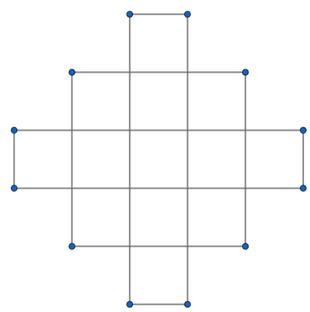
Intermediate Test 1 Solutions

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. *You are given the following shape:*



You need to tile this with L-shapes made of 3 blocks. Which single blocks could you shade out of the original diagram to make this possible?

Note that any L-block covers at most 1 shaded square. To cover all of them, we would need 5 L-blocks, covering a total of 15 squares, when we only have 13 squares. So without removing one of these, we will not be able to tile the image. On the other hand, removing one of these squares always results in a possible tiling. This is left as an exercise.

2. *Let $ABCD$ be a trapezoid with $AD \parallel BC$. The angle bisector of $\angle DAB$ intersects the angle bisectors of $\angle ABC$ and $\angle CDA$ at points P and S respectively, and the angle bisector of $\angle BCD$ intersects the angle bisectors of $\angle ABC$ and $\angle CDA$ at points Q and R respectively. Furthermore, $PS \parallel RQ$. Prove that $AB = CD$.*

Extend CQ onto AD. Let T be the intersection of CQ and side AD. Since we have $AD \parallel BC$, $\angle DTC = \angle QCB$ with alternating angles. Also, since $PS \parallel QR$, we also have corresponding angles $\angle DAP = \angle DTC \implies \angle DAP = \angle QCB$. We now have $\angle BAD = 2 \cdot \angle DAP = 2 \cdot \angle QCB = \angle BCD$. Since the opposite angles of the trapezoid are now equal, we know this trapezoid is a parallelogram. Since the opposite sides of a parallelogram are equal in length, we will have $AB = CD$.

3. *Find all natural numbers x, y and z satisfying*

$$x + \frac{1}{y + \frac{1}{z}} = \frac{850862}{421}$$

Note $\frac{850862}{421} = 2021\frac{21}{421}$ where 2021 is the integer part of the fraction.

Since y and z are given as natural numbers, we have that $y + \frac{1}{z} > 1$ which implies $\frac{1}{y+\frac{1}{z}} < 1$. Since x is a natural number, $\frac{1}{y+\frac{1}{z}}$ will need to be the fractional part of $2021\frac{21}{421}$. This will make x the integer part of $2021\frac{21}{421}$ which means $x = 2021$. Now we are left with $\frac{1}{y+\frac{1}{z}} = \frac{21}{421}$ which we can rewrite as $y + \frac{1}{z} = \frac{421}{21} = 20\frac{1}{21}$. Since $z \geq 1$, we will have $\frac{1}{z} \leq 1$. If $z = 1$, $y + \frac{1}{z} = y + \frac{1}{1} = y + 1 = 20\frac{1}{21}$ which would make $y = 19\frac{1}{21}$. This would be a contradiction since y is given as a natural number. From this we conclude that $z \neq 1$ and that $\frac{1}{z} < 1$. This means that $\frac{1}{z}$ will be the fractional part and y the integer part of $20\frac{1}{21}$. This makes $y = 20$ and $z = 21$. This means the only possible solution for (x, y, z) is $(2021, 20, 21)$.

4. *Find all possible real numbers k such that the values of x satisfying*

$$k(2-k)x^2 - (k+4)x + 6 = 0$$

are positive integers.

Firstly, notice that if $k(2-k) = 0$, then $x = \frac{6}{k+4}$. The only positive integer solution of this is when $k = 2$: $x = 1$. Assuming $k(2-k) \neq 0$, we can use the quadratic formula. We see that the solutions of the equation in x are

$$\begin{aligned} x &= \frac{(k+4) \pm \sqrt{(k+4)^2 - 24k(2-k)}}{2k(2-k)} \\ &= \frac{(k+4) \pm \sqrt{25k^2 - 40k + 16}}{2k(2-k)} \\ &= \frac{(k+4) \pm \sqrt{(5k-4)^2}}{2k(2-k)} \\ &= \frac{(k+4) \pm (5k-4)}{2k(2-k)} \end{aligned}$$

The two solutions are thus $x_1 = \frac{6k}{2k(2-k)} = \frac{3}{2-k}$ and $x_2 = \frac{8-4k}{2k(2-k)} = \frac{2}{k}$. Now we must have that $\frac{2}{k}$ and $\frac{3}{2-k}$ are integers simultaneously. $\frac{2}{k} \in \mathbb{Z} \iff k = \frac{2}{n}$ where $n \in \mathbb{Z}$. Hence we must have the following is an integer

$$\frac{3}{2-k} = \frac{3}{2-\frac{2}{n}} = \frac{3n}{2n-2} = \frac{3n}{2(n-1)}$$

Having $n-1|3$ yields $n \in \{-2, 0, 2, 4\}$. If $n-1 \nmid 3$, we must have $2(n-1)|n$. If $n > 2$, then $2(n-1) > n$. If $n < 0$, then $2(n-1) < n < 0$. Thus we only need to check $n \in \{0, 1, 2\}$. The values of k that we get are $k \in \{-1, 1, \frac{1}{2}, 2\}$. It can then be checked that the only k values that provide positive integers solutions are

$$k \in \{\frac{1}{2}, 1, 2\}$$

5. *Let O be the circumcentre of $\triangle ABC$. Let X, Y and Z be the reflections of O over BC, CA and AB respectively. Prove that $\triangle XYZ$ is congruent to $\triangle ABC$ and the corresponding sides are parallel.*

Let X' be the intersection of OX and BC and Z' be the intersection of OZ and AB. Since X is the reflection of O across BC we have $OX' = X'X$ as well as $OX \perp BC$ and since X' is on OX we also have $OX' \perp BC$. BC is a chord on the circumcircle with centre O and OX' is a perpendicular on BC, thus $BX' = X'C$. By similar arguments we have $OZ' = Z'Z$ and $AZ' = Z'B$.

By midpoint theorem on $\triangle ABC$, we have $X'Z'||CA$ and $2XX' = AC$. By midpoint theorem on $\triangle OZX$, we have $X'Z'||XZ$ and $2X'Z' = XZ$. Therefore, $XZ||X'Z'||CA$ and $XZ = 2X'Z' = AC$. Applying similar arguments to prove $XY = AB$, and $YZ = BC$ gives $\triangle ABC \equiv \triangle XYZ$. The correspoonding sides are parallel part follows with similar arguments as well.

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