CH 9 (continued)

Note Title 5/12/2017

Def: A Riemannian structure (or Riemannian metric) on a manifold M^n is a smooth assignment of inner-products (billinear, symmetric, positive definite)

$$\langle\, ,\rangle_p \text{ on } T_pM \quad \forall p \in M.$$

Here, 'smooth' means for any local parametrization

$$f : W \to M \quad \text{and} \quad v_1, v_2 \in \mathbb{R}^n$$

$$\underset{W}{\underset{\cap}{x}} \mapsto \underset{\mathbb{R}}{\underset{\cap}{\langle D_xf(v_1), D_xf(v_2)\rangle_{f(x)}}} \text{ is smooth.}$$

Being billinear, knowing $\langle\, ,\rangle_p$ is the same as knowing it on a basis.

The functions

$$g_{ij}(x) := \langle D_xf(e_i), D_xf(e_j)\rangle_{f(x)} \quad 1 \le i, j \le n$$

completely determines $\langle\, ,\rangle$ on $f(W)$. In particular these functions are smooth on any local parametrization $\Leftrightarrow$ $\langle\, ,\rangle_p$ satisfies the smoothness condition.

"first fundamental form"

Note

$[g_{ij}(x)]$ is a smooth function of $n \times n$ symmetric positive definite matrices.

Abstract setting: $M + \langle\, ,\rangle_p \to$ Riemannian manifold

Concrete setting: $M^n \subseteq \mathbb{R}^l$ submanifold $(n \le l)$
let
$\langle \cdot, \cdot \rangle_p$ be the restriction to $T_pM \subseteq \mathbb{R}^l$ of the usual inner product on $\mathbb{R}^l$.

Note : not every manifold has an orientation form

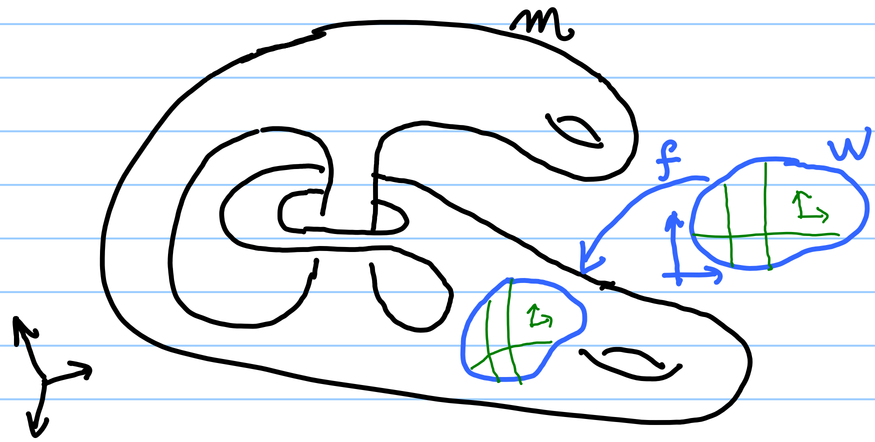
BUT

every manifold can be given a (in fact many) Riemannian metric (Not hard to prove using the same partition of unity idea.)

The Riemannian volume form

Recall from [Math538, Lecture 5 and HW4], assume

$f: W \to \mathcal{M} \subset \mathbb{R}^l$ is a local parametrization of a regular surface in $\mathbb{R}^l$ (i.e. $\mathcal{M} \subset \mathbb{R}^l$ is a submanifold).



$$\text{Volume } f(W) = \int_W \cdots \int \sqrt{\det \left[\langle D_x f(e_i), D_x f(e_j) \rangle_{f(x)} \right]}\, dx_1 \cdots dx_n$$

If $\mathcal{M}$ is now an abstract manifold with a Riemannian metric, the same formula makes sense, except

(✲) $$\text{Volume } f(W) = \int_W \cdots \int \sqrt{\det \left[\langle \underbrace{D_x f(e_i)}, \underbrace{D_x f(e_j)} \rangle_{f(x)} \right]}\, dx_1 \cdots dx_n$$

abstract tangent vectors in $T_{f(x)}\mathcal{M}$

abstract inner product

The two abstractions cancel each other

$$= \int_W \cdots \int \sqrt{\det [g_{ij}(x)]}\, dx_1 \cdots dx_n$$

This formula makes sense regardless of the orientability of $\mathcal{M}$.

