Advanced Test 4 Solutions

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. *Consider a $3 \times 3 \times 3$ 3-dimensional chess cube with some hyperrooks. Hyperrooks can move along any direction parallel to an edge of the cube (like a normal rook, but also up and down). What is the maximum number of hyperrooks you can place in the chess cube without any of them attacking each other?*

   First, we prove that 9 hyperrooks is the maximum amount of hyperrooks you can place on a chess cube. Consider the 9 columns (in the up-down direction) of the chess cube. If a column contains 2 hyperrooks, the hyperrooks would attack each other. Therefore, each column has at most 1 hyperrook. Since there are 9 columns, there are at most 9 hyperrooks. We show that placing 9 hyperrooks is possible by construction.

   Construction for 9 rooks:

   Top layer

R		
	R	
		R

   Middle layer

	R	
		R
R		

   Bottom layer

		R
R		
	R	

2. *Find all positive integers a, b and c satisfying*

$$a + b - c = 14$$
$$a^2 + b^2 - c^2 = 14.$$

   Rearranging the first equation, we have $c = a + b - 14$. We can substitute this into the second equation to get

$$\begin{aligned} a^2 + b^2 - c^2 &= 14 \\ \implies a^2 + b^2 - (a + b - 14)^2 &= 14 \\ \implies -2ab + 28a + 28b - 196 &= 14 \\ \implies ab - 14a - 14b + 98 &= -7 \\ \implies (a - 14)(b - 14) - 98 &= -7 \\ \implies (a - 14)(b - 14) &= 91 = 7 \cdot 13 \end{aligned}$$

   If a and b are integers, then so are $a - 14$ and $b - 14$. So we have that $a - 14$ and $b - 14$ must simply be the paired factors of 91. We can not check every case

$$\begin{aligned} (a - 14, b - 14) &= (91, 1) \implies (a, b) = (105, 15) \\ &= (13, 7) \implies (a, b) = (27, 21) \\ &= (-13, -7) \implies (a, b) = (1, 7) \\ &= (-91, -1) \implies (a, b) = (-77, 13) \end{aligned}$$

   Notice that swapping a and b will produce more solutions which are just the same as the ones already generated, but also swapped. If we know a and b, we can find c. Hence, using the condition that a, b, $c \in \mathbb{N}$, the only solutions are

$$\begin{aligned} (a, b, c) &= (105, 15, 106) \\ &= (27, 21, 34) \end{aligned}$$

where a and b may also be swapped in each triple.

3. *You are given nine real numbers, a_1, a_2, $\cdots$, a_9 with an average of m. What is the minimum possible number of triples (i, j, k) with $1 \leq i < j < k \leq 9$ and $a_i + a_j + a_k \geq 3m$?*

   Call a triple (a_i, a_j, a_k) *good* if $i < j < k$ and $a_i + a_j + a_k \geq 3m$. Let $s(X)$ denote the sum of the elements of the set X, and $S = \{a_1, a_2, \ldots, a_9\}$. Consider all of the ways of dividing S into three non-overlapping triples A, B, and C. Since $s(A) + s(B) + s(C) = s(S) = 9m$, we know that at least one of $s(A)$, $s(B)$, and $s(C)$ is at least $3m$, and so at least one of A, B, or C is good. We calculate the number of ways of dividing S into 3 non-overlapping triples A, B, and C. There are $\binom{9}{3}$ ways to choose A, and $\binom{6}{3}$ ways of choosing B once A has been chosen. C is then uniquely determined, and so there are $\binom{9}{3}\binom{6}{3}$ such partitions overall. Now any triple X appears in $3\binom{6}{3}$ such triples. This is because there are 3 options for which of A, B, or C X is equal to, and there are then $\binom{6}{3}$ ways to choose the first of the remaining sets. We see that if we add up the number of good triples that appear in some partition of S into triples A, B, and C, then on the one hand we obtain at least $\binom{9}{3}\binom{6}{3}$ since every partition contains a good triple, and also that every good triple has been counted precisely $3\binom{6}{3}$ times. It follows that

   $$3\binom{6}{3}(\#\text{ good triples}) \geq \binom{9}{3}\binom{6}{3}$$

   and so the number of good triples is at least 28.

   On the other hand, if we let $a_1 = a_2 = \cdots = a_9 = 0$ and $a_9 = 1$, then the good triples are precisely those that contain a_9. There are thus $\binom{8}{2} = 28$ such triples, showing that 28 is indeed the minimum.

