CH 9

Note Title 5/6/2017

$M = M^m$ – differentiable manifold of dimension m

We shall need 3 types of "smooth tensor fields":

- vector field : a smooth assignment of tangent vector at each point

$$X_p \in T_pM \,, \; p \in M$$

- Riemannian metric : a smooth assignment of inner product at each point

$$\langle , \rangle_p \in \{ \beta : T_pM \times T_pM \to \mathbb{R} \text{ billinear, symmetric, positive definite} \} \qquad p \in M$$

- differential k-forms :

a smooth assignment of alternating k-forms on T_pM,

$$\omega_p \in \mathrm{Alt}^k(T_pM) \quad , \; p \in M$$

Recall T_pM is the tangent space of M at $p \in M$.

Recall (from Math 538) what it means by a smooth vector field :

$$X : M^m \to \bigcup_{p \in M} T_pM^m \quad X(p) \in T_pM^m \text{ is smooth}$$

if

the component functions of X in any chart are smooth.

Recall :

any chart $(\mathcal{U}, h)$ of M induces a frame of basis

$$\frac{\partial}{\partial x_1}\Big|_p, \cdots, \frac{\partial}{\partial x_m}\Big|_p \text{ for each } T_pM, p \in \mathcal{U},$$

so the vector field can be written as

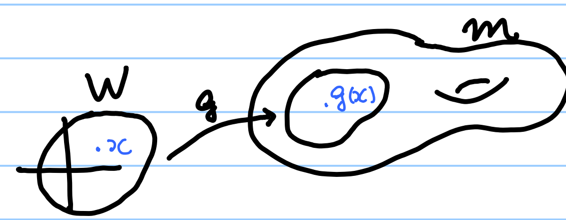
$$X(p) = \sum_{i=1}^{m} \underbrace{a_i(p)}_{\text{component functions of } X \text{ in the chart.}} \frac{\partial}{\partial x_i}\Big|_p \qquad \text{for } p \in \mathcal{U}$$

It's the same idea for defining smoothness of Riemannian metric and differential p-forms.

Let's first spell out that for the latter.

Let $\quad g : W \overset{\text{open}}{\subseteq} \mathbb{R}^m \to M$ be a local parameterization
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ inverse of a chart

For $x \in W$,
$\qquad D_x g : \mathbb{R}^m \to T_{g(x)} M$
is an isomorphism ("basis theorem")



and induces an isomorphism

$$\mathrm{Alt}^k(D_x g) : \mathrm{Alt}^k(T_{g(x)} M) \to \mathrm{Alt}^k(\mathbb{R}^m).$$

(Recall from CH2 : For a linear map $f : V \to W$,

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V) \quad \text{is defined by}$$

$$\mathrm{Alt}^p(f)(\omega)(\xi_1, \cdots, \xi_p) = \omega(f(\xi_1), \cdots, f(\xi_p)).$$

Easy to check : $\begin{cases} \mathrm{Alt}^p(g \circ f) = \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g) \\ \mathrm{Alt}^p(id) = id \end{cases}$

This also means $f : V \to W$ is invertible $\Rightarrow$ $\mathrm{Alt}^p(f)$ is invertible
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\mathrm{Alt}^p(f)]^{-1} = \mathrm{Alt}^p(f^{-1})]$)

Now, define the "pullback of ω to W"

$$g^*(\omega) : W \to \mathrm{Alt}^k(\mathbb{R}^m) \quad \text{by}$$

$$g^*(\omega)_x := \mathrm{Alt}^k(D_x g)(\omega_{g(x)}).$$

For $k=0$ (0-form = "scalar field"),

$$g^*(\omega)_x := \omega_{g(x)}.$$

Just like the case of vector field, a chart provides a local coordinate representation of the "field of k-forms" $(\omega_p)_{p \in M}$, which we can talk about its smoothness.

<u>Def</u> A family $\omega = \{\omega_p : p \in M\}$ of alternating k-forms

$$\omega_p : \underbrace{T_pM \times \cdots \times T_pM}_{k \text{ times}} \to \mathbb{R}$$

is said to be <u>smooth</u> if $g^*(\omega)$ is smooth for every local parametrization g.

