CH 10

Note Title 5/21/2017

Integration on manifolds

To put things in perspective, consider a 0-form on M, ie.

$$f : M \to \mathbb{R} \quad C^\infty$$

The "integral"

$$\int_M f$$

would not make sense if M is just an abstract manifold. If M has some extra structure, e.g. a (Riemannian) volume form, or a measure, then an integral can be defined that has a similar meaning of an "area integral" in Calculus when $\dim M = 2$ and "volume integral" when $\dim M = 3$.

What we do now is a little more general and abstract.

We show that <u>n-forms</u> on an abstract manifold M^n have just the correct <u>"transformation properties"</u> so that we can define its integral on M invariantly.

We do not need a Riemannian metric or measure, but we do need M to be oriented.

Let M^n be an oriented manifold.

$$\Omega^n_c(M^n) = \{\omega \in \Omega^n(M^n) : \text{supp}\,\omega \text{ is compact}\}$$

$$\text{supp}\,\omega = \overline{\{p \in M : \omega_p \neq 0\}}$$

We define $\int_M : \Omega^n_c(M^n) \longrightarrow \mathbb{R}$

a vector subspace of $\Omega(M^n)$

First we define it in the special case $M^n = \mathbb{R}^n$ (with the standard orientation).

$\omega \in \Omega_c^n(\mathbb{R}^n)$ can be uniquely espressed in the form

$$\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$$

$f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ has compact support. Define

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f(x)\, dx_1 \wedge \cdots \wedge dx_n := \underbrace{\int_{\mathbb{R}^n} f(x)\, dx_1 \cdots dx_n}_{\text{the usual Riemann integral of } f.}$$

$$\underbrace{\int_{\mathbb{R}^n} f(x)\, d\mu_n}_{\text{the usual Lebesgue integral in } \mathbb{R}^n.} =$$

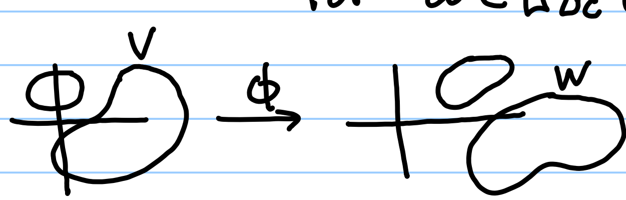
The same definition can be used when $\omega \in \Omega_c^n(V)$ for $V \underset{\text{open}}{\subseteq} \mathbb{R}^n$.

<u>Lemma 10.1</u> Let $\phi : V \to W$ be a diffeomorphism between open sets of $\mathbb{R}^n$, assume that

$$\operatorname{sgn}(\det(D_x\phi)) = \delta \;(\in +1 \text{ or } -1) \quad \forall x \in V.$$

[Note: this condition is automatically satisfied if V and W are connected.]

For $\omega \in \Omega_c^n(W)$, we have



$$\int_V \phi^*(\omega) = \delta \cdot \int_W \omega.$$

<u>Proof</u>: This is where the transformation property of

n-forms comes in :

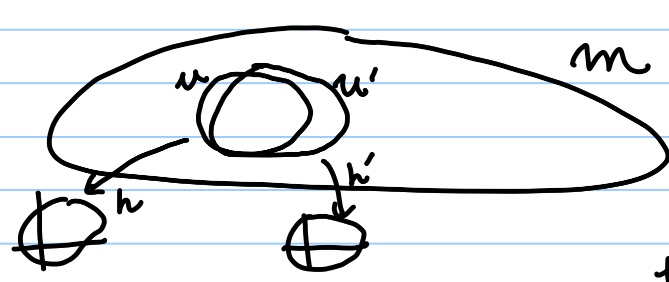
if $\omega = f(x)\, dx_1 \wedge \cdots \wedge dx_n$, $f \in C_c^\infty(W, \mathbb{R})$,
then
$$\phi^*(\omega) = f(\phi(x)) \det(D_x\phi)\, dx_1 \wedge \cdots \wedge dx_n$$

But
$$\begin{array}{ccc} \int_W f(x)\, d\mu_n & = & \int_V f(\phi(x))\, |\det(D_x\phi)|\, d\mu_n \\ \| & & \| \\ \int_W \omega & & \& \int_V \phi^*(\omega) \end{array}$$
$\square$

