CH2

Note Title 4/4/2017

The alternating algebra (the "pointwise" theory)

V - vector space over $\mathbb{R}$

Def A k-linear map

$$\omega : \overbrace{V \times V \times \cdots \times V}^{k \text{ times}} \to \mathbb{R}$$

is said to be alternating if

$$\omega(\xi_1, \dots, \xi_k) = 0$$

whenever

$$\xi_i = \xi_j \text{ for some pair } i \neq j.$$

Def The set of all alternating, k-linear maps, with its obvious vector space structure, is denoted by

$$\mathrm{Alt}^k(V).$$

Claim: If $k > \dim V$, then $\mathrm{Alt}^k(V) = 0$.

Proof: Let $e_1, \dots, e_n$ be a basis of V

$$\omega \in \mathrm{Alt}^k(V).$$

By multilinearity,

$$\omega(\xi_1, \dots, \xi_k) = \omega(\textstyle\sum \lambda_{i,1} e_i, \dots, \sum \lambda_{i,k} e_i)$$

$$= \sum_{j_1 \cdots j_k} \underbrace{\lambda_J}_{\lambda_{j_1,1} \cdots \lambda_{j_k,k}} \omega(e_{j_1}, \dots, e_{j_k})$$

Since $k > n$, there must be at least one repetition among the elements $e_{j_1}, \dots, e_{j_k}$.

Hence, $\omega(\xi_1, \dots, \xi_k) = 0$. $\square$

$S(k)$ = the permutation group of $\{1, \dots, k\}$

transpositions : $(i,j) \in S(k)$

$\sigma = (i,j)$ is defined by $\sigma(k) = \begin{cases} j & k=i \\ i & k=j \\ k & k \neq i,j \end{cases}$

Every permutation has a sign

$$\text{sign} : S(k) \to \{+1, -1\}$$

$\text{sign}(\sigma) = \begin{cases} +1 & \text{if } \sigma \text{ is the composition of an even \# of transpositions} \\ -1 & \text{if } \sigma \text{ is the composition of an odd \# of transpositions} \end{cases}$

Fact : the decomposition is not unique, but the parity is.

$$\text{sign}(\sigma \circ \tau) = \text{sign}(\sigma) \cdot \text{sign}(\tau)$$

(so sign is a group homomorphism)

Every permutation can be written as a composition of transpositions of the type $(i, i+1)$

e.g. $(i, i+2) = (i, i+1) \circ (i+1, i+2) \circ (i, i+1)$

Lemma If $\omega \in Alt^k(V)$, $\sigma \in S(k)$, then

$$\omega(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}) = \operatorname{sign}(\sigma)\, \omega(\xi_1, \cdots, \xi_k).$$

Proof: Suffices to prove the formula for $\sigma =$ a transposition (i,j).

Write $\omega_{ij}(\xi, \xi') = \omega(\xi_1, \cdots, \underset{\substack{\uparrow \\ i}}{\xi}, \cdots, \underset{\substack{\uparrow \\ j}}{\xi'}, \cdots, \xi_k)$

With ξ_ℓ, $\ell \neq i, j$, fixed,

$$\omega_{ij} \in Alt^2(V).$$

$$\omega_{ij}(\xi_i + \xi_j, \xi_i + \xi_j) = 0$$

$$\| $$

$$0 = \omega_{ij}(\xi_i, \xi_i) + \omega_{ij}(\xi_i, \xi_j)$$

$$+$$

$$\omega_{ij}(\xi_j, \xi_i) + \omega_{ij}(\xi_j, \xi_j) \overset{\|}{} 0$$

so

$$\omega_{ij}(\xi_i, \xi_j) = -\omega_{ij}(\xi_j, \xi_i) \qquad \square$$

E.g. (Recall Lecture 2, Math 538)

$$V = \mathbb{R}^k, \quad D : \underbrace{\mathbb{R}^k \times \cdots \times \mathbb{R}^k}_{k} \to \mathbb{R},$$

$$D(\xi_1, \cdots, \xi_k) = \det([\xi_1, \cdots, \xi_k])$$

is alternating, ie $D \in Alt^k(\mathbb{R}^k)$.