Such a smooth ω is called a differential p-form on M.

$$\Omega^k(M) := \{ \text{all differential k-forms on } M)$$

$$\Omega^0(M) := C^\infty(M, \mathbb{R}) = \{ \text{all smooth } f : M \to \mathbb{R} \}.$$

Again like the case of vector field, a basic question You must ask is :

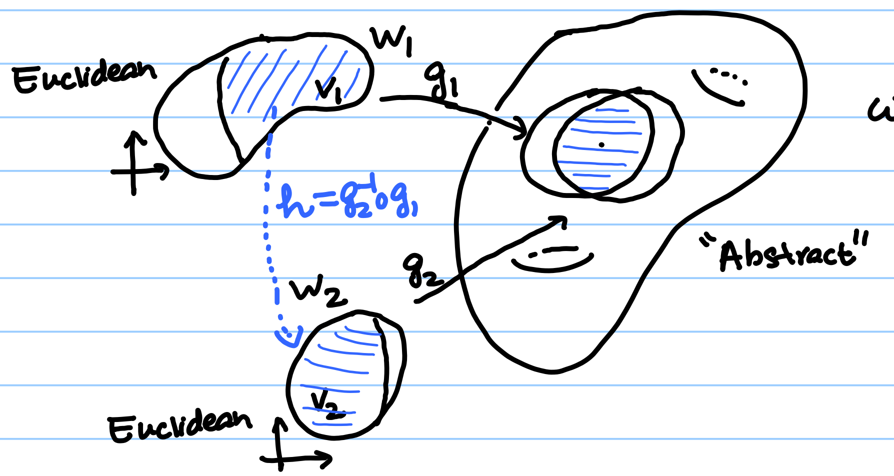
Can ω be smooth in one set of coordinates but not another?

Precisely, if $g_1 : W_1 \to M$, $g_2 : W_2 \to M$ are two local parametrizations $g_1(W_1) \cap g(W_2) \neq \emptyset$, can it be that

$g_1^* \omega$ is smooth on $g_1^{-1}(g_1(W_1) \cap g_2(W_2))$

but $g_2^* \omega$ is not smooth on $g_2^{-1}(g_1(W_1) \cap g_2(W_2))$?

Work out the formula for how ω tranforms under a change of coordinates, then we will know :



write
$V_i = g_i^{-1}(g_1 W_1 \cap g_2 W_2)$
$i = 1, 2.$

An expression we wrote many times in Math 538 :

$$g_1 = g_2 \circ \underbrace{(g_2^{-1} \circ g_1)}_{\text{change of coordinate map}} =: h \leftarrow \text{smooth with smooth inverse (assumption of a manifold)}$$

$$g_1^* \omega = (g_2 \circ h)^* \omega \qquad\qquad g_2^* = (g_1 \circ h^{-1})^* \omega$$

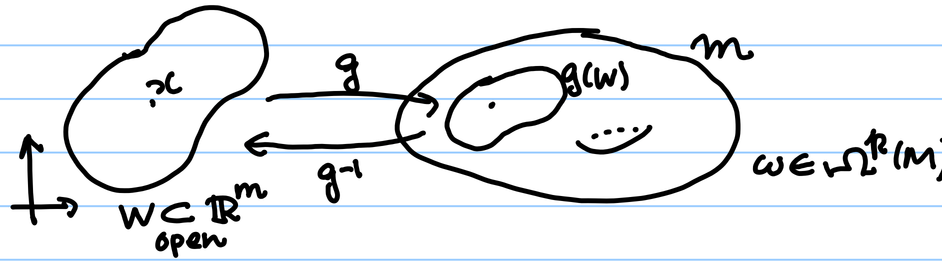
$$= h^* (g_2^* \omega) \qquad\qquad = (h^{-1})^* (g_1^* \omega)$$

so $g_2^* \omega$ is smooth on V_2 $\Longleftrightarrow$ $g_1^* \omega$ is smooth on V_1

$\uparrow$
h is a diffeomorphism

In particular, if we just check that ω is smooth on a set of charts that cover M, then ω is smooth.
(Lemma 9.6)

