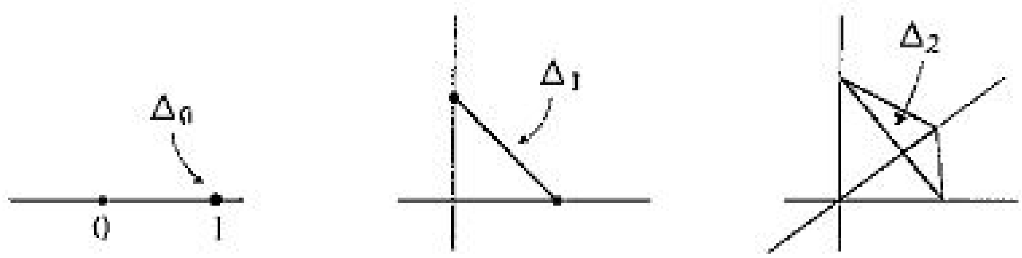
Math T680 Topics in Geometry

$\boxed{\text{HW \#7}}$

Due: Friday, June 16, 2017

The standard n-simplex Δ_n is defined as the set

$$\Delta_n = \{x \in \mathbb{R}^{n+1} : 0 \leqslant x^i \leqslant 1,\ \sum_{i=1}^{n+1} x^i = 1\}.$$



The subset of Δ_n obtained by setting $n-k$ of the coordinates x^i equal to 0 is homeomorphic to Δ_k, and is called a k-face of Δ_n.

Let M^n be a compact manifold. If $\Delta \subset M$ is a diffeomorphic image of some Δ_m, then the image of a k-face of Δ_m is called a k-face of Δ. Now by a **triangulation** of M we mean a finite collection $\{\sigma_i^n\}$ of diffeomorphic images of Δ_n which cover M and which satisfy the following condition:

If $\sigma_i^n \cap \sigma_j^n = \emptyset$, then for some k the intersection $\sigma_i^n \cap \sigma_j^n$ is a k-face of both σ_i^n and σ_j^n.

A smooth manifold can be triangulated – a theorem that turns out to be not very surprising but difficult to prove in general. For this homework let's take this fact for granted. Assuming that our (compact) manifold M has a triangulation $\{\sigma_i^n\}$. We call each σ_i^n an **n-simplex of the triangulation** and any k-face of any σ_i^n a k-simplex of the triangulation. Write

$$\alpha_k = \text{ number of } k\text{-simplices in the triangulation.}$$

1. Prove

$$\sum_{i=0}^{n} (-1)^i \alpha_i = \sum_{i=0}^{n} (-1)^i \dim H^i(M^n) := \text{Euler characteristics } \chi(M) \text{ of } M.$$

Hint: Let U be the disjoint union of open balls, one within each n-simplex σ_i^n, and let V_{n-1} be the complement of the set consisting of the centers of these balls, so that V_{n-1} is a neighborhood of the union of all $n-1$-simplexes of M. Then $M = U \cup V_{n-1}$ and $U \cap V_{n-1}$ has the same cohomology as a disjoint union of α_n copies of S^{n-1}. Use Mayer-Vietoris. Along the way, you will need the result from HW#3.

Let's call this result the '**generalized Descartes-Euler theorem**'.

2. If M is homeomorphic to S^2, use 1. to prove $\alpha_0 - \alpha_1 + \alpha_2 = 2$. If M is homeomorphic to the torus $S^1 \times S^1$, prove $\alpha_0 - \alpha_1 + \alpha_2 = 0$.

   The big results:

   - Poincaré-Hopf (relating $\chi(M)$ to properties of singularties of smooth vector fields on M)
   - Morse's theorems (relating $\chi(M)$ to properties of critical points of smooth functions $M \to \mathbb{R}$)
   - Gauss-Bonnet (relating $\chi(M)$ to the Gauss curvature of M, here M is a 2-dimensional manifold embedded in $\mathbb{R}^3$)
   - Generalized Descartes-Euler theorem (relating $\chi(M)$ to number of simplices of different dimensions in a triangulation of M)
   - triangulation theorem

   are all related. For a triangulated compact manifold M^n

   $$\chi(M) = \mathrm{Index}(X) = \sum_{\lambda=0}^{n} (-1)^\lambda c_\lambda = \sum_{i=0}^{n} (-1)^i \alpha_i.$$

   Chapter 12 of M&T establishes the first two connections. There is no coverage of the third (M&T wants to avoid triangulation.) This HW fills in this gap.

   In the two-dimension case, we also have

   $$\chi(M) = \frac{1}{2\pi} \int K \, dA.$$

   S.S. Chern extended this result to general even dimensional manifolds. The second half of M&T covers this.

3. We can use Poincaré-Hopf to prove the generalized Descartes-Euler's theorem. Can you recall the trick covered in class?

   With the same trick, we can use the generalized Descartes-Euler's theorem – instead of Morse's theory as it is done in M&T – to prove Poincaré-Hopf. Please outline such an argument.

   (The trick is easiest to picture in dimension $n = 2$. Let's focus on this case for this HW.)