METHODS OF NONLINEAR OPTIMIZATION: HW#3

(1) Problem 6.11 of [Beck] on Page 114.

(In (iv), if the 1-norm is replaced by the usual Euclidean 2-norm, then K is a typical ice-cream cone. I wonder if you have ever seen ice-cream served in such a 1-norm cone.)

We shall cover some duality theory at the end of the course. It goes approximately like this: given a convex (or even non-convex) optimization problem, one can always define another optimization problem called its dual. Properties of one problem tell you something about the other. The intimate relationship between the two problems can help solving the original problem more efficiently.

(2) Consider

$$P = \{x \in \mathbb{R}^n : Ax \leq b,\, x \geq 0\}. \tag{0.1}$$

Here A is an arbitrary $m \times n$ matrix. If the i-th row of A is zero, then the corresponding inequality either holds for all x (when $b_i \geq 0$) or holds for no x (when $b_i < 0$); otherwise the inequality represents a half-space. Without loss of generality, we can assume that none of the rows of A is zero. In this case, P is the intersection of $m + n$ half-spaces. We call P a convex polytope.

In the text, we talk about **basic feasible solutions** and **extreme points** for convex polytopes expressed in the form of

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^N : \mathcal{A}\mathbf{x} = \mathbf{b},\, \mathbf{x} \geq 0\}. \tag{0.2}$$

Here $\mathcal{A}$ is not an arbitrary matrix, but one with linearly independent rows. The theorem on "basic feasible solutions $\equiv$ extreme points" is stated and proved for polytope of the form (0.2).

Here, I want to relate the concept of basic feasible solutions/extreme points for (0.2) to the concept of 'vertices' for (0.1).

(i) First, let's see that we can 'lift dimension' and identify a polytope $P \subset \mathbb{R}^n$ of the form (0.1) with a polytope $\mathcal{P} \in \mathbb{R}^{m+n}$ of the form (0.2).

Define the 'slack variables' $x_S := b - Ax$. Then define $\mathcal{A} := [A,\, I_{m\times m}] \in \mathbb{R}^{m\times(n+m)}$, and $\mathbf{b} := b$, and use them to define a $\mathcal{P}$ as in (0.2). Show that when $\mathcal{P}$ is defined this way, the map

$$x \overset{\iota}{\mapsto} [x; b - Ax]$$

is a one-to-one correspondence between P and $\mathcal{P}$.[1]

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[1]The semicolon in '$[x; y]$' here means stacking a column vector x on top of another column vector y to form a longer column vector.

(ii) For a convex polytope P of the form (0.1), define a **vertex** as a point $x \in P$ which makes n of the $m+n$ inequalities in (0.1) active (i.e. become equalities), and such that the corresponding $n \times n$ linear system is non-singular.

Give an example of a convex polytope of the form (0.1) with $m = 5$ and $n = 2$, and so that P is a convex hexagon in $\mathbb{R}^2$. (Note: $5 + 2 = 7$, how do we get a 6-gon?) Please jot down your choice of A and b and please draw a detailed figure with labels. Call this hexagon H. Also, determine carefully the coordinates of the six vertices of H.

If we call the identification map in (i) ι, how do you describe the set $\iota(H)$ geometrically? (I am only asking for an intuitive answer for this one.)

(iii) Prove that, under the identification in (i), the concept of vertex in (ii) is equivalent to the concept of basic feasible solution in our text, i.e. show that x is a vertex of P if and only if $\iota(x)$ is a basic feasible solution of $\iota(P)$.

(3) Problem 6.16 of [Beck] on Page 114. How do you describe this set geometrically?

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