LAW OF LARGE NUMBERS & KELLY RULE OF BETTING

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Steve is both a good tennis player and a professional trader. He has $C = \$10,000$ spare money to gamble with his ball pals in tennis matches. In each match, he would wager $\$x$ for a match he plays against someone, meaning that he would win $\$x$ from his opponent if he wins, and lose $\$x$ if he loses.

Steve is a relatively strong player among his peers, he knows he has a 65% chance ($p = 0.65$) of winning each match against any of them. Yet, his (pretty well-off) pals are happy to take his wager for any amount up to $10,000.

He adopts a trader's point of view: to maximize his expected return and minimize his risk.

A constraint here is that **he must decide in advance how many matches he is going to play**, because he has to schedule his matches in advance given his and his friends' busy schedule. In particular, he must decide in advance the number of matches he is going to play, denoted by n, and wager exactly $\$10,000/n$ in each match.

How should Steve choose n in order to achieve his goal?

Assume n is chosen and $X_1, \ldots, X_n$ are the returns of Steve's n matches. We assume Steve's performance in different matches are independent. So $X_1, \ldots, X_n$ are i.i.d. with a distribution of $P[X_i = +\$10000/n] = 0.65$ and $P[X_i = -\$10000/n] = 0.35$. In you like, you can think of each X_i as a *scaled* Bernoulli random variable:

$$X_i \sim \frac{C}{n} \underbrace{(2 \cdot \text{Bernoulli}(p) - 1)}_{\substack{=1 \text{ or } -1 \text{ with} \\ \text{probability } p = 0.65 \\ \text{and } p = 0.35, \text{ resp.}}}. \tag{0.1}$$

The total return is $X_1 + \ldots + X_n$.

The expected return is: $E[X_1 + \ldots + X_n] = E[X_1] + \ldots + E[X_n] = n(10000/n \cdot 0.65 - 10000/n \cdot 0.35) =$ \$3000, which is independent of n.

The 'risk' level can be modelled by how much his actual return can deviate from his expected return on average. If $n = 1$, his return is -\$10,000 with probability 0.35 and \$10,000 with probability 0.65. In contrast, if $n = 2$, the return is -\$10,000 with probability 0.35^2, \$0 with probability $2 \cdot 0.35 \cdot 0.65$, and \$10,000 with probability 0.65^2. Given that $0.35^2 \ll 0.35$, it is clear that Steve is much less likely to lose (all) his money by choosing $n = 2$ than $n = 1$. Moreover, with $n = 2$ his actual return has a smaller average deviation from the expected \$3000 gain when compared to $n = 1$.

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Formally, if we model 'risk' by the average *square*[1] deviation of the actual return from the expected return, then the risk becomes $E[(\text{total return} - 3000)^2]$, which happens to be the same as the *variance* of $X_1 + \cdots + X_n$. So,

$$\begin{aligned}\text{`Risk'} = V[X_1 + \cdots + X_n] &= nV[X_1] = n \cdot V[X_1] \\ &= n\left[(C/n)^2 4p(1-p)\right] \quad \text{by (0.1)} \\ &= \frac{4C^2 p(1-p)}{n}.\end{aligned}$$

(Note: In above I used the fact that $V[\sum_i \alpha_i Y_i] = \sum_i \alpha_i^2 V[Y_i]$, if $Y_1, \ldots Y_n$ are **independent** random variables and $\alpha_1, \ldots, \alpha_n$ are constants. I also used the fact the variance of a Bernoulli(p) random variable is $p(1-p)$.)

The variance shrinks linearly with n, and it has an important practical consequence: Despite the inherent uncertainty in the outcome of any match, by choosing n big Steve can almost surely win \$3000 using his tennis skills.

This is a manifestation of the law of large number.[2] The basic results we use from probability are: if $X_1, \ldots, X_n$ are any random variables, then $E[\sum_i \alpha_i X_i] = \sum_i \alpha_i E[X_i]$. If $X_1, \ldots, X_n$ are independent, then $\text{Var}[\sum_i \alpha_i X_i] = \sum_i \alpha_i^2 \text{Var}[X_i]$. In particular, if $X_1, \ldots, X_n$ are i.i.d. with the common variance σ^2, then

$$\text{Var}\Big[\frac{1}{n}\sum_i X_i\Big] = \frac{\sigma^2}{n}.$$

Of course, choosing n big means Steve has to play a very large number of matches, and make an embarrassingly small bet each time. Steve loves tennis so the former is not a problem for him, but his friends may not want him to bring such a 'high frequency trading' scheme into their tennis circle.

Exercise: If Steve improves his tennis skills so that his winning probability p increases, what does it mean to the risk level?

Exercise: If Steve is risk-seeking instead of risk-averse, he would choose the smallest possible n, namely, $n = 1$. But what does it really mean in practice? What, if any, is the advantage of choosing $n = 1$ from the point of view of financial reward?

