made choices of other players. The following elements must be part of any extensive-form game's representation:

1. Set of players, N.
2. Players' payoffs as a function of outcomes, $\{v_i(\cdot)\}_{i \in N}$.
3. Order of moves.
4. Actions of players when they can move.
5. The knowledge that players have when they can move.
6. Probability distributions over exogenous events.
7. The structure of the extensive-form game represented by $1-6$ is common knowledge among all the players.

5.1.1. The Game Tree

The sequential Battle of the Sexes game, in which Alex moves first, is represented in Figure 14. It starts at a node denoted x_0, at which player 1 can choose between O and F. Then, depending on the choice of player 1, player 2 gets to move at either node x_1 or x_2 and makes a choice between o and f. (We are now using lowercase letters to denote the choices of player 2, a convenient way to distinguish between the players.) The sequence of choices will result in one of the outcomes at the bottom of the tree. However, how can we distinguish between the sequential case in which player 2 knows the move of player 1 and the simultaneous case in which player 2 moves after player 1 but is ignorant about player 1's move?

To address this concern formally, and to complete the structure of a game tree, a certain amount of detail and notation needs to be introduced. This is the main objective of this section; later, once the concepts are clear, we will focus our efforts on a variety of examples to master the use of game trees.

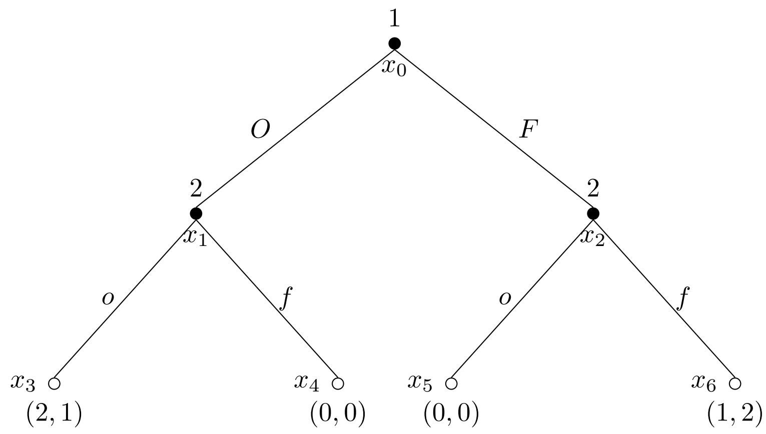

Figure 14.: The sequential-move Battle of the Sexes game

Definition 35 *A game tree is a set of nodes $x \in X$ with a precedence relation $x > x'$, which means "x precedes x'." Every node in a game tree has only one predecessor. The precedence relation is transitive ($x > x'$, $x' > x'' \Rightarrow x > x''$), asymmetric ($x > x' \Rightarrow$ not $x' > x$), and incomplete (not every pair of nodes x, y can be ordered). There is a special node called the root of the tree, denoted by x_0, that precedes any other $x \in X$. Nodes that do not precede other nodes are called terminal nodes, denoted by the set $Z \subset X$. Terminal nodes denote the final outcomes of the game with which payoffs are associated. Every node x that is not a terminal node is assigned either to a player, $i(x)$, with the action set $A_i(x)$, or to Nature.*

Consider the extensive form game given in Figure 14. There are two players $N = \{1, 2\}$. The terminal nodes are $Z = \{x_3, x_4, x_5, x_6\}$, and payoffs are defined over terminal nodes: $v_i : Z \to \mathbb{R}$, where $v_i(z)$ is i's payoff if terminal node $z \in Z$ is reached.

The precedence relation, together with the way in which players are assigned to nodes, describes the way in which the game unfolds. For example, in 14 player 1 is assigned to the root, so $i(x_0) = 1$ and $i(x_1) = i(x_2) = 2$. Player 1's action set at the root, $A_1(x_0)$, includes two choices that determine whether the game will terminate at nodes x_3 or x_4 or at nodes x_5 or x_6. Player 2's action set at x_1 and x_2 also includes two choices. Node x_0 precedes nodes x_1 and x_2.

We proceed to put structure on the information that a player has when it is his turn to move. A player can have very fine information and know exactly where he is in the game tree, or he may have coarser information and not know what has happened before his move, therefore not knowing exactly where he is in the game tree. We introduce the following definition:

Definition 36 *Every player i has a collection of information sets $h_i \in H_i$ that partition the nodes of the game at which player i moves with the following properties:*

1. *If h_i is a singleton that includes only x then player i who moves at x knows that he is at x.*
2. *If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$ then player i who moves at x does not know whether he is at x or x'.*
3. *If $x \neq x'$ and if both $x \in h_i$ and $x' \in h_i$ then $A_i(x') = A_i(x)$.*

Property 3 is essential to maintain the logic of information. If instead $x \in h_i$ and $x' \in h_i$ but $A_i(x) \neq A_i(x')$, then by the mere fact that player i has different actions from which to choose at each of the nodes x and x', he should be able to distinguish between these two nodes. It would therefore be illogical to assume that he cannot distinguish between them.

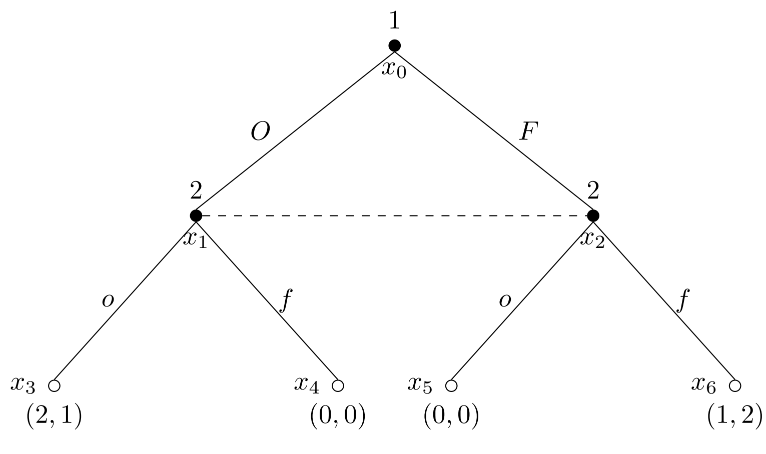

Figure 15.: The simultaneous-move Battle of the Sexes game

Figure 15 depicts the simultaneous move Battle of the Sexes game. Player 2 cannot distinguish between x_1 and x_2, so $h_2 = \{x_1, x_2\}$. The dashed line that connects x_1 and x_2 represents player 2's information set.

5.1.2. Imperfect versus Perfect Information

We defined games of complete information as the situation in which each player i knows the action set and the payoff function of each and every player $j \in N$, and this itself is common knowledge. This definition sufficed for the normal-form representation. For extensive-form games, however, it is useful to distinguish between two different types of complete-information games:

Definition 37 *A game of complete information in which every information set is a singleton and there are no moves of Nature is called a game of perfect information. A game in which some information sets contain several nodes or in which there are moves of Nature is called a game of imperfect information.*

A game will be one of imperfect information when a player must make a move either without knowing the move of another player or without knowing the realization of a choice of Nature. Uncertainty over the choice of Nature, or exogenous uncertainty, is at the heart of the single-person decision problems. Uncertainty over the choice of another player, or endogenous uncertainty, is the subject of simultaneous-move games like the simultaneous-move Battle of the Sexes game. Notice, however, that both situations share a common feature: occurrences that some player does not know are captured by uncertainty over where he is in the game, be it from exogenous or endogenous uncertainty. In either case a player must form beliefs about the unobserved actions, of Nature or of other players, in order to analyze his situation.

5.2. Strategies and Nash Equilibrium

Now that we have the structure of the extensive-form game well defined, and have developed game trees to represent this structure, we move to the next important step of describing strategies. In the normal-form game it was very easy to define a strategy for a player: a pure strategy was some element from his set of actions, and a mixed strategy was some probability distribution over these actions. As we will now see, a strategy is more involved in extensive-form games.

5.2.1. Pure Strategies

Definition 38 *A pure strategy for player i is a complete plan of play that describes which pure action player i will choose at each of his information sets.*

In extensive form games a strategy for player i is a *complete contingent plan*. For example, in the simultaneous-move Battle of the Sexes game described in Figure 15 the strategy sets are $S_1 = \{O, F\}$ and $S_2 = \{o, f\}$. But in the sequential-move Battle of the sexes game described in Figure 14 the strategy sets are $S_1 = \{O, F\}$ and $S_2 = \{oo, of, fo, ff\}$. The information sets of player 2 are the two nodes, x_1 and x_2. At each node player 2 has two actions o and f. So, oo is a strategy where player 2 chooses o regardless of the choice of player 1, while of means that player 2 will play o if player 1 played O and will play f if player 1 player F.

Let H_i be the collection of all information sets at which player i plays, and let $h_i \in H_i$ be one of i's information sets. In the sequential-move Battle of the Sexes game $H_2 = \{\{x_1\}, \{x_2\}\}$, but in the simultaneous-move Battle of the Sexes game $H_2 = \{x_1, x_2\}$. Let $A_i(h_i)$ be the actions that player i can take at h_i, and let A_i be the set of all actions of player i, $A_i = \cup_{h_i \in H_i} A_i(h_i)$. We can now define a pure strategy as follows:

Definition 39 *A pure strategy for player i is a mapping $s_i : H_i \to A_i$ that assigns an action $s_i(h_i) \in A_i(h_i)$ for every information set $h_i \in H_i$. We denote by S_i the set of all pure-strategy mappings $s_i \in S_i$.*

5.2.2. Mixed versus Behavioral Strategies

Now that we have defined pure strategies, the definition of mixed strategies follows immediately, just like in the normal-form game:

Definition 40 *A mixed strategy for player i is a probability distribution over his pure strategies $s_i \in S_i$.*

How do we interpret a mixed strategy? In exactly the same way that we did for the normal form: a player randomly chooses between all his pure strategies–in this case all the complete contingent plans of play–and once a particular plan is selected the player follows it.

This description of mixed strategies was sensible for normal-form games because there it was a once-and-for-all choice to be made. In a game tree, however, the player may want to randomize at some nodes, independently of what he did at earlier nodes. In other words, the player may want to "cross the bridge when he gets there." This cannot be captured by mixed strategies as previously defined because once the randomization is over, the player is choosing a pure plan of action.

To illustrate this point, consider again the sequential-move Battle of the Sexes game. The previous definition of a mixed strategy implies that player 2 can randomize among any of his four pure strategies in the set $S_2 = \{oo, of, fo, ff\}$. It does not, however, allow him to choose strategies of the form "If player 1 plays O then I will play f, while if he plays F then I will mix and play f with probability 1/3."

To allow for strategies that let players randomize as the game unfolds we define a new concept as follows:

Definition 41 *A behavioral strategy specifies for each information set $h_i \in H_i$ an independent probability distribution over $A_i(h_i)$ and is denoted by $\sigma_i : H_i \to \Delta A(h_i)$, where $\sigma_i(a_i(h_i))$ is the probability that player i plays action $a_i(h_i) \in A_i(h_i)$ in information set h_i.*

Arguably a behavioral strategy is more in tune with the dynamic nature of the extensive-form game. When using such a strategy, a player mixes among his actions whenever he is called to play. This differs from a mixed strategy, in which a player mixes before playing the game but then remains loyal to the selected pure strategy.

As it turns out, mixed and behavioral strategies are equivalent under a rather mild condition, that is the game is of perfect recall.

Definition 42 *A game of perfect recall is one in which no player ever forgets information that he previously knew.*

5.2.3. Normal-Form Representation of Extensive-Form Games

The sequential-move Battle of the Sexes game can be represented by a 2×4 matrix as follows

		Player 2			
		oo	of	fo	ff
Player 1	O	$2, 1$	$2, 1$	$0, 0$	$0, 0$
	F	$0, 0$	$1, 2$	$0, 0$	$1, 2$

Figure 16.: Normal-form representation of the sequential Battle of the Sexes game

Any extensive-form game can be transformed into a normal-form game by using the set of pure strategies of the extensive form as the set of pure strategies in the normal form, and the set of payoff functions is derived from how combinations of pure strategies result in the selection of terminal nodes. Furthermore every extensive-form game will have a unique normal form that represents it, which is not true for the reverse transformation (see the following remark).

Clearly this exercise of transforming extensive-form games into the normal form seems to miss the point of capturing the dynamic structure of the extensive-form game. Why then would we be interested in this exercise? The reason is that the concept of a Nash equilibrium is static in nature, in that the equilibrium posits that players take the strategies of others as given, and in turn they play a best response. Therefore the normal-form representation of an extensive form will suffice to find all the Nash equilibria of the game. This is particularly useful if the extensive form is a two-player game with a finite number of strategies for each player, because we can write its normal form as a matrix and solve it with the simple techniques developed earlier. As we will now see, this approach has some useful implications.

Remark Though every extensive form has a unique normal-form representation, it is not true that every normal form can be represented by a unique extensive-form game.

Consider the following matrix for a normal-form game and notice that it is a consistent representation of either of the two game trees depicted in Figure 17. Notice that the extensive form on the left is a game of perfect information while the game on the right is one of imperfect information. However, in terms of their "strategic" character, the two extensive-form games are identical, because a game's strategic essence is captured by its normal form.

		Player 2	
		C	D
Player 1	N	$0, 0$	$0, 0$
	E	$1, 1$	$-1, 2$

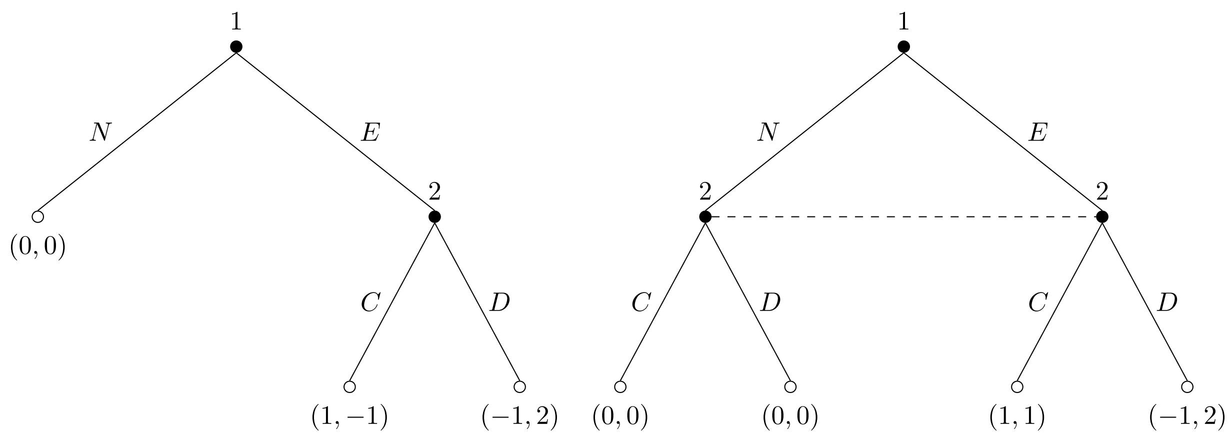

Figure 17.: Two extensive forms with the same normal form

5.3. Nash Equilibrium and Paths of Play

In the sequential-move Battle of the Sexes example there are three Nash equilibria (O, of), (O, oo) and (F, ff), the first two result in the exact same outcome of both players going to the opera, see Figure 16. Is there an important difference between these two predictions? Indeed there is–the difference between these two equilibria is not in what the players actually play in equilibrium, but instead what player 2 plans to play in an information set that is not reached in equilibrium.

Definition 43 *Let $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ be a Nash equilibrium profile of behavioral strategies in an extensive-form game. We say that an information set is on the equilibrium path if given*

σ^* it is reached with positive probability. We say that an information set is off the equilibrium path if given σ^* it is never reached.

Using this definition we can reinterpret the forces that make a Nash equilibrium prediction self-enforcing. In a Nash equilibrium players choose to proceed on the equilibrium path because of their beliefs about what the other players are doing both on and off the equilibrium path. The Nash equilibrium (F, ff) of the sequential-move Battle of the Sexes game is supported by player 1's correct belief that if he would deviate from the equilibrium path and play O then he would receive 0 because player 2 will proceed to play f in the information set x_1. In other words, the "threat" imposed by player 2's strategy of how he will proceed off the equilibrium path is supporting the actions of player 1 on the equilibrium path.

5.4. Summary

- In addition to the set of players, their possible actions, and their payoffs from outcomes, the extensive-form representation captures the order in which they play as well as what each player knows when it is his turn to move. Game trees provide a useful tool to describe extensive-form games.
- If a player cannot distinguish between two or more nodes in a game tree, then they belong to the same information set. Care has to be taken to specify correctly each player's information sets.
- In games of perfect information, every player knows exactly what transpired before each of his turns to move, so each information set is a singleton. If this is not the case then the players are playing a game of imperfect information.
- A pure strategy defines a player's deterministic plan of action at each of his information sets. A mixed strategy is a probability distribution over pure strategies, while a behavioral strategy is a plan of probability distributions over actions in every information set.
- Every extensive-form game has a unique normal-form representation, but the reverse

is not necessarily true.

6. Credibility and Sequential Rationality

Consider the sequential move Battle of the Sexes game depicted in Figure 14 and its normal form representation in Figure 16. There are three Nash equilibria (O, of), (O, oo) and (F, ff). The equilibria (O, oo) and (F, ff) have player 2 committing to a strategy that, despite being a best response to player 1's strategy, would not have been optimal were player 1 to deviate from his strategy and cause the game to move off the equilibrium path. In what follows we will introduce a natural requirement that will result in more refined predictions for dynamic games. These will indeed rule out equilibria such as (O, oo) and (F, ff) in the sequential-move Battle of the Sexes game.

6.1. Sequential Rationality and Backward Induction

To address the critique we posed regarding the incredible nature of the equilibria (O, oo) and (F, ff) in the sequential-move Battle of the Sexes game, we will directly criticize the behavior of player 2 in the event that player 1 did not follow his prescribed strategy. We will insist that a player use strategies that are optimal at every information set in the game tree. We call this principle sequential rationality, because it implies that players are playing rationally at every stage in the sequence of the game, whether it is on or off the equilibrium path of play. Formally, we have

Definition 44 *Given strategies $\sigma_{-i} \in \Delta S_{-i}$ of i's opponents, we say that σ_i is sequentially rational if and only if i is playing a best response to σ_{-i} in each of his information sets.*

In the sequential-move Battle of the Sexes game of Figure 14, player 2 should play a best response at each of his information sets. So, at x_1 he should play o and at x_2 he should play

f. Now applying sequential rationality to player 1 requires that player 1, who is correctly predicting the behavior of player 2, should choose O. Hence the unique prediction from this process is the path of play that begins with player 1 choosing O followed by player 2 choosing o. Furthermore the process predicts what would happen if players deviate from the path of play: if player 1 chooses F then player 2 will choose f. We conclude that the Nash equilibrium (O, of) is the unique pair of strategies that survives our requirement of sequential rationality.

This type of procedure, which starts at nodes that directly precede the terminal nodes at the end of the game and then inductively moves backward through the game tree, is also known as backward induction in games. This is exactly a multiperson dynamic programming version of the single decision maker's process. As the example just given suggests, when we apply this procedure to finite games of perfect information it will result in a prescription of strategies for each player that are sequentially rational, as the following proposition states:

Proposition 45 *Any finite game of perfect information has a backward induction solution that is sequentially rational. Furthermore if no two terminal nodes prescribe the same payoffs to any player then the backward induction solution is unique.*

Remark This proposition is often referred to as Zermelo's Theorem, named after the mathematician Ernst Zermelo. In 1913 Zermelo published a paper roughly stating that in games like chess, a player can guarantee at least a tie in a finite number of moves. Interestingly neither backward induction nor the notion of strategies–and surely not sequential rationality–was used in Zermelo's original paper, despite widespread belief to the contrary.

6.2. Subgame-Perfect Nash Equilibrium: Concept

Things become a bit trickier when we try to expand our reach to suggest solutions for games of imperfect information, in which backward induction as previously defined encounters some serious problems. Consider, for example, the extensive form game described in Figure 18.

If player 1 chooses A then the two players play a simultaneous-move game where player

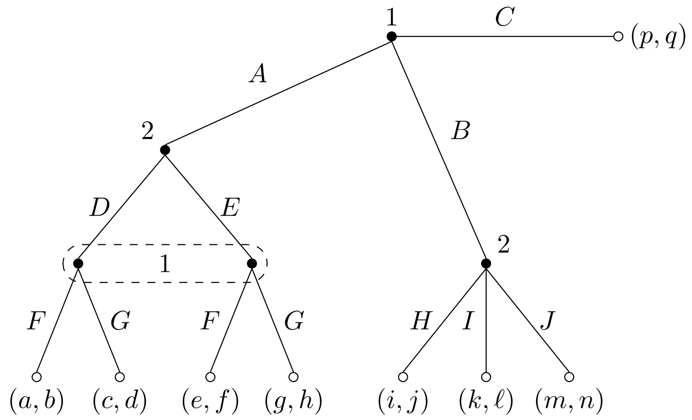

Figure 18.

1 chooses either F or G and player 2 chooses either D or E. To try to solve this game using backward induction we need to first identify the set of "last players" that precede terminal nodes and then choose actions that would maximize their payoff at this stage. In this game, however, this is not possible, because player 1 has an information set before the terminal nodes that is not a singleton. His best response is therefore not well defined without assigning a belief to this player about what player 2 actually chose to do, and these beliefs are not part of the backward induction process.

This example shows that backward induction cannot be applied to games of imperfect information. Interestingly there is another important class of games for which we cannot apply the procedure of backward induction, and these are games that do not necessarily end in finite time. Intuitively games that do not necessarily end in a finite number of moves may not have a finite set of terminal nodes, and without such a set we cannot begin the backward induction procedure. At first thought such games may seem a bit bizarre–what kind of game will never end? As we will see later, such games not only are interesting in their own right but also will offer critical ways to model realistic situations. Hence our goal is to find a natural way in which to extend the concept of sequential rationality to games of imperfect information and to games that have an infinite sequence of moves.

To analyze games of imperfect information we need to introduce the following definition:

Definition 46 *A proper subgame G of an extensive-form game Γ consists of only a single node and all its successors in Γ with the property that if $x \in G$ and $x' \in h(x)$ then $x' \in G$. The subgame G is itself a game tree with its information sets and payoffs inherited from Γ.*

The idea of a proper subgame (which we will often just call a subgame) is simple and allows us to "dissect" an extensive-form game into a sequence of smaller games, an approach that in turn will allow us to apply the concept of sequential rationality to games of imperfect information.

In every game of perfect information, every node is a singleton and hence can be a root of a subgame. This implies that in games of perfect information every node, together with all the nodes that succeed it, forms a proper subgame.

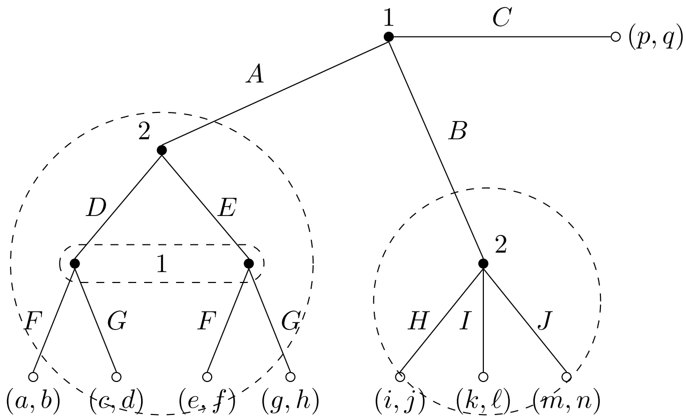

Figure 19.: Proper subgames

The extensive-form game of Figure 18 has two proper subgames which are depicted in Figure 19 (in the dashed circles).

A subgame is a stand-alone game within the whole game, and it is a useful to apply the concept of sequential rationality to extensive-form games. In particular, at any node or information set within a subgame G, a player's best response depends only on his beliefs about what the other players are doing within the subgame G, and not at nodes that are outside the subgame.

Definition 47 *Let Γ be an n-player extensive-form game. A behavioral strategy profile $\sigma^* =$*

$(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ *is a subgame-perfect (Nash) equilibrium if for every proper subgame* G *of* Γ *the restriction of* σ^* *to* G *is a Nash equilibrium in* G.

To find the subgame-perfect Nash equilibrium of the game in Figure 19 we start from the end of the game. We find the best action for player 2 in the subgame in the smaller circle and the Nash equilibria of the 2×2 game in the subgame in the bigger circle. Then player 1, knowing how the game will be played, chooses the best action action among A, B and C.

