CH 7 : Applications

Note Title 5/1/2017

Any result established in this chapter can be shown using homology. The arguments are all about using H^p or H_p (they contain the same information) to distinguish topological spaces. As such, you won't see any mention of differential forms.

Standard notations

$$D^n = \{x \in \mathbb{R}^n : \|x\| \le 1\} \qquad \text{n-ball}$$

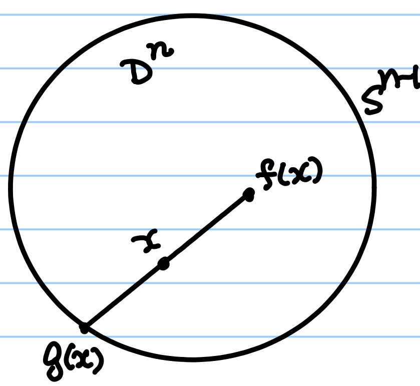
$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \qquad \text{(n-1)-sphere}$$

Brouwer's fixed point theorem (1912):

Every continuous map $f: D^n \to D^n$ has a fixed point.

(i.e. $\exists\, x \in D^n$ s.t. $f(x) = x$.)

Proof: If not, $\exists$ a continuous map $g: D^n \to S^{n-1}$ that fixes the boundary, i.e. $g(x) = x \ \forall x \in S^{n-1}$.



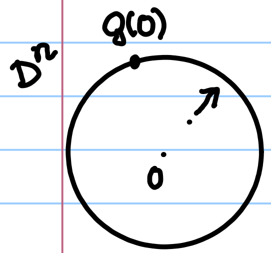
The next lemma shows this is not possible.

Lemma There is no continuous $g: D^n \to S^{n-1}$ with $g|_{S^{n-1}} = id_{S^{n-1}}$.

Proof The $n=1$ case follows easily from the connectedness of $D^1 = [-1,1]$.

Assume $n \geq 2$. Consider:

$$\mathrm{id}_{\mathbb{R}^n - \{0\}} : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\} \qquad x \mapsto x$$



$$r : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\} \qquad x \mapsto x/\|x\|$$

They are homotopic via $F(x,t) = (1-t)x + t\, x/\|x\|$.

If g is of the indicated type, consider

$$G(x,t) := g(t \cdot r(x)), \quad t \in [0,1], \; x \in \mathbb{R}^n - \{0\}$$

constant map $\to$ $G(x,0) = g(0) \quad \forall x \in \mathbb{R}^n - \{0\}$

$$G(x,1) = r(x) = x/\|x\|.$$

G is continuous, so $r \simeq$ a constant map

This implies $\mathbb{R}^n - \{0\}$ is contractible, so has trivial cohomology $\forall p \geq 1$. But we proved in Ch6 that this is not the case: $H^{n-1}(\mathbb{R}^n - \{0\}) = \mathbb{R}$.

$\square$

Remark: The first homotopy group (a.k.a. the fundamental group) can be used to prove the Brouwer's fixed pt theorem for $n=2$. (see Munkres' standard textbook on Pointset topology.) The fundamental group, based only on loop homotopy, cannot handle the theorem for any dimension $n > 2$.

Hairy Ball theorem

$$S^n \subset \mathbb{R}^{n+1}$$

$$T_x S^n = \{\alpha'(0) : \alpha : (-\varepsilon, \varepsilon) \to S^n, \alpha(0) = x\}$$

$$\alpha(t) \in S^n \iff \langle \alpha(t), \alpha(t) \rangle = 1 \Rightarrow \langle \alpha'(t), \alpha(t) \rangle = 0$$
$$\Rightarrow \langle \alpha'(0), x \rangle = 0$$

so

$$T_x S^n = \{x\}^\perp.$$

A (continuous) vector field on S^n is a continuous map

$$v : S^n \to \mathbb{R}^{n+1}$$

s.t.

$$v(x) \in T_x S^n \quad \forall x \in S^n.$$

$$\boxed{\text{Theorem: The sphere } S^n \text{ has a tangent vector field with } v(x) \neq 0 \ \ \forall x \in S^n \iff n \text{ is odd.}}$$

<u>Proof</u>: Assume that such a non-vanishing vector field exists.

