CH 3 : De Rham Cohomology

Note Title 4/11/2017

$\mathcal{U} \underset{\text{open}}{\subset} \mathbb{R}^n$ $\{e_1, \cdots, e_n\}$ – standard basis of $\mathbb{R}^n$

$\{\varepsilon_1, \cdots, \varepsilon_n\}$ – dual basis of $\mathrm{Alt}^1(\mathbb{R}^n)$

Def A differential p-form on $\mathcal{U}$ is a smooth map

$$\omega : \mathcal{U} \to \mathrm{Alt}^p(\mathbb{R}^n)$$

$\Omega^p(\mathcal{U}) :=$ the vector space of all such maps.

Convention : $p = 0$, $\mathrm{Alt}^0(\mathbb{R}^n) = \mathbb{R}$ (consistent with the formula $\dim \mathrm{Alt}^p(\mathbb{R}^n) = \binom{n}{p}$)

$\Omega^0(\mathcal{U}) = C^\infty(\mathcal{U}, \mathbb{R})$

Exactly what do we mean by smoothness of a function

$$\omega : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \underset{W}{\underset{\shortparallel}{(\text{a vector space over } \mathbb{R})}} \; ?$$

Def : ω is smooth if the component functions of ω in some basis of W are smooth function from $\mathcal{U}$ to $\mathbb{R}$ in the traditional sense in calculus.

i.e. if $w_1, \cdots, w_m$ is a basis of W,

$$\omega(x) = \sum_{i=1}^{m} \underbrace{c_i(x)}_{\text{component functions } \mathcal{U} \to \mathbb{R}} w_i$$

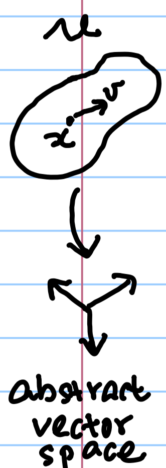
then saying ω is smooth is the same as saying $c_1(x), \cdots, c_m(x)$ smooth functions.

Note : ① this definition does not depend on the choice of basis. If the component functions of ω are smooth in one basis of W, the component functions in any other basis of W are smooth.

② Hardly surprisingly, there is an equivalent way to redefine this smoothness without refering to any basis of W. (A hint of how it works is given below.)

$$\omega : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathrm{Alt}^p(\mathbb{R}^n)$$

$$x \in \mathcal{U}$$



$D_x\omega : \mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n)$ is the linear map

$$D_x\omega(v) = \frac{d}{dt}\,\omega(x+tv)\Big|_{t=0}$$ the "usual derivative"

$$\parallel$$

$$\lim_{t\to 0} \frac{1}{t}\left[\omega(x+tv) - \omega(x)\right]$$

vector space structure of $\mathrm{Alt}^p(\mathbb{R}^n)$

think about this limit using any norm in $\mathrm{Alt}^p(\mathbb{R}^n)$ (all norms are equivalent)

If we use the basis $\{\varepsilon_I = \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} : I = (i_1, \cdots, i_p),\ i_1 < \cdots < i_p\}$ for $\mathrm{Alt}^p(\mathbb{R}^n)$,

then every $\omega \in \Omega^p(\mathcal{U})$ can be written as

$$\omega(x) = \sum_I \omega_I(x)\, \varepsilon_I$$

and

$$D_x\omega(e_j) = \sum_I \frac{\partial \omega_I}{\partial x_j}(x)\, \varepsilon_I, \qquad j = 1, \cdots, n.$$

$$\Leftrightarrow \quad D_x\omega(v) = \sum_I \underbrace{D_v\omega_I(x)}\, \varepsilon_I$$

the usual directional derivative of $\omega_I : \mathcal{U} \to \mathbb{R}$

Directional derivative is linear in the direction:

$$D_{\alpha v+\beta w} f(x) = \alpha D_v f(x) + \beta D_w f(x)$$

The function

$$\begin{array}{ccc} x & \longmapsto & D_x\omega \\ \cap & & \cap \\ \mathcal{U} & & L(\mathbb{R}^n, \mathrm{Alt}^p(\mathbb{R}^n)) \end{array}$$

← Yet another vector space

is a map from $\mathcal{U}$ to the linear space of all linear maps $\mathbb{R}^n \to \mathrm{Alt}^p(\mathbb{R}^n)$

$$\omega : \mathcal{U} \to \mathrm{Alt}^p(\mathbb{R}^n) = W$$

$$D\omega : \mathcal{U} \to L(\underbrace{\mathbb{R}^n}_{=V}, \underbrace{\mathrm{Alt}^p(\mathbb{R}^n)}_{=W})$$

$$D^2\omega : \mathcal{U} \to L(V, L(V, W)) \approx L^2(V \times V, W)$$ ← bilinear maps

$$D^3\omega : \mathcal{U} \to L(V, L(V, L(V, W))) \approx L^3(V \times V \times V, W)$$ ← trilinear maps

$$\vdots$$

Def: The exterior differential

$$d : \Omega^p(\mathcal{U}) \to \Omega^{p+1}(\mathcal{U})$$

is the linear operator

$$d_x\omega(\xi_1, \dots, \xi_{p+1}) = \sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)\underbrace{(\xi_1, \dots, \hat{\xi}_l, \dots, \xi_{p+1})}$$

"exterior derivative" "usual derivative"

the p-tuple made up of $(\xi_1, \dots, \xi_{p+1})$ with ξ_l skipped

Why is $d_x\omega$ a $(p+1)$-linear map?