It will be convenient to think that

- there is an n-form in $\circledast$, called the volume form of $\mathcal{M}$, to be defined next,

  and

- $\circledast$ is about "integrating this volume form" (integration of forms will be defined in CH 10)

Replace $(e_1, \dots, e_n)$ in the integrand of $\circledast$ by n arbitrary vectors in $\mathbb{R}^n$, and consider

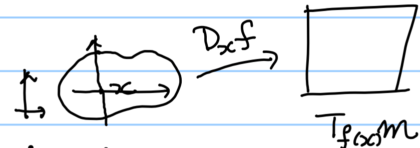
$$(v_1, \dots, v_n) \mapsto \sqrt{\cancel{\det \langle D_xf(v_i), D_xf}(v_j) \rangle_{f(x)}}$$

always positive, cannot be an n-form

$$(v_1, \dots, v_n) \overset{\eta}{\mapsto} \operatorname{sgn}(\det V) \cdot \sqrt{\det[\langle D_xf(v_i), D_xf(v_j)\rangle]}$$

$V = [v_1, \dots, v_n] \in \mathbb{R}^{n \times n}$

Claim: η is an n-form pointwise



(the determinant is suggestive, but the square root and stuff isn't helping.....)

Key: $\sqrt{\det X^T X} = |\det X|, \quad X \in \mathbb{R}^{n \times n}$

[LHS makes sense for $X \in \mathbb{R}^{n \times k}$, $k \le n$, but the RHS makes no sense for $k < n$.]

Proof: Pick o.n. basis $b_1, \dots, b_n$ on $(T_{f(x)}M, \langle \, , \rangle_{f(x)})$.

Let $A \in \mathbb{R}^{n \times n}$ be the matrix that represents

$D_x f : \mathbb{R}^n \to T_{f(x)}M$ in this basis.

So $$D_x f(e_i) = \sum_{k=1}^{n} A_{ik} b_k \qquad i = 1, \cdots, n$$

$$\underbrace{D_x f(v)}_{v = \sum_{l=1}^{n} v_l e_l} = \sum_l v_l D_x f(e_l)$$

$$= \sum_l v_l \sum_k A_{lk} b_k$$

$$= \sum_k \sum_l v_l A_{lk} b_k$$

$$\langle D_x f(v_i), D_x f(v_j) \rangle_{f(x)} = \langle \sum_k \sum_l (v_i)_l A_{lk} b_k, \sum_{k'} \sum_{l'} (v_j)_{l'} A_{l'k'} b_{k'} \rangle$$

$$= \sum_{k,k'} \sum_{l,l'} (v_i)_l A_{lk} A_{l'k'} (v_j)_{l'} \underbrace{\langle b_k, b_{k'} \rangle}_{\delta_{kk'}}$$

$$= \sum_k \sum_{l,l'} (v_i)_l A_{lk} A_{l'k} (v_j)_{l'}$$

or

$$\left[\langle D_x f(v_i), D_x f(v_j) \rangle_{f(x)} \right]_{1 \le i,j \le n} = V^T A A^T V$$

$$V = [v_1, \cdots, v_n] \in \mathbb{R}^{n \times n}$$

So

$$\eta(v_1, \cdots, v_n) = \operatorname{sgn}(\det V) \operatorname{sgn}(\det A^T V) \det A^T V$$

$$= \operatorname{sgn}(\det A^T) \det A^T V,$$

which is n-linear and alternating, i.e. an n-form.

I hope this derivation explains why "volume" would have anything to do with n-form. Notice that the derivation is done 'pointwise' and it shouldn't be surprising η is a (smooth) differential n-form on the parameter domain

$$W \overset{\text{open}}{\subset} \mathbb{R}^n,$$

which can be pulled-back by $(D_x f^{-1})^*$ to an n-form on

$$f(W) \overset{\text{open}}{\subset} M,$$

which we call $vol_{M,f}$. It has the property

$$vol_{M,f}(b_1, \cdots, b_n) = +1, \text{ or } -1$$

for any o.n. basis $(b_1, \cdots, b_n)$.

[check it, it is basically a tautology.]