<u>Proposition</u> For an oriented n-dimensional smooth manifold M^n, $\exists!$ linear map
$$\int_M : \Omega_c^n(M^n) \to \mathbb{R}$$
with the following property :

if $\omega \in \Omega_c^n(M^n)$ has support contained in U, where (U, h) is a positively oriented chart, then
$$\int_M \omega = \int_{h(U)} (h^{-1})^* \omega . \quad \text{—} (*)$$

Key Idea: ① The transformation property of n-forms means $(*)$ is invariant under change of coordinates



$\mathrm{supp}(\omega) \subset U \cap U'$
$\Downarrow$
$$\int_{h(U)} (h^{-1})^* \omega = \int_{h'(U')} (h'^{-1})^* \omega$$

② But what if $\mathrm{supp}(\omega)$ cannot be covered by a single chart ? use a partition of unity $(\rho_\alpha)_{\alpha \in A}$ subordinate to a positive atlas $(\mathcal{U}_\alpha)_{\alpha\in A}$:

$$\omega = \sum_{\alpha\in A} \rho_\alpha \omega \quad , \quad \int_M \omega := \sum_{\alpha \in A} \int_M \rho_\alpha \omega .$$

Lemma (10.3) (i) $\int_M \omega$ changes sign when the orientation of M^n is reversed.

(ii) If $\omega \in \Omega_c^n(M^n)$ has support contained in an open set $W \subseteq M^n$, then

$$\int_M \omega = \int_W \omega,$$

when W is given the orientation induced by M.

(iii) If $\phi : N^n \to M^n$ is an orientation-preserving differeomorphism, then we have that

$$\int_M \omega = \int_N \phi^* \omega \quad , \quad \forall\, \omega \in \Omega_c^n(M^n).$$

Idea: use a partition of unity, restrict to the case where $\mathrm{supp}_M(\omega)$ is contained in a chart.

<u>Remark</u>: Orientation form (and in particular a Riemannian volume form) induces a measure on M in the following way:

If the orientation of M is given by the orientation form $\sigma \in \Omega^n(M)$, then any n-form (smooth or merely continuous) can be written uniquely as

$$f\sigma, \quad f \in C^\infty(M, \mathbb{R}) \ (\text{or } C^0(M,\mathbb{R})).$$

$\mathrm{supp}(f\sigma) = \mathrm{supp}(f)$.

Our definition of $\int_M f \underbrace{\sigma}_{\text{smooth } n\text{-form}}$

extends to

$$\int_M f \underbrace{\sigma}_{\text{continuous } n\text{-form}},$$ so we have a map

$$I_\sigma : \underbrace{C_c^0(M, \mathbb{R})}_{\substack{\text{space of continuous} \\ \text{compactly supported} \\ \text{functions } M \to \mathbb{R}}} \to \mathbb{R}, \quad I_\sigma(f) = \int_M f\sigma.$$

Easy to check: I_σ is linear and positive
(i.e. $I_\sigma(f) \geq 0$ for $f \geq 0$)

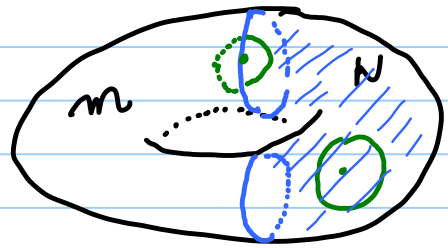
According to Riesz's representation theorem, I_σ determines a positive measure μ_σ on M s.t.

$$\int_M \underbrace{f}_{\text{cont. 0-form}} d\mu_\sigma = \int_M \underbrace{f\sigma}_{\text{cont. } n\text{-form}}, \quad f \in C_c^0(M, \mathbb{R})$$

We only need a tiny bit of measure theory here, but it seems helpful to see that an n-form is something like "a 0-form with a measure / volume element built-in".

If M^n is an oriented <u>Riemannian</u> manifold, then the associated volume form determines a measure μ_M on M^n. (If $M^n = \mathbb{R}^n$ with the usual metric, μ_M = the Lebesgue measure on $\mathbb{R}^n$.)

