Lecture 8 : SO(3)

Note Title 2/26/2017

We have proved (Lecture 4 + Lecture 7) that it is a submanifold of $\mathbb{R}^9 \approx \mathbb{R}^{3\times 3}$. Also, it is a Lie group (Lecture 5.)

The reason why we still have more to say about SO(3) is :

- it is important on its own, so would be useful to know more details

- it exemplifies some of the important structures of Lie groups in general.

Recall at the end of Lecture 4, we showed that the tangent vectors of SO(3) (represented by matrices in $\mathbb{R}^{3\times 3}$) are exactly the skew-symmetric matrices.

$$T_{id} SO(3) = \{ A \in \mathbb{R}^{3\times 3} : A^T = -A \}.$$

This must be a subspace of $\mathbb{R}^{3\times 3}$ ($\approx \mathbb{R}^9$), let's check it explicitly :

$$\alpha, \beta \in \mathbb{R}, \quad A^T = -A, \quad B^T = -B$$
$$\Rightarrow$$
$$(\alpha A + \beta B)^T = \alpha A^T + \beta B^T = -\alpha A - \beta B = -(\alpha A + \beta B)$$

On top of the vector space structure, there is another structure that is very important called the Lie-bracket, which is something I want you to see briefly :

If $A, B \in T_{id} SO(3)$, then so is their <u>matrix commutator</u>:

$$[A,B] := AB - BA.$$

<u>Proof</u>: If $A^T = -A$, $B^T = -B$,

$$\begin{aligned} [A,B]^T = (AB-BA)^T &= B^T A^T - A^T B^T \\ &= (-B)(-A) - (-A)(-B) \\ &= BA - AB \\ &= -[A,B]. \end{aligned}$$

$\blacksquare$

<u>Def</u>: A real vector space V is a Lie algebra if there is defined on it an operation

$$[\ ,\ ] : V \times V \to V$$

(called bracket) that satisfies

1. (Bilinearity) $[au + bv, w] = a[u,w] + b[v,w]$
   $[u, av + bw] = a[u,v] + b[u,w]$
2. (Skew-symmetry) $[u,v] = -[v,w]$
3. (Jacobi identity) $[u,[v,w]] + [w,[u,v]] + [v,[w,u]] = 0$

There is also something called:

"Lie bracket of vector fields on a manifold"

whose definition is irrelevant to Lie group:

It goes like this: any vector field V on a manifold M can be thought of as a map from $C^\infty(M)$ to $C^\infty(M)$.

For $f \in C^\infty(M)$ $\quad V(f)(p) := V(p)(f)$

Easy to check $V(f): M \to \mathbb{R}$ is smooth, so $V(f) \in C^\infty(M)$. As such, V maps every $f \in C^\infty(M)$ to a function in $C^\infty(M)$.

Let $V, W : C^\infty(M) \to C^\infty(M)$ be vector fields.

Define the (vector field) Lie bracket of V and W, denoted

$$[V, W] : C^\infty(M) \to C^\infty(M)$$

by

$$[V, W]f = V(Wf) - W(Vf).$$

[Note: Every vector field be viewed as a map from $C^\infty(M)$ to $C^\infty(M)$, but not any map from $C^\infty(M)$ to $C^\infty(M)$ is a vector field. The map has to satisfy linearity and the Leibniz product rule. It's not hard to check that $[V, W] : C^\infty(M) \to C^\infty(M)$ is linear and Leibnizian.]

Easy to check: matrix commutator is a bracket on $T_{id}SO(3)$.

Not hard to check: the (vector field) Lie bracket is a bracket on $\mathcal{T}(M) =$ the vector space of all vector fields on M.

Define $v \in T_{id}SO(3) \mapsto V \in \mathcal{T}(SO(3))$
$$V_g = L_{g*}(v) \in T_g SO(3)$$

(such a vector field is called a left-invariant vector field.)

This correspondence is a vector space isomorphism.

$T_{id}SO(3) \longleftrightarrow \{$left invariant vector fields of $SO(3)\} \subset \mathcal{T}(SO(3))$

Under this correspondence, the matrix Lie bracket on $T_{id}SO(3)$ is "the same" as the vector field Lie bracket.

What's the point of all these Lie bracket / commutator stuff in the last three pages?

