Math 538 Differential Geometry and Manifolds

$\boxed{\text{HW \#3}}$

Due: Friday October 30, 2020

1. Let
$$S = \left\{ [\cos(u), \sin(u), \cos(v)]/(\sqrt{2} - \sin(v)) : (u, v) \in [0, 2\pi] \times [0, 2\pi] \right\},$$
$$\widetilde{S} = \left\{ [(\sqrt{2} + \cos(u)) \cos(v), \sqrt{2} + \cos(u)) \sin(v), \sin(u)] : (u, v) \in [0, 2\pi] \times [0, 2\pi] \right\}.$$

   (i) Prove $S = \widetilde{S}$.

   (ii) Prove that S $(= \widetilde{S})$ is a regular surface. How many parameterizations/charts do you need to cover S?

   (iii) Derive formulas for the mean and Gauss curvatures and principle directions of this surface. Be careful about how you present the answers.

   (iv) Identify the points of S that are elliptic ($K > 0$), hyperbolic ($K < 0$), and parabolic ($K = 0$).

   (v) Prove that *exactly* one of the two parametrizations in S and $\widetilde{S}$ has the property that the induced tangent vectors (i.e. X_u and X_v) are orthogonal and have the same length. Such a parametrization is called a *conformal* parametrization.

   For the one that is conformal, do the two equal length (and orthogonal) tangent vectors have the same length at all (u, v)?

2. (i) Using the 'global parameterization' $X(u, v)$ of a Möbius band provided in Lecture 3, derive formulas for the mean and Guass curvatures, $H(u, v)$, $K(u, v)$, of the Möbius band.

   (ii) Prove that $S := \{X(u, v) : u \in [0, 2\pi], v \in (-1, 1)\}$ is a regular surface in $\mathbb{R}^3$. How many parameterizations/charts do you need to cover S_1?

   (iii) Check:
$$H(0, v) \overset{??}{=} H(2\pi, v), \quad K(0, v) \overset{??}{=} K(2\pi, v), \quad \forall\, v \in (-1, 1).$$

   (iv) One of the two curvature functions is not continuous on the surface. Why is that? It may help if you use your formulas to color code the Möbius band and spin it around on your screen. For this purpose, consider using my computer demo in http://www.math.drexel.edu/~tyu/Math538/Demos.html.