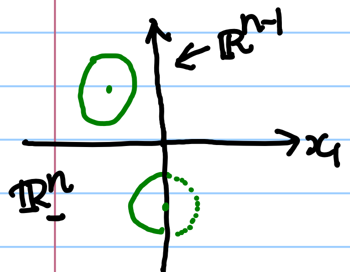
Def (Domain with smooth boundary)



Let M^n be a manifold.

$N \subseteq M^n$ is called a domain with (smooth) boundary if

$\forall p \in N$, $\exists$ chart (U, h) around p s.t.



$$h(U \cap N) = h(U) \cap \mathbb{R}^n_-,$$

where $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}$.

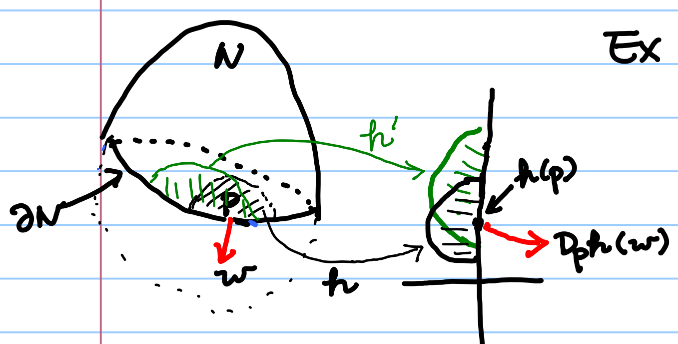
Ex: prove:

(a) $p \in \text{interior}(N) \iff \exists$ chart (U, h) around p s.t. $h(p)_1 < 0$

(b) $p \in \partial N \iff \exists$ chart (U, h) around p s.t. $h(p)_1 = 0$

A tangent vector $w \in T_pM$ at a boundary point $p \in \partial N$ is said to be outward directed, if there exists a C^∞-chart (U, h) around p with $h(U \cap N) = h(U) \cap \mathbb{R}^n_-$ and s.t.

$D_ph(w)$ has a positive first coordinate.



Ex: (c) Prove that the same is true for any other chart around p.

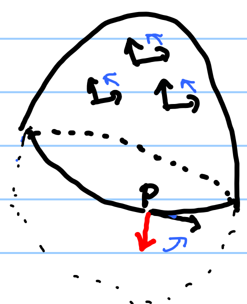
(d) The change of coordinate calculation you need here also proves part (i) of the lemma below. Work out the details.

Lemma (i) Let $N \subseteq M^n$ be a domain with smooth boundary. Then ∂N is an $(n-1)$-dimensional smooth submanifold of M^n.

(ii) Suppose M^n is oriented. There is an induced orientation of ∂N with the following property: if

$p \in \partial N$ and $v_1 \in T_pM$ is an outward directed tangent vector,

then



an ordered basis $v_2, \dots, v_n$ for $T_p\partial N$ is positively oriented

$\Leftrightarrow$

the ordered basis $v_1, v_2, \dots, v_n$ for T_pM is positively oriented.

- For the Stoke's theorem, we want to integrate n-forms

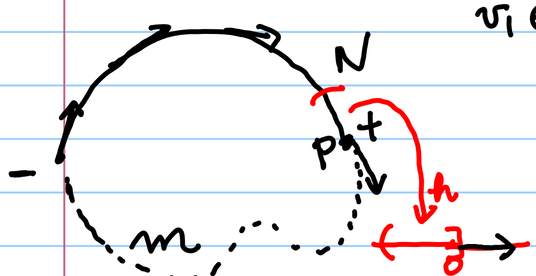
  $\omega \in \Omega_c^n(M)$ over domains N with boundary.

  We can formally define

$$\int_N \omega = \int_M 1_N \omega$$

  based on the measure on M induced by the orientation form on M.

- If $n=1$, ∂N is 0-dimensional. An orientation of ∂N consists of a choice of sign, $+$ or $-$, for every point $p \in \partial N$. For any positively oriented vector $v_1 \in T_pM$,



$$\mathrm{sgn}(p) = \begin{cases} + & \text{if } v_1 \text{ is outward directed} \\ - & \text{otherwise} \end{cases}$$

$$\int_{\partial N} f := \sum_{p \in \partial N} \mathrm{sgn}(p) f(p)$$

Stokes' Theorem

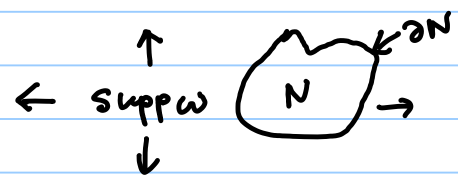
Let $N \subset M^n$ be a domain with smooth boundary in an oriented manifold. Let ∂N have the induced orientation.