Every Lie group G has a tangent space at the identity that, together with the Lie bracket, is called the Lie algebra $\mathfrak{g}$. The key idea is that there is a so-called exponential map from $\mathfrak{g}$ to G
$$\exp : \mathfrak{g} \to G,$$
that is not only a local diffeomorphism, but also specifies the group structure of G based on the Lie algebra on $\mathfrak{g}$. One way to see this connection is via the Baker-Campell-Hausdorff (BCH) formula:

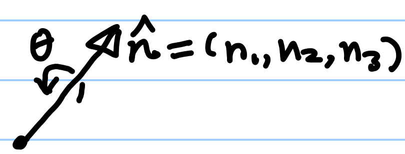
$$\exp(A)\exp(B) = \exp(A+B + \tfrac{1}{2}[A,B] + \tfrac{1}{12}[A,[A,B]] + \cdots)$$

Note: In general, $\exp(A+B) \neq \exp(A)\exp(B)$, and it is the commutator (or Lie bracket) that accounts for the difference, via the BCH formula.

We shall define and use the exp map for $SO(3)$ (in fact $GL(n)$) below.

Euler's rotation theorem

In three-dimensional space, any displacement of a rigid body such that a point on the rigid body remains fixed, is equivalent to a single rotation about some axis that runs through the fixed point.



i.e. For any $R \in SO(3)$,

$R =$ rotation of $\mathbb{R}^3$ through $\theta \in [0, \pi]$ radians about an axis in $\mathbb{R}^3$ along $\hat{n} = (n_1, n_2, n_3)$ in the sense determined by the right-hand rule from the direction of $\hat{n}$.

Moreover, the angle $\theta \in [0, \pi]$ is uniquely determined by R, and

a) If $\theta \in (0, \pi)$, then $\hat{n}$ is unique

b) If $\theta = 0$, then any $\hat{n}$ will do.

c) If $\theta = \pi$, then $\hat{n}$ is unique up to sign.

This way of representing SO(3) is called the axis-angle representation.

Given $R \in SO(3)$, the existence of rotation axis is equivalent to the existence of a direction which R leaves invariant, i.e.

$$\exists\, n \in \mathbb{R}^3,\ n \neq 0 \quad \text{s.t.} \quad R \cdot n = n,$$

which is equivalent to saying that

(*) 1 is an eigenvalue of every $R \in SO(3)$.

Once the direction determined by n is established, R has to behave like a 2-D rotation in the remaining $2\,(=3-1)$ dimensions. The uniqueness part of the theorem should be clear.

<u>Proof of (*)</u>:

Let $R \in SO(3)$, i.e. $R^T R = I$, $\det(R) = 1$

$$(*) \iff \det(R - 1 \cdot I) = 0$$

$$\begin{aligned}
\det(R - I) &= \det((R-I)^T) = \det(R^T - I) \\
&= \det(R^{-1} - R^{-1}R) \\
&= \det(R^{-1}(I - R)) \\
&= \det(R^{-1}) \det(-(R-I)) \\
&= \det(-(R-I)) \qquad (\text{note: } \det(R^{-1}) = 1) \\
&= -\det(R - I) \qquad (\text{we are 3-D.})
\end{aligned}$$

So $\det(R - I) = 0$ $\blacksquare$

Note: Without the help of linear algebra, Euler's original proof is purely geometric (and quite tricky.)

How to go back and forth the matrix representation (of 3-D rotation) and the axis-angle rotation?

$$SO(3) \ni R \xleftrightarrow{?} (\hat{n}, \theta)$$

We showed $\lambda_1 = 1$ is an eigenvalue of R, and hence $R\hat{n} = \hat{n}$ for some unit vector $\hat{n} \in \mathbb{R}^3$.
Since

$$\det(R) = 1 = \text{product of the 3 eigenvalues, } 1 = \lambda_1, \lambda_2, \lambda_3, \text{ of } R$$

so we must have

$$\lambda_2 \lambda_3 = 1,$$

and it can be shown (some details omitted):

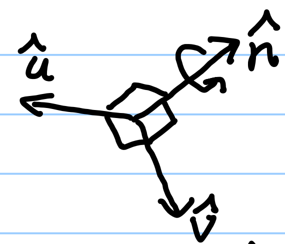
$$\theta = \begin{cases} 0 & \text{if } (\lambda_2, \lambda_3) = (1, 1) \\ \pi & \text{if } (\lambda_2, \lambda_3) = (-1, -1) \\ \cos^{-1}(\alpha) \in (0, \pi) & \text{if } \lambda_1, \lambda_2 = \alpha \pm i\beta \\ & \quad \alpha^2 + \beta^2 = 1, \ \beta \neq 0 \end{cases}$$