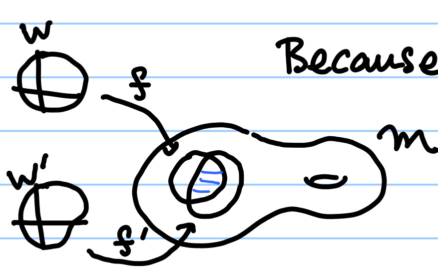
If we assume M is oriented, $(b_1, \cdots, b_n)$ is positively oriented, and the parametrization is orientation preserving, then

$$vol_{M,f}(b_1, \cdots, b_n) = +1. \quad \text{—— } (\star)$$

Can these "local volume forms" be 'stitched together' into a global n-form vol_M?

Answer: They are already stitched!

Because $(\star)$ means $vol_{M,f} = vol_{M,f'}$ on $f(W) \cap f(W')$. [why?]



So we have a differential n-form vol_M on M satisfying

$$vol_M(b_1, \cdots, b_n) = 1$$

for every positively oriented orthonormal basis of a tangent space T_pM.

M&T say all these in an entirely opposite order: (Little is said about what vol_M has to do with volume though.)

Proposition (orientation form meets Riemannian metric)
(9.16)

If M^n is an oriented Riemannian manifold then M^n has a uniquely determined orientation form vol_M with

$$vol_M(b_1, \cdots, b_n) = 1$$

for every positively oriented orthonormal basis of a tangent space T_pM.

vol_M is called the volume form on M.

In an orientation preserving local coordinate system $f: W \to M^n$,

$$f^*(vol_M) = \sqrt{\det(\underbrace{g_{ij}(x)}_{\langle D_xf(e_i), D_xf(e_j)\rangle_{f(x)}})}\, dx_1 \wedge \cdots \wedge dx_n$$

See the proof in M&T. (It uses the same sort of algebra as in my exposition.)

vol_M is called the Riemannian volume form of M.

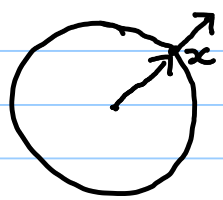
Example 9.18, 9.19

It must be that S^{n-1} is orientable?

What is a good way to write down an orientation form? the volume form?

How about the closely related $\mathbb{RP}^{n-1}$?

$S^{n-1} \subset \mathbb{R}^n$



Consider the $(n-1)$-form $\omega_0 \in \Omega^{n-1}(\mathbb{R}^n)$

$$\omega_{0x}(w_1, \cdots, w_{n-1}) := \det(x, w_1, \cdots, w_{n-1}).$$

- $\omega_{0x} \in \mathrm{Alt}^{n-1}(\mathbb{R}^n)$ for every fixed $x \in \mathbb{R}^n$
- varies smoothly with x, and
- If $x \in S^{n-1}$, $w_1, \cdots, w_{n-1}$ is a basis of $T_x S^{n-1}$, then

$x \perp w_i$ so
$x, w_1, \cdots, w_{n-1}$ is a basis for $\mathbb{R}^n$ and

$$\omega_{0x}(w_1, \cdots, w_{n-1}) \neq 0.$$

$i: S^{n-1} \to \mathbb{R}^n$ inclusion

$i^*\omega_0$ is a non-vanishing $(n-1)$-form on S^{n-1}.

For the orientation of S^{n-1} given by ω_0, the basis $w_1, \cdots, w_{n-1}$ of $T_x S^{n-1}$ is positively oriented
iff
the basis $x, w_1, \cdots, w_{n-1}$ for $\mathbb{R}^n$ is positively oriented.

We give S^{n-1} the Riemannian structure induced by $\mathbb{R}^n$. Then

$$\mathrm{vol}_{S^{n-1}} = i^* \omega_0$$

[Since if $w_1, \dots, w_{n-1}$ is a positively oriented o.n. basis of $T_x S^{n-1}$, then

$$(i^*\omega_0)(w_1, \dots, w_{n-1}) = \det(x, w_1, \dots, w_{n-1}) = 1.$$

This property is satisfied only by the volume form.]

Claim: $\boxed{\mathbb{RP}^{n-1} \text{ is orientable} \iff n \text{ is even}}$

I'll put the details in the next HW.