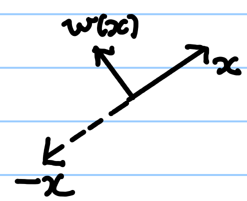
Set

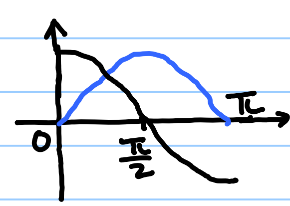
$$w(x) := v(x/\|x\|), \quad x \in \mathbb{R}^{n+1} - \{0\}.$$

So

$$w(x) \neq 0 \quad \text{and} \quad w(x) \perp x.$$

Consider $F(x,t) = \cos(\pi t)\, x + \sin(\pi t)\, w(x)$





$$F(x,t) \neq 0 \quad \forall x \in \mathbb{R}^n - \{0\}, \quad t \in [0,1]$$

$$F(x,0) = x, \quad F(x,1) = -x.$$

so $f_0 := \mathrm{id}_{\mathbb{R}^{n+1} - \{0\}} \simeq f_1 =$ the antipodal map

$$f_1(x) = -x.$$

$$\text{Recall} \quad \overset{\text{id}}{\overset{\shortparallel}{f_0}} \simeq f_1 : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$$

$$\Rightarrow \quad \underset{\text{id}}{\underset{\shortparallel}{f_0^*}} = f_1^* : H^p(\mathbb{R}^{n+1} - \{0\}) \to H^p(\mathbb{R}^{n+1} - \{0\})$$

Choose $p = n$

$$\begin{array}{ccccc} f_1^* : & H^n(\mathbb{R}^{n+1} - \{0\}) & \to & H^n(\mathbb{R}^{n+1} - \{0\}). & \text{— ①} \\ \shortparallel & \cong & & \cong & \\ \text{id} & \mathbb{R} & & \mathbb{R} & \end{array}$$

On the other hand, it is shown in ch6 that

if $A \in GL(n+1)$, $f_A : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}$
$f_A(x) = Ax$ is a diffeom.

then

$$\begin{array}{ccccc} f_A^* : & H^n(\mathbb{R}^{n+1} - \{0\}) & \to & H^n(\mathbb{R}^{n+1} - \{0\}) & (n \geq 1) \\ & \cong & & \cong & \\ & \mathbb{R} & & \mathbb{R} & \end{array}$$

is multiplication by $\det A / |\det A| \in \{\pm 1\}$.

In our case, the antipodal map f_1 is the same as f_{-I}, so

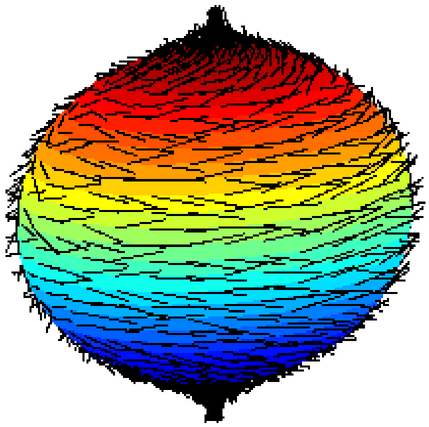
f_1^* is multiplication by $(-1)^{n+1}$ — ②

①, ② $\Rightarrow$ $(-1)^{n+1} = 1$ $\Rightarrow$ n is odd.

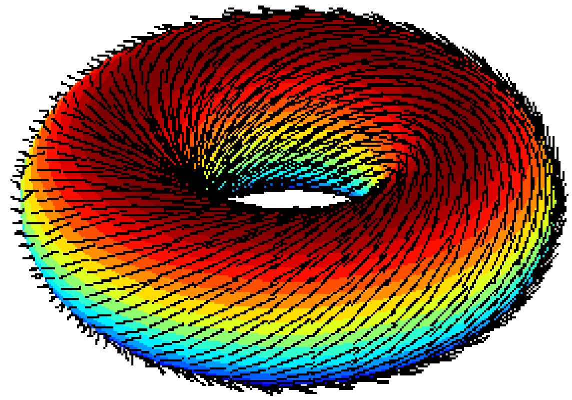
Conversely, if n is odd, it is easy to see that a non-vanishing tangent vector field exists:

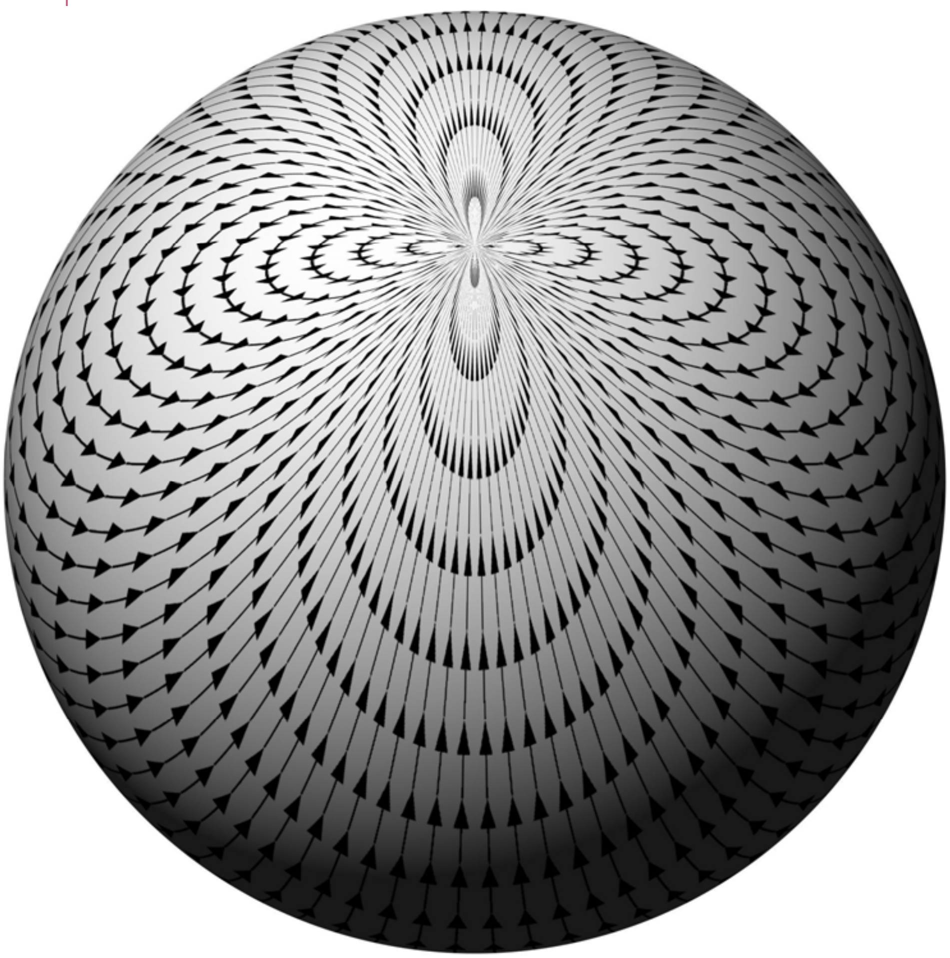
$$v(x_1, x_2, \cdots, x_{2m-1}, x_{2m}) := (-x_2, x_1, \cdots, -x_{2m}, x_{2m}). \qquad \square$$

"You can't comb the hair on a sphere."



But you can on a torus.





a vector field on S^2 with only one zero

Index of the zero
$=$
2
$=$
Euler-char. of S^2.

Later we will see that any such isolated zero of a tangent vector field on a manifold M can be assigned an integer called the index. The Poicaré-Hopf theorem says

$$\text{Euler characteristics of } M = \sum \text{index}.$$

For S^n, n odd, there are non-vanishing vector fields and there is the well-known "vector-field problem":

What is the maximal number of linearly independent tangent vector fields one may have on S^n?

Answer (Adams' theorem):
$2^b + 8a - 1$, where $n+1 = (2c+1)\cdot 2^{4a+b}$, $0 \leq b < 4$

n	1	3	5	7	9	11	13	15	17	19	21	23	25	27
#	1	3	1	7	1	3	1	8	1	3	1	7	1	3

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VECTOR FIELDS ON SPHERES

BY J. F. ADAMS

(Received November 1, 1961)