Kelly Rule of Betting. The setup above means Steve must predetermine the number of games played and he cannot compound his profits earned from earlier matches. We now change the setting to that of a money manager: Steve will play essentially an unlimited number of games (unless his

[1]The choice of using the square is up to debate here. If the actual return is lower than the expected return, it is of course a bad thing and should be accounted for as risk. But there is nothing wrong when the actual return is higher than the expected return. Let me just say that the use of a square here is a famous trick of the mathematicians and makes the math easier. It may not be the best way to quantify risk but yet it leads to conclusions with practical relevance.

[2]In statistics terms, the law of large numbers states that the sample mean approaches the population means when the sample size gets large.

account goes bankrupt), and his goal is to maximize the **long term growth** of his capital, by **compounding his earlier gains**.[3]

Imagine that Steve has been betting \$2000 on each game for a year, and managed to grow his account from \$10,000 to \$30,000. With three times more capital, it seems to make sense to make larger bets. In general, the size of his bets should be proportional to his current capital. The question then becomes:

What percentage of his *current* capital should he bet on each game in order to maximize the **long term growth** of his capital?

Assume that the money manager bet $100r\%$ of his capital in each game. Then, after each game, his capital goes up by a factor of $1+r$ with probability p, and shrinks by a factor of $1-r$ with probability $1-p$. This can be viewed as a random variable, which we denote by R_i for the i-th game. Then $R_1, R_2, \ldots$ are i.i.d. random variables with the probability mass function

$$P[R_i = 1+r] = p, \quad P[R_i = 1-r] = 1-p.$$

After n games, the fortune of this money manager is:

$$CR_1R_2\cdots R_n.$$

We would like to choose r so that the (long term) average growth per game, quantified by

$$(R_1R_2\cdots R_n)^{1/n}, \tag{0.2}$$

is maximized.

Next, notice that

$$G := \log(R_1R_2\cdots R_n)^{1/n} = \frac{1}{n}\sum_{i=1}^{n}\log(R_i). \tag{0.3}$$

Since log is a monotonic function, maximizing (0.2) is the same as maximizing (0.3). But, wait, what does it even mean to maximize the random variable G?

Here is the big idea, using again the law of large number. As we are thinking of long-term growth, n is large. When n is large, the variance of G, being the average of a bunch of i.i.d. random variables, is negligible. Remember that

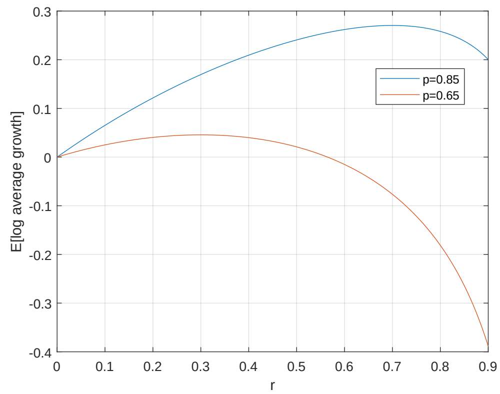
$$\mathrm{Var}(G) = \frac{\mathrm{Var}(\log R_i)}{n} \to 0, \qquad \text{as } n \to \infty.$$

Hence, for large n, we may think of G as being very close to its expected value, which is

$$E[G] = E[\log R_i] = p\log(1+r) + (1-p)\log(1-r).$$

You may sense that there is a clever choice of r, dependent on the value of $p(> 1/2)$, that maximizes $E[G]$. If you gamble too much, you can go bankrupt easily; if you bet too little, you grow your capital too slowly. In the extreme, we have $E[G] = 0$ if $r = 0$, and, worse, $E[G] \to -\infty$ if $r \to 1$.

[3]It is said that compounding is the 8th wonder of the world. However, compounding also means you are not supposed to spend a single penny you make.



The plot above already gives you the ballpark of that clever choice of r. We can obtain an analytic formula of the maximizer by solving the equation $\frac{d}{dr}p\log(1+r)+(1-p)\log(1-r)=0$. Since

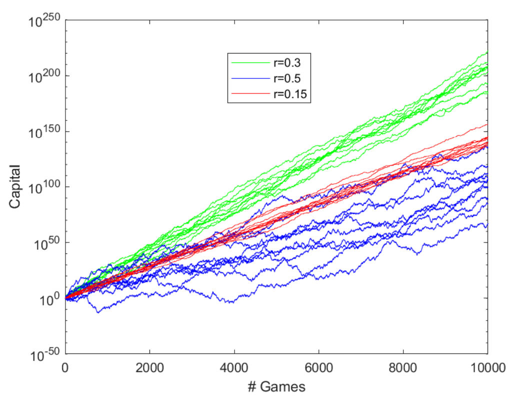
$$\frac{d}{dr}p\log(1+r)+(1-p)\log(1-r)=\frac{p}{1+r}-\frac{1-p}{1-r}=\frac{2p-1-r}{(1+r)(1-r)}.$$

The maximizing r is

$$r^*=2p-1.$$

This is known as the Kelly rule of betting.

For $p=0.65$, here are 10 simulations of the growth of capital after 10000 games, with $r=2\cdot 0.65-1=0.3$, $r=0.5$ and $r=0.15$.



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