For every $\omega \in \Omega^{n-1}(M^n)$ with $N \cap \operatorname{supp}_M(\omega)$ compact we have

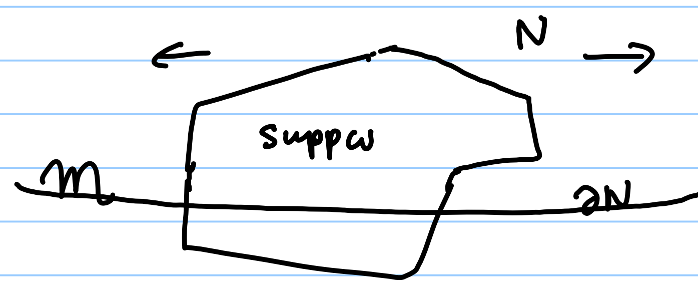
$$\int_{\partial N} i^*(\omega) = \int_N d\omega$$

where

$i: \partial N \to M$ is the inclusion map.



N compact, suppω is not



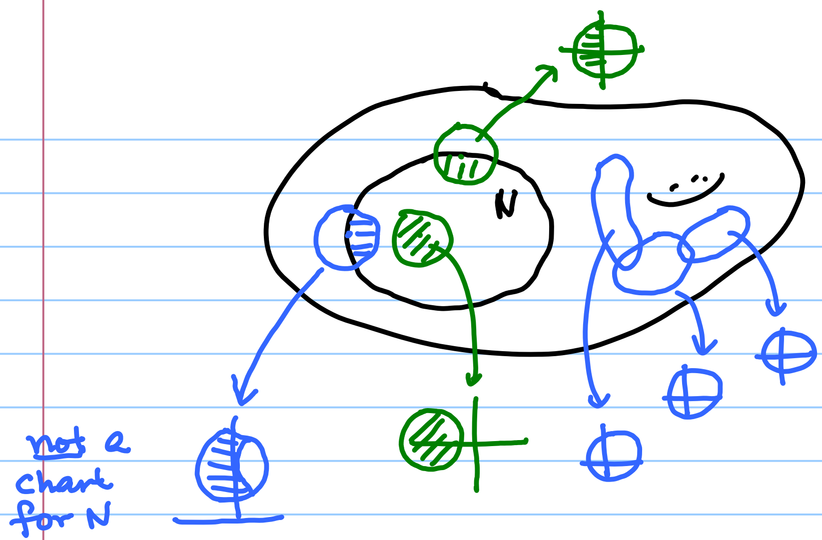
supp ω is compact, N is not

Proof (i) We assume $n \geq 2$. The $n=1$ can be handled quite easily with minor changes.

The assumption implies that $i^*\omega$ has compact support on ∂N, as $\operatorname{supp} i^*\omega \subset N \cap \operatorname{supp}_M(\omega)$.

And we may assume ω is compactly supported: choose $f \in \Omega_c^0(M)$ s.t. $f \equiv 1$ on $N \cap \operatorname{supp}_M(\omega)$, so $f\omega = \omega$ on N, and the two integrals are unchanged when ω is replaced by $f\omega$.

Choose an atlas of M consisting of the special types of charts for N.



$(U_\alpha)_{\alpha \in A}$

$(\rho_\alpha)_{\alpha \in A}$

Then choose a subordinate partition of unity $(\rho_\alpha)_{\alpha \in A}$.