Why is it alternating?

<u>Lemma</u> (2.7) : A k-linear map ω is alternating $\iff$ $\omega(\xi_1, \cdots, \xi_k) = 0$ for all k-tuples with $\xi_i = \xi_{i+1}$ for some $i \in \{1, \cdots, k-1\}$.

This is a simple consequence of the fact that $S(k)$ is generated by transpositions of the form $(i, i+1)$.

If $\xi_i = \xi_{i+1}$, then

$$\sum_{l=1}^{p+1} (-1)^{l-1} D_x\omega(\xi_l)(\xi_1, \cdots, \hat{\xi}_l, \cdots, \xi_{p+1})$$

$$= (-1)^{i-1} D_x\omega(\xi_i)(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{p+1})$$

$$+ (-1)^{i} D_x\omega(\xi_{i+1})(\xi_1, \cdots, \hat{\xi}_{i+1}, \cdots, \xi_{p+1})$$

$$= 0$$

This proves $d_x\omega$ is alternating.

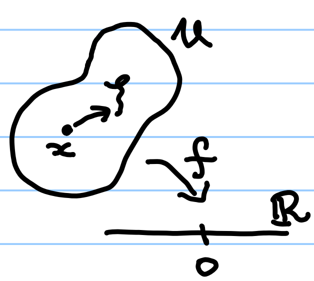
<u>Example</u> $\quad f : U \to \mathbb{R} \quad$ differential 0-form

$df \in \Omega^1(U)$

when $p=0$, $d = D$

$d_xf(\xi) = D_xf(\xi)$ (by definition of d)

$$= \frac{\partial f}{\partial x_1}(x)\xi_1 + \cdots + \frac{\partial f}{\partial x_n}(x)\xi_n$$



Special case: $f = x_i$ "i-th projection"

then $d_x x_i(\xi) = \xi_i$ (independent of x)

so $dx_i \in \Omega^1(U)$ is the constant map

$$x \mapsto \varepsilon_i \qquad (\varepsilon_i(\xi) = \xi_i)$$

So we can write

$$df = \frac{\partial f}{\partial x_1} \overset{= \varepsilon_1}{dx_1} + \cdots + \frac{\partial f}{\partial x_n} \overset{= \varepsilon_n}{dx_n}.$$

The next lemma gives a connection of $\wedge$ with d.

Lemma (3.4) If $\omega(x) = f(x)\, \underset{= \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}}{\varepsilon_I}$

then

$$d_x \omega = d_x f \wedge \varepsilon_I.$$

Proof Recall

$$D_x \omega(\xi) = \underbrace{D_x f(\xi)}_{d_x f(\xi)} \varepsilon_I$$

$\leftarrow$ from previous example

$$d_x \omega(\xi_1, \ldots, \xi_{p+1}) = \sum_{k=1}^{p+1} (-1)^{k+1} d_x f(\xi_k)\, \varepsilon_I(\xi_1, \ldots, \hat{\xi}_k, \ldots, \xi_{p+1})$$

$$= (d_x f \wedge \varepsilon_I)(\xi_1, \ldots, \xi_{p+1}) \qquad \square$$

The following formula is useful:

$$\underset{\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}}{\varepsilon_k \wedge \varepsilon_I} = \begin{cases} 0 & \text{if } k \in I \\ (-1)^r \varepsilon_J & \text{if } k \notin I \end{cases}$$

with r s.t. $i_r < k < i_{r+1}$

$J = (i_1, \ldots, i_r, k, \ldots, i_p)$.

