Regular Surfaces: brief discussions on global and local issues

Note Title 1/17/2017

From regular curves to regular surfaces:

(I) a change in dimension, of course

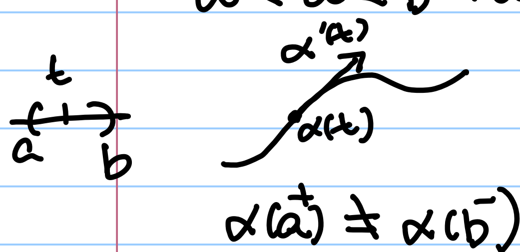
curves — (smooth) 1-D objects living in n-D, $n \geq 2$

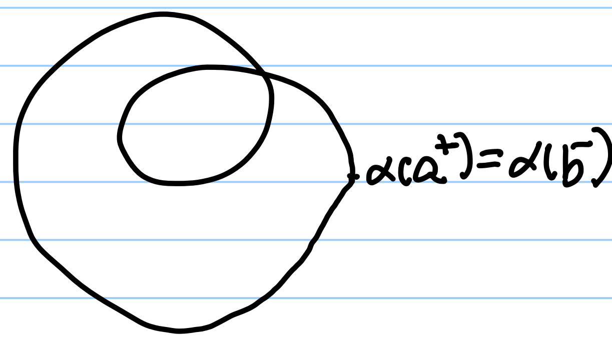
surfaces — (smooth) 2-D objects living in n-D, $n \geq 3$

regular parameterized curve

$\alpha : (a,b) \to \mathbb{R}^n$, $\alpha'(t) \neq 0$, $\forall t$

$-\infty < a < b < \infty$





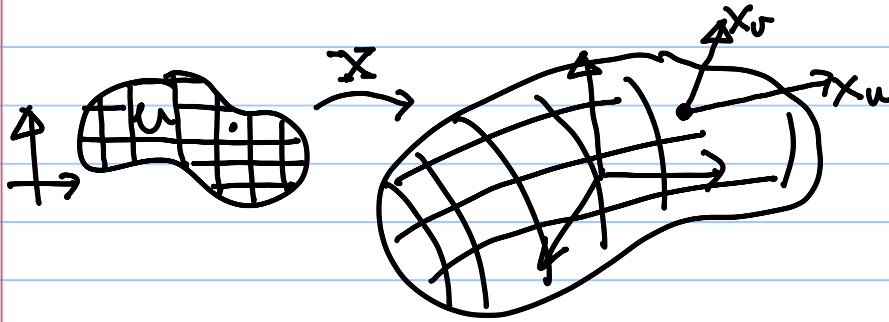
regular parameterized surface $\parallel$ $[X_u \; X_v]$

$X : \mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^2 \to \mathbb{R}^n$, $dX(u,v)$ rank 2

"full rank"

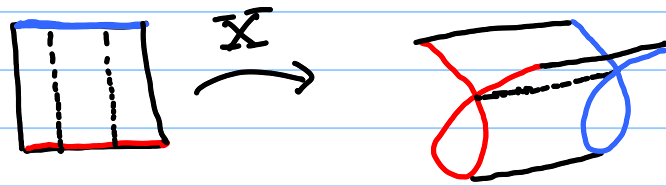
"injective"

∂_u, ∂_v are 2 linearly independent vectors in $\mathbb{R}^3$



Such a map α is "<u>locally injective</u>", but may not be (globally) injective.

Global Issues :

e.g. 

<u>Ex</u>: turn this picture into a rigorous example.

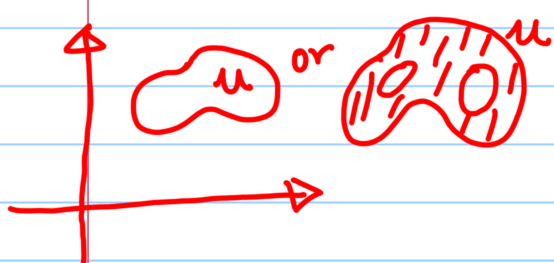
Allowing for surfaces that can self-intersect, as we allowed for curves that can self-intersect, is perfectly natural.

A much bigger problem :

any curve can be expressed as the image of some $\alpha : (a,b) \to \mathbb{R}^n$

<u>but</u>

not every surface can be expressed as the image of a single $\mathbf{X} : \mathcal{U} \to \mathbb{R}^n$



no matter how complicated you make this open set $\mathcal{U}$ is!

e.g.

etc.

we can only parameterize a surface <u>locally</u>.

Subtle point: the topology of an open set in $\mathbb{R}^2$ can be complicated, but even so that is <u>not</u> enough to deal with the different topologies of surfaces.

Have to generalize the concept of parameterized regular surface .....

Ⓘ Definition:

A subset $S \subset \mathbb{R}^3$ is a ~~parameterized~~ regular surface if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map

$$\underset{\text{open in } \mathbb{R}^2}{\mathbf{X}: \mathcal{U}} \to \underset{\text{open in } \mathbb{R}^3}{V \cap S}$$

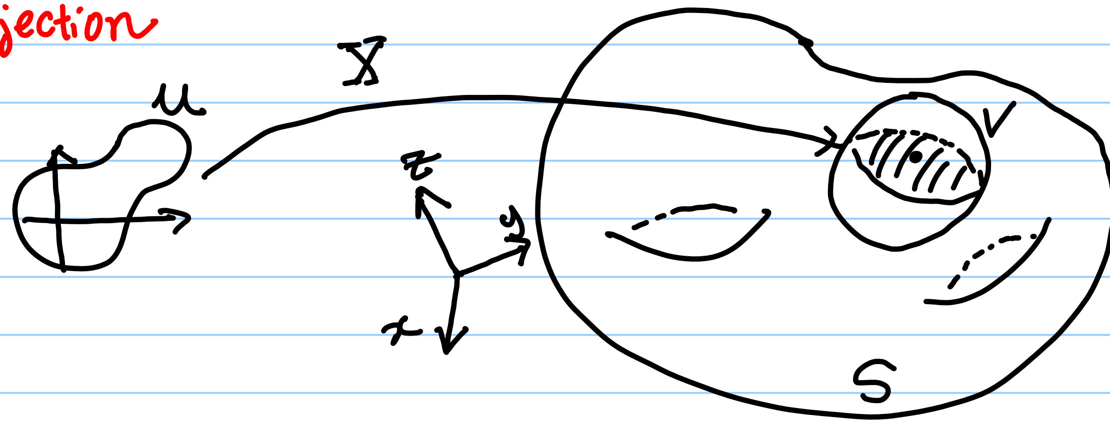
"a local parameterization"

s.t.

1. $\mathbf{X}$ is C^∞
2. $\mathbf{X}$ is a homeomorphism
3. $d\mathbf{X}(u,v)$ is injective for all $(u,v) \in \mathcal{U}$.

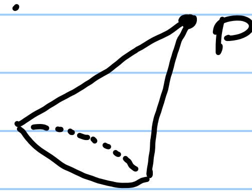
{ Not a easy condition to work with. Fortunately 2' is enough.

2'. $\mathbf{X}$ is a bijection

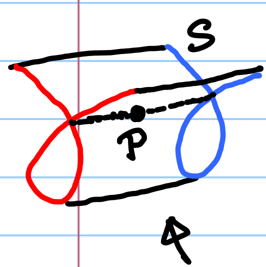


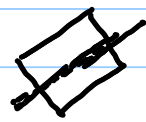
Thm: $1+2+3 \iff 1+2'+3$.

