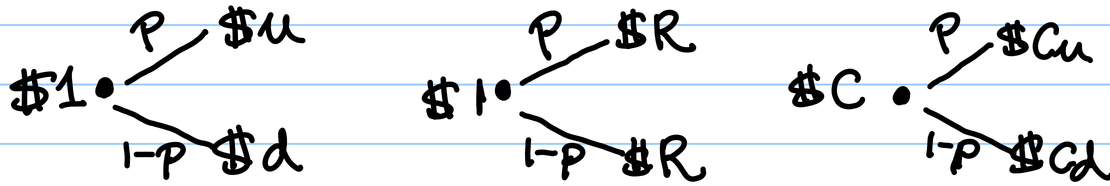
Week 4

Note Title 3/30/2015



any security | bond | any derivative of the security

the "underlying" | | the "derivative"

$$\boxed{\begin{aligned} C &= \frac{1}{R}\left[\frac{R-d}{u-d}C_u + \frac{u-R}{u-d}C_d\right] \\ &= \frac{1}{R}\left[\tilde{p}\,C_u + (1-\tilde{p})\,C_d\right] \\ & \qquad\qquad \tilde{p} = \frac{R-d}{u-d} \end{aligned}}$$

Notice the logic:

irrelevant	relevant
• p	• u, d
• exact type of the underlying	• R
• exact type of the derivative (call, put, future, exotic derivatives etc.)	• (C_u, C_d): the payoff of the derivative contingent upon the payoff of the underlying

no p, but $\tilde{p}$!

Aside: The underlying itself can be a derivative product. A standard example is "option on future".

- Does this (probability?) $\tilde{p}$ have a deeper meaning?

Notice: $\tilde{p}$ is the unique value such that

$$\underbrace{\tilde{p}u + (1-\tilde{p})d}_{\shortparallel} = R \quad — \quad (*)$$

Check: $\dfrac{R-d}{u-d}u + \dfrac{u-R}{u-d}d$ //

Financial interpretation:

$\tilde{p}$ is the probability so that the expected return of the stock is the same as the risk-free (bond) return.

Would you invest in such a stock?

p \ you	risk-adverse	risk-seeking	risk-neutral
$p > \tilde{p}$	hmm... maybe	Yea!	Yes
$p = \tilde{p}$	no!	hmm... maybe	don't care
$p < \tilde{p}$	no way!	hmm... maybe	no

i.e "risk-neutral"

If $p = \tilde{p}$ and you don't care about risk, then the stock is just as good an investment as the risk-free bond.

Hence the terminology:

$\boxed{\tilde{p} \text{ is called the risk-neutral probability (of the stock.)}}$

A suggestive notation:

$$C(0) = \frac{1}{R} \tilde{E}[C(1)]$$

"risk neutral expectation"

In above, we use the property that the market is linear and <u>complete</u>, i.e. every payoff (C_u, C_v) can be replicated by a linear combination of stock and bonds.

See:

[Luenberger, 2nd edition, Ch 11]

for a more in-depth discussion of these concepts.

Perhaps at this point the more important thing to remember is:

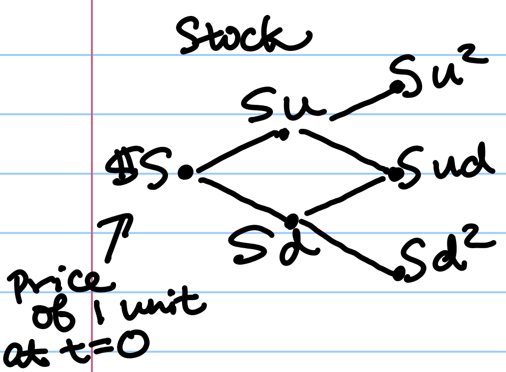
long

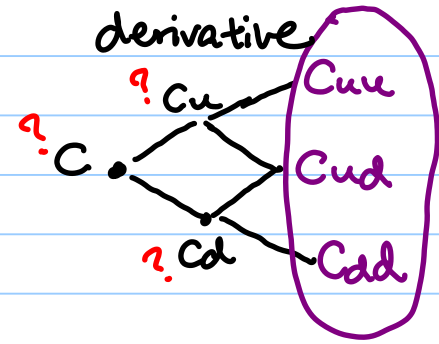
$$\boxed{\begin{array}{c} \$x \left(= \dfrac{C_u - C_d}{u-d}\right) \\ \text{worth of Stock} \\ + \\ \$y \left(= \dfrac{-dC_u + uC_d}{R(u-d)}\right) \\ \text{worth of bond} \end{array}}$$

"hedges" a short position in the derivetive

Note: when $y < 0$ and $x > 0$ (which is the case for a call option), we are borrowing $\$y$ from the bank to finance part of the stock position.

Multiperiod options:





Bond
rate $= R$

again, define the risk-neutral probability

$$\tilde{p} = \frac{R-d}{u-d}$$

The no-arbitrage price of the derivative is determined by:

$$C_u = \frac{1}{R}\left[\tilde{p}\, C_{uu} + (1-\tilde{p})\, C_{ud}\right]$$

$$C_d = \frac{1}{R}\left[\tilde{p}\, C_{ud} + (1-\tilde{p})\, C_{dd}\right]$$

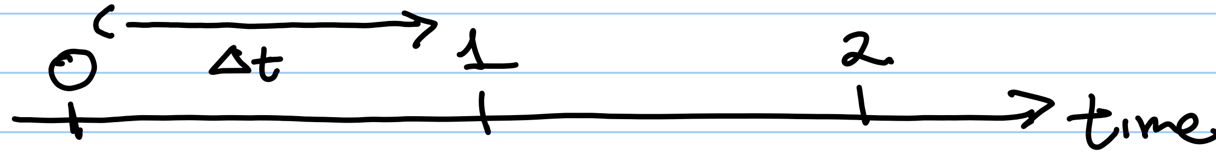
$$C = \frac{1}{R}\left[\tilde{p}\, C_u + (1-\tilde{p})\, C_d\right]$$

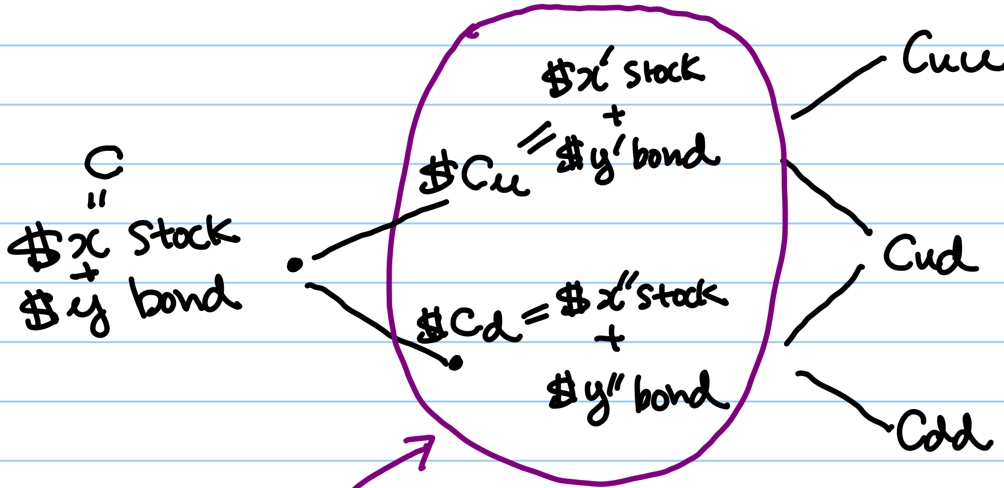
Note: the derivative price is determined backward in time.

Similar for any number of periods.

Why does it guarantee no-arbitrage?