Lemma $d \circ d = 0$

Let $\omega = f \varepsilon_I \in \Omega^p(\mathcal{U})$, $f \in \Omega^0(\mathcal{U})$

$$d\omega = df \wedge \varepsilon_I = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \varepsilon_j \wedge \varepsilon_I$$

$$d^2\omega = \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} \varepsilon_i \wedge \underbrace{(\varepsilon_j \wedge \varepsilon_I)}_{(\varepsilon_i \wedge \varepsilon_j)\wedge \varepsilon_I}$$

1-form

$\varepsilon_i \wedge \varepsilon_j = (-1)^{1\cdot 1} \varepsilon_j \wedge \varepsilon_i$

$$= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_j \partial x_i} \right) \varepsilon_i \wedge \varepsilon_j \wedge \varepsilon_I$$

$\varepsilon_i \wedge \varepsilon_i = 0$

$$= 0$$

$\square$

The exterior product $\wedge$ for forms extends pointwise to differential forms

$$\wedge : \mathrm{Alt}^p(\mathbb{R}^n) \times \mathrm{Alt}^q(\mathbb{R}^n) \to \mathrm{Alt}^{p+q}(\mathbb{R}^n)$$

$$\hookrightarrow \wedge : \Omega^p(\mathcal{U}) \times \Omega^q(\mathcal{U}) \to \Omega^{p+q}(\mathcal{U})$$

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)$$

For a smooth function $f \in C^\infty(\mathcal{U}, \mathbb{R}) \overset{\text{by convention}}{=} \Omega^0(\mathcal{U})$,

$$(f\omega_1) \wedge \omega_2 = f(\omega_1 \wedge \omega_2) = \omega_1 \wedge (f\omega_2), \quad \forall\, \omega_1 \in \Omega^p(\mathcal{U}),\ \omega_2 \in \Omega^q(\mathcal{U})$$

$$f \wedge \omega_1 = f\omega_1$$

<u>Lemma</u> (product rule for diff. forms) For $\omega_1 \in \Omega^p(\mathcal{U})$, $\omega_2 \in \Omega^q(\mathcal{U})$,

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$$

<u>Proof</u>: By bilinearity of $\wedge$, it suffices to check this formula for

$$\omega_1 = \underset{\substack{\uparrow \\ \text{smooth fcn.}}}{f}\, \varepsilon_{\underset{(i_1,\dots,i_p)}{I}}, \quad \omega_2 = \underset{\substack{\uparrow \\ \text{smooth fcn.}}}{g}\, \varepsilon_{\underset{(j_1,\dots,j_q)}{J}}.$$

$$\omega_1 \wedge \omega_2 = f\,\varepsilon_I \wedge g\,\varepsilon_J$$
$$= fg\, \underbrace{\varepsilon_I \wedge \varepsilon_J}$$

$\leftarrow$ constant $(p+q)$-form

$$d(\omega_1 \wedge \omega_2) = d(fg) \wedge \varepsilon_I \wedge \varepsilon_J$$

usual product rule + earliear comment $\longrightarrow$

$$= (g\,df + f\,dg) \wedge \varepsilon_I \wedge \varepsilon_J$$

$$= g\,df \wedge \varepsilon_I \wedge \varepsilon_J + f\,\underbrace{dg \wedge \varepsilon_I}_{=(-1)^p \varepsilon_I \wedge dg} \wedge \varepsilon_J$$

lemma 3.4 + anti-commutativity of $\wedge$ $\longrightarrow$

$$= d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2 \qquad \square$$

$$\Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \rightarrow \cdots \xrightarrow{d} \Omega^n(U)$$

$$d \circ d = 0$$

"de Rham complex of U"

Thm (uniqueness of d) There is only one such linear $d : \Omega^p(U) \to \Omega^{p+1}(U)$ $p = 0, 1, \cdots$, s.t.

(i) $df = \frac{\partial f}{\partial x_1}\varepsilon_1 + \cdots + \frac{\partial f}{\partial x_n}\varepsilon_n$, $f \in \Omega^0(U)$

(ii) $d \circ d = 0$

(iii) $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge d\omega_2$, $\omega_1 \in \Omega^p(U)$,

We have proved that the exterior derivative satisfies (i) – (iii). Property (i) says d has to be the usual derivative when applied to 0-forms. Properties (ii)–(iii) force d to do the same as the exterior derivatve to the higher order forms. See M&T for details.

When $U \subset \mathbb{R}^3$ (ie. $n=3$),

$$\begin{array}{ccccccc} C^\infty(U,\mathbb{R}) & \xrightarrow{\text{grad}} & C^\infty(U,\mathbb{R}^3) & \xrightarrow{\text{curl}} & C^\infty(U,\mathbb{R}^3) & \xrightarrow{\text{div}} & C^\infty(U,\mathbb{R}) \\ \wr\wr & & \wr\wr & & \wr\wr & & \wr\wr \\ \Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) \end{array}$$

the uniqueness theorem and the dimension count

$$\dim \mathrm{Alt}^p(\mathbb{R}^3) = \binom{3}{p}$$

suggests that if we use the right vector space isomorphisms

$$\mathbb{R}^1 \overset{?}{\longleftrightarrow} \mathrm{Alt}^0(\mathbb{R}^3) = \mathbb{R}^1$$

$$\mathbb{R}^3 \overset{?}{\longleftrightarrow} \mathrm{Alt}^1(\mathbb{R}^3)$$

$$\mathbb{R}^3 \overset{?}{\longleftrightarrow} \mathrm{Alt}^2(\mathbb{R}^3)$$

$$\mathbb{R}^1 \overset{?}{\longleftrightarrow} \mathrm{Alt}^3(\mathbb{R}^3)$$

then the operators grad, curl, div are exactly the same as the exterior derivative operators.

