Week 2

Note Title 3/30/2015

The law of one price :

If two portfolios are guaranteed to have the same value at a future time $\tau > t$ regardless of the state of the market at time τ, then they must have the same value at time t.

Connection to no-arbitrage :

P_i = price of Portfolio i , $i = 1, 2$

If $P_1 > P_2$ at time t
then

@ time t : Short sell portfolio 1 at \$ P_1
Buy portfolio 2 at \$ P_2

@ time τ : sell portfolio 2 at whatever its price is
buy back portfolio 1 at that same price.

No-arbitrage return : \$ $P_1 - P_2 > 0$

If $P_1 < P_2$, do the opposite to pocket

\$ $P_2 - P_1 > 0$ risk-free.

Conclusion: no-arbitrage $\Longleftrightarrow$ $P_1 = P_2$

Note: need the short-selling mechanism in the financial system in order for the argument to work.

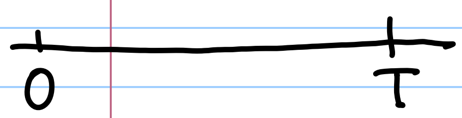
- see class demo.

Theorem Suppose an asset can be stored at zero cost and also sold short.

Suppose : current spot price ($t=0$) is S

Then : the no-arbitrage forward price (for delivery at $t=T$)

is $$F = S e^{rT} \quad \text{———} \quad \bigstar$$



e^{rT}: growth factor if money is deposited in an ideal bank offering continuous time compound interest with annualized rate r for a period of time T (years)

Proof:

Assume $F < S e^{rT}$:

At time 0 : ① Short 1 unit of the asset
② invest the cash (\$$S$) into the bank
③ long a forward contract

At time T : ②' collect \$$Se^{rT}$ from bank

③' Exercise the contract to buy 1 unit of the asset at \$$F$

①' return the asset

NET-RETURN : $Se^{rT} - F$, risk free.

Assume $F > Se^{rT}$:

At time 0: ① long 1 unit of the asset
② borrow $S from the bank
③ short a forward contract

At time T: ②' pay $ Se^{rT} to bank

③' Exercise the contract to sell 1 unit
+ of the asset at $F

①'

NET-RETURN : $Se^{rT} - F$, risk free.

Conclusion : no arbitrage $\Longleftrightarrow$ $F = Se^{rT}$.

Alternative proof (no fundamental difference):

Portfolio 1: · Long 1 forward contract
· short 1 unit of underlying asset

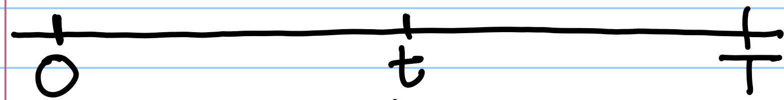
value at $(t=0) = 0 + S(0)$

value at $(t=T) = -F$

Portfolio 2: Borrow Fe^{-rT} from bank

value at $(t=0) = Fe^{-rT}$
value at $(t=T) = -F$

By the law of one price: $S(0) = Fe^{-rT}$
or $F = S(0)e^{rT}$



At time t, how much should the forward contract worth? (assume the delivery price at time T is \$K)

Portfolio 1:
- Long 1 forward contract
- short 1 unit of underlying asset

value at t $= -F(t) + S(t)$ ($F(t)$ not 0 anymore) $t>0$

value at T $= -K$

Portfolio 2: Borrow $Ke^{-r(T-}$ from bank

value at t $= Ke^{-r(T-t)}$

value at $(t=T)$ $= -K$

By the law of one price:

$$-F(t) + S(t) = Ke^{-r(T-t)}$$

$$\boxed{F(t) = S(t) - Ke^{-r(T-t)}}$$

Put - Call Parity

Theorem : Let $C(t)$ and $P(t)$ be the values at time t of a European call and put option, respectively, with maturity T and strike K, on the same <u>non-dividend</u> paying asset with spot price $S(t)$. Then

$$\boxed{P(t) + S(t) - C(t) = Ke^{-r(T-t)}}$$

or

else arbitrage opportunity exists.

