Week 6

Note Title 4/19/2015

Black-Scholes formula:

$$C = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \quad -①$$
$$P = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) \quad -②$$

where

$$d_1 = \left[\ln(S/K) + (r-q+\sigma^2/2)(T-t)\right] / \sigma\sqrt{T-t}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$
$$N(z) = \text{cdf of } N(0,1) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-t^2/2}dt$$

Based on the geometric brownian motion model on the underlying asset price:

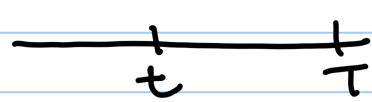
$$\ln\left(\frac{S_{t_2}}{S_{t_1}}\right) = \mu(t_2-t_1) + \sigma\sqrt{t_2-t_1}\,\underset{\uparrow}{Z}$$

Standard normal

C and P vary with 6 parameters:

underlying
- S_t, or simply S – the spot price of the underlying at time t
- q – annualized (continuous time) dividend rate
- σ – volitility of the underlying

option
- $T-t$ – time to maturity



$T \leftarrow$ maturity date

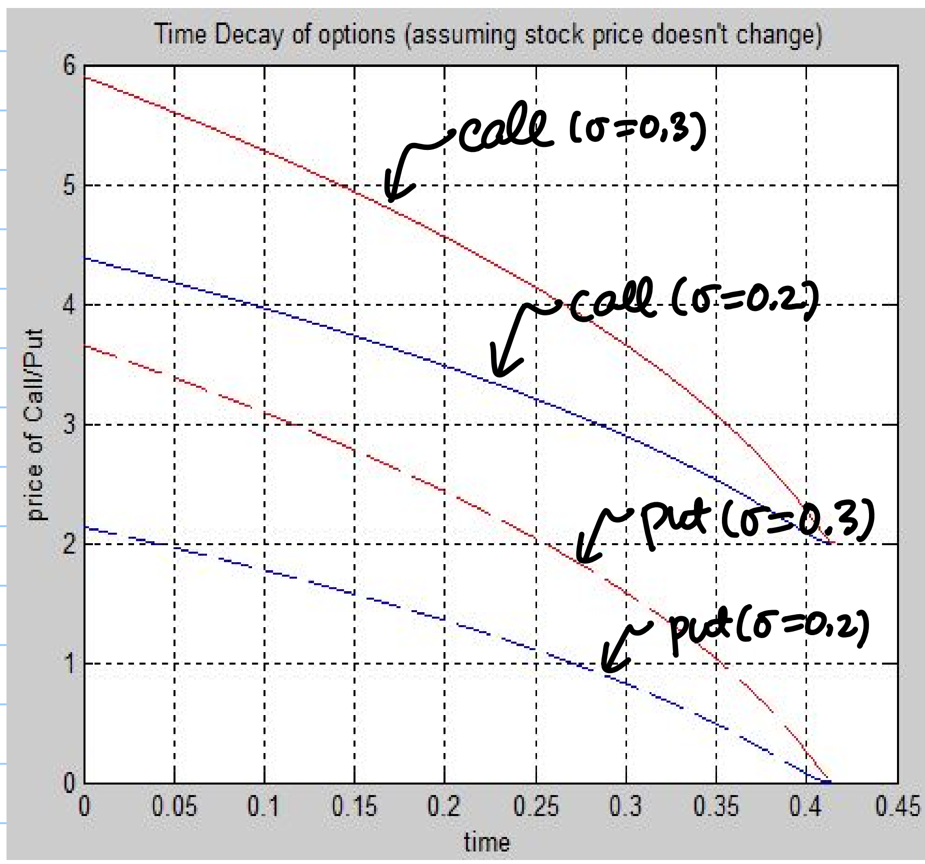
- K – strike price of the option

bond
- r – annualized (continuous time) interest rate

Note : ② follows from ① and the put-call parity :

$$P = Ke^{-r(T-t)} - Se^{-q(T-t)} + \underbrace{C}_{\substack{\parallel \text{ by } ① \\ Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)}}$$

