Lecture 4: Higher Dimensional Regular Surfaces

Note Title 1/8/2017

$SO(n) = \{A \in \mathbb{R}^{n \times n} : A^T A = I, \det(A) > 0\}$,

when viewed as a subset of

$$\mathbb{R}^{n \times n} \approx \mathbb{R}^{n^2}$$

is certainly <u>not</u> a linear subspace

Sure enough,

$$A, B \in SO(n) \not\Rightarrow A+B \in SO(n),$$

and $\quad 0 \notin SO(n)$.

Here, you see that:

- there are 'nonlinearities in linear maps',

and

- there are 'linearities in nonlinear maps'.

My excuse for making the last comment is simply that a key mathematical tool for dealing with nonlinearities is the idea of Local Linear Approximation.

$f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable means:

$\mathcal{U}$ ← some open neighborhood of $a \in \mathbb{R}^n$

$$f(x) = f(a) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} (x-a) + o(\|x-a\|)$$

$f(x)$ ↑ nonlinear

← linear

The goal of this lecture is to

Define "k-dimensional regular surfaces in $\mathbb{R}^n$"

Give a mental picture of what is a "k-dimensional manifold"

And, as an important example

Prove : $SO(n)$ is a $\frac{n(n-1)}{2}$-dimensional regular surface in $\mathbb{R}^{n^2}$.

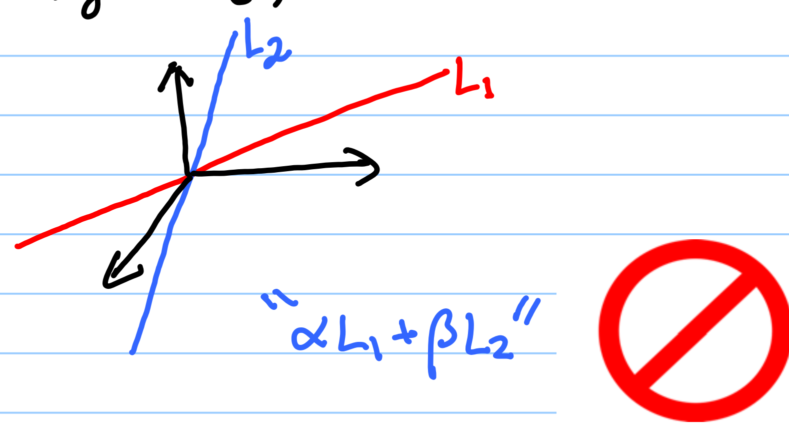
discuss : $SO(n)$ is a $\frac{n(n-1)}{2}$-dimensional manifold.

Another interesting example :

$G(n,k)$ = the set of all k-dimensional linear subspaces of $\mathbb{R}^n$

<u>Like $SO(n)$</u>, $G(n,k)$ is made up of objects in linear algebra, but, as a space by itself, it is not a linear space.

e.g. $n=3$, $k=1$



Unlike $SO(n)$, $G(n,k)$ does not naturally sit in some Euclidean space.

$$SO(n) \subset \mathbb{R}^{n^2}$$

$$G(n,k) \subset \mathbb{R}^{??}.$$

We shall later prove:

- $G(n,k)$ is a $k(n-k)$-dim. manifold.
- "$G(n,k) \hookrightarrow \mathbb{R}^{n^2}_{sym}$"

Definition:
A subset $S \subset \mathbb{R}^n$ is a k-dimensional regular surface in $\mathbb{R}^n$ if, for each $p \in S$ there exists a neighborhood V in $\mathbb{R}^n$ and a map

$$\underset{\text{open in } \mathbb{R}^k}{X : \mathcal{U}} \to \underset{\text{open in } \mathbb{R}}{V \cap S}$$

"a local parameterization"

s.t.

1. X is C^∞
2. X is a homeomorphism
3. $dX(q)$ is injective for all $q \in \mathcal{U}$.

Not a easy condition to work with. Fortunately $2'$ is enough.

$2'$. X is a bijection

Note : nothing exciting here, all we did was changing :

2 to k $\leftarrow$ "intrinsic dimension" and

3 to n. "ambient dimension"

$\underbrace{n-k}$ "co-dimension"

However : notice that the concepts of

- height over tangent plane
- Gauss map

rely on the concept of normal vector

that only generalizes easily to the case when

$k = n-1$.

In this case, we call S a hypersurface.

The most 'boring' type of k-dimensional regular surfaces in $\mathbb{R}^n$ are the ones that do not curve, i.e.

the k-dimensional planes in $\mathbb{R}^n$.

$$S = \operatorname{span}(v_1, \ldots, v_k) + x_0$$

$$= \{ x_0 + t_1 v_1 + t_2 v_2 + \cdots + t_k v_k : t_1, \ldots, t_k \in \mathbb{R} \}$$

$x_0 \in \mathbb{R}^n$, $v_1, \ldots, v_k$ - k linearly independent vectors in $\mathbb{R}^n$.

To show that S is a regular surface, for any $p \in S$, simply choose $\mathcal{U} = \mathbb{R}^k$, $V = \mathbb{R}^n$

$$\tilde{t} = (t_1, \dots, t_k)^T \mapsto x_0 + t_1 v_1 + \dots + t_k v_k$$

is a perfectly smooth map from $\underset{\mathcal{U}}{\mathbb{R}^k}$ to $\underset{V}{\mathbb{R}^n}$.

parameterization = explicit representation

In this case, we can also choose $n-k$ independent vectors orthogonal to S, i.e.

$$v_{k+1}, \dots, v_n \perp S - x_0$$

then

$$S = \{x \in \mathbb{R}^n : [v_{k+1}, \dots, v_n]^T (x - x_0) = 0\}$$

$$= x_0 + \mathrm{null}([v_{k+1}, \dots, v_n]^T).$$

This gives an <u>implicit representation</u> of S

"$x \in S$ if (some condition on x) is satisfied."

span / image $\longleftrightarrow$ explicit representation

null / kernel $\longleftrightarrow$ implicit representation

Important to master this 'boring example', because <u>locally</u> a k-dim. regular surface / manifold is not very different from a k-dim. plane.

