Spin Networks for Perfect State Transfer

dln-dev

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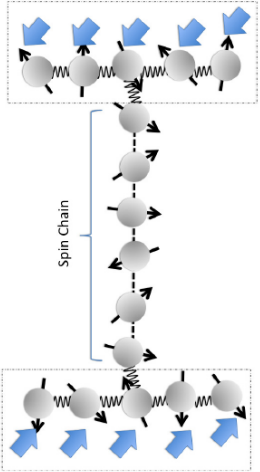
Overview

HARDWARE

Processors



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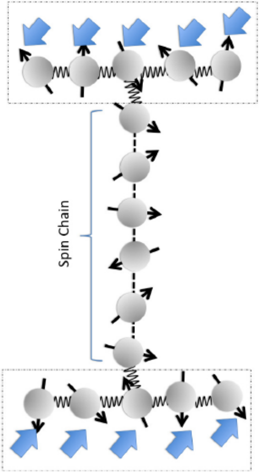
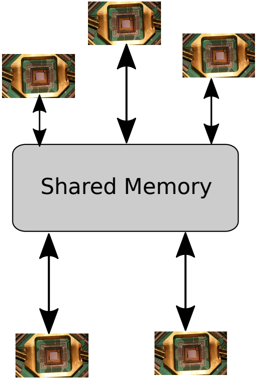
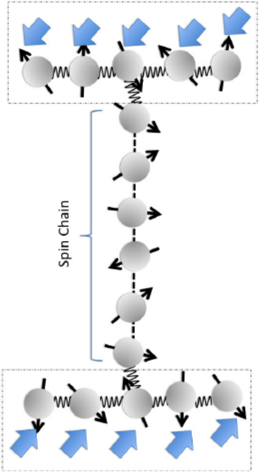


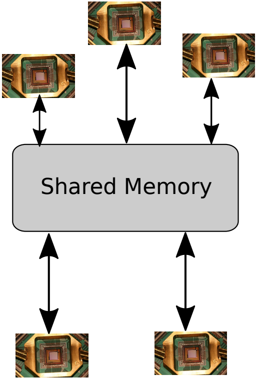
Computers

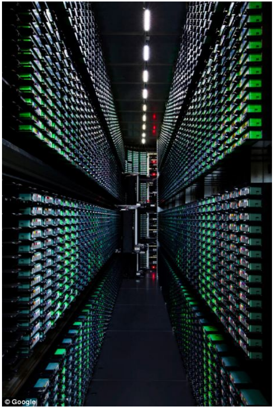


Hardware

Processors	Computers	Computing







Requirements

- Separation of highly controlled regions
- Spacers
  - (Perfect) state transfer
  - Fixed interactions
  - Natural dynamics
  - Control single qubit at each end only (for now)

$\rightarrow$ Spin networks!

MODEL

$$H_{XX} = \frac{1}{2} J \sum_{i=1}^{N} \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right]$$

- $J \equiv const.$
- No z coupling
- $[H_{XX}, \sigma_{\text{tot}}^z] = 0$
- Eigenstates: $|\tilde{k}\rangle = \sqrt{\frac{2}{N+1}} \sum_{s=1}^{N} \sin\left(\frac{\pi k s}{N+1}\right) |s\rangle$
- Eigenvalues: $E_k = 2\cos\left(\frac{\pi k}{N+1}\right)$

Math setup

Ground state of ferromagnetic spin-$\frac{1}{2}$ chain:

$$|\uparrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes \cdots \otimes |\uparrow\rangle_N = |\uparrow_1\uparrow_2 \ldots \uparrow_N\rangle := |0\rangle$$

Excitation operator $\sigma^- = \frac{1}{2}\left(\sigma^x - \mathrm{i}\sigma^y\right)$:

$$1_1 \otimes 1_2 \otimes \cdots \otimes 1_{s-1} \otimes \sigma^- \otimes 1_{s+1} \otimes \cdots \otimes 1_N := \sigma_s^-$$