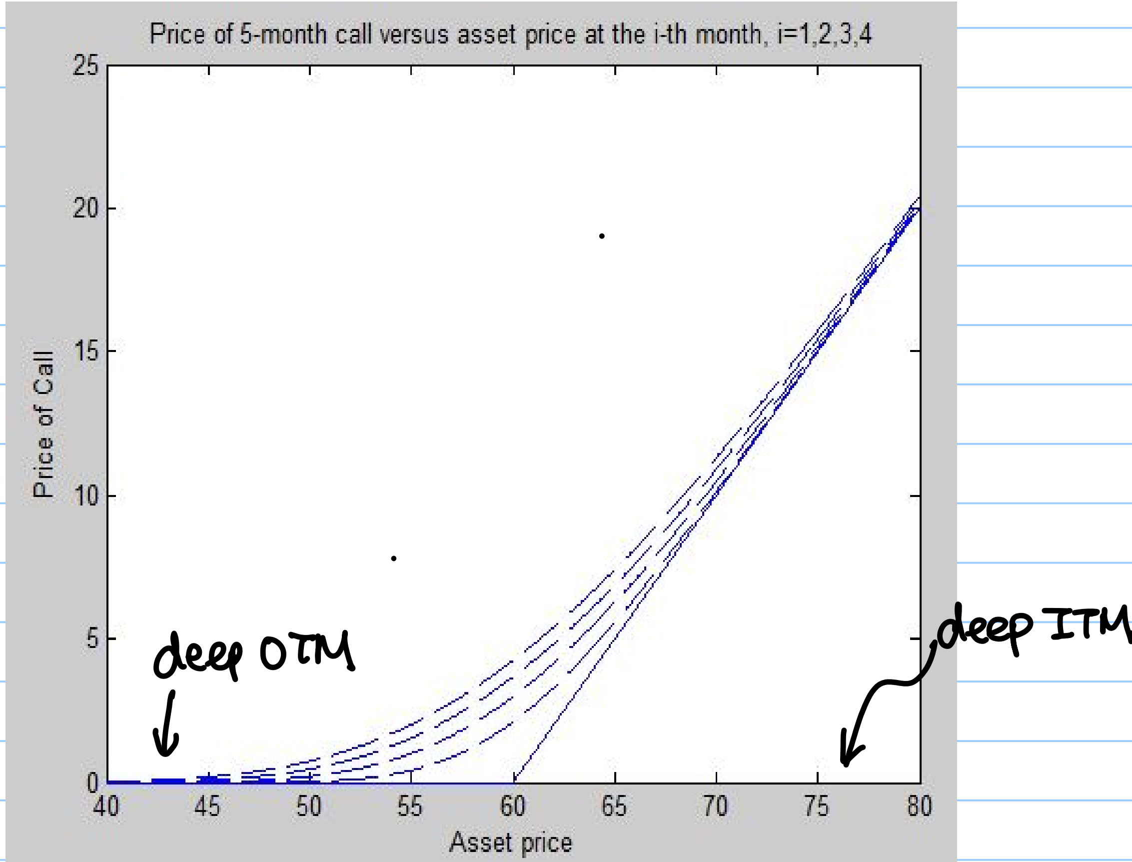
$$= Ke^{-r(T-t)} \underbrace{[1-N(d_2)]}_{=N(-d_2)} - Se^{-q(T-t)} \underbrace{[1-N(d_1)]}_{=N(-d_1)}$$

