Week 6

Note Title 4/10/2021

- Go through Ex 4-2-2 in class.

- Reading : Section 4.3.

Duality theory

Theorem 4.4.1 (Weak Duality Theorem). *If x is primal feasible and u is dual feasible, then the dual objective function evaluated at u is less than or equal to the primal objective function evaluated at x, that is,*

$$\left.\begin{array}{ll} Ax \geq b, & x \geq 0 \\ A'u \leq p, & u \geq 0 \end{array}\right\} \Longrightarrow b'u \leq p'x.$$

This is obvious from the "druggist" interpretation of a primal-dual pair.
But let's write out a self-contained proof:

$$p^T x = x^T p \geq x^T (A^T u) \underset{\substack{\uparrow \\ \text{why?}}}{=} u^T (Ax) \geq u^T b = b^T u. \qquad \blacksquare$$

Note: $\boxed{\text{If the primal is unbounded, the dual must be infeasible.}}$

Proof by contradiction: Assume $\bar{u}$ is dual feasible. By the weak duality theorem,

$$p^T x \geq \underbrace{b^T \bar{u}}, \quad \forall \text{ primal feasible } x$$

serves as a (finite) lower bound for the primal objective

This means the primal LP must be bounded (from below). $\Longrightarrow\!\!\!\!\!\times\!\!\Longleftarrow$!

$\boxed{\text{Similarly, if the dual is unbounded, the primal must be infeasible.}}$

Ex 4-4-1

Ex 4-4-2

Theorem 4.4.2 (Strong Duality or Trichotomy Theorem). *Exactly one of the following three alternatives holds:*

(i) Both *primal and dual problems are* feasible *and consequently both have optimal solutions with equal extrema.*

(ii) *Exactly* one *of the problems is* infeasible *and consequently the other problem has an unbounded objective function in the direction of optimization on its feasible region.*

(iii) Both *primal and dual problems are* infeasible.

Note that while this is a result about any primal-dual pair of LP, and the simplex method is a particular algorithm for solving LP. Yet our proof of this theorem uses the simplex method.

Recall: **Theorem 3.5.3.** *If a linear program is* feasible, *then starting at any feasible tableau, and using the smallest-subscript anticycling rule, the simplex method terminates after a finite number of pivots at an* optimal *or* unbounded *tableau.*

Proof of the strong duality theorem:

Logically speaking, there are three possibilities for any primal-dual pair:

(i) Both LPs are feasible.

(ii) Exactly one is feasible and the other is infeasible

(iii) Both are infeasible.

We have seen from examples that (i) and (ii) can actually happen. Here is an easy example for (iii):

$$A = [0], \quad b = [1], \quad p = [-1]$$

$S = \{x : Ax \geq b, x \geq 0\} = \{x \in \mathbb{R} : 0 \cdot x \geq 1, x \geq 0\}$ is clearly empty,

so is $S' = \{u : u^T A \leq p, u \geq 0\} = \{u \in \mathbb{R} : u \cdot 0 \leq -1, u \geq 0\}$.

(i) If both problems are feasible, then both objectives are bounded by weak duality. Hence the simplex method with the smallest-subscript rule applied to

$$
\begin{array}{cc|c|c|}
 & & \begin{matrix} u_{m+1}= & \cdots & u_{m+n}= \\ x_1 & \cdots & x_n \end{matrix} & \begin{matrix} w= \\ 1 \end{matrix} \\ \hline
-u_1 & x_{n+1}= & & \\
\vdots & \vdots & A & -b \\
-u_m & x_{n+m}= & & \\ \hline
1 & z= & p^T & 0 \\ \hline
\end{array}
$$

cannot terminate and must terminate at a primal optimal tableau :

$$
\begin{array}{cc|c|c|}
 & & \begin{matrix} u_{\hat{B}} \\ x_N \end{matrix} & \begin{matrix} w= \\ 1 \end{matrix} \\ \hline
-u_{\hat{N}} & x_B= & H & h \geq 0 \\ \hline
1 & z= & \underset{0}{\underset{\vee|}{c^T}} & \alpha \\ \hline
\end{array}
$$

$x_N = 0$, $x_B = h$ (≥ 0) is an optimal solution for the primal LP, with $z = \alpha$.

By Theorem 4.1.1,

$u_{\hat{N}} = 0$, $u_{\hat{B}} = c$ (≥ 0) is an optimal solution for the dual LP, with $w = \alpha$.

(ii) In the case when exactly one of the problems is infeasible, we want to argue that the other problem must be unbounded.