<u>Proof</u> : consider the portfolio :

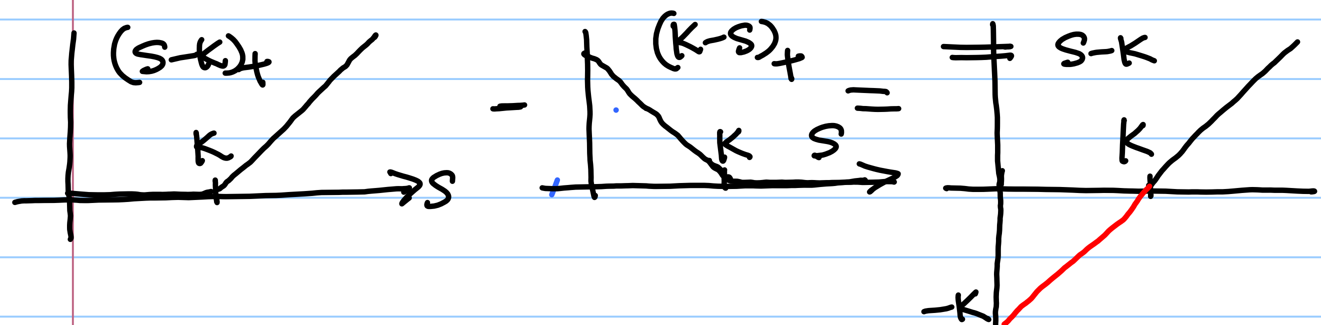
1. short a call
2. long a put

$\Big\}$ both expired at time T strike \$K

3. long 1 unit of asset

value at time t = $C(t) - P(t) - S(t)$

value at time $T = -(S(T)-K)_+ + (K-S(T))_+ + S(T) = K$



Consider another portfolio :

- invest $Ke^{-r(T-t)}$ in the bank

value at time $t = -Ke^{-r(T-t)}$

value at time $T = +K$

So by the law of one price,

$$P(t) + S(t) - C(t) = Ke^{-r(T-t)}$$

Can also derive this based on

long a call + short a put } expiry: T, strike: K $\equiv$ long a forward contract with maturity T and delivery price K

What if the stock pays dividends continuously at the rate q?

consider the portfolio:

1. short a call } both expired at time T
2. long a put } strike \$$K$
3. long ~~1 unit of asset~~ $S(t)e^{-q(T-t)}$ worth of asset

$$\text{value at time } t = C(t) - P(t) - S(t)e^{-q(T-t)}$$

$$\text{value at time } T = -\left[(S(T)-K)_+ + (K-S(T))_+ + S(T)\right] = -K$$

Then the same argument gives

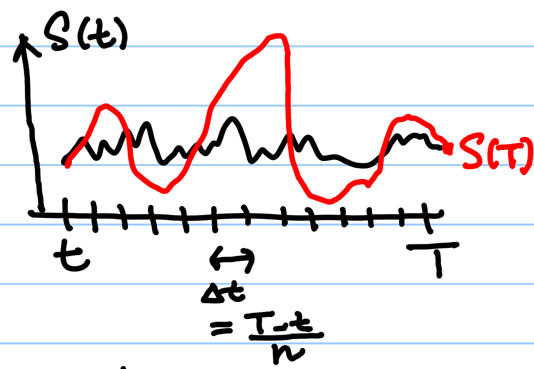
$$\boxed{P(t) + S(t)e^{-q(T-t)} - C(t) = Ke^{-r(T-t)}}$$

Dividend-paying assets

In above, I used the following assumption without any justification:

> If you have 1 unit of an asset at time t, and this asset pays dividend continuously at the (annualized) rate q, then at time T, you have
> $$\$ S(T) e^{q(T-t)}$$
> worth of the asset.

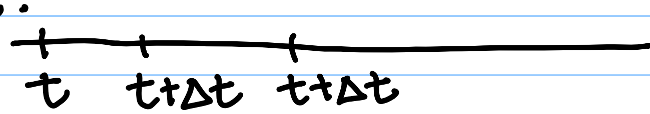
($S(t)$ = price of 1 unit of the asset @ time t)



Here is the reasoning:

If we reinvest all the dividends earned into the same asset, assuming that any proportion of unit of the asset can be purchased at any time

then:



Assume dividend is paid at the end of each of these (tiny) time intervals,

assume the dividend is calculated based on the asset price at end of the same time interval, so $\$S(t)$ (1 unit) @ time t

becomes $\underbrace{\$ S(t+\Delta t)(1+q\Delta t)}$ @ time $t+\Delta t$

$\downarrow$ ← invest dividend to asset

$$\frac{\$ S(t+\Delta t)(1+q\Delta t)}{\$ S(t+\Delta t)/\text{unit}} = (1+q\Delta t) \text{ units}$$

at subsequent time intervals :

time	# of units owned	price per unit
t	1	$S(t)$
$t+\Delta t$	$1+q\Delta t$	$S(t+\Delta t)$
$t+2\Delta t$	$(1+q\Delta t)^2$	$S(t+2\Delta t)$
$\vdots$	$\vdots$	$\vdots$
$T=t+n\cdot\Delta t$	$(1+q\Delta t)^n$	$S(T)$

$$(1+q\Delta t)^n = \left(1+q\frac{T-t}{n}\right)^n \to e^{q(T-t)}, \; n\to\infty.$$