Recap:



$$D_x g : \mathbb{R}^m_x (\simeq \mathbb{R}^m) \to T_{g(x)}M$$

$$D_{g(x)} g^{-1} : T_{g(x)}M \to \mathbb{R}^m$$

inverse

A k-form on M can be "pulled backed" by $g: W \to M$ to a k-form on W

$$g^*\omega \in \Omega^k(W).$$

$$(g^*\omega)_x = \mathrm{Alt}^k(D_x g)(\omega_{g(x)}).$$

Similarly, a k-form on W can be pulled backed by $g^{-1}: g(W) \to W$ to a k-form on $\underbrace{g(W)}_{\text{an open submanifold of } M}$:

$$\eta \in \Omega^k(W),$$

$$((g^{-1})^*\eta)_{g(x)} := \mathrm{Alt}^k(D_{g(x)} g^{-1})(\eta_x)$$

More generally, any smooth map between manifolds

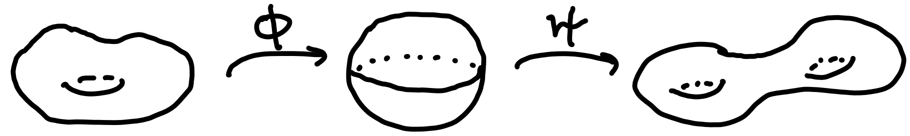
$$\phi : M^m \to N^n$$

induces

$$D_x\phi : T_xM \to T_{\phi(x)}N, \quad \phi^* : \Omega^k(N^n) \to \Omega^k(M^m)$$

$$(\phi^*\omega)_x := \mathrm{Alt}^k(D_x\phi)(\omega_{\phi(x)}) \qquad x \in M.$$

Easy to check: $\phi^*\omega$ is smooth, hence in $\Omega^k(M)$.



As expected, if $\phi : M \to N$, $\psi : N \to R$ smooth maps between manifold, we have (the easy-to-check)

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* : \Omega^k(R) \to \Omega^k(M)$$

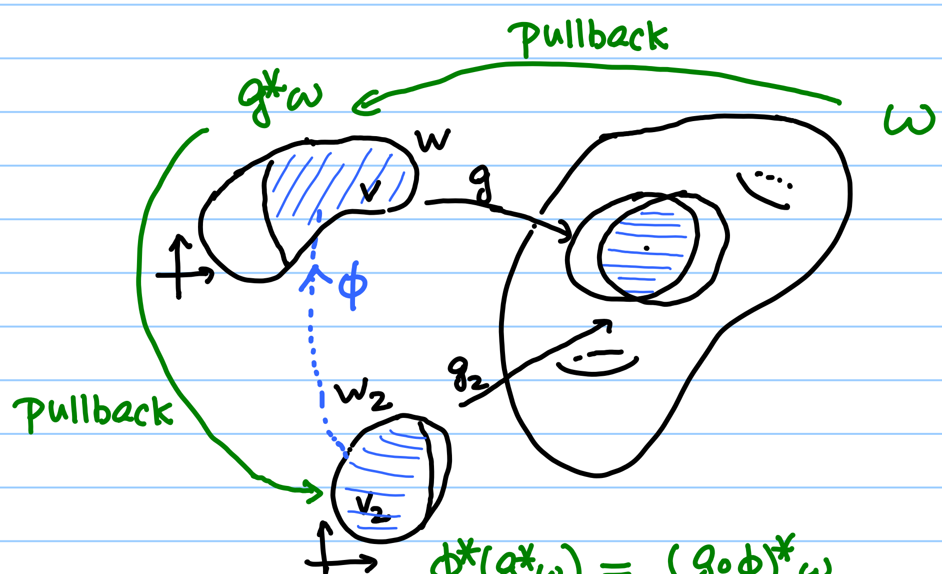
Exterior derivative

$$d : \Omega^k(M) \to \Omega^{k+1}(M)$$

can be defined via local parametrizations $g : W \to M$ by

$$d\omega := (g^{-1})^* d(g^* \omega)$$

Q:
Why wouldn't it depends on the parametrization?



$$\phi^*(g^*\omega) = \underbrace{(g \circ \phi)}_{= g_2}{}^* \omega$$

If $g_2 : W_2 \to M$ is another parametrization, we have

$$g_2 = g \circ \phi.$$

we must check

$$(g^{-1})^* d\, g^* \omega = \underbrace{(g_2^{-1})^*}_{= (\phi^{-1} \circ g^{-1})^*} d \underbrace{g_2^* \omega}_{\phi^*(g^* \omega)}$$

$$\iff (g^{-1})^* \circ d \circ g^* \omega = (g^{-1})^* \circ (\phi^{-1})^* \circ d \circ \phi^* \circ g^* \omega$$

$$\iff \underset{\overset{\shortparallel}{d_V}}{d} = (\phi^{-1})^* \circ \underset{\overset{\shortparallel}{d_{V_2}}}{d} \circ \phi^* \omega$$

This boils down to a property of 'd' back in the Euclidean setting.