We have $\omega = \sum_\alpha \rho_\alpha \omega$, $\omega \in \Omega_c^{n-1}(M^n)$,

$$\int_{\partial N} \omega = \sum_\alpha \int_{\partial N} \rho_\alpha \omega, \qquad \int_N d\omega = \sum_\alpha \int_N d(\rho_\alpha \omega).$$

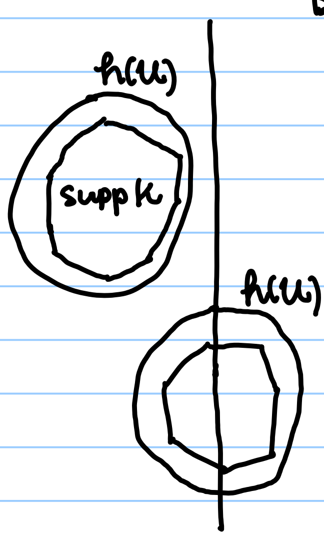
So it is enough to prove the theorem in the case where

$$\omega \in \Omega_c^{n-1}(M^n), \quad \operatorname{supp}_M(\omega) \subseteq U \quad \text{and}$$

(U,h) is a positively oriented chart with

$$h(U \cap N) = h(U) \cap \mathbb{R}^n_-.$$

Let $K \in \Omega_c^{n-1}(\mathbb{R}^n)$ be the $(n-1)$-form that is $(h^{-1})^*(\omega)$ on $h(U)$ and 0 on $\mathbb{R}^n \setminus h(U)$.



By diffeomorphism invariance we have

$$\int_{\partial N} i^*\omega = \int_{h(U) \cap \partial \mathbb{R}^n_-} (h^{-1})^*(\omega) = \int_{\partial \mathbb{R}^n_-} K$$

$$\int_N d\omega = \int_{h(U) \cap \mathbb{R}^n_-} (h^{-1})^*(\omega) = \int_{\mathbb{R}^n_-} dK$$

So the proof further reduces to the special case where $M = \mathbb{R}^n$, $N = \mathbb{R}^n_-$ and $\omega \in \Omega_c^{n-1}(\mathbb{R}^n)$.

(ii) Let $\omega = \sum_{i=1}^{n} f_i(x)\, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$

$R>0$ s.t. $\operatorname{supp} f_i \subseteq [-R,R]^n$

$i : \partial\mathbb{R}^n_- \to \mathbb{R}^n$ inclusion, $x_1, \cdots, x_n \in \Omega^0(\mathbb{R}^n)$

$$i^* x_i = \begin{cases} 0 & i=1 \\ x_i & i>1 \end{cases}$$

$$i^*\omega = \sum_{i=1}^{n} i^* f_i \; d(i^* x_1) \wedge \cdots \wedge \widehat{d(i^* x_i)} \wedge \cdots d(i^* x_n)$$

$$= f_1(0, x_2, \cdots, x_n)\, dx_2 \wedge \cdots \wedge dx_n$$

Hence,

$$\int_{\partial\mathbb{R}^n_-} \omega = \int f_1(0, x_2, \cdots, x_n)\, d\mu_{n-1}$$

lebesgue measure of $\mathbb{R}^{n-1}$.

Next,

$$d\omega = \sum_{i=1}^{n} df_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}\, \underbrace{dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}_{\begin{cases} = 0 & \text{if } j \neq i \\ = (-1)^i dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n \end{cases}}$$

$$= \sum_{i=1}^{n} (-1)^i \frac{\partial f_i}{\partial x_i}\, dx_1 \wedge \cdots \wedge dx_n$$

Hence

$$\int_{\mathbb{R}^n_-} d\omega = \sum_{i=1}^{n} (-1)^{i-1} \int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}\, d\mu_n$$

Lebesgue measure of $\mathbb{R}^n$

For $i = 2, \cdots, n$,

$$\int_{\mathbb{R}^n_-} \frac{\partial f_i}{\partial x_i}\, d\mu_n = \int_{-R}^{0} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial f_i}{\partial x_i}(x)\, dx_1 \cdots dx_n$$

Fubini's thm

$$= \int_{-R}^{0} \int_{-R}^{R} \cdots \underbrace{\int_{-R}^{R} \frac{\partial f_i}{\partial x_i}(x)\, dx_i}_{\overset{\text{F.T.C.}}{=} \left[f_i(x) \right]_{x_i=-R}^{x_i=R} = 0}\, dx_1 \cdots \widehat{dx_i} \cdots dx_n$$