Idea: Dynamic hedging





adaptively readjust the portfolio, based on the price action of the underlying

$$x' = \frac{C_{uu} - C_{ud}}{u-d}$$

$$y' = \frac{-dC_{uu} + uC_{ud}}{R(u-d)}$$

$$x'' = \frac{C_{ud} - C_{dd}}{u-d}$$

$$y'' = \frac{-dC_{ud} + uC_{dd}}{R(u-d)}$$

The (dynamically readjusted) portfolio perfectly replicates the payoff of the derivative.

So the no-arbitrary price of the derivative must be the price of the portfolio at time 0.

[See class demo]

Example:

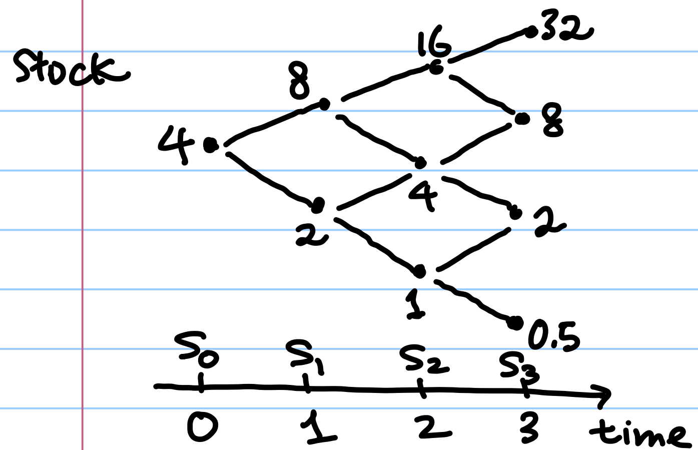
Consider a 3-period stock with

$S_0 = \$4$ (initial stock price)

$u = 2$, $d = 1/2$

Interest rate: 25% per period, so $R = 1 + 1/4$

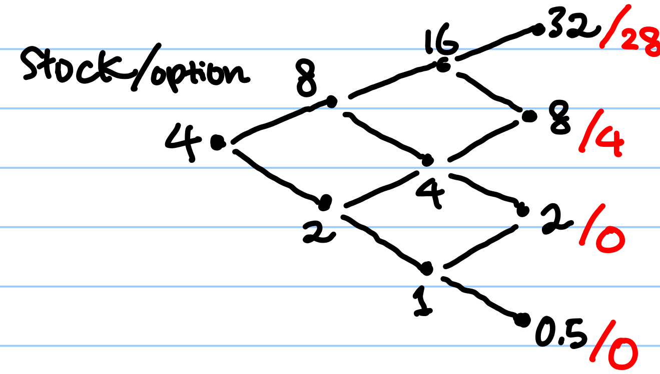
Let's price an ATM European <u>call</u> option on this stock, i.e. $K = \$4$



$S_i :=$ price of stock at time i

Payoff of option:

$\max(S_3 - K, 0)$



Aside: The 2^3 paths of the stock market at time 3:

8 "

uuu

uud

udu

duu

udd

dud

ddu

ddd

We may write

$S(u) = 8$, $S(d) = 2$

$S(uu) = 16$

$S(ud) = S(du) = 4$

$S(dd) = 1$

$S(uuu) = 32$

$S(uud) = S(udu) = S(duu) = 8$

$S(udd) = S(dud) = S(ddu) = 2$

$S(ddd) = 0.5$

risk-neutral probability $\tilde{p} = \dfrac{R-d}{u-d} = \dfrac{5/4 - 1/2}{2 - 1/2} = \dfrac{1}{2}$