But

$$(\phi^{-1})^* \circ (d_{V_2} \circ \phi^*)$$

$$= (\phi^{-1})^* \circ (\phi^* \circ d_V) \qquad \text{(Thm 3.12)}$$

$$= ((\phi^{-1})^* \circ \phi^*) \circ d_V$$

$$= d_V$$

So d is indeed parametrization independent and

$$d : \Omega^k(M) \to \Omega^{k+1}(M) \text{ is well-defined.}$$

A whole bunch of properties pertaining to differential forms and de Rham cohomology generalize easily to manifolds:

- $d \circ d = 0$

  Hence we have defined a chain complex:

  $$0 \to \cdots \to \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \to \cdots \to 0$$

  $$\Omega^k(M) = 0 \text{ if } k < 0 \text{ or } k > \dim M.$$

- For a smooth map $\phi : M \to N$, the "pullbacks"

$$\phi^* : \Omega^p(N) \to \Omega^p(M)$$

commute with d:

$$\phi^* \circ d_N = d_M \circ \phi^*,$$

i.e. ϕ^* is a chain map:

$$\begin{tikzcd}
\to \Omega^{k-1}(N) \arrow[r, "d"] \arrow[d, "\phi^*"] & \Omega^k(N) \arrow[r, "d"] \arrow[d, "\phi^*"] & \Omega^{k+1}(N) \to \arrow[d, "\phi^*"] \\
\to \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \to
\end{tikzcd}$$

- wedge product has always been well-defined for any abstract vector space V

$$\wedge : \mathrm{Alt}^k(V) \times \mathrm{Alt}^l(V) \to \mathrm{Alt}^{k+l}(V)$$

Just apply it pointwise to differential forms on M

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

$$(\omega \wedge \tau)_p := \omega_p \wedge \tau_p$$

the wedge product on T_pM

Easy to check: $\omega \wedge \tau$ is smooth.

- $$d(\omega \wedge \tau) = d\omega \wedge \tau + (-1)^k \omega \wedge d\tau$$

$$\omega \wedge \tau = (-1)^{kl} \tau \wedge \omega$$

for $\omega \in \Omega^k(M)$, $\tau \in \Omega^l(M)$.

- And of course, we have

$$H^p(M) := \frac{\ker(d: \Omega^p(M) \to \Omega^{p+1}(M))}{\operatorname{Im}(d: \Omega^{p-1}(M) \to \Omega^p(M))}$$

the pth cohomology vector space of $\Omega^*(M)$

- wedge product "descends" to the cohomology spaces :

$$\underset{H^p(M)}{\underset{\cap}{[\omega_1]}} \times \underset{H^q(M)}{\underset{\cap}{[\omega_2]}} := \underset{H^{p+q}(M)}{\underset{\cap}{[\omega_1 \wedge \omega_2]}}$$

[This is not obvious even in the Euclidean setting.]
[But generalizing it to manifold is trivial.]

- The chain map ϕ^* "descends" to linear maps

$$H^p(\phi) : H^p(N) \to H^p(M)$$

$$\phi : M \to N$$

$$\rightsquigarrow$$

$$\phi^* \text{ or } \Omega^p(\phi) : \Omega^p(N) \to \Omega^p(M)$$

$$\rightsquigarrow$$

$$H^p(\phi) : H^p(N) \to H^p(M) \quad \text{Linear}$$

with the "contravariant functor" property

$$M \xrightarrow{\phi} N \xrightarrow{\psi} R \quad \text{smooth maps}$$

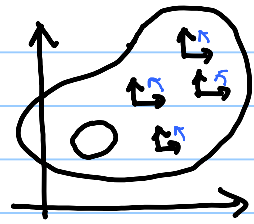
$$H^p(M) \xleftarrow{H^p(\phi)} H^p(N) \xleftarrow{H^p(\psi)} H^p(R)$$

$$H^p(\psi \circ \phi) = H^p(\phi) \circ H^p(\psi).$$

Orientation

Now, a property, called orientibility, that is much more complicated for manifolds than open sets in $\mathbb{R}^n$.