We now extend the concept of local linear approximation / derivative to the case of

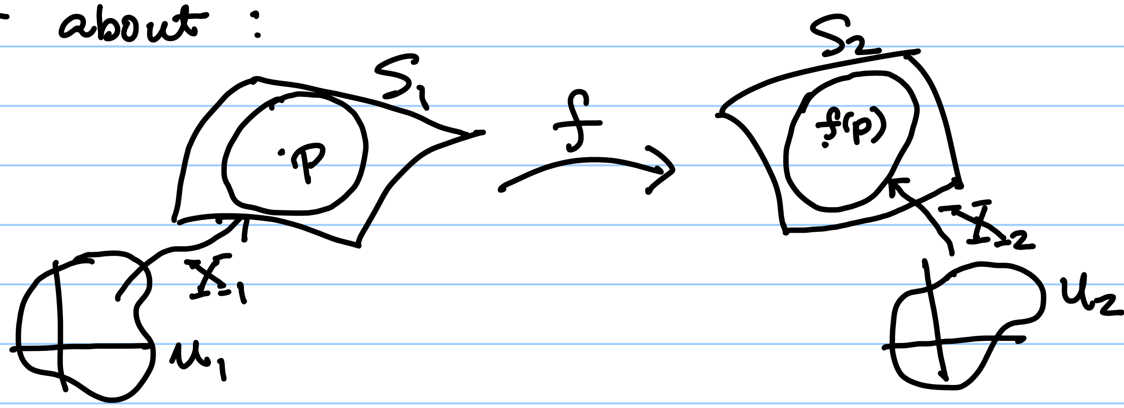
$$f : S_1 \to S_2$$

regular surfaces.

$\boxed{\text{But before we do so, what does it mean by "}f\text{ is differentiable"?}}$

Note : Local parameterization provides 'coordinate neighborhood' of a point.

How about :



<u>Def</u> : f is <u>defined</u> to be differentiable at $p \in S_1$

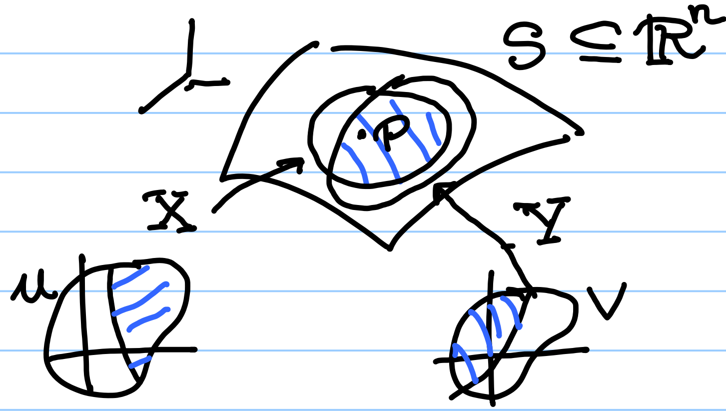
if

$$X_2^{-1} \circ f \circ X_1 : U_1 \to U_2$$

$U_1 \subset \mathbb{R}^{k_1}$, $U_2 \subseteq \mathbb{R}^{k_2}$

is differentiable in the usual sense in advanced Calculus.

Ex: what is the problem with this definition?



Proposition: If $X : U \to S$, $Y : V \to S$ are two C^k local parameterizations (a.k.a "coordinate neighborhoods") around $p \in S$, then the change of coordinates map

$$X^{-1} \circ Y : Y^{-1}(X(U) \cap Y(V)) \to X^{-1}(X(U) \cap Y(V))$$

is C^k with a C^k inverse. (aka a C^k diffeomorphism.)

[For simplicity, assume $k = \infty$, i.e. all local parameterizations are infinitely smooth.]

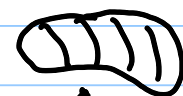
Ex: why does this result fix the problem in the previous ex.?

Discussions:

What needs to be proved?

Didn't we assume those local parameterizations are smooth, so composition of C^∞ maps are C^∞.

The stinging technicality is that it makes no sense at this point to say that

X^{-1} :  $\to \mathbb{R}^k$ is C^∞ smooth.

curved surface

" $\frac{\partial X^{-1}}{\partial(?)}$ " !?

It is, however, sensible to talk about the continuity of X^{-1}. Indeed, condition 2. in the def. of regular surfaces requires X^{-1} to be continuous.

$X^{-1} \circ Y$: (Euclidean) $\to$ (Euclidean)

$$\underset{\mathbb{R}^k}{(\text{Euclidean})} \xrightarrow{Y} \underset{\underset{\underset{\mathbb{R}^n}{(\text{Euclidean})}}{\cap}}{(\text{curved object})} \xrightarrow{X^{-1}} \underset{\mathbb{R}^k}{(\text{Euclidean})}$$

"smoothness of Y" o.k.

"smoothness of $X^{-1} \circ Y$" o.k.

"continuity of X^{-1}" o.k.

"smoothness of X^{-1}" not ok. !

Proof:

Trick: Get help from the Euclidean structure of the <u>ambient space</u> ($\mathbb{R}^n$).