Single spin excitation:

$$\sigma_s^- |0\rangle := |s\rangle$$

Single spin superposition: Prepare spin at s as $\alpha|\uparrow\rangle_s + \beta|\downarrow\rangle_s \rightarrow$

$$|\Psi\rangle := \alpha|0\rangle + \beta|s\rangle$$

PERFECT STATE TRANSFER

$$|\Psi_{in}\rangle = \cos\left(\frac{\Theta}{2}\right)|0\rangle + e^{\mathrm{i}\Phi}\sin\left(\frac{\Theta}{2}\right)|s\rangle$$

$$|\Psi_{out}\rangle = \frac{1}{\sqrt{P(t)}}\left(\cos\left(\frac{\Theta}{2}\right)|0\rangle + e^{\mathrm{i}\Phi}\sin\left(\frac{\Theta}{2}\right)f^N_{r,s}(t)\,|r\rangle\right)$$

$$\rho_{out} = P(t)\,|\Psi_{out}\rangle\langle\Psi_{out}| + (1-P(t))\,|\uparrow\rangle\langle\uparrow|$$

Transition amplitude: $f^N_{r,s}(t) = \langle r|\, e^{\mathrm{i}Ht}\,|s\rangle$

Fidelity:

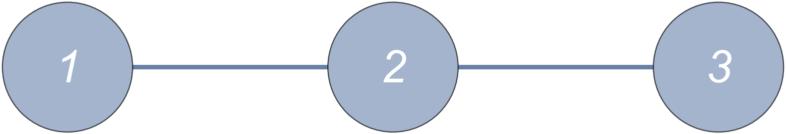
$$\mathfrak{F}(t) = \frac{1}{4\pi}\int \langle\Psi_{in}|\,\rho_{out}\,|\Psi_{in}\rangle\;\mathrm{d}\Omega = \frac{|f^N_{r,s}|\cos\lambda}{3} + \frac{|f^N_{r,s}|^2}{6} + \frac{1}{2}$$

Graphs

$$G = (V, E)$$

- Edges undirected $\rightarrow E \subseteq \{\{i, j\} : i, j \in V\}$
- Single edges only $\rightarrow \{i, j\}$ unique

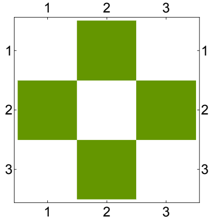
Example: $V = \{1, 2, 3\}, E = \{\{1, 2\}, \{2, 3\}\}$

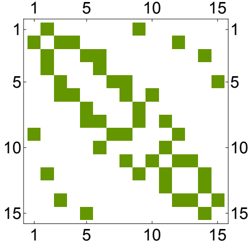


ADJACENCY MATRICES

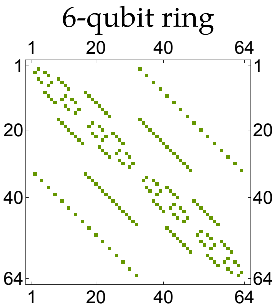
$$a_{ij} = \begin{cases} 1 & \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$$

- No multi-edges $\rightarrow$ all elements either 0 or 1
- Unordered Pairs $\rightarrow$ symmetric
- Orthogonal basis exists, eigenvalues are roots of $det(A - \lambda 1) = 0$





Correspondence to Hamiltonians



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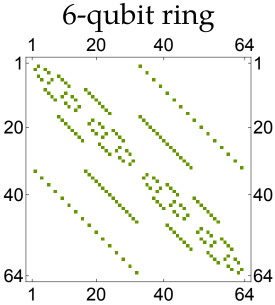
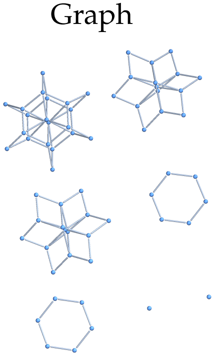
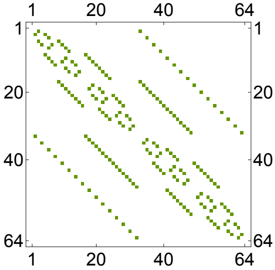