For a smooth submanifold $M^n \subset \mathbb{R}^{n+k}$, we have

$$T_pM^\perp = \{v \in \mathbb{R}^{n+k} : v \perp T_pM\} \quad \text{for each } p \in M.$$

A smooth normal vector field Y on an open set $U \subset M$ is a smooth map

$$Y : U \to \mathbb{R}^{n+k} \quad \text{with } Y(p) \in T_pM^\perp, \ \forall p \in U.$$

When the co-dimension $k = 1$, and $\|Y(p)\| = 1 \ \forall p$, Y is called a Gauss map on U.

$\bigcup_{p \in M} T_pM$ is called the tangent bundle of M
(well-defined without any Riemannian metric or embedding)

$\bigcup_{p \in M} T_pM^\perp$ is called the normal bundle of M

– definition depends on both an embedding and a Riemannian metric

Lemma For every $p_0 \in M^n \subseteq \mathbb{R}^{n+k}$, $\exists$ an open neighborhood W of p_0 and smooth vector fields Y_j $(1 \le j \le k)$ on W s.t.

$Y_1(p), \cdots, Y_k(p)$ form an o.n. basis of $T_pM^\perp$ for every $p \in M$.

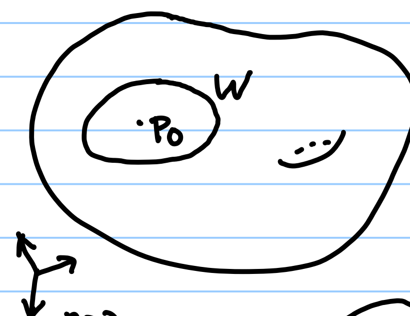
Issue at hand:

use a coordinate patch W around p_0 to induce a smooth tangent vector fields

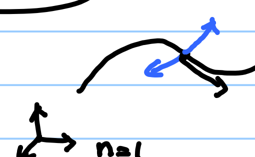
$$X_1(p), \cdots, X_n(p), \quad p \in W.$$

For each $p \in W$, of course we can find vectors $Y_1(p), \cdots, Y_k(p)$ that form an o.n. basis for T_pM.

But how would you pick them so that they vary smoothly with p?



$n=2$
$k=1$



$n=1$
$k=2$

Trick:

Pick an arbitrary o.n. basis $Y_1, \cdots, Y_k$ of $T_{p_0}M^\perp$ and just at p_0.

Use the same $Y_1, \cdots, Y_k$ at the nearby points p.

They won't be orthogonal to T_pM for $p \neq p_0$, but should be linear independent to $X_1(p), \cdots, X_n(p)$ by continuity.

Orthogonalize $X_1(p), \cdots, X_n(p), Y_1, \cdots, Y_k$

$$\Big\downarrow \text{ Gram-Schmidt}$$

$\left.\tilde{X}_1(p), \cdots, \tilde{X}_n(p), Y_1(p), \cdots, Y_k(p)\right\}$ argue that these vary smoothly with p.

Proposition 9.22

Let $M^n \subseteq \mathbb{R}^{n+1}$ be a smooth submanifold of co-dimension 1.

(i) The map

$$\{\text{smooth normal vector fields on } M\} \to \Omega^n(M)$$

defined by $Y \mapsto \omega = \omega_Y$,

$$\omega_p(W_1, \cdots, W_n) = \det(Y(p), W_1, \cdots, W_n)$$

is a 1-1 correspondence.

(ii) This induces a 1-1 correspondence between Gauss maps $Y : M \to S^n$ and orientations on M.

