Linear Programming: Shadow price and Game Theory (brief introduction)

Note Title 2/19/2022

Consider again the diet problem

One seeks the diet with lowest cost that achieves all the nutritional requirements:

$$\min\ c_1x_1+\cdots+c_nx_n \quad \text{s.t. } A_{11}x_1+\cdots+A_{1n}x_n \geq b_1$$
$$\vdots$$
$$A_{m1}x_1+\cdots+A_{mn}x_n \geq b_m$$
$$x_1,\ldots,x_n \geq 0$$

where

x_j = # of units of food j to be included in the diet
c_j = cost of 1 unit of food j
b_i = minimum daily requirement of nutrient i
A_{ij} = amount of nutrient i contained in one unit of food j

Notice that here the customers' only concern are the raw nutrients.

What if now there is a supplier who sells the raw nutrients directly?

y_i = price of 1 unit of nutrient i.

Food manufacturer

sushi
pizza
⋮

"Supplier"

sugar
salt

beef
pork
chicken

brocolli
cabbage
⋮

consumer

you
me
⋮

The food manufacturer would not buy from the supplier if the "raw nutrients" (beef, pork, chicken, brocolli, cabbage, ⋯,) that go into the food cost more than how much he can sell the food.

It means $y_1, \cdots, y_m \geq 0$ should satisfy:

$$[y_1, \cdots, y_m] \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \leq [p_1, \cdots, p_n].$$

Note: $\sum_{i=1}^{m} y_i A_{ij}$ = price of the equivalence of nutrients in 1 unit of food j.

Within these constraints, this sneaky supplier, knowing that the raw nutrients are what the customers truly care, also wants to steal the customers from the food manufacturer. So he solves:

$$\max\ b_1 y_1 + \cdots + b_m y_m \quad \text{s.t.} \quad [y_1, \cdots, y_m] \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \leq [p_1, \cdots, p_n], \quad [y_1, \cdots, y_m] \geq 0.$$

This happens to the dual LP of the LP that the consumer faces.

By strong duality, (the min food cost of a consumer) = (the max revenue of the supplier).

Shadow price / sensitivity analysis

Problem data is often uncertain, modelers may wish to know how the solution will be affected if, say, a constraint value is perturbed.

E.g. $\min\ x_1 + 1.5x_2 + 3x_3$ s.t.
$$\begin{aligned} x_1 + x_2 + 2x_3 &\geq 6 \\ x_1 + 2x_2 + x_3 &\geq 10 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

convert it into the standard form by adding slack variables x_4, x_5:

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}, \quad c = [1, 1.5, 3, 0, 0]^T.$$

Ex: solve this LP using the dual simplex method, and see that $B = \{1, 2\}$ is an optimal basis.

$$\begin{array}{r|c|c|} & x_N & 1 \\ \hline x_B = & -B^{-1}N & B^{-1}b \\ \hline c^T x = & (c_N - N^T B^{-T} c_B)^T & c_B^T B^{-1} b \\ \hline \end{array} \quad = \quad \begin{array}{r|ccc|c|l} & x_3 & x_4 & x_5 & 1 & \\ \hline x_1 = & & & & 2 & \\ x_2 = & & & & 4 & \geq 0 \\ \hline & 1.5 & 0.5 & 0.5 & 8 & \\ \hline & & \geq 0 & & & \end{array}$$

Q: If b changes a little, how would the optimal value changes?

A: $\dfrac{\partial c^T x^*(b)}{\partial b_i} = \lambda_i^*$ ← sol. of dual

Here is why:

Note that b only shows up in the last column of the tableau, meaning that if b is perturbed to, say, $\tilde{b}$, the basis B stays optimal as long as $B^{-1}b \geq 0$.

But since at the current b, $B^{-1}b > 0$, by continuity $B^{-1}\tilde{b} > 0$ if $\tilde{b} - b$ is small enough.

Let λ^* be the solution of the dual $\max b^T\lambda$ s.t. $A^T\lambda \leq c$, $\lambda \geq 0$

The same λ^* is of course feasible for $\max \tilde{b}^T\lambda$ s.t. $A^T\lambda \leq c$, $\lambda \geq 0$, and continues to be optimal, provided $B^{-1}\tilde{b} \geq 0$.