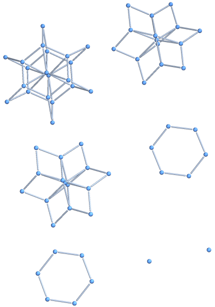


Correspondence to Hamiltonians

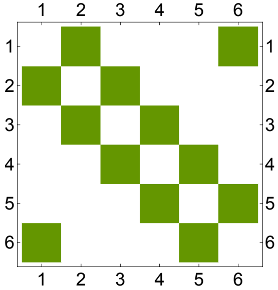
6-qubit ring



Graph

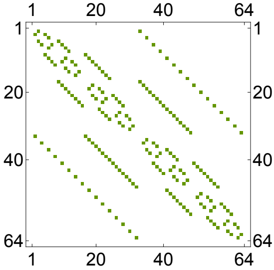


Single spin subspace



Correspondence to Hamiltonians

6-qubit ring



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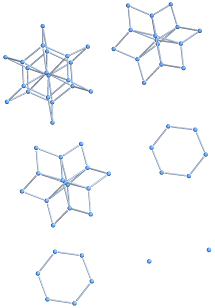
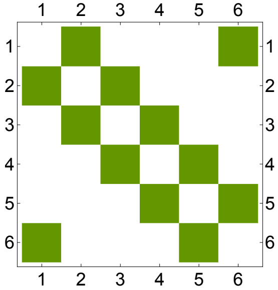


Single spin subspace



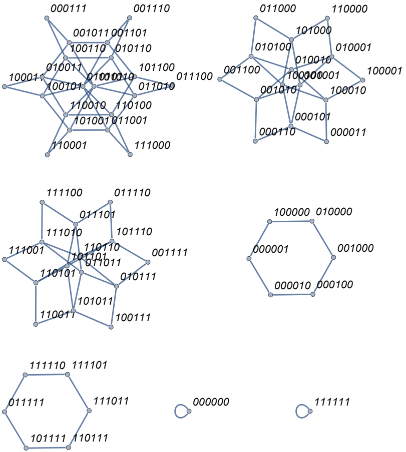
This is the adjacency matrix for the 6-qubit ring!

How to identify subspaces

Recipe:

- Matrix labels -1
- Convert labels to binary
- Identify rows and columns with equal number of 1s (e.g. 1)
- Strip off all other rows and columns
- Compile matrix from rows and columns left
- $\binom{N}{s}$ vertices (states) in graph

Graph



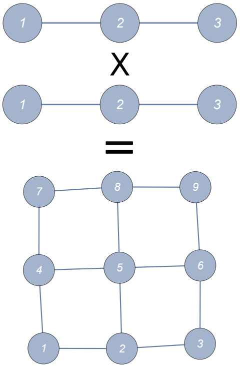
Cartesian Product of Graphs

$$G \times H = (V_{G\times H}, E_{G\times H})$$

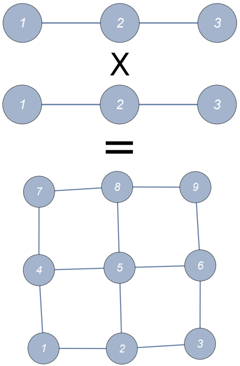
- $V_{G\times H} = V(G) \times V(H)$
- $\{(u, u'), (v, v')\} \in E_{G\times H}$ if
  - $u = v$ and $\{u', v'\} \in E_H$ or
  - $u' = v'$ and $\{u, v\} \in E_G$

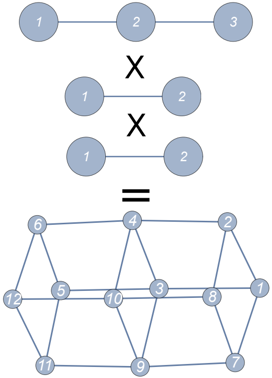
$$A_{G\times H} = A_G \otimes 1_{|V_H|} + 1_{|V_G|} \otimes A_H$$