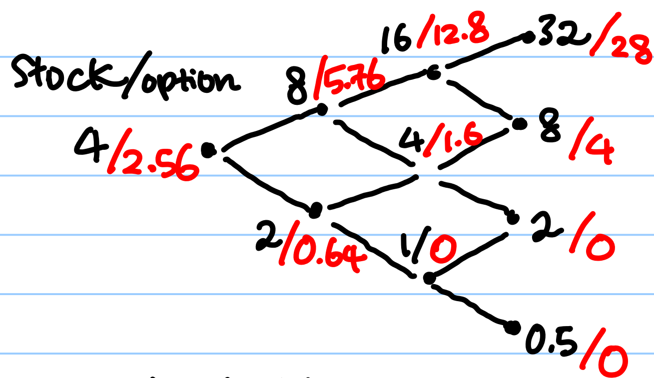
payoff of call option at time 3 :

$C(uuu) = 28$

$C(uud) = C(udu) = C(duu) = 4$
$C(udd) = C(dud) = C(ddu) = 0$

"path-independent" c.f. the (path-dependent 'exotic' options in HW #3)

$C(ddd) = 0$



Pricing calculation:

$$C_2(\underline{uu}) = \frac{1}{R}\left[\tilde{p}\, C_3(\underline{uu}u) + (1-\tilde{p})\, C_3(\underline{uu}d)\right]$$
$$= 12.8$$

$$C_2(\underline{ud}) = \frac{1}{R}\left[\tilde{p}\, C_3(\underline{ud}u) + (1-\tilde{p})\, C_3(\underline{ud}d)\right]$$

$||$ $\qquad\qquad$ 1.6

$$C_2(\underline{du}) = \frac{1}{R}\left[\tilde{p}\, C_3(\underline{du}u) + (1-\tilde{p})\, C_3(\underline{du}d)\right]$$

$$C_2(\underline{dd}) = \frac{1}{R}\left[\tilde{p}\, C_3(\underline{dd}u) + (1-\tilde{p})\, C_3(\underline{dd}d)\right]$$
$$= 0$$

$$C_1(\underline{u}) = \frac{1}{R}\left[\tilde{p}\, C_2(\underline{u}u) + (1-\tilde{p})\, C_2(\underline{u}d)\right]$$
$$= 5.76$$

$$C_1(d) = \frac{1}{R}\left[\tilde{p}\, C_2(du) + (1-\tilde{p})\, C_2(dd)\right]$$
$$= 0.64$$

$$C_0 = \frac{1}{R}\left[\tilde{p}\, C_1(u) + (1-\tilde{p})\, C_1(d)\right] = 2.56$$

If we buy this call on a unit of 100 shares of stock,

max profit $= \$2800 - \$256$

max loss $= \$256$

Now, the trickiest :

why should the option be priced in this way?

- If the call is priced above \$2.56, you short the call, collect the premium. Use \$256 to long the dynamic hedging portfolio

  You would not gain or loss anything in the positions no matter what happens to the stock market.