By strong duality, $c^Tx^*(b) = b^T\lambda^*$, $c^Tx^*(\tilde{b}) = \tilde{b}^T\lambda^*$ ($\tilde{b} - b$ small enough).

So $c^Tx^*(\tilde{b}) - c^Tx^*(b) = (\tilde{b} - b)^T\lambda^*$ so $\boxed{\dfrac{\Delta c^Tx^*(b)}{\Delta b_i} = \lambda_i^*}$

just need to be small, no need to take the limit. (Everything is linear here.)

Interpretation: $\boxed{\text{If a consumer consumes one more unit of nutrient } i \text{, that sneaky supplier would increase the price of 1 unit of nutrient } i \text{ by } \$\lambda_i^*.}$

For this reason, λ_i^* is called the shadow price of nutrient i.

Game Theory

Prisoner's dilemma Two criminals are caught. During plea bargaining, the District Attorney urges both criminals to confess and plead guilty.

They are separately offered the same deal:

Jail time of criminal X

$$A = \begin{array}{c} \begin{bmatrix} 5 & 0 \\ 10 & 1 \end{bmatrix} \begin{array}{l} \leftarrow X \text{ confesses} \\ \leftarrow X \text{ not confesses} \end{array} \\ \begin{array}{ll} \uparrow Y \text{ confesses} & \uparrow Y \text{ not confesses} \end{array} \end{array}$$

Jail time of criminal Y

$$B = \begin{array}{c} \begin{bmatrix} 5 & 10 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \leftarrow X \text{ confesses} \\ \leftarrow X \text{ not confesses} \end{array} \\ \begin{array}{ll} \uparrow Y \text{ confesses} & \uparrow Y \text{ not confesses} \end{array} \end{array}$$

If a (wise) Godfather advises his mob to never confess when caught, then there would be no dilemma, each would spend a year in jail.

Notice that it is a form of cooperation, and is a good compromise in this case.

What if these criminals are unorganized?

Not knowing what the other criminal has in mind, and each being rational, each try to optimize the worst scenario:

X solves : $\min_i \max_j A_{ij} = \min \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 = A_{11}$ so X chooses to confess

Y solves : $\min_j \max_i B_{ij} = \min [5, 10] = 5 = B_{11}$ Y chooses to confess also

A different (but subtly related) viewpoint :

$\begin{bmatrix} 5 & 0 \\ \wedge & \wedge \\ 10 & 1 \end{bmatrix}$ ← X chooses to confess, because he is better off regardless of Y's choice — (I)

$\begin{bmatrix} 5 < 10 \\ 0 < 1 \end{bmatrix}$ Y chooses to confess, because he is better off regardless of X's choice — (II)

Yet another viewpoint :

Y thinks X would confess because of (I), so Y confesses also.

X thinks Y would confess because of (II), so X confesses also.

Conclusion : In a non-cooperative setting (and assuming each player is perfectly "rational"), each criminal chooses to confess.

This is a trivial form of a Nash equilibrium : $\exists\, i^*, j^*$ (namely $i^*=1$, $j^*=1$)
st. $A_{i^*,j^*} \leq A_{ij^*}\ \forall i$, $B_{i^*j^*} \leq B_{i^*j}\ \forall j$

And, in this case, $\exists\, i^*$ s.t. $A_{i^*j} \leq A_{ij}\ \forall j$

Also (the game being fair for both players, reflected by $B=A^T$), $\exists\, j^*$ st. $B_{ij^*} \leq B_{ij}$, $\forall i$

Fact : Let $A, B \in \mathbb{R}^{m \times n}$ (think of them as payoff matrices, instead of jail time, for X and Y, respectively).

If $\exists\, i^*$ st $A_{i^*j} \geq A_{ij}$, $\forall i,j$ then $\exists$ a corresponding j^* s.t. the "trivial Nash equilibrium property"

$$\boxed{A_{i^*,j^*} \geq A_{ij^*}\ \forall i\ ,\ B_{i^*,j^*} \geq B_{i^*j}\ \forall j} \quad — \ (\star)$$

Similarly, if $\exists\, j^*$ st $B_{ij^*} \geq B_{ij}\ \forall i,j$ then $\exists$ a corresponding i^* st. $(\star)$ holds.

This is called a <u>dominant strategy</u> of X, it means regardless of the strategy of Y, X has the highest payoff applying strategy i^*.