[M&T has all the right materials, but in an order I disagree.]



open set in $\mathbb{R}^n$ always orientible

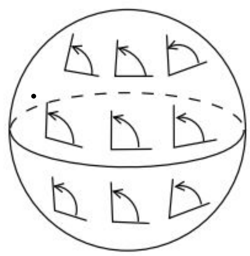


Fig. 15.1 A sphere is orientable

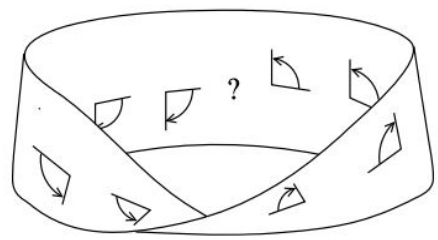


Fig. 15.2 A Möbius band is not orientable

(J. Lee: Intro. to Smooth Manifolds, 2nd edition)

To describe this property in precise mathematical terms, recall:

(I) [Lecture 2, Math 538]

The orientation of an ordered basis in $\mathbb{R}^n$ $E_1, \cdots, E_n$ is captured by the sign of $\det[E_1, \cdots, E_n]$.

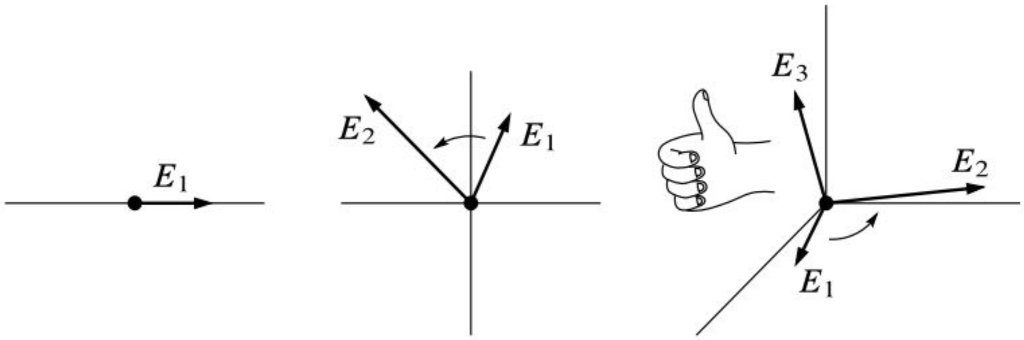
Put differently, the linear map $\mathbb{R}^n \to \mathbb{R}^n$ that maps

$$e_i \to E_i \qquad i = 1, \cdots, n$$

e_i: the standard basis of $\mathbb{R}^n$

is orientation preserving if $\det[E_1, \cdots, E_n] > 0$.

(II) What would this mean if we are in an abstract vector space V (e.g. the tangent space of an abstract manifold) for which there is no canonical basis?



$\dim V = 1$: there is no left or right
$\dim V = 2$: there is no clockwise or anti-clockwise
$\dim V = 3$: there is no 'right-handedness' or 'left-handedness'
etc.

But, by leveraging (I), we can still say two ordered basis of V

$$(E_1, \cdots, E_n) \qquad (\tilde{E}_1, \cdots, \tilde{E}_n)$$

are <u>consistently oriented</u> if the transition matrix (B_i^j)

$$\left(E_i = \sum_{j=1}^{n} B_i^j \tilde{E}_j \quad , i=1,\cdots,n \right)$$

has positive determinant.

The notion of two ordered bases of V being consistently oriented is clearly an equivalence relation. And there are precisely two equivalence classes of ordered basis.

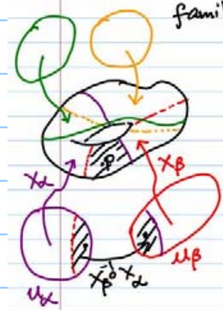
Any choice of the two is called an orientation of V.