6.3. Subgame-Perfect Nash Equilibrium: Examples

This section presents some well-known examples of games and their corresponding subgame-perfect equilibria.

6.3.1. The Centipede Game

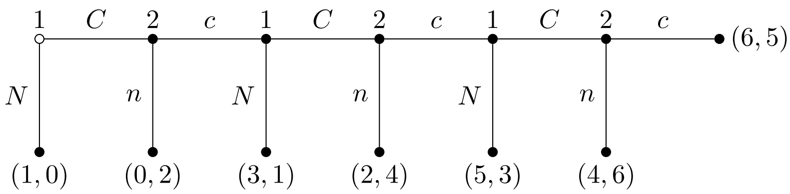

Figure 20.: The Centipede Game

Consider the perfect-information game depicted in Figure 20. The game should be read from left to right as follows: Player 1 can terminate the game immediately by choosing N in his first information set or can continue by choosing C. Then player 2 faces the same choice (using lowercase letters for his choices), and if player 2 chooses to continue then the ball is back in player 1's court, who again can terminate or continue to player 2, at which stage player 2 concludes the game by choosing n or c for the second time.

It would be nice for the players to be able to continue through the game to reach the payoffs of (6, 5). However, backward induction indicates that this will not happen. At his last information set, player 2 will choose n to get 6 instead of 5. Anticipating this a step earlier, player 1 will choose N to get 5 instead of 4, and the logic follows until player 1's first information set, at which he will choose N and players receive $(1, 0)$.

Notice that this game has an interesting structure: as long as the players continue, the sum of their payoffs increases. Nevertheless the "curse of rationality," so to speak, predicts a unique outcome: at the last stage the last player will, by being selfish, prefer to stop short of the payoff that maximizes the sum of the players' payoffs, and backward induction implies that this decision is anticipated and acted on by the player before him, and so on as the actions unravel to the bleak outcome of $(1, 0)$.

Remark The Centipede Game was first introduced by Rosenthal (1981), who used it as an example of a game in which the unique backward induction solution is extremely unappealing and goes against every intuition one might have about how the players will play. Indeed this game has been taken into the lab by letting pairs of players play it, and the experimental evidence of many studies, starting with McKelvey and Palfrey (1992), suggests that players do not play in the way predicted by backward induction. There are at least two reasons that players will not play the backward induction solution. One is that they actually care about each other's payoffs. This explanation may be less convincing for cases in which the players are anonymous, but it is not easy to rule out. The other reason is that players do not share a common knowledge of rationality. Palacios-Huerta and Volij (2009) try to put this second hypothesis to a test by taking the game to a chess tournament and having highly ranked chess players play it. Contrary to previous evidence, their results show that 69% of chess players choose to end the game immediately, and when grand masters are playing, all of them end the game immediately! This striking result suggests that when players are expected to share common knowledge of rationality they indeed play the backward induction solution.

6.3.2. Stackelberg Competition

The Stackelberg duopoly is a game of perfect information that is a sequential-moves variation on the Cournot duopoly model of competition. It was introduced and analyzed by Heinrich von Stackelberg (1934). It illustrates an important point: the order of moves might matter, and rational actors will take this into account.

Consider the Cournot duopoly of Section 3.2.2, but assume that one firm is the leader, in the sense that it chooses its quantity first. The other firm is the follower, chooses its quantity after it observes the choice of the leader. Suppose firm 2 is the leader and 1 is the follower.

We solve the game backwards starting from the follower. We know that firm 1's best-response is $q_1 = \frac{1-q_2}{2}$. Then firm 2's constrained maximization problem becomes

$$\max_{\{q_2\}} \pi_2 = (1 - q_1 - q_2)q_2$$
$$\text{subject to: } q_1 = \frac{1-q_2}{2}.$$

This is equivalent to maximizing

$$\pi_2 = \left(1 - \frac{1-q_2}{2} - q_2\right) q_2 = \left(\frac{1-q_2}{2}\right) q_2. \tag{6.3.1}$$

Figure 21 presents the leader's constrained maximization problem. The leader is choosing his highest isoprofit subject to his constraint, which is the best-response of the follower. The optimal choice is $q_2^* = \frac{1}{2}$. Given this, player 1 chooses $q_1^* = \frac{1}{4}$. The equilibrium profits are: $\pi_2 = \frac{1}{8}$ and $\pi_1 = \frac{1}{16}$. We conclude that when two firms are competing by setting quantities, if one firm can somehow commit to move first, it will enjoy a first-mover advantage.

In sum, the subgame-perfect Nash equilibrium is: $q_2^* = \frac{1}{2}$ and $q_1^* = \frac{1-q_2}{2}$. Note that the equilibrium strategy of the follower is a complete contingent plan that specifies an optimal q_1 for any choice of the leader.

6.3.3. Advertising in the Cournot Model

Consider the following modification of the Cournot model. The inverse demand function is $p = a_1 - q_1 - q_2$, where a_1 is the level of advertising done by firm 1. Advertising increases the consumers' willingness to pay for the product. In stage 1 firm 1 chooses a_1 and in stage 2 the two firms compete in quantities. The cost of advertising is $\frac{a_1^3}{3}$.

The profit functions are: $\pi_1 = pq_1 - \frac{a_1^3}{3}$ and $\pi_2 = pq_2$.

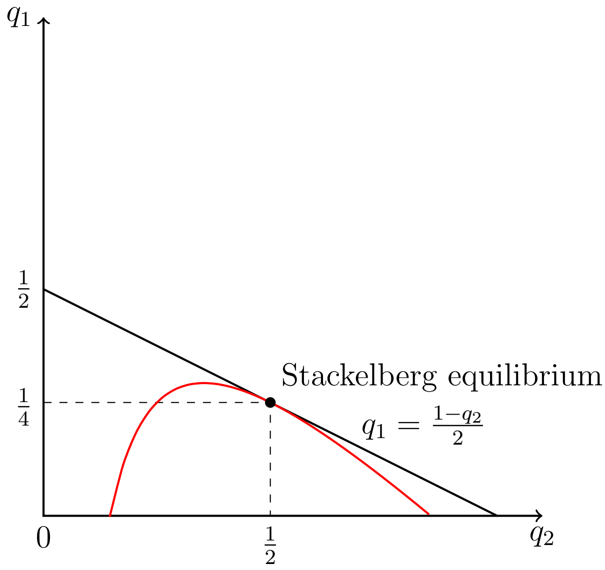

Figure 21.: Stackelberg equilibrium

We solve the game backwards starting from stage 2. Every choice a_1 initiates a subgame. It is easy to compute the equilibrium for each a_1

$$q_1 = q_2 = \frac{a_1}{3}. \tag{6.3.2}$$

In stage 1 firm 1 chooses a_1 to maximize $\pi_1 = \frac{a_1^2}{9} - \frac{a_1^3}{3}$. The first order condition is $\frac{2a_1}{9} - a_1^2 = 0$, which is satisfied at $a_1 = 0$ and $a_1 = \frac{2}{9}$. The level that maximizes π_1 is $a_1^* = \frac{2}{9}$.

In sum, the subgame-perfect Nash equilibrium is $a_1^* = \frac{2}{9}$ and $q_1^* = q_2^* = \frac{a_1}{3}$.

6.3.4. Dynamic Monopoly

A monopolist is selling a durable good over a two-period horizon. If a consumer buys the good in the first period, he or she can use over both periods. There are four consumers: Hal, Hilbert, Laurie and Lauren with the willingnesses to pay given in Figure 22.

The monopolist charges a price p_1 in period 1 and a price p_2 in period 2 to maximize his profits. We assume that the discount factor is one. We will be looking for a subgame-perfect Nash equilibrium.

Consumers are not strategic: The equilibrium prices are: $p_1 = 1700$ and $p_2 = 200$. In

	Period 1	Period 2
Hal & Hilbert	1200	500
Laurie & Lauren	500	200

Figure 22.: Willingnesses to pay

period 1 Hal and Hilbert buy and enjoy the good over two periods and in period 2 Laurie and Lauren buy and consume the good for one period. The monopolist's profits are 3800.

Consumers are strategic: Let's see if the above prices are an equilibrium. Suppose a high type buyer deviates and buys in period 2. The monopolist still has an incentive to charge $p_2 = 200$. The alternative would be $p_2 = 500$ and only the high type who deviated buys. But this price yields lower profits ($500 < 600$). The high type who deviated enjoys a consumer surplus $500 - 200 = 300 > 0$. So, such a deviation is profitable. An incentive compatibility constraint is not satisfied.

High types can wait, buy the product later and become better off. Hence, they must receive a discount in period 1. The discount should be equal to the amount by which the high type becomes better off when he deviates, i.e., 300. The incentive compatible prices are: $p_1 = 1700 - 300 = 1400$ and $p_2 = 200$. The profit is 3200.[1] Effectively, the monopolist is trying to separate consumers with different valuations by offering different prices across time. In the process the monopolist competes with himself. Coase (1972) argued that under certain conditions the monopoly price will converge to a very low price even from the first period, Coase conjecture.

Price guarantees as commitment to high prices. Suppose the seller commits to refund any price difference between the two periods (a best price guarantee policy). Then the profit of the seller is $3200 - 2 \times 1200 = 800$, because the high types will demand the refund.

If the seller does not lower its prices then his profit will be $1700 \times 2 = 3400$. The seller is better off committing to such a policy. Consumers are worse off.

[1] There are other pricing strategies but they yield lower profits.

6.3.5. Strategic Effect of Commitments

Consider the following two-stage game: in stage 1 firm 1 makes an investment K_1 and in stage 2 the two firms, 1 and 2, compete by choosing the strategic variables x_1 and x_2. K_1 has a direct effect on firm 1's profit (i.e., holding x_2 fixed) and a strategic effect through x_2. Let $x_1^*(K_1)$ and $x_2^*(K_1)$ be the stage 2 equilibrium.[2] Firm 1's profit function is

$$\pi_1(K_1, x_1^*(K_1), x_2^*(K_1)).$$

Using the envelope theorem, the overall effect of K_1 on firm 1's profit is

$$\underbrace{\frac{d\pi_1}{dK_1}}_{\text{Direct effect}} + \underbrace{\frac{\partial \pi_1}{\partial x_2}\frac{dx_2^*}{dK_1}}_{\text{Strategic effect}}.$$

The sign of the strategic effect can be related to the investment making firm 1 tough or soft and to the slope of the second-stage reaction curve. Assume that the second-period actions of both firms have the same nature, in the sense that $\partial\pi_1/\partial x_2$ and $\partial\pi_2/\partial x_1$ have the same sign. Using the fact that

$$\frac{dx_2^*}{dK_1} = \left(\frac{dx_2^*}{dx_1}\right)\left(\frac{dx_1^*}{dK_1}\right) = R_2'(x_1^*)\left(\frac{dx_1^*}{dK_1}\right)$$

(where R_2 is the reaction function of firm 2) by the chain rule and arranging we obtain

$$\text{sign}\left(\frac{\partial \pi_1}{\partial x_2}\frac{dx_2^*}{dK_1}\right) = \text{sign}\left(\frac{\partial \pi_2}{\partial x_1}\frac{dx_1^*}{dK_1}\right) \times \text{sign}(R_2').$$

If the commitment makes firm 1 tough, $\frac{\partial \pi_2}{\partial x_1}\frac{dx_1^*}{dK_1}$ is negative, while if it makes it soft it is positive. Also, if the strategic variables are substitutes, $\text{sign}(R_2')$ is negative, while if they are complements it is positive.

Therefore, when the commitment is tough, x_1 should increase under strategic substitutability and decrease under strategic complementarity. In these cases the strategic effect is positive. Examples: Cournot competition with substitutes and Bertrand competition with complements. If a firm makes an investment that reduces its marginal cost (and hence quantity will increase in Cournot, or price will decrease in Bertrand) then strategic effect will be positive.

[2] See Fudenberg and Tirole (1984).

When the commitment is soft, x_1 should increase under strategic complementarity and decrease under strategic substitutability. In these cases the strategic effect is positive. Examples: Cournot competition with complements and Bertrand competition with substitutes. If a firm makes an investment that reduces its marginal cost (and hence quantity will increase in Cournot, or price will decrease in Bertrand) then strategic effect will be negative.

6.3.5.1. An Example: Strategic Leadership in Price Competition

Two firms, 1 and 2, compete in prices P_1 and P_2, facing the following demand functions:

$$P_1 = 1 - Q_1 - \gamma Q_2 \text{ and } P_2 = 1 - Q_2 - \gamma Q_1,$$

where $\gamma \in [0, 1]$ measures the degree of product differentiation. If $\gamma = 0$ the two products are independent, while if $\gamma = 1$ the two products are homogeneous (Cournot). Each firm's marginal cost is 0.

Invert the demand system using Cramer's rule in order to get a system of own-quantities in terms of both firms' prices. The demand system in matrix notation is

$$\begin{pmatrix} 1 & \gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 1 - P_1 \\ 1 - P_2 \end{pmatrix}.$$

Solving for the inverted system yields

$$\begin{aligned}
Q_1 = \frac{\begin{vmatrix} 1 - P_1 & \gamma \\ 1 - P_2 & 1 \end{vmatrix}}{1 - \gamma^2} &= \frac{1}{1 + \gamma} - \frac{1}{1 - \gamma^2} P_1 + \frac{\gamma}{1 - \gamma^2} P_2 \\
Q_2 = \frac{\begin{vmatrix} 1 & 1 - p_1 \\ \gamma & 1 - p_2 \end{vmatrix}}{1 - \gamma^2} &= \frac{1}{1 + \gamma} - \frac{1}{1 - \gamma^2} P_2 + \frac{\gamma}{1 - \gamma^2} P_1.
\end{aligned}$$

Let $a \equiv 1/(1+\gamma)$, $b \equiv 1/(1-\gamma^2)$ and $c \equiv \gamma/(1-\gamma^2)$. The profit functions can be expressed as follows

$$\begin{aligned}
\pi_1 &= P_1 Q_1 = P_1(a - bP_1 + cP_2) = aP_1 - bP_1^2 + cP_1 P_2 \\
\pi_2 &= P_2 Q_2 = P_2(a - bP_2 + cP_1) = aP_2 - bP_2^2 + cP_1 P_2.
\end{aligned}$$

Suppose the firms move simultaneously.

The derivative of π_1 with respect to P_1 is: $a - 2bP_1 + cP_2$. Set it equal to zero and solve for P_1 to obtain the best-response function of firm 1

$$P_1 = \frac{a}{2b} + \frac{cP_2}{2b}.$$

The derivative of π_2 with respect to P_2 is: $a - 2bP_2 + cP_1$. Set it equal to zero and solve for P_2 to obtain the best-response function of firm 2

$$P_2 = \frac{a}{2b} + \frac{P_1}{2b}.$$

Solve the best-response functions with respect to P_1 and P_2 to obtain the Nash equilibrium

$$(P_1^*, P_2^*) = \left(\frac{a}{2b-c}, \frac{a}{2b-c}\right) = \left(\frac{1-\gamma}{2-\gamma}, \frac{1-\gamma}{2-\gamma}\right).$$

The equilibrium profits are

$$\pi_1 = \pi_2 = \left(\frac{a}{2b-c}\right)^2.$$

Now suppose firms move sequentially with firm 2 being the leader and firm 1 the follower.

The leader will choose its price to maximize its profits knowing how the follower will respond. Hence the profit function of the leader, using the best-response function of the follower, is

$$\pi_L = P_L Q_L = P_L \left(a - bP_L + cP_F\right) = P_L \left(a - bP_L + c\left(\frac{a}{2b} + \frac{cP_L}{2b}\right)\right).$$

Taking the derivative of π_L with respect to P_L setting equal to zero and solving for P_L we obtain

$$P_L = \frac{a(2b+c)}{2(2b^2-c^2)} = \frac{(1-\gamma)(2+\gamma)}{2(2-\gamma^2)}.$$

The follower's price is

$$P_F = \frac{a(4b^2 - c^2 + 2cb)}{4b(2b^2-c^2)} = \frac{(1-\gamma)(4+2\gamma-\gamma^2)}{4(2-\gamma^2)}.$$

It can be verified that $P_L > P_F$ and moreover both prices are higher than the Nash equilibrium prices from the simultaneous move game. In addition, profits from the sequential game are higher than the profits from the simultaneous game and the follower's profit is higher than the leader's profit.

Figure 23 depicts the equilibrium under both simultaneous and sequential moves on part of the firms. Note that equilibrium prices when one firm assumes the role of the leader are higher than when firms move simultaneously. Profits are also higher for both firms, although the follower's profit is higher than the leader's profits (second-mover advantage). This highlights the role of strategic commitments. There are two effects on the leader's profits: a direct effect and an indirect (strategic) effect. The direct effect refers to the change in the leader's profits when it increases its price, holding the price of the rival fixed. This effect is negative, because a unilateral change in price, from the Nash equilibrium, lowers profits. The indirect (strategic) effect refers to the change in the leader's profit due to the rival's response. This response entails an increase in price. Prices are strategic complements (upward sloping best-responses) and the leader's commitment (higher price) is soft (because it increases the follower's profit). According to the taxonomy, the strategic effect on the leader's profit is positive. This is because the leader, by committing to a higher price, induces the follower to also increase its price (albeit not as much as the leader). Moreover, the positive strategic effect outweighs the negative direct effect for the leader's profits, since as we mentioned above the leader's profit increases relative to the simultaneous-move case. It is a form of an implicit collusion, since the firms succeed in raising their prices.

6.3.6. Time-Inconsistent Preferences

Consider a player who will receive a stream of payments $x_1, x_2, ..., x_T$ over the periods $t = 1, 2, ..., T$, and who evaluates per-period payments using the utility function $u(x)$. His discounted sum of future payoffs in period $t = 1$ is

$$v(x_1, ..., x_T) = u(x_1) + \delta u(x_2) + \delta^2 u(x_3) + \cdots + \delta^{T-1} u(x_T),$$

where $\delta \in (0, 1)$ is the discount factor. Consider a player with $u(x) = \ln(x)$ who needs to allocate a fixed budget K across three periods. Since he will not waste any of his budget, it

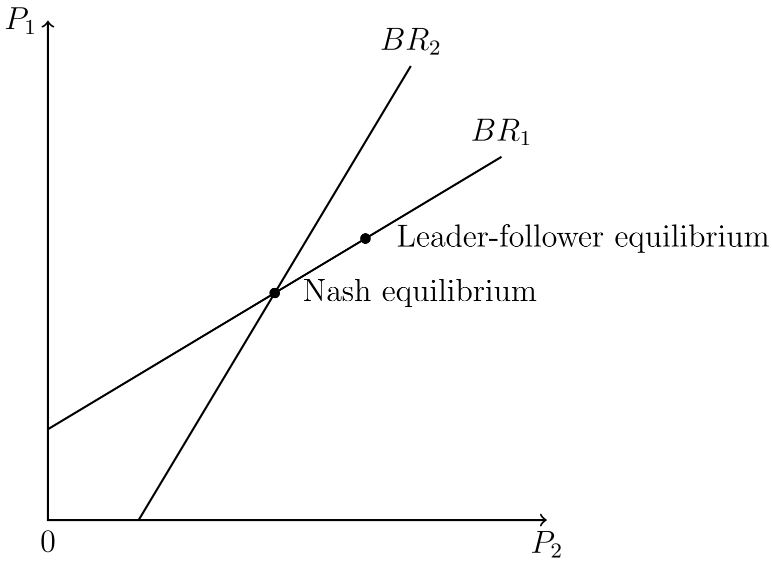

Figure 23.: Leader-Follower vs. Simultaneous Moves in a Differentiated Products Price Competition Duopoly. Firm 2 is the Leader.

follows that $x_1 = K - x_2 - x_3$ and the player solves the following problem:

$$\max_{\{x_2,x_3\}} \ln(K - x_2 - x_3) + \delta \ln(x_2) + \delta^2 \ln(x_3). \tag{6.3.3}$$

The solution is

$$x_1 = \frac{K}{1+\delta+\delta^2}, \quad x_2 = \frac{\delta K}{1+\delta+\delta^2}, \quad x_3 = \frac{\delta^2 K}{1+\delta+\delta^2}.$$

The player chooses to consume more in earlier periods because he is equating the marginal utility from consumption across time, taking into account that future periods are discounted. An interesting question is the following: if the player planned out his consumption over time, would he choose to stick to his plan of action after he consumes x_1? The answer is yes (verify it).

We call this kind of behavior time consistent because a player will choose to follow his original plan. The discounting rule described by (6.3.3) is called exponential discounting, and it is generally known in economic analysis that a player who uses this formula to evaluate his present-value utility will always stick to his plan of action; therefore his behavior will be time consistent.

Consistency of choice makes sense and turns out to be important for individual and national savings behavior for example, but is it borne out empirically? A simple illustration of this is a typical pure-time-preference experiment from the psychology literature. Participants are asked to choose among the following monetary prizes:

- Question 1: Would you prefer \$100 today or \$200 in 2 years?
- Question 2: Would you prefer \$100 in 6 years or \$200 in 8 years?

Respondents often prefer the \$100 in Question 1 and the \$200 in Question 2, not realizing that Question 2 involves the same choice as Question 1 but with a 6-year delay. If these people behave true to their answers, they will be time inconsistent: in the case of Question 2, although they state their preference now for the \$200 prize in 8 years, when year 6 arrives they will take the \$100 and run!

The above example suggests a time-inconsistent behavior in that, as time goes by, people will want to deviate from plans of action that they originally set. To model this idea consider a different discounting rule known as hyperbolic discounting. The player uses the discount rate δ for the future periods as in exponential discounting, but he uses an additional discount factor $\beta \in (0, 1)$ to discount all of the future compared to present consumption. Using our three-period example, the player's modified discounted present-value problem is given by

$$\max_{\{x_2, x_3\}} \ln(K - x_2 - x_3) + \beta\delta \ln(x_2) + \beta\delta^2 \ln(x_3).$$

Simply put, when the player is looking toward the future, the discount factor he uses between periods $t = 1$ and $t = 2$ is stronger than the one he uses between periods $t = 2$ and $t = 3$. In particular, at any period and looking forward, the player uses $\beta\delta < \delta$ to discount between the current period and the next, while he uses only δ to discount between any two consecutive future periods. As we will now see, hyperbolic discounting will cause problems of self-control, in that a player will plan to do one thing but later choose to revise his plan.

To simplify the analysis set $\delta = 1$ and $\beta = \frac{1}{2}$. With exponential discounting the optimal consumption plan is $x_1 = x_2 = x_3 = \frac{K}{3}$.

Now consider a sophisticated player with hyperbolic discounting. Looking forward to his future self, he knows that player 2, who is left with a budget of K_2, will solve the following problem

$$\max_{\{x_2\}} \ln(x_2) + \frac{1}{2}\ln(K_2 - x_2).$$

The solution is $x_2 = \frac{2K_2}{3}$ and $x_3 = \frac{K_2}{3}$. So, in the future the player will not resist the temptation to consume the remaining budget unequally across periods. Given that the player is sophisticated, he knows that in period 1 maximizes the following

$$\max_{\{x_1\}} \ln(x_1) + \frac{1}{2}\ln\left(\frac{2}{3}(K - x_1)\right) + \frac{1}{2}\ln\left(\frac{1}{3}(K - x_1)\right).$$

The optimal solution is $x_1 = \frac{K}{2}$. Therefore, under hyperbolic discounting the consumption across time is $x_1 = \frac{K}{2}$, $x_2 = \frac{K}{3}$ and $x_3 = \frac{K}{6}$, very different from the equal consumption of the budget under exponential discounting.

Problems of self-control and time-inconsistent behavior are at the center of the field known as behavioral economics. Models of hyperbolic discounting have been used to explain insufficient savings (Laibson, 1997), procrastination (O'Donoghue and Rabin, 1999), and other related phenomena. Scholars in this literature often distinguish between players who are not aware of their self-control problem (they do not perform backward induction), referring to them as naive, and players who are aware of their self-control problem, referring to them as sophisticated. What is interesting about sophisticated players is that they would choose to constrain their own future behavior in order to achieve a higher net present value from their stream of consumption.

6.4. Summary

- Extensive-form games will often have some Nash equilibria that are not sequentially rational, yet we expect rational players to choose sequentially rational strategies.
- In games of perfect information backward induction will result in sequentially rational Nash equilibria. If there are no two payoffs at terminal nodes that are the same then

there will be a unique sequentially rational Nash equilibrium.