When R is given, its eigenvalues are uniquely determined, so $\theta \in [0, \pi]$ is uniquely determined as above. But the eigenvector associated with eigenvalue 1 is, of course, non-unique and we must choose the correct unit eigenvector $\hat{n}$ so that $(\hat{n}, \theta)$ represents R according to the right-hand convention. (There are only two possible choices of $\hat{n}$, it is not hard to come

up with a way to determine which.)
Once this is done, we have:

$$(\ast) \text{ — } R = [\hat{n}\ \hat{u}\ \hat{v}] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} [\hat{n}\ \hat{u}\ \hat{v}]^T$$

where $\hat{u}$ and $\hat{v}$ are two arbitrary orthogonal unit vectors s.t. $\hat{u} \times \hat{v} = \hat{n}$.



Note: the choice of $\hat{u}, \hat{v}$ would not affect $(\ast)$.

When the axis-angle representation $(\hat{n}, \theta)$ is given, $(\ast)$ can be used to determine the rotation matrix R.

But it seems like the approach above of going back and forth the two representations is neither aesthetically pleasing nor convenient to compute with.

Here is a better way called the Rodrigues' rotation formula:

$$R = I + (\sin\theta) K + (1 - \cos\theta) K^2, \text{ where}$$

$$K = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}.$$

The way we prove it will involve the following ideas

- flow of a velocity vector fields
- ODE and exponential maps.

It seems like an overkill to use these techniques for proving a linear algebra result. Indeed, there is a more elemantary proof. But I found the the following proof really useful for understanding a nonlinear version of rotation called the <u>curl of a vector field</u>.

Let's begin with a linear algebra exercise:

<u>Ex</u> : Show

$$\hat{n} \times v = \underbrace{\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}}_{=K} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}_{=v}$$

Show

$$\left(I + (\sin\theta) K + (1-\cos\theta) K^2\right) v$$
$$= v \cos\theta + (\hat{n} \times v) \sin\theta + \hat{n}(\hat{n} \cdot v)(1-\cos\theta)$$

(This is how Rodrigues formula can be used in practice; no 3×3 matrix is even needed.)

We divide the proof of Rodrigues formula into 3 parts:

(1) Imagine a linear (time-independent) velocity vector field in $\mathbb{R}^3$:

$$\text{velocity vector} \longrightarrow \underbrace{V(}_{}\underset{\substack{\uparrow \\ \text{location}}}{x}) = \underset{\substack{\uparrow \\ \text{an arbitrary } 3\times 3 \text{ matrix}}}{A}x$$

If a particle truly moves in $\mathbb{R}^3$ according to this velocity vector field, what is the trajectory of the particle, assuming the particle starts at $x_0 \in \mathbb{R}^3$ at time 0? This is equivalent to following question:

If $x : [0, \infty) \to \mathbb{R}^3$ satisfies

$$x'(t) = V(x(t)) = Ax(t), \qquad x(0) = x_0$$

what is $x(t)$?

This is a linear, constant coefficient system of ordinary differential equations of which the solution is well-known:

$$\underset{3\times 1}{x(t)} = \underset{3\times 3}{\exp(At)} \cdot \underset{3\times 1}{x(0)}, \quad \text{where}$$

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$$

I assume that you have seen this part of ODE (you sure have seen the 1-D case since Calculus I). And you know that the series above converge absolutely for any A and uniformly on any bounded set in $\mathbb{R}^{n^2}$ or $\mathbb{C}^{n^2}$.

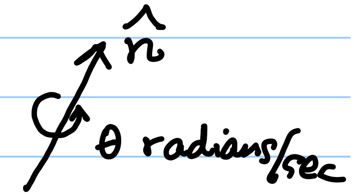
(2) Armed with (1), we need one more observation to show that:

$$\boxed{R = \exp(K\theta)}.$$

(same notations in the Rodrigues formula.)