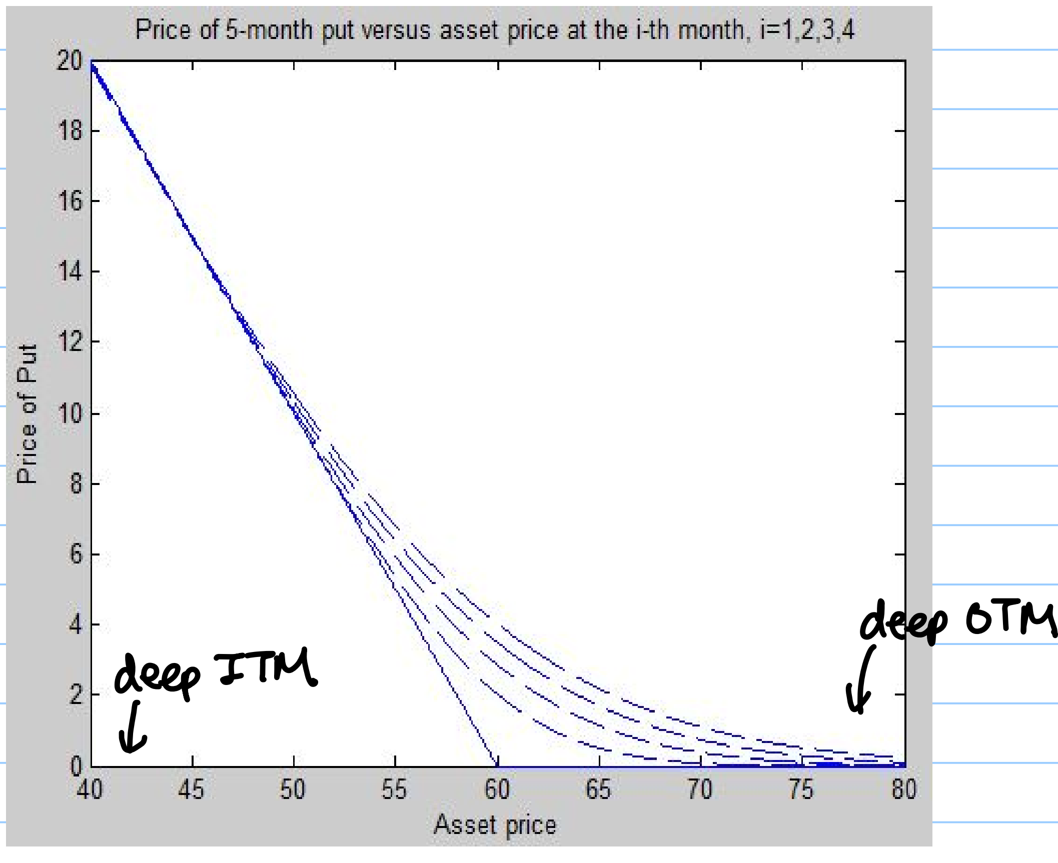


Assume $S \equiv 62$ at all time

$K = 60$

Note: For a fixed strike price K and a fixed time to expiration (T-t) (even if large), when S<<K (deep out of the money), C->0 (delta~=0 here)
when S>>K (deep in the money), C~=S (little advantage in owning the option over owning the stock itself, delta~=1 here)





Option Greeks

Δ Γ Θ Vega implied volatility

	LAST	DELTA	GAMMA	THETA	VEGA	IMPLD VL. %	CHANGE %	
AAPL May22'15 124 CALL	2.75	0.4856	0.0542	-0.0931	0.1031	28.804%	-0.70	-20.29%
AAPL May22'15 124 PUT	2.90	-0.5146	0.0542	-0.0922	0.1031	27.396%	+0.83	40.10%
AAPL	124.01						-1.79	-1.42%

V = value of a portfolio of derivative on one underlying
(e.g. a naked call / put
a butterfly, or any
option combo.)

$$\Delta(V) = \frac{\partial V}{\partial S}, \quad \Theta(V) = \frac{\partial V}{\partial t}$$

$$\Gamma(V) = \frac{\partial^2 V}{\partial S^2}, \quad \text{vega}(V) = \frac{\partial V}{\partial \sigma}, \quad \rho(V) = \frac{\partial V}{\partial r}$$

GBM model assumes σ does not change, we shall change our viewpoint later

$$C = C(S, t; \sigma, \underbrace{r, q}_{\text{assumed fixed for now}}, K)$$

K: fixed

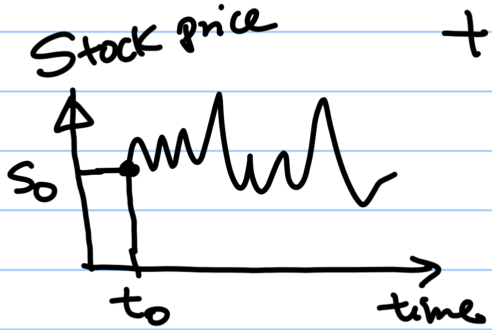
The greeks tell you how an option price changes when the underlying parameters change:

Taylor's theorem:

$$C(S_0 + \Delta S,\ t_0 + \Delta t\ ;\ \underbrace{\sigma, r, q, K}_{\text{assumed fixed for now}})$$

$$\approx C(S_0, t_0) + \underbrace{\left.\frac{\partial C}{\partial S}\right|_{S=S_0}}_{\text{Delta } (>0)} \Delta S + \underbrace{\left.\frac{\partial C}{\partial t}\right|_{t=t_0}}_{\text{Theta } (<0)} \Delta t \qquad \Delta S, \Delta t \text{ small}$$

$$+ \frac{1}{2} \underbrace{\left.\frac{\partial^2 C}{\partial S^2}\right|_{S=S_0}}_{\text{Gamma } (>0)} (\Delta S)^2$$



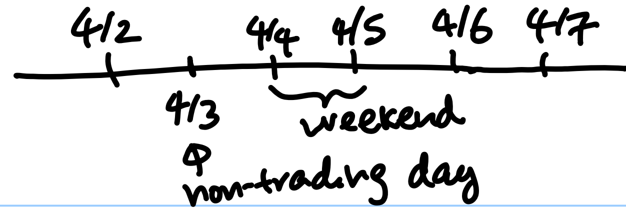
[why worry about this 2nd order term but not others (such as $\Delta S\,\Delta t$, or Δt^2 terms)? will address this later....]

Similarly,

$$P(S_0 + \Delta S,\ t_0 + \Delta t\ ;\ \sigma, r, q, K)$$

$$= P(S_0, t_0) + \underbrace{\left.\frac{\partial P}{\partial S}\right|_{S=S_0}}_{\text{Delta } (<0)} \Delta S + \underbrace{\left.\frac{\partial P}{\partial t}\right|_{t=t_0}}_{\text{Theta } (<0)} \Delta t$$

$$+ \frac{1}{2} \underbrace{\left.\frac{\partial^2 P}{\partial S^2}\right|_{S=S_0}}_{\text{Gamma } (>0)} (\Delta S)^2 + \cdots$$



Recall our near-the-money IBM call

$t_0 =$ April 2, 2015, $T =$ April 17, 2015

$S_{4/2/2015} = \$160$, $S_{4/6/2015} = \$162.5$, $\Delta S = \$2.5$

$C_{4/2/2015} = \$2.36$ $\quad C_{4/6/2015} = \$3.70$ $\quad \Delta t = 1$ trading day

- near-the-money delta $\gtrsim 0.5$
- theta decay negligible in this case

$$\Delta C \gtrsim 0.5 \times \$2.5 = \$1.25$$

This is about the same magnitude we saw in the option market

$$(\Delta C)_{real} = \$1.34$$

Note: The option value goes up by more than 50% while the underlying only goes up by $\frac{2.5}{160} = 1.56\%$.

Formulas for the Greeks

$$C = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$
$$P = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1)$$

where

$$d_1 = \left[\ln(S/K) + (r-q+\sigma^2/2)(T-t)\right] / \sigma\sqrt{T-t}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- $\Delta(C) = \frac{\partial C}{\partial S} = e^{-q(T-t)} N(d_1) \in (0,1)$

  $\Delta(P) = \frac{\partial P}{\partial S} = -e^{-q(T-t)} N(-d_1) \in (-1,0)$

If $r=q=0$, $K=S$ (ATM)

$$d_1 = \frac{\sigma}{2}\sqrt{T-t} \quad \begin{matrix} > 0 \\ \approx 0 \end{matrix}$$

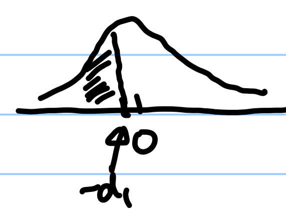
Typically this number is not significantly bigger than 0 (e.g. $\sigma = 0.2$,
$T-t = 2$ weeks
$= \frac{14}{252}$ years)

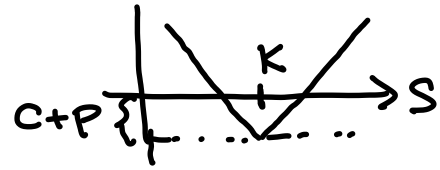
So

$$\Delta(C) = N(d_1) \gtrsim 0.5.$$

Similarly,

$$\Delta(P) = -N(-d_1) \gtrsim -0.5$$





Straddle : long one ATM call + long one ATM put } same underlying, same expiry

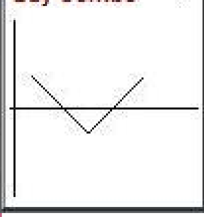
CONTRACT	LAST	DELTA	GAMMA	THETA	VEGA	IMPLD VL. %
IBM	170.16					
IBM May 170 Straddle		0.0385	0.1573	-0.2389	0.2079	
IBM May15'15 170 CALL	2.03	0.5196	0.0814	-0.1159	0.1040	18.899%
IBM May15'15 170 PUT	2.16	-0.4811	0.0760	-0.1230	0.1040	20.257%

