Intermediate Test 2 Solutions

January Camp 2021

1. *Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let H be the orthocentre of $\triangle ABC$ and let K be the reflection of H across BC. Prove that $HBKC$ is a parallelogram.*

   By reflection $\angle BHC = \angle BKC$, since $AB = AC$ and the diagram is symmetric with respect to AH, $\angle HBK = \angle HCK$. Hence in quadrilateral $BHCK$, opposite angles are equal, making it a parallelogram.

2. *Let a and r be real numbers such that*

$$a + ar + ar^2 + ar^3 + \cdots + ar^{2020} = 200 \qquad \textit{and} \qquad a + ar + ar^2 + ar^3 + \cdots + ar^{4041} = 380.$$

   Find the value of

$$a + ar + ar^2 + ar^3 + \cdots + ar^{6062}.$$

   Solution 1

   We can factorise the equations into $\frac{a(r^{2021}-1)}{r-1} = 200$, $\frac{a(r^{4042}-1)}{r-1} = 380$ and $\frac{a(r^{6063}-1)}{r-1}$. Dividing the second equation by the first gives

$$\frac{r^{4202}-1}{r^{2021}-1} = \frac{380}{200} = \frac{19}{10}$$

   Let $x = r^{2021}$. The equation then becomes $\frac{x^2-1}{x+1} = \frac{19}{10}$. This simplifies to $x + 1 = \frac{19}{10}$, so $r^{2021} = x = \frac{9}{10}$. We finish off with the following

$$\begin{aligned}
\frac{a(r^{2021}-1)}{r-1} &= 200 \\
\frac{a(r^{2021}-1)}{r-1} \cdot \frac{r^{6063}-1}{r^{2021}-1} &= 200 \cdot \frac{r^{6063}-1}{r^{2021}-1} \\
\frac{a(r^{6063}-1)}{r-1} &= 200 \cdot \frac{x^3-1}{x-1} \\
\frac{a(r^{6063}-1)}{r-1} &= 200 \cdot \frac{(9/10)^3-1}{(9/10)-1} \\
\frac{a(r^{6063}-1)}{r-1} &= 542
\end{aligned}$$

   Solution 2

   This solution does not use factorisations, and is the one I originally came up with.

$$\begin{aligned}
380 &= a + ar + \cdots + ar^{4041} \\
380 &= (a + ar + \cdots + ar^{2020}) + (ar^{2021} + ar^{2022} + \cdots + ar^{4041}) \\
380 &= (200) + r^{2021}(a + ar + \cdots + ar^{2020}) \\
380 &= (200) + r^{2021}(200)
\end{aligned}$$

So $r^{2021} = \frac{9}{10}$. We apply the same technique.

$$\begin{aligned}
a + ar + \cdots + ar^{6062} &= a + ar + \cdots + ar^{6062} \\
&= (a + ar + \cdots + ar^{4041}) + (ar^{4042} + ar4043 + \cdots + ar^{6062}) \\
&= (380) + r^{4042}(a + ar + \cdots + ar^{2020}) \\
&= (380) + \left(\frac{9}{10}\right)^2 \cdot (200) \\
&= 542
\end{aligned}$$

3. *Isaac is planning a nine-day holiday. Every day he will go surfing, or water skiing, or he will rest. On any given day he does just one of these three things. He never does different water-sports on consecutive days. How many schedules are possible for the holiday?*

   Let a_n be the amount of schedules possible in an n day holiday. We wish to find a_9. Let s_n be the amount of schedules in an n day holiday where Isaac's first activity must be surfing, simirlarly let w_n be the amount of schedules in an n day holiday that starts with water skiing. We will find a recurrence relation between these sequences.

   Consider a schedule that starts with swimming. If you swap all the swimming days with skiing days, and vice versa, you get a valid schedule that starts with water skiing. This shows that $s_n = w_n$.

   We find a recurrence for w_n by considering a schedule n days long. By definition the schedule must start with water skiing. The activity after that could either be water skiing again, or a rest day. There are w_{n-1} ways of having a schedule with water skiing twice in a row, and if a rest day is taken the other $n-2$ days can be filled freely, which gives a_{n-2} ways. Thus

$$w_n = w_{n-1} + a_{n-2} \text{ for } n > 2$$

   We find a recurrence for a_n by considering a schedule n days long. The day can either start with water skiing or swimming, there are w_n and s_n ways of doing this respectively. If a rest day is taken, the other $n-1$ days can be filled in freely, giving a_{n-1} more ways of filling the schedule. Thus $a_n = w_n + s_n + a_{n-1}$, but recall that $s_n = w_n$ so

$$a_n = 2w_n + a_{n-1} \text{ for } n > 1$$

   It is easy to check that $a_1 = 3, w_1 = 1, a_2 = 7, w_2 = 2$. We do the rest by table

n	a_n	b_n
1	3	1
2	7	2
3	17	5
4	41	12
5	99	29
6	239	70
7	577	169
8	1393	408
9	3363	985

   Which gives us $a_9 = 3363$

4. *A positive integer N has exactly 2021 positive divisors (including 1 and N itself), and it is divisible by 2021. Prove that N is not divisible by 2021^{43}.*

We recall that if

$$N = {p_1}^{a_1} {p_2}^{a_2} \cdots {p_k}^{a_k}$$

is the prime factorisation of N, then the number of divisors of N is given by

$$(a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

We thus investigate the solutions to

$$(a_1 + 1)(a_2 + 1) \cdots (a_k + 1) = 2021.$$

We know that $2021 = 43 \times 47$, which are both prime, and so the only ways of factorising 2021 as a product of some number of integers are 2021 and 43×47.

Since N is divisible by 2021, N has at least the two primes factors 43 and 47, and so it is the second factorisation that is relevant: we must have that

$$a_1 + 1 = 43 \qquad \text{and} \qquad a_2 + 1 = 47$$

or vice versa. Thus the only options for N are $43^{42} \times 47^{46}$, or $43^{46} \times 47^{42}$, neither of which is divisible by $2021^{43} = 43^{43} \times 47^{43}$.

5. *Let a, b, c, x, y and z be positive real numbers with $a + b + c = x + y + z$. Prove that*

$$\frac{a}{x+y} + \frac{b}{y+z} + \frac{c}{z+x} + \frac{x}{a+b} + \frac{y}{b+c} + \frac{z}{c+a} > 2.$$

Increasing the value of each of the denominators decreases the value of each fraction, and so

$$\begin{aligned}
&\frac{a}{x+y} + \frac{b}{y+z} + \frac{c}{z+x} + \frac{x}{a+b} + \frac{y}{b+c} + \frac{z}{c+a} \\
&\quad > \frac{a}{x+y+z} + \frac{b}{y+z+x} + \frac{c}{z+x+y} + \frac{x}{a+b+c} + \frac{y}{b+c+a} + \frac{z}{c+a+b} \\
&\quad = \frac{a+b+c}{x+y+z} + \frac{x+y+z}{a+b+c} \\
&\quad = 2.
\end{aligned}$$

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