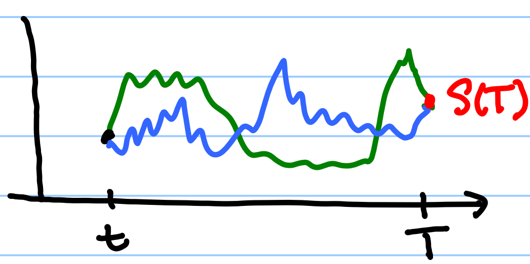
Therefore, the "continuous dividend re-investing portfolio" has

$e^{q(T-t)}$ units of the asset,

which worths a total of

$\$ S(T)\, e^{q(T-t)}$.

Note : the path of price fluctuation is irrelevant !



Note: the put-call parity only tells you $P(t)$ if $C(t)$ is known.

(It reduces two difficult problems into one.)

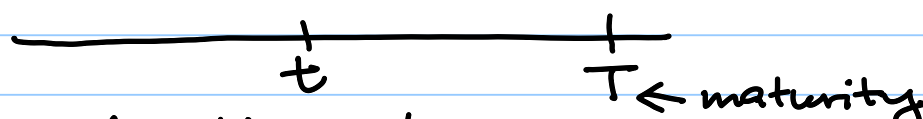
Early exercise

Recall: an american option offers the possibility of early exercise

- see class demo
- note: selling a call/put option $\neq$ exercising the option

Proposition: For call options on a stock that pays no dividends prior to expiration, early exercise is never optimal.

Proof:



Assume at time t, a call option is in the money, ie $S(t) > K$.

You are tempted to exercise the call, especially if you need cash rightaway...

Caveat: the concern that you may need the cash rightaway cannot be an excuse for early exercise, it is because we assume we can always borrow money from a bank, we can also short-sell some asset for cash.

If you exercise and sell the asset immediately
- you collect $\$ S(t)-K$, and the cash worths
$$\$ (S(t)-K)e^{r(T-t)} \text{ at time } T.$$

Instead, use a "static hedging strategy":
- keep the American call
- short 1 unit of the underlying
- put cash $\$ S(t)$ in bank

At time T, how much does this portfolio worth?

[ITM]
If $S(T)>K$:
$$S(t)e^{r(T-t)}-K \quad \text{(why?)}$$
$$> S(t)e^{r(T-t)}-Ke^{r(T-t)} \quad \text{if } r>0$$

[OTM]
If $S(T)\leq K$:
$$S(t)e^{r(T-t)}-S(T) \quad \text{(why?)}$$
$$\geq S(t)e^{r(T-t)}-K$$
$$> S(t)e^{r(T-t)}-Ke^{r(T-t)}$$

Either case, the static hedging strategy gives a better return.

Consequence : having the flexibility to exercise early does not add any value to an american call option over an European one.

$$\therefore \quad C_{American}(t) = C_{European}(t), \ \forall t \leq T$$

if q = dividend rate of the underlying $= 0$

and K, T, r are the same.

- what if $q > 0$?

The static hedging portfolio may not be better than early exercise if
$S(T) > K$ (ITM),
because when we short a dividend paying stock we are responsible for paying the dividend to the lender, so the return at maturity is not
$S(t)e^{r(T-t)} - K$ anymore, but smaller.

- what about put options?

assume a put option is in the money at time t, ie
$S(t) < K$

Early exercise : \$ $(K - S(t))$ @ time t

\$ $(K - S(t))e^{r(T-t)}$ @ time T

Static hedging :
- keep the put
- long 1 unit of the underlying
- borrow \$ $S(t)$ from bank

At time T :