Exterior product

$$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \longrightarrow \mathrm{Alt}^{p+q}(V)$$

$p = q = 1$, $\omega_1 \wedge \omega_2$ is given by

$$(\omega_1 \wedge \omega_2)(\xi_1, \xi_2) = \omega_1(\xi_1)\,\omega_2(\xi_2) - \omega_2(\xi_1)\,\omega_1(\xi_2)$$

In general, define

$$(\omega_1 \wedge \omega_2)(\xi_1, \cdots, \xi_{p+q})$$
$$:= \frac{1}{p!\,q!} \sum_{\sigma \in S(p+q)} \mathrm{sign}(\sigma)\, \omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)})\, \omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)})$$

It's clear that $\omega_1 \wedge \omega_2$ is a $(p+q)$-linear map, the fact that it is alternating follows from the following simple fact (not found in M&T - in fact the treatment in M&T on this particular matter seems unnatural to me):

Lemma : for any k-linear map $L : V \times \cdots \times V \to \mathbb{R}$, the k-linear map

$$\pi_k L : V \times \cdots \times V \to \mathbb{R},$$

$$(\pi_k L)(\xi_1, \cdots, \xi_k) := \sum_{\sigma \in S(k)} \mathrm{sign}(\sigma)\, L(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)})$$

is alternating.

In fact, π_k is a linear map from the space of k-linear maps to $\mathrm{Alt}^k(V)$. It isn't quite a projection, but close :

$$L \in \mathrm{Alt}^k(V) \implies \pi_k(L) = k!\, L$$

By applying this lemma to the $(p+q)$-linear map

$$(\xi_1, \cdots, \xi_{p+q}) \mapsto \omega_1(\xi_1, \cdots, \xi_p)\, \omega_2(\xi_{p+1}, \cdots, \xi_{p+q})$$

we establish that $\omega_1 \wedge \omega_2$ is alternating.

<u>Proof of lemma</u>

Let $\tau \in S(k)$

$$(\pi_k L)(\xi_{\tau(1)}, \cdots, \xi_{\tau(k)}) = \sum_{\sigma} \operatorname{sign} \sigma \; L(\xi_{\tau\sigma(1)}, \cdots, \xi_{\tau\sigma(k)})$$

$$= \operatorname{sign}(\tau) \sum_{\sigma \in S(k)} \operatorname{sign}(\sigma\tau)\; L(\xi_{\tau\sigma(1)}, \cdots, \xi_{\tau\sigma(k)})$$

As σ runs over all permutations of $\{1, \cdots, k\}$, so does $\tau\sigma$. Therefore

$$(\pi_k L)(\xi_{\tau(1)}, \cdots, \xi_{\tau(k)}) = \operatorname{sign}(\tau)\, (\pi_k L)(\xi_1, \cdots, \xi_k)$$

thus $\pi_k L$ is alternating. $\square$

The definition of $\omega_1 \wedge \omega_2$ consists of a sum with $(p+q)!$ terms, but since if $\sigma, \sigma' \in S(p+q)$ are such that

$$\{\sigma(1), \cdots, \sigma(p)\} = \{\sigma'(1), \cdots, \sigma'(p)\}$$
$$\{\sigma(p+1), \cdots, \sigma(p+q)\} = \{\sigma'(p+1), \cdots, \sigma'(p+q)\}$$

then the two corresponding summands are the same (not just the same up to a sign, but exactly the same — why?)

Once you fill in this detail, we can compress the above summation:

<u>Proposition</u> $(\omega_1 \wedge \omega_2)(\xi_1, \cdots, \xi_{p+q})$

$$= \sum_{\sigma \in S(p,q)} \operatorname{sign}(\sigma)\, \omega_1(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(p)})\, \omega_2(\xi_{\sigma(p+1)}, \cdots, \xi_{\sigma(p+q)})$$

where

$$S(p,q) := \{\sigma \in S(p+q) : \sigma(1) < \cdots < \sigma(p),\ \sigma(p+1) < \cdots < \sigma(p+q)\}$$

Permutations in $S(p,q)$ are called "(p,q)- shuffles".

The compressed summation has:

$$|S(p,q)| = \binom{p+q}{p} \quad \text{terms}$$

as opposed to the original $(p+q)!$ terms.

$\mathrm{Alt}^p(V)$ is a linear space over $\mathbb{R}$

$\wedge : \mathrm{Alt}^p(V) \times \mathrm{Alt}^q(V) \to \mathrm{Alt}^{p+q}(V)$ is billinear

Lemma (anti-commutativity) $\omega_1 \in \mathrm{Alt}^p(V)$, $\omega_2 \in \mathrm{Alt}^q(V)$

$$\omega_1 \wedge \omega_2 = (-1)^{pq} \omega_2 \wedge \omega_1$$

Lemma (associativity)

$$\omega_1 \wedge (\omega_2 \wedge \omega_3) = (\omega_1 \wedge \omega_2) \wedge \omega_3$$

Convention : $\mathrm{Alt}^0(V) := \mathbb{R}$

$\mathrm{Alt}^k(V)$, $k = 0, 1, 2, \cdots$ together with $\wedge$ is an example of a graded $\mathbb{R}$-algebra which is also anti-commutative and connected.