Condition 3 guarantees tangent planes, disallows cusps :

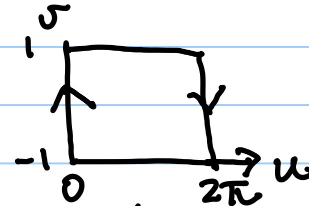


Condition 2/2' disallows self-intersection :





$S \cap$ (any open ball around P) does not 'look like' (i.e. not <u>homeomorphic</u>) an open set in $\mathbb{R}^2$



$$\begin{bmatrix} (2 - v\sin\frac{u}{2})\sin u \\ (2 - v\sin\frac{u}{2})\cos u \\ v\cos(\frac{u}{2}) \end{bmatrix}$$

möbius band : allowed
Klein bottle : not allowed

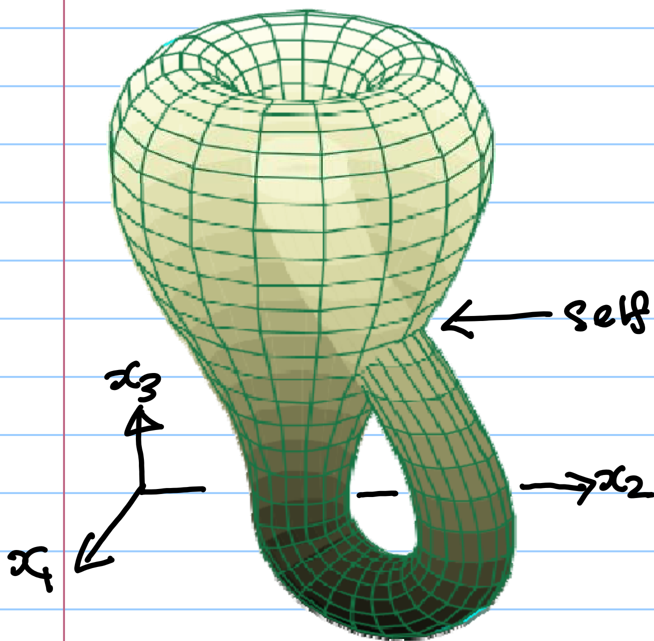
Still can't deal with all topologies !

🚩 This problem will go away when we work with manifolds.

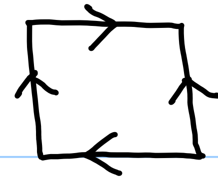
Teaser : Klein bottle can be first described as a <u>2-dimension manifold</u>, and subsequently be shown that it

(i) can be embedded in $\mathbb{R}^4$
(ii) can be immersed in $\mathbb{R}^3$
(iii) cannot be embedded in $\mathbb{R}^3$

(ii)



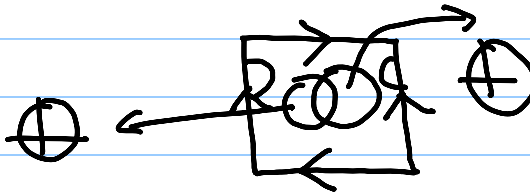
self intersection



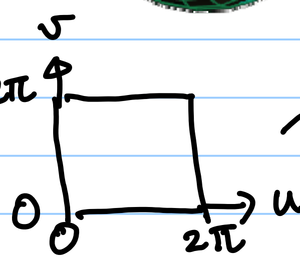
Klein bottle as an abstract <u>topological</u> <u>space</u>.

extra structure

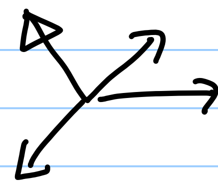
2 – <u>manifold</u>



(i)



$\longrightarrow \mathbb{R}^4$



$$(u,v) \mapsto \begin{bmatrix} (r\cos v + a)\cos u \\ (r\cos v + a)\sin u \\ r\sin v \cos u/2 \\ r\sin v \sin u/2 \end{bmatrix}$$

[This concludes my teaser, will come back to this example with rigor.]

Associated with (i) is the (easy) extension of regular surface in $\mathbb{R}^3$ to :

" a k-dimensional regular surface in $\mathbb{R}^N$ "

intrinsic dimension

$(2,3) \longrightarrow (k, N)$

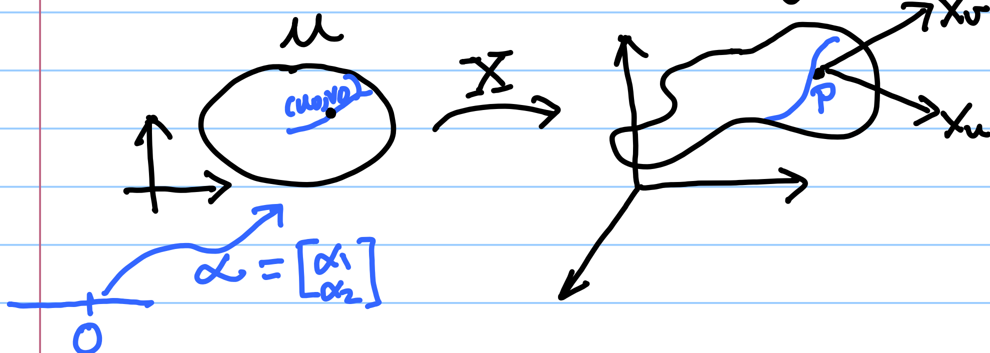
ambient dimension

Local Issues :

$$\mathbf{X} : \mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^2 \to \mathbb{R}^3 , \quad \mathbf{X}(u,v) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

$$d\mathbf{X}(u_0, v_0) = \begin{array}{c} \begin{array}{cc} \mathbf{X}_u & \mathbf{X}_v \end{array} \\ \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}\Bigg|_{(u_0,v_0)} \end{array}$$

has rank $= 2$



tangent plane at $p = \mathbf{X}(u_0, v_0)$ is :

$$\mathrm{span}(\mathbf{X}_u, \mathbf{X}_v) = \{\text{all linear combinations } a\mathbf{X}_u + b\mathbf{X}_v\}$$

Proof:

$$= \{\text{all } (\mathbf{X} \circ \alpha)'(0) \text{ with } \alpha : (-\varepsilon, \varepsilon) \to \mathcal{U},\ \alpha(0) = (u_0, v_0)\}$$

$$\| \\ T_pS$$

$(\supseteq)$ $(\mathbf{X} \circ \alpha)' = D\mathbf{X}(\alpha(0)) \cdot \alpha'(0)$

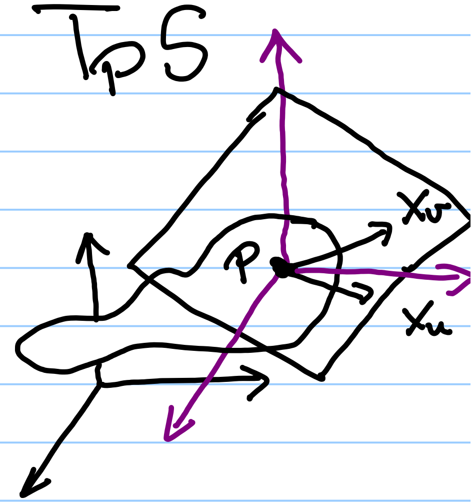
$= \alpha_1'(0)\mathbf{X}_u + \alpha_2'(0)\mathbf{X}_v .$

$(\subseteq)$ For given $a, b \in \mathbb{R}$, choose $\alpha(t) = \begin{pmatrix} u_0 + at \\ v_0 + bt \end{pmatrix}$

then

$(\mathbf{X} \circ \alpha)'(0) = a\mathbf{X}_u + b\mathbf{X}_v .$ ∎

"T_pS is a 2-D subspace of $\mathbb{R}^3_p$."



Idea: As in the study of curves, find a better way to describe the surface locally.