In the case prisoner's dilemma, being a fair game, each player has a dominant strategy.

If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B=\left[\begin{array}{ccc}1 & > & 0\\ 3 & < & 4\end{array}\right]$, there is a dominant strategy for X, but no dominant strategy for Y.

Ex: Prove the fact above. (It's easy.)

Perhaps less obvious is that the converse is <u>not</u> true (even in the zero-sum setting.)

And, way less obvious, when even $(\ast)$ does not hold for the game, there is a randomized version of $(\ast)$ that always holds.

called Nash equilibrium.

The prisoner's dilemma is an example of a non-zero sum game ($A+B \neq 0$)

In a non-zero sum game, the two players need not be hostile to each other. Cooperation may lead to a "win-win" situation, much better than that offered by a (non-cooperative) Nash equilibrium.

In a zero sum game ($A+B=0$), there is no point to cooperate.

Examples of zero-sum games:

① A = payoff matrix for $X = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ (A_{ij} = payoff to X if X applies strategy i and Y applies strategy j)

payoff matrix for $Y = -A$.

Notice $A = \begin{bmatrix} 2 \leq 2 \\ \vee \quad \wedge \\ 1 \leq 3 \end{bmatrix}$. Clearly Y would always choose strategy $j^*=1$ to minimize his loss. X, being perfectly rational, knows Y thinks this way, would always choose $i^*=1$ to maximize her gain. (And it would not matter that Y knows X thinks that way.)

The (trivial) Nash equilibrium is $(i^*, j^*) = (1,1)$.

② $A = \begin{bmatrix} 1 & < & 2 & < & 3 \\ \wedge & & \| & & \vee \\ 3 & > & 2 & = & 2 \end{bmatrix}$. There is neither a dominating row (for X), nor a dominating column (for Y).

Nonetheless, the "trivial Nash equilibrium" property holds:

$$\max_i \left(\min_j A_{ij} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 = A_{22}$$

$$\min_j \left(\max_i A_{ij} \right) = [3; 2, 3] = 2 = A_{22}, \quad \text{with } (i^*, j^*) = (2,2)$$

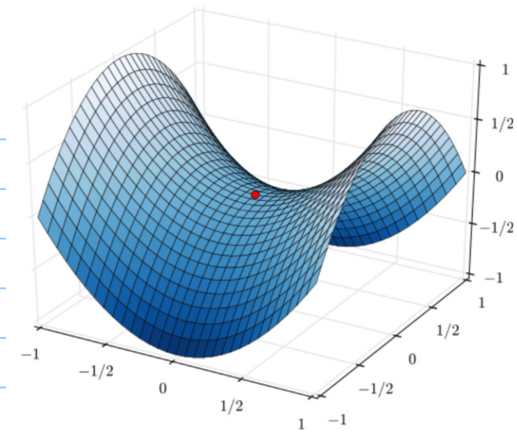
And $\underbrace{A_{ij^*} \leq A_{i^*j^*} \leq A_{i^*j}}$

if X keeps wielding strategy i^*, Y is never better off using any strategy other than j^*
if Y ———————— j^*, X ———————————————— i^*.

In general,

(i) $\max_i \left(\min_j A_{ij} \right) \leq \min_j \left(\max_i A_{ij} \right)$ (Proof: $\max_i \left(\min_j A_{ij} \right) \leq \max_i A_{ij} \ \forall j$)

(ii) $\underbrace{\max_i \left(\min_j A_{ij} \right) = \min_j \left(\max_i A_{ij} \right)}_{\text{the "max min = min max" property}} \iff \underbrace{\exists\, i^*, j^* \text{ s.t. } A_{ij^*} \leq A_{i^*j^*} \leq A_{i^*j},\ \forall i, j}_{\text{a saddle point property (or what I called a "trivial Nash equilibrium")}}$



A more general observation:

Let $f : A \times B \to \mathbb{R}$ (A, B can be any sets). Then:

(i) $\max\limits_{a\in A} \min\limits_{b\in B} f(a,b) \leq \min\limits_{b\in B} \max\limits_{a\in A} f(a,b)$

(ii) $\max\limits_{a\in A} \min\limits_{b\in B} f(a,b) = \min\limits_{b\in B} \max\limits_{a\in A} f(a,b) \iff \exists (a^*, b^*) \in A\times B$ s.t. $f(a,b^*) \leq f(a^*,b^*) \leq f(a^*,b)$ $\forall a\in A,\ b \in B$.