Examples of Product Graphs



Examples of Product Graphs





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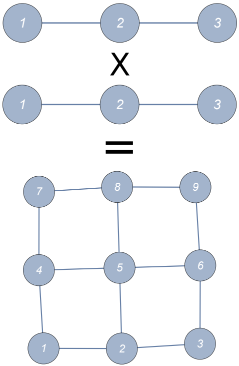
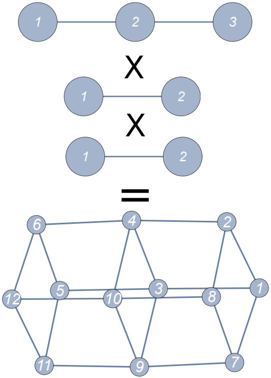
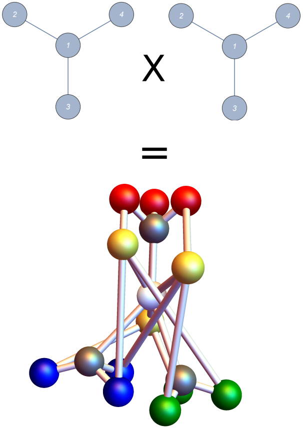






FIDELITY ON PRODUCT GRAPHS

$$
\begin{aligned}
f^{|V_{G\times H}|}_{(r,r'),(s,s')}(t) &= \left(\langle r| \otimes \langle r'|\right) e^{-\mathrm{i}A_{G\times H}t} \left(|s\rangle \otimes |s'\rangle\right) \\
&= \left(\langle r| \otimes \langle r'|\right) e^{-\mathrm{i}(A_G \otimes 1_{|V_H|})t} e^{-\mathrm{i}(1_{|V_G|}\otimes A_H)t} \left(|s\rangle \otimes |s'\rangle\right) \\
&= \sum_{k=1}^{K} \left[\left(\left(\langle r| \otimes \langle r'|\right) |g'_k\rangle \langle g'_k| e^{-\mathrm{i}G_k{}'t} |h'_k\rangle \langle h'_k| e^{-\mathrm{i}H_k{}'t} \left(|s\rangle \otimes |s'\rangle\right)\right) \right]
\end{aligned}
$$

- $K = |V_{G\times H}|, L = |V_G|, M = |V_H| \to K = L \cdot M$
- $\langle g'_k| = \langle g_l| \otimes \langle e_m|, \langle h'_k| = \langle e_l| \otimes \langle h_m|$ etc.
- $G_k{}' = G_l \cdot E_m, H_k{}' = E_l \cdot H_m$
- $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$

$$
\begin{aligned}
f^{|V_{G\times H}|}_{(r,r'),(s,s')}(t) &= \sum_{l=1}^{L} \left[\langle r \,|\, g_l\rangle \langle g_l \,|\, s\rangle e^{-\mathrm{i}G_l t} \right] \sum_{m=1}^{M} \left[\left\langle r' \,\middle|\, h_m \right\rangle \left\langle h_m \,\middle|\, s' \right\rangle e^{-\mathrm{i}H_m t} \right] \\
&= \left\langle r \,\middle|\, e^{-\mathrm{i}A_G t} \,\middle|\, s \right\rangle \left\langle r' \,\middle|\, e^{-\mathrm{i}A_H t} \,\middle|\, s' \right\rangle
\end{aligned}
$$

Fidelity on Product Graphs

$$f^{|V_{G\times H}|}_{(r,r'),(s,s')}(t) = f^{|V_G|}_{r,s}(t)\cdot f^{|V_H|}_{r',s'}(t)$$

$$f^{|V_G|}_{r,s}(t) = 1 \text{ for } t = \tau_G$$
$$f^{|V_H|}_{r',s'}(t) = 1 \text{ for } t = \tau_H$$
$$f^{|V_{G\times H}|}_{(r,r'),(s,s')}(t) = 1 \text{ for } t = \tau_{G\times H}$$

iff $\frac{t_G}{t_H} \in \mathbb{Q}$

Overview

Spin Chains

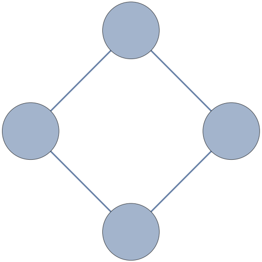
$$f^N_{N,1}(t) = \frac{2}{N+1} \sum_s^N \sin\left(\frac{\pi s}{N+1}\right) \sin\left(\frac{\pi s N}{N+1}\right) e^{-\mathrm{i} E_s t}$$