(III) [Lecture 3, Math 538] explains the notion of orientability for a regular surface in $\mathbb{R}^3$:

Def A regular surface is called orientable if it has a <u>continuous Gauss map</u>.

<u>Proposition</u>:
S has a continuous Gauss map iff it is possible to cover S with a family of coordinate neighborhoods, i.e. $X_\alpha : U_\alpha \to S$ (regular local parameterizations) s.t. $\bigcup_\alpha X_\alpha(U_\alpha) = S$ in such a way that if $p \in X_\alpha(U_\alpha) \cap X_\beta(U_\beta)$ then $X_\beta^{-1} \circ X_\alpha$ has a positive Jacobian @ $X_\alpha^{-1}(p)$

i.e. $\det\big(\underbrace{d(X_\beta^{-1}\circ X_\alpha)}_{2\times 2}\big|_{X_\alpha^{-1}(p)}\big) > 0$



Proof : see Do Carmo C & S, sec. 2-6.

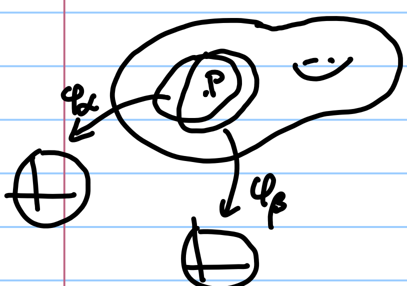
It seems natural to define :

Def A manifold M is orientible if it can be covered by an atlas that is consistently oriented, i.e.

$\exists$ charts $(U_\alpha, \varphi_\alpha)$ of M s.t.

$$\bigcup U_\alpha = M$$

$$\det\left(d(\varphi_\alpha^{-1}\circ\varphi_\beta)\big|_{\varphi_\beta^{-1}(p)}\right) > 0, \ \forall p \in U_\alpha \cap U_\beta$$

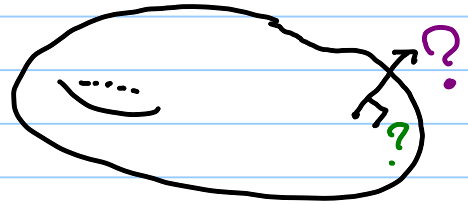


[This is the same as saying that the ordered bases of $T_p(M)$ induced by the two charts are consistently oriented.

But note that this is more than a pointwise statement, those ordered bases vary smoothly as p moves.]

This means : given an orientation of T_pM at any $p \in M$, we can "propagate" it to the whole connected component of M containing p.

Note: we cannot talk about a Gauss map of a manifold M without introducing additional structure.



no ambient space
no notion of angle

no Gauss map

Redefining orientibility / orientation using n-forms:

"Pointwise level" $\dim V = n \geq 1$

choosing an ordered basis to define an orientation on V = choosing a non-zero n-form $\omega \in \mathrm{Alt}^n(V)$, $\omega \neq 0$

Precisely, for any $\omega \in \mathrm{Alt}^n(V)$, $\omega \neq 0$

$$(E_1, \cdots, E_n) \sim (\tilde{E}_1, \cdots, \tilde{E}_n) \iff \omega(E_1, \cdots, E_n),\ \omega(\tilde{E}_1, \cdots, \tilde{E}_n)$$

consistently oriented

are of the same sign.

This follows from the following formula (easily followed from Lemma 2.13, Theorem 2.15 in CH 2):

$$\omega(\underbrace{\tilde{E}_1}_{= BE_1}, \cdots, \underbrace{\tilde{E}_n}_{= BE_n}) = \det(B)\, \omega(E_1, \cdots, E_n).$$

This leads to M&T's definition at the "manifold level":

Def (i) A smooth manifold M^n of dimension n is called orientable if there exists an

$$\omega \in \Omega^n(M^n) \quad \text{with} \quad \omega_p \neq 0 \ \forall p \in M.$$

Such an ω is called an orientation form on M.

(ii) Two orientation forms ω, τ on M are equivalent if

$$\tau = f \cdot \omega \quad \text{for some } f \in \Omega^0(M)$$
$$f(p) > 0 \quad \forall p \in M.$$

An orientation of M is an equivalence class of orientation forms on M.