By renaming the axes if necessary, we can assume

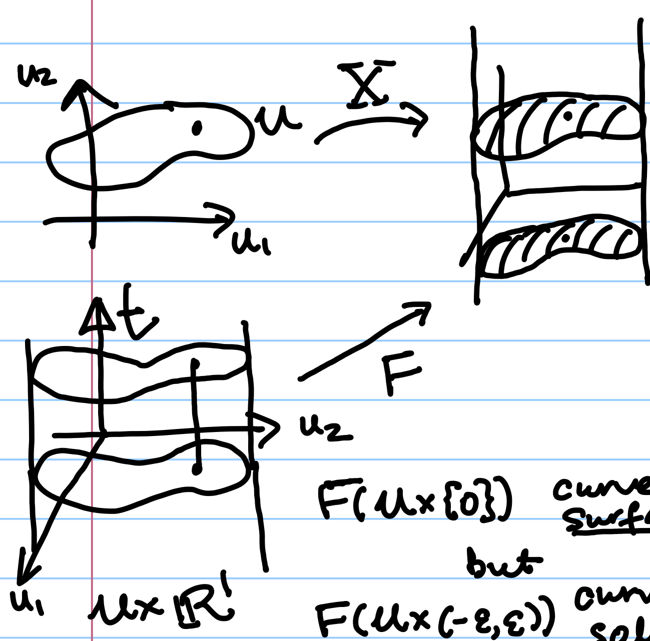
$$\left.\begin{bmatrix} \boxed{\begin{matrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_k} \\ \vdots & & \vdots \\ \frac{\partial x_k}{\partial u_1} & \cdots\cdots & \frac{\partial x_k}{\partial u_k} \end{matrix}} \\ \begin{matrix} \vdots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots\cdots & \frac{\partial x_n}{\partial u_k} \end{matrix} \end{bmatrix}\right|_{X^{-1}(p)} \leftarrow \text{invertible}$$

(full rank $= k$)

Extend X to a map $F: \overbrace{U \times \mathbb{R}^{n-k}}^{\text{open in } \mathbb{R}^n} \to \mathbb{R}^n$

$$\underset{\substack{\| \\ (u_1,\dots,u_k)}}{u},\ \underset{\substack{\| \\ (t_1,\dots,t_{n-k})}}{t} \quad (u, t) \longmapsto \begin{bmatrix} x_1(u) \\ \vdots \\ x_k(u) \\ x_{k+1}(u) + t_1 \\ \vdots \\ x_n(u) + t_{n-k} \end{bmatrix}$$

easiest to picture when $k=2$, $n=3$



$F(U \times \{0\})$ curved <u>surface</u>

but

$F(U \times (-\varepsilon, \varepsilon))$ curved <u>solid</u>

How does F behave near

$$(\underbrace{X^{-1}(p)}_{\in\, U}, \underbrace{0}_{\in\, \mathbb{R}^{n-k}}) \quad ?$$

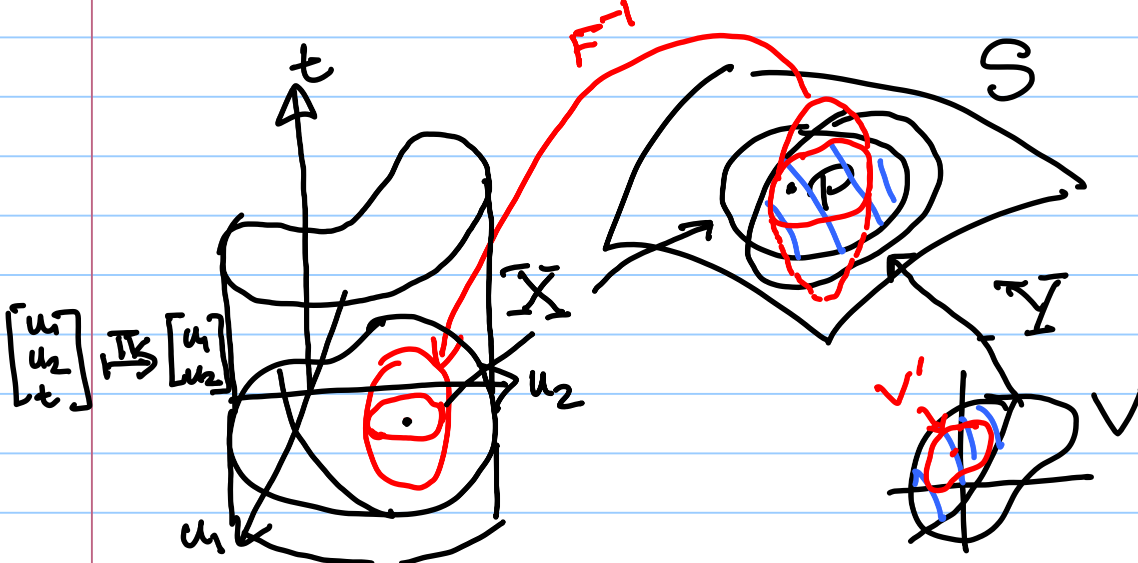
$$dF\Big|_{(X^{-1}(p),0)} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \cdots & \frac{\partial x_1}{\partial u_k} & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \\ \frac{\partial x_k}{\partial u_1} & \cdots & \frac{\partial x_k}{\partial u_k} & 0 & \cdots & 0 \\ \vdots & & \vdots & 1 & & 0 \\ & & & & \ddots & \\ \frac{\partial x_n}{\partial u_1} & \cdots & \frac{\partial x_n}{\partial u_k} & 0 & & 1 \end{bmatrix}\Bigg|_{(X^{-1}(p),0)}$$

$n=3,\ k=2$

$$dF\Big|_{(X^{-1}(p),0)} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & 0 \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & 0 \\ \frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \boxed{1} \end{bmatrix}$$

is an $n \times n$ invertible matrix.