Recall: graph of a function from calculus

$$h: \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^1 \quad \leftarrow \text{not } 3$$

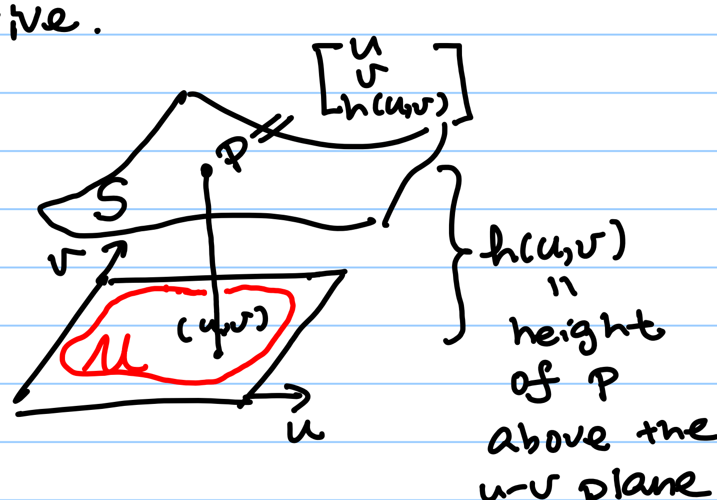
$$S = \text{graph}(h)$$
$$= \left\{ \begin{bmatrix} u \\ v \\ h(u,v) \end{bmatrix} : (u,v) \in \mathcal{U} \right\}$$

$(u,v) \overset{X}{\longmapsto} \begin{bmatrix} u \\ v \\ h(u,v) \end{bmatrix}$ is always a regular parameterization of S.

$$DX = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ h_u & h_v \end{bmatrix}$$ has rank 2 everywhere

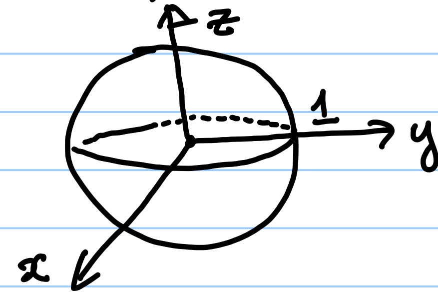
Not only is X locally injective, it is globally injective.

$S = \text{graph}(h)$ is always a regular surface (For any $p \in S$, simply choose $V_p = \mathbb{R}^3$)



Is it true that every regular surface is the graph of a function $h : \mathcal{U} \to \mathbb{R}$.

Definitely <u>not</u> globally



Maybe locally ?

For any given point $p \in S$, can we always find a neighborhood of p, V, s.t.

$$V \cap S = \text{graph}(h)$$

$$= \left\{ \begin{bmatrix} x \\ y \\ h(x,y) \end{bmatrix} : (x,y) \in \mathcal{U} \right\}$$

for some $h : \mathcal{U} \to \mathbb{R}$?

E.g. S = unit sphere in $\mathbb{R}^3$,
Seems possible for most p.

What about $p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$?

Doesn't work, as any neighborhood of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ involves points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big\} \sim \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

← both positive and negative

What to do ?

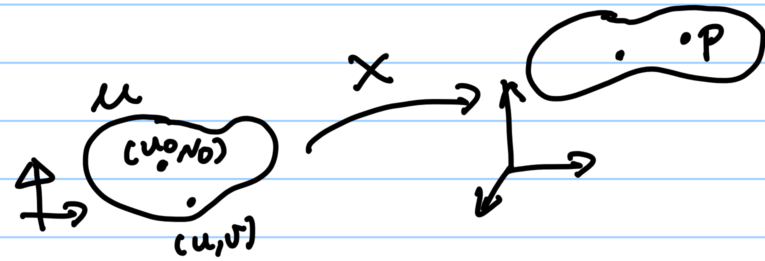
How about $\begin{bmatrix} h(y,z) \\ y \\ z \end{bmatrix}$ or $\begin{bmatrix} x \\ h(x,z) \\ z \end{bmatrix}$?

When to use which and
how to argue rigorously?

🚩 Local linear approximation:

When $(u,v) \approx (u_0, v_0)$,

$$\mathbf{X}(u,v) \approx \overbrace{\mathbf{X}(u_0,v_0)}^{P} + d\mathbf{X}(u_0,v_0)\begin{bmatrix} u-u_0 \\ v-v_0 \end{bmatrix}$$



$d\mathbf{X} = \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix}$ is rank 2 means at least one of
$\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix}$, $\begin{bmatrix} x_u & x_v \\ z_u & z_v \end{bmatrix}$, $\begin{bmatrix} y_u & y_v \\ z_u & z_v \end{bmatrix}$
is invertible.

Depending on which one, compose $\mathbf{X}$ with one of the following projection maps:

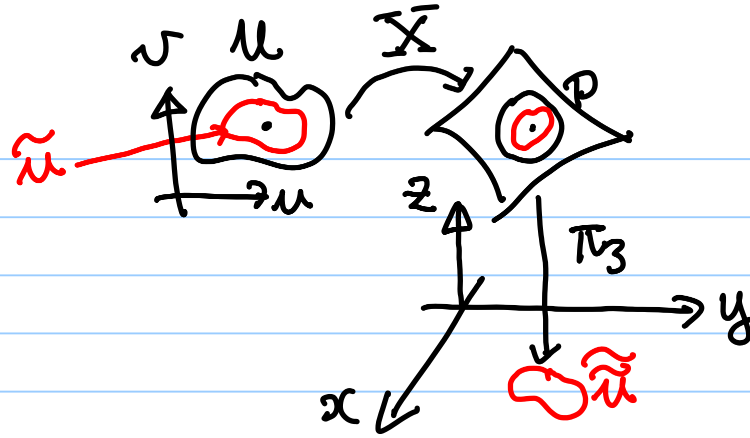
$\pi_3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}$, $\pi_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x \\ z \end{bmatrix}$, $\pi_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} y \\ z \end{bmatrix}$

$$\pi_i \circ \mathbf{X} : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}^{\cancel{3}\,2}$$

$$\begin{aligned} d(\pi_i \circ \mathbf{X}) &= d\pi_i \cdot d\mathbf{X} \quad \text{(chain rule)} \\ &= \pi_i \cdot d\mathbf{X} \quad (\pi_i \text{ is already linear}) \end{aligned}$$

which is <u>invertible</u>.

By the inverse function theorem, the nonlinear $\pi_i \circ \mathbf{X}$ is <u>locally invertible.</u>



i.e.

$\exists\ \tilde{\mathcal{U}} \subset \mathcal{U}$
$\quad \ni (u_0, v_0)$

so that

$\pi_i \circ \bar{X} : \tilde{\mathcal{U}} \to \underbrace{\pi_i \circ \bar{X}(\tilde{\mathcal{U}})}_{\tilde{\tilde{\mathcal{U}}}}$ has a differentiable inverse.

so

$X_i(u,v)$ can now be reparameterized as a height function over the

x-y plane	$i=3$
y-z plane	$i=1$
x-z plane	$i=2$

i.e. $\quad X_i \circ (\pi_i \circ \bar{X})^{-1} : \tilde{\tilde{\mathcal{U}}} \to \mathbb{R}$

$$\bar{X}(\tilde{\mathcal{U}}) = \begin{cases} \left\{ \begin{bmatrix} x \\ y \\ X_3 \circ (\pi_3 \circ \bar{X})^{-1}(x,y) \end{bmatrix} : (x,y) \in \tilde{\tilde{\mathcal{U}}} \right\} & \text{if } i=3 \\ \left\{ \begin{bmatrix} x \\ X_2 \circ (\pi_2 \circ \bar{X})^{-1}(x,z) \\ z \end{bmatrix} : (x,z) \in \tilde{\tilde{\mathcal{U}}} \right\} & \text{if } i=2 \\ \left\{ \begin{bmatrix} X_1 \circ (\pi_1 \circ \bar{X})^{-1}(y,z) \\ y \\ z \end{bmatrix} : (y,z) \in \tilde{\tilde{\mathcal{U}}} \right\} & \text{if } i=1 \end{cases}$$