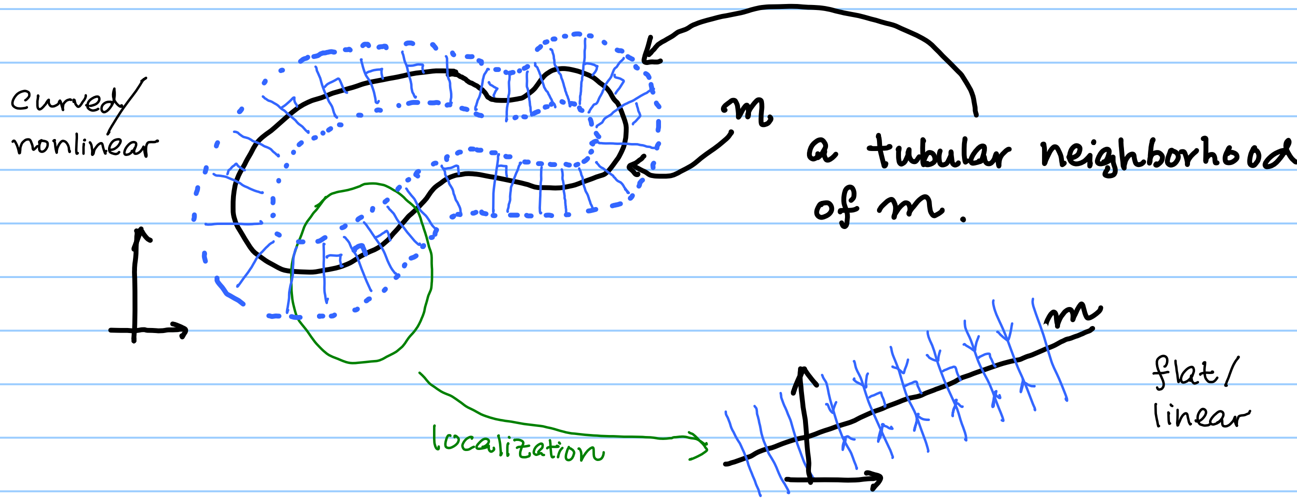
[The generic case is when M is connected and orientable, then M has exactly two Gauss maps that correspond to two orientations.]

In part (i), M need not be orientable, Y is allowed to vanish at some (or all) points of M, ω is allowed to vanish. In fact the first fact you need for the proof is:

$$Y(p) = 0 \iff \omega_p = 0.$$

[See MBT for the complete proof. It's quite easy.]

The last result we need is pretty obvious intuitively, but it is arduous to prove.



Theorem.

Let $M^n \subseteq \mathbb{R}^{n+k}$ be a smooth submanifold. There exists an open set $V \subseteq \mathbb{R}^{n+k}$ with $M \subseteq V$ and an extension of id_M to a smooth map

$$r : V \to M \qquad \text{"retraction map"}$$

s.t.

(i) For $x \in V$ and $y \in M$

$$\|x - r(x)\| \leq \|x - y\| \qquad \text{equality} \iff y = r(x).$$

(ii) For $p \in M$, $r^{-1}(p)$ is an open ball in the affine subspace $p + T_pM^\perp$ with center at p and radius $\rho(p)$,

$$\rho : M \to \mathbb{R}^+ \text{ is a smooth positive fcn.}$$

If M is compact then ρ can be chosen to be constant.

(iii) If $\varepsilon : M \to \mathbb{R}$ is smooth and

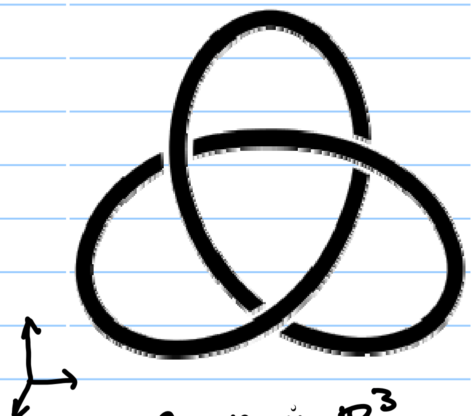
$$0 < \varepsilon(p) < \rho(p), \quad \forall p \in M,$$

then the "offset surface"

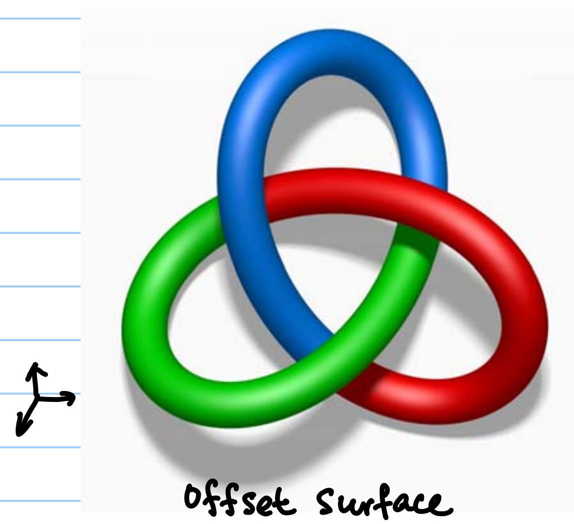
$$S_\varepsilon := \{ x \in V : \| x - r(x) \| = \varepsilon(r(x)) \}$$

is a smooth submanifold of codimension 1 in $\mathbb{R}^{n+k}$

E.g.