1. Results

The question of vector fields on spheres arises in homotopy theory and in the theory of fibre bundles, and it presents a classical problem, which may be explained as follows. For each n, let S^{n-1} be the unit sphere in euclidean n-space R^n. A vector field on S^{n-1} is a continuous function v assigning to each point x of S^{n-1} a vector $v(x)$ tangent to S^{n-1} at x. Given r such fields $v_1, v_2, \cdots, v_r$, we say that they are linearly independent if the vectors $v_1(x), v_2(x), \cdots, v_r(x)$ are linearly independent for all x. The problem, then, is the following: for each n, what is the maximum number r of linearly independent vector fields on S^{n-1}? For previous work and background material on this problem, we refer the reader to [1, 10, 11, 12, 13, 14, 15, 16]. In particular, we recall that if we are given r linearly independent vector fields $v_i(x)$, then by orthogonalisation it is easy to construct r fields $w_i(x)$ such that $w_1(x), w_2(x), \cdots, w_r(x)$ are orthonormal for each x. These r fields constitute a cross-section of the appropriate Stiefel fibering.

The strongest known positive result about the problem derives from the Hurwitz-Radon-Eckmann theorem in linear algebra [8]. It may be stated as follows (cf. James [13]). Let us write $n = (2a+1)2^b$ and $b = c + 4d$, where a, b, c and d are integers and $0 \leq c \leq 3$; let us define $\rho(n) = 2^c + 8d$. Then there exist $\rho(n) - 1$ linearly independent vector fields on S^{n-1}.

It is the object of the present paper to prove that the positive result stated above is best possible.

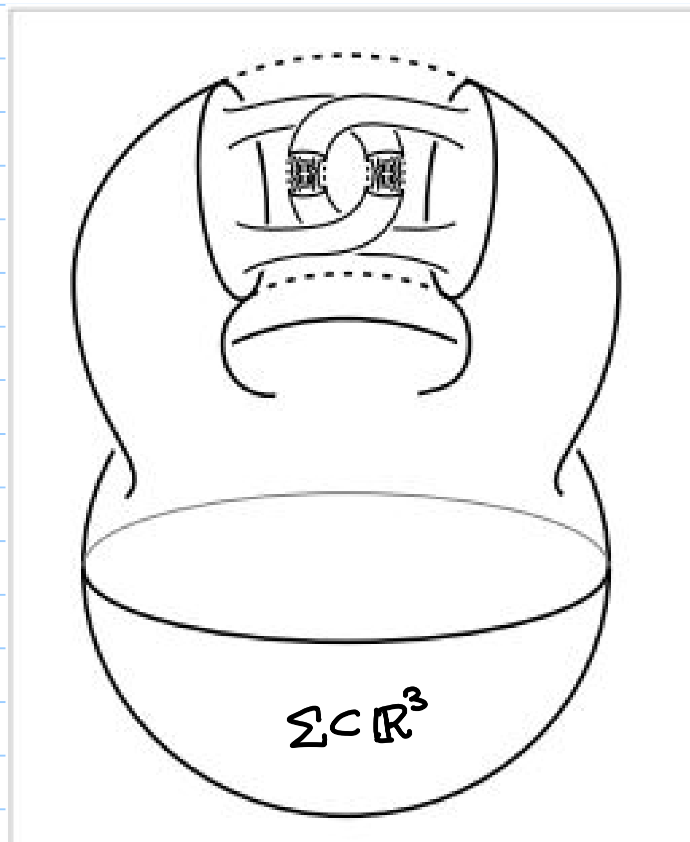
THEOREM 1.1. *If $\rho(n)$ is as defined above, then there do not exist $\rho(n)$ linearly independent vector fields on S^{n-1}.*

Besides the fixed point theorem and the hairy ball theorem, Brouwer also proved:

(Jordan-Brouwer separation theorem) If $\Sigma \subseteq \mathbb{R}^n$ $(n \geq 2)$ is homeomorphic to S^{n-1} then

(i) $\mathbb{R}^n - \Sigma$ has precisely 2 connected components $\mathcal{U}_1$ and $\mathcal{U}_2$, where $\mathcal{U}_1$ is bounded and $\mathcal{U}_2$ is unbounded.

(ii) $\partial \mathcal{U}_1 = \Sigma = \partial \mathcal{U}_2$.



(Invariance of domain) If $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is injective and continuous, then $f(\mathcal{U})$ is open in $\mathbb{R}^n$ and f maps $\mathcal{U}$ homeomorphically to $f(\mathcal{U})$.