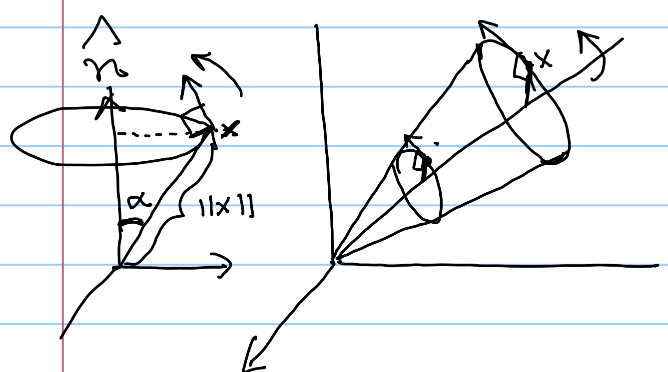
Consider the velocity vector field $V(x)$ generated by rotating about $\hat{n}$ at a <u>constant</u> angular velocity of

θ radians/second.



What is $V(x)$?

[Note : It is a dynamics, not static, problem.]



① $V(\vec{x}) \perp \vec{x}$

② $V(\vec{x}) \perp \hat{n}$

③ $\underbrace{\|V(\vec{x})\|}_{\text{speed at } \vec{x}} = (\|x\| \sin\alpha) \cdot \theta$ (unit length/sec)

①, ②, ③ and right-hand convention $\Rightarrow$

$$V(\vec{x}) = \theta\, \hat{n} \times \vec{x}$$
$$= \theta \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \theta K \vec{x}$$

This shows $V(\vec{x})$ is the kind of linear vector field considered in (1), so :

$\phi(t, x)$ = where the particle is after t secs if its beginning location is x
$= \exp(\theta K t) \cdot \vec{x}$

$$R\vec{x} = \phi(1, \vec{x}) = \exp(\theta K)\vec{x}, \text{ or } \boxed{R = \exp(\theta K)}$$

(3) From $R = \exp(\theta K)$ to Rodrigues formula:

First note that $K^3 = -K$.
So, $K^4 = -K^2$, $K^5 = K$, $K^6 = K^2$, $K^7 = -K$, etc.

One way is see this is to check it directly (by writing $n_3 = \sqrt{1 - n_1^2 - n_2^2}$ and verify that $K^3 + K = 0$.

Another way is to observe that the characteristic polynomial of K is:

$$p(\lambda) = \det(K - \lambda I) = \det \begin{bmatrix} -\lambda & -n_3 & n_2 \\ n_3 & -\lambda & -n_1 \\ -n_2 & n_1 & -\lambda \end{bmatrix}$$
$$= -\lambda^3 - \lambda(n_1^2 + n_2^2 + n_3^2)$$
$$= -\lambda^3 - \lambda$$

So, by the Hamilton-Cayley theorem, $p(K) = 0$, or $-K^3 - K = 0$, or $K^3 = -K$.

Consequently,

$$R = \exp(\theta K) = \sum_{k=0}^{\infty} \frac{(\theta K)^k}{k!}$$
$$= I + \frac{\theta}{1!} K + \frac{\theta^2}{2!} K^2 + \frac{\theta^3}{3!} K^3 + \cdots$$
$$= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - + \cdots\right) K + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right) K^2$$
$$= I + \sin\theta\, K + (1 - \cos\theta) K^2. \quad \blacksquare$$

Ex Directly check that $I + (\sin\theta) K + (1-\cos\theta) K$ is in $SO(3)$.

You need 1. below to do this Ex.

0. If $A = UDU^{-1}$, $\exp(A) = U \exp(D) U^{-1}$.
1. $\det(\exp(A)) = \exp(\mathrm{Tr}(A))$
2. $\exp(A+B) = \exp(A)\exp(B)$ if $AB = BA$

Proof of 1.: Let $A = UDU^{-1}$ be the Jordan canonical decomposition of A.

$$\exp(A) = U \exp(D) U^{-1},$$

$$\det(\exp(A)) = \det(\exp(D))$$

$$D = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad D^k = \begin{bmatrix} \lambda_1^k & & *' \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$\exp(D) = \begin{bmatrix} \exp(\lambda_1) & & *'' \\ & \ddots & \\ 0 & & \exp(\lambda_n) \end{bmatrix}$$

$$\text{so} \quad \det(\exp(D)) = \prod_{i=1}^{n} \exp(\lambda_i) = \exp\left(\sum \lambda_i\right) = \exp(\mathrm{Tr}(A)).$$

Let us summarize what we have proved earlier:

$\exp : so(3) \to SO(3)$ maps the axis-angle representation to the matrix representation (in the sense discussed) so it has a intuitive geometric interpretation. Moreover, it shows that

$\exp : so(3) \to SO(3)$ is <u>surjective</u>.