For $i=1$,

$$\int_{\mathbb{R}^n_-} \frac{\partial f_1}{\partial x_1}\, d\mu_n = \int_{-R}^{0}\int_{-R}^{R}\cdots\int_{-R}^{R} \frac{\partial f_1}{\partial x_1}(x)\, dx_1 \cdots dx_n$$

$$= \int_{-R}^{R}\cdots\int_{-R}^{R} \underbrace{\int_{-R}^{0} \frac{\partial f_1}{\partial x_1}(x)\, dx_1}_{f_1(0,x_2,\dots,x_n)-0}\, dx_2\cdots dx_n$$

$$= \int_{\substack{\mathbb{R}^{n-1} \\ \shortparallel \\ \partial\mathbb{R}^n_-}} f_1(0,x_2,\dots,x_n)\, d\mu_{n-1}$$

we have proved $\displaystyle\int_{\partial\mathbb{R}^n_-} \omega = \int_{\mathbb{R}^n_-} d\omega.$ $\square$

Note: If M^n is an oriented smooth manifold (so $\partial M=\phi$) and $\omega\in\Omega_c^{n-1}(M^n)$,

then

$$\int_M d\omega = 0.$$

Example:

Let $\omega_0 \in \Omega^{n-1}(\mathbb{R}^n - \{0\})$ be defined by

$$\omega_{0x}(w_1, \dots, w_{n-1}) = \det(x, w_1, \dots, w_{n-1}) \in \mathrm{Alt}^{n-1}(\mathbb{R}^n)$$

$$= \sum_{i=1}^{n} \underbrace{\omega_{0x}(e_1, \dots, \hat{e}_i, \dots, e_n)}_{= \det(x, e_1, \dots, \hat{e}_i, \dots, e_n)} dx_1 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n$$

$$= (-1)^{i-1} x_i \qquad \text{Ex: (check it.)}$$

$i : S^{n-1} \subset \mathbb{R}^n \to \mathbb{R}^n$ inclusion

$i^*\omega_0$ is an orientation form on S^{n-1}

If S^{n-1} inherits the Riemannian structure from $\mathbb{R}^n$, $i^*\omega_0$ is also the corresponding volume form, since

if $w_1, \dots, w_{n-1}$ is a positively oriented o.n. basis of $T_x S^{n-1}$, then $x, w_1, \dots, w_{n-1}$ is a positively oriented o.n. basis of $T_x \mathbb{R}^n$ ($\cong \mathbb{R}^n$), so

$$\omega_{0x}(w_1, \dots, w_{n-1}) = \det(x, w_1, \dots, w_{n-1}) = 1$$

$i^*\omega_0 = \mathrm{vol}_{S^{n-1}}$, being a top-dimensional form in S^{n-1}, is, of course, closed. ($d : \Omega^n(M^n) \to \{0\}$ $= \Omega^{n+1}(M^n)$)

Trick: consider the map

$$r : \mathbb{R}^n - \{0\} \to S^{n-1}, \quad r(x) = x/\|x\|$$

consider $\omega = r^*(\mathrm{vol}_{S^{n-1}})$,

which must be closed also, since $d\omega = d\, r^*(\mathrm{vol}_{S^{n-1}})$
$= r^*(d\,\mathrm{vol}_{S^{n-1}}) = 0$

By a calculation on P74-75 of MBT:

$$\omega = \frac{1}{\|x\|^n} \sum_{i=1}^{n} (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

or $\omega_x = \|x\|^{-n} \omega_{0x}$.

Note: ω is closed, but ω_0 is not.

Recall:

$$\dim \overbrace{H^{n-1}(S^{n-1})}^{\ni [\mathrm{Vol}_{S^{n-1}}]}$$
$$\|$$
$$\dim \underbrace{H^{n-1}(\mathbb{R}^n - \{0\})}_{\ni [\omega]} = 1 \qquad n \geq 2$$

(ω being closed is used here.)

Claim: $[\omega] \neq 0$ in $H^{n-1}(\mathbb{R}^n - \{0\})$, $[\mathrm{Vol}_{S^{n-1}}] \neq 0$ in $H^{n-1}(S^{n-1})$

<u>Proof:</u> It suffices to show that ω cannot be written as $d\tau$ for any $\tau \in \Omega^{n-2}(\mathbb{R}^n - \{0\})$.

