Advanced Test 5

January Camp 2021

Time: 4 hours

1. A square-based pyramid has all of its edges the same length. A cube is placed inside so that one of its faces lies on the base of the pyramid, and the opposite face has an edge along each side of the pyramid (think – the natural way to put a cube in a pyramid). Show that the sum of any of the cube's edge lengths with any of the cube's face diagonal lengths is the same as the edge length of the pyramid.

2. There is a soccer competition with five teams where each team plays exactly one match against each other team. If one team wins against another team, that team gets 5 points whereas the losing team gets 0 points. If two teams draw, they each get 1 point if neither team scored a goal and they each get 2 points if they scored at least one goal. At the end of the competition it is found that the total points for the teams form five consecutive nonnegative integers. What is the minimum number of goals scored in this competition?

3. Given a sequence a_0, a_1, a_2, $\cdots$, of natural numbers with $a_0 = 1$ and $a_n^2 > a_{n-1}a_{n+1}$ for all $n > 0$, show that

   a) $a_n < a_1^n$ for all $n > 1$.

   b) $a_n > n$ for all n.

4. There are 20 people at a party. Each person holds some number of coins. Every minute, each person who has at least 19 coins simultaneously gives one coin to every other person at the party. What is the smallest number of coins that could be at the party such that at least one coin is given out in every minute?

5. Given a function $f : \mathbb{Q} \to \mathbb{Q}$ satisfying

   i) $f(0) = 2$ and $f(1) = 3$

   ii) For all rational numbers x and all integers n, we have
   $$f(x + n) - f(x) = n(f(x + 1) - f(x))$$

   iii) For all non-zero rational numbers x, we have $f(x) = f(\frac{1}{x})$

   Find all possible rational values of x such that $f(x) = 2021$.

6. Let $\triangle ABC$ have $\angle B > \angle C$. The internal angle bisector of A intersects BC at D and the external angle bisector of A intersect BC at E. P is a variable point on EA such that A lies on the segment EP. DP intersects AC at M and ME intersects AD at Q. Prove that all lines PQ intersect at a unique point as P changes.

- Submit your solutions at `https://forms.gle/M1L9KgbwzDxCKEjD9`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

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