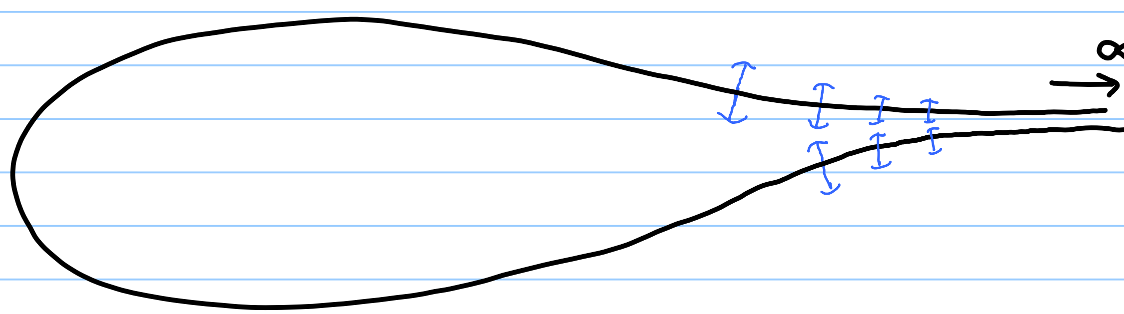


curve in $\mathbb{R}^3$
codimension $k=2$

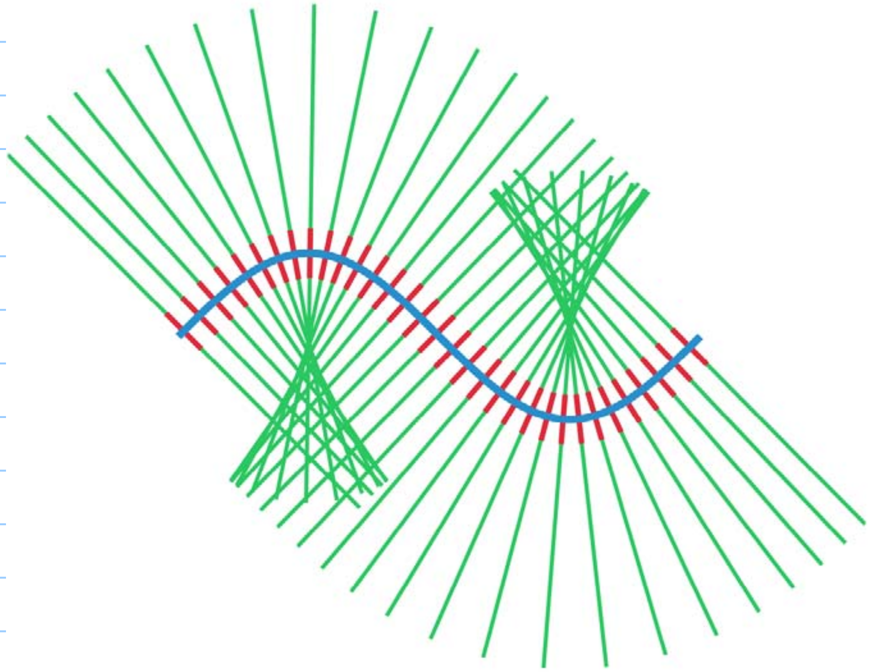
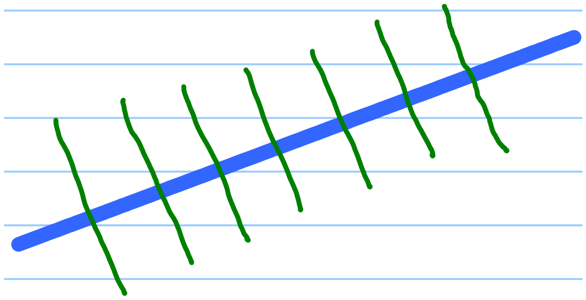


offset surface
codimension 1

In (ii), $\rho(p)$ cannot be made constant when M is not compact, e.g.