The following isomorphisms make this happens:

$$\mathbb{R} \ni a \quad \longleftrightarrow \quad a \in \mathrm{Alt}^0(\mathbb{R}^3)$$

$$\mathbb{R}^3 \ni (f_1, f_2, f_3) \longleftrightarrow f_1\varepsilon_1 + f_2\varepsilon_2 + f_3\varepsilon_3 \in \mathrm{Alt}^1(\mathbb{R}^3)$$

$$\mathbb{R}^3 \ni (g_1, g_2, g_3) \longleftrightarrow g_1\varepsilon_2\wedge\varepsilon_3 + g_2\varepsilon_3\wedge\varepsilon_1 + g_3\varepsilon_1\wedge\varepsilon_2 \in \mathrm{Alt}^2(\mathbb{R}^3)$$

$$\mathbb{R} \ni a \quad \longleftrightarrow \quad a\,\varepsilon_1\wedge\varepsilon_2\wedge\varepsilon_3 \in \mathrm{Alt}^3(\mathbb{R}^3)$$

If $f = (f_1, f_2, f_3) : \mathcal{U} \to \mathbb{R}^3$ is a smooth function then

$$f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3 \in \Omega^1(\mathcal{U})$$

and

$$\begin{aligned}
d(f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3)
&= df_1 \wedge \varepsilon_1 + df_2 \wedge \varepsilon_2 + df_3 \wedge \varepsilon_3 \\
&= \left(\frac{\partial f_1}{\partial x_1} \varepsilon_1 + \frac{\partial f_1}{\partial x_2} \varepsilon_2 + \frac{\partial f_1}{\partial x_3} \varepsilon_3 \right) \wedge \varepsilon_1 \\
&\quad + \left(\frac{\partial f_2}{\partial x_1} \varepsilon_1 + \frac{\partial f_2}{\partial x_2} \varepsilon_2 + \frac{\partial f_2}{\partial x_3} \varepsilon_3 \right) \wedge \varepsilon_2 \\
&\quad + \left(\frac{\partial f_3}{\partial x_1} \varepsilon_1 + \frac{\partial f_3}{\partial x_2} \varepsilon_2 + \frac{\partial f_3}{\partial x_3} \varepsilon_3 \right) \wedge \varepsilon_3 \\
&= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \right) \varepsilon_2 \wedge \varepsilon_3 + \left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \right) \varepsilon_3 \wedge \varepsilon_1 \\
&\quad + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \varepsilon_1 \wedge \varepsilon_2
\end{aligned}$$

$= \operatorname{curl} f$ under the identification of 2-forms and $\mathbb{R}^3$.

If $g = (g_1, g_2, g_3) : \mathcal{U} \to \mathbb{R}^3$ is smooth then

$$g_1 \varepsilon_2 \wedge \varepsilon_3 + g_2 \varepsilon_3 \wedge \varepsilon_1 + g_3 \varepsilon_1 \wedge \varepsilon_2 \in \Omega^2(\mathcal{U})$$

and

$$\begin{aligned}
d(g_1 \varepsilon_2 \wedge \varepsilon_3 + g_2 \varepsilon_3 \wedge \varepsilon_1 + g_3 \varepsilon_1 \wedge \varepsilon_2)
&= \frac{\partial g_1}{\partial x_1} \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 + \frac{\partial g_2}{\partial x_2} \varepsilon_2 \wedge \varepsilon_3 \wedge \varepsilon_1 + \frac{\partial g_3}{\partial x_3} \varepsilon_3 \wedge \varepsilon_1 \wedge \varepsilon_2 \\
&= \left(\frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3} \right) \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3 .
\end{aligned}$$

$= \operatorname{div} g$ under the identification of 3-forms and $\mathbb{R}$.

Def The pth de Rham cohomology group is the quotient vector space

$$H^p(U) = \frac{\ker(d: \Omega^p(U) \to \Omega^{p+1}(U))}{\operatorname{Im}(d: \Omega^{p-1}(U) \to \Omega^p(U))}$$

"closed p-forms"

"exact p-forms"

$H^p(U) = 0 \quad p < 0$

$$\begin{aligned} H^0(U) &= \ker(d: C^\infty(U, \mathbb{R}) \to \Omega^1(U)) \\ &= \{f: U \to \mathbb{R} \text{ smooth fcns with vanishing derivatives}\} \\ &= \{f: U \to \mathbb{R} \text{ locally constant}\} \end{aligned}$$

Lemma $H^0(\Omega) = \{f: U \to \mathbb{R} \mid f \text{ is constant on each connected component}\}$

$\dim H^0(\Omega) = \#$ of connected components of U (can be ∞)

Technicalities: For a general topological space, "connected components" and "path (connected) components" may be different. But no difference for an open set in $\mathbb{R}^n$.