Assume the contrary, and recalling

$$i^*\omega = \mathrm{Vol}_{S^{n-1}}, \quad i: S^{n-1} \to \mathbb{R}^n - \{0\},$$

$$\int_{S^{n-1}} i^*\omega = \int_{S^{n-1}} \mathrm{Vol}_{S^{n-1}} \neq 0 \quad \text{on the one hand.}$$

On the other hand,

$$\int_{S^{n-1}} i^*\omega = \int_{S^{n-1}} i^* d\tau = \int_{S^{n-1}} d(i^*\tau) \overset{\text{Stoke's theorem}}{=} 0$$

$$\Rightarrow\!\Leftarrow$$

Similarly, if $\mathrm{Vol}_{S^{n-1}} = d\tau$ for some $\tau \in \Omega^{n-2}(S^{n-1})$, we have a similar contradiction. $\square$

Consequence: For $n \geq 2$, $[\omega]$ is a basis of $H^{n-1}(\mathbb{R}^n - \{0\})$

$[\mathrm{vol}_{S^{n-1}}]$ is a basis of $H^{n-1}(S^{n-1})$.

moreover, we can think of

$$\int_{S^{n-1}} : H^{n-1}(S^{n-1}) \longrightarrow \mathbb{R}$$
$$[\eta] \mapsto \int_{S^{n-1}} \eta$$ as a linear isomorphism.

We show that this is a general phenomenon for connected oriented compact manifolds:

Thm (Corollary 10.14) For a connected oriented compact manifold M^n, integration over M induces an isomorphism

$$\int_M : H^n(M^n) \to \mathbb{R}.$$

This follows from the following (not so trivial) result:

Theorem (10.13) If M^n is connected and oriented (not necessarily compact), then the sequence

$$\Omega_c^{n-1}(M) \xrightarrow{d} \Omega_c^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0$$

is exact.

If M is compact, then $\dim H^n(M^n) < \infty$,

$$\Omega^{n-1}(M) \xrightarrow{d} \Omega^n(M) \xrightarrow{\int_M} \mathbb{R} \to 0 \text{ is exact}$$

$$H^n(M^n) = \Omega^n(M) / \mathrm{Im}(d : \Omega^{n-1}(M) \to \Omega^n(M))$$
$$\underset{\substack{\uparrow \\ \text{Thm} \\ 10.13}}{=} \Omega^n(M) / \{\eta \in \Omega^n(M) : \int_M \eta = 0\}$$

This means every element in $H^n(M^n)$,

$$[\omega] = [\omega'] \iff \omega = \omega' + \eta, \ \int_M \eta = 0$$
$$\iff \int_M \omega - \omega' = 0$$
$$\iff \int \omega = \int \omega'$$

In other words, $\int_M \omega$ depends only on the cohomology class $[\omega]$,

hence

$$\underset{H^n(M)}{\underset{\cap}{[\omega]}} \mapsto \underset{\mathbb{R}}{\underset{\cap}{\int_M \omega}}$$ is well-defined and is a linear isomorphism.

Proof of Thm 10.13:

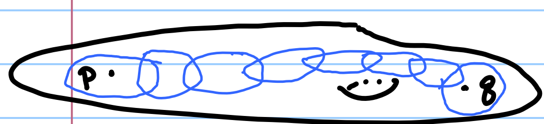
① First prove it in the case of $M = \mathbb{R}^n$.

② Let M - connected manifold, $(U_\alpha)_{\alpha \in A}$ an open cover.

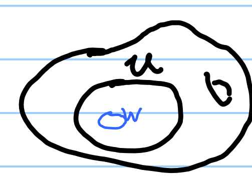
For any $p, q \in M$, $\exists\ \alpha_1, \dots, \alpha_k$ s.t

(i) $p \in U_{\alpha_1}$, $q \in U_{\alpha_k}$

(ii) $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$, $i = 1, \dots, k-1$.



③ Let $U \subseteq M$ be an open set diffeomorphic to $\mathbb{R}^n$, $W \subseteq U$ nonempty and open.



$\forall\ \omega \in \Omega^n_c(M)$, $\operatorname{supp} \omega \subseteq U$

$\exists\ K \in \Omega^{n-1}_c(M)$ st $\operatorname{supp}(K) \subseteq U$ and $\operatorname{supp}(\omega - dK) \subseteq W$

(i.e $\omega = dK$ on $U \setminus W$)

④ Assume M is connected, $W \subseteq M$ is non-empty and open.

$\forall\, \omega \in \Omega^n_c(M)$, $\exists\, K \in \Omega^{n-1}_c(M)$ with $\mathrm{supp}(\omega - dK) \subseteq W$.