By the inverse function theorem, F has a local inverse, which is as smooth as F.



i.e. $\exists$ open neighborhood N of p in $\mathbb{R}^n$ s.t. $F^{-1}: N \to \mathcal{U} \times \mathbb{R}^{n-k}$ is well-defined and

$$F \circ F^{-1} = F^{-1} \circ F = id$$

$$X^{-1} \circ Y \Big|_{\substack{\text{some smaller}\\ \text{nbhd of}\\ Y^{-1}(p),\\ \text{call it } V'}} = \underset{\text{linear}}{\pi} \circ \underset{\text{smooth}}{F^{-1}} \circ \underset{\text{smooth}}{Y} \Big|_{V'}$$

$$V' = Y^{-1}(N \cap S)$$

$$\pi(u, t) = u$$

Note: the composition with π on the r.h.s. is only for argument sake, it does not have any "real effect" as $F^{-1}(Y(v))$ is always of the form $(u, 0)$.

We have argued that $X^{-1} \circ Y$ is C^k smooth in the neighborhood of any point in its domain. $\blacksquare$

Comment: Note that this proof relies on the Euclidean structure of the ambient space, which is something we want to dispense with in the development of manifolds. It is, therefore, necessary to <u>impose</u> the smoothness of change of coordinates in the definition of a smooth manifold. (See Lecture 5.)

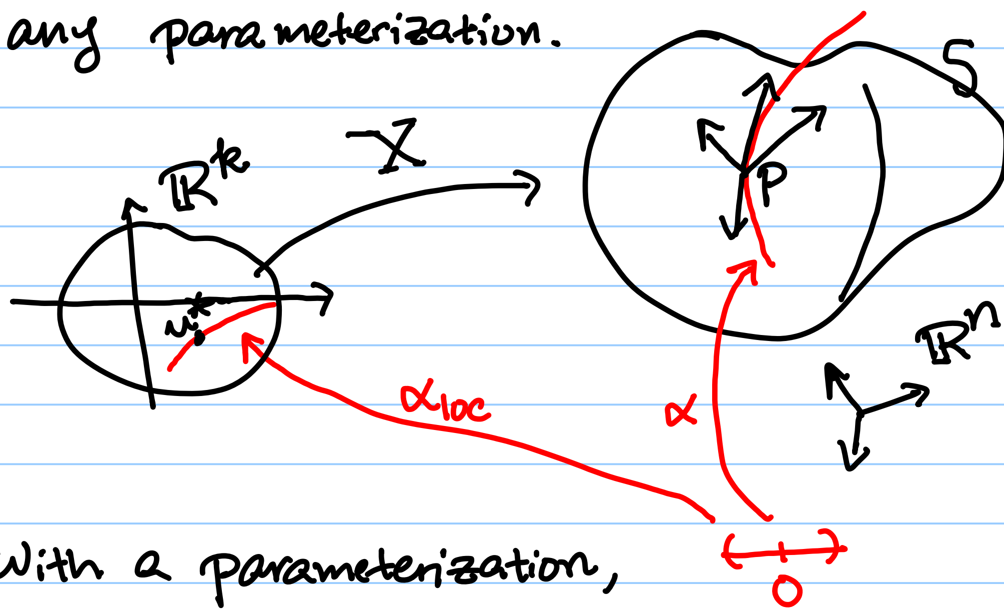
Tangent Plane

S - k-dimensional regular surface in $\mathbb{R}^n$

__Def:__

$$T_pS := \{ \alpha'(0) : \alpha : (-\varepsilon, \varepsilon) \to S, \ \alpha(0) = p \}$$
$$\subseteq \mathbb{R}^n_p$$

This definition is nicer than than the one given before (even for the $(k,n) = (2,3)$ case), because it looks simpler and does not involve any parameterization.



With a parameterization,

$$\underset{n\times 1}{\alpha'(0)} = \Big[X \circ \underbrace{(X^{-1} \circ \alpha)}_{\alpha \text{ in local coordinates, call it } \alpha_{loc}} \Big]'(0)$$

$$= \underset{n \times k}{\big[dX(u_0) \big]} \cdot \underset{k \times 1}{\alpha'_{loc}(0)}$$

$$= (\alpha'_{loc}(0))_1 \left.\frac{\partial X}{\partial u_1}\right|_{u^*} + \cdots + (\alpha'_{loc}(0))_k \left.\frac{\partial X}{\partial u_k}\right|_{u^*}$$

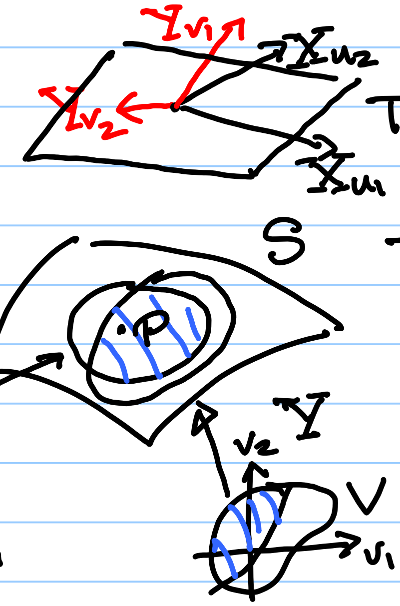
$$\mathcal{B} = \left\{ \frac{\partial X}{\partial u_1}\Big|_{u^*}, \ldots, \frac{\partial X}{\partial u_k}\Big|_{u^*} \right\}$$

is an ordered basis of T_pS.