Proof:

(i) $\max\limits_{a\in A} (\min\limits_{b\in B} f(a,b)) \leq \max\limits_{a\in A} f(a,b)$ for any fixed $b \in B$

so $\max\limits_{a\in A} (\min\limits_{b\in B} f(a,b)) \leq \min\limits_{b\in B} \max\limits_{a\in A} f(a,b)$.

(ii) $(\Rightarrow)$ Assume a^* solves the 'maxmin', b^* solves the 'minmax'.

$$f(a^*,b^*) \leq \max_{a\in A} f(a,b^*) = \min_{b\in B} \max_{a\in A} f(a,b) = \max_a \min_b f(a,b) = \min_b f(a^*,b) \leq f(a^*,b)$$

Similarly, $f(a^*,b^*) \geq \min\limits_b f(a^*,b) = \max\limits_a \min\limits_b f(a,b) = \min\limits_b \max\limits_a f(a,b) = \max\limits_a f(a,b^*) \geq f(a,b^*)$

$(\Leftarrow)$ Assume that the saddle point property holds with (a^*, b^*).

We argue by contradiction that a^* solves the 'maxmin', b^* solves the 'minmax'.

Assume the contrary that a^* does not solve the maxmin, then

$$\max_a \min_b f(a,b) > \min_b f(a^*, b) = f(a^*, b^*) = \max_a f(a, b^*) \geq \min_b \max_a f(a,b),$$

which contradicts maxmin $\leq$ minmax.

So a^* solves the maxmin. Similarly, b^* solves the minmax. And we have

$$\max_a \min_b f(a,b) = f(a^*, b^*) = \min_b \max_a f(a,b)$$

Q.E.D.

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③ $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$ $\max_i \min_j A_{ij} = \max \begin{bmatrix} -2 \\ -2 \end{bmatrix} = -2$, $\min_j \max_i A_{ij} = \min [3 \;\; 1] = 1$

So there isn't a 'trivial' Nash equilibrium in this case.

Assume that you are alone, and you are lonely, a very attractive person invites you to be the X player (or the "row player").

An additional rule: If you agree to play, you have to play many games with this attractive person.

Can you resist?

Answer: (assuming you can stay rational), you should not play this game.

If such a zero-sum game is to be played many times, and assuming each player is perfectly rational, then

- neither player wants to be predictable, meaning that the choice of strategy in each game should be made independent from any previous game.

- so the decision to be made for each player is how often he/she should employ each of his/her strategies.

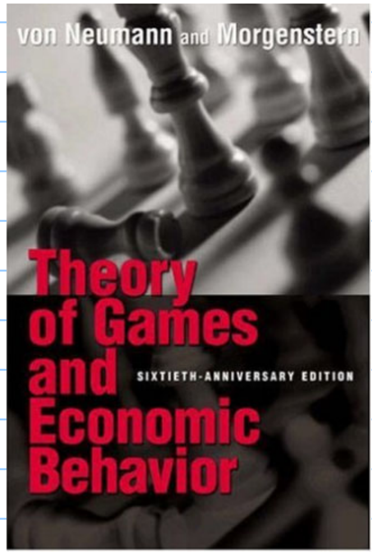
Let x_i = frequency/probability that player X uses strategy i, $i=1,\cdots,m$
y_j = ______________________________ j, $j=1,\cdots,n$

Prob[X uses i <u>and</u> Y uses j] $= x_i y_j$
(in each game)

So, in the long run, player X's average payoff (= player Y's average loss) is:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_i y_j \quad \text{per game.}$$

$$= X^T A Y \qquad X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$



Player X wants to maximize $x^T A y$ by choosing x, but he has no control of y.
Player Y ———— minimize ———————— y, but she ———————— x.

However,
X expects that whatever x he chooses, Y would pick y to hurt him the most.
Likewise,
Y expects that whatever y she chooses, X would pick x to hurt her the most.

So, ideally, X would like to solve $\max\limits_{x} \min\limits_{y} x^T A y$

Y ———————— $\min\limits_{y} \max\limits_{x} x^T A y$.

But if Y is busy hurting X, can she still afford to minimize her loss?
Same concern for X.

These problems look ill-defined, let alone having meaningful solutions.