Recap :

graph of a bivariate function $\underset{\text{"locally"}}{\overset{\text{always}}{\rightleftarrows}}$ regular surface in $\mathbb{R}^3$

[good to be able to encode information with one number instead of three]

But

$z = f(x,y)$, $y = g(x,z)$ or $x = h(y,z)$

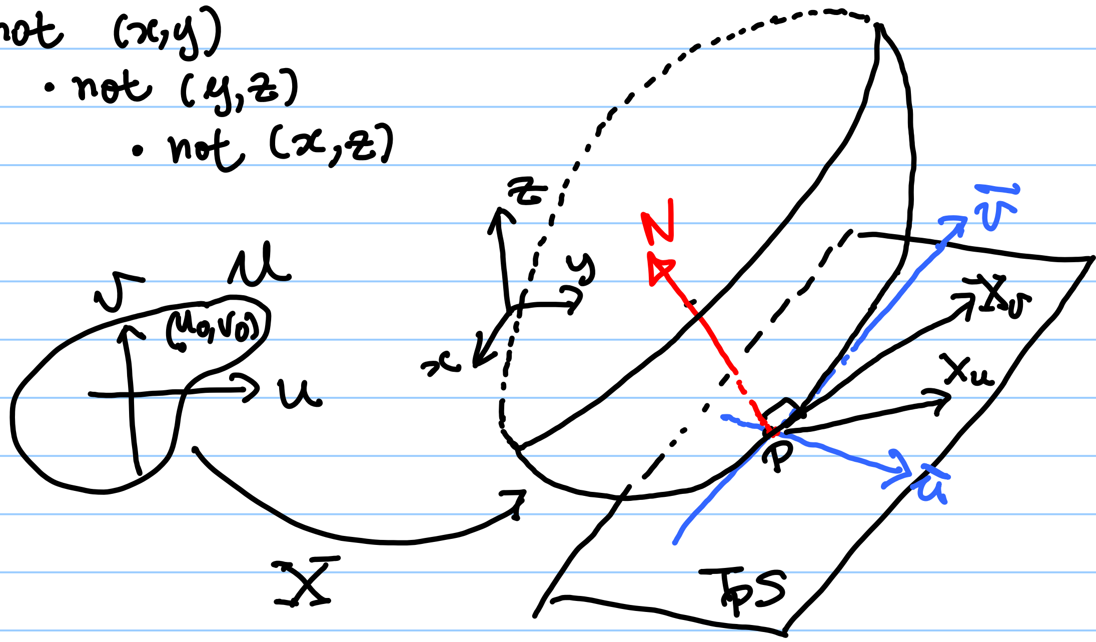
are still not the most convenient height function to describe the local geometry of S. Well, why should they, afterall? The x, y, z-axes can be quite meaningless to the shape of S.

Wouldn't it be better to locally parameterize S (near a point $p \in S$) by the height of S over the tangent plane T_pS?

[We shall see what 'better' means.]

parameters to use:

- not (u,v)
  - not (x,y)
    - not (y,z)
      - not (x,z)



- but $(\bar{u}, \bar{v})$.

Note: While $\operatorname{span}(X_u, X_v) = T_pS$, $\{X_u, X_v\}$ is not an o.n. basis.

Choose an o.n. basis $\{e_1, e_2\}$, so

$$N = e_1 \times e_2 = X_u \times X_v / \|X_u \times X_v\|$$

Height of $\mathbf{X}(u,v)$ above T_pS

$$= \langle N, \mathbf{X}(u,v) - \overset{=p}{\mathbf{X}(u_0,v_0)} \rangle$$

$$\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{X}(u,v) - p, e_1 \rangle \\ \langle \mathbf{X}(u,v) - p, e_2 \rangle \end{bmatrix}$$

Note : The point p has been chosen and fixed, the frame $\{e_1, e_2, N\}$ does not move.

<u>NOT</u> a "moving frame" (for now.)

Instead, (u, v) moves around (u_0, v_0)
$(\bar{u}, \bar{v})$ moves around $(0, 0)$
$X(u,v)$ moves around $p = X(u_0, v_0)$

[But, we shall change this point of view later.]

In the following, assume

$$X(u_0, v_0) = p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

just so that I can write "$X(u,v)$" instead of "$X(u,v) - X(u_0, v_0)$".

Aside :
If $X_u = a\, e_1 + b\, e_2$
$X_v = c\, e_1 + d\, e_2$

then the "change of basis matrix" is :

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \langle X_u, e_1 \rangle, & \langle X_u, e_2 \rangle \\ \langle X_v, e_1 \rangle, & \langle X_v, e_2 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} e_1 & e_2 \end{bmatrix}^T \begin{bmatrix} X_u, & X_v \end{bmatrix}.$$

---

① Height $= \langle \bar{X}(u,v), N \rangle = N^T X(u,v)$

② $$\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \langle \bar{X}(u,v), e_1 \rangle \\ \langle \bar{X}(u,v), e_2 \rangle \end{bmatrix} =: \chi(u,v)$$

or
$$\begin{array}{ccc} (u_0,v_0) & & (0,0) \\ (u,v) & \underset{?}{\overset{\chi}{\rightleftarrows}} & (\bar{u},\bar{v}) \end{array}$$

want to parameterize S in a neighborhood of p by :

Height $(\bar{u},\bar{v})$

Need : solve the (nonlinear) 2x2 system ②, write (u,v) in terms of $(\bar{u},\bar{v})$

i.e. $(u,v) = \chi^{-1}(\bar{u},\bar{v})$

Then "plug into" ① to define :

$$h(\bar{u},\bar{v}) := \langle \bar{X}(\chi^{-1}(\bar{u},\bar{v})), N \rangle.$$

Does χ^{-1} "exist locally"?

Inverse function theorem again,

$$\underset{2\times 2}{d\chi(u_0,v_0)} = \begin{bmatrix} \langle \bar{X}_u, e_1 \rangle & \langle \bar{X}_v, e_1 \rangle \\ \langle \bar{X}_u, e_2 \rangle & \langle \bar{X}_v, e_2 \rangle \end{bmatrix}\Bigg|_{(u_0,v_0)}$$

(why?)

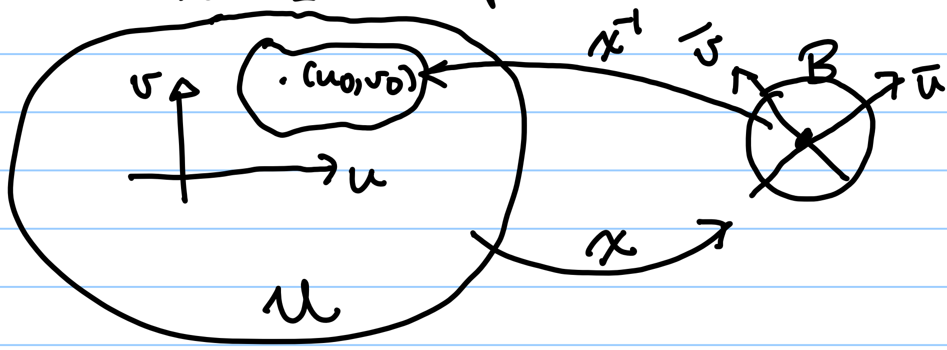
the change of basis matrix

$$= \underset{2\times 3}{\begin{bmatrix} e_1 & e_2 \end{bmatrix}^T} \underset{3\times 2}{\begin{bmatrix} X_u & X_v \end{bmatrix}} \leftarrow \text{rank } 2$$

rank 2 →

So $dx(x_0, v_0)$ is invertible,

so $\exists$ an open ball around $(0,0)$, B,



s.t. $x^{-1}: B \to \mathcal{U}$ differentiable exists, i.e.

$$x(x^{-1}(\bar{u},\bar{v})) = (\bar{u},\bar{v}) \quad \forall (\bar{u},\bar{v}) \in B.$$

and

$$x^{-1}(x(u,v)) = (u,v) \quad \forall (u,v) \in x^{-1}(B).$$

$h: B \to \mathbb{R}^1$

$$h(\bar{u},\bar{v}) = \langle \mathbf{X} \circ x^{-1}(\bar{u},\bar{v}), N \rangle = N^T \cdot \mathbf{X} \circ x^{-1}(\bar{u},\bar{v})$$

parameterize $\mathbf{X}(x^{-1}(B)) \subset S$

Why is this parameterization better than the previous ones?