- Subgame-perfect equilibrium is the more general construct of backward induction for games of imperfect information.
- At least one of the Nash equilibria in a game will be a subgame-perfect equilibrium.

7. Repeated Games

A repeated game is a multistage game in which the same stage-game is being played at every stage. These games have been studied for two primary reasons. The first is that repeated games seem to capture many realistic settings. These include firms competing in the same market over long periods of time, politicians engaging in pork-barrel negotiations in session after session of a legislature, and workers on a team production line who perform some joint task day after day. The second reason is that repeating the same game over time results in a very convenient mathematical structure that makes the analysis somewhat simple and elegant. As such, social scientists were able to refine this analytical tool and provide a wide range of models that can be applied to understand different social phenomena. This chapter provides a glimpse into the analysis of repeated games and shows some of the extreme limits to what we can do with reward-and-punishment strategies.

7.1. Finitely Repeated Games

A finitely repeated game is, as its name suggests, a stage-game that is repeated a finite number of times. We can define a finitely repeated game as follows:

Definition 48 *Given a stage-game G, $G(T,\delta)$ denotes the finitely repeated game in which the stage-game G is played T consecutive times, and δ is the common discount factor.*

As an example, consider the two-stage repeated game in which the game in Figure 24 is repeated twice with a discount factor of δ.

There are two pure-strategy Nash equilibria, and they are Pareto ranked, that is, (R,r) is better than (F,f) for both players. We have a "carrot" and a "stick" to use in an attempt

		Player 2		
		m	f	r
	M	$4,4$	$-1,5$	$0,0$
Player 1	F	$5,-1$	$1,1$	$0,0$
	R	$0,0$	$0,0$	$3,3$

Figure 24.

to discipline first-period behavior. This implies that for a high enough discount factor we may be able to find subgame-perfect equilibria that support behavior in the first stage that is not a Nash equilibrium of the one-shot stage-game.

For a discount factor $\delta \geq \frac{1}{2}$ the following strategies constitute a subgame-perfect equilibrium of this two-stage game:

Player 1: Play M in stage 1. In stage 2 play R if (M, m) was played in stage 1, and play F if anything but (M, m) was played in stage 1.

Player 2: Play m in stage 1. In stage 2 play r if (M, m) was played in stage 1, and play f if anything but (M, m) was played in stage 1.

Let's see why. In stage 2, if (M, m) was played in stage 1 player 1 expected player 2 to play r and player 2 expects player 1 to play R. No player has an incentive to unilaterally deviate, since (R, r) is a Nash equilibrium. If a different than (M, m) action profile was chosen, then player 1 expects player 2 to choose f and player 2 expects player 1 to choose F. Since (F, f) is also a Nash equilibrium, no player has an incentive to deviate. In stage 1 player 1 can choose M or deviate. If he chooses M the present value of his utility is $4 + 3\delta$. If he deviates he chooses F and the present value of his utility is $5 + \delta$. He will not deviate if and only if $4 + 3\delta \geq 5 + \delta \Rightarrow \delta \geq \frac{1}{2}$.

In the two-stage repeated game, if any one of the players deviates from the proposed path of play (M, m) then both players will incur a loss of -2 in the second-period stage-game because the players will switch from the "good" second-stage equilibrium, (R, r), to the "bad" one, (F, f). The temptation to deviate from (M, m) for either player is the added payoff of $+1$ in the first period that a player gets by deviating from M (or m) to F (or f). It is the multiplicity of equilibria in the stage-game that is giving the players the leverage to use conditional second-stage strategies of the reward-and-punishment kind.

The conclusion must be that we need multiple continuation Nash equilibria to be able to support behavior that is not a Nash equilibrium of the stage-game in earlier stages. This immediately implies the following result.

Proposition 49 *If the stage-game of a finitely repeated game has a unique Nash equilibrium, then the finitely repeated game has a unique subgame-perfect equilibrium.*

What are the consequences of this proposition? It immediately applies to a finitely repeated game like the Prisoner's Dilemma or any of the simultaneous-move market games (Cournot and Bertrand competitions). If such a stage-game is played T times in a row as a finitely repeated game, then there will be a unique subgame-perfect equilibrium that is the repetition of the static noncooperative Nash equilibrium play in every stage. This result may seem rather disturbing, but it follows from the essence of credibility and sequential rationality. If two players play the Prisoner's Dilemma, say, 500 times in a row, then intuition suggests that they can use the future to discipline behavior and try to cooperate by playing mum in early periods, with deviations from mum being punished by a later move to fink. However, the "unraveling" process that proves proposition 49 will apply to this finitely repeated Prisoner's Dilemma precisely because it is common knowledge that the game will end after 500 periods. In the last stage-game at T = 500, the players must each play fink because it is the unique Nash equilibrium. (In fact it is the unique dominant strategy!) As a consequence, in the stage-game at T = 499 the players cannot provide reward-and-punishment incentives, so in this stage they must again play fink. This argument will continue, thus proving that it is impossible to support a subgame-perfect equilibrium in which the players can cooperate and play mum in early stages of the game.

7.2. Infinitely Repeated Games

The problem identified with the finitely repeated Prisoner's Dilemma is a consequence of the common knowledge among the players that the game has a fixed and finite number of periods. If a repeated game is finite, and if its stage-game has a unique Nash equilibrium, then starting in the last period T the "Nash unraveling" of the static Nash equilibrium follows from sequential rationality.

What would happen if we eliminated this problem by assuming that the game does not have a final period? That is, what if, regardless of the stage at which the players find themselves and whatever they have played, there is always a "long" future ahead of them? As we will see, this slight but critical modification will give the players the ability to support play that is not a static Nash equilibrium of the stage-game, even when the stage-game has a unique Nash equilibrium. In fact we will see that once the stage-game is repeated infinitely often, the players will have the freedom to support a wide range of behaviors that are not consistent with a static Nash equilibrium in the stage-game. First we need to define payoffs for an infinitely repeated game.

7.2.1. Payoffs

Let G be the stage-game, and denote by $G(\delta)$ the infinitely repeated game with discount factor δ. The natural extension of the present-value concept that we used in the previous chapter is given by the following:

Definition 50 *Given the discount factor* $0 < \delta < 1$, *the present value of an infinite sequence of payoffs* $\{v_i^t\}_{t=1}^{\infty}$ *for player* i *is*

$$v_i = v_i^1 + \delta v_i^2 + \delta^2 v_i^3 + \cdots ... = \sum_{t=1}^{\infty} \delta^{t-1} v_i^t.$$

The question is whether it is reasonable to assume that players will engage in an infinite sequence of play. Intuitively a sensible answer is that they won't. We can, however, motivate this idea by interpreting discounting as a response to an uncertain future.

Imagine that the players are playing a given stage-game today and with some probability $\delta < 1$ they will play the game again tomorrow. With probability $1 - \delta$, however, their relationship ends for some exogenous reason and the players will no longer interact with each other. If the relationship continues for one more day then again, with probability $\delta < 1$, they will continue for another period, and so on.

Now imagine that in the event these players continue playing the game, the payoffs of player i are given by the sequence $\{v_i^t\}_{t=1}^{\infty}$. If there is no additional discounting we can think of the expected payoff from this sequence as

$$Ev = v_i^1 + \delta v_i^2 + (1-\delta) \times 0 + \delta^2 v_i^3 + \delta(1-\delta) \times 0 + \delta^3 v_i^4 + \delta^2(1-\delta) \times 0 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1} v_i^t.$$

That is, we can think of the present value from the sequence of payoffs as the expected value from playing this stochastic sequence of stage-games. There is something appealing about the interpretation of $\delta < 1$ as the probability that the players continue their ongoing relationship for one more period. In particular we can ask a simple question: what is the probability that the players will play this game infinitely many times? The answer is clearly $\delta^{\infty} = 0$! What makes this interesting is that the game will end in finite time with probability 1, but the potential future is always present at any given period.

7.2.2. Strategies

Consider the extensive-form representation of an infinitely repeated game. In particular imagine a tree with a root, from which it continues to expand both in "length"–because of the added stages–and in "width"–because more and more information sets are created after each period. Because the stage-game is repeated infinitely many times, there will be an infinite number of information sets! Because a player's strategy is a complete contingent plan that specifies what the player will do in each information set, it may seem that defining strategies for players in an infinitely repeated game will be quite cumbersome, if not outright impossible.

It turns out that there is a convenient and rather natural way to describe strategies in this setup. Notice that every information set of each player is identified by a unique path of play or history that was played in the previous sequences. For example, if we consider the infinitely repeated Prisoner's Dilemma, then in the fourth stage each player has 64 information sets.[1] Each of these 64 information sets corresponds to a unique path of play, or history, in the first three stages. For example, the players playing (M, m) in each of the three previous stages is one such history. Any other combination of three consecutive plays will be identified with a different and unique history.

This observation implies that there is a one-to-one relationship between information sets and histories of play. Because of this relationship, from now on we use the word "history" to describe a particular sequence of action profiles that the players have chosen up until the stage that is under consideration (for which past play is indeed the history). To make things precise, we define histories, and history-contingent strategies, as follows:

Definition 51 *Consider an infinitely repeated game. Let H_t denote the set of all possible histories of length t, $h_t \in H_t$, and let $H = \cup_{t=1}^{\infty} H_t$ be the set of all possible histories (the union over t of all the sets H_t). A pure strategy for player i is a mapping $s_i : H \to S_i$ that maps histories into actions of the stage-game. Similarly a behavioral strategy of player i, $\sigma_i : H \to \Delta S_i$ maps histories into stochastic choices of actions in each stage.*

7.3. Subgame-Perfect Equilibria

Now that we have identified a precise and concise way of describing strategies for the players in an infinitely repeated game, it is straightforward to define a subgame-perfect equilibrium:

Definition 52 *A profile of pure strategies $(s_1^*(\cdot), s_2^*(\cdot), ..., s_n^*(\cdot))$, $s_i : H \to S_i$ for all $i \in N$, is a subgame-perfect equilibrium if the restriction of $(s_1^*(\cdot), s_2^*(\cdot), ..., s_n^*(\cdot))$ is a Nash equilibrium in every subgame. That is, for any history of the game h_t, the continuation play dictated by $(s_1^*(\cdot), s_2^*(\cdot), ..., s_n^*(\cdot))$ is a Nash equilibrium. (Similarly for behavioral strategies $(\sigma_1^*(\cdot), \sigma_2^*(\cdot), ..., \sigma_n^*(\cdot))$.)*

As with the case of strategies, this may at first seem like an impossible concept to implement. How could we check that a profile of strategies is a Nash equilibrium for any history,

[1] Because the stage-game has four distinct outcomes, after three repetitions of the stage-game there will be $4^3 = 64$ distinct outcomes, or histories.

especially because each strategy is a mapping that works on any one of the infinitely many histories? As it turns out, one familiar result will be a useful benchmark:

Proposition 53 *Let $G(\delta)$ be an infinitely repeated game, and let $(\sigma_1^*, \sigma_2^*, ..., \sigma_n^*)$ be a (static) Nash equilibrium strategy profile of the stage-game G. Define the repeated-game strategy for each player i to be the history-independent Nash strategy, $\sigma_i^*(h) = \sigma_i^*$ for all $h \in H$. Then $(\sigma_1^*(h), \sigma_2^*(h), ..., \sigma_n^*(h))$ is a subgame-perfect equilibrium in the repeated game for any $\delta < 1$.*

If player i believes that his opponents' behavior is independent of the history of the game then there can be no role for considering how current play affects future play. As a consequence, if he believes that his opponents' current play coincides with their play in the static Nash equilibrium of the stage-game, then by the definition of a Nash equilibrium his best response must be to choose his part of the Nash equilibrium.

This proposition demonstrates that for the infinitely repeated Prisoner's Dilemma playing fink unconditionally in every period by each player is a subgame-perfect equilibrium. The more interesting question that remains is whether or not we can support other types of behavior as part of a subgame-perfect equilibrium. In particular can the players choose mum for a significant period of time in equilibrium? The answer is yes.

Before discussing some general results, we start with an example of the infinitely repeated Prisoner's Dilemma with discount factor $\delta < 1$.

		Player 2	
		m	f
Player 1	M	$4, 4$	$-1, 5$
	F	$5, -1$	$1, 1$

Figure 25.: Prisoner's dilemma game

Consider the path of play in which the two players choose (M, m) in every period. Following this path the players' average payoffs are $(v_1, v_2) = (4, 4)$. We now ask whether this path of play can be supported as a subgame-perfect equilibrium. Clearly the history-independent strategies of "play M (or m) regardless of the history" cannot be a subgame-perfect equilibrium for a simple reason: following a deviation of any player from mum to fink at any stage, the other player's strategy suggests that he is committed to continue to play mum, making the deviation profitable. This implies that if we can support (M, m) in each and every period then we need to find some way to "punish" deviations.

To support "good" behavior as a subgame-perfect equilibrium, first we need a "carrot" continuation equilibrium to reward good behavior and a "stick" continuation equilibrium to punish bad behavior. Second we need a high enough discount factor for the reward-and-punishment strategies to be effective.

An obvious candidate for a "stick" continuation equilibrium is the unconditional repeated play of the (F, f) static Nash equilibrium, which results in the players receiving an average payoff $(v_1, v_2) = (1, 1)$. What then can be the "carrot" continuation equilibrium? This is where things get interesting, and in a very ingenious way. Recall that we are trying to support the play (M, m) in every period as a subgame-perfect equilibrium. If we can indeed do this, it means that playing (M, m) forever would be a consequence of some subgame-perfect equilibrium with appropriately defined strategies. But if this is true then we have our answer: the subgame-perfect equilibrium that implies the play of (M, m) forever will be the "carrot" continuation equilibrium that supports itself.

Now this may at first come across as a circular argument: the reason that playing (M, m) forever can be a subgame-perfect equilibrium is that it can be supported as a subgame-perfect equilibrium! Indeed this is a circular argument, but it is not inconsistent. It is for this reason that an equilibrium of this nature is often called a bootstrap equilibrium, in that we are using the proposed equilibrium to support itself. This argument is consistent precisely because any infinitely repeated game has the same set of possible future plays at any stage of the game. It is this observation that makes the analysis manageable and offers us the ability to use this circular logic in a consistent way.

To see how this works consider the Prisoner's Dilemma previously described. Assuming that playing (M, m) in every future period can be the reward, or "carrot" continuation equilibrium, we need to determine whether playing (F, f) unconditionally forever can be a sufficient "stick." To do this consider the following strategies for each player:

Player 1: In the first stage play $s_1^1 = M$. For any stage $t > 1$, play $s_1^t(h_{t-1}) = M$ if and only if the history h_{t-1} is a sequence that consists only of (M, m), that is, $h_{t-1} = \{(M, m), (M, m), ..., (M, m)\}$. Otherwise, if some player ever played fink and $h_{t-1} \neq \{(M, m), (M, m), ..., (M, m)\}$ then play $s_1^t(h_{t-1}) = F$.

Player 2: Play the mirror strategy to player 1: $s_2^1 = m$. For any stage $t > 1$, play $s_2^t(h_{t-1}) = m$ if and only if $h_{t-1} = \{(M, m), (M, m), ..., (M, m)\}$, while if $h_{t-1} \neq \{(M, m), (M, m), ..., (M, m)\}$ then play $s_2^t(h_{t-1}) = f$.

Before checking if this pair of strategies is a subgame-perfect equilibrium, it is worth making sure that they are well defined. We have to define the players' actions for any history, and we have done this as follows: in the first stage they both intend to be "good" and play mum. Then at any later stage they look back at history and ask: was there ever a deviation in any previous stage from always playing mum? If the answer is no, meaning that both players always cooperated and played mum, then in the next stage each player chooses mum. If, however, there was any deviation from cooperation in the past, be it once or more than once, be it by one player or both, the players will revert to playing fink, and by the definition of the strategies they will stick to fink thereafter.

These strategies are commonly referred to as grim-trigger strategies because they include a natural trigger: once someone deviates from mum, this is the trigger that causes the players to revert their behavior to fink forever, resulting in a very grim future. More generally the idea behind grim-trigger strategies is as follows. If the stage-game has a Nash equilibrium, and we are trying to support subgame-perfect equilibrium behavior that results in outcomes that are better than any Nash equilibrium, we use the established subgame-perfect equilibrium of playing the grim static Nash outcome forever to support the more desirable outcomes that are not supported by static best responses being played. Hence we use the grim trigger to provide incentives for the players to stick to behavior for which short-run temptations to deviate are present.

To verify that the grim-trigger strategy pair is a subgame-perfect equilibrium we need to check that there is no profitable deviation in any subgame. This may seem like an impossible mission because we have an infinite number of subgames. It is not that tedious thanks to the power of the one-stage deviation principle. Indeed the following powerful proposition is extremely useful.

Proposition 54 *In an infinitely repeated game $G(\delta)$ a profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ is a subgame-perfect equilibrium if and only if there is no player i and no single history h_{t-1} for which player i would gain from deviating from $\sigma_i^*(h_{t-1})$.*

The one-stage deviation principle implies that in order to confirm that a profile of strategies is a subgame-perfect equilibrium, we need only to check that no player has a single history from which he would want to deviate unilaterally.[2] Applying this to our repeated Prisoner's Dilemma may not seem helpful because there are still an infinite number of histories. Nonetheless the task is not a hard one because even though there are an infinite number of histories, they fall into one of two relevant categories: either there was no deviation and the players intend to play (M, m) or there was some previous deviation and the players intend to play (F, f). These two categories of histories are common to any application of grim-trigger strategies in repeated games, which makes them relatively easy to check as part of a subgame-perfect equilibrium. That is, whenever we are trying to support one kind of behavior forever with the threat of resorting to another kind of behavior, we have two "states" in which the players can be, and we need only to check that they would not want to deviate from each of these two states.

Consider first the category of histories that are not consecutive sequences of only (M, m), implying that play ought to be the grim Nash equilibrium of the stage-game. Notice that

[2]Even in a finitely repeated game the number of strategies for each player can be huge. To show that a strategy is part of a subgame-perfect equilibrium, one does not have to check for a profitable deviation to all the strategies. Showing that there is no profitable unilateral deviation at each stage of the game, or for any given history, is enough. This is much easier, as the number of stages is smaller. For instance, if the Prisoner's Dilemma game is repeated 5 times each player has $2^5 = 32$ strategies but the number of stages in only 5. The one-stage deviation principle comes in to simplify what seems like a rather daunting task. Interestingly the principle is based on an idea that was originally formulated by David Blackwell (1965) in the context of dynamic programming for a single-player decision problem. The relation to a single-player decision problem comes from the fact that when we want to check if a player i is playing a best response to σ_{-i} in every subgame, then all we are really doing is checking to see whether he is playing an optimal action in each of his information sets, taking the actions of all other players σ_{-i} as given. Thus once the strategies σ_{-i} of other players in the extensive form are considered to be fixed, player i solves a standard dynamic programming problem.

these histories are off the equilibrium path because following the proposed strategies would result in (M, m) being played forever. Thus in any subgame that is off the equilibrium path, the proposed strategies recommend that the players play (F, f) in this period, and in any subsequent period, because a deviation from (M, m) has occurred sometime in the past. Clearly if this is the case then no player would want to choose mum instead of fink because, given his belief that his opponent will play fink, such a deviation from fink to mum will cause him a loss of -2 at this stage (getting -1 instead of 1), together with no gains in subsequent stages because the previous deviation will keep the players on the grim-trigger Nash equilibrium path. Thus in any subgame that is off the equilibrium path no player would ever want to deviate unilaterally from fink to mum.

Now consider the other category of histories that are consecutive sequences of (M, m), which are all the histories that are on the equilibrium path. If a player chooses to play mum, his current payoff is 4, and his continuation payoff from the pair of strategies is an infinite sequence of 4s starting in the next period. Therefore his payoff from following the strategy and not deviating will be

$$v_i^* = 4 + \delta 4 + \delta^2 4 + \cdots = 4 + \frac{4\delta}{1-\delta}.$$

The division of the player's payoff into today's payoff of 4 and the continuation payoff from following his strategy of $\frac{4\delta}{1-\delta}$ is useful. It recognizes the idea that a player's actions have two effects: The first is the direct effect on his immediate payoffs today; the second is the indirect effect on his continuation equilibrium payoff, which results from the way in which the equilibrium continues to unfold.

If the player deviates from mum and chooses fink instead, then he gets 5 instead of 4 in the immediate stage of deviation, followed by his continuation payoff, which is an infinite sequence of 1s. Indeed we had already established that in the continuation game following a deviation off the equilibrium path, every player will stick to playing fink because his opponent is expected to do so forever. Therefore his payoff from deviating from the proposed strategy will be

$$v_i' = 5 + \delta 1 + \delta^2 1 + \cdots = 5 + \frac{\delta}{1-\delta}.$$

Thus we can easily see that the trade-off between sticking to the proposed strategy and deviating boils down to a simple comparison. A deviation will yield an immediate gain of 1 because the player gets 5 instead of 4 in the deviation stage, which needs to be weighed against the loss in continuation payoffs, which will drop from $\frac{4\delta}{1-\delta}$ to $\frac{\delta}{1-\delta}$. We conclude that the player will not want to deviate if $v_i^* \geq v_i'$, which holds if and only if $\delta \geq \frac{1}{4}$.

If the players are sufficiently patient, so that the future carries a fair amount of weight in their preferences, then there is a reward-and-punishment strategy that will allow them to cooperate forever. The loss in continuation payoffs will more than offset the gains from immediate defection, and this will keep the players on the cooperative path of play.

7.4. Application: Tacit Collusion

One of the most celebrated applications of repeated-game equilibria with reward- and-punishment strategies has been to the study of tacit collusion among firms. Most developed countries forbid firms from entering into explicit contracts to restrict competition. In the United States section 1 of the Sherman Antitrust Act of 1890 proclaims as illegal any "conspiracies in restraint of trade," including such contracts to fix prices. However, even if there have been no actual meetings or discussions between competitors, can we be certain that they are not setting prices through some implicit understanding or agreement?

The use of repeated-game strategies can come in handy for firms, which through their beliefs about collusion-supporting strategies can support anticompetitive behavior without ever discussing it or reaching explicit agreements. Instead they employ implicit or tacit collusion. To demonstrate this point consider the Cournot duopoly problem in which each of the two firms has costs of production equal to zero and demand is given by $p = 1 - q_1 - q_2$, and imagine that this market game is played repeatedly in periods $t = 1, 2, ...$ with some discount factor $\delta \in (0, 1)$.

The Nash profits are $\pi_1^* = \pi_2^* = \frac{1}{9}$. Firms can try to sustain the monopoly profit $\pi_i^c = \frac{1}{8}$ as a subgame-perfect equilibrium using grim-trigger strategies (c for cooperate). Each firm on the equilibrium path produces half of the monopoly quantity, $q = \frac{1}{4}$. Given that firm i's best response is $q_i = \frac{1-q_j}{2}$, a deviating firm produces $q = \frac{3}{8}$. The profits from a one time deviation are $\pi_i^d = \frac{9}{64}$. Firms can sustain monopoly profits if and only if the following inequality is satisfied

$$\underbrace{\pi_i^c}_{\substack{\text{Current payoff}\\\text{from cooperate}}} + \underbrace{\frac{\delta \pi_i^c}{1-\delta}}_{\substack{\text{Future payoff}\\\text{if cooperate}}} \geq \underbrace{\pi_i^d}_{\substack{\text{Current payoff}\\\text{from best defect}}} + \underbrace{\frac{\delta \pi^*}{1-\delta}}_{\substack{\text{Future payoff}\\\text{if defect}}},$$

which is equivalent to

$$\delta \geq \frac{\pi_i^d - \pi_i^c}{\pi_i^d - \pi_i^*} = \frac{9}{17}.$$

As long as firms are patient enough, or the probability of future interactions is high enough, firms can sustain (tacit) collusion forever using grim-trigger strategies. But we know from history that even when such collusive agreements are in place there are occasional price wars between the firms, in which collusive behavior fails for some amount of time and is later restored. Our analysis suggests that price wars are instances of off-the-equilibrium-path behavior and hence are puzzling when they do occur. That is, if the collusive behavior is sustainable as a subgame-perfect equilibrium then no firm would want to deviate at any stage on the equilibrium path. Why then would the firms engage in a price war? One explanation may be that they cannot easily detect deviations because, for example, actual produced quantities are not readily observable.

To demonstrate the idea behind this argument, consider the case of OPEC, a cartel that really does manipulate prices through the quantities that are produced (a' la Cournot). Arguably it is prohibitively difficult to observe the quantities produced by each country (each "firm"), which in turn implies that using trigger strategies is problematic. Indeed if the players cannot observe the choices made by other players then how will they know if they should move from the "carrot" equilibrium to the "stick" equilibrium?