DU215710 Combo Selection (Simulated Trading)

Multiple | Strategy | Pair or Leg-by-leg

Underlying: IBM Type: OPT Leg Exch: SMART Multiplier: 100
Strategy: Straddle

Buy Combo



Action	Ratio	Expiry	Strike	Right
Buy	1	MAY 15 '15	170.0	Call
Buy	1	MAY 15 '15	170.0	Put

May 7, 2015

If $T-t$ small (ie near-term options),
$\Delta(\text{straddle}) \simeq 0$ (approximately "delta-neutral")

i.e. portfolio is not sensitive to small change in the underlying.

Recall: If $f'(x_0) = 0$, then the 2nd degree term becomes important:

$$f(x) = f(x_0) + \cancel{f'(x_0)(x-x_0)}^{=0} + \frac{1}{2} f''(x_0)(x-x_0)^2 + \cdots$$

So the Gamma is important for delta-neutral portfolio.

Strangle (similar to straddle):

- long one (slightly) out-of-money call
  +
  long one (slightly) out-of-money put

	LAST	DELTA	GAMMA	THETA	VEGA	IMPLD VL. %
IBM	170.16					
IBM May 167.5/172.5 Strangle		0.0159	0.1385	-0.2137	0.1917	
IBM May15'15 167.5 PUT	1.20	-0.3062	0.0643	-0.1118	0.0951	20.981%
IBM May15'15 172.5 CALL	0.97	0.3220	0.0742	-0.1018	0.0967	18.557%

May 7, 2015

- Approximately delta-neutral

- Gamma (Γ)

$$\Gamma(C) = \frac{e^{-q(T-t)}}{S\sigma\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \quad (>0)$$

$$\Gamma(P) = \Gamma(C) \qquad (>0)$$

What does it mean to a delta-neutral Straddle or Strangle?

"Gamma gain"

$$V(S_0+\Delta S,\ t_0+\Delta t)$$

$$= V(S_0,t_0) + \cancel{\underbrace{\frac{\partial V}{\partial S}\Big|_{S_0} \Delta S}_{=0}} + \overbrace{\underbrace{\frac{\partial^2 V}{\partial S^2}}_{\substack{= \Gamma(C;K_1)>0 \\ +\Gamma(P;K_2)>0}} (\Delta S)^2}^{>0}$$

"Theta decay" $\rightarrow$ $$+ \underbrace{\frac{\partial V}{\partial t}\Big|_{t_0}}_{<0} \Delta t$$

fight!

When we long options (as in straddle or strangle), gamma is our friend:

regardless of the movement of the underlying (i.e. ΔS positive or negative) "Gamma gain" is positive.

However, the portfolio only gains value if the "Gamma gain" beats the "Theta decay".

- Theta (Θ)

$$\Theta(C) = -\frac{S\sigma e^{-q(T-t)}}{2\sqrt{2\pi(T-t)}} e^{-d_1^2/2} + qSe^{-q(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2)$$

$$\Theta(P) = -\frac{S\sigma e^{-q(T-t)}}{2\sqrt{2\pi(T-t)}} e^{-d_1^2/2} - qSe^{-q(T-t)}N(-d_1) + rKe^{-r(T-t)}N(-d_2)$$

more difficult to analyze, but most of the time

$$\Theta(C) < 0, \quad \Theta(P) < 0$$

For instance, if $q=r=0$, then it's clear that $\Theta(C) < 0$, $\Theta(P) < 0$.

[Note: if $S=K$

$$\lim_{t \to T} \Theta(C) = -\infty$$

$$\lim_{t \to T} \Theta(P) = -\infty$$
]

When you are selling options, Theta is your friend (most of the time.)

- $\text{vega}(C) = \frac{\partial C}{\partial \sigma} = S e^{-q(T-t)} \sqrt{T-t} \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2}$

  $\parallel$

  $\text{vega}(P)$

But what does it mean to think of σ varying?

will discuss:
"Implied Volatility"