Advanced Test 3 Solutions

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. *There is a book with n chapters where chapter i has i pages. The probability of opening the book in the same chapter twice in a row is p. Is it possible for p to be $1/k$ for some integer k?*

   Notice that the book has $\frac{n(n+1)}{2}$ pages $(1+2+3+...+n)$. Now, the number of ways that we can land in the same chapter twice is $1^2+2^2+3^2+...+n^2$, since each chapter has i pages that we could have landed in each time. This can be simplified as: $1^2+2^2+3^2+...+n^2 = \frac{n(n+1)(2n+1)}{6}$. The total number of ways to open the book twice is just $(\frac{n(n+1)}{2})^2$, since we can land on any page, then any page again. So $p = \frac{n(n+1)(2n+1)}{6}/(\frac{n(n+1)}{2})^2 = \frac{(2n+1)}{3}/(\frac{n(n+1)}{2}) = \frac{2(2n+1)}{3n(n+1)}$. Since $n(n+1)$ is always divisible by 2, the 2 on the top will cancel. Now, we seek n such that $2n+1$ will cancel i.e. since $2n+1$ is odd, we seek n with $2n+1 \mid 3n(n+1)$.

   $$\begin{aligned}
   &2n+1 \mid 3n^2+3n \\
   &2n+1 \mid 2(3n^2+3n) - 3n(2n+1) = 3n \\
   &2n+1 \mid 2(3n) - 3(2n+1) \\
   &2n+1 \mid -3
   \end{aligned}$$

   Finally, we get $2n+1 = 1, 3, -1, -3$ which gives $n = 0, 1, -1, -2$, none of which are valid numbers of chapters. So there is no $n > 1$ giving $p = \frac{1}{k}$.

2. *Points D, E, and F lie respectively on sides BC, CA, and AB of triangle ABC such that $BDEF$ is a parallelogram. Prove that the area of $BDEF$ is maximal when D, E, and F are the midpoints of the sides.*

   Let $BF = \alpha AB$ and $BD = \beta BC$. Then the area of the parallelogram is $BD.BF \sin\hat{B} = \alpha\beta AB.BC \sin\hat{B}$. Since $AB.BC\sin\hat{B}$ does not depend on D, E, F, we seek to maximise $\alpha\beta$. Note that $DE \parallel BA$ so $\triangle CAB \sim \triangle CED$. Then $\frac{DE}{BA} = \frac{CD}{CB}$, with $DE = BF = \alpha AB$ (parallelogram) and $CD = BC - DB = (1-\beta)BC$, so $\frac{\alpha AB}{AB} = \frac{(1-\beta)BC}{BC}$, giving $\alpha = 1-\beta$. Finally, we are left with maximising $\beta(1-\beta) = \beta - \beta^2 = \frac{1}{4} - (\beta - \frac{1}{2})^2$, which is clearly maximised by $\beta = \frac{1}{2}$. This also gives $\alpha = \frac{1}{2}$, which means D and F are midpoints, so E is a midpoint. So D, E, F being midpoints gives us $\alpha = \beta = \frac{1}{2}$ and a maximal area.

3. *Find all polynomials P with real coefficients such that $(x+1)P(x-1) - (x-1)P(x)$ is constant.* Letting $x = 1$, we see that the constant is $2P(0)$. Thus the equation is simply

   $$(x+1)P(x-1) - (x-1)P(x) = 2P(0)$$

   Letting $x = 0$, we get $P(-1) = P(0)$. Consider now the polynomial $Q(x) = P(x) - P(0)$. Substituting $P(x) = Q(x) + P(0)$ into the equation, we get

   $$\begin{aligned}
   (x+1)(Q(x-1)+P(0)) - (x-1)(Q(x)+P(0)) &= 2P(0) \\
   \implies (x+1)Q(x-1) - (x-1)Q(x) &= 0
   \end{aligned}$$

   Now, we know that $Q(0) = Q(-1) = 0$, so Q can be written as $Q(x) = kx(x+1)R(x)$ where k is some constant, and $R(x)$ is some polynomial. Thus gives

   $$k(x+1)(x-1)(x)R(x-1) - k(x-1)(x)(x+1)R(x) = 0$$

which can be simplified to

$$kx(x^2-1)(R(x-1)-R(x))=0$$

when $x(x^2-1)\neq 0$, we must have $R(x)=R(x-1)$. This is true for infinitely many x and since R is a polynomial, we then have that R must be a constant polynomial. Thus, we know that Q is exactly $Q(x)=kx(x+1)$. From here we can see that P must be

$$P(x)=kx(x+1)+c$$

so only polynomials of the form $P(x)=kx(x+1)+c$ can satisfy the constraint.

We can then check that this does indeed satisfy the original condition:

$$\begin{aligned}(x+1)P(x-1)-(x-1)P(x)&=(x+1)\cdot(k(x-1)x+c)-(x-1)\cdot(kx(x+1)+c)\\&=kx(x^2-1)-kx(x^2-1)+c\cdot(x+1)-c\cdot(x-1)\\&=2c\end{aligned}$$

4. *Does there exist an infinite set A of natural numbers such that any finite sum of distinct elements of A is not a perfect power, where a perfect power is a number of the form a^b with $b>1$ and $a\in\mathbb{N}$.*

Solution The answer is **yes**. We construct the set A inductively. Let $a_0=2$ (or any other number that is not a perfect power).

Suppose that we have already chosen values $A_n=\{a_0,a_1,\ldots,a_n\}$ such that any finite sum involving only elements of A_n is not a perfect power. We will choose a_{n+1} such that the same is true of $A_{n+1}=A_n\cup\{a_{n+1}\}$.