A clever observer will make the following argument: In the repeated Cournot model we do not have to observe quantities–as long as no firm deviates, the market price will remain fixed (at $p = \frac{1}{2}$ in the example previously solved). If the price suddenly falls, it is immediately apparent that someone deviated from the proposed quantities, and the grim-trigger strategy results in a price war.

Such an observation is clever indeed, but it fails in the potential instance of random prices. In the market for oil, for example, the price is determined by supply (quantity produced) and demand. In the real world, however, demand is not constant as in our simplistic example. In some weeks there is a greater need for energy (say, in colder weeks when homes need heating) and in others the need is less. As a consequence prices will no longer be "clean" indicators of deviations by firms. Instead they can act only as "noisy" signals that imply either a reduction in demand or a deviation in quantities by some player. This in turn implies that players will be tempted to deviate and offer the excuse "I didn't deviate; demand must have been low!" So it turns out that occasional price wars are an inherent reality of trying to collude when detection is not perfect.

This interesting idea was first put forth in a seminal paper by Green and Porter (1984), which led to a large literature aimed at analyzing repeated games with imperfect monitoring. In such games players cannot perfectly monitor the behavior of other players, and as a consequence they must rely on signals that are not perfect indicators of the behavior of those players.

7.5. Summary

- If a stage-game that has a unique Nash equilibrium is repeated for finitely many periods then it will have a unique subgame-perfect equilibrium regardless of the discount factor's value.
- If a stage-game that has a unique Nash equilibrium is repeated infinitely often with a discount factor $0 < \delta < 1$, it will be possible to support behavior in each period that is not a Nash equilibrium of the one-shot stage-game.
- The "carrot" and "stick" incentives are created by bootstrapping the repetition of the stage-game's unique equilibrium, which becomes a more potent threat as the discount factor approaches 1.
- Repeated games are useful frameworks to understand how people cooperate over time, how firms manage to collude in markets, and how reputations for good behavior are

sustained over time even when short-run temptations are present.

Part III.

STATIC GAMES OF INCOMPLETE INFORMATION

8. Bayesian Games

In all the examples and appropriate tools for analysis that we have encountered thus far, we have made an important assumption: that the game played is common knowledge. In particular we have assumed that the players are aware of who is playing, what the possible actions of each player are, and how outcomes translate into payoffs. Furthermore we have assumed that this knowledge of the game is itself common knowledge. These assumptions enabled us to lay the methodological foundation for such solution concepts as iterated elimination of dominated strategies, rationalizability, and most importantly Nash equilibrium and subgame-perfect equilibrium.

But are these assumptions always reasonable? Is it reasonable, for example, to assume that the production technologies in the Cournot or Bertrand games are indeed common knowledge? And if they are, should we believe that the productivity of workers in each firm is known to the other firm? More generally, is it reasonable to assume that the cost function of each firm is precisely known to its opponent?

Perhaps it is more convincing to believe that firms have a reasonably good idea about their opponents' costs but do not know exactly what they are. Yet the analysis toolbox we have developed so far is not adequate to address such situations. How do we think of situations in which players have some idea about their opponents' characteristics but don't know for sure what these characteristics are?

In the mid-1960s John Harsanyi realized the similarity between beliefs over a player's actions and beliefs over his other characteristics, such as costs and preferences. Harsanyi proceeded to develop an elegant and extremely operational way to capture the idea that beliefs over the characteristics of other players–their types–can be embedded naturally into the framework of game theory that we have already developed. This advancement set Harsanyi up to be the third Nobel Laureate to share the prestigious prize with John Nash and Reinhard Selten in 1994.

We call games that incorporate the possibility that players could be of different types (a concept soon to be well defined) games of incomplete information. As with games of complete information, we will develop a theory of equilibrium behavior that requires players to have beliefs about their opponents' characteristics and their actions, and furthermore requires that these beliefs be consistent or correct. It should be no surprise that this will require very strong assumptions about the cognition of the players: we assume that common knowledge reigns over the possible characteristics of players and over the likelihood that each type of player is indeed part of the game.

To operationalize this idea and endow players with well-defined beliefs over the types of other players, Harsanyi (1967-68) suggested the following framework. Imagine that before the game is played Nature chooses the preferences, or type, of each player from his possible set of types. Another way to think about this approach is that Nature is choosing a game from among a large set of games, in which each game has the same players with the same action sets, but with different payoff functions. If Nature is randomly choosing among many possible games, then there must be a well-defined probability distribution over the different games. It is this observation, together with the requirement that everything about a game must be common knowledge, that will make this setting amenable to equilibrium analysis.

At this stage an example is useful. Consider the following simple "entry game," depicted in Figure 26, in which an entrant firm, player 1, decides whether or not to enter a market. The incumbent firm in that market, player 2, decides how to respond to an entry decision of player 1 by either fighting or accommodating entry. The payoffs show that if player 1 enters, player 2's best response is to accommodate, which in turn implies that the unique subgame-perfect equilibrium is for player 1 to enter and for player 2 to accommodate entry.

Now imagine that there is one type of player 1 with payoffs as given in Figure 26, and there are two types of player 2. The first type, called "rational," has payoffs as shown in

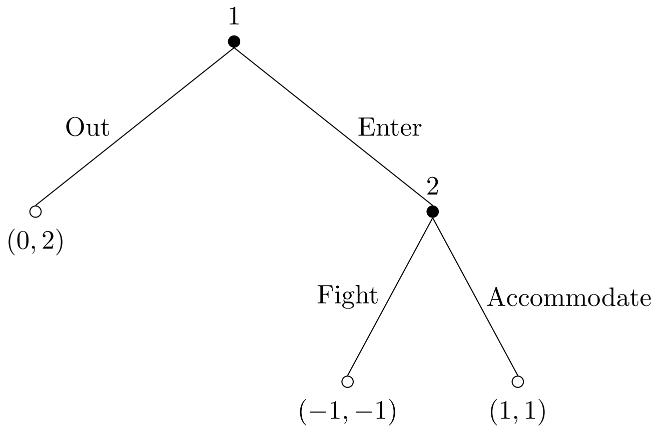

Figure 26.: A simple entry game

Figure 26. The second type, called “crazy,” enjoys fighting and the payoff he gets from (Enter, Fight) is 2 instead of -1. The structure of the game is fixed by the set of players N and the action spaces A_i for each player $i \in N$, yet Nature chooses which type of player 2 is playing the game with player 1. To complete the structure of this game we need to state the likelihood or probability of each type being selected by Nature. Let p denote the probability that Nature chooses the rational type. We can now depict the extensive form of this incomplete-information entry game in Figure 27.

Player 2 has four strategies $\{AA, AF, FA, FF\}$. The strategy set for player 1 is simply $\{E, O\}$. The normal form representation of the game is depicted in Figure 28.

Though our analysis of this game seems complete (ignoring mixed strategies), it may still be confusing to appreciate fully the strides we took from the start of describing a world in which players were not sure about certain elements of the game. First, we modeled this situation as one in which players have uncertainty about the preferences of other players. Second, we assumed that players share the same beliefs about this uncertainty, which allowed us to create a new game for which the standard equilibrium analysis applied. In particular, the matrix game has two players with expected payoffs derived from the probability distribution over the different types of each player (in this case only player 2 had types). This was John Harsanyi's ingenious solution: we cannot perform equilibrium analysis unless we assume that each player knows the distribution of his opponents' types (the common prior

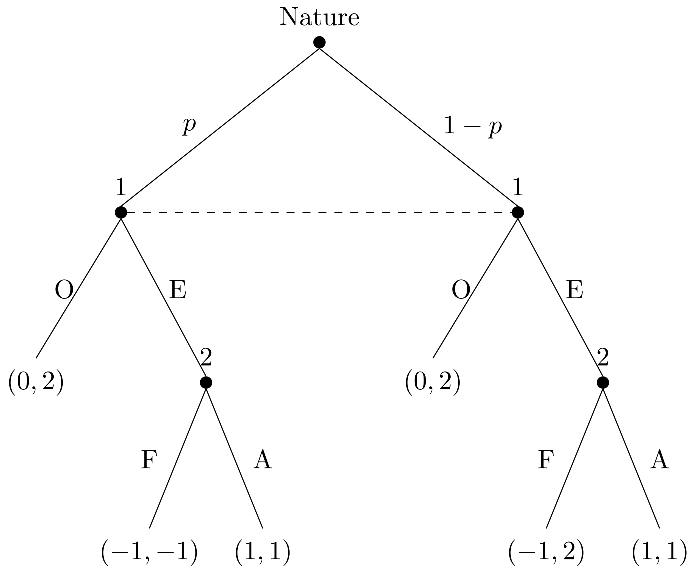

Figure 27.: An incomplete information entry game

		Player 2			
		AA	AF	FA	FF
Player 1	O	$0, 2$	$0, 2$	$0, 2$	$0, 2$
	E	$1, 1$	$2p-1, 2-p$	$1-2p, 1-2p$	$-1, 2-3p$

Figure 28.: Normal form representation of the incomplete information entry game

assumption). Then, with this requirement in place, once a player assumes some behavior of the different types of his opponents then he can calculate his expected payoff from his own different actions. In this way Harsanyi changed the complex and challenging concept of incomplete information into a well-known game of imperfect information, in which Nature chooses the players' types and we can then use our standard tools of analysis.

8.1. Strategic Representation of Bayesian Games

8.1.1. Players, Actions, Information, and Preferences

Definition 55 *The normal-form representation of an n-player static Bayesian game of incomplete information is*

$$\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \rangle$$

where $N = \{1, 2, ..., n\}$ *is the set of players;* A_i *is the action set of player* i*;* $\Theta_i = \{\theta_{i1}, \theta_{i2}, ..., \theta_{ik_i}\}$ *is the type space of player* i*;* $v_i : A \times \Theta_i \to \mathbb{R}$ *is the type-dependent payoff function of player* i*, where* $A = A_1 \times A_2 \times \cdots \times A_n$*; and* ϕ_i *describes the belief of player* i *with respect to the uncertainty over the other players' types, that is,* $\phi_i(\theta_{-i}|\theta_i)$ *is the (posterior) conditional distribution on* θ_{-i} *(all other types but* i*) given that* i *knows his type is* θ_i*.*

It is convenient to think about a static Bayesian game as one that proceeds through the following steps:

1. Nature chooses a profile of types $(\theta_1, \theta_2, ..., \theta_n)$.
2. Each player i learns his own type, θ_i, which is his private information, and then uses his prior ϕ_i to form posterior beliefs over the other types of players.
3. Players simultaneously (hence this is a static game) choose actions $a_i \in A_i$, $i \in N$.
4. Given the players' choices $a = (a_1, a_2, ..., a_n)$, the payoffs $v_i(a; \theta_i)$ are realized for each player $i \in N$.

In the foregoing definition a player's payoff $v_i(a; \theta_i)$ depends on the actions of all players and only on i's type, but it does not depend on the types of the other players θ_{-i}. This particular assumption is known as the private values case because each type's payoff depends only on his private information. This case is not rich enough to capture all the interesting examples that we will analyze, and for this reason we will later introduce the case of common values, in which $v_i(a_1, a_2, ..., a_n; \theta_1, \theta_2, ..., \theta_n)$ is possible. For expositional clarity, we first explore the private values setting.

8.1.2. Deriving Posteriors from a Common Prior: A Player's Beliefs

Conditional probabilities follow a mathematical rule that derives the way in which a player or decision maker should change a prior (initial) belief in the light of new evidence, resulting in a posterior (updated) belief. In our application the idea can be described as follows. First, before Nature chooses the actual type of each player, imagine that every player does not yet know what his type will be; but he does know the probability distribution that Nature uses to choose the types for all the players. Later, after Nature has chosen a type for each player, they all independently and privately learn their types. This new piece of information for each player, his type, may provide some new piece of evidence about how the other players? types may have been chosen. It is in this respect that a player may derive new beliefs about the other players once he learns his type.

As a concrete example, imagine that there are two of many possible states of Nature or events (the possible choices that Nature can make). One is that it will be sunny (S) and the other is that the waves will be high (H). These two states can occur exclusively or together according to some prior distribution $\phi(\cdot)$. That is, $\phi(\cdot)$ describes the probabilities assigned to any combination of these states being true. Let $\phi(S)$ be the prior probability that it will be sunny, $\phi(H)$ be the prior probability that the waves will be high, and $\phi(S \cap H)$ be the prior probability that it will be sunny and that the waves will be high. Let's imagine that you wake up and see that it is sunny; without seeing them, what can you infer about the probability that the waves are high? It is not necessarily true that it is $\phi(H)$, because you just learned that it is sunny and this new information may be relevant. This is where the conditional probability formula applies, because it precisely computes the probability of state H given that you know that state S happened. Formally we have

Definition 56 *Conditional on event S being true, the conditional probability that event H is true is given by*

$$\Pr(H|S) = \frac{\phi(S \cap H)}{\phi(S)}.$$

8.1.3. Strategies and Bayesian Nash Equilibrium

Recall that in the static normal-form games of complete information we did not make a distinction between actions and strategies because choices were made once and for all. For games of incomplete information, however, we need to be a bit more careful to specify strategies correctly. The representation of a Bayesian game described earlier has action sets, A_i , for each player $i \in N$. However, each player i can be one of several types $\theta_i \in \Theta_i$, and each type θ_i may choose a different action from the set A_i. Thus to define a strategy for player i we need to specify what each type of player i will choose when Nature calls upon this type to play the game. For this we define strategies as follows:

Definition 57 *Consider a static Bayesian game*

$$\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \rangle.$$

A pure strategy for player i is a function $s_i : \Theta_i \to A_i$ that specifies a pure action $s_i(\theta_i)$ that player i will choose when his type is θ_i. A mixed strategy is a probability distribution over a player's pure strategies.

Now that we have completely defined what a static Bayesian game is, and what the strategies for each player are, it is easy to define a solution concept that is derived from Nash equilibrium as follows:

Definition 58 *In the Bayesian game*

$$\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \rangle$$

a strategy profile $s^* = (s_1^*(\cdot), s_2^*(\cdot), ..., s_n^*(\cdot))$ is a pure-strategy Bayesian Nash equilibrium if, for every player i, for each of player i's type $\theta_i \in \Theta_i$, and for every $a_i \in A_i$, $s_i^*(\cdot)$ satisfies

$$\sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i) \geq \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i(a_i, s_{-i}^*(\theta_{-i}); \theta_i).$$

To restate the definition, a Bayesian Nash equilibrium has each player choose a type-contingent strategy $s_i^*(\cdot)$ so that given any one of his types θ_i, and his beliefs about the strategies of his opponents $s_{-i}^*(\cdot)$, his expected payoff from $s_i^*(\theta_i)$ is at least as large as that from any one of his actions $a_i \in A_i$.

If there is a continuum of types θ_i, then the summation in the above definition is replaced by an expectation $E_{\theta_{-i}}(\cdot|\theta_i)$.

8.2. Examples

This section offers two examples as illustrations of Bayesian games of incomplete information. In the first example each player has a finite number of actions and types, while in the second each player has a continuum of types.

8.2.1. Teenagers and the Game of Chicken

Two teenagers, players 1 and 2, have borrowed their parents' cars and decided to play the game of chicken as follows: They drive toward each other in the middle of a street, and just before impact they must simultaneously choose whether to be chicken and swerve to the right, or continue driving head on. If both are chicken then both gain no respect from their friends but suffer no losses; thus both get a payoff of 0. If i continues to drive while $j \neq i$ plays chicken then i gains all the respect, which is a payoff of R, and j gets no respect, which is worth 0. In this case both players suffer no additional losses. Finally if both continue to drive head on then they split the respect (because respect is considered to be relative), but an accident is bound to happen and they will at least be reprimanded by their parents, if not seriously injured. An accident imposes a personal loss of k on each player, so the payoff to each one is $\frac{R}{2} - k$.

There is, however, a potential difference between these two youngsters: The punishment, k, depends on the type of parents they have. For each kid, parents can be either harsh (H) or lenient (L) with equal probability, and the draws from Nature on the types of parents are independently distributed. This means that the likelihood that player i's parents are harsh is equal to $\frac{1}{2}$ and is independent of the type of parents that player j has. If player i's parents are harsh then the cost of an accident is high, denoted by $k = H$. If instead the parents are lenient then cost of an accident is low, denoted by $k = L < H$. Each kid knows the type of his parents but does not know the type of his opponent's parents. The distribution of types is common knowledge (this is the common prior assumption).

There are four states of nature $\{LL, LH, HL, HH\}$. The action sets of players are $A_1 = \{C, D\}$ and $A_2 = \{c, d\}$, for "chicken" and "drive". A strategy for player 1 is denoted by $xy \in S_1 = \{CC, CD, DC, DD\}$, where x is what he does if he is an L type and y is what he does if he is an H type. Similarly $S_2 = \{cc, cd, dc, dd\}$.

The expected payoffs are calculated as follows. For example, if player 1 chooses CD and player 2 chooses dd then the expected payoffs for player 1 are

$$Ev_1(CD, dd) = \frac{1}{4} \times 0 + \frac{1}{4} \times 0 + \frac{1}{4} \times \left(\frac{R}{2} - H\right) + \frac{1}{4} \times \left(\frac{R}{2} - H\right) = \frac{R}{4} - \frac{H}{2}.$$

This results in the matrix-form Bayesian game depicted below:

Player 2

Player 1	cc	cd	dc	dd
CC	$0, 0$	$0, \frac{R}{2}$	$0, \frac{R}{2}$	$0, R$
CD	$\frac{R}{2}, 0$	$\frac{3R}{8} - \frac{H}{4}, \frac{3R}{8} - \frac{H}{4}$	$\frac{3R}{8} - \frac{H}{4}, \frac{3R}{8} - \frac{L}{4}$	$\frac{R}{4} - \frac{H}{2}, \frac{3R}{4} - \frac{L}{4} - \frac{H}{4}$
DC	$\frac{R}{2}, 0$	$\frac{3R}{8} - \frac{L}{4}, \frac{3R}{8} - \frac{H}{4}$	$\frac{3R}{8} - \frac{L}{4}, \frac{3R}{8} - \frac{L}{4}$	$\frac{R}{4} - \frac{L}{2}, \frac{3R}{4} - \frac{L}{4} - \frac{H}{4}$
DD	$R, 0$	$\frac{3R}{4} - \frac{L}{4} - \frac{H}{4}, \frac{R}{4} - \frac{H}{2}$	$\frac{3R}{8} - \frac{L}{4} - \frac{H}{4}, \frac{R}{4} - \frac{L}{2}$	$\frac{R}{2} - \frac{L}{2} - \frac{H}{2}, \frac{R}{2} - \frac{L}{2} - \frac{H}{2}$

To solve for the Bayesian Nash equilibria we need to have more information about the parameters R, H, and L. Assume that $R = 8$, $H = 16$, and $L = 0$, which results in the matrix-form game of Figure 29.

Player 2

Player 1	cc	cd	dc	dd
CC	$0, 0$	$0, 4$	$0, 4$	$0, 8$
CD	$4, 0$	$-1, -1$	$-1, 3$	$-6, 2$
DC	$4, 0$	$3, -1$	$3, 3$	$2, 2$
DD	$8, 0$	$2, -6$	$2, 2$	$-4, -4$

Figure 29.

A quick analysis reveals that the game has a unique pure-strategy Bayesian Nash equilibrium: (DC, dc). That is, the children of lenient parents will continue driving head on, while those of harsh parents will swerve to avoid the costly consequences.

8.2.2. Study Groups

Let's consider a situation with two students, players 1 and 2, who have to hand in a joint lab assignment. Each student i can either put in the effort ($e_i = 1$) or shirk ($e_i = 0$), where the cost of putting in the effort is the same for each student and is given by some $c < 1$, while shirking involves no cost. If either one or both of the students put in the effort then the lab assignment is a success, while if both shirk then it is a failure.

Students vary in how much they care about their educational success, which is described by the following specification. Each student has a type $\theta_i \in [0,1]$, which is independently and uniformly distributed over this interval. That is, for each player i the distribution of his type is $F(\theta_i) = \theta_i$ for $\theta_i \in [0,1]$, and the density is $f(\theta_i) = 1$ for $\theta_i \in [0,1]$. Student i's personal value from a successful assignment is given by the square of his type, θ_i^2. Hence if student i chooses to put in the effort then his payoff is guaranteed to be $\theta_i^2 - c$. If he chooses to shirk, however, then his payoff depends on what his partner does. If his partner j puts in the effort then student i's payoff is θ_i^2, while if his partner j shirks as well then student i's payoff is 0. Each student knows only his own type before choosing his effort e. It is also common knowledge that the types are distributed independently and uniformly on $[0,1]$ and that the cost of effort is $c < 1$.

This is an example of a Bayesian game with continuous type spaces and discrete sets of actions. It is not too useful to try to draw the game tree that represents this game because of the continuous types, nor can we derive a matrix because, despite the two possible actions, $e \in \{0,1\}$, the continuous types imply that there is a continuum of strategies: a pure strategy for player i is a function $s_i : [0,1] \to \{0,1\}$ that identifies a choice of $e_i \in \{0,1\}$ for every type $\theta_i \in [0,1]$.

Given a belief of player i about the strategy of player j, the only factor that affects i's payoff is the probability that player j chooses $e_j = 1$. The best response of player i will be to choose effort $e = 1$ if this is (weakly) better than choosing $e = 0$, which will be true if and only if

$$\theta_i^2 - c \geq \theta_i^2 \Pr(s_j(\theta_j) = 1)$$

which can be rewritten as

$$\theta_i \geq \sqrt{\frac{c}{1 - \Pr(s_j(\theta_j) = 1)}}.$$

This means that we are looking for a Bayesian Nash equilibrium in which each student has a threshold type $\theta_i \in [0,1]$ such that

$$s_i(\theta_i) = \begin{cases} 0, & \text{if } \theta_i < \hat{\theta}_i \\ 1, & \text{if } \theta_i \geq \hat{\theta}_i. \end{cases}$$

This observation lets us calculate what a player's best-response function is given his belief about the threshold strategy of his opponent. If player j is using a threshold $\hat{\theta}_j$ so that $e = 1$ if and only if $\theta_j \geq \hat{\theta}_j$ then it follows that $\Pr(s_j(\theta_j) = 1) = 1 - \hat{\theta}_j$. This in turn means that player i will choose $e = 1$ if and only if

$$\theta_i \geq \sqrt{\frac{c}{\hat{\theta}_j}}.$$

In the symmetric Bayesian Nash equilibrium $\hat{\theta}_1 = \hat{\theta}_2 = c^{1/3} < 1$. Also, there is no other (asymmetric) equilibrium. Hence the unique Bayesian Nash equilibrium is the symmetric threshold choices $\theta^* = c^{1/3}$, which are implemented by the following strategies for each player $i \in \{1, 2\}$:

$$s_i^*(\theta_i) = \begin{cases} 0, & \text{if } \theta_i < c^{1/3} \\ 1, & \text{if } \theta_i \geq c^{1/3}. \end{cases}$$

8.3. Inefficient Trade and Adverse Selection

One of the main conclusions of competitive market analysis in economics is that markets allocate goods to the people who value them the most. The simple intuition behind this conclusion works as follows: If a good is misallocated so that some people who have it value it less than people who do not, then so-called market pressures will cause the price of that good to increase to a level at which the current owners will prefer to sell it rather than hold on to it, and the people who value it more will be willing to pay that price. The determination of such a price is not clearly specified, but various mechanisms such as bargaining, auctions, or market intermediaries may help obtain it.

This powerful argument is based on some assumptions, one of which is that the value of the good is easily understood by all market participants, or in our terminology, there is perfect information about the value of the good. It is important to understand the extent to which these arguments stand or fall in the face of incomplete information, when some people are better informed about the value of goods than others. To address this question we will develop a simple example that follows in the spirit of the important contribution made by George Akerlof (1970), a contribution that introduced the idea of adverse selection into economics and earned its author a Nobel Prize.

Consider a buyer who wants to buy a used car. The quality of the car can be either H, or M, or L with equal probability. The seller knows the car's true quality but not the buyer who only knows the distribution of qualities. We assume that when the quality is L is seller values the car at \$10000 while the buyer values it at \$14000; when the quality is M is seller values the car at \$20000 while the buyer values it at \$24000; and when the quality is H is seller values the car at \$30000 while the buyer values it at \$34000.