Terminology: a choice by X of a probability vector $x \in \Delta_m = \{x \in \mathbb{R}_+^m : \sum x_i = 1\}$ is called a *mixed strategy* of X.

Similarly, a $y \in \Delta_n$ is a mixed strategy of Y.

If $x = [0, \cdots, 1, \cdots 0]^T \in \Delta_m$, $y = [0, \cdots, 1, \cdots 0]^T \in \Delta_n$, that's called a pure strategy.

Applying our earlier result to $f: \Delta_m \times \Delta_n \to \mathbb{R}$, $(x,y) \mapsto x^T A y$, we have

- $\max\limits_x \min\limits_y x^T A y \leq \min\limits_y \max\limits_x x^T A y$
- saying $\min\limits_y x^T A y = \min \max\limits_x x^T A y$ is equivalent to $\exists\, x^* \in \Delta_m, y^* \in \Delta_n$ s.t.

$$x^T A y^* \leq x^{*T} A y^* \leq x^{*T} A y \quad \forall x \in \Delta_m, y \in \Delta_n.$$

In the earlier (non-randomized, pure strategy) setting:

$$f: \{1, \cdots, m\} \times \{1, \cdots, n\} \to \mathbb{R}, \quad f(i,j) = A_{ij} = [0 \cdots \overset{\overset{i}{\downarrow}}{1} \cdots 0] A \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow j \\ \\ \\ \end{array},$$

for most payoff matrix A, $\max \min < \min \max$.

$\Leftrightarrow$ $f(i,j)$ has no saddle point

But in the randomized, mixed strategy setting,

$$f : \underset{\substack{\uparrow \\ \text{convex}}}{\Delta_m} \times \underset{\substack{\uparrow \\ \text{convex}}}{\Delta_n} \to \mathbb{R}, \quad (x,y) \mapsto x^T A y, \quad \leftarrow \begin{array}{l} \text{linear (hence concave) in } x \text{ for fixed } y \\ \text{linear (hence convex) in } y \text{ for fixed } x \end{array}$$

So, from this point of view, the landscape of this function resembles that of

$$y^2 - x^2 \quad \leftarrow \text{concave in } x, \text{ convex in } y, \text{ saddle point at } (0,0).$$

And perhaps f always has a saddle point.

And this is exactly what J. von Neumann discovered in 1928:

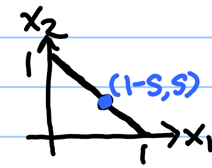
> For any $A \in \mathbb{R}^{m \times n}$, $\exists\, x^* \in \Delta_m, y^* \in \Delta_n$ s.t.
>
> $$x^T A y^* \le \underbrace{x^{*T} A y^*}_{\substack{\shortparallel \\ \max_x \min_y x^T A y \,=\, \min_y \max_x x^T A y}} \le x^{*T} A y \quad \forall\, x \in \Delta_m, y \in \Delta_n$$

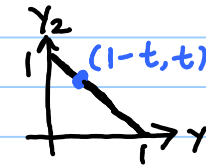
called the "value of the game"

(The equilibrum mixed strategies x^* and y^* need not be unique, but the value of the game is unique.)

For our earlier example,

$$x^T A y = [1-t,\ t] \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1-s \\ s \end{bmatrix}$$