In general, unless M is a linear subspace, $\rho(p)$ cannot be chosen to be infinity:

Applications of tubular neighborhoods for calculating de Rham cohomology of manifolds

$M^n \subseteq \mathbb{R}^{n+k}$ V tubular neighborhood of M

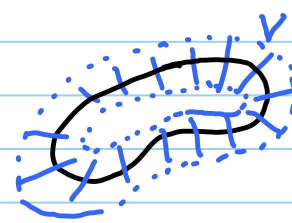
$r: V \to M$ retraction map

$i: M \to V$ inclusion

$$r \circ i = id_M$$

so

$$H^d(i) \circ H^d(r) = id_{H^d(M)}$$



$$H^d(M) \xleftarrow{H^d(r)} H^d(V) \xleftarrow{H^d(i)} H^d(M)$$

so it must be that $H^d(r)$ is surjective and $H^d(i)$ is injective.

Theorem (5.5) Assume that $\mathcal{U} \overset{\text{open}}{\subset} \mathbb{R}^N$ is covered by a finite collection of convex open sets $\mathcal{U}_1, \cdots, \mathcal{U}_r$. Then
$$\dim H^p(\mathcal{U}) < \infty \quad \forall p.$$

[The proof is an elegant (and quite easy) application of Mayer-Vietoris.]

Proposition. For any compact manifold $\mathcal{M}^n$ all cohomology spaces $H^p(\mathcal{M})$ are finite dimensional.

Proof We may assume that $\mathcal{M}^n \underset{\substack{\text{smooth}\\ \text{submanifold}}}{\subseteq} \mathbb{R}^{n+k}$.

Let V be a tubular neighborhood of $\mathcal{M}$ with

retraction $r: V \to \mathcal{M}$, inclusion $i: \mathcal{M} \to V$.

For any $p \in \mathcal{M}$, $\exists$ open ball in $\mathbb{R}^{n+k}$, $B(p, r_p)$, s.t.

$$p \in B(p, r_p) \subset V.$$

So we have an open covering $\bigcup_p B(p, r_p)$ of $\mathcal{M}$. By compactness there is a finite sub-covering

$$\mathcal{M} \subseteq \underbrace{B(p_1, r_{p_1}) \cup \cdots \cup B(p_r, r_{p_r})}_{\mathcal{U}} \subseteq V$$

We have a smooth inclusion $i: \mathcal{M} \to \mathcal{U}$ and smooth retraction $r|_{\mathcal{U}}: \mathcal{U} \to \mathcal{M}$

The argument above shows that $H^d(i): \underbrace{H^d(\mathcal{U})} \to H^d(\mathcal{M})$ is surjective. But

$\dim H^d(\mathcal{U}) < \infty$ by Theorem 5.5, and so is $H^d(\mathcal{M})$. $\square$

Proposition 9.25 Let M_1 and M_2 be smooth submanifolds of Euclidean spaces.

(i) If $f_0, f_1 : M_1 \to M_2$ are two homotopic smooth maps, then

$$H^d(f_0) = H^1(f_1) : H^d(M_2) \to H^d(M_1)$$

(ii) Every continuous map $M_1 \to M_2$ is homotopic to a smooth map.

Proof: We established in CH6 this proposition in the special case when M_1 and M_2 are open sets in Euclidean spaces.

Use the tubular neighborhood theorem to "bulk up" the manifolds M_1, M_2 to V_1, V_2.

$$\begin{tikzcd} M_1 \arrow[r, shift left, "f_0"] \arrow[r, shift right, "f_1"'] \arrow[d, shift right, "i_1"'] & M_2 \arrow[d, shift right, "i_2"'] \\ V_1 \arrow[u, shift right, "r_1"'] & V_2 \arrow[u, shift right, "r_2"'] \end{tikzcd}$$

$$f_0 \simeq f_1$$
$$\Downarrow$$
$$i_2 \circ f_0 \circ r_1 \simeq i_2 \circ f_1 \circ r_1 \quad : V_1 \to V_2$$