Let's assume that the buyer makes a take-it-or-leave-it offer to the seller. The assumptions on the payoffs of the two players imply that from an efficiency point of view the buyer should buy the car. Indeed if the quality of the car were common knowledge then there would be many prices at which the buyer and the seller would be happy to trade the car.

The buyer knows that the car is valued by the seller at \$20000 on average. So, he could make an offer of \$20000. But then he knows that only the L and M type sellers would accept such an offer, because an H type seller values the car at \$30000. But when the buyer faces only the L and M types sellers, the average value of the car for these two types is \$15000. But at this price the M type seller does not want to sell.

Proposition 59 *Trade can occur in a Bayesian Nash equilibrium only if it involves the lowest type of seller trading. Furthermore any price* $p^* \in [\$10000, \$14000]$ *can be supported as a Bayesian Nash equilibrium.*

The conclusion of this result is that trade will occur only if the quality of the car is the lowest. This happens because of what is called adverse selection. When the buyer is willing to pay a price equal to his average value, then the type of seller who is willing to sell at this price is below average, because the best types select not to sell for an average price, hence the adverse selection of lower-than-average sellers. In the example this unraveling causes traded quality to drop to its lowest level, preventing the market from implementing efficient trade outcomes.

It is also worth mentioning that this scenario falls into the category of games with common values. In this category of games the type of one player affects the payoffs of another player. In this example the type of seller affects the payoffs of both players. It is precisely this feature of the game that causes the adverse effects of equilibrium.

8.4. Committee Voting

Many decisions are made by committees. Examples include legislatures, firms, membership clubs, and juries. Each member of the committee will have different information, or different ways of interpreting information, and the goals of the committee members may be congruent or diverse. Because each committee member has private information about his own preferences or about the value of different decisions to other players, committee votes can be modeled as Bayesian games.

As an example, consider a jury made up of two players (jurors) who must collectively decide whether to acquit (A) or to convict (C) a defendant. The process calls for each player to cast a sealed vote, and the defendant is convicted only if both vote C. The problem is that there is uncertainty about whether the defendant is guilty (G) or innocent (I). The prior probability that the defendant is guilty is given by $q > 1/2$ and is common knowledge. Assume that each player cares about making the right decision, so that if the defendant is guilty then each player receives a payoff of 1 from a conviction and 0 from an acquittal. If instead the defendant is innocent then each player receives a payoff of 0 from a conviction and 1 from an acquittal. If the only information available to the players is the probability q then the game can be described by the matrix in Figure 30.

Clearly because $q > 1/2$ each player has a weakly dominant strategy, which is to vote C and convict the defendant. Now imagine that things are a bit more complex. Each player i has a different expertise and, when observing the evidence, gets a private signal $\theta_i \in \{\theta_G, \theta_I\}$

		Player 2	
		A	C
Player 1	A	$1-q, 1-q$	$1-q, 1-q$
	C	$1-q, 1-q$	q, q

Figure 30.: Committee voting

that contains valuable information. In particular, when the defendant is guilty player i is more likely to receive the signal θ_G than when the defendant is innocent, and vice versa for θ_I. Assume that the signals are independent: the probability of receiving the signal θ_G when the defendant is guilty is equal to the probability of receiving the signal θ_I when the defendant is innocent, so that

$$\Pr(\theta_G|G) = \Pr(\theta_I|I) = p > \frac{1}{2}.$$

The signal each player receives is his private information. With this information structure in place we have a Bayesian game of incomplete information in which each player i will choose a signal-dependent action to maximize the probability that a guilty defendant is convicted while an innocent one is acquitted. Because each player has two types, given by the signal he observes, each player will have four pure strategies, $s_i \in \{AA, AC, CA, CC\}$, where strategy xy means that he chooses x when his signal is θ_G and he chooses y when his signal is θ_I. Notice that, like the adverse selection game, this is a common-values game because both players want the right judgment to be rendered.

A natural first question is the following: if each player would be able to convict or acquit the defendant by himself, how would his signal determine his choice? Consider the decision problem with just one player. Without receiving the signal, the player knows that the defendant is guilty with probability $q > 1/2$, so that he would choose to convict the defendant. After receiving the signal, however, the player will update his beliefs about the defendant as follows.

When player i receives signal θ_G his updated belief is

$$\Pr(G|\theta_i = \theta_G) = \frac{\Pr(G \text{ and } \theta_i = \theta_G)}{\Pr(\theta_i = \theta_G)} = \frac{qp}{qp + (1-q)(1-p)} > q$$

which means that the player is even more convinced that the defendant is guilty, and hence will choose to convict him. If, however, the signal was θ_I then the updated belief is

$$\Pr(G|\theta_i = \theta_I) = \frac{\Pr(G \text{ and } \theta_i = \theta_I)}{\Pr(\theta_i = \theta_I)} = \frac{q(1-p)}{q(1-p)+(1-q)p} < q.$$

This means that the player is less sure of the defendant's guilt than he was before the signal, and whether this is enough to persuade him to acquit the defendant depends on the value of p. In particular he will choose to acquit the defendant if and only if

$$\frac{q(1-p)}{q(1-p)+(1-q)p} < \frac{1}{2}.$$

which reduces to $p > q$. Indeed the reason p has to be "high enough" is that the signal has to be informative enough about the defendant's actual condition. If, for example, $p = 1/2$ the signal contains no information and the posterior $\Pr(G|\theta_i = \theta_I) = q$. As p increases above $1/2$, the signal becomes more informative and the posterior, $\Pr(G|\theta_i = \theta_I) = q$, decreases. Once p increases above the critical level of q then the signal is strong enough to reverse the player's prior conviction, and his posterior belief is that the defendant is more likely to be innocent than guilty.

The observation that if $p > q$ then each player would choose to vote according to his signal in the one-person decision problem leads to the obvious next question: is the pair of strategies in the game, $s_i = CA$, where each votes according to his signal, a Bayesian Nash equilibrium?

Notice that given the rule that unanimity is needed to convict the defendant, a player is decisive, or "pivotal," only if the other player chooses C. The reason is that if player j chooses A then the defendant will be acquitted for sure, regardless of what player i chooses, while if player j chooses C then the decision of whether the defendant will be convicted or not depends on the choice of player i. For this reason, if player i believes that player j is playing according to the strategy CA, then he must also believe that his own vote matters only when player j observes a signal $\theta_j = \theta_G$. We therefore need to calculate the posterior belief that player i has about whether the defendant is guilty conditional on his own signal and on the belief that player j's signal is θ_G.

If player i's signal was $\theta_i = \theta_G$ then his updated belief is

$$\Pr(G|\theta_i = \theta_G \text{ and } \theta_j = \theta_G) = \frac{\Pr(G \text{ and } \theta_i = \theta_G \text{ and } \theta_j = \theta_G)}{\Pr(\theta_i = \theta_G \text{ and } \theta_j = \theta_G)} = \frac{qp^2}{qp^2 + (1-q)(1-p)^2} > q$$

which means that the player is even more convinced that the defendant is guilty and hence will choose C to guarantee that the defendant is convicted when both have the signal θ_G.

If player i's signal was $\theta_i = \theta_I$ then his updated belief is

$$\Pr(G|\theta_i = \theta_I \text{ and } \theta_j = \theta_G) = \frac{\Pr(G \text{ and } \theta_i = \theta_I \text{ and } \theta_j = \theta_G)}{\Pr(\theta_i = \theta_I \text{ and } \theta_j = \theta_G)} = q.$$

Let's examine whether CA is a Bayesian Nash equilibrium. Suppose player j votes C if $\theta_j = \theta_G$ and A if $\theta_j = \theta_I$. Suppose now player i has received $\theta_i = \theta_G$. If he votes C his expected utility is

$$\Pr(\theta_j = \theta_G|\theta_i = \theta_G)\Pr(G|\theta_j = \theta_G, \theta_i = \theta_G)\times 1 + \Pr(\theta_j = \theta_I|\theta_i = \theta_G)\Pr(I|\theta_j = \theta_I, \theta_i = \theta_G)\times 1.$$

If he votes A the defendant is acquitted regardless of j's signal and i's expected utility is

$$\Pr(\theta_j = \theta_G|\theta_i = \theta_G)\Pr(I|\theta_j = \theta_G, \theta_i = \theta_G)\times 1 + \Pr(\theta_j = \theta_I|\theta_i = \theta_G)\Pr(I|\theta_j = \theta_I, \theta_i = \theta_G)\times 1.$$

Voting C gives a higher expected utility because the second terms in the above two expressions are equal and the probability of G is higher than the probability of I when both jurors have received guilty signals.

Next consider the expected utilities of player i when $\theta_i = \theta_I$. If he votes A his expected utility is

$$\Pr(\theta_j = \theta_G|\theta_i = \theta_I)\Pr(I|\theta_j = \theta_G, \theta_i = \theta_I)\times 1 + \Pr(\theta_j = \theta_I|\theta_i = \theta_I)\Pr(I|\theta_j = \theta_I, \theta_i = \theta_I)\times 1.$$

If he votes C his expected utility is

$$\Pr(\theta_j = \theta_G|\theta_i = \theta_I)\Pr(G|\theta_j = \theta_G, \theta_i = \theta_I)\times 1 + \Pr(\theta_j = \theta_I|\theta_i = \theta_I)\Pr(I|\theta_j = \theta_I, \theta_i = \theta_I)\times 1.$$

Voting C gives a higher expected utility because the second terms in the above two expressions are equal and the probability of G is higher than the probability of I when one signal is G and the other is I.

This means that when player i is conditioning the value of his signal on the event when his vote actually counts, which is when player j chooses C, then his signal $\theta_i = \theta_I$ becomes less convincing about the defendant's innocence. In fact, given the symmetric informational structure of the signals, if player i is in the situation in which he believes that he is pivotal then observing a signal of innocence is canceled out by his belief that player j saw a signal of guilt. That is, both signals cancel out, leaving the posterior equal to the prior q.

We conclude therefore that playing $s_i = CA$ for both players is not a Bayesian Nash equilibrium. In fact both players choosing CC, always convict regardless of the signal, is a Bayesian Nash equilibrium. This means that when both players receive a signal $\theta_i = \theta_I$, which occurs with probability $q(1-p)^2 + (1-q)p^2 > 0$, despite the fact that it is more likely that the defendant is innocent, he will still be convicted.

This example is a simple case of what Feddersen and Pesendorfer (1996) refer to as the "swing voter's curse" and is also closely related to independent work by Austen-Smith and Banks (1996). When a voter believes that his vote counts, he must condition this on the situation in which his vote counts. But then it means that he will interpret his information differently and not use it in an unbiased way. The jury game here is a simple two-player example of the more general analysis in Feddersen and Pesendorfer (1998), who show that with more than two players there will sometimes be conditions under which players will vote based on their information, but that the problem of the swing voter's curse is generally present.

This observation is quite important for scholars of voting in political science because it is in tension with the famous Condorcet jury theorem. That theorem states that under certain assumptions, the likelihood that a group of individuals will choose the correct alternative by majority voting exceeds the likelihood that any individual member of the group will choose that alternative. One of the assumptions is that individuals in the voting game behave exactly as they would if choosing alone-they will follow their information in voting. However, the insights from our example show that in strategic situations, individuals must form beliefs about when their vote counts and incorporate these beliefs into their own information.

8.5. Summary

- In most real-world situations players will not know how much their opponents value different outcomes of the game, but they may have a good idea about the range of their valuations.
- It is possible to model uncertainty over other players' payoffs by introducing types that represent the different possible preferences of each player. Adding this together with Nature's distribution over the possible types defines a Bayesian game of incomplete information.
- Using the common prior assumption on the distribution of players' types, it is possible to adopt the Nash equilibrium concept to Bayesian games, renamed a Bayesian Nash equilibrium.
- Markets with asymmetric information can be modeled as games of incomplete information, resulting in Bayesian Nash equilibrium outcomes with inefficient trade outcomes.

9. Auctions and Competitive Bidding

The use of auctions to sell goods has become commonplace thanks to the Internet auction platform eBay, which has become a popular shopping destination for over 100 million households across the globe. Before the age of the Internet, the thought of an auction raised visions of the sale of a Picasso or a Renoir in one of the prestigious auction houses, such as Sotheby's (founded in 1744) or Christie's (founded in 1766). In fact the use of auctions dates back much further. For a history of auctions see Cassidy (1967). Auctions are also used extensively by private- and public-sector entities to procure goods and services.

The use of game theory to analyze both behavior in auctions and the design of auctions themselves, was introduced by the Nobel Laureate William Vickrey (1961), whose work spawned a large and still-expanding literature. The "big push" of game theoretical research on auctions happened after the successful use of game theory to advise both the U.S. government and the bidding firms when the Federal Communications Commission first decided to auction off portions of the electromagnetic spectrum for use by telecommunication companies in 1994. This auction was considered so successful that a reference to the work of many game theorists appears in an article in The Economist titled "Revenge of the Nerds" (July 23, 1994, page 70).

As we will soon see, auctions have many desirable properties, and these have made them a favorite choice of the U.S. Federal Acquisition Regulation as the legally preferred form of procurement in the public sector. They are very transparent, they have well-defined rules, they usually allocate the auctioned good to the party who values it the most, and, if well designed, they are not too easy to manipulate. Generally speaking there are two common types of auctions. The first type, as we will refer to it, is the open auction, in which the bidders observe some dynamic price process that evolves until a winner emerges. There are two common forms of open auctions:

The English Auction: This is the classic auction we often see in movies (e.g.,The Red Violin), in which the bidders are all in a room (or nowadays sitting by a computer or a phone) and the price of the good goes up as long as someone is willing to bid it higher. Once the last increase is no longer challenged, the last bidder to increase the price wins the auction and pays that price for the good. (The price may start at some minimum threshold, which would be the seller's reserve price.)

The Dutch Auction: This less familiar auction almost turns the English auction on its head. As with the English auction, the bidders observe the price changes in real time, but instead of starting low and rising by pressure from the bidders, the price starts at a prohibitively high value and the auctioneer gradually drops the price. Once a bidder shouts 'buy,' the auction ends and the bidder gets the good at the price at which he cried out. This auction was and still is popular in the flower markets of the Netherlands, hence its name.

The second common type of auction is the sealed-bid auction, in which participants write down their bids and submit them without knowing the bids of their opponents. The bids are collected, the highest bidder wins, and he then pays a price that depends on the auction rules. As with open auctions, there are two common forms of sealed-bid auctions:

The First-Price Sealed-Bid Auction: This very common auction form has each bidder write down his bid and place it in an envelope; the envelopes are opened simultaneously. The highest bidder wins and then pays a price equal to his own bid. A mirror image of this auction, sometimes referred to by practitioners as a reverse auction, is used by many governments and businesses to award procurement contracts. For example, if the government wants to build a new building or highway, it will present plans and specifications together with a request for bids. Each potential builder who chooses to participate will submit a sealed bid; the lowest bidder wins and receives the amount of its bid upon completion of the project (or possibly incremental amounts upon the completion of agreed-upon milestones).

The Second-Price Sealed-Bid Auction: As with the first-price sealed-bid auction, each bidder writes down his bid and places it in an envelope; the envelopes are opened simultaneously and the highest bidder wins the auction. The difference is that although the highest bidder wins, he does not pay his bid but instead pays a price equal to the second-highest bid or the highest losing bid. This auction may not seem common or familiar, but it turns out that it has very appealing properties and shares a strong connection to the very common English auction.

Regardless of the type of auction that is being administered, two things should be obvious. First, auctions are games in which the players are the bidders, the actions are the bids, and the payoffs depend on whether or not one receives the good and how much one pays for it (and possibly how much one pays for participating in the auction in the first place). Second, it is hard to believe that bidders know exactly how much the good being sold is worth to the other bidders. Hence auctions have all the characteristics we have specified as appropriate for modeling as Bayesian games of incomplete information.

9.1. Independent Private Values

Buyers can have two reasons to purchase a good. They may wish to consume it, like some specialty food, thus using it for their own consumption benefit. Or they may wish to buy it as an investment, like gold or stocks, and possibly sell it at a later date. Motives can also be mixed, like those for buying a rare painting or a home, which can be both enjoyed by the buyer and considered an investment.

In the first case, buying a good for immediate consumption, the only valuation of the good that should matter to each potential buyer is how much it is worth from his own private perspective, with no consideration of how much others value the good. Consider a seller who offers a prime but extremely ripe piece of filet mignon that will go bad within the hour. With no time to turn around and resell it, the only thing one can do with the filet is to quickly grill it and eat it. The value one imputes to it, or one's willingness to pay for it, should depend only on how much one will enjoy this particular dish. This situation is referred to as one of private values, in which each person's willingness to pay depends only on his own type, and this in turn is private information. This differs from the common-values setting, in which the preferences of some players may depend on the types of other players.

A single risk-neutral seller wishes to sell an indivisible object to one of n risk neutral buyers. Buyer i's value for the object, v_i, is drawn from the interval $[0,1]$ according to the distribution function $F_i(v_i)$ with density function $f_i(v_i)$, where $F' = f$. We assume that the buyers' values are mutually independent. Each buyer knows his own value but not the values of the other buyers. However, the density functions, $f_1, ..., f_n$, are common knowledge. If buyer i's value is v_i, then if he wins the object and pays p, his payoff (i.e., his Bernoulli utility function) is $v_i - p$.

An auction defines a game of incomplete information, where $\Theta_i = [0,1]$, $\Theta = [0,1]^n$, $F(\cdot) = \{F_1(\cdot), ..., F_n(\cdot)\}$, and $u_i = v_i - p$ if player i wins the auction and $u_i = 0$ otherwise (the auctioneer is player $n+1$, but he is not included in the above list).

9.1.1. Equilibrium Behavior in a Second-Price, Sealed-Bid Auction

Theorem 60 *It is a (weakly) dominant strategy of every bidder in a second-price, sealed-bid auction to bid* $b_i(v_i) = v_i$.

This famous result was first discovered by Vickrey (1961). As we will see, it is also quite straightforward once the analysis is laid out. We will prove this result by showing that for any valuation θ_i bidding θ_i weakly dominates both higher bids and lower bids. The analysis will argue the case for not bidding $b_i < \theta_i$, and the other argument is practically the mirror image.

Consider the case of a bidder who is bidding below his valuation, $b_i < \theta_i$. There are three possible cases of interest with respect to the other $n-1$ bids:

Case 1: Player i is the highest bidder, in which case i wins and pays a price $p < b_i$. This corresponds to the situation in which all the other $n-1$ bids are below b_i, including the second-highest bid. If instead of bidding b_i player i would have bid θ_i then he would still win and pay the same price, so in case 1 bidding his valuation is as good as bidding b_i.

Case 2: The highest bidder j bids $b_j^* > \theta_i$, in which case i loses. If instead of bidding b_i player i would have bid θ_i then he would still lose to b_j^*, so in case 2 bidding his valuation is as good as bidding b_i.

Case 3: The highest bidder j bids $b_i < b_j^* < \theta_i$. If instead player i would have bid θ_i, he would have won the auction and received a payoff of $v_i = \theta_i - b_j^* > 0$, making this a profitable deviation, so in case 3 bidding his valuation is strictly better than bidding b_i.

Since cases 1-3 cover all the relevant situations, we conclude that bidding θ_i weakly dominates any lower bid because it is never worse and sometimes better. A similar argument shows that a bid $b_i > \theta_i$ will also be weakly dominated by bidding θ_i.

The fact that every player has a weakly dominant strategy, $s_i(\theta_i) = \theta_i$, implies that each player bidding his valuation is a Bayesian Nash equilibrium in weakly dominant strategies. This result is noteworthy not only because of its simple prescription–that players bid their valuations truthfully in a second-price sealed-bid auction–but also because it implies three other attractive attributes of this auction format.

First, in the IPV setting, bidders in a second-price sealed-bid auction do not care about the probability distribution over their opponents' types, and therefore the assumption of common knowledge of the distribution of types can be relaxed when such auctions are analyzed. In particular it means that we can apply this result even when we think that players have no idea about their opponents' valuations. This is a very nice feature of the second-price sealed-bid auction.

Second, in a second-price sealed-bid auction, even if types are correlated but values are private (a player's value depends only on his own type and not the types of other players), then it is a weakly dominant strategy to bid truthfully. This implies that if we are correct about our private values assumption, but we incorrectly assume that values are independent, it is still true that bidding your valuation is a weakly dominant strategy.

Last but not least, in the private values setting, the outcome of a second-price sealed-bid auction is Pareto optimal because the person who values the good most will be the one who gets it.

9.1.2. Equilibrium Behavior in a First-Price, Sealed-Bid Auction

For simplicity, we assume that the n bidders are ex-ante symmetric, i.e., $f_i(v) = f(v)$, $i = 1, ..., n$. A bidder's optimal bid will depend on how the other bidders bid. Let's consider the problem from bidder 1's perspective whose private value is v_i. We may view bidder 1's strategy as a bidding function $b : [0,1] \to \mathbb{R}_+$.

We search for a symmetric Bayesian-Nash equilibrium given by the strictly increasing bidding function $b(\cdot)$. Bidder i wins the auction if and only if his bid is the highest. Define the maximum valuation among the remaining $n-1$ bidders by $Y_1 = \max(V_2, ..., V_n)$. Y_1 is a random variable and is distributed according to the following distribution function,

$$\begin{aligned}
G(y_1) &= \Pr ob(Y_1 \leq y_1) = \Pr ob(V_2 \leq y_1, ..., V_n \leq y_1) = \\
\text{(independence assumption)} &= \Pr ob(V_2 \leq y_1) \times \cdots \times \Pr ob(V_n \leq y_1) \\
&= \underbrace{F(y_1) \times \cdots \times F(y_1)}_{\text{n-1 times}} = F^{n-1}(y_1).
\end{aligned}$$

Theorem 61 *There exists a unique equilibrium in the first-price sealed-bid auction game. It is given by,*

$$b(v) = E(Y_1 | v \geq Y_1) = \frac{1}{G(v)} \int_0^v y_1 g(y_1) \, dy_1.$$

Proof. Suppose that bidders $j \neq 1$ follow the symmetric, increasing and differentiable equilibrium strategy b. Suppose bidder 1 has $V_1 = v$ and bids β. The expected payoff of bidder 1 if he submits bid β is,

$$\begin{aligned}
U_1(\beta | v) &= (v - \beta) \Pr ob(b(V_j) \leq \beta, \forall j = 2, ..., n | V_1 = v) \\
&= (v - \beta) \Pr ob\left(V_j \leq b^{-1}(\beta), \forall j = 2, ..., n | V_1 = v\right) \\
\text{(independence)} &= (v - \beta) \Pr ob\left(V_j \leq b^{-1}(\beta), \forall j = 2, ..., n\right) \\
\text{(independence)} &= (v - \beta) F^{n-1}\left(b^{-1}(\beta)\right) \\
&= (v - \beta) G\left(b^{-1}(\beta)\right).
\end{aligned}$$

Maximizing with respect to β yields the first order condition,[1]

$$-G\left(b^{-1}(\beta)\right)+\left(v-b\left(b^{-1}(\beta)\right)\right)g\left(b^{-1}(\beta)\right)\frac{1}{b'\left(b^{-1}(\beta)\right)}=0.$$

At the symmetric equilibrium, $\beta=b(v)$, and the above becomes,

$$G(v)\,b'(v)+g(v)\,b(v)=vg(v),$$

or equivalently,

$$\frac{d}{dv}\left(G(v)\,b(v)\right)=vg(v)$$

and since $b(0)=0$ we have,

$$b(v)=\frac{1}{G(v)}\int_0^v y_1 g(y_1)\,dy_1=E\left(Y_1|v\geq Y_1\right). \tag{9.1.1}$$

The first order condition is only a necessary condition. We need to formally establish that if the other $n-1$ bidders follow b, then it is indeed optimal for bidder 1 with value v to bid $b(v)$.

First note that $b(v)$ is an increasing and continuous function. From (9.1.1) we have,

$$\begin{aligned} b(v)\,G(v) &= \int_0^v y_1 g(y_1)\,dy_1 \Rightarrow \text{ (differentiate w.r.t. to } v) && (9.1.2)\\ b'(v)\,G(v)+b(v)\,g(v) &= vg(v)\Rightarrow \\ b'(v)\,G(v) &= g(v)\,(v-b(v)). && (9.1.3)\end{aligned}$$

Integrating (9.1.2) by parts we obtain,

$$\begin{aligned} b(v)\,G(v) &= vG(v)-\int_0^v G(y_1)\,dy_1 \text{ (divide both sides by } G(v)\text{) } \Rightarrow \\ b(v) &= v-\frac{1}{G(v)}\int_0^v G(y_1)\,dy_1 \Rightarrow b(v)<v. && (9.1.4)\end{aligned}$$

By combining (9.1.3) and (9.1.4) we can show that $b'(v)>0$, i.e., the bidding function is strictly increasing.