We shall see the usefulness of this reformulation.

It's not very hard to see why the two definitions are equivalent. (see proposition 9.14 below.)

Assume M^n is orientable and an orientation form ω is chosen.

Then an ordered basis $b_1, \dots, b_n$ of T_pM is said to be

positively or negatively oriented w.r.t ω

if

$$\omega_p(b_1, \dots, b_n) > 0 \quad \text{or} \quad \omega_p(b_1, \dots, b_n) < 0, \text{ resp.}$$

Clearly the sign depends only on the orientation determined by ω.

How many different orientations can M have?

For any two orientation forms ω and τ on M

$\tau = f \cdot \omega$ for a uniquely determined smooth function $f: M \to \mathbb{R}$
$f(p) \neq 0 \quad \forall p$

If M is connected, such an f ($\in \Omega^0(M)$) is either positive or negative throughout M, so we have

<u>Lemma</u> On a connected orientable manifold there are precisely 2 orientations.

We had "pointwise orientation", i.e. an orientation on T_pM for a point $p \in M$.

We can also have "local orientation":

If $\mathcal{U} \overset{\text{open}}{\subset} M^n$, then if M already has an orientation you can restrict it to $\mathcal{U}$ to give $\mathcal{U}$ an orientation.

Conversely, $\mathcal{U}$ (being a manifold itself) can have its own orientation, e.g.

if $\varphi: \mathcal{U} \to \varphi(\mathcal{U}) \subset \mathbb{R}^n$ is a chart we can always use the orientation of $\mathbb{R}^n$ to induce an orientation on $\mathcal{U}$ (details later.)

Q: If we have an open cover $(V_i)_{i \in g}$ of M and each V_i has an orientation, can we stitch these "local orientations" together to give a (global) orientation for M?

The answer is clearly negative in general, any small enough V_i is orientable but M may not be orientable.

Techical questions: exactly when can we 'stitch orientations'? how?

Answer: - when the 'local orientations' coincide at overlaps
- use a partition of unity

(I never went through it in details :) The topological requirements (Hausdorff, and countable) ensure that the following is true:

Theorem : (Existence of partition of unity subordinate to any open cover)

For any open cover $(V_i)_{i \in J}$ of a smooth manifold M, there exists smooth functions

$$\varphi_i : M \to \mathbb{R} \qquad i \in J \qquad \text{s.t.}$$

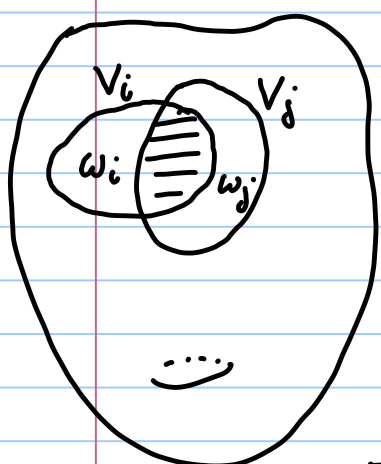
(i) $\varphi_i(p) \in [0,1] \quad \forall i \in J, p \in M$

(ii) $\mathrm{supp}(\varphi_i) \subset V_i$

(iii) $\forall p \in M$, only a finite # of $\varphi_i(p)$ is positive (local finiteness)

(iv) $\sum_{i \in J} \varphi_i(p) = 1 \quad \forall p \in M.$

[Basically Thm 9.11 of M&T, except that Thm 9.11 assumes M is a submanifold of $\mathbb{R}^l$. It is not hard to show that every manifold can be embedded in some $\mathbb{R}^l$ (if keeping l as small as possible is not a concern.) See, e.g., John Lee's book for a proof this partition of unity result without assuming an embedding.]



Lemma 9.10. Let $\mathcal{V} = (V_i)_{i \in \mathcal{I}}$ be an open cover of a smooth manifold M. Suppose all V_i have orientations and that the restrictions of the orientations from V_i and V_j to $V_i \cap V_j$ coincide $\forall i \neq j$.

Then M has a uniquely determined orientation with the given restriction to V_i for all $i \in \mathcal{I}$.

