Introduction to Optimization. HW#1. **Due: Wednesday, April 9, 2025**

1. (10 points) You are curious about the annual salary of three of your close friends (I just name them X, Y and Z.) You got the 'insider information' that the sum of the three salaries is \$$a$, that X earns \$$b$ more than Y annually, and Y earns \$c more than Z.

   Can you determine the salaries of X, Y and Z from a, b and c? If so, determine them; otherwise explain why.

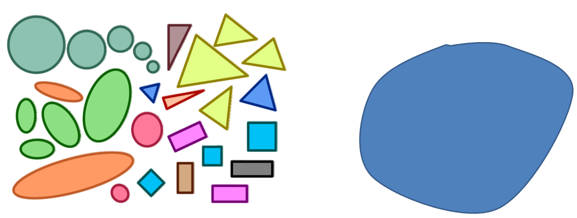
2. (20 points) Again, there are three quantities of interest which I call x, y and z. And again we cannot directly acquire these three quantities, but this time you are given the indirect information that x is a units more than y, x is b units more than z, and y is c units more than z.

   Under what condition(s) on a, b and c are there values of x, y and z that satisfy the above information? In this case, can you determine x, y and z **uniquely**? If the solution is not unique, determine the set of all the solutions.

   Under what condition(s) on a, b and c would there be no values of x, y and z that satisfy the above information?

3. (30 points) You have a fixed amount of fencing material, and you would like to use it to enclose the largest possible area. This is known as the isoperimetric problem.

   This optimization problem is not easy; the difficulty hinges on the fact that there are too many possible shapes: a triangle, a pentagon, a hexagon, an ellipse, or any shape with a curved boundary. The dimensionality of this problem is not only big, but infinite!

   - (5 points) The problem becomes solvable by freshman calculus (in fact you only need high-school algebra in this case) if we restrict ourselves to rectangular shapes. Among all rectangles with a fixed perimeter, which one has the largest area? Set up the optimization problem by defining appropriate variable(s) and show how to solve it analytically.
   - (15 points) Among all the triangles with a fixed perimeter, which triangle has the largest area? Hint: You can assume the three sides of a triangle have lengths a, b, c, with $a + b + c = 2$. So you can write c in terms of a and b and we are left with only two degrees of freedom. Use Heron's formula.

     What are the constraints on (a, b)? Use the triangle inequality to determine the feasible region for (a, b). Show that the constraints are given by the same kind of linear equalities as seen in a linear program. *Carefully graph the feasible region (a.k.a. constraint set) on the a-b plane.*

- (10 points) Is the optimum value of the problem attained at a *vertex* of the feasible region, as you may expect from a linear program?

4. (40 points) As in class, we have data (x_i, y_i), $i = 1, \ldots, m$. But this time, we speculate that the data follows a quadratic model:

$$y_i \approx \alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2, \quad i = 1, \ldots, m.$$

Two ways to determine the model parameters α_0, α_1, α_2 are to solve:

- [**Least L^2 regression**] $\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^m (\alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 - y_i)^2$
- [**Least L^1 regression**] $\min_{\alpha_0,\alpha_1,\alpha_2} \sum_{i=1}^m |\alpha_0 + \alpha_1 x_i + \alpha_2 x_i^2 - y_i|$

By generalizing/mimicking what we do in class for the linear regression problem, show that:

(i) For the least L^2 regression method, the optimization problem can be solved based on solving a 3×3 system of linear equations.

(ii) For the least L^1 regression method, the optimization problem can be formulated as a linear program. Note: you must present the final LP in the standard form, using *matrix notations*, in order to receive full credits.