(i) Hence $$H^d(i_2 \circ f_0 \circ r_1) = H^d(i_2 \circ f_1 \circ r_1) \quad : H^d(V_2) \to H^d(V_1)$$

$$\underset{\substack{\uparrow \\ \text{injective}}}{H^d(r_1)} \circ H^d(f_0) \circ \underset{\substack{\uparrow \\ \text{surjective}}}{H^d(i_2)} = H^d(r_1) \circ H^d(f_1) \circ H^d(i_2)$$

$$\Rightarrow \qquad H^d(f_0) = H^d(f_1).$$

(ii) If $\phi : M_1 \to M_2$ is continuous,

$$i_2 \circ \phi \circ r_1 : V_1 \to V_2 \text{ is continuous.}$$
$$\simeq$$
$$\text{some } g : V_1 \to V_2 \text{ smooth (CH6).}$$

Since $i_2 \circ \phi \circ r_1 \simeq g$

$$\underbrace{r_2 \circ (i_2 \circ \phi \circ r_1) \circ i_1}_{\substack{\shortparallel \\ \phi}} \simeq \underbrace{r_2 \circ g \circ i_1}_{\text{smooth}} \quad : M_1 \to M_2$$

$\square$

As in CH6, we can now speak of

$$\boxed{H^p(\phi) \overset{\text{def}}{:=} H^p(f) \quad \text{for any choice of smooth } f \simeq \phi \text{, the choice doesn't matter.}}$$

The following result from CH6 now generalizes to manifolds:

Thm For $p \in \mathbb{Z}$, ~~open sets~~ smooth manifolds U, V, W ~~in Euclidean spaces~~, we have

(i) If $\phi_0, \phi_1 : U \to V$ are homotopic continuous maps, then

$$\phi_0^* = \phi_1^* : H^p(V) \to H^p(U).$$

(ii) If $\phi : U \to V$, $\psi : V \to W$ both continuous then

$$(\psi \circ \phi)^* = \phi^* \circ \psi^* \; : H^p(W) \to H^p(U)$$

(iii) If the continuous map $\phi : U \to V$ is a homotopy equivalence, then

$$\phi^* : H^p(V) \to H^p(U) \text{ is an isomorphism.}$$

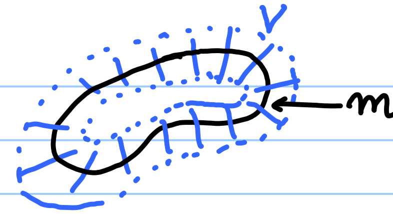
Corollary: A homoemorphism $h : U \to V$ between ~~open sets in Euclidean spaces~~ smooth manifolds induces isomorphisms

$$h^* : H^p(U) \to H^p(V) \text{ for all } p.$$

A "bootstrap":

$r: V \to M$ retraction map

$i: M \to V$ inclusion



$$\boxed{\begin{array}{l} H^d(r) \text{ is surjective} \\ H^d(i) \text{ is injective} \end{array}} \quad (\ast)$$

$$\Downarrow$$

Proposition 9.26

Proposition 9.26, in turn, implies a stronger version of $(\ast)$.

Corollary: $H^d(i): H^d(V) \to H^d(M)$ is an isomorphism with $H^d(r)$ as its inverse.

<u>Proof</u>: $r \circ i = id_M \Rightarrow H^d(i) \circ H^d(r) = id_{H^d(M)}$

Also, $\underbrace{i \circ r}_{= r.} \simeq id_V$, because V contains the line segment between x and $r(x)$, $\forall x \in V$.

By Proposition 9.26 (i),

$$H^d(r) \circ H^d(i) = id_{H^d(V)}.$$

So $H^d(r)$ and $H^d(i)$ are inverse of each other. $\square$

<u>Example</u>: $\mathbb{R}^{n+1} - \{0\}$ can be thought of as a (very big) tubular neighborhood of S^n, with

$$r: \mathbb{R}^{n+1} - \{0\} \to S^n, \quad r(x) = x/\|x\|.$$

So $$H^d(S^n) \cong H^d(\mathbb{R}^{n+1} - \{0\}) \cong \begin{cases} \mathbb{R} & \text{if } d = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

The Mayer-Vietoris theorem has a natural extension to manifolds. I let you explore it in HW #6.