[1] We use the following: The derivative of f^{-1} is the reciprocal of the derivative of f , with argument and value reversed. So, if $y=f(x)$, then $x=f^{-1}(y)$ and,

$$\frac{df^{-1}(y)}{dy}=\frac{1}{f'(x)}.$$

Denote by $z = b^{-1}(\beta)$ the value for which β is the equilibrium bid. The expected payoff of bidder 1 with value v if he bids $b(z)$ is calculated as follows,

$$\begin{aligned}
U(b(z), v) &= G(z)(v - b(z)) \\
&= G(z)v - G(z)E(Y_1 | z \geq Y_1) \\
&= G(z)v - \int_0^z y_1 g(y_1)\, dy_1 \\
\text{(integration by parts)} &= G(z)v - G(z)z + \int_0^z G(y_1)\, dy_1 \\
& G(z)(v - z) + \int_0^z G(y_1)\, dy_1.
\end{aligned}$$

Thus we have,

$$\begin{aligned}
U(b(v), v) - U(b(z), v) &= \int_0^v G(y_1)\, dy_1 - G(z)(v - z) - \int_0^z G(y_1)\, dy \\
&= G(z)(z - v) - \int_v^z G(y_1)\, dy_1 \geq 0
\end{aligned}$$

regardless of whether $z \geq v$ or $v \geq z$. This shows that bidding an amount $b(z') > b(v)$ results in loss. Similarly, bidding an amount $b(z'') < b(v)$ also results in loss. ■

Hence, in the unique symmetric equilibrium of a first-price, sealed-bid auction, each bidder bids the expectation of the second-highest bidder's value conditional on winning the auction.

Example. Suppose that $V_i \sim U[0,1]$, $\forall i$. The distribution function of V is, $F = v$, the distribution function of Y_1 is, $G = F^{n-1} = v^{n-1}$ and the density function of Y_1 is, $g = (n-1)v^{n-2}$. The equilibrium bidding function is,

$$b(v) = \frac{1}{G(v)} \int_0^v y g(y)\, dy = v - \frac{v}{n}.$$

So, if for example the numbers of bidders is 3, then the bidding function is, $b(v) = \frac{2}{3}v$. Thus, each bidder shades its bid from his true value. As the number of bidders increases, the competition between the bidders increases and the bid shading decreases.

9.1.3. Revenue Equivalence

The revenue equivalence theorem states that any auction game that satisfies four conditions will yield the seller the same expected revenue, and will yield each type of bidder the same expected payoff. These conditions are as follows: (1) each bidder's type is drawn from a "well-behaved" distribution; (2) bidders are risk neutral; (3) the bidder with the highest type wins; and (4) the bidder with the lowest possible type $\underline{\theta}$ has an expected payoff of zero.

First note that both the first and the second-price auctions are efficient, in the sense that the object always goes to the bidder with the highest valuation for the object.

The expected revenue from the first-price auction is,

$$R_{FPA} = n \int_0^1 b(v) f(v) F^{n-1}(v)\, dv \tag{1}$$

where nfF^{n-1} is the density of the first order statistic, $Y_{(1)} = \max(V_1, ..., V_n)$ and $b(v) = \frac{1}{G(v)} \int_0^v yg(y)\, dy$.

The expected revenue of the second price auction is,

$$R_{SPA} = \int_0^1 vh(v)\, dv = n(n-1) \int_0^1 vF^{n-2}(v) f(v)(1 - F(v))\, dv,$$

where $h = n(n-1) F^{n-2} f(1 - F)$ is the density of the second highest of n independent random variables with common density f. This can be understood as follows. The distribution of the second-highest order statistic is,

$$\begin{aligned} F_2(v) &= F(v)^n + nF(v)^{n-1}[1 - F(v)] \\ &= nF(v)^{n-1} - (n-1) F(v)^n . \end{aligned}$$

The probability that the second highest realization is less than v is the union of the following two disjoint events: i) either all realizations are less than v, or ii) $n-1$ are less than v and one is higher than v. There are n different ways in which ii) can occur. The density is

$$f_2(v) = n(n-1) F(v)^{n-2} f(v)(1 - F(v)).$$

We will now compare R_{FPA} with R_{SPA},

$$\begin{aligned}
R_{FPA} &= n\int_0^1 \left[\frac{1}{F^{n-1}(v)}\int_0^v (n-1)\, yf(y)\, F^{n-2}(y)\, dy\right] f(v)\, F^{n-1}(v)\, dv \\
&= n(n-1)\int_0^1 \int_0^v \left[yf(y)\, f(v)\, F^{n-2}(y)\right] dydv \\
&\overset{\text{(interchange the order of integration)}}{=} n(n-1)\int_0^1 \int_y^1 \left[yf(y)\, f(v)\, F^{n-2}(y)\right] dvdy \\
&= n(n-1)\int_0^1 yF^{n-2}(y)\, f(y)\,(1-F(y))\, dy \\
&= R_{SPA}.
\end{aligned}$$

The two auctions yield the same expected revenue to the auctioneer.

Example. It can be easily computed that if the distribution is uniform the expected revenue is,

$$R_{FPA} = R_{SPA} = \frac{n-1}{n+1}.$$

So, when $n = 2$, the expected revenue is $\frac{1}{3}$, when $n = 3$, it is $\frac{1}{2}$ and so on.

9.2. Common Values and the Winner's Curse

Recall that the private values setting describes the situation in which each player's payoff depends on the profile of actions of all players and on his own type, but not on the types of other players. This setting is useful in describing such scenarios as how much different people value a hamburger or a bag of chips, but in many cases the payoff of one player will depend on the private information of other players.

Consider, for instance, a house that is on the market-how much would you be willing to pay for it? The answer will depend on two major components: first, your own private value of living in that house, and second, what you expect to get for the house if you choose to sell it at a later date. Other people may have investigated the house and hired inspectors who discovered different shortcomings that affect the value of the house. Hence the information of other people would enter into your willingness to pay for a house if you knew it, yet other people?s information, or types, will generally be their own private information. This same argument will apply to a piece of art, a car, or even a movie-you may value a movie more if you think other people value it more, so you can later talk to them about it and all agree on how good (or bad) it was.

We refer to such scenarios as having a common-values component, similar to the adverse selection model of Section 8.3. To illustrate an extreme example of pure common values in which each player has the same value from winning the auction, imagine that two identical oil firms are considering the purchase of a new oil field. It is common knowledge that the amount of oil is either small, worth \$10 million of net profits; medium, worth \$20 million of net profits; or large, worth \$30 million of net profits. Thus the oil field has one of three values, $v \in \{10, 20, 30\}$. Imagine that it is also common knowledge that these values are distributed so that it is equally likely that the amount is small or large, and twice as likely that it is medium, so that

$$\Pr(v = 10) = \Pr(v = 30) = \frac{1}{4} \text{ and } \Pr(v = 20) = \frac{1}{2}.$$

Now assume that the government, which currently owns the field, will auction it off in a second-price sealed-bid auction, and that before the auction each of the two firms will perform a (free) exploration that will provide some signal about the quantity of oil in the field. Specifically each firm $i \in \{1, 2\}$ receives a low or high signal, $\theta_i = \{L, H\}$, which is correlated with the amount of oil as follows:

1. If $v = 10$ then $\theta_1 = \theta_2 = L$.
2. If $v = 30$ then $\theta_1 = \theta_2 = H$.
3. If $v = 20$ then either $\theta_1 = L$ and $\theta_2 = H$, or $\theta_1 = H$ and $\theta_2 = L$, where each of these two events (conditional on $v = 20$) occurs with equal probability.

The signal outcomes are not independent-they are correlated with the actual amount of oil. Can it be a Bayesian-Nash equilibrium (BNE) in a second-price sealed-bid auction that players submit truthful bids. Formally, is $s_i(\theta_i) = E(v_i|\theta_i)$ for $i \in \{1, 2\}$ a BNE?

Let's assume player 2 is playing according to this prescription and check player 1's best-response. Suppose $\theta_1 = L$. Then

$$E(v_1|\theta_1 = L) = \frac{1}{2} \times 10 + \frac{1}{2} \times 20 = 15.$$

So player 1's expected value is 15. Then with probability $\frac{1}{2}$ player 2 also bids 15, in which case each wins with probability $\frac{1}{2}$, but the oil field is worth only 10. With probability $\frac{1}{2}$ player 2 bids $E(v_2|\theta_2 = H) = 25$ and player 1 loses. So, player 1's expected profit is -1.25, which means that he would rather bid less than 15 and never win.

This perhaps surprising outcome is a direct consequence of the common-values setting in which the types are correlated. When player 1 wins the oil field it is because his opponent's bid was low (in this case they tie), a consequence of player 2 having a low signal. But if player 1 bids his average value given his signal, then he is not taking into account the fact that he wins only when player 2's signal is low. This is "bad news" because it implies that the quantity of oil is lower than the player thinks it is on average. This phenomenon, which occurs in common-values settings, is known as the "winner's curse": a player wins when his signal is the most optimistic, which in the common-values setting means that he has overestimated the value of the good and is overpaying if he does not take this conditioning into account. In an equilibrium, therefore, players will have to choose their bids conditioning on the fact that when they win, this means that their signal is higher than everyone else's, which is likely to imply that the winner is too optimistic about the good's value. The reasoning is very similar to that behind the "swing voter's curse" described in Section 8.4.

9.3. General Model of Auction

9.3.1. Affiliation

Let $x \in \mathbb{R}^n$ and $x' \in \mathbb{R}^n$

$$x \vee x' = \{\max\{x_1, x_1'\}, ..., \max\{x_n, x_n'\}\}.$$

$$x \wedge x' = \{\min\{x_1, x_1'\}, ..., \min\{x_n, x_n'\}\}.$$

Let $X_1, ..., X_n$ be n random variables. Let $f(x_1, ..., x_n)$ be their joint density. We say that $X = (X_1, ..., X_n)$ are affiliated if and only if for all x and x' realizations of X

$$f(x \vee x')f(x \wedge x') \geq f(x)f(x').$$

If $n = 2$ and X_1 and X_2 are affiliated then

$$E(X_2|X_1 = x)$$

is increasing in x.

Theorem 62 *If f is strictly positive and twice continuously differentiable then f is affiliated if and only iff f is log-supermodular*

$$\frac{\partial \log f}{\partial x_i \partial x_j} \geq 0.$$

9.3.2. First- and second-price auctions

There are n bidders with valuations $V_1, ..., V_n$. Bidders do not necessarily know the realization of their valuations. This is in contrast to the private value auction. Each bidder i obtains his own sample and draws a signal X_i before submitting his bid. Given $X_i = x_i$ the new estimate of i for the object is $E(V_i|X_i = x_i)$. Let $F(x)$ be the joint distribution of $X = (X_1, ..., X_n)$, and $f(x)$ be the density.

We make the following assumptions.

1. Each variable X_i takes values in $[0, 1]$.
2. Bidders are symmetric (F is invariant under permutations of bidders).
3. $X_1, ..., X_n$ are affiliated.

Special cases:

1. Common value: $V_1 = V_2 = \cdots = V_n$.
2. Private value $V_i = X_i$.

Denote

$$v(x, y) = E(V_1|X_1 = x, Y_1 = y), \tag{9.3.1}$$

where $Y_1 = \max\{X_2, ..., X_n\}$.

Theorem 63 *A symmetric equilibrium of the second price auction is*

$$b^2(x) = v(x, x).$$

In the private value case $b(x) = x$.

Theorem 64 *A symmetric equilibrium of the first price auction is*

$$b^1(x) = \int_0^x v(y, y) dL(y|x),$$

where

$$L(y|x) = \exp\left[-\int_y^x \frac{f_{y_1|x_1}(t|t)}{F_{y_1|x_1}(t|t)} dt\right].$$

9.4. Summary

- Auctions are commonly used games that allocate scarce resources among several potential bidders.
- There are two extreme settings of auction games. The first case is that of private values, in which the information of each player is enough for him to infer his value from winning the object. The second case is that of common values, in which the information of other players will determine how much the object is worth to any player.
- Auctions often differ in their rules, such as open or sealed bidding, or first or second price. Different rules will result in different equilibrium bidding behavior.
- In the private values setting, the second-price sealed-bid auction is strategically equivalent to the English auction. In both auctions each player has a simple weakly dominant strategy of bidding his true value for the object. The highest-value player wins and pays the second-highest value.

- In the private values setting, the first-price sealed-bid auction is strategically equivalent to the Dutch auction. In both auctions each player's best response depends on the strategies of other players, and calculating a Bayesian Nash equilibrium is not straightforward. In equilibrium each bidder shades his valuation when bidding in order to obtain a positive expected value from the auction.
- The revenue equivalence theorem identifies conditions under which each of the four kinds of auctions yields the seller the same expected revenues and results in the same outcomes for the participating bidders.
- If the auction is one of common values then players must take into account the downsides of the winner's curse and bid accordingly to avoid overpaying for the object.

Part IV.

DYNAMIC GAMES OF INCOMPLETE INFORMATION

10. Sequential Rationality with Incomplete Information

As we argued in Chapter 5, static (normal-form) games do not capture important aspects of dynamic games in which some players respond to actions that other players have previously made. Furthermore, as we demonstrated with the introduction of backward induction and subgame-perfect equilibrium in Chapter 6, we need to pay attention to the familiar problem of credibility and sequential rationality. This chapter applies the idea of sequential rationality to dynamic games of incomplete information and introduces equilibrium concepts that capture these ideas. That is, we want to focus attention on equilibrium play in which players play best-response actions not only on the equilibrium path but also at points in the game that are not reached, which we referred to previously as off the equilibrium path.

As we saw in the examples in Section 8.2, in games of incomplete information some players will have information sets that correspond to the set of types that their opponents may have, because every player does not know which types Nature chose for the other players. Regardless of whether a player observes his opponents' past behavior (which in games of complete information would imply perfect information), there will always be uncertainty about which types the opponents are when incomplete information is present. This in turn implies that structurally there will be many information sets that are not singletons, and this will lead to many fewer proper subgames. As we will now see, this impedes the applicability of subgame perfection as a solution concept that guarantees sequential rationality. We will have to deal more rigorously with the idea that players hold beliefs, and that these beliefs need to be consistent with the environment (Nature) and the strategies of all other players.

10.1. The Problem with Subgame Perfection

Consider the entry game represented in Figure 26. It has two Nash equilibria (O, F) and (E, A) but only the second one is subgame-perfect. Now consider a straightforward variant of this game that includes incomplete information. In particular imagine that the entrant may have a technology that is as good as that of the incumbent, in which case the game above describes the payoffs. However, the entrant may also have an inferior technology, in which case he would not gain by entering and the incumbent would lose less if fighting occurred. A particular case of this story can be captured by the following sequence of events:

1. Nature chooses the entrant's type,which can be weak (W) or competitive (C), so that $\theta_1 \in \{W, C\}$, and let $\Pr(\theta_1 = C) = p$. The entrant knows his type but the incumbent knows only the probability distribution over types.
2. The entrant chooses between E and O as before, and the incumbent observes the entrant's choice.
3. After observing the action of the entrant, and if the entrant enters, the incumbent can choose between A and F as before.

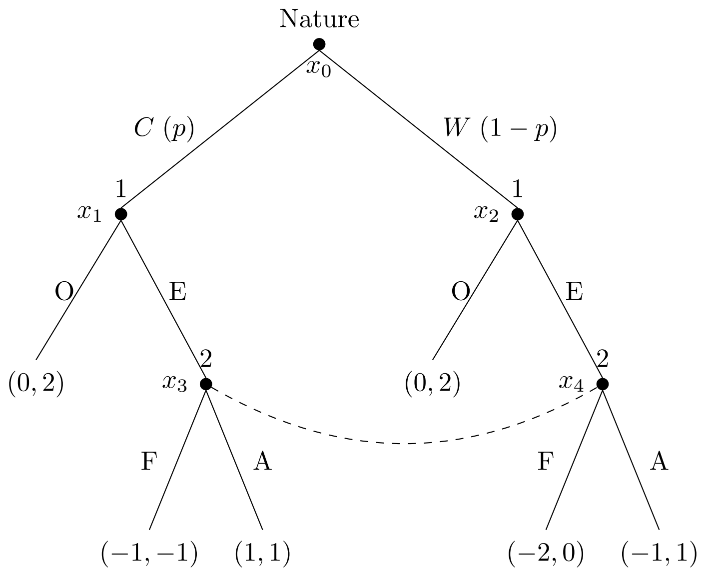

Figure 31.: An incomplete information entry game

The extensive form game is depicted in Figure 31. The pure-strategy set of player 1 is $s_1 \in S_1 = \{OO, OE, EO, EE\}$. For example, $s_1 = OE$ means that player 1 chooses O when his type is C, and he chooses E when his type is W. Because player 2 has only one information set that follows entry, and two actions in that information set, he has two pure strategies, $s_2 \in \{A, F\}$.

		Player 2	
		F	A
Player 1	OO	$0, 2$	$0, 2$
	OE	$-1, 1$	$-1/2, 2/3$
	EO	$-1/2, 1/2$	$1/2, 3/2$
	EE	$-3/2, -1/2$	$0, 1$

Figure 32.

We set $p = 0.5$, in which case the matrix representation of the Bayesian game is given in Figure 32. Therefore both (OO, F) and (EO, A) are pure-strategy Bayesian Nash equilibria of the Bayesian game. Interestingly these two Bayesian Nash equilibria are tightly related to the two Nash equilibria of the complete-information game in Figure 26. The equilibrium (OO, F) is one in which the incumbent "threatens" to fight, which causes the entrant to stay out regardless of his type, similar to the (O, F) equilibrium in the game of complete information. The equilibrium (EO, A) is one in which the incumbent accommodates entry, which causes the strong entrant to enter (getting 1 instead of 0) and the weak entrant to stay out (getting 0 instead of -1), similar to the (E, A) equilibrium in the game in Figure 26.

Not only are the equilibria similar, but there is a similar problem of credibility with the equilibrium (OO, F): player 2 threatens to fight, but if he finds himself in the information set that follows entry, he has a strict best response, which is to accommodate entry. Thus the Bayesian Nash equilibrium (OO, F) involves noncredible behavior of player 2 that is not sequentially rational.

Now comes the interesting question: which of these two equilibria survives as a subgame-perfect equilibrium in the extensive-form game? Recall that the definition of a subgame-perfect equilibrium is that in every proper subgame, the restriction of the strategies to that subgame must be a Nash equilibrium in the subgame. This, as we saw in Chapter 6, means that players are playing mutual best responses both on and off the equilibrium path. However, looking at the extensive-form game in Figure 31, it is easy to see that there is only one subgame, which is the complete game. Therefore, both (OO, F) and (EO, A) survive as subgame-perfect equilibria.

This example demonstrates that the very appealing concept of subgame-perfect equilibrium may have no bite for some games of incomplete information. At first this may seem somewhat puzzling. However, the problem is that the concept of subgame- perfect equilibrium restricts attention to best responses within subgames, but when there is incomplete information the induced information sets over types of other players often cause the only proper subgame to be the complete game.

The reason this happens is that in the modified entry game we analyzed, even though player 2 observes the actions of player 1, the fact that player 1 has several types implies that there is no proper subgame that starts with player 2 making a move. This is because whenever player 2 makes a move, he does not know the type of player 1, which implies that there are no proper subgames except for the whole game. Indeed this problem of action nodes being linked through information sets that represent the uncertainty players have over the types of their opponents will carry over to all games of incomplete information.

In order to extend the logic of subgame-perfect equilibrium to dynamic games of incomplete information, we need to impose a more rigorous structure on our solution concept in order for sequential rationality to be well defined. Our goal is to identify a structure of analysis that will cause the elimination of equilibria that involve noncredible threats such as those in the modified entry game, which we do in the next section.

10.2. Perfect Bayesian Equilibrium

To address the problem with subgame perfection demonstrated in the previous section we need to identify a way in which to express the fact that following entry, it is not sequentially rational for player 2 to play fight. In particular we need to make statements about the sequential rationality of player 2 within each of his information sets even though the information set is not itself the first node of a proper subgame. We need to be able to make statements like "in this information set player 2 is not playing a best response, and therefore his behavior is not sequentially rational."

This kind of reasoning is precisely what is missing from the definition of subgame-perfect equilibrium–we were not able to isolate player 2 within his information set. However, to describe a player's best response within his information set, we will have to ask what the player is playing a best response to. The answer, of course, must follow from the definition of what a best response is: we must include beliefs in the analysis. This will allow us to consider the beliefs of player 2 in his information sets and then analyze his best response to these beliefs.

More generally, to introduce sequential rationality at information sets that are not singletons, we need to insist that players form beliefs in every information set they have. These will include information sets that are reached with some positive probability given the proposed actions of the players as well as information sets that are not reached at all. Recall from Section 3.1 the notion of being on and off the equilibrium path, which is introduced again for Bayesian games as follows:

Definition 65 *Let $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies in a game of incomplete information. We say that an information set is on the equilibrium path if given σ^* and given the distribution of types, it is reached with positive probability. We say that an information set is off the equilibrium path if given σ^* and the distribution of types, it is reached with zero probability.*

We now formally introduce the notion that players must have beliefs in each of their information sets:

Definition 66 *A system of beliefs μ of an extensive-form game assigns a probability distribution over decision nodes to every information set. That is, for every information set $h \in H$ and every decision node $x \in h$, $\mu(x) \in [0,1]$ is the probability that player i who moves in information set h assigns to his being at x, where $\sum_{x \in h} \mu(x) = 1$ for every $h \in H$.*

For example, in the entry game of Figure 31 the beliefs of player 1 are trivially defined because his information sets are singletons, while the belief of player 2 is a probability distribution over the two nodes in his unique information set. Denote by $\mu(x_3)$ player 2's belief that he is at the node corresponding to player 1 being competitive (C) and playing E, and let $1-\mu(x_3)$ be his belief that he is at the node corresponding to player 1 being weak (W) and playing E. We are now ready to lay out our first requirement of sequential rationality in games of incomplete information:

Requirement 1. Every player will have a well-defined belief over where he is in each of his information sets. That is, the game will have a system of beliefs.

Now that we have established the need to have a system of beliefs, we need to ask the following question: how should the beliefs in a system of beliefs be determined? That is, can players have any beliefs they want, or should the beliefs be restricted by some elements that are not controlled by the player himself?

Just as we required the beliefs of players about the strategies of their opponents to be correct in order to define a Nash equilibrium, here too we will add a very similar requirement. In games of incomplete information, two constraints will influence whether a player's beliefs are correct. The first is the behavior of the other players, which we can consider as an endogenous constraint on beliefs. It is endogenous in the sense that it is determined by the strategies of the players, which are the "variables" that the players control. The second constraint on beliefs comes from the choices of Nature through the distribution of types. This is an exogenous constraint on beliefs. It is exogenous in the sense that it is determined by Nature, which is not something that the players control but rather part of the environment.

Let's illustrate this using the entry game. Following the above requirement, we defined player 2's belief in his information set as $\mu(x_3) = \Pr(\text{player 1 is competitive}|E)$ and $1 - \mu(x_3) = \Pr(\text{player 1 is weak}|E)$.

Imagine now that player 2 believes that player 1 is choosing the strategy EO, so that if he is competitive he enters and if he is weak he stays out. What should player 2 believe if his information set is reached? The only consistent belief would be that player 1 is competitive, and therefore it must be that $\mu(x_3) = 1$. This follows because with probability p Nature chooses a competitive type for player 1, and with the strategy EO a competitive player 1 always enters. The probability of reaching the node "weak followed by entry" inside the information set of player 2 is 0 because with probability $1 - p$ Nature chooses a weak type for player 1, and with the strategy EO a weak player 1 never enters. Any other belief would not be consistent with the belief that player 1 is playing EO.