  Riskless profit : $\text{option Premium} - 256 > 0$ ← >256

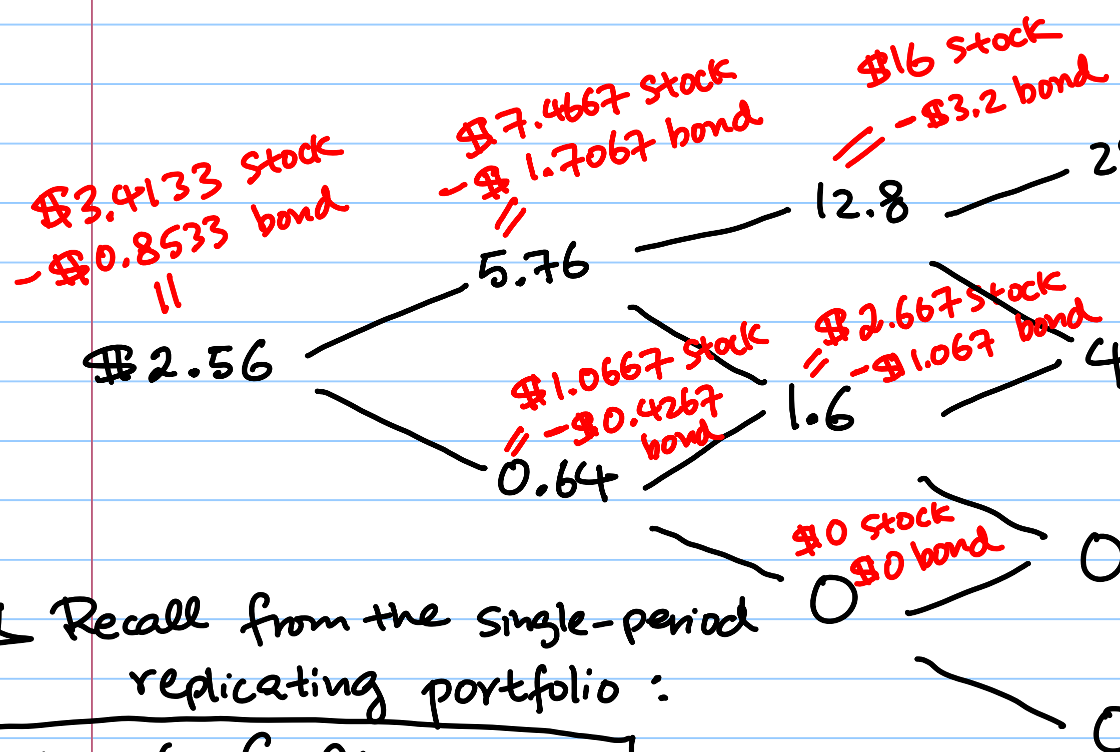
- If the call is priced below \$2.56, you long the call and short the dynamic hedging portfolio (You collect \$256 here to finance the long call position)

  You would not gain or loss anything in the positions no matter what happens to the stock market.

  Riskless profit : $256 - \text{option Premium} > 0$

  ↑ <256

The dynamic hedging portfolio :



★ Recall from the single-period replicating portfolio :

$$\boxed{\begin{array}{l} \$x \left(= \dfrac{C_u - C_d}{u-d}\right) \text{ worth of stock} \\ + \\ \$y \left(= \dfrac{-dC_u + uC_d}{R(u-d)}\right) \text{ worth of bond} \end{array}}$$

Note:

① In the case of a call option, we are always shorting bonds (ie borrowing money) to finance part of the long stock position

② The dynamic hedging strategy keeps a trader very busy : We do not just hold the \$3.4133 stock and −\$0.8533 bond positions till the end of time, we need to dynamically adjust the positions.

③ computation is done backward in time, the dynamic hedging strategy should, of course, be interpreted forward in time.

With a twist, this binomial lattice model can also be used to price American (ie. allowing early exercise) put option.

The logic may be a bit tricky, but the algorithm is very simple:

At each node:

1. Calculate the value of the put using the discounted risk-neutral formula.

2. Calculate the value that would be obtained by immediate exercise of the put

3. select the larger of the 2 values.

Note: If the stock price drops to 0 and $K > 0$, then exercising the put now is clearly optimal.

Payoff of a put option:

You can make \$K (=maximum payoff of the option) now, why wait?

This clearly shows that when the price of the underlying drops significantly, then early exercising is actually optimal.

Example:

Consider again a 2-period stock with

$S_0 = \$4$ (initial stock price)

$u = 2$, $d = 1/2$

Interest rate: 25% per period, so $R = 1 + 1/4$

Let's price an <u>European</u> and an <u>American</u> put option on this stock, i.e. $K = \$5$

Payoff of option:

$\max(K - S_3, 0)$

stock / option:

$S_0 = 4$

$S_1(u) = 8$

$S_1(d) = 2$

$S_2(uu) = 16 \,/\, 0 = P_2(uu)$

$S_2(ud) = S_2(du) = 4 \,/\, 1 = P_2(ud) = P_2(du)$

$S_2(dd) = 1 \,/\, 4 = P_2(dd)$

European put prices:

$$P_1(u) = \frac{1}{R}\left[\tilde{p} P_2(uu) + (1-\tilde{p}) P_2(ud)\right] = 0.4$$

$$P_1(d) = \frac{1}{R}\left[\tilde{p} P_2(du) + (1-\tilde{p}) P_2(dd)\right] = 2$$

$$P_0 = \frac{1}{R}\left[\tilde{p} P_1(u) + (1-\tilde{p}) P_1(d)\right] = 0.96$$

$$\frac{1}{R} = \frac{1}{1.25} = \frac{4}{5}, \quad \tilde{p} = \tilde{q} = \frac{1}{2}$$

$$\begin{tikzcd}[column sep=small]
 & & P_2(uu) = 0 \\
 & \begin{matrix} P_1(u) \\ = 0.4 \end{matrix} \arrow[ur] \arrow[dr] & \\
\begin{matrix} P_0 \\ = 0.96 \end{matrix} \arrow[ur] \arrow[dr] & & P_2(ud) = P_2(du) = 1 \\
 & \begin{matrix} P_1(d) \\ = 2 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & P_2(dd) = 4
\end{tikzcd}$$

→ Exercise: what is the dynamic hedging portfolio that replicates the payoff of the option?

American put prices:

$$V_1(u) = \max\left(0, \tfrac{4}{5}\left(\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 1\right)\right) = 0.4$$

$$V_1(d) = \max\left(3, \tfrac{4}{5}\left(\tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot 4\right)\right) = 3$$

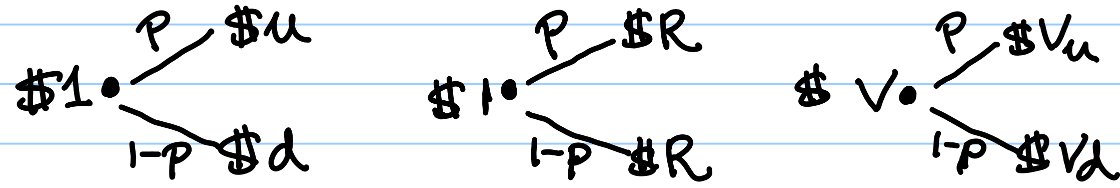
$$V_0 = \max\left(1, \tfrac{4}{5}\left(\tfrac{1}{2}\cdot 0.4 + \tfrac{1}{2}\cdot 3\right)\right) = 1.36$$

$$\begin{tikzcd}[column sep=small]
 & & V_2(uu) = 0 \\
 & \begin{matrix} V_1(u) \\ = 0.4 \end{matrix} \arrow[ur] \arrow[dr] & \\
\begin{matrix} V_0 \\ = \\ 1.36 \end{matrix} \arrow[ur] \arrow[dr] & & V_2(ud) = V_2(du) = 1 \\
 & \begin{matrix} V_1(d) \\ = 3 \end{matrix} \arrow[ur] \arrow[dr] & \\
 & & V_2(dd) = 4
\end{tikzcd}$$

Q: What is the rationale behind this pricing algorithm?

What if the option is priced above or below \$1.36?