② – easy point-set topology argument

① – reduces to a Calculus problem :

Let $\omega = f(\overbrace{x_1, \dots, x_n}^{x})\, dx_1 \wedge \dots \wedge dx_n$ with

$$\int \omega = \int_{\mathbb{R}^n} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n = 0 \ , \ f \in \Omega^0_c(\mathbb{R}^n)$$

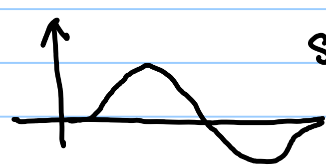
Note:

$$K = \sum_{j=1}^{n} (-1)^{j-1} f_j(x)\, dx_1 \wedge \dots \wedge \widehat{dx_j} \wedge \dots \wedge dx_n$$

$$\Rightarrow dK = \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j}\, dx_1 \wedge \dots \wedge dx_n$$

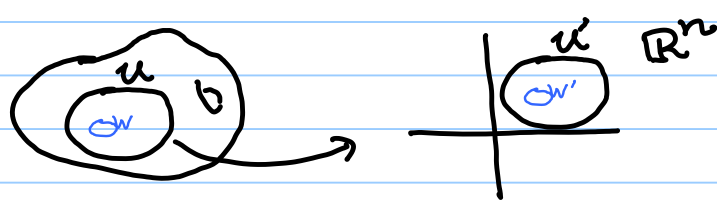
So to find $K \in \Omega^{n-1}_c(\mathbb{R}^n)$ s.t. $dK = \omega$, it suffices to find

$$f_1, \dots, f_n \in \Omega^0_c(\mathbb{R}^n) \text{ s.t. } \sum_{j=1}^{n} \frac{\partial f_j}{\partial x_j} = f$$

$n = 1$  set $f_1(x) := \int_{-\infty}^{x} f(t)\, dt$

$n > 1$: induction on n (see M&T pg 92-93)
(tricky)

③ follows from ① easily : suffices to assume $M = \mathcal{U} = \mathbb{R}^n$ (thanks to diffeomorphism invariance)

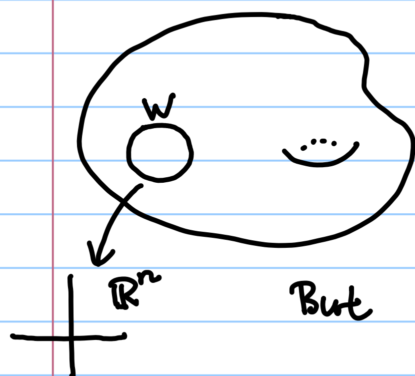


④ follows from ③ and ② : Here W may not be contained in an open set diffeomorphic to $\mathbb{R}^n$, we can choose an open cover of M, $\{U_\alpha\}_{\alpha\in A}$, so that each U_α is diffeomorphic to $\mathbb{R}^n$, and a subordinate partition of unity to write

$$\omega = \sum_{j=1}^{n} \omega_j .$$

④ can be established by working on each piece.

Finally, suppose $\omega \in \Omega_c^n(M)$ with $\int_M \omega = 0$.



choose $W \overset{\text{open}}{\subseteq} M$, W diffeomorphic to $\mathbb{R}^n$.

By ④, $\exists$ $K \in \Omega_c^{n-1}(M)$ with

$$\operatorname{supp}(\omega - dK) \subseteq W.$$

But

$$\int_W \omega - dK = \int_M \omega - dK \underset{\substack{\uparrow \\ \int_M \omega = 0}}{=} -\int_M dK \underset{\substack{\uparrow \\ \text{Stoke's thm.}}}{=} 0$$

W is where ω and dK may not agree, so consider

$$(\omega - dK)|_W \quad \text{and use the assumption}$$

that W is diffeomorphic to $\mathbb{R}^n$, so by ① we can find $\tau_0 \in \Omega^{n-1}(W)$ s.t.

$$(\omega - dK)|_W = d\tau_0 .$$

Let $\tau \in \Omega_c^{n-1}(M)$ be the extension of τ_0 which vanishes outside of $\operatorname{supp}_W(\tau_0)$.

Then $\quad \omega - dK = d\tau \quad$ (on the whole manifold M)

or

$$\omega = d(\tau + K). \qquad \square$$