An open set in $\mathbb{R}^n$ can have at most countably many connected components.

Proof: Exercise.

A closed p-form $\omega \in \Omega^p(\mathcal{U})$ gives a cohomology class, denoted by

$$[\omega] = \omega + d\Omega^{p-1}(\mathcal{U}) \in H^p(\mathcal{U}),$$

$[\omega] = [\omega'] \iff \omega - \omega'$ is exact.

Typically, $\dim \operatorname{Ker}(d : \Omega^p(\mathcal{U}) \to \Omega^{p+1}(\mathcal{U}))$,
$\dim \operatorname{Im}(d : \Omega^{p-1}(\mathcal{U}) \to \Omega^p(\mathcal{U})) = \infty$

but

$$\dim H^p(\Omega) < \infty.$$

$\wedge$ on forms descends to cohomology classes:

$$[\omega_1][\omega_2] := [\omega_1 \wedge \omega_2]$$

happens to be well-defined because:

$$(\omega_1 + d\eta_1) \wedge (\omega_2 + d\eta_2)$$

$$= \underbrace{\omega_1 \wedge \omega_2}_{\text{closed?}} + \underbrace{d\eta_1 \wedge \omega_2 + \omega_1 \wedge d\eta_2 + d\eta_1 \wedge d\eta_2}_{\text{exact?}}$$

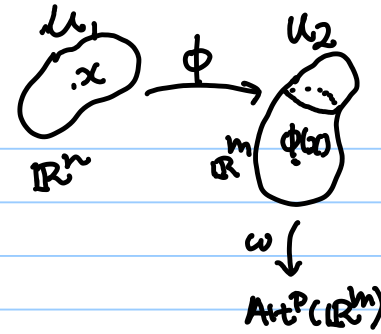
$$\text{Note: } d(\overset{p-1}{\eta_1} \wedge \overset{q}{\omega_2} + (-1)^p \overset{p}{\omega_1} \wedge \overset{q-1}{\eta_2} + \overset{p-1}{\eta_1} \wedge \overset{q}{d\eta_2})$$

$$= d\eta_1 \wedge \omega_2 + (-1)^{p-1} \eta_1 \wedge \underbrace{d\omega_2}_{=0} + (-1)^p \underbrace{d\omega_1}_{=0} \wedge \eta_2 + (-1)^p (-1)^p \omega_1 \wedge d\eta_2 + d\eta_1 \wedge d\eta_2 + (-1)^{p-1} d\eta_1 \wedge \underbrace{d^2\eta_2}_{=0}$$

$$d(\omega_1 \wedge \omega_2) = \underbrace{d\omega_1}_{=0} \wedge \omega_2 + (-1)^p \omega_1 \wedge \underbrace{d\omega_2}_{=0} = 0$$

Pullback

Def: Let $\mathcal{U}_1 \overset{\text{open}}{\subset} \mathbb{R}^n$, $\mathcal{U}_2 \overset{\text{open}}{\subset} \mathbb{R}^m$



$$\phi: \mathcal{U}_1 \to \mathcal{U}_2 \quad \text{smooth}$$

The induced morphism

$$\Omega^p(\phi) \ (\text{or } \phi^*) : \Omega^p(\mathcal{U}_2) \to \Omega^p(\mathcal{U}_1)$$

is defined by

$$\Omega^p(\phi)(\omega)_x = \underbrace{\mathrm{Alt}^p(\overset{:\mathbb{R}^n \to \mathbb{R}^m}{D_x\phi})}_{:\mathrm{Alt}^p(\mathbb{R}^m) \to \mathrm{Alt}^p(\mathbb{R}^n)} \circ \omega(\phi(x))$$

$$\Omega^0(\phi)(\omega)_x = \omega(\phi(x))$$

$\phi^*(\omega)$ is usually called the **pullback of ω** by ϕ.

Write:

$$\phi^*(\omega)_x(\underset{\xi_i \in \mathbb{R}^n}{\xi_1, \cdots, \xi_p}) = \omega_{\phi(x)}(D_x\phi(\xi_1), \cdots, D_x\phi(\xi_p))$$

If $\mathcal{U}_1 \xrightarrow{\phi} \mathcal{U}_2 \xrightarrow{\psi} \mathcal{U}_3$ smooth

by chain rule $D_x(\psi \circ \phi) = D_{\phi(x)}\psi \circ D_x\phi$,

easy to check that

$$(\psi \circ \phi)^* = \phi^* \circ \psi^*,$$

$$(id_{\mathcal{U}})^* = id_{\Omega^p(\mathcal{U})}.$$

Q: Why are we doing all these?