ITM if $S(T) < K$, return is $\underbrace{K - S(t)e^{r(T-t)}}$

not better than early exercise

OTM if $S(T) \geq K$, $S(T) - S(t)e^{r(T-t)}$

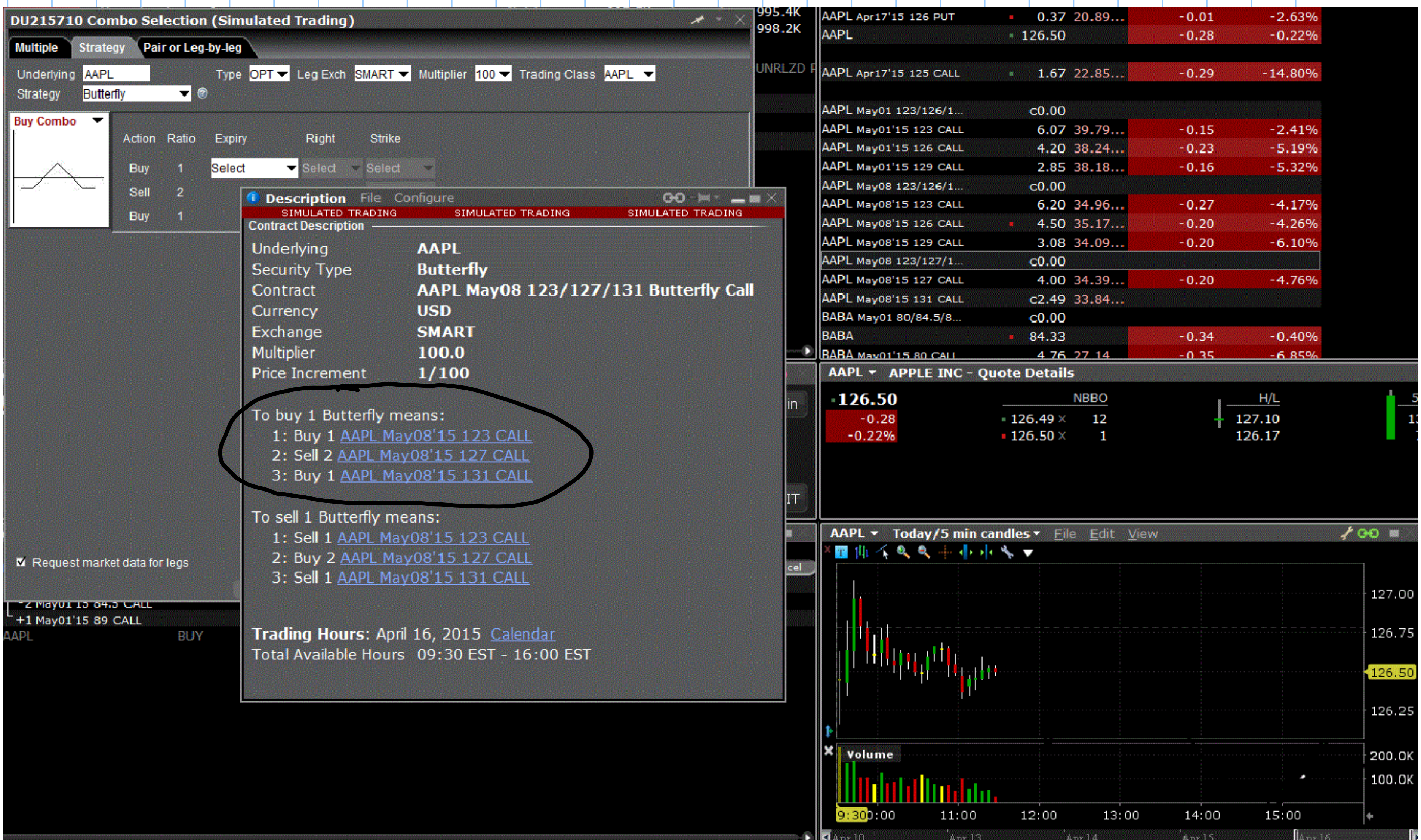
better than early exercise if $S(T)$ is big enough

The argument just does not work anymore.

Some standard options combos :

- covered call - covered put
- bull call spread
- bull put spread
- bear call spread
- bear put spread
- straddle - strangle
- butterfly ← HW 2
- calendar spread
- iron condor

(see Matlab Demos and HW#2)