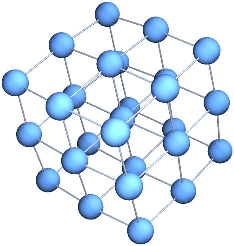
- $N = 2$: $f^2_{2,1}(t) = -\mathrm{i}\sin(t)$ with $\tau_2 = \pi/2$
- $N = 3$: $f^3_{3,1}(t) = -\left[\sin(t/\sqrt{2})\right]^2$ with $\tau_3 = \pi/\sqrt{2}$
- $N \geq 4$: $f^N_{N,1}(t) \neq 1 \quad \forall\ t$

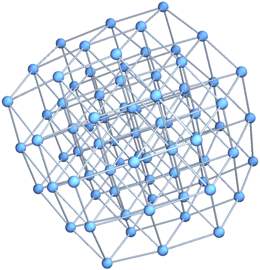
Proof shows that $f^N_{N,1}(t) = 1$ implies $\frac{\cos\left(\frac{2\pi}{N+1}\right)}{\cos\left(\frac{\pi}{N+1}\right)} \in \mathbb{Q}$

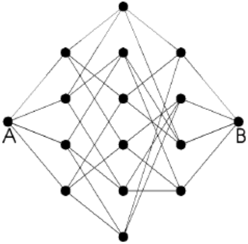
$$H = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Spin Networks





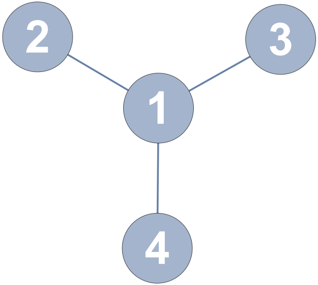




Spin Switch

$$H = \frac{1}{2} J \sum_{i=2}^{N} \left[\sigma_1^x \sigma_i^x + \sigma_1^y \sigma_i^y \right] + \sum_{i=1}^{N} h_i \sigma_i^z$$

- Introduce local potentials
- Slight variations only
- Diagonal entries
- E.g. magnetic offset field



SPIN SWITCH

- Start: $|\Psi_1\rangle = \alpha\,|0\rangle + \beta\,|2\rangle$
- Target: $|\Psi_2\rangle = \alpha\,|0\rangle + \beta\,|3\rangle$
- $\langle 3|U(\tau)|2\rangle = \langle 2|U(\tau)|3\rangle \overset{!}{=} 1$
- Set $h_2 = h_3$, since $[H, P_{23}] = 0$

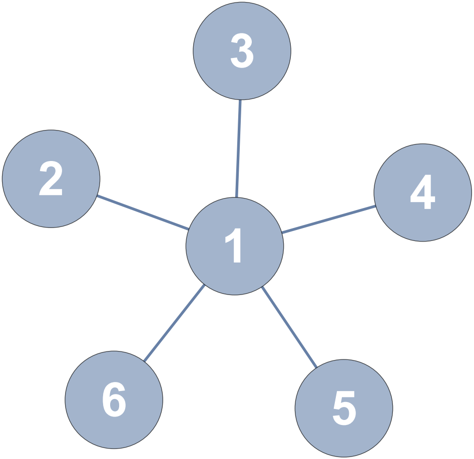
$$H = \begin{pmatrix} h_1 & 1 & 1 & 1 \\ 1 & h_2 & 0 & 0 \\ 1 & 0 & h_3 & 0 \\ 1 & 0 & 0 & h_4 \end{pmatrix}$$

$$\sum_{k=1}^{4} \langle 2|\left(P_{23}\,|e_k\rangle\right)\langle e_k|2\rangle\, e^{-\mathrm{i}E_k\tau} \overset{!}{=} 1$$

Choose E_k so they eliminate the phase introduced by P_{23}:
$E_k \in \{0, h_2, \pm\eta h_2\}$, η even.
Compare $\det(H - \lambda 1_4) = 0$ and $\det(\mathrm{diag}(0, h_2, \pm\eta h_2) - \lambda 1_4) = 0$
Real roots of the resulting polynomial are solutions for h_2,
$\tau = \frac{\pi}{h_2}$.