The idea of the proof is to stitch the orientation forms

$$\omega_i \quad \text{on} \quad V_i$$

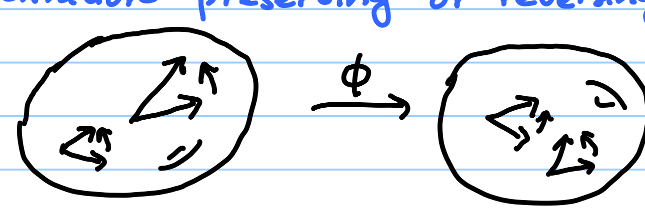
together by using a partition of unity subordinate to (V_i) in the following way:

$$\omega := \sum_{i \in \mathcal{I}} \underbrace{\phi_i \omega_i}_{\substack{\text{extended to an } n\text{-form on all of} \\ M \text{ by letting it vanish on} \\ M - \operatorname{supp}(\phi_i)}} .$$

Ex: Try to fill in the details (or look it up.)

For maps between oriented (orientable) manifolds we can speak of them being orientation-preserving or reversing

$$\underset{\omega_1}{\phi : M_1^n} \rightarrow \underset{\omega_2}{M_2^n}$$



Orientation preserving means :

$\forall p \in M_1 \qquad v_1, \cdots, v_n \in T_p M_1$

$\omega_1(v_1, \cdots, v_n)$, $\underbrace{\omega_2(D_p\phi\, v_1, \cdots, D_p\phi\, v_n)}_{(\phi^*\omega_1)(v_1,\cdots,v_n)}$ have the same sign

But this is the same as saying ω_1 and $\phi^*\omega_1$ determine the same orientation on M. This is the definition you find in MBT:

<u>Def</u> : $\quad \underset{\omega_1 \qquad \omega_2}{\phi : M_1^n \to M_2^n}$ is orientation preserving (resp. reversing)

if ω_1 and $\phi^*\omega_2$ determine the same orientation on M_1 as ω_1 (resp. $-\omega_1$).

Back to the Euclidean setting,

$$\phi : U_1 \overset{\text{open}}{\subset} \mathbb{R}^n \longrightarrow U_2 \overset{\text{open}}{\subset} \mathbb{R}^n \text{ is}$$

orientation preserving $\iff$ $\det(D_x\phi) > 0 \;\; \forall x \in U_1$.
resp. reversing $\qquad\qquad\qquad$ resp. <0

(Here U_1 and U_2 are equipped with the standard orientation of $\mathbb{R}^n$.)

This follows immediately from the following identity established in CH 3 :

$$\phi^*(dx_1 \wedge \cdots \wedge dx_n) = \det(D_x\phi)\, dx_1 \wedge \cdots \wedge dx_n .$$

This also implies that our two definitions of orientability are equivalent.

For a manifold M^n with an orientation form ω, we can then speak of oriented chart :

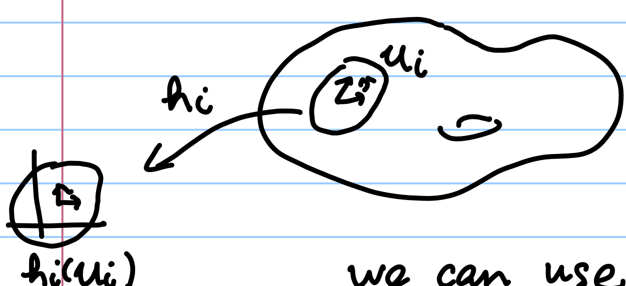
A chart $h : U \to h(U) \subseteq \mathbb{R}^n$ is oriented if it is orientation preserving

Two charts are both oriented $\Rightarrow$ their transition function is orientation preserving

A <u>positive atlas</u> of M is an atlas such that all Jacobi determinants of all transition functions are positive.

<u>Proposition 9.14</u> If M^n has a positive atlas $(h_i : U_i \to h_i(U_i))_{i \in J}$ then M has a uniquely determined orientation so all h_i are oriented charts.

<u>Proof</u> : Use the standard orientation on $h_i(U_i) \subset \mathbb{R}^n$ to induce an orientation ω_i on U_i so that h_i is orientation preserving.



Then by assumption, all ω_i are consistent at intersections and we can use Lemma 9.10 to 'stitch' them together into a global orientation for M. $\square$