Elements in $\mathrm{Alt}^p(V)$ are called p-forms.

Subtle point (to be elaborated)

the associativity has to do with the $1/p!q!$ factor in the definition of $\wedge$.

Lemma (2.13) For 1-forms $\omega_1, \dots, \omega_p \in \mathrm{Alt}^1(V)$,

$$(\omega_1 \wedge \cdots \wedge \omega_p)(\xi_1, \dots, \xi_p) = \det\left(\omega_i(\xi_j)\right)_{1 \le i,j \le p}$$

Proof $p=2$ is obvious

Proceed by induction in p.

$$\omega_1 \wedge (\omega_2 \wedge \cdots \wedge \omega_p)(\xi_1, \xi_2, \dots, \xi_p)$$

$$= \sum_{\sigma \in S(1,p-1)} \operatorname{sign}(\sigma)\, \omega_1(\xi_{\sigma(1)})\, (\omega_2 \wedge \cdots \wedge \omega_p)(\xi_{\sigma(2)}, \dots, \xi_{\sigma(p)})$$

$$= \sum_{j=1}^{p} (-1)^{j+1} \omega_1(\xi_j)\, (\omega_2 \wedge \cdots \wedge \omega_p)(\underbrace{\xi_1, \dots, \hat{\xi}_j, \dots, \xi_p}_{(p-1)\text{ tuple with } \xi_j \text{ omitted}})$$

which, by induction hypothesis, is the desired determinant (expanded by the first row). $\square$

$$\det \begin{bmatrix} \omega_1(\xi_1) & \cdots\cdots & \omega_1(\xi_p) \\ \omega_2(\xi_1) & \cdots\cdots & \omega_2(\xi_p) \\ \vdots & & \\ \omega_p(\xi_1) & \cdots\cdots & \omega_p(\xi_p) \end{bmatrix} \leftarrow$$

$$= \omega_1(\xi_1) \det \begin{bmatrix} \omega_2(\xi_2) \cdots & \omega_2(\xi_p) \\ \vdots & \\ \omega_p(\xi_2) \cdots & \omega_p(\xi_p) \end{bmatrix} - + \cdots$$

$\omega_1, \dots, \omega_p \in \underbrace{Alt^1(V)}_{= V^*}$ are linearly independent

$\Leftrightarrow$ $\exists\ \xi_1, \dots, \xi_p \in V$ s.t.

$$\omega_i(\xi_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

$$\Rightarrow (\omega_1 \wedge \cdots \wedge \omega_p)(\xi_1, \dots, \xi_p) \underset{\substack{\uparrow \\ \text{previous} \\ \text{lemma}}}{=} \det(I_{p \times p}) = 1$$

$$\Rightarrow \omega_1 \wedge \cdots \wedge \omega_p \neq 0$$

Conversely, if $\omega_1, \dots, \omega_p \in Alt^1(V)$ are linearly dependent

then we can write, e.g.

$$\omega_p = \sum_{i=1}^{p-1} r_i \omega_i \quad , \text{so}$$

$$\omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_p = \sum_{r=1}^{p-1} r_i \omega_1 \wedge \cdots \wedge \omega_{p-1} \wedge \omega_i \overset{\substack{\text{again by} \\ \text{previous lemma} \\ \downarrow}}{=} 0$$

We have proved:

<u>Lemma</u> For $\omega_1, \dots, \omega_p \in Alt^1(V)$

$$\omega_1 \wedge \cdots \wedge \omega_p \neq 0 \Leftrightarrow \text{they are linearly independent}$$

Theorem Let $e_1, \cdots, e_n$ be a basis of V

$\varepsilon_1, \cdots, \varepsilon_n \in V^* = \mathrm{Alt}^1(V)$ be the dual basis.

Then $\{\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} : \sigma \in S(p, n-p)\}$

is a basis of $\mathrm{Alt}^p(V)$. In particular

$$\dim \mathrm{Alt}^p(V) = \binom{n}{p}.$$

Proof: Since $\varepsilon_i(e_j) = \delta_{i,j}$, lemma 2.13 gives

$$(\ast) \quad \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}(e_{j_1}, \cdots, e_{j_p}) = \begin{cases} 0 & \text{if } \{i_1, \cdots, i_p\} \neq \{j_1, \cdots, j_p\} \\ \mathrm{sign}(\sigma) & \text{if } \{i_1, \cdots, i_p\} = \{j_1, \cdots, j_p\} \end{cases}$$

where σ is the permutation $\sigma(i_k) = j_k$.