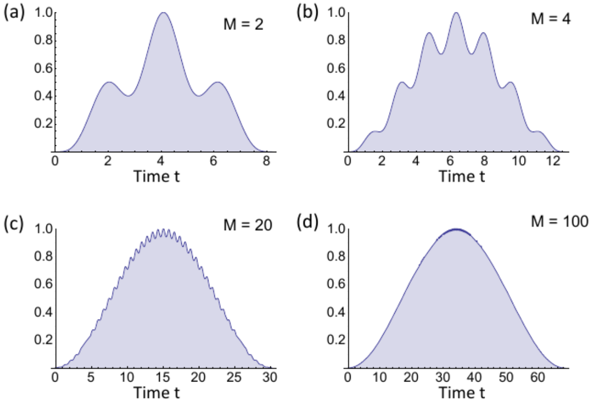
Spin Switch with more Arms

- Impose further symmetry condition: $\lambda_4 = \lambda_5 = \cdots = \lambda_N$
- Renormalize $|4\rangle_{new} = (|4\rangle_{old} + |5\rangle + \cdots + |N\rangle)/\sqrt{M}$, $M = N - 3$



$$\tilde{H} = \begin{pmatrix} h_1 & 1 & 1 & \sqrt{M} \\ 1 & h_2 & 0 & 0 \\ 1 & 0 & h_3 & 0 \\ \sqrt{M} & 0 & 0 & h_4 \end{pmatrix}$$

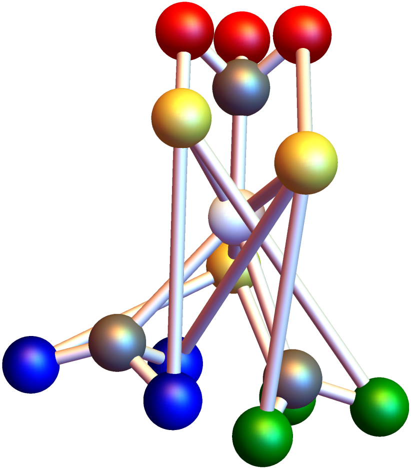
Fidelity



(d)

M	a	b	c	d	e
2	1.84	1.41	1	-2.61	0.766
4	2.84	2.00	1	-3.33	0.495
20	6.75	4.47	1	-6.96	0.208
100	15.3	10.0	1	-15.4	0.092

SWITCH^2



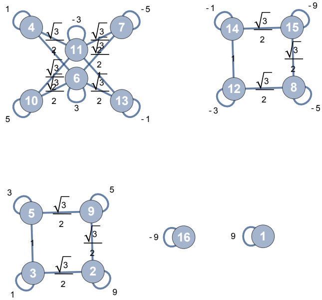
- $h_{u,v} = h_u + h_v \to$ local potentials are added
- State at vertex $(2,2)$ is transferred to vertex $(3,3)$ or $(4,4)$ at $t = \frac{\pi}{h_2}$
- Symmetry of simple switch $\to$ transfer from $(2,3)$ to $(3,2)$ or $(2,4)$ to $(4,2)$ with same configuration
- Higher products similar

Overview

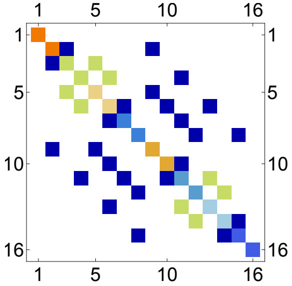
Varying Couplings

$$H_{XX} = \frac{1}{2}\sum_{i=1}^{N} J_i \left[\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right]$$