Player 1's behavior can of course be more complex than choosing EO. More generally in the entry game, the probability of reaching the node "player 1 is type C and he chose E" inside the information set of player 2 is p times the probability that a type C player 1 chose E. By definition, $\mu(x_3)$ is the belief that player 2 assigns to "player 1 is type C conditional on entry occurring." Hence we need to consider the beliefs of player 2 as being conditional on the fact that his information set was indeed reached. Assume that player 1 is playing the following strategy: If $\theta_1 = C$ then he plays E with probability σ_C and O with probability $1 - \sigma_C$. Similarly if $\theta_1 = W$ then he plays E with probability σ_W and O with probability $1 - \sigma_W$. Now by Bayes' rule we must have

$$\mu(x_3) = \Pr(C|E) = \frac{p\sigma_C}{p\sigma_C + (1-p)\sigma_W}.$$

Following this argument, we are ready to state our second requirement for sequential rationality: given a conjectured profile of strategies, and given Nature's choices, we require players' beliefs to be correct in information sets that are reached with positive probability. Formally, we have

Requirement 2. Let $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies. We require that in all information sets beliefs that are on the equilibrium path be consistent with Bayes' rule.

This is precisely how beliefs are formed; they include both the exogenous constraints that follow from Nature's probability distribution and the endogenous constraints that follow from beliefs about the other players' strategies from $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$.

Now consider the pure strategy OO (or $\sigma_C = \sigma_W = 0$). The probability of reaching player 2's information set is not positive because there is no way in which it is reached. That is, no type of player 1 plays E, and this implies that E is never chosen. If player 2 believes that player 1 chooses OO, and suddenly finds himself in his information set that follows entry, then Bayes' rule does not apply because given the suggested strategy both the numerator and the denominator in Bayes' rule are zero, and thus $\mu(x_3)$ is not well defined. What then should determine $\mu(x_3)$? In other words, if we cannot apply Bayes' rule because of the beliefs over strategies, what will we use to determine beliefs? For this we introduce the third requirement.

Requirement 3. At information sets that are off the equilibrium path any belief can be assigned to which Bayes' rule does not apply.

This means that when the moves of Nature combined with the belief over the strategies of the other players do not impose constraints on beliefs, then indeed beliefs could be whatever the player chooses them to be. Looking back at the case in which player 1 chooses the pure strategy OO in the entry game, then $\mu(x_3)$ can be any number in the interval $[0, 1]$ because it is not constrained by Bayes' rule.

All of the first three requirements were imposed to introduce well-defined beliefs for every player at each of his information sets. Now we come to the final requirement of sequential rationality:

Requirement 4. Given their beliefs, players' strategies must be sequentially rational. That is, in every information set players will play a best response to their beliefs.

To write down the above requirement formally, consider player i with beliefs over information sets derived from the belief system μ, given player i's opponents playing σ_{-i}. It says that if h is an information set for player i then it must be true that he is playing a strategy σ_i that satisfies

$$E[v_i(\sigma_i, \sigma_{-i}, \theta_i)|h, \mu) \geq E[v_i(s_i', \sigma_{-i}, \theta_i)|h, \mu), \forall s_i' \in S_i.$$

We can now incorporate requirements 1-4 to show that the noncredible equilibrium in the entry game with incomplete information is fragile. To see this, notice that once we specify a belief $\mu(x_3)$ for player 2, then for any $\mu(x_3) \in [0, 1]$, it is a best response for player 2 to play A. This means that once we endow player 2 with a well-defined belief (requirement

1) then despite the fact that these beliefs are not restricted if player 1 chooses OO (by requirement 3), the Bayesian Nash equilibrium (OO, F) has player 2 not playing a best response to any belief, which violates requirement 4. Now that we have defined beliefs together with sequential rationality, we need to combine all these components to define a coherent equilibrium concept:

Definition 67 *A Bayesian Nash equilibrium profile $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$ together with a system of beliefs μ constitutes a perfect Bayesian equilibrium for an n-player game if they satisfy requirements 1-4.*

10.3. Sequential Equilibrium

Perfect Bayesian equilibrium has become the most widely used solution concept for dynamic games with incomplete information. There are, however, examples of games in which the perfect Bayesian equilibrium solution concept allows for equilibria that seem unreasonable. The reason for this is that restriction 3 of the perfect Bayesian equilibrium concept places no restrictions on beliefs that are off the equilibrium path.

To see this, consider the game shown in Figure 33. If player 1 plays D with a positive probability then by requirement 2 the beliefs of player 2 are completely determined by Bayes' rule, which implies that $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$. With these beliefs player 2 must play L. If player 2 plays L then player 1's best response is to play D, which implies that the pair of strategies (D, L) together with the implied beliefs $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$ form a perfect Bayesian equilibrium.

Note, however, that the pair of strategies (U, R) can also be supported as a perfect Bayesian equilibrium. If player 1 plays U then by requirements 2 and 3 beliefs are not restricted in player 2's information set. In particular, player 2 can believe that $\mu_2(x_3) > \frac{2}{3}$, in which case he believes that by playing R his expected payoff will be $3\mu_2(x_3) > 2$, for which playing R is a best response. As a consequence, playing U for player 1 is also a best response!

This example may seem unusual, but it turns out to be quite common. It is for this reason

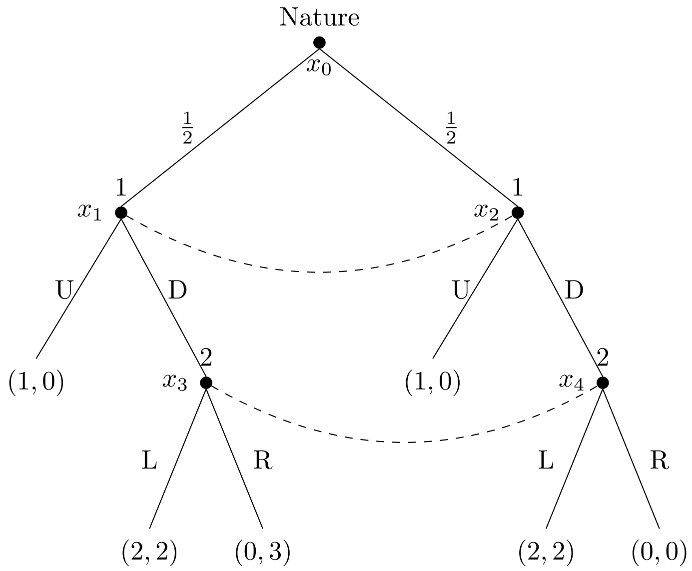

Figure 33.: An game with an unreasonable perfect Bayesian equilibrium

that many applications of dynamic games of incomplete information use stronger concepts than perfect Bayesian equilibrium. These are often referred to as "equilibrium refinements," as their goal is to refine the set of outcomes that can be supported in equilibrium. The key idea behind these refinements is to put restrictions on the sorts of beliefs that players can hold in information sets that are off the equilibrium path.

One commonly used refinement is called sequential equilibrium, and it was introduced by Kreps and Wilson (1982). They define the notion of "consistent" beliefs as follows:

Definition 68 *A profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$, together with a system of beliefs μ^*, is consistent if there exists a sequence of nondegenerate mixed strategies, $\{\sigma^k\}_{k=1}^{\infty}$, and a sequence of beliefs that are derived from each σ^k according to Bayes' rule, $\{\mu^k\}_{k=1}^{\infty}$, such that $\lim_{k\to\infty}(\sigma^k, \mu^k) = (\sigma^*, \mu^*)$.*

To see the power behind this definition, observe that in the game described in Figure 33 the only consistent beliefs for player 2 are $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$. The reason is that any belief that is part of a consistent pair of strategies and beliefs must be derived from a sequence of strategies that cause every information set to be reached with positive probability on the equilibrium path. This follows from the requirement that $\{\sigma^k\}_{k=1}^{\infty}$ be a sequence of nondegenerate mixed strategies, which implies that each player is mixing among all his actions with positive probability. As we argued earlier, if player 1 plays D with positive probability then the beliefs of player 2 are completely determined by Bayes' rule to be $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$, so in any sequence of the form required by consistency the limit of these beliefs must be $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$.

Definition 69 *A profile of strategies $\sigma^* = (\sigma_1^*, ..., \sigma_n^*)$, together with a system of beliefs μ^*, is a sequential equilibrium if (σ^*, μ^*) is a consistent perfect Bayesian equilibrium.*

This definition implies of course that every sequential equilibrium is a perfect Bayesian equilibrium, but the reverse is not true. Indeed of the two perfect Bayesian equilibria we identified earlier, (U, R) is not part of a sequential equilibrium because the only beliefs that can be part of a consistent strategy profile-belief pair are $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$, which implies that (D, L) together with the beliefs $\mu_2(x_3) = \mu_2(x_4) = \frac{1}{2}$ is the unique sequential equilibrium.

Sequential equilibrium is a bit harder to apply than perfect Bayesian equilibrium in practice, so as with most applications in the social sciences we will stick to perfect Bayesian equilibrium as our solution concept. It is interesting to note that there are games in which even sequential equilibrium allows for unreasonable outcomes, which suggests that one might want to consider a stronger refinement for a solution concept. Indeed we will see one such refinement for signaling games.

10.4. Summary

- Because games of incomplete information have information sets that are associated with Nature's choices of types, it will often be the case that the only proper subgame is the whole game. As a consequence, subgame-perfect equilibrium will rarely restrict the set of Bayesian Nash equilibria to those that are sequentially rational.
- By requiring that players form beliefs in every information set, and requiring these beliefs to be consistent with Bayes' rule, we can apply the concept of sequential rationality

to Bayesian games.

- In a perfect Bayesian equilibrium, beliefs are constrained on the equilibrium path but not off the equilibrium path. It is important, however, that beliefs off the equilibrium path support equilibrium behavior.
- In some games the concept of perfect Bayesian equilibrium will not rule out play that seems sequentially irrational. Equilibrium refinements, such as sequential equilibrium, have been developed to address these situations.

11. Signaling Games

In games of incomplete information there is at least one player who is uninformed about the type of another player. In some instances it will be to the benefit of players to reveal their types to their opponents. For instance, if a potential rival to an incumbent firm or an incumbent politician knows that he is strong, he may want to reveal that information to the incumbent, to suggest "I am strong and hence you should not waste time and energy fighting me." Of course even a weak player would like to try to convince his opponent that he is strong, so merely stating "I am strong" will not do. There has to be some credible means, beyond such "cheap talk," through which the player can signal his type and make his opponent believe him.

Games in which such signaling is possible in equilibrium are called signaling games; they originated in the Nobel Prize–winning contribution of Michael Spence (1973), which he developed in his Ph.D. thesis. Spence investigated the role of education as an instrument that signals information to potential employers about a person's intrinsic abilities, but not necessarily what he has learned. Signaling games share a structure that includes the following four components:

1. Nature chooses a type for player 1 that player 2 does not know, but cares about (common values).
2. Player 1 has a rich action set in the sense that there are at least as many actions as there are types, and each action imposes a different cost on each type.
3. Player 1 chooses an action first, and player 2 then responds after observing player 1's choice.
4. Given player 2's belief about player 1's strategy, player 2 updates his belief after ob-

serving player 1's choice. Player 2 then makes his choice as a best response to his updated beliefs.

These games are called signaling games because of the potential signal that player 1?s actions can convey to player 2. If in equilibrium each type of player 1 is playing a different choice then in equilibrium the action of player 1 will fully reveal player 1?s type to player 2. That is, even though player 2 does not know the type of player 1, in equilibrium player 2 fully learns the type of player 1 through his actions.

Of course, it need not be the case that player 1's type is revealed. If, for instance, in equilibrium all the types of player 1 choose the same action then player 2 cannot update his beliefs at all. Because of this variation in the signaling potential of player 1's strategies, these games have two important classes of perfect Bayesian equilibria:

1. Pooling equilibria: These are equilibria in which all the types of player 1 choose the same action, thus revealing nothing to player 2. Player 2's beliefs must be derived from Bayes' rule only in the information sets that are reached with positive probability. All other information sets are reached with probability zero, and in these information sets player 2 must have beliefs that support his own strategy. The sequentially rational strategy of player 2 given his beliefs is what keeps player 1 from deviating from his pooling strategy.

2. Separating equilibria: These are equilibria in which each type of player 1 chooses a different action, thus revealing his type in equilibrium to player 2. Player 2's beliefs are thus well defined by Bayes' rule in all the information sets that are reached with positive probability. If there are more actions than types for player 1, then player 2 must have beliefs in the information sets that are not reached (the actions that no type of player 1 chooses), which in turn must support the strategy of player 2. Player 2?s strategy supports the strategy of player 1.

The choice of terms is not coincidental. In a pooling equilibrium all the types of player 1 pool together in the action set, and thus player 2 can learn nothing from the action of player 1. Player 2's posterior belief after player 1 moves must be equal to his prior belief over the distribution of Nature's choices of types for player 1. In a separating equilibrium each type of player 1 separates from the others by choosing a unique action that no other type chooses. Thus after observing what player 1 did, player 2 can infer exactly what type player 1 is.

Remark. There is a third class of equilibria called hybrid or semi-separating equilibria, in which different types choose different mixed strategies. As a consequence some information sets that belong to the uninformed player can be reached by different types with different probabilities. Thus Bayes' rule implies that in these information sets player 2 can learn something about player 1 but cannot always infer exactly which type he is. We will explore these kinds of equilibria in Chapter ??. See Fudenberg and Tirole (1991, Chapter 8) for a more advanced treatment.

The incomplete-information entry game of Figure 31 can be used to illustrate these two classes. In the Bayesian Nash equilibrium (OO, F), both types of player 1 chose "out," so player 2 learns nothing about player 1's type (in this case he has no active action following player 1's decision to stay out). Thus (OO, F) is a pooling equilibrium (though it is not a perfect Bayesian equilibrium, as demonstrated earlier). In the Bayesian Nash equilibrium (EO, A), which is also a perfect Bayesian equilibrium, player 1's action perfectly reveals his type: if player 2 sees entry, he believes with probability 1 that player 1 is strong, while if player 1 chooses to stay out then player 2 believes with probability 1 that player 1 is weak. Therefore (EO, A) is a separating equilibrium.

11.1. Education Signaling: The MBA Game

This section analyzes a very simple version of an education signaling game in the spirit of Spence's work that sheds some light on the signaling value of education. To focus attention on the signaling value of education, we will ignore any productive value that education may provide. That is, we assume that a person learns nothing productive from education but has to "suffer" the loss of time and the hard work of studying to get a diploma, in this case an MBA degree. The game proceeds in the following steps:

1. Nature chooses player 1's skill (productivity at work), which can be high (H) or low

(L), and only player 1 knows his skill. Thus his type set is $\Theta = \{H, L\}$. The probability that player 1's type is H is given by $\Pr(\theta = H) = p > 0$, and it is common knowledge that this is Nature's prior distribution.

2. After player 1 learns his type, he can choose whether to get an MBA degree (D) or be content with his undergraduate-level degree (U), so that his action set is $A_1 = \{D, U\}$. Getting an MBA requires some effort that is type dependent. Player 1 incurs a private cost c_θ if he gets an MBA, and a cost of 0 if he does not. We assume that high-skilled types find it easier to study, captured by the assumption that $c_H < c_L$. We assume in particular that $c_H = 2$ and $c_L = 5$.

3. Player 2 is an employer, who can assign player 1 to one of two jobs. Specifically player 2 can assign player 1 to be either a manager (M) or a blue-collar worker (B), so that his action set is $A_2 = \{M, B\}$. The employer will retain the profit from the project and must pay a wage to the worker depending on the job assignment. The market wage for a manager is w_M and that for a blue-collar worker is w_B , where $w_M > w_B$. We assume in particular that $w_M = 10$ and $w_B = 6$.

4. Player 2's payoff (the employer's profit) is determined by the combination of skill and job assignments. It is assumed that the MBA degree adds nothing to productivity. A high-skilled worker is relatively better at managing, while a low-skilled worker is relatively better at blue-collar work. The employer's net profits from the possible skill-assignment matches are given in the following table:

		Assignment	
		M	B
Skill	H	10	5
	L	0	3

The complete game tree is represented in Figure 34.

We look for a separating perfect-Bayesian Nash equilibrium (BNE) where type H chooses D and type L chooses U. Given this strategy of player 1 the beliefs of player 2 are: $\mu_U = 0$ and $\mu_D = 1$. Player 2 when he observes D chooses M and when he observes U chooses B. Player 1 has no incentive to deviate. If type L deviates to D, then, given player 2's belief,

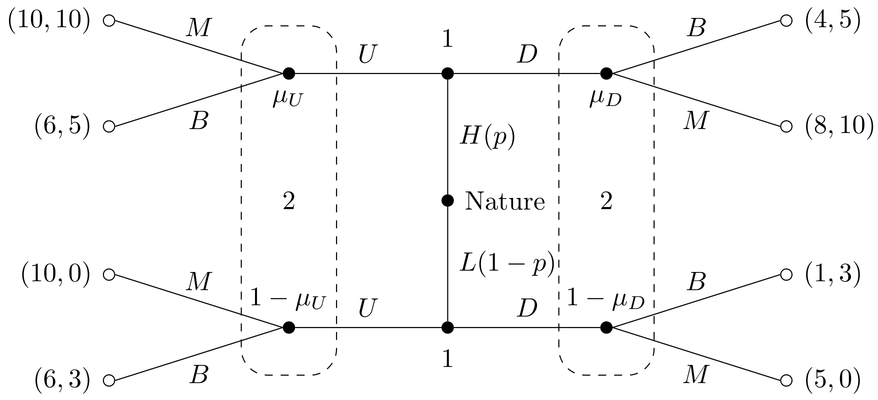

Figure 34.: Education signaling game

his payoff is $5 < 6$. Similarly, if type H deviates to U his payoff is $6 < 8$. Hence, (DU, BM) together with $\mu_U = 0$ and $\mu_D = 1$ constitute a perfect BNE.

Next, we look for a pooling BNE. Suppose both types choose U. Then $\mu_U = p$. Player 2: If he observes U, then $M \succ B$ iff $10p > 5p + 3(1-p) \Rightarrow p > \frac{3}{8}$. If M is chosen then both types get 10, if B is chosen both types get 6. If player 2 observes D (out-of-equilibrium), then $M \succ B$ iff $10\mu_D > 5\mu_D + 3(1-\mu_D) \Rightarrow \mu_D > \frac{3}{8}$.

So if $p > \frac{3}{8}$, player 2 chooses M when he sees U and no type of player 1 wants to deviate to D, no matter what player 2's beliefs are off-the-equilibrium path. In this case (UU, MM) with $\mu_D > \frac{3}{8}$ and (UU, MB) with $\mu_D < \frac{3}{8}$, with $\mu_U = p$ in both cases are pooling perfect BNE.

If $p < \frac{3}{8}$, then player 2 chooses B when he sees U and both players get 6. Player L would not want to deviate to D, no matter what player 2 chooses. Type H would want to deviate if player 2 chooses M, which happens if $\mu_D > \frac{3}{8}$. So (UU, BB) with $\mu_D < \frac{3}{8}$ and $\mu_U = p$ is a pooling perfect BNE.

11.2. Limit Pricing and Entry Deterrence

We turn to another application of the analysis of dynamic games of incomplete information, this time to an important antitrust issue. Bain (1949) argued that an incumbent firm can engage in limit pricing to deter entry, and thus try to gain control as a long-run monopolist. This practice is defined as one in which a firm prices below marginal costs, which is obviously a losing strategy in the short run. But if by doing this the firm runs competitors out of business, deters other firms from entering the market, and thus becomes a monopolist, then this might be a winning strategy in the long run. Limit pricing is considered anticompetitive behavior and is illegal in the United States.

However, this argument should make one feel a bit uncomfortable: Is it reasonable to punish firms for charging low prices? How can we convince ourselves that Bain's intuition is correct and that a firm may indeed choose to incur short-run losses to generate long-run profits? These questions were answered by Milgrom and Roberts (1982). We present a simplified version of their analysis in the following game.

Consider a market for some good that will last for two periods with demand $P = 5 - q$ in each period. In period $t = 1$ only one firm, the incumbent (player 1), is in the market, and this incumbent will continue to produce in period $t = 2$. A potential entrant (player 2) observes what happened in period $t = 1$ and then makes a choice of whether to enter the market at a fixed cost of $\frac{1}{2}$ and compete against the incumbent in period $t = 2$, or to remain out of the market. If player 2 stays out of the market then player 1 remains a monopolist in period 2, while if player 2 enters the market then they compete in a Cournot duopoly game. Each firm has marginal costs c_i, where it is common knowledge that the entrant's marginal costs are $c_2 = 2$. There is asymmetric information with respect to the costs (the type) of the incumbent. In particular the entrant knows only that $c_1 \in \{1, 2\}$, so that the incumbent can be either more efficient than the entrant (have low [L] costs of $c_1 = 1$) or as efficient (have high [H] costs of $c_1 = 2$). It is common knowledge that $\Pr(c_1 = 1) = \frac{1}{2}$, but only the incumbent knows his true costs. The timing of this game, which captures the story, is described as follows:

1. Nature chooses the type of player 1, $c_1 \in \{1, 2\}$, each type with equal probability.
2. Player 1 observes his costs and decides how much to produce in the first period, q_1, and the price in the first period is then $P = 5 - q_1$.

3. Player 2 observes q_1 and decides whether or not to enter at a fixed cost of $F = \frac{1}{2}$.
4. If player 2 stays out then in period $t = 2$ player 1 chooses q_1^2 and the price is $P = 5 - q_1^2$.
5. If player 2 enters then in period $t = 2$ each player i simultaneously chooses q_i^2 and the price is $P = 5 - q_1^2 - q_2^2$ (Cournot competition).

We will look for a perfect BNE.

11.2.1. Separating Equilibrium

If the entrant does not enter the monopoly profit when the cost is H is 2.25 (the price is 3.5) and when the cost is L is 4 (the price is 3).

If entry takes place, the equilibrium quantities are:

- Incumbent's cost is L: $(q_1^2, q_2^2) = \left(\frac{5}{3}, \frac{2}{3}\right)$ and the profits are $\pi_1^2 = \frac{25}{9}$ and $\pi_2^2 = -\frac{1}{18}$.
- Incumbent's cost is H: $(q_1^2, q_2^2) = (1, 1)$ and the profits are $\pi_1^2 = 1$ and $\pi_2^2 = \frac{1}{2}$.

In a separating equilibrium, the entrant learns the type of the incumbent by observing the first period price and it enters if and only if the incumbent's type is H. Let p_1^{1L} be the first period price the L type charges. The H type charges a price 2 in the first period, since entry will take place anyway.

The incentive compatibility constraint of the L type is

$$(5 - q_1^{1L} - 1)q_1^{1L} + 4 \geq 4 + 2.25 \Rightarrow q_1^{1L} \leq 3.32(\text{ or } p_1^{1L} \geq 1.68). \tag{ICC L}$$

The LHS of (ICC L) is the profit is the L type does not deviate, while the RHS is the deviation profit. If the L type charges any price other than p_1^{1L} the entrant thinks it is of H type and enters. In this case the incumbent plays a best response to $q_2^2 = 1$ and earns profits 2.25 So if type L deviates from p_1^{1L} the price it will charge will be the monopoly price 3.

The incentive compatibility constraint of the H type is

$$2.25 + 1 \geq (5 - q_1^{1L} - 2)q_1^{1L} + 2.25 \Rightarrow q_1^{1L} \geq 2.62(\text{ or } p_1^{1L} \leq 2.38). \tag{ICC H}$$

The LHS of (ICC H) is the profit of H type if it does not deviate, while the RHS is the deviation profit. If the H type deviates, it mimics the L type.

Hence, in a separating perfect BNE the L type must choose a price in the first period p_1^{1L} in $[1.68, 2.38]$. The H type chooses $p_1^{1H} = 3.5$. The entrant's belief $\mu(p_1^1) = \Pr(L|p_1^1)$ is 1 if $p_1^1 = p_1^{1L}$ and 0 otherwise.

It is worth noting something special about all these perfect Bayesian equilibria: for a low-cost firm to deter entry, it must choose a price that is lower than its short-run profit-maximizing (monopoly) profits. Why? Because to scare off opponents credibly, it must choose something that a high-cost firm would not find profitable to do. In other words, for a signal to be credible, it must be the case that no other type would want to use the signal.

11.2.2. Pooling Equilibrium

A pooling equilibrium will be defined by $\{q_1^{1*}, q_1^{2L}, q_1^{2H}, q_2^2(q_1^1), \mu(q_1^1)\}$ because both types of firm 1 choose the same quantity q_1^{1*} in the first period. This immediately implies that beliefs must satisfy $\mu(q_1^1) = \frac{1}{2}$ because it is common knowledge that Nature chooses $\Pr(c_1 = 1) = \frac{1}{2}$ and pooling behavior implies that firm 2 does not update its beliefs. This implies that firm 2, if it would enter following q_1^{1*}, will play a static game of incomplete information. We therefore need to find the Bayesian Nash equilibrium for this second-period game, which is an incomplete-information version of the Cournot model. Thus we find q_1^{2L}, q_1^{2H} and $q_2^2(q_1^{1*})$ by simultaneously solving three maximization problems: one for the entrant and two for the incumbent's two types.