DU215710 Combo Selection (Simulated Trading)
Multiple
Strategy
Pair or Leg-by-leg
Underlying AAPL
Type OPT
Leg Exch SMART
Multiplier 100
Trading Class AAPL
Strategy Butterfly
Buy Combo
Action Ratio Expiry Right Strike
Buy 1
Sell 2
Buy 1
Request market data for legs
Description File Configure
SIMULATED TRADING
Contract Description
Underlying AAPL
Security Type Butterfly
Contract AAPL May08 123/127/131 Butterfly Call
Currency USD
Exchange SMART
Multiplier 100.0
Price Increment 1/100
To buy 1 Butterfly means:
1: Buy 1 AAPL May08'15 123 CALL
2: Sell 2 AAPL May08'15 127 CALL
3: Buy 1 AAPL May08'15 131 CALL
To sell 1 Butterfly means:
1: Sell 1 AAPL May08'15 123 CALL
2: Buy 2 AAPL May08'15 127 CALL
3: Sell 1 AAPL May08'15 131 CALL
Trading Hours: April 16, 2015 Calendar
Total Available Hours 09:30 EST - 16:00 EST
AAPL Apr17'15 126 PUT 0.37 20.89... -0.01 -2.63%
AAPL 126.50 -0.28 -0.22%
AAPL Apr17'15 125 CALL 1.67 22.85... -0.29 -14.80%
AAPL May01 123/126/1... c0.00
AAPL May01'15 123 CALL 6.07 39.79... -0.15 -2.41%
AAPL May01'15 126 CALL 4.20 38.24... -0.23 -5.19%
AAPL May01'15 129 CALL 2.85 38.18... -0.16 -5.32%
AAPL May08 123/126/1... c0.00
AAPL May08'15 123 CALL 6.20 34.96... -0.27 -4.17%
AAPL May08'15 126 CALL 4.50 35.17... -0.20 -4.26%
AAPL May08'15 129 CALL 3.08 34.09... -0.20 -6.10%
AAPL May08 123/127/1... c0.00
AAPL May08'15 127 CALL 4.00 34.39... -0.20 -4.76%
AAPL May08'15 131 CALL c2.49 33.84...
BABA May01 80/84.5/8... c0.00
BABA 84.33 -0.34 -0.40%
AAPL APPLE INC - Quote Details
126.50
-0.28
-0.22%
NBBO
126.49 x 12
126.50 x 1
H/L
127.10
126.17
AAPL Today/5 min candles File Edit View
Volume
127.00
126.75
126.50
126.25
200.0K
100.0K
11:00
12:00
13:00
14:00
15:00
+1 May01'15 89 CALL
AAPL
BUY

Pricing a single-period binomial option

Assume the initial price of a stock is \$$S$ per unit.

At the end of one-period of time:

$$\$1 \begin{cases} \nearrow \$u & \text{(with probability } p) \\ \searrow \$d & \text{(with probability } 1-p) \end{cases}$$

$\boxed{0<d<u}$ if we invest \$1 in the stock.

Assume the one-period interest rate is r, i.e.

$$\$1 \longrightarrow \$\underset{1+r}{\underset{\parallel}{R}} \quad \text{with probability } 1$$

if we invest \$1 in the bond.

Proposition: $d < R < u$

$\iff$ no arbitrage opportunity exists in this (over-simplified) market.

Proof: exercise.

(Assume $R \geq u$, construct a portfolio to make you some money risk-free.
Similar for $R \leq d$.)

(By now we have seen this sort of no-arbitrage argument numerous times already.)

Now, assume this market wants to issue a call option with strike price K and expiration at the end of the period.

Q1 : How much should this option worth?

Q2 : How can such an over-simplified setting be useful?

- assume binomial !
- assume one-period !!
- assume u, d, p are known !!!

A2 : You will be surprised ....

A1 : To price the call option, we once again use a no-arbitrage argument (or the law of one price) :

return of a call option on 1 unit of the stock

$$\$C \begin{cases} \nearrow \underbrace{\max(uS-K, 0)}_{=C_u} & \text{if the stock goes up} \\ \searrow \underbrace{\max(dS-K, 0)}_{=C_d} & \text{if the stock goes down} \end{cases}$$

Key observation: Assuming any fractional unit of stocks can be traded, we can create a portfolio consisting only of stock and bond to replicate the return of the option.

Then, by the law of one price,

(★) —— C = value of such a portfolio

What is this replicating portfolio then?

$\$x$ worth of stock $\quad x = ?$
$\$y$ worth of bond $\quad y = ?$

s.t.

$$\begin{cases} ux + Ry = C_u \\ dx + Ry = C_d \end{cases}$$

$$\begin{bmatrix} u & R \\ d & R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & R \\ d & R \end{bmatrix}^{-1} \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{u-d} & -\frac{1}{u-d} \\ -\frac{d}{R(u-d)} & \frac{u}{R(u-d)} \end{bmatrix} \begin{bmatrix} C_u \\ C_d \end{bmatrix}$$

By (★)

$$C = x + y = \frac{R-d}{R(u-d)} C_u + \frac{u-R}{R(u-d)} C_d$$

$$\boxed{C = \frac{1}{R}\left[\frac{R-d}{u-d} C_u + \frac{u-R}{u-d} C_d \right]}$$

This result is full of meanings, as we will see....