Ex: Fill in any logical gap. (Compare with the discussion in Lecture 3.)

Note:

(I)



Two coordinate neighborhoods induce two different bases for the same tangent space T_pS.

(II) $\alpha'(0) = \tilde{\alpha}'(0)$

$$\Leftrightarrow \underbrace{\begin{bmatrix} dX(u^*) \end{bmatrix}}_{\text{injective}} \begin{bmatrix} \alpha'_{loc}(0) \end{bmatrix} = \begin{bmatrix} dX(u^*) \end{bmatrix} \begin{bmatrix} \tilde{\alpha}'_{loc}(0) \end{bmatrix} \quad (\text{any } X)$$

$$\Leftrightarrow \alpha'_{loc}(0) = \tilde{\alpha}'_{loc}(0).$$

Def: Let

$$f : S_1 \to S_2 \text{ be differentiable.}$$

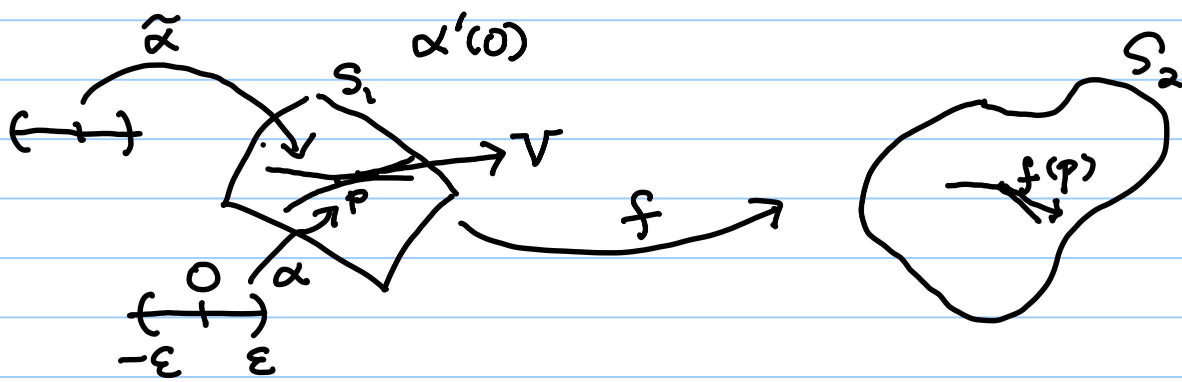
Its <u>differential</u> at $p \in S_1$

$$df_p : T_p S_1 \longrightarrow T_{f(p)} S_2$$

is defined by :

$$T_p S_1 \ni \underset{\alpha'(0)}{\overset{}{v}} \longmapsto (f \circ \alpha)'(0) \in T_{f(p)} S_2$$

$v = \alpha'(0)$



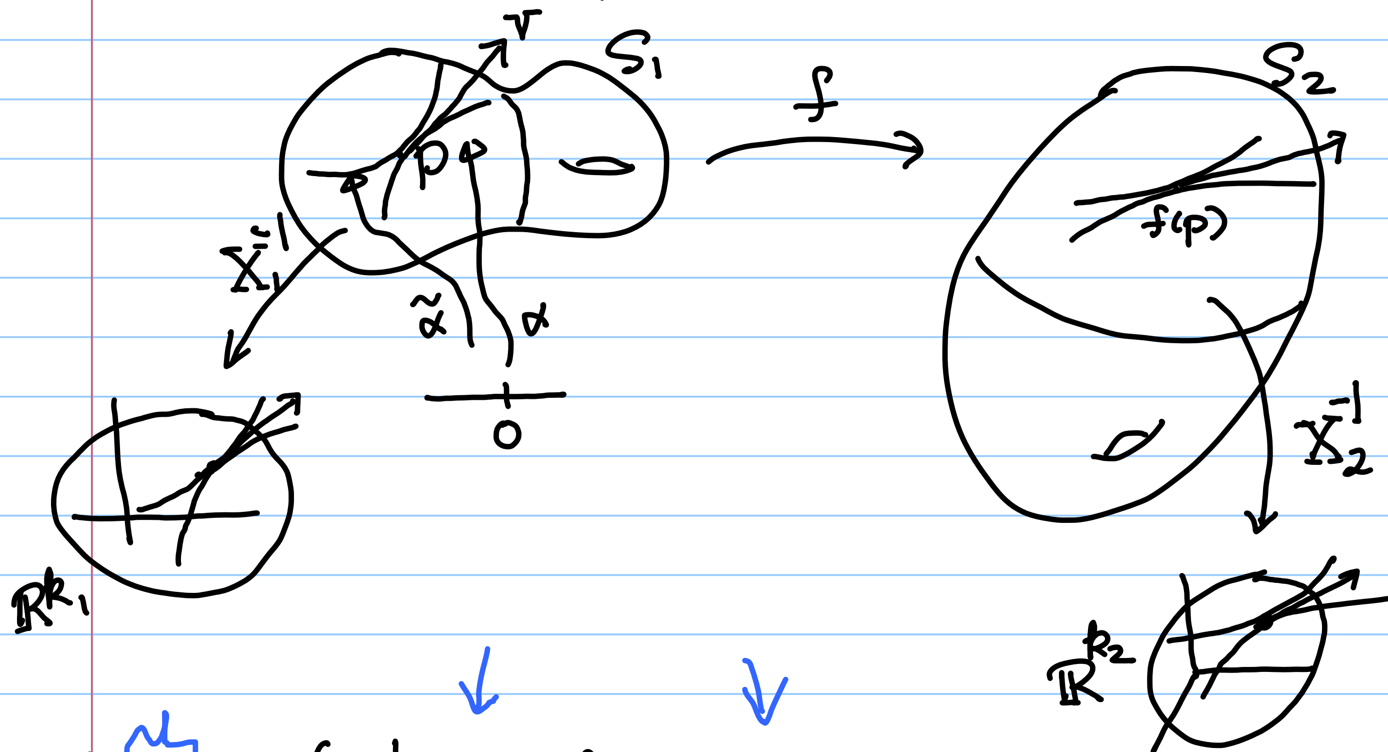
But what if $\tilde{\alpha}'(0) = v$, then