$h(0,0) = 0$

$$dh(0,0) \underset{\substack{\uparrow \\ \text{chain rule}}}{=} N^T \cdot \left(d\mathbf{X}(p) \cdot \underbrace{dx^{-1}(0,0)}_{?} \right)$$

$$dh_{(0,0)} = \left[\langle N, \underset{T_pS}{\underset{\in}{\cdot}} \rangle , \ \langle N, \underset{T_pS}{\underset{\in}{\cdot}} \rangle \right] = [0,0].$$

$$h(\bar u, \bar v) = 0 + [0,0]\begin{bmatrix} \bar u \\ \bar v \end{bmatrix} + \frac{1}{2}[\bar u \ \bar v]\begin{bmatrix} h_{\bar u \bar u} & h_{\bar u \bar v} \\ h_{\bar v \bar u} & h_{\bar v \bar v} \end{bmatrix}\begin{bmatrix} \bar u \\ \bar v \end{bmatrix} + O\left(\left\| \begin{bmatrix} \bar u \\ \bar v \end{bmatrix} \right\|^3\right)$$

so near $(0,0)$

$$h(\bar u, \bar v) \approx \frac{1}{2}[\bar u \ \bar v]\begin{bmatrix} h_{\bar u \bar u} & h_{\bar u \bar v} \\ h_{\bar v \bar u} & h_{\bar v \bar v} \end{bmatrix}\begin{bmatrix} \bar u \\ \bar v \end{bmatrix}$$

$$(= A\bar u^2 + B\bar u \bar v + C\bar v^2)$$

How does the graph of such a bivariate quadratic function look like?

🤔

🚩 The innocent fact that "order does not matter when taking mixed partials" means

$$\begin{bmatrix} h_{\bar u \bar u} & h_{\bar u \bar v} \\ h_{\bar v \bar u} & h_{\bar v \bar v} \end{bmatrix}$$ is a symmetric matrix.

so it has orthogonal eigenvectors.

By choosing $\{e_1, e_2\}$ to be the eigen-directions, the Hessian matrix becomes <u>diagonal</u>,

$$h(\bar u, \bar v) \approx \frac{1}{2}(\lambda_1 \bar u^2 + \lambda_2 \bar v^2)$$

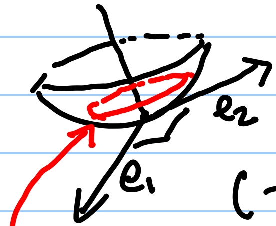
real

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

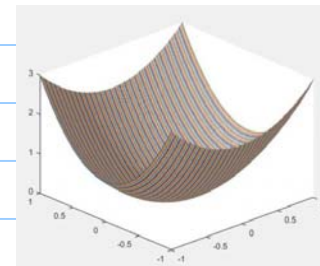
How does the graph of such a bivariate quadratic function look like?

- If $\lambda_1 \lambda_2 > 0$ (ie. λ_1, λ_2 both +ve or −ve)

then

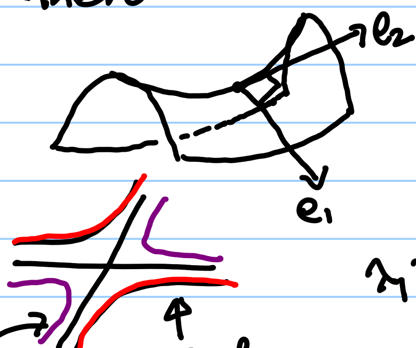


P is called <u>Elliptic</u>.
(The local level curves are ellipses: $\lambda_1 \bar{u}^2 + \lambda_2 \bar{v}^2 = c$)



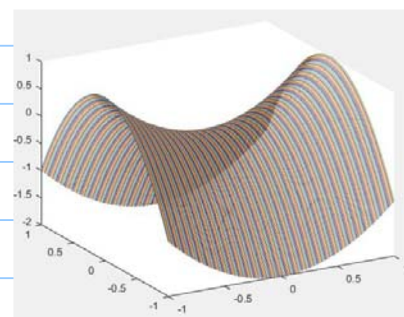
- If $\lambda_1 \lambda_2 < 0$ (one +ve, one −ve)

then P is called <u>Hyperbolic</u>.
The local level curves are hyperbola $\lambda_1 \bar{u}^2 + \lambda_2 \bar{v}^2 = c$



c is of the other sign

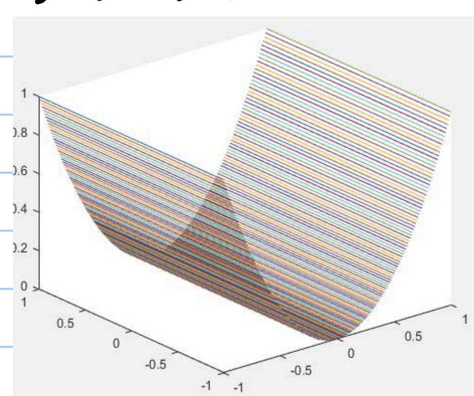
c is of one sign



- If $\lambda_1 \lambda_2 = 0$, but $(\lambda_1, \lambda_2) \neq (0,0)$

then P is called <u>Parabolic</u>.

But not because of the level curves this time.



- If $(\lambda_1, \lambda_2) = (0,0)$ then <u>Planar</u>.