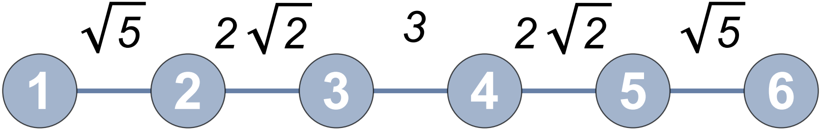
Weighted graphs



Weighted adjacency matrices



PERFECT COUPLINGS

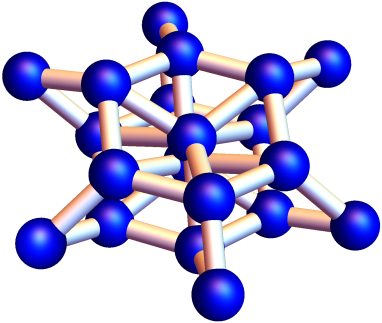


- Let $J_i = \frac{\lambda}{2}\sqrt{i(N-i)}$
- N qubit spin chain with $J_i \leftrightarrow$ Particle with spin $s = \frac{N-1}{2}$
- $H = \lambda S_x \rightarrow U(t) = e^{-\mathrm{i}\lambda t S_x}$
- $\langle N|U(t)|1\rangle = \left(-\mathrm{i}\sin\left(\frac{\lambda t}{2}\right)\right)^{N-1}$

$$H = \begin{pmatrix} 0 & \sqrt{5} & 0 & 0 & 0 & 0 \\ \sqrt{5} & 0 & \sqrt{8} & 0 & 0 & 0 \\ 0 & \sqrt{8} & 0 & \sqrt{9} & 0 & 0 \\ 0 & 0 & \sqrt{9} & 0 & \sqrt{8} & 0 \\ 0 & 0 & 0 & \sqrt{8} & 0 & \sqrt{5} \\ 0 & 0 & 0 & 0 & \sqrt{5} & 0 \end{pmatrix}$$

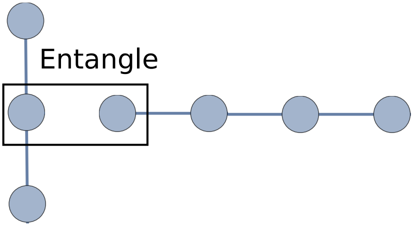
Higher Excitation Subspaces

- PST in larger networks
- Control more than one qubit
  - Quantum error correction and fault tolerance
- Prepare multiple states, send to different target sites



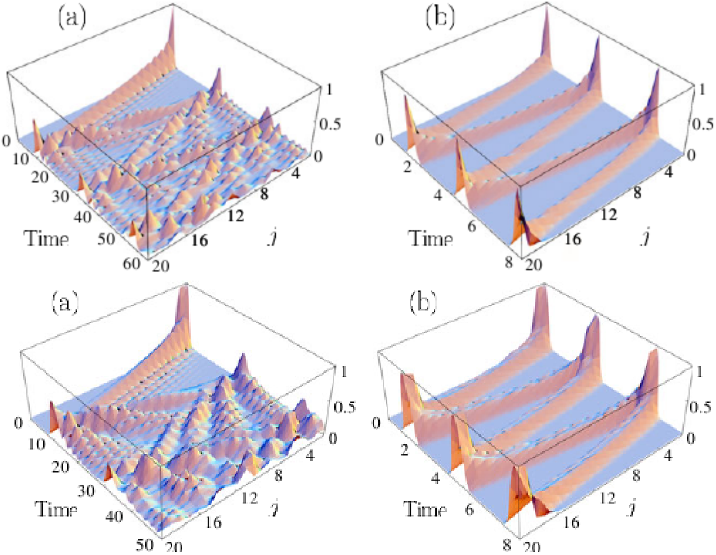
Calculating higher spin graphs from single spin graphs is an easy task.

Entanglement



- Transport part of an entangled pair along the network
- Entangle separated parts of a system
- Quantum teleportation as transportation scheme

Experimental Verification



Overview

Goal

- Framework for engineering spin chains with given properties
- Define certain constraints for hamiltonian
- Single spin subgraph should yield a network capable of PST