Recall that in the single period setting



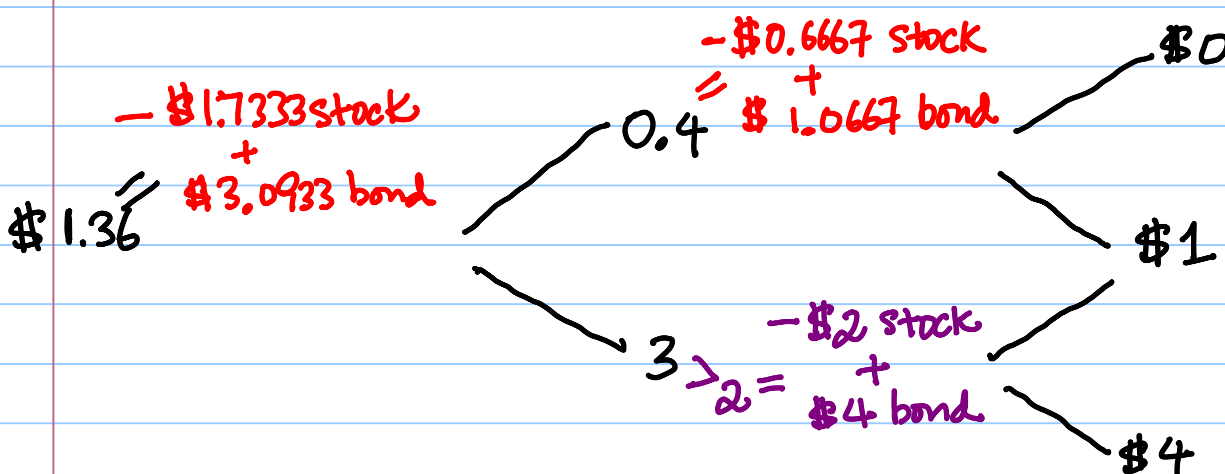
any security | bond | any __derivative__ of the security

$$\boxed{\begin{array}{c} \$x \ \left(= \dfrac{V_u - V_d}{u-d}\right) \\ \text{worth of stock} \\ + \\ \$y \ \left(= \dfrac{-dV_u + uV_d}{R(u-d)}\right) \\ \text{worth of bond} \end{array}}$$

"hedges" a short position in the derivative

$$\Downarrow$$

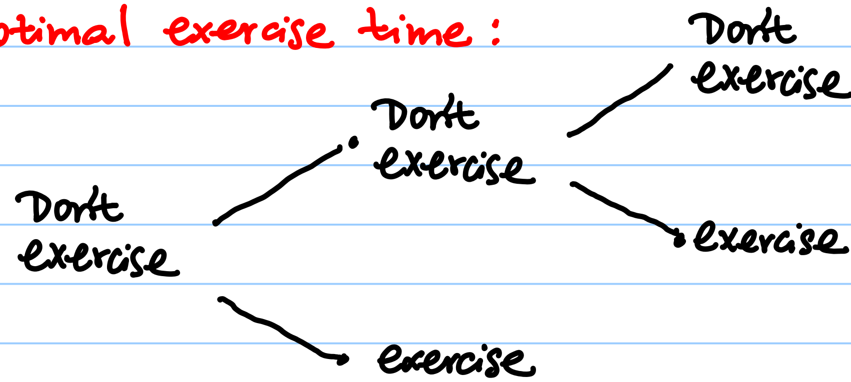
$$V = x + y = \frac{1}{R}\left[\tilde{p}V_u + (1-\tilde{p})V_d\right], \qquad \tilde{p} = \frac{R-d}{u-d}$$



Claim:

- If you long the American put, it is to your best interest to exercise the put at time 1 if the stock goes down at time 1,

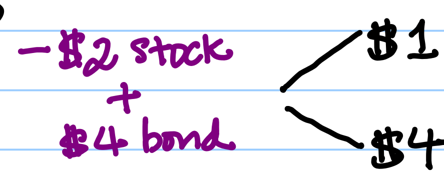
optimal exercise time:



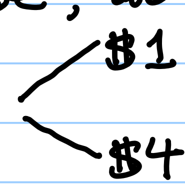
You don't need a crystal ball to see the the optimal exercise time.

If the stock is down at time 1, compare

① You exercise (early), you pocket \$3.
Now, if you just keep the \$3, you may regret that you exercise too early in case the stock goes down again. But: you can use 2 of the 3 dollars to build the replicating portfolio



② You don't exercise, at time 2 you get



Which one is better ?

- If you know that there are option traders out there who don't understand this logic and don't exercise at the optimal time, then you can make a riskless profit out of them.

How?

See : Example_Binomial_Lattice.m

uses binprice() in the financial toolbox.

Pressing issue :

How can the binomial lattice model be useful for modelling real asset prices?

This is to be addressed next.