Assume the contrary. Then the simplex method with the smallest-subscript rule (applying to the feasible and bounded problem) must terminate at a primal-dual optimal tableau (as in (i)). This contradicts the assumed infeasibility.

Conclusion : the feasible problem must be unbounded. ∎

KKT Optimality condition

(P) $\min p^T x$ s.t. $Ax \geq b,\ x \geq 0$ (D) $\max b^T u$ s.t. $A^T u \leq p,\ u \geq 0$

Let $\bar{x}$ be a solution of (P) and $\bar{u}$ a solution of (D).

Notice: $\mathbb{R}^n \ni \bar{x}$ $\bar{u} \in \mathbb{R}^m$

$\mathbb{R}^m \ni A\bar{x} - b$ $-A^T\bar{u} + p \in \mathbb{R}^n$ are all non-negative vectors.

Consider: $0 \leq \underset{\geq 0}{\bar{u}^T}\,(\underset{\geq 0}{A\bar{x}-b}) + \underset{\geq 0}{\bar{x}^T}(\underset{\geq 0}{-A^T\bar{u}+p}) = \cancel{u^T A x} - \bar{u}^T b - \cancel{\bar{x}^T A^T \bar{u}} + \bar{x}^T p$

$= p^T \bar{x} - b^T \bar{u} = 0$ ← from strong duality

This means: $u^T(A\bar{x}-b) = 0$ and $\bar{x}^T(-A^T\bar{u}+p) = 0$,

which also means: $\boxed{u_i(A\bar{x}-b)_i = 0 \quad \forall i, \qquad \bar{x}_j(-A^T\bar{u}+p)_j = 0 \quad \forall j}$ ← KKT conditions

The argument above shows that the KKT conditions are necessary conditions for $\bar{x}$ and $\bar{u}$ being solutions of (P) and (D), respectively.

Turns out the KKT conditions are also sufficient conditions for optimality.

For a detailed proof, see Theorem 4.5.1 in the textbook.

To conclude :

(KKT conditions) aka complimentary slackness conditions

$$\boxed{\begin{array}{l} \bar{x} \text{ solves } (P) \\ \text{and } \bar{u} \text{ solves } (D) \end{array} \iff \begin{array}{ll} u_i (A\bar{x}-b)_i = 0 & \forall i \\ \bar{x}_j (-A^T\bar{u}+p)_j = 0 & \forall j \end{array}}$$

$$\boxed{\begin{bmatrix} 0 \\ * \\ * \\ 0 \end{bmatrix} \quad \begin{bmatrix} * \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} * \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ * \end{bmatrix}}$$

at least one of 2 corresponding entries has to be 0

also expressed as

$$\boxed{\begin{array}{l} 0 \leq A\bar{x}-b \perp \bar{u} \geq 0 \\ 0 \leq p-A^T\bar{u} \perp \bar{x} \geq 0 \end{array}}$$

Example 4-5-1, See the script I prepare for you.

$$(P)\ \min p^T x \quad \text{st } Ax \geq b,\ x \geq 0 \qquad (D)\ \max b^T u \quad \text{st. } A^T u \leq p,\ u \geq 0 \ \rightarrow\ \min (-b)^T u \quad \text{st. } -A^T u \geq -p,\ u \geq 0$$

Dual Simplex Method

As explained earlier, the dual Simplex method is really just the simplex method applied to the dual LP.

Recall: If $b \leq 0$, simplex method applied to (P) does not require Phase I.

If $p \geq 0$, simplex method applied to (D) does not require Phase I.

To apply the simplex method to (D), you may set up a tableau for the "standardization" of (D)

$$T = \text{totlb}(-A^T, -p, -b);$$

followed by the usual simplex steps.

Alternatively, we may work with the original tableau for the primal-dual pair:

$$T = \text{totlb}(A, b, p);$$
$$T = \text{dualbl}(T);$$

and perform Jordan exchanges based on the following pivot selection rules:

1. (Pivot Row Selection): The pivot row is any row r with $h_r < 0$. If none exist, the current tableau is dual optimal.

2. (Pivot Column Selection): The pivot column is any column s such that

$$c_s / H_{rs} = \min_j \left\{ c_j / H_{rj} \mid H_{rj} > 0 \right\}.$$

If $H_{rj} \leq 0$ for all j, the dual objective is unbounded above.

I let you see for yourself why this is indeed "the simplex method applied to the dual".

Example 4-6-1

Recall our "L^1- regression" problem from Week 1, the LP looks like

$$\min \quad y_1 + \cdots + y_m$$

$$\text{s.t.} \quad Ax \geq b \leftarrow \text{contains both positive and negative entries}$$
$$x \geq 0$$

Using the dual simplex method would avoid Phase I.