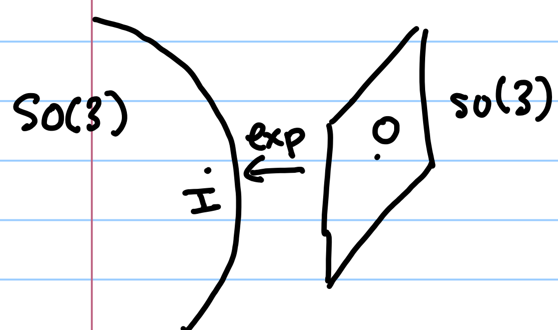
So it seems like using the three parameters (a, b, c) in $\begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ -b & a & 0 \end{bmatrix}$ is a very good way to parameterize $SO(3)$.

It can also be shown that

$$\underset{\substack{\uparrow \\ \text{a (linear)} \\ \text{submanifold} \\ \text{of } \mathbb{R}^9}}{\exp : so(3)} \to \underset{\substack{\uparrow \\ \text{a submanifold} \\ \text{of } \mathbb{R}^9}}{SO(3)}$$

is smooth (in fact <u>real analytic</u>.)

Claim: $\exp_{*0} = id_{so(3)}$, and hence an atlas on $SO(3)$ can be obtained from $so(3)$.



<u>Proof</u>: Let $A \in so(3)$. Then $\alpha : (-\varepsilon, \varepsilon) \to so(3)$ given by $\alpha(t) = 0 + tA$ is a smooth curve in $so(3)$ with $\alpha(0) = 0$, $\alpha'(0) = A$.

Thus,

$$\exp_{*0}(A) = \exp_{*0}(\alpha'(0)) = (\exp \circ \alpha)'(0)$$

$$= \frac{d}{dt} \underbrace{\exp(tA)}_{I + tA + \frac{1}{2}t^2A^2 + \cdots} \Big|_{t=0} = A.$$

This shows $\exp_{*0} = id_{so(3)}$.

The inverse function theorem therefore implies that $\exp : so(3) \to SO(3)$ is a local diffeomorphism near $0 \in so(3)$. Since $\exp(0) = I \in SO(3)$ we can find an open neighborhood $\mathcal{U}$ of I in $SO(3)$ on which $\exp^{-1}$ exists and is smooth and maps onto a neighborhood of 0 in $so(3) \cong \mathbb{R}^3$. Thus

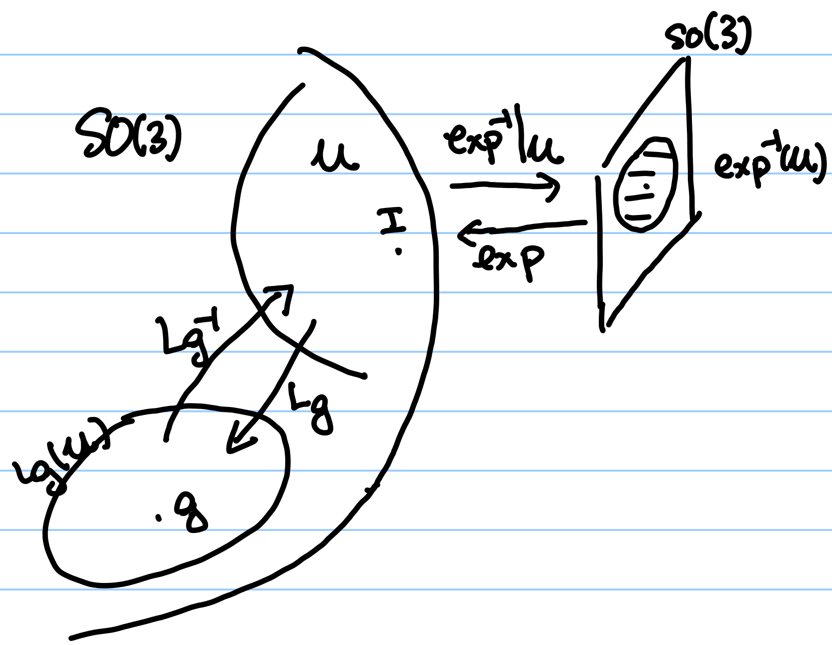
$$(\mathcal{U}, \exp^{-1}) \text{ is a chart at } I \in SO(3).$$

To get an atlas, notice that the chart at I can be translated to a chart at any $g \in SO(3)$ because

$$L_g : SO(3) \to SO(3), \quad L_g(h) = gh$$

is a diffeomorphism, so

$$(L_g(\mathcal{U}), \exp^{-1} \circ L_{g^{-1}}) \text{ is a chart at } g \in SO(3).$$