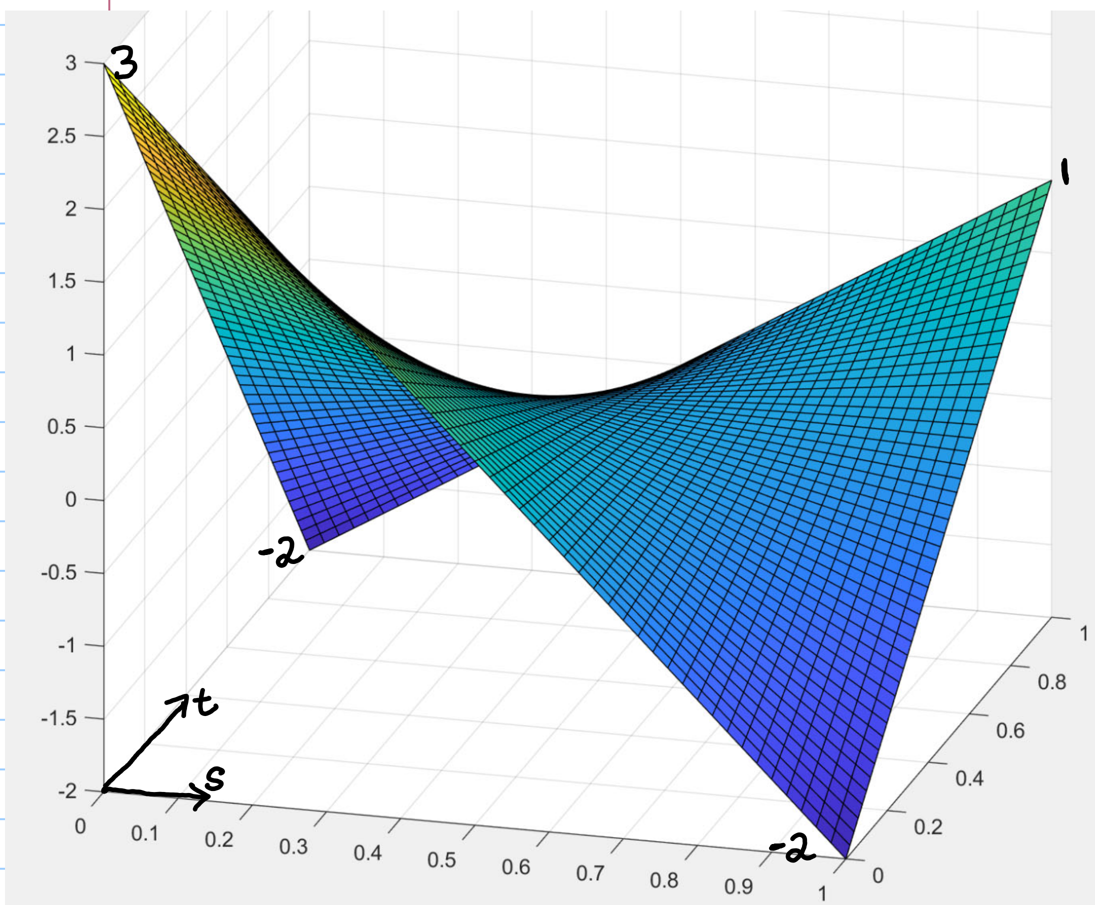


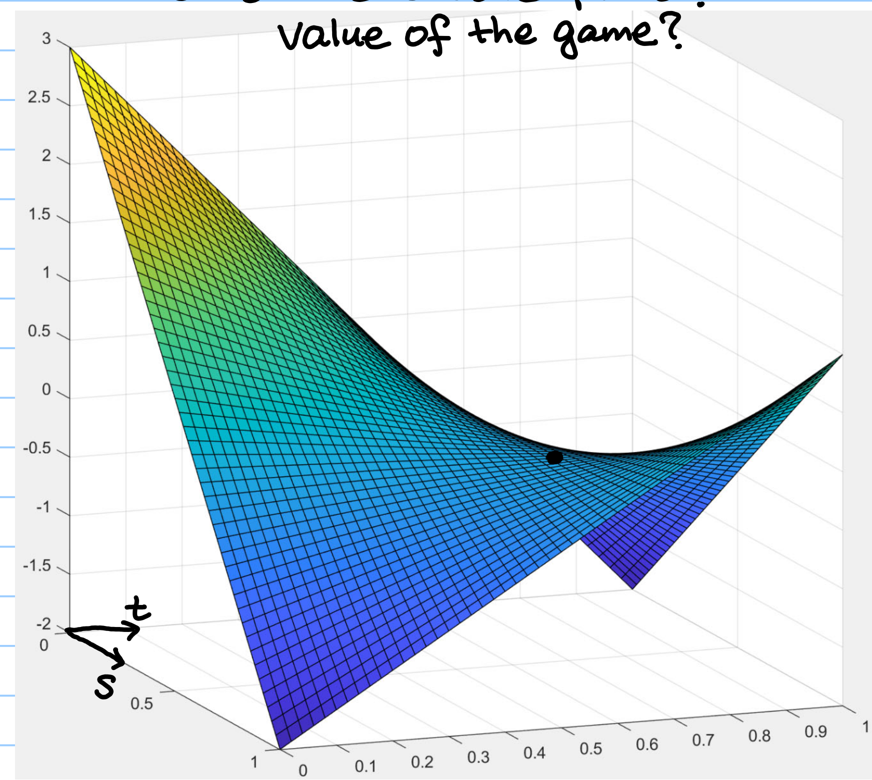


$\Delta_2 \cong [0,1]$

Where is the saddle point?