The Bayesian Nash equilibrium of this incomplete-information Cournot game can be found by solving the three best-response functions (which are just three first-order conditions) simultaneously, which yields

$$q_1^{2L} = \frac{19}{12}, q_1^{2H} = \frac{13}{12} \text{ and } q_2^2(q_1^{1*}) = \frac{5}{6}.$$

This in turn results in second-period expected profits for firm 2 equal to $v_2^2 = \frac{7}{18} > 0$, for the H type of firm 1 equal to $v_1^{2H} = \left(\frac{13}{12}\right)^2$ and for the L type of firm 1 equal to $v_1^{2L} = \left(\frac{19}{12}\right)^2$. This implies that firm 2 will choose to enter in a pooling equilibrium in which it does not learn the type of firm 1.

To proceed we have to set beliefs for off-the-equilibrium-path choices of firm 1 ($q_1^1 \neq q_1^{1*}$), which will determine the best-response function of firm 2 $q_2^2(q_1^1)$ in information sets that are off the equilibrium path. We will then need to find the value of q_1^{1*} to complete the equilibrium. As we have seen in our analysis of the separating perfect Bayesian equilibrium, the best way to keep the different types of firm 1 on the equilibrium path is by making deviation off the equilibrium path very undesirable. This is done by setting the "worse" beliefs that firm 2 could have about firm 1, which are $\mu(q_1^1) = 0$ for $q_1^1 \neq q_1^{1*}$ (i.e., the firm's cost is H).

Using these off-the-equilibrium-path beliefs together with the on-the-equilibrium-path beliefs implies that the sequentially rational best-response function of firm 2 is

$$q_2^2(q_1^1) = \begin{cases} 1 & \text{if } q_1^1 \neq q_1^{1*} \\ \frac{5}{6} & \text{if } q_1^1 = q_1^{1*}. \end{cases}$$

We are left to find a level of q_1^{1*} from which neither type of firm 1 would want to 1 deviate. In the analysis of the separating equilibrium we calculated the "best" first-period deviation of each type, which is to choose its monopoly quantities followed by the best response to $q_2^2 = 1$ in the second period. These yielded a profit of 3.25 to the H type and 6.25 to the L type, as we calculated earlier. We therefore need to satisfy two inequalities for q_1^{1*} to be a best response for both types. For each type the best deviation is to its monopoly quantity, which is 1.5 for the H type and 2 for the L type. Thus for each type the range is an interval that is symmetric around its monopoly quantity. For the H type

$$(5 - q_1^{1*} - 2)q_1^{1*} + \left(\frac{13}{12}\right)^2 \geq 3.25,$$

which reduces to

$$1.083 \leq q_1^{1*} \leq 1.917.$$

Similarly, for the L type

$$(5 - q_1^{1*} - 1)q_1^{1*} + \left(\frac{19}{12}\right)^2 \geq 6.25,$$

which reduces to

$$1.493 \leq q_1^{1*} \leq 2.507.$$

Conclusion: In the limit pricing-entry deterrence game there is a continuum of pooling perfect Bayesian equilibria. Any profile of strategies $\{q_1^{1*}, q_1^{2L}, q_1^{2H}, q_2^2(q_1^1)\}$ together with beliefs $\mu(q_1^1)$ that satisfies $q_1^{1*} \in [1.493, 1.917]$, $q_1^{2L} = \frac{19}{12}$, $q_1^{2H} = \frac{13}{12}$ and

$$q_2^2(q_1^1) = \begin{cases} 1 & \text{if } q_1^1 \neq q_1^{1*} \\ \frac{5}{6} & \text{if } q_1^1 = q_1^{1*} \end{cases}$$

$$\mu(q_1^1) = \begin{cases} 0 & \text{if } q_1^1 \neq q_1^{1*} \\ \frac{1}{2} & \text{if } q_1^1 = q_1^{1*} \end{cases}$$

is a pooling perfect Bayesian equilibrium.

11.3. Refinements of Perfect Bayesian Equilibrium in Signaling Games

In both the MBA game and the limit pricing-entry deterrence game we had a plethora of perfect Bayesian equilibria. This outcome suggests that when we applied sequential rationality to the Bayesian Nash equilibrium concept we still did not manage to get rid of many equilibria. It implies that for the games we analyzed the predictive power of the perfect Bayesian equilibrium solution concept is not as sharp as we would hope.

Let's consider the MBA game first and focus our attention on the pooling equilibrium in which both types of worker choose U, and then regardless of the education choice the employer assigns the worker to B. Now consider the following deviation and the speech that an H type could deliver:

I am an H type. To convince you I am going to deviate and choose D. If you believe me, and put me in the M job instead of a B job, I will get 8 instead of 6. The reason you should believe me is that if I were an L type who chose D and you were to promote me then I would get 5 instead of 6. Therefore you should believe me when I tell you that I am an H type because no L type in his right mind would do what I am about to do.

What should the employer think? The argument makes sense because if the candidate were an L type then there is no way in which he could gain from this move, and in fact he will lose. In contrast an H type can gain if he is believed by the employer. This logic suggests that the employer should be convinced by this deviation combined with the speech. Now if we take this a step further, the employer can make these kinds of logical deductions himself:

Let me see which type can gain from this deviation. If neither can or if both can, I will keep my off-the-equilibrium-path beliefs as before. But if only one type of worker can benefit and other types can only lose, then I should update my beliefs accordingly and act upon these new, more "sophisticated" beliefs.

This logical process leads to the intuitive criterion that was developed by Cho and Kreps (1987). For any given set of beliefs of player 2, player 1 (who has private information) can use his action to send a message to player 2 in the spirit of "only an x type would benefit from this move; therefore I am an x type." The intuitive criterion is a way of ruling out less reasonable equilibria, those for which such messages can be profitably sent, and hence it acts to refine the equilibrium predictions. That is, take a perfect Bayesian equilibrium and see if it survives the intuitive criterion. If it does not–that is, a player can make a deviation with such a convincing message–then it is ruled out by the intuitive criterion.

We now define the intuitive criterion more formally. Consider a signaling game in which player 1 has private information $\theta \in \Theta$ and chooses actions $a_1 \in A_1$ in the first period, after which player 2 observes his action, forms a posterior belief over player 1's type, and then chooses action $a_2 \in A_2$. Imagine that the set of player 1's types is finite, and let $\hat{\Theta} \subset \Theta$ be a subset of player 1's types. Let $BR_2(\hat{\Theta}, a_1)$ be the set of best-response actions of player 2 if player 1 has chosen action $a_1 \in A_1$ and the belief μ of player 2 puts positive probability only on types in the set $\hat{\Theta}$.

Definition 70 *A perfect Bayesian equilibrium σ^* fails the intuitive criterion if there exist $a_1 \in A_1$, $\theta \in \Theta$ and $\hat{\Theta} \subset \Theta$ such that*

1. $v_1(\sigma^*; \theta) > \max_{a_2 \in BR_2(\Theta, a_1)} v_1(a_1, a_2; \theta)$, *for all* $\theta \in \hat{\Theta}$
2. $v_1(\sigma^*; \theta) < \min_{a_2 \in BR_2(\Theta / \hat{\Theta}, a_1)} v_1(a_1, a_2; \theta)$.

From the definition we see that a perfect Bayesian equilibrium fails the intuitive criterion if two conditions hold. Condition (1) states that any type in the subset of types $\hat{\Theta}$ would never choose to play a_1 because regardless of which type player 2 believes him to be, he would do strictly worse than if he stuck to the equilibrium. Condition (2) states that type θ will do strictly better than the equilibrium by playing a_1 if he can convince player 2 that his type is not in $\hat{\Theta}$.

We can now go back to the MBA game and see why the pooling equilibrium fails the intuitive criterion. In this equilibrium both types expect a payoff of 6. Using the definition for the pooling equilibrium, let $a_1 = D$, $\hat{\Theta} = \{L\}$, and $\theta = H$. Condition (1) is satisfied because by choosing D the L type will be worse off: the most he can get is 5 (10 from a manager's wage less 5 for the cost of education). Condition (2) is satisfied because if player 2 is convinced that player 1 is indeed an H type then the H type will receive 8 (10 from a manager's wage less 2 for the cost of education) instead of 6. The separating equilibrium, however, does not fail the intuitive criterion because no type can be made better off by convincing player 2 that he is truly that type.

For the limit pricing entry deterrence game the analysis is more complex. To see why the pooling equilibrium fails, we first need to define $BR_2(\Theta, a_1)$ for any a_1 (which for the limit pricing game was q_1^1). We saw from our analysis of the separating equilibrium that at one extreme belief, when the entrant believes that he is facing the high-cost incumbent, $\mu(q_1^1) = 1$, he will enter and in equilibrium produce $q_2^2 = 1$. At the other extreme, when he believes that the incumbent has low costs, $\mu(q_1^1) = 0$, then he will stay out (effectively producing $q_2^2 = 0$). We also saw that in the intermediate case of a pooling equilibrium, the entrant's belief is $\mu(q_1^1) = \frac{1}{2}$, and in equilibrium he chooses to produce $q_2^2 = \frac{5}{6}$. It should not be hard to realize that the lower the belief $\mu(q_1^1)$, the lower will be the equilibrium quantity chosen by player 2, until some lower bound on beliefs is reached for which the incomplete-information Cournot Bayesian Nash best response will be some level $q_2^2 = \underline{q} < \frac{5}{6}$ that yields player 2 an expected payoff of zero. Any lower quantity cannot be part of an equilibrium in which player 2 chooses to enter. Hence for the limit pricing-entry deterrence game we will have $BR_2(\Theta, a_1) = \{0\} \cup [\underline{q}, 1]$ regardless of a_1.

Now to see that pooling fails the intuitive criterion, let $a_1 = 3$, $\hat{\Theta} = \{H\}$, and $\theta = L$. Condition (1) is satisfied because the pooling equilibrium yields the H type a payoff of at least 3.25, while by deviating to $q_1^1 = 3$ the H type will at most get monopoly profits in period 2 equal to 2.25. So this deviation is always worse for him because

$$(5 - 3 - 2)3 + 2.25 = 2.25 < 3.25.$$

Condition (2) is satisfied because if player 2 is convinced that player 1 is indeed an L type then player 2 will stay out and the L type will receive

$$(5 - 3 - 1)3 + 4 = 7 > 4 + \left(\frac{19}{12}\right)^2 .$$

Notice that the right side of the above inequality is the highest pooling equilibrium payoff that the type can receive, because it includes his monopoly profits in the first period followed by the pooling equilibrium profits in the second. We conclude that the pooling equilibrium fails the intuitive criterion.

In the limit pricing-entry deterrence game we are still left with a continuum of separating equilibria, in which $2.62 \leq q_1^{1L} \leq 3.32$ was undetermined. It turns out that of all these equilibria, the only one that does not fail the intuitive criterion is the best separating equilibrium for player 1, which is $q_1^{1L} = 2.62$. The intuition is that an L type would never benefit from choosing $q_1^1 = 2.62$, so the H type can use this deviation from any other separating equilibrium quantity. For signaling games of the type we have seen, the intuitive criterion will always select the best separating equilibrium as the unique prediction.

11.4. Summary

- In games of incomplete information some types of players would benefit from conveying their private information to the other players.
- Announcements or cheap talk alone cannot support this in equilibrium, be- cause then disadvantaged types would pretend to be advantaged and try to announce “I am this type” to gain the anticipated benefits. This strategy cannot be part of an equilibrium because by definition players cannot be fooled in equilibrium.

- For advantaged types to be able to separate themselves credibly from dis- advantaged types there must be some signaling action that costs less for the advantaged types than it does for the disadvantaged types.
- Signaling games will often have many perfect Bayesian and sequential equilibria because of the flexibility of off-the-equilibrium-path beliefs. Refinements such as the intuitive criterion help pin down equilibria, often resulting in the least-cost separating equilibrium.

12. Building a Reputation

It is common to hear descriptions of some ruthless businesspeople as having “a reputation for driving a tough bargain” or “a reputation for being greedy.” Others are referred to as having “a reputation for being trustworthy” or “a reputation for being nice.” What does it really mean to have a reputation for being a certain type of person? Would people put in the effort to build a reputation for being someone they really are not? Some of the most interesting applications of dynamic games of incomplete information are in modeling and understanding how reputational concerns affect people’s behavior.

This chapter provides some of the central insights of the game theoretic literature that deals with incentives of players to build or maintain reputations for being someone they are not, in the sense of being nice, tough, or any other adjective that comes to mind and can enhance their reputations in the eyes of others. The insights described below have spawned a large literature, and the curious (and more technically inclined) reader is encouraged to consult Mailath and Samuelson (2006).

12.1. Cooperation in a Finitely Repeated Prisoner’s Dilemma

The main idea that drives most game theoretic models of reputation building was first developed by Kreps, Milgrom, Roberts, and Wilson (1982) in a seminal paper that became known as the “gang of four” paper. In it they considered a finitely repeated Prisoner’s Dilemma game for which backward induction implies that there is a unique subgame-perfect equilibrium in which the players always defect. However, it is somewhat counterintuitive that players will play this way–is there no chance that if the game is finite but sufficiently long then by cooperating today they can sustain some quid-pro-quo cooperative behavior?

As the gang of four demonstrated, if there is even just a little bit of incomplete information about the players' types then for a long enough horizon of T periods there can actually be quite a lot of cooperation. To understand the idea behind their arguments, consider the following version of the Prisoner's Dilemma:

		Player 2	
		c	d
Player 1	C	$1, 1$	$-1, 2$
	D	$2, -1$	$0, 0$

Imagine that player 2 has standard "strategic" Prisoner's Dilemma payoffs given by the matrix, so that for him the dominant strategy is to defect (d). Player 1, however, is one of two types: he is either a strategic type or a grim-trigger type. The grim-trigger type uses the following simple strategy: In the first period he always cooperates (C); in every period $t > 1$ thereafter he continues to cooperate as long as player 2 has cooperated (c). If in any period t player 2 defects then from period $t+1$ onward the grim-trigger type will defect (D). Let $p > 0$ be the probability that Nature chooses player 1 to be the grim-trigger type.

If this game is played only once then the outcome is obvious: both player 2 and the strategic player 1 will choose to defect as it is their dominant strategy. If the game is played for two periods, however, things become a bit more subtle. We have the following:

Claim. If the game is played for two periods then there is a unique perfect Bayesian equilibrium. In period $t = 2$ player 2 and the strategic type of player 1 choose to defect. In period $t = 1$ the strategic type of player 1 chooses to defect while player 2 chooses to cooperate if and only if $p > \frac{1}{2}$.

Proof. It's clear that in period $t = 2$ both player 2 and the strategic player 1 will choose to defect regardless of what happened in period $t = 1$ and of player 2's beliefs about player 1's type. In period $t = 1$ the strategic type of player 1 will choose D because regardless of what he plays in period $t = 1$ he expects player 2 to play d in period $t = 2$. We are left to show that player 2 chooses to cooperate in period $t = 1$ if and only if $p > \frac{1}{2}$. The grim-trigger type of player 1 will cooperate in period $t = 1$, and in period $t = 2$ he will cooperate if and only if player 2 cooperated in period $t = 1$. As a consequence player 2's payoff from cooperating is

$$v_1(C) = p(1) + (1-p)(-1) + p(2) = 4p - 1,$$

while his payoff from defecting is

$$v_1(D) = p(2) + (1-p)(0) + p(0) = 2p.$$

It follows that player 2's best response is to play c in period $t = 1$ if and only if $p > \frac{1}{2}$ ■

The above claim shows that when there is a high enough chance that player 1 is a grim-trigger type then player 2 will choose to cooperate in the first period. The reason is simply that if he defects, he guarantees himself a payoff of 0 in the second period, while if he cooperates, he may get a payoff of 2 from defecting. He is, however, taking a risk that he plays against a strategic type. Hence if that risk is not too high and if the expected reward is high enough then he will cooperate. Notice, however, that this has nothing to do with reputation–the strategic type of player 1 always plays defect and hence reveals himself immediately after the first period. With longer horizons, however, things change quite dramatically, as the next claim shows.

Claim If the game is played for three periods and if $p > \frac{1}{2}$ then in the unique perfect Bayesian equilibrium both player 2 and the strategic type of player 1 choose to cooperate in period $t = 1$.

Proof. We begin by considering how the game will proceed after period $t = 1$. Observe that if player 2 defects in the first period then both players will defect thereafter. Now imagine that player 2 cooperates in period $t = 1$ and let $\mu(GT)$ denote player 2's belief in period $t = 2$ that player 1 is a grim-trigger type. From the previous claim we know how the perfect Bayesian equilibrium will proceed in the last two periods, so that player 2 will choose c in period $t = 2$ if $\mu(GT) > \frac{1}{2}$ and d otherwise. We now will see if the perfect Bayesian equilibrium can support both player 2 and the strategic type of player 1 cooperating in period $t = 1$.

Now imagine that the strategic type of player 1 believes that player 2 will choose c in period

$t = 1$. If the strategic type chooses C then he pools with the grim-trigger type, implying by Bayes' rule that $\mu(GT) = p > \frac{1}{2}$, and his payoff from choosing C is $v_2(C) = 1+2+0 = 3$. (He will gain 1 in period $t = 1$ from cooperating, and because $\mu(GT) > \frac{1}{2}$ then from the previous claim player 2 will choose c and the strategic type of player 1 will choose D in period $t = 2$, followed by (d, D) in period $t = 3$.) If the strategic type chooses d then he separates from the grim-trigger type, implying by Bayes' rule that $\mu(GT) = 0$, and his payoff from choosing D is $v_2(D) = 2 + 0 + 0$. (Because he reveals himself to be a strategic type then $\mu(GT) = 0$ and the play continues with both players choosing to defect in the following two periods.) This implies that a strategic type of player 1 should pool with the grim-trigger type and cooperate in the first period so as not to reveal his true type.

Turning to player 2, if he believes that both types of player 1 cooperate in $t = 1$ then his payoff from choosing c is $v_1(c) = 1+p(1+2)+(1-p)(-1) = 4p$ while his payoff from choosing d is $v_1(d) = 2+0+0$. Because $p > \frac{1}{2}$ this implies that player 2 should also cooperate in period $t = 1$. The reason this is the unique perfect Bayesian equilibrium is that even if player 2 believes that the strategic type of player 1 is playing D then he should still choose c because in this case $v_1(c) = p(1 + 1 + 2) + (1 - p)(-1) = 5p - 1$ while $v_1(d) = p(2) + (1 - p)(0) = 2p$, which is less than $v_1(c)$ if $p > \frac{1}{3}$. ■

The above claim demonstrates that with three periods a strategic type of player 1 has incentives to behave as if he is a grim-trigger type, thus "building a reputation" for cooperation. Of course, for this to be the unique perfect Bayesian equilibrium we had to assume that $p > \frac{1}{2}$. It turns out, however, that the gang of four proved a much more striking result: if the game is repeated for long enough then the players will cooperate in almost all periods. More formally, we have the following:

Proposition 71 *Consider a T-period repeated Prisoner's Dilemma game in which T is large. The number of periods in which either player 2 or the strategic player 1 defects is bounded above by a constant $M < T$ that depends on p and not on T.*

A formal proof is not provided, and the interested reader can refer to the original paper by Kreps et al. (1982) or to Mailath and Samuelson (2006, Chapter 17). The intuition follows from the fact that if the strategic type of player 1 chooses D in some period $t < T$ then it becomes common knowledge that player 1 is strategic, and backward induction implies that both players will defect until the end of the game. For this reason the strategic player 1 has an incentive to mimic the grim-trigger type and maintain a reputation for not being strategic. These reputational incentives only diminish as the game gets close to T. Anticipating these incentives, it is in the best interest of player 2 to cooperate as well. What makes the gang of four result remarkable is that it does not require specific assumptions on p. Even for very small values of p, for a large enough T cooperation will be sustained for all but a finite number of periods.

12.1.1. An Investment Game

The next example shows that the presence of a 'cooperative' type helps the 'ordinary' type establish a cooperative reputation even in a one-shot game.

The extensive form game is depicted in Figure 35. Player 1 and 2 need to make sequential investments to make an asset useable. Each player chooses whether to invest I or not N, with player 1 moving first. If one player does not make an investment both players' payoffs are zero. Player 1 owns the asset, so he controls how it will be used. He can share the asset with player 1 (be benevolent B) or not (be selfish S). Player 1 can be one of two types: ordinary (O) or cooperative (C). The cooperative type derives higher utility from not being selfish. The type of player 1 is unobservable by player 2.

We search for a perfect Bayesian equilibrium of this game.

First note that every PBE specifies S, I' and B' for player 1. Only the behavior of the ordinary type (O) at the beginning of the game has yet to be determined and depends on what player 2 does. Let q denote player 2's belief that the type is (C), given an investment from player 1. When player 2 chooses I its expected payoff is $4q - 2$, while when it chooses N the payoff is 0. Hence, $I \succeq N \Rightarrow q \geq 1/2$.

Can type (O) select N in a PBE? NO. Here is why.

If (O) selects N and, as we know, (C) always selects I', then $q = 1$. Player 2 then must choose I. But then type (O) has a profitable deviation to I, which contradicts the initial assumption.

Can type (O) select I in a PBE? NO. Here is why.

If (O) selects I, then $q = 1/4$. Player 2 then must choose N, making I a bad choice for (O).

Therefore, there does not exist a PBE in pure strategies. We search for a mixed strategy PBE.

Let r denote the probability that (O) chooses I and s the probability that player 2 selects I.

Type (O) must be indifferent between I and N, implying that its expected payoff must be zero: $6s - 2(1-s) = 0 \Rightarrow s = 1/4$.

If $s = 1/4$ player 2's expected payoff from I should be zero, because it gets zero from N: $2q - 2(1-q) = 0 \Rightarrow q = 1/2$.

We need to determine q using Bayes' rule. Let IS stand for information set. Then

$$q = \Pr(C|IS) = \frac{\Pr(IS|C)\Pr(C)}{\Pr(IS|C)\Pr(C) + \Pr(IS|O)\Pr(O)} = \frac{1 \times \frac{1}{4}}{1 \times \frac{1}{4} + r \times \frac{3}{4}}.$$

Since $q = 1/2$, we obtain $r = 1/3$.

In sum, player 1 uses the strategy specifying S, I' and B' with certainty and action I with probability 1/3. Player 2's belief is $q = 1/2$ and chooses I with probability 1/4.

With some probability the players invest in the asset and it creates value. The presence of the cooperative type helps the ordinary type to establish a reputation for cooperative behavior.

12.2. Summary

- Games of incomplete information can shed light on incentives for rational strategic players to behave in ways that help them build a reputation for having certain behavioral characteristics.

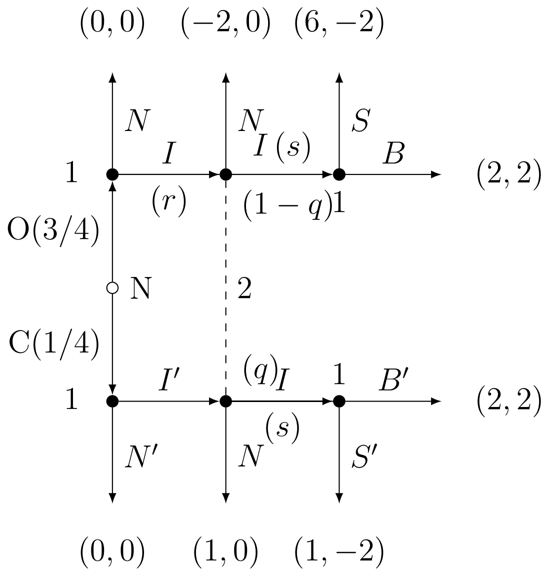

Figure 35.: Reputation game

- In equilibrium models players are, by definition, never fooled. However, if there is incomplete information then players will have rational uncertainty about whether players they face are set in their ways.
- This rational uncertainty is what gives strategic players an incentive to imitate behavioral "types" and act in ways that are not short-run best-response actions, but that in turn give rise to long-run benefits, thus providing reputational incentives.
- The incomplete information and the resulting reputational incentives cause finitely repeated games and other finite dynamic games not to unravel to the often grim backward-induction outcome, but instead to result in high-payoff behavior that can persist on very long time horizons.
- These game theoretic models help us understand concepts such as apparently "crazy" behavior that in turn results in long-run benefits for the player acting in this way.