Let S be the set of values of all possible sums involving distinct elements of A_n. We are done if we can find a_{n+1} that is not a perfect power, and such that $a_{n+1}+s$ is not a perfect power for every element $s\in S$.

Let B be the largest element of S. (i.e. $B=a_0+a_1+\cdots+a_n$) If we could find an interval $[x,x+B]$ (where $x\notin A_n$) that does not contain any perfect powers, then we could set $a_{n+1}=x$. Then since $a_{n+1}+s$ falls in this interval for every $s\in S$, we know that $a_{n+1}+s$ is not a perfect power for every $s\in S$.

We claim that there are in fact arbitrarily long intervals of natural numbers which do not contain any perfect powers. Let $p(N)$ be the number of perfect powers that are at most N. We claim that $p(N)/N$ becomes arbitrarily small as N gets larger. This completes the proof, because if every interval of length $B+1$ always contained a perfect power, then we would have

$$p(N)\geq\left\lfloor\frac{N}{B+1}\right\rfloor>\frac{N}{B+1}-1,$$

and so $p(N)/N$ would tend to something at least as large as $\frac{1}{B+1}$ as N tends to infinity.

We now derive an estimate for $p(N)/N$. Let k be the largest natural number such that there exists a natural number a such that $a^k\leq N$. Since $2^k\leq a^k\leq N$, we know that $k\leq\log_2 N$.

Since there are at most $\sqrt{N}$ perfect squares that are at most N, at most $\sqrt[3]{N}$ perfect cubes that are at most N, and so on, we see that

$$p(N)\leq\sqrt{N}+\sqrt[3]{N}+\sqrt[4]{N}+\cdots+\sqrt[k]{N}$$

and since

$$\sqrt[i]{N}\leq\sqrt{N}$$

for each i, we obtain that

$$p(N)\leq k\sqrt{N}\leq\sqrt{N}\log_2 N\implies\frac{p(N)}{N}\leq\frac{\sqrt{N}\log_2 N}{N}$$

which does indeed tend to 0 as $N\to\infty$.

Alternative Solution Consider the set

$$S = \left\{2^{k+1} \cdot 3^k \middle| k \in \mathbb{N}_0\right\}$$

containing the numbers $2, 12, 72, 78, \ldots$. We will show that no finite sum of elements of S is a perfect power.

Consider any finite sum of elements of S

$$n = 2^{a_1+1} \cdot 3^{a_1} + 2^{a_2+1} \cdot 3^{a_2} + \cdots + 2^{a_k+1} \cdot 3^{a_k}$$

where $a_1 < a_2 < \cdots < a_k$. We see that the largest power of 2 that divides n is 2^{a_1+1}, and the largest power of 3 that divides n is 3^{a_1}. If n is the m^{th} power of an integer, then we need that $m \mid a_1 + 1$ and also $m \mid a_1$. But then $m \mid 1$, and so n can only be the first power of an integer, but not any higher power.

5. *Let ABC be an acute non-isosceles triangle with altitudes BB_1 and CC_1 intersecting at H. The angle bisectors of $\angle B_1AC_1$ and B_1HC_1 intersect the line B_1C_1 at points L_1 and L_2, respectively. Let P and Q be the second points of intersection of the circumcircles of triangles AHL_1 and AHL_2 with the line B_1C_1 respectively. Prove that the points B, C, P, and Q lie on a circle.*

Note A, B_1, C_1 and H are concyclic. Let W_1 and W_2 be the midpoints of arcs C_1HB_1 and C_1AB_1 of this circle respectively. Note line AW_1 passes through L_1 and line AW_2 passes through L_2. Let us prove that point P lies on HW_1 and point Q lies on AW_2. Indeed, suppose that HW_1 intersects B_1C_1 at some point P'. Then

$$\angle L_1PH = \angle W_1AB_1 - \angle C_1AH = \angle HAL_1,$$

so PHL_1A is inscribed and hence $P = P'$. Analogously one can prove that Q lies on AW_2. Note that AW_1HW_2 is a rectangle. It follows that

$$\angle W_2QP = \angle L_1PH = \angle L_1AH.$$

Let PQ intersect BC at point T, M be the midpoint of BC and K be the second point of intersection of MH with the circle AB_1HC_1. It is well-known that K lies on the circumcircle of triangle ABC and on line AT. Then $TK \times TA = TB \times TC$. It is now sufficient to prove that $TP \times TQ = TB \times TC$. We will prove that $TK \times TA = TP \times TQ$, i.e. $PKAQ$ is inscribed. For this it is enough to prove that P,K and W_2 are collinear. Indeed from this condition:

$$\angle TKP = \angle W_2KA = \angle W_2HA = \angle HAL_1 = \angle AQP.$$

Note that MB_1 and MC_1 are the tangents to the circle AB_1HC_1. Let P'' be a point of intersection of B_1C_1 and KW_2. Since KH and the tangents to the circle with diameter AH are concurrent at the midpoint of BC, the quadruple of lines $W_2H, W_2K, W_2B_1, W_2C_1$ is harmonic. On the other hand, the quadruple of points L_2, P, B_1, C_1 is harmonic from the properties of internal and external bisectors of $\angle B_1HC_1$ in triangle B_1HC_1. Since the triple of lines W_2H, W_2B_1, W_2C_1 intersects line B_1C_1 at points L_2, B_1, C_1 respectively, the points P and P'' coincide.

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