Value of the game?





We may think of $x^TAy = [x_1,\cdots,x_{m-1},\underset{}{\overset{1-x_1-\cdots-x_{m-1}}{\overset{\shortparallel}{x_m}}}]\,A\begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix}$ as a degree 2 polynomial in $(m-1)+(n-1)$ variables.

$$y_n = 1-y_1-\cdots-y_{n-1}$$

This polynomial always has a saddle point $(x_1,\cdots,x_{m-1},y_1,\cdots,y_{n-1})$ in $\mathbb{R}^{(m-1)+(n-1)}$. However, this saddle point will not be the one guaranteed by von Neumann's minimax theorem when

$$[x_1,\cdots,x_{m-1},1-x_1-\cdots-x_{m-1}]^T \notin \Delta_m \quad \text{or} \quad [y_1,\cdots,y_n,1-y_1-\cdots-y_{n-1}]^T \notin \Delta_n .$$

When $m=n=2$, write $[x_1,x_2]=[1-s,s]$, $[y_1,y_2]=[1-t,t]$

$$p(s,t) = [x_1,x_2]A\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = [1-s,s]\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} 1-t \\ t \end{bmatrix} = A_{11} + (A_{21}-A_{11})s + (A_{12}-A_{11})t + (A_{11}-A_{12}-A_{21}+A_{22})st$$

Critical point : $s^* = (A_{11}-A_{12})/(A_{11}-A_{12}-A_{21}+A_{22})$

$t^* = (A_{11}-A_{21})/(A_{11}-A_{12}-A_{21}+A_{22})$

$$\parallel$$

$$\text{const.} + \underbrace{\frac{1}{2}\begin{bmatrix} s-s^* \\ t-t^* \end{bmatrix}^T \begin{bmatrix} 0 & A_{11}-A_{12}-A_{21}+A_{22} \\ A_{11}-A_{12}-A_{21}+A_{22} & 0 \end{bmatrix}\begin{bmatrix} s-s^* \\ t-t^* \end{bmatrix}}_{=(s-s^*)(t-t^*)(A_{11}-A_{12}-A_{21}+A_{22})}$$

As long as $A_{11}-A_{12}-A_{21}+A_{22} \neq 0$, the Hessian is negative definite, and the critical point is a saddle point. In fact,

$$p(s,t^*) = p(s^*,t^*) = p(s^*,t), \quad \forall s,t.$$

This does not prove von Nemann's theorem even when $m=n=2$, because s^* or t^* may not be in $[0,1]$.

But when $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$, $s^* = t^* = 5/8 \in [0,1]$.

The value of the game is $\begin{bmatrix} 3/8 & 5/8 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} = -1/8$.

Interpretation: if the Y player holds her hand up 3/8 of the time and hand down 5/8 of the time, the X player losses at least \$1/8 per game on average regardless of what he does.

$A^T \neq -A$, so the game is by definition not fair. And indeed it is not fair! (ie. its value $\neq 0$)

(It is easy to construct a game with value 0, but $A^T \neq -A$:

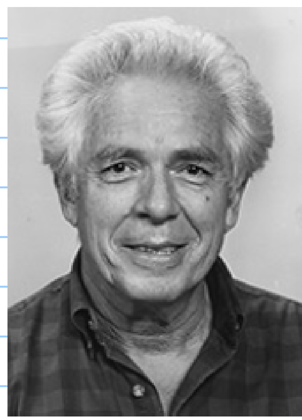
Take an arbitrary zero-sum game with payoff matrix A, subtract from every entry of A the value of the game. The resulted matrix $\tilde{A}$ must be the payoff matrix of a game with 0 value.)

Von Neumann's original proof (1928) is based on a non-constructive argument, using a fixed point theorem.