Example $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ $\quad \varepsilon_1, \cdots, \varepsilon_n \in \Omega^1(\mathcal{U}_1)$

$\mathcal{U}_1 \subset \mathbb{R}^n$, $\mathcal{U}_2 \subset \mathbb{R}^m$ $\quad \varepsilon_1, \cdots, \varepsilon_m \in \Omega^1(\mathcal{U}_2)$

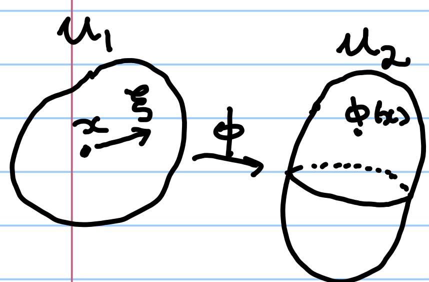
constant 1-forms

Claim: $\phi^*(\varepsilon_i) := d\phi_i$

ϕ_i ← ith component function of ϕ

d: exterior derivative (= usual derivative for 0-form)

Let $\xi \in \mathbb{R}^n$ and let the 1-form $\phi^*(\varepsilon_i)$ at x act on it.



$\phi^*(\varepsilon_i)$: pullback 1-form on $\mathcal{U}_1$ is not constant anymore, as the map ϕ is not constant

ε_i: constant 1-form on $\mathcal{U}_2$

$$\phi^*(\varepsilon_i)_x(\xi) = \varepsilon_i(D_x\phi(\xi))$$

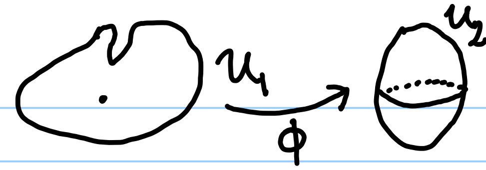
$$= \text{ith component of } D_x\phi(\xi) \ (\in \mathbb{R}^m)$$

$$= \sum_{k=1}^{n} \left.\frac{\partial \phi_i}{\partial x_k}\right|_x \underbrace{\xi^k}_{= \varepsilon_k(\xi)}$$

So if you don't mind the abuse of notation, we can write (as in M & T):

$$\phi^*(\varepsilon_i) = \sum_{k=1}^{n} \frac{\partial \phi_i}{\partial x_k} \varepsilon_k = d\phi_i .$$

pointwise $\wedge$, $\underset{\text{local}}{d}$, $*$ ← pullback



<u>Thm</u>

(i) $\phi^*(\omega \wedge \tau) = \phi^*(\omega) \wedge \phi^*(\tau)$

(ii) $\phi^*(f) = f \circ \phi$ if $f \in \Omega^0(\mathcal{U}_2) = C^\infty(\mathcal{U}_2, \mathbb{R})$

(iii) $d\phi^*(\omega) = \phi^*(d\omega)$

The proof is pretty routine (but not short and uses all the basic properties of $\wedge$ and d), see M&T.

The authors state without proof that the converse of this theorem is also true in the following sense:

Let $\phi : \mathcal{U}_1 \to \mathcal{U}_2$ be a fixed smooth map.

If $\phi' : \Omega^*(\mathcal{U}_2) \to \Omega^*(\mathcal{U}_1)$ ('$*$' means ϕ' maps any p-form in $\Omega^p(\mathcal{U}_2)$ to a p-form in $\Omega^p(\mathcal{U}_1)$, for any p.)

satisfies

(i) $\phi'(\omega \wedge \tau) = \phi'(\omega) \wedge \phi'(\tau)$

(ii) $\phi'(f) = f \circ \phi$ if $f \in \Omega^0(\mathcal{U}_2) = C^\infty(\mathcal{U}_2, \mathbb{R})$

(iii) $d\phi'(\omega) = \phi'(d\omega)$

then $\phi' = \phi^*$.

Idea: (ii) says ϕ' has to do the same as ϕ^* for 0-forms. Then (i), (iii) together will force ϕ' to also do the same as ϕ^* for the higher order forms. In fact, the proof of the orignal theorem is in the same spirit. (HW #2)

Notation:

Recall $dx_i = \varepsilon_i$

write $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, $I = (i_1, \cdots, i_p)$

instead of $\varepsilon_I = \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}$.