$$v \begin{cases} \nearrow (f \circ \alpha)'(0) \\ \qquad \| \quad \longleftarrow ? \\ \searrow (f \circ \tilde{\alpha})'(0) \end{cases}$$

Below, we answer this question affirmatively, it requires a pretty standard calculation in basic manifold theory. This calculation also illustrates that

- $df_p : T_p S_1 \to T_{f(p)} S_2$ is a <u>linear</u> map

To check (?), write everything in local coordinates and use the chain rule from advanced calculus:



$$\underbrace{X_2^{-1} \circ f \circ \alpha}_{(f\circ\alpha) \text{ in local coordinates}} = \underbrace{(X_2^{-1} \circ f \circ X_1)}_{f \text{ in local coordinates}} \circ \underbrace{(X_1^{-1} \circ \alpha)}_{\alpha \text{ in local coordinates}}$$

$$(X_2^{-1} \circ f \circ \alpha)'(0) = \frac{d}{dt} (X_2^{-1} \circ f \circ X_1) \circ (X_1^{-1} \circ \alpha) \Big|_{t=0}$$

$$= \underbrace{d(X_2^{-1} \circ f \circ X_1)\Big|_{X_1^{-1}(p)}}_{k_2 \times k_1} \cdot \underbrace{d(X_1^{-1} \circ \alpha)}_{k_1 \times 1}\Big|_0$$

But

$$\alpha'(0) = \tilde{\alpha}'(0) \underset{\text{Note (II)}}{\Longrightarrow} d(X_1^{-1} \circ \alpha)\big|_{t=0} = d(X_1^{-1} \circ \tilde{\alpha})\big|_{t=0}$$

So $(X_2^{-1} \circ f \circ \alpha)'(0) = (X_2^{-1} \circ f \circ \tilde{\alpha})'(0)$, and (again by Note II earlier)

$$(f \circ \alpha)'(0) = (f \circ \tilde{\alpha})'(0).$$

Q.E.D.

Ex: Based on the above derivation, argue that df_p is linear.

Recap:

curved / nonlinear objects

$$f : S_1 \to S_2$$

$$df_p : T_p S_1 \to T_{f(p)} S_2$$

linear map that serves as first order local approximation of f

linear spaces that serve as first order local approximations of S_1 and S_2, resp.

Example : Gauss map

$$N : S \to S^2$$

$\leftarrow$ 2-sphere

$\uparrow$ surface in $\mathbb{R}^3$

$p \in S$, $\quad dN_p : T_pS \to T_{N(p)}S^2$

goes under several names

- shape operator
- 2nd fundamental form
- Weingarten map

But most importantly we need to first identify

$$T_{N(p)}S^2 \quad \text{with} \quad T_pS$$

and write instead

$$dN_p : T_pS \to T_pS$$

Note that N is a unit vector ($\Leftrightarrow$ $N \in S^2$)

$$\langle N, N \rangle = 1$$

In local coordinates, $\langle N(u,v), N(u,v) \rangle = 1$ $\forall u, v$

so $\langle N, N_u \rangle = 0$

$\langle N, N_v \rangle = 0$

which means $N_u, N_v \in T_pS$.

So reinterpret

$$dN_p(\underset{=\alpha'(0)}{v}) \in T_{N(p)}S^2$$

as

$$d(\underbrace{N \circ \alpha}_{\underbrace{N\circ X}\circ\underbrace{X^{-1}\circ\alpha}_{\alpha_{loc}}}) = [N_u, N_v]\,\alpha_{loc}'(0) \in T_pS^2$$

Proposition:

$dN_p : T_pS \to T_pS$ is <u>self-adjoint</u>.

<u>Proof</u>: The condition

$$\langle dN_p(v), w\rangle = \langle v, dN_p(w)\rangle \quad \forall v, w$$

$\Updownarrow$ in local coordinates (u,v)

$$\langle N_u, X_v\rangle = \langle X_u, N_v\rangle \quad -(*)$$

But (why?)

$$\langle N, X_u\rangle = 0 = \langle N, X_v\rangle,$$

(not just @ p, but the whole neighborhood)

So

$$\langle N_v, X_u\rangle + \langle N, X_{uv}\rangle = 0$$
$$\langle N_u, X_v\rangle + \langle N, X_{vu}\rangle = 0$$

$\Longrightarrow (*)$ using $X_{uv} = X_{vu}$ and the symmetry of $\langle , \rangle$. ∎

Ex (needed for HW #3 and for the previous proof):

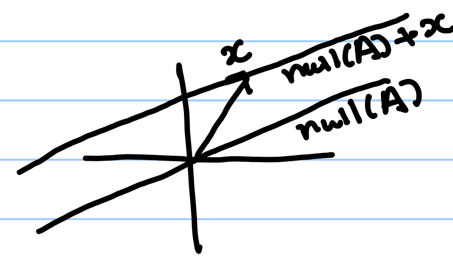
Explain: $\boxed{\begin{aligned} dN_p(X_u) &= N_u \\ dN_p(X_v) &= N_v \end{aligned}}$ [Note: notations abused]

Recall

If $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map,

$n > m$

$\text{rank}(A) = m$



then $A^{-1}\{0\} = \text{null}(A)$

$A^{-1}\{y\} = \text{null}(A) + \begin{pmatrix} \text{any point} \\ \text{in } A^{-1}\{y\} \end{pmatrix}$

are $(n-m)$-dimensional planes in $\mathbb{R}^n$

When $m=1$, these are called <u>hyperplanes</u>.