4. *Let ABC be an acute-angled triangle with orthocentre H. Let the point H' be the reflection of H over AB. Let N be the intersection of HH' and AB. The circumcircle of $\triangle ANH'$ intersects AC again at M. The circumcircle of $\triangle BNH'$ intersects BC again at P. Show that the points M, N and P are collinear.*

   Lemma:
   Let H be an orthocentre in triangle ABC and H' be the reflection point of H with respect to AB. Then H' lies on the circumcircle of triangle ABC.

   Proof of Lemma
   Let AB and HH' intersect at point N and let A_1 be the foot of the perpendicular from A onto BC. Then triangle ANH is congruent to triangle ANH'. From

   $$\angle H'AB = \angle H'AN = \angle NAH = \angle BAA_1 = 90^\circ - \angle ABC = \angle NCB = \angle H'CB$$

   it follows that $AH'BC$ is cyclic.

   Now we turn back to the question. From $\angle ANH' = 90^\circ$ it follows that AH' is a diameter of a circle around triangle ANH'. Then $\angle H'MA = 90^\circ$, i.e. $H'M \perp AC$. Similarly, $H'P \perp BC$. We also have $H'N \perp AB$ and from the lemma we have $AH'BC$ cyclic. Hence by the theorem on a Simson's Line, we have M, N and P collinear.

5. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

   $$f(xy) = \max\{f(x+y), f(x)f(y)\}$$

   for all x, $y \in \mathbb{R}$.

   Substituting $(x, y) = (0, 0)$, we get $f(0) = \max\{f(0), f(0)^2\}$, so we must have

   $$f(0) \geq f(0)^2 \implies f(0) \in [0, 1]$$

Now we substitute $(x,y) = (x,0)$ giving $f(0) = \max\{f(x), f(x)f(0)\}$. Lastly, substituting $(x,y) = (x,-x)$, we get $f(-x^2) = \max\{f(0), f(x)f(-x)\}$. If we let $f(0) = a$, then these three identities read as follows

$$a \in [0,1] \tag{1}$$
$$a = \max\{f(x), af(x)\} \tag{2}$$
$$f(-x^2) = \max\{a, f(x)f(-x)\} \tag{3}$$

Identity (1) with (2) tells us that if $f(k) \geq 0$, then $f(k) \geq af(k)$, so $f(k) = a$. We shall now consider two separate cases:

- Case 1: $a = 0$
  Identity (2) gives $0 = \max\{f(x), 0\}\ \forall x \in \mathbb{R} \implies f(x) \leq 0\ \forall x \in \mathbb{R}$. Identity (3) gives $f(-x^2) = 0$ and $f(x)f(-x) \leq 0$. Thus, for $k \leq 0$, we must have $f(k) = 0$. Consider now the substitution $(x,y) = (x,-1)$; we get

  $$f(-x) = \max\{f(x-1), f(x)f(-1)\} = \max\{f(x-1), 0\} = 0\ \forall x \in \mathbb{R}$$

  so in particular we then have $f(x) = 0\ \forall x \in \mathbb{R}$.
- Case 2: $a \in (0,1]$
  Suppose $f(k) \leq 0$. We must then have $f(k) \leq af(k) \implies af(k) = a$ by Identity (2). Solving this we get

  $$a(f(k) - 1) = 0$$

  So either $a = 0$ or $f(k) = 1$. The assumption in this case was that $a > 0$ and $f(k) \leq 0$, so we have found a contradiction. Hence there cannot be any $k \in \mathbb{R}$ such that $f(k) \leq 0$. This means that $f(x) > 0\ \forall x \in \mathbb{R}$, but we already have shown that this must give $f(x) = a\ \forall x \in \mathbb{R}$.

In summary: $f(0) = 0 \implies f(x) = 0\ \forall x \in \mathbb{R}$ and $f(0) = a \in (0,1] \implies f(x) = a\ \forall x \in \mathbb{R}$. Hence we can write this concisely as $f(x) = a\ \forall x \in \mathbb{R}$ where $a \in [0,1]$.

Check:

$$\begin{aligned} &\text{LHS} = f(xy) = a \\ &\text{RHS} = \max\{f(x+y), f(x)f(y)\} = \max\{a, a^2\} = a \\ \implies &\text{LHS} = \text{RHS} \end{aligned}$$

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