An arbitrary p-form can be written as

$$\omega(x) = \sum \omega_I(x)\, dx_I.$$

Ex:

(i) $\gamma : (a,b) \to \mathcal{U} \subset \mathbb{R}^n$ smooth curve

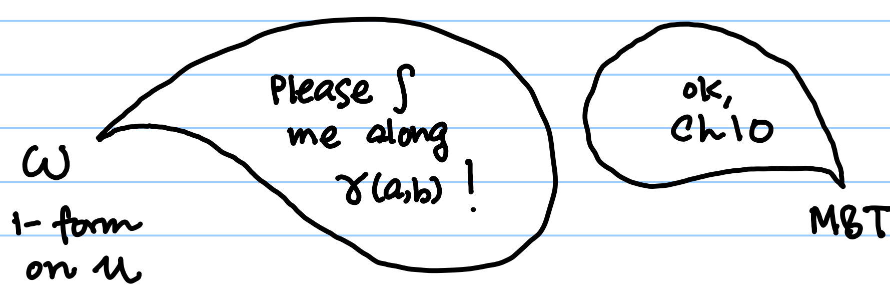
$\omega = f_1 dx_1 + \cdots + f_n dx_n$ 1-form on $\mathcal{U}$

Pullback $$\gamma^*(\omega)_t = \underbrace{\gamma^*(f_1)}_{f_1\circ\gamma} \wedge \underbrace{\gamma^*(dx_1)}_{d(\underbrace{\gamma^*(x_1)}_{\gamma_1}))} + \cdots + \underbrace{\gamma^*(f_n)}_{f_n\circ\gamma} \wedge \underbrace{\gamma^*(dx_n)}_{d(\underbrace{\gamma^*(x_n)}_{\gamma_n}))}$$

$$= \sum_{i=1}^{n} (f_i \circ \gamma)\, \underbrace{d\gamma_i}_{\gamma_i'(t)dt}$$

$$= \langle f(\gamma(t)), \gamma'(t) \rangle\, dt$$

$\uparrow$ usual inner-product in $\mathbb{R}^n$



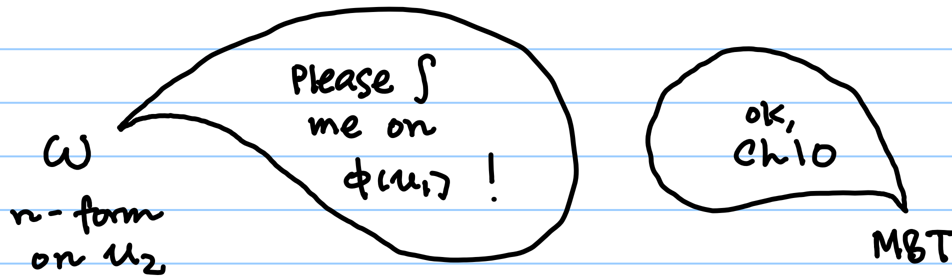
(ii) $\phi : \underset{\subset \mathbb{R}^n}{U_1} \to \underset{\subset \mathbb{R}^n}{U_2}$ smooth map between open sets in $\mathbb{R}^n$

$$\phi^*(dx_1 \wedge \cdots \wedge dx_n)$$

$$= \phi^*(dx_1) \wedge \cdots \wedge \phi^*(dx_n)$$

Lemma 2.13

$$= d(\underbrace{\phi^* x_1}_{\phi_1}) \wedge \cdots \wedge d(\underbrace{\phi^* x_n}_{\phi_n}) = \det(D_x\phi)\, dx_1 \wedge \cdots \wedge dx_n$$



The pullback on forms also descends to cohomology class :

For $\phi : U_1 \to U_2$

$H^p(\phi) : H^p(U_2) \to H^p(U_1)$ is defined by

$$H^p(\phi)[\omega] = [\phi^*\omega]$$

It is well-defined because

if $\omega \sim \omega'$ i.e. $\omega' = \omega + d\nu$

then $\phi^*\omega' = \phi^*\omega + \underbrace{\phi^* d\nu}_{d\phi^*\nu}$, so $[\phi^*\omega'] = [\phi^*\omega]$.

Furthermore,

$$H^{p+q}(\phi)(\underbrace{[\omega_1][\omega_2]}_{= [\omega_1 \wedge \omega_2]}) \text{ (tricky)}$$

$$= [\phi^*(\omega_1 \wedge \omega_2)]$$

$$= [\phi^*\omega_1 \wedge \phi^*\omega_2] = [\phi^*\omega_1][\phi^*\omega_2].$$

Theorem (Poincare's lemma — a partial converse of $d \circ d = 0$)

If $U \overset{\text{open}}{\subset} \mathbb{R}^n$ is star-shaped, then

$$H^p(U) = 0, \quad p = 1, \cdots, n$$

$$H^0(U) = \mathbb{R}.$$

<u>Proof</u>: Assume U is star-shaped w.r.t. the origin. We find a linear operator

$$S_p : \Omega^p(U) \to \Omega^{p-1}(U)$$

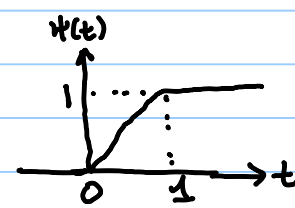
s.t.