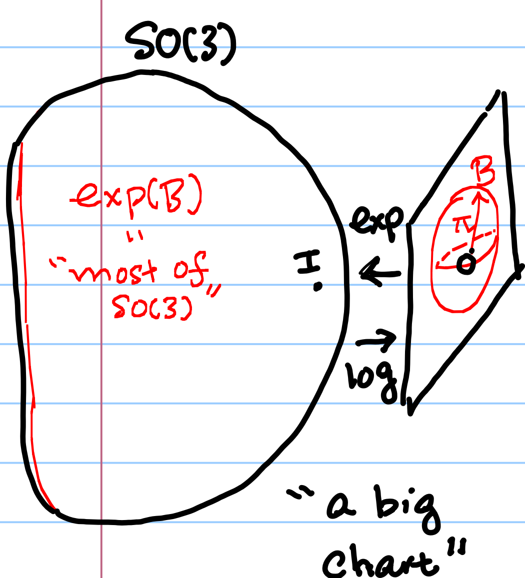
Remark: The inverse function theorem only tells us $\exp: so(3) \to SO(3)$ is locally invertible near 0. And it is enough for the purpose of creating an atlas for $SO(3)$. But from Euler's theorem and what is proved in this lecture, we also know that:

$$\exp: so(3) \to SO(3)$$

is injective when restricted to

$$B = \left\{ \begin{bmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{bmatrix} : a^2+b^2+c^2 < \pi^2 \right\}$$

and $\exp(B)$ is exactly missing those elements in $SO(3)$ that "rotate by 180° about some axis."



So $\exp|_B : B \to SO(3)$ is injective. Moreover, it can be shown that

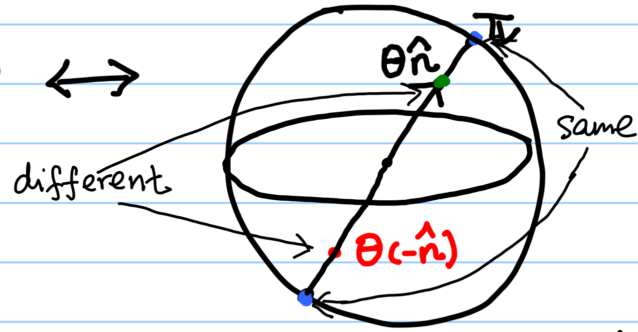
$$\exp^{-1} : \exp(B) \to B$$

is smooth and is usually called the "matrix logarithm".

Global Issues:

The uniqueness part of Euler's theorem actually suggests a correspondence between $SO(3)$ and $\mathbb{RP}^3 = G(4,1) = S^3/\{+,-\}$

$SO(3) \longleftrightarrow$



ONLY antipodal pairs on the boundary are identified.

$$R \longleftrightarrow \underbrace{\theta\hat{n}}_{\substack{\shortparallel \\ (x,y,z)}} \in \left(\begin{array}{l} \text{the solid ball in } \mathbb{R}^3 \text{ with} \\ \text{radius } \pi \text{, but with} \\ \text{antipodal points on the} \\ \text{boundary identified.} \end{array} \right)$$

$$\longleftrightarrow \pm\left(x, y, z, \sqrt{\pi^2 - x^2 - y^2 - z^2}\right) \in \mathbb{RP}^3$$

Ex: Show that this is not only a bijection, but also a diffeomorphism between $SO(3)$ and $\mathbb{RP}^3$.

From this you see that S^3 = unit sphere in $\mathbb{R}^4$ forms a double cover of $\mathbb{RP}^3 \approxeq SO(3)$. But more is true: S^3 is not just a sphere but also a Lie group if you identify it with either

$$SU(2) = \{A \in \mathbb{C}^{2\times 2} : A^*A = I, \det(A) = 1\}$$

or the "unit quaternions". This double cover happens to be very meaningful to the physicists.

Related to this double cover is the fact that the first fundamental group of $SO(3) = \mathbb{Z}_2$

This means there are two kinds of continuous loops in SO(3), one that is homotopic to the trivial loop, and the other that is not. To get a hint that this topological fact is of significance to the physicists, take a look at the famous

"Dirac belt trick",

credited to Physics Nobel laureate Paul Dirac for illustrating the two kinds of loops above.

See, for example,

https://vimeo.com/62228139