called principle curvatures

HW #2 : If $\lambda_1 \geq \lambda_2$,

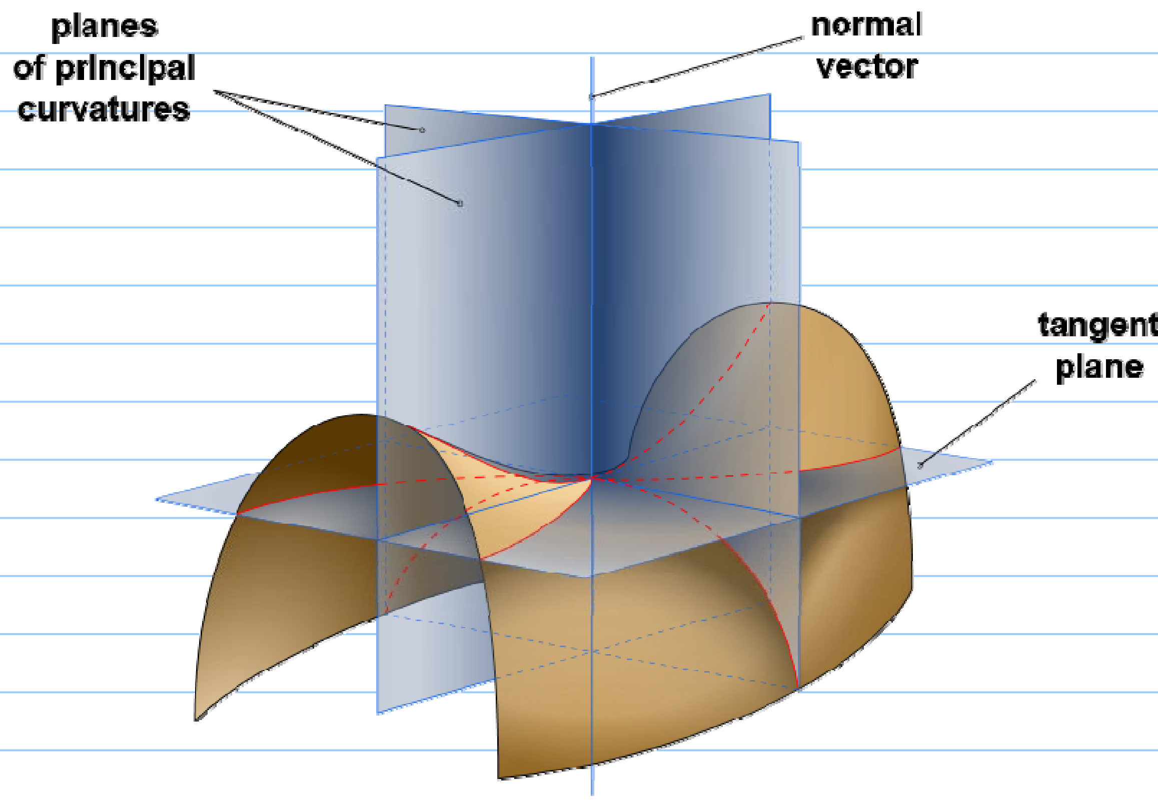
λ_1 (resp. λ_2) = max (resp. min) curvature among all regular curves passing through p.

I always feel :

① NOT intuitively clear why at every point of a regular surface there should be two orthogonal directions e_1, e_2 — Principle Directions @ p s.t. the surface 'curves' the most and the least!

② It's unfortunate that NOT every linear map is diagonalizable, but the symmetric (self-adjoint) ones always are. Not only that, their eigenvectors are orthogonal, and eigenvalues are real.

Good thing is that these two mathematical phenomena are almost the same one.


planes
of principal
curvatures
normal
vector
tangent
plane

Can we write λ_1, λ_2 in terms of X?
e_1, e_2

If for no better reasons, at least we want to have formulas to compute the principle curvatures and directions from formulas of X?

Again, for convenience, assume $p = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \begin{bmatrix} \langle X(u,v), e_1 \rangle \\ \langle X(u,v), e_2 \rangle \end{bmatrix} =: x(u,v)$$

$$h(\bar{u},\bar{v}) = \langle N, X(u,v) \rangle = N^T X(\underbrace{u(\bar{u},\bar{v}), v(\bar{u},\bar{v})}_{\text{a standard abuse of notations}})$$

$$h_{\bar{u}} = N^T \left(X_u \frac{\partial u}{\partial \bar{u}} + X_v \frac{\partial v}{\partial \bar{u}} \right)$$

$$h_{\bar{v}} = N^T \left(X_u \frac{\partial u}{\partial \bar{v}} + X_v \frac{\partial v}{\partial \bar{v}} \right)$$

$$h_{\bar{u}\bar{u}} = N^T \begin{bmatrix} \left(X_{uu} \frac{\partial u}{\partial \bar{u}} + X_{uv} \frac{\partial v}{\partial \bar{u}} \right) \frac{\partial u}{\partial \bar{u}} + X_u \frac{\partial^2 u}{\partial \bar{u}^2} \\ + \\ \left(X_{vu} \frac{\partial u}{\partial \bar{u}} + X_{vv} \frac{\partial v}{\partial \bar{u}} \right) \frac{\partial v}{\partial \bar{u}} + X_v \frac{\partial^2 v}{\partial \bar{u}^2} \end{bmatrix}$$

$$\Big\downarrow \quad \text{@ } (\bar{u},\bar{v}) = (0,0)$$
$$(u,v) = (u_0, v_0)$$
$$N^T X_{uu}(u_0,v_0) = e$$
$$N^T X_{uv}(u_0,v_0) = f$$
$$N^T X_{vv}(u_0,v_0) = g$$

$$h_{\bar{u}\bar{u}}(0,0) = \left(e \frac{\partial u}{\partial \bar{u}} + f \frac{\partial v}{\partial \bar{u}} \right) \frac{\partial u}{\partial \bar{u}} + \cancel{N^T X_u(u_0,v_0)} \frac{\partial^2 u}{\partial \bar{u}^2}$$
$$+$$
$$\left(f \frac{\partial u}{\partial \bar{u}} + g \frac{\partial v}{\partial \bar{u}} \right) \frac{\partial v}{\partial \bar{u}} + \cancel{N^T X_v(u_0,v_0)} \frac{\partial^2 v}{\partial \bar{u}^2}$$

all derivatives $\underbrace{\begin{bmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{bmatrix}}_{} = dx^{-1}(0,0)$ are evaluated at $(0,0)$

Similarly,

$$h_{\bar{u}\bar{v}}(0,0) = \left(e \frac{\partial u}{\partial \bar{v}} + f \frac{\partial v}{\partial \bar{v}}\right) \frac{\partial u}{\partial \bar{u}} + \left(f \frac{\partial u}{\partial \bar{v}} + g \frac{\partial v}{\partial \bar{v}}\right) \frac{\partial v}{\partial \bar{u}}$$

$$h_{\bar{v}\bar{v}}(0,0) = \left(e \frac{\partial u}{\partial \bar{v}} + f \frac{\partial v}{\partial \bar{v}}\right) \frac{\partial u}{\partial \bar{v}} + \left(f \frac{\partial u}{\partial \bar{v}} + g \frac{\partial v}{\partial \bar{v}}\right) \frac{\partial v}{\partial \bar{v}}$$

$$\mathrm{Hess}(h)(0,0) = \underbrace{\begin{bmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{u}} \\ \frac{\partial u}{\partial \bar{v}} & \frac{\partial v}{\partial \bar{v}} \end{bmatrix}}_{[dx^{-1}(0,0)]^T} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \underbrace{\begin{bmatrix} \frac{\partial u}{\partial \bar{u}} & \frac{\partial u}{\partial \bar{v}} \\ \frac{\partial v}{\partial \bar{u}} & \frac{\partial v}{\partial \bar{v}} \end{bmatrix}}_{dx^{-1}(0,0)}$$

Still not a good formula to compute with, as x, x^{-1} involve $\{e_1, e_2\}$.

$$dx(u,v) = \begin{bmatrix} \langle x_u, e_1 \rangle, & \langle x_v, e_1 \rangle \\ \langle x_u, e_2 \rangle, & \langle x_v, e_2 \rangle \end{bmatrix}$$

$$\begin{aligned} dx^{-1}(0,0) &= [dx(u_0, v_0)]^{-1} \\ &= \begin{bmatrix} \langle x_u, e_1 \rangle, & \langle x_v, e_1 \rangle \\ \langle x_u, e_2 \rangle, & \langle x_v, e_2 \rangle \end{bmatrix}^{-1}\Bigg|_{(u_0,v_0)} \\ &= \Big(\underbrace{[e_1\ e_2]^T [x_u, x_v]}_{dx(u_0,v_0)}\Big)^{-1} \end{aligned}$$