We now discuss a useful nonlinear generalization of the above.

Regular Level Set Theorem (Euclidean version):

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ a smooth map, $n > m$.

If $y \in F(\mathbb{R}^n)$ and F is a submersion at each $x \in F^{-1}(y)$, i.e.

$$\boxed{dF_x : T_x\mathbb{R}^n \to T_y\mathbb{R}^m \text{ is of rank } m, \ \forall x \in F^{-1}(y)}.$$

then

$F^{-1}(y)$ is a regular surface in $\mathbb{R}^n$ with dimension $n-m$.

- We shall state and prove this result in a more general context, in which

🚩 $\mathbb{R}^n$ is replaced by N – an n-dim. manifold

🚩 $\mathbb{R}^m$ is replaced by M – an m-dim. manifold

🚩 "regular surface in $\mathbb{R}^n$" is replaced by 'submanifold of M'

Now, we apply this result to prove

Proposition:

$$\boxed{O(n) \text{ and } SO(n) \text{ are } \tfrac{n(n-1)}{2}\text{-dimensional regular surfaces in } \mathbb{R}^{n^2}.}$$

Proof:

(I) Note that

$$\det : O(n) \to \mathbb{R}, \quad O(n) \subseteq \mathbb{R}^{n^2}$$

is continuous.

And $SO(n) = O(n) \cap \det^{-1}(\mathbb{R}_+)$ is an <u>open subset</u> of $O(n)$

[In fact, $O(n)$ consists of two connected components, $SO(n)$ is one of them.]

Hence, if we can show that $O(n)$ is a regular surface in $\mathbb{R}^{n^2}$ with a certain intrinsic dimension, then so is $SO(n)$.

(II) Recall $O(n) = \{A \in \mathbb{R}^{n \times n} : A^T A = I\}$

Plausible strategy:

consider $F : \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$

$$A \mapsto A^T A \quad (C^\infty\text{-smooth})$$

then $O(n) = F^{-1}(I)$

and apply the regular level set theorem.

But this is not going to fit into the setting of the theorem, as

dF_A can never be full rank,

since $F(A)$ is always symmetric, meaning that, as a map from $\mathbb{R}^{n^2}$ to $\mathbb{R}^{n^2}$, $n(n-1)/2$ pairs of the component functions are the same, so

$$dF_A = \begin{array}{cc} n^2 & \\ \left[\begin{matrix} \\ \\ \\ \end{matrix}\right] & n^2 \end{array}$$

would also have $n(n-1)/2$ pairs of rows that are identical.

e.g.

$$n=2, \quad F(A) = \begin{bmatrix} F_{11}(A) & F_{12}(A) \\ F_{21}(A) & F_{22}(A) \end{bmatrix}$$

$$dF(A) = \begin{bmatrix} \partial F_{11}/\partial A_{11} & \partial F_{11}/\partial A_{12} & \partial F_{11}/\partial A_{21} & \partial F_{11}/\partial A_{22} \\ \partial F_{12}/\partial A_{11} & \partial F_{12}/\partial A_{12} & \partial F_{12}/\partial A_{21} & \partial F_{12}/\partial A_{22} \\ \partial F_{21}/\partial A_{11} & \partial F_{21}/\partial A_{12} & \partial F_{21}/\partial A_{21} & \partial F_{21}/\partial A_{22} \\ \partial F_{22}/\partial A_{11} & \partial F_{22}/\partial A_{12} & \partial F_{22}/\partial A_{21} & \partial F_{22}/\partial A_{22} \end{bmatrix}$$

has <u>rank at most 3</u>.

So, consider instead

$$F: \mathbb{R}^{n^2} \to \mathbb{R}^{n^2 - \frac{n(n-1)}{2}}$$

$$A \overset{F}{\longmapsto} \left[\cancel{A^T A} \right]$$

remove these repetitive components

e.g. $n=2 \quad \mathbb{R}^4 \to \mathbb{R}^3$

$n=3 \quad \mathbb{R}^9 \to \mathbb{R}^6$

By the regular level set theorem, we are done if we can show that dF_A is full rank for any $A \in O(n)$.

(III) We observe the structure of dF_A in the case of $n=3$, the pattern holds for any n.

$$A_i = i\text{th column of } A = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix}$$

$$A^T A = \left[\langle A_i, A_j \rangle \right]_{i,j=1,\dots,n}$$

e.g. $n=3$

$$(a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33})^T \overset{F}{\longmapsto} \begin{bmatrix} \langle A_1, A_1 \rangle \\ \langle A_1, A_2 \rangle \\ \langle A_1, A_3 \rangle \\ \langle A_2, A_2 \rangle \\ \langle A_2, A_3 \rangle \\ \langle A_3, A_3 \rangle \end{bmatrix}$$