$$(*) \quad \begin{cases} dS_p\omega + S_{p+1}d\omega = \omega, & \omega \in \Omega^p(U),\ p>0 \\ S_1 d\omega = \omega - \omega(0), & \omega \in \Omega^0(U) \end{cases}$$

This means

$$d\omega = 0 \begin{array}{l} \Rightarrow dS_p\omega = \omega \\ \Rightarrow \omega \equiv \omega(0) \end{array} \Bigg\} \ \omega \in \Omega^p(U) \quad \begin{array}{l} p>0 \\ p=0 \end{array}$$

$$\phi : U \times \mathbb{R} \to U, \quad (x,t) \mapsto \psi(t)x$$



$$\omega \in \Omega^p(U), \quad \omega = \sum_I h_I(x)\, dx_I$$

$$\phi^*(\omega) = \sum_I h_I(\psi(t)x)\,(x_{i_1}\psi'(t)dt + \psi(t)dx_{i_1}) \wedge \cdots \wedge (x_{i_p}\psi'(t)dt + \psi(t)dx_{i_p})$$

$\in$ $\Omega^p(U \times \mathbb{R})$

$$= \sum_{\substack{I \\ \shortparallel \\ (i_1,\cdots,i_p)}} \overbrace{f_I(x,t)}^{h_I(\psi(t)x)\,\psi(t)^p}\, dx_I + \sum_{\substack{J \\ \shortparallel \\ (j_1,\cdots,j_{p-1})}} g_J(x,t)\, dt \wedge dx_J$$

Define $\hat{S}_p : \Omega^p(\mathcal{U}\times\mathbb{R}) \to \Omega^{p-1}(\mathcal{U})$ by

$$\hat{S}_p\Big(\sum_I f_I(x,t)\,dx_I + \sum_J g_J(x,t)\,dt\wedge dx_J\Big)$$

$$:= \sum_J\Big(\int_0^1 g_J(x,t)\,dt\Big)\,dx_J$$

and $S_p : \Omega^p(\mathcal{U}) \to \Omega^{p-1}(\mathcal{U})$ by

$$S_p(\omega) := \hat{S}_p(\phi^*(\omega)).$$

Then we have for $p>0$

$$d\,S_p(\omega) + S_{p+1}\,d(\omega)$$

first think of it as a general p-form on $\Omega\times\mathbb{R}$

$$\sum_I f_I(x,t)\,dx_I + \sum_J g_J(x,t)\,dt\wedge dx_J$$

$$= \underbrace{d\,\hat{S}_p(\phi^*\omega)}_{=\ \sum_{J,i}\left(\frac{\partial}{\partial x_i}\int_0^1 g_J(x,t)\,dt\right)dx_i\wedge dx_J} + \underbrace{\hat{S}_{p+1}(d\,\phi^*\omega)}_{=\ \sum_I\left[\int_0^1 \frac{\partial f_I(x,t)}{\partial t}\,dt\right]dx_I \ -\ \sum_{J,i}\left[\int_0^1 \frac{\partial g_J(x,t)}{\partial x_i}\,dt\right]dx_i\wedge dx_J}$$

$$= \sum_I \underbrace{\Big[\int_0^1 \frac{\partial f_I(x,t)}{\partial t}\,dt\Big]}_{f_I(x,1)-f_I(x,0)}\,dx_I$$

$$\big(f_I(x,t) = h_I(\psi(t)x)\,\psi(t)^p\big) \quad p>0$$

$$= \sum_I h_I(x)\,dx_I = \omega$$

If ω is a 0-form,

$$S_1(d\omega) = \hat{S}_1(d\underbrace{\phi^*\omega}_{\omega(\psi(t)x)}) = \int_0^1 \frac{d}{dt}\,\omega(\psi(t)\cdot x)\,dt$$

$$= \omega(x) - \omega(0).$$

$\square$

Remark: It seems more natural to directly define $S_p : \Omega^p(U) \to \Omega^{p-1}(U)$ as

$$\boxed{\begin{aligned} S_p\omega(x) := \sum_{\substack{I \\ \| \\ (i_1,\dots,i_p) \\ i_1<\dots<i_p}} \sum_{\alpha=1}^{p} (-1)^{\alpha-1} \left[\int_0^1 t^{p-1}\,\omega_I(tx)\,dt\right] x_{i_\alpha} \\ dx_{i_1}\wedge\cdots\wedge\widehat{dx_{i_\alpha}}\wedge\cdots\wedge dx_{i_p} \\ \text{when } \omega = \sum_I \omega_I\, dx_I \quad (dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_p}), \end{aligned}}$$

and check that this S_p satisfies (*).