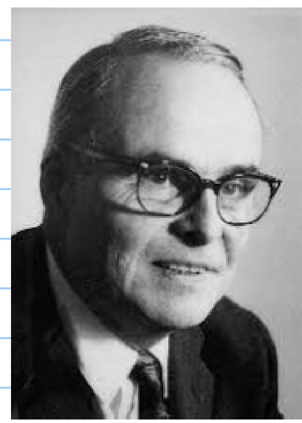
It inspired J. Nash to extend the result to multiple players, non-zero sum, non-cooperative games (1950), again using a non-constructive fixed point argument.

LP began before 1947, Gale, Kuhn, Tucker published the first proof of duality in 1951, again inspired by von Neumann's minimax theorem.

J. von Neumann

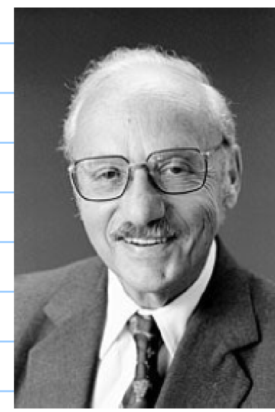


H. Kuhn



A. Tucker

KKT



G. Dantzig

J. Nash

We now reverse history by proving the minimax theorem — constructively — using LP duality.

Consider the primal-dual pair of LPs:

$$\text{(P)} \min_{x \in \mathbb{R}^m} [1,\cdots,1]x \text{ st. } A^T x \geq \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, \; x \geq 0 \qquad \text{(D)} \max_{y \in \mathbb{R}^n} [1,\cdots,1]y \text{ st } Ay \leq \begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}, y \geq 0$$

Note that (D) is always feasible, as $y = [0,\cdots,0]^T$ is a feasible point for (D).

(P), however, is not always feasible.

Ex (tricky): Prove that for a fair game (i.e. $A^T = -A$), (P) is never feasible.

Nonetheless, if we change the payoff matrix A to $A + \alpha \begin{bmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{bmatrix}$ for a big enough $\alpha > 0$, the resulted (P) must be feasible. (why?)

Also, adding a constant α to all entries of A only increases the expected payoff by the same α and cannot affect the optimal mixed strategies of either player, since

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j (A_{ij} + \alpha) = \sum_i \sum_j x_i y_j A_{ij} + \alpha \big(\underbrace{\sum_i \sum_j x_i y_j}_{=1} \big).$$

With such a fix, both (P) and (D) are feasible, and hence, by strong duality, have the same finite optimal value. In other words,

$$\exists\, \bar{x} \in \mathbb{R}^m,\ \bar{y} \in \mathbb{R}^n \text{ s.t. } \bar{x} \text{ solves (P)},\ \bar{y} \text{ solves (D) and}$$

$$\sum \bar{x}_i = \sum \bar{y}_j \leftarrow \text{call this common value } \theta \qquad (\theta > 0, \text{ as } x=0 \text{ is infeasible for (P)})$$

Let $x^* = \frac{1}{\theta}\bar{x} \in \Delta_m$, $y^* = \frac{1}{\theta}\bar{y} \in \Delta_n$. (These are then probability distributions.)

We now show that x^*, y^* have the desired saddle point property.

$$\text{Since } A^T\bar{x} \geq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \text{ for any } y \in \Delta_n,\ y^T A^T \bar{x} \geq y^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1.$$

Similarly, since $A\bar{y} \leq \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, for any $x \in \Delta_m$, $x^T A \bar{y} \leq x^T \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 1$.

$$\text{So,} \quad x^T A \bar{y} \leq 1 \leq \bar{x}^T A y, \text{ or } x^T A y^* \leq \underset{\substack{\uparrow \\ \text{value of the game}}}{1/\theta} \leq x^{*T} A y, \quad \forall x \in \Delta_m, y \in \Delta_n$$

QED.

Ex: For $A = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$, compute the value and the optimal mixed strategies for each playing by solving the primal-dual pair. Use the dual-simplex method.

Side note:

The graph of $p(x,y) = -x^2+y^2 = 2\left(-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right)\left(\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}\right) = [x,y]\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$

is just a 45° rotation of that of $q(s,t) = 2st = [s,t]\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}s\\ t\end{bmatrix}$.

Appealing to the spectral theorem, $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$.

Both have a saddle point at $(0,0)$.

But there is a qualitative difference from the point of view of "unilateral changes":

$$p(x,0) \leq p(0,0) \leq p(0,y) \quad \forall x,y,$$

whereas $$q(s,0) = q(0,0) = q(0,t) \quad \forall s,t.$$