$$
dF_A = \begin{array}{r}
\\
2A_1^T = \\
A_2^T = \\
\\
\\
\\
\\
\end{array}
\begin{array}{c}
\begin{array}{ccccccccc}
 & & & & A_1^T & & & &
\end{array}\\
\left[\begin{array}{ccc:ccc:ccc}
2a_{11} & 2a_{21} & 2a_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\
a_{12} & a_{22} & a_{32} & a_{11} & a_{21} & a_{31} & 0 & 0 & 0 \\
a_{13} & a_{23} & a_{33} & 0 & 0 & 0 & a_{11} & a_{21} & a_{31} \\
0 & 0 & 0 & 2a_{12} & 2a_{22} & 2a_{32} & 0 & 0 & 0 \\
0 & 0 & 0 & a_{13} & a_{23} & a_{33} & a_{12} & a_{22} & a_{32} \\
0 & 0 & 0 & 0 & 0 & 0 & 2a_{13} & 2a_{23} & 2a_{33}
\end{array}\right]
\end{array}
$$

when $A^TA = I$, i.e. $\langle A_i, A_j \rangle = \delta_{ij}$, the rows of dF_A are also orthogonal, therefore

$$\text{rank } dF_A = 6 \quad \text{when } A \in O(3).$$

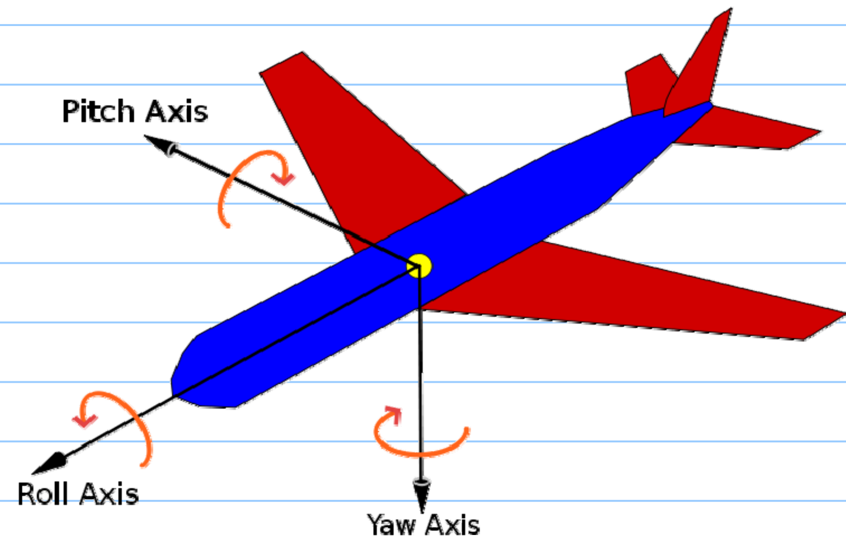
In general, any row of dF_A is of the form:

$$
d\langle A_i, A_j \rangle = \begin{cases} [0 \cdots 0 \ \overset{\text{ith block}}{\overset{\downarrow}{2A_i^T}} \ 0 \cdots 0], & i = j \\[2ex] [0 \cdots \underset{\text{ith block}}{A_j^T} \cdots \underset{\text{jth block}}{A_i^T} \cdots 0], & i \neq j \end{cases}
$$

And the rows are orthogonal (in $\mathbb{R}^{n^2}$), (hence linearly independent) when evaluated at an $A \in O(n)$.

$$\text{so} \quad \text{rank } dF_A = n^2 - \frac{n(n-1)}{2}, \ \forall A \in O(n).$$

∎



$$\dim SO(2) = 1$$
$$\dim SO(3) = 3$$

$SO(3)$ is also a group, generated by

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

Ex: Label the three axes in the figure so that 'pitch', 'yaw', 'row' correspond to the three matrices above.

Now you know that $SO(3)$ is a regular surface in $\mathbb{R}^9$, it would have a well-defined tangent space at each element.

How would $T_A SO(3)$ look like?

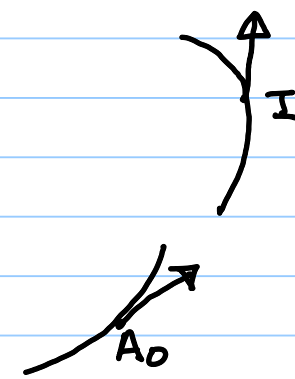
Let $A(t)$ be a curve in $SO(3)$, with $A(0) = A_0$

$$A(t)^T A(t) = I$$

$$\dot{A}(t)^T A(t) + A(t)^T \dot{A}(t) = 0$$

$$\dot{A}(0)^T A(0) + A(0)^T \dot{A}(0) = 0$$

$$A_0^T \dot{A}(0) = -(A_0^T \dot{A}(0))^T$$



i.e. $A_0^T \dot{A}(0)$ is a skew-symmetric matrix.

Recall Frenet-frame from Lecture 1:

$$\Big[\underbrace{t(s), n(s), b(s)}_{\in SO(3)}\Big]'$$ is a tangent vector of $SO(3)$ @ $[t(s), n(s), b(s)]$

so

$$\Big[t(s), n(s), b(s)\Big]^T \Big[t(s), n(s), b(s)\Big]' = \text{a skew-symmetric matrix}$$

or

$$\Big[t(s), n(s), b(s)\Big]' = \Big[t(s), n(s), b(s)\Big]\begin{bmatrix} \text{a skew-} \\ \text{symmetric matrix} \\ \text{depending on } s \end{bmatrix}$$

This is how the Frenet-frame equation looks like ; again see the comments in Lecture 1.

But, a 3x3 skew symmetric matrix has three degrees of freedom

$$\begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{bmatrix}$$

The skew-symmetric matrix that shows up in the Frenet-frame equation however, only has 2 degrees of freedom.

$$\begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$$

Ex: Explain what is going on here.

[more on SO(3) and SO(n) later.]