Trick:

$$[dx(u_0,v_0)]^{-1}\,\mathrm{Hess}(h)(0,0)\cdot[dx(u_0,v_0)]$$

$$= \underbrace{\big[dx(u_0,v_0)\big]^{-1}\big[dx(u_0,v_0)\big]^{-T}}_{\big[dx(u_0,v_0)^T\cdot dx(u_0,v_0)\big]^{-1}}\begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

$$\big[dx(u_0,v_0)^T\cdot dx(u_0,v_0)\big]^{-1} = \Big[[x_u,x_v]^T\underset{3\times 2}{[e_1\, e_2]}\,\underset{2\times 3}{[e_1\, e_2]^T}[x_u,x_v]\Big]^{-1}$$

$$\overset{\leftarrow \text{ why?}}{=} \Big[[x_u,x_v]^T[e_1,e_2,N]\,[e_1,e_2,N]^T[x_u,x_v]\Big]^{-1}$$

$$[x_u,x_v]^T[e_1,e_2,N] = \begin{bmatrix} \langle x_u,e_1\rangle & \langle x_u,e_2\rangle, & 0 \\ \langle x_v,e_1\rangle & \langle x_v,e_2\rangle, & 0 \end{bmatrix}, \qquad [e_1,e_2,N]\,[e_1,e_2,N]^T = I_{3\times 3}$$

$$= \begin{bmatrix} \overset{E}{\langle x_u,x_u\rangle}, & \overset{F}{\langle x_u,x_v\rangle} \\ \underset{F}{\langle x_v,x_u\rangle}, & \underset{G}{\langle x_v,x_v\rangle} \end{bmatrix}^{-1}$$

$$[dx(u_0,v_0)]^{-1}\,\mathrm{Hess}(h)(0,0)\cdot[dx(u_0,v_0)]$$

$$= \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{\text{called the first fundamental form of } S.}^{-1}\underbrace{\begin{bmatrix} e & f \\ f & g \end{bmatrix}}_{\text{called the second fundamental form of } S}.$$

[Q: What's so fundamental about E, F, G !?]

Note : $dX(u,v) = \begin{bmatrix} \langle X_u, e_1\rangle, & \langle X_v, e_1\rangle \\ \langle X_u, e_2\rangle, & \langle X_v, e_2\rangle \end{bmatrix}$

is the change of basis matrix on T_pS :

$$\underbrace{\begin{bmatrix} X_u & X_v \end{bmatrix}}_{\substack{3\times 2 \\ \text{NOT o.n.}}} = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{\substack{3\times 2 \\ \text{o.n.}}} \underbrace{\begin{bmatrix} \langle X_u, e_1\rangle, & \langle X_v, e_1\rangle \\ \langle X_u, e_2\rangle, & \langle X_v, e_2\rangle \end{bmatrix}}_{\substack{2\times 2 \\ \text{not orthogonal}}}$$

and

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$ not symmetric (unless $[X_u, X_v]$ is o.n.)

$$= \frac{1}{EG-F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

$$= \frac{1}{EG-F^2} \begin{bmatrix} eG - fF & fG - gF \\ -eF + fE & -fF + Eg \end{bmatrix}$$

$$\lambda_1 + \lambda_2 = \text{trace} = \frac{1}{EG-F^2}(eG - 2fF + Eg)$$

$$\lambda_1 \lambda_2 = \det = \frac{1}{EG-F^2}(eg - f^2)$$

$$\boxed{\begin{array}{ll} H := (\lambda_1 + \lambda_2)/2 & \text{mean curvature} \\ K := \lambda_1 \lambda_2 & \text{Gauss curvature} \end{array}}$$

characteristic polynomial :

$$\lambda^2 - \underbrace{(\text{trace})}_{2H} \lambda + \underbrace{(\det)}_{K} = 0$$

principal curvatures $\lambda_1, \lambda_2 = H \pm \sqrt{H^2 - K}$.

Def: When $\lambda_1 = \lambda_2$, p is called an umbilical point.

E.g. Every point of a sphere is umbilical.

Note:

$$N = X_u \times X_v / \| X_u \times X_v \|$$

depends on the "orientation of the parameterization"

E.g. If we change the parameterization from $\mathbf{X}(u,v)$ to $\mathbf{X}(v,u)$ then the order of the basis of $T_p(S)$ is changed

$$\{X_u, X_v\} \to \{X_v, X_u\},$$

consequently

$$N \to -N$$

Ex: • Check that magnitudes of $\lambda_1, \lambda_2, K, H$ are invariant under reparameterization (as they should.)

• λ_1, λ_2, H change signs under a change of orientation of the parameterization.

K is insensitive to orientation.

Yet another observation:

$$\langle N, X_u \rangle = 0$$

$$\Rightarrow \langle N_u, X_u \rangle + \langle N, X_{uu} \rangle = 0$$

$$\Rightarrow e = \langle N, X_{uu} \rangle = -\langle N_u, X_u \rangle$$

Similarly

$$f = \langle N, X_{uv} \rangle = -\langle N_v, X_u \rangle$$

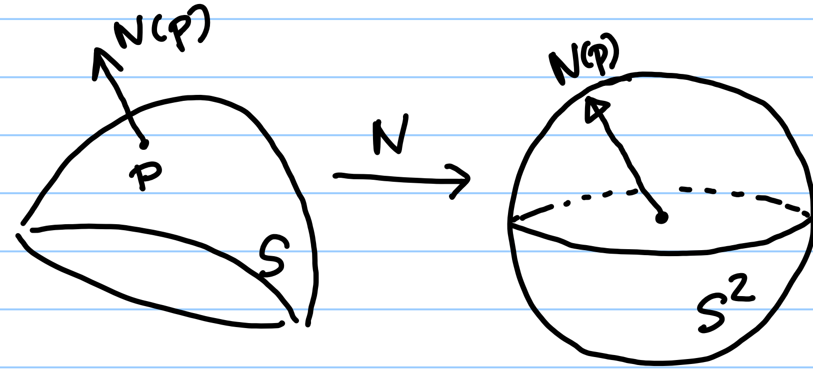
$$g = \langle N, X_{vv} \rangle = -\langle N_v, X_v \rangle.$$

🚩 The moment we write N_u, N_v, we begin to think of N as moving.

$\{X_u, X_v, N\}$ is thought of as a "moving frame" on S.

<u>Def</u>: $N : S \longrightarrow \{\text{unit sphere}\} =: S^2$

is called the <u>Gauss map</u> of S.



Def A regular surface is called orientable if it has a <u>continuous Gauss map</u>.

E.g. It can be shown that the Mobius band is not orientable

(see Do Carmo C&S, sec 2-6)

<u>Proposition</u>:

S has a continuous Gauss map

iff

it is possible to cover S with a family of coordinate neighborhoods,

i.e

$X_\alpha : U_\alpha \to S$ (regular local parameterizations)

s.t.

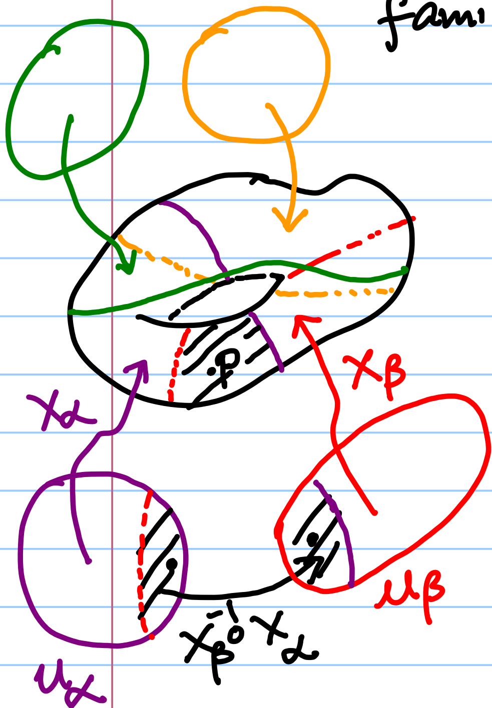
$\bigcup_\alpha X_\alpha(U_\alpha) = S$ in such a way that

if

$p \in X_\alpha(U_\alpha) \cap X_\beta(U_\beta)$

then

$X_\beta^{-1} \circ X_\alpha$ has a positive Jacobian @ $X_\alpha^{-1}(p)$



i.e. $\det\Big(\underbrace{d(X_\beta^{-1}\circ X_\alpha}_{2\times 2}\big|_{X_\alpha^{-1}(p)})\Big) > 0$

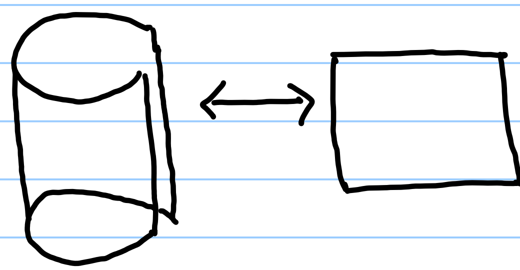
Proof : see Do Carmo C&S, sec. 2-6.