For any $\omega \in \mathrm{Alt}^p(V)$, $\xi_1, \cdots, \xi_p \in V$

$$\xi_1 = \sum_{i_1=1}^{n} \varepsilon_{i_1}(\xi_1) e_{i_1}, \cdots, \xi_p = \sum_{i_p=1}^{n} \varepsilon_{i_p}(\xi_p) e_{i_p}$$

$$\omega(\xi_1, \cdots, \xi_p) \underset{\substack{\uparrow \\ \text{multilinearity}}}{=} \underbrace{\sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n}}_{\substack{n^p \text{ terms,} \\ \text{but many are } 0}} \underbrace{\omega(e_{i_1}, \cdots, e_{i_p})}_{\substack{0 \text{ unless} \\ i_1, \cdots, i_p \text{ are} \\ \text{distinct}}} \varepsilon_{i_1}(\xi_1) \cdots \varepsilon_{i_p}(\xi_p)$$

$\downarrow$ compress the sum $\begin{cases} \text{1. remove } (i_1, \cdots i_p) \text{ not distinct} \\ \text{2. use only the ordered tuples } i_1 < \cdots < i_p \end{cases}$

$$\overset{\text{uses } (\ast)}{=} \sum_{\sigma \in S(p, n-p)} \omega(e_{\sigma(1)}, \cdots, e_{\sigma(p)}) \, \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)}(\xi_1, \cdots, \xi_p)$$

So, $$\omega = \sum_{\sigma \in S(p, n-p)} \omega(e_{\sigma(1)}, \cdots, e_{\sigma(p)}) \, \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)}$$

This shows the $\binom{n}{p}$ elements $\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} \in \mathrm{Alt}^p(V)$ span $\mathrm{Alt}^p(V)$.

To show that they are linearly independent, assume

$$\sum_{\sigma \in S(p,n-p)} \lambda_\sigma \varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} = 0 , \quad \lambda_\sigma \in \mathbb{R}$$

For any $\pi \in S(p,n-p)$

$$\varepsilon_{\sigma(1)} \wedge \cdots \wedge \varepsilon_{\sigma(p)} (e_{\pi(1)}, \cdots, e_{\pi(p)}) = \begin{cases} 0 , & \sigma \neq \pi \\ 1 , & \sigma = \pi \end{cases}$$

so $\lambda_\pi \cdot 1 = 0$. And $\lambda_\pi = 0$, $\forall \pi \in S(p, n-p)$. $\square$

A linear map $f: V \to W$ induces the linear map

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(W) \to \mathrm{Alt}^p(V)$$

by setting

$$\mathrm{Alt}^p(f)(\omega)(\xi_1, \cdots, \xi_p) = \omega(f(\xi_1), \cdots, f(\xi_p))$$

If $V \xrightarrow{f} W \xrightarrow{g} X$

then $\mathrm{Alt}^p(V) \xleftarrow{\mathrm{Alt}^p(f)} \mathrm{Alt}^p(W) \xleftarrow{\mathrm{Alt}^p(g)} \mathrm{Alt}^p(X)$

and $\mathrm{Alt}^p(g \circ f) = \mathrm{Alt}^p(f) \circ \mathrm{Alt}^p(g)$

any abstract vector space over $\mathbb{R}$

$\dim V = n$

Recall that for any linear map $f : V \to V$

- $\det(f)$ — determinant of f
- $\mathrm{tr}(f)$ — trace of f
- $\lambda_1, \ldots, \lambda_n$ — eigenvalues of f

are well-defined (i.e. independent of choice of basis)

$$\left(\begin{aligned} \det(V^{-1} A V) &= \det(A) = \prod \lambda_i \\ \mathrm{tr}(V^{-1} A V) &= \mathrm{tr}(A) = \sum \lambda_i \end{aligned} \right).$$

Same comment applies to any

$$\mathrm{Alt}^p(f) : \mathrm{Alt}^p(V) \to \mathrm{Alt}^p(V).$$

We have the following:

<u>Theorem</u> The characteristic polynomial of $f : V \to V$ linear is given by

$$\det(f - t\,\mathrm{id}) = \sum_{i=0}^{n = \dim V} (-1)^i \,\mathrm{tr}(\mathrm{Alt}^{n-i}(f))\, t^i$$

In particular,

$$\det(f) = \mathrm{tr}(\mathrm{Alt}^n(f)).$$

Proof: see M&T.