[more about the Gauss map later...]

The grand challenge questions:

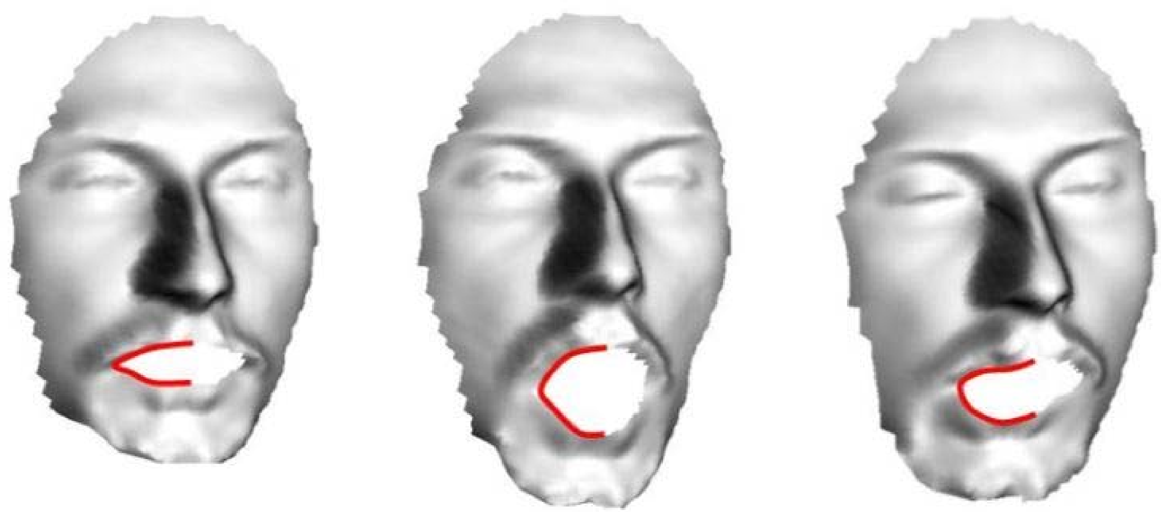
- what determines the surface up to rigid motion in $\mathbb{R}^3$?

- what determines the surface up to isometry?



not congruent

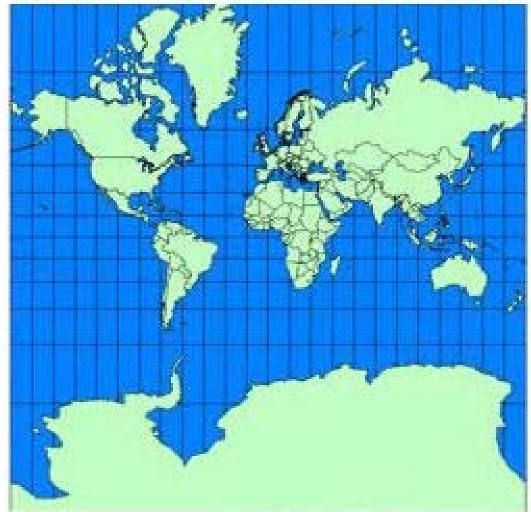
but isometric
(recall lecture 2)



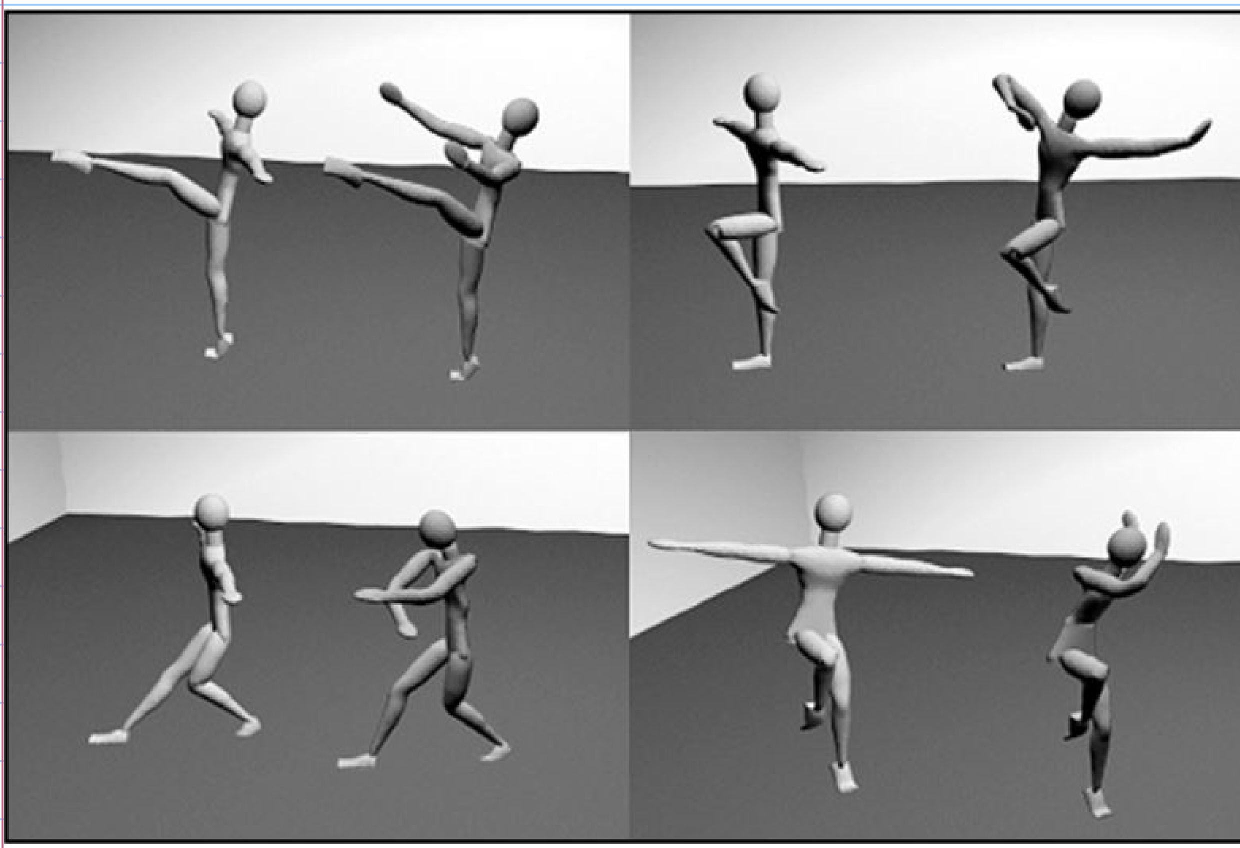
- same face (approximate isometries)
  - different expressions

isometry

possible?



[But let's see some higher dimensional surfaces / manifolds first.]



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