Advanced Test 2

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. A positive integer N has exactly 2021 positive divisors (including 1 and N itself), and it is divisible by 2021. Prove that N is not divisible by 2021^{43}.

2. Let a, b, c, x, y and z be positive real numbers with $a + b + c = x + y + z$. Prove that

$$\frac{a}{x+y} + \frac{b}{y+z} + \frac{c}{z+x} + \frac{x}{a+b} + \frac{y}{b+c} + \frac{z}{c+a} > 2.$$

3. Let circles Γ_1 and Γ_2 intersect at A and B. One of the tangents to Γ_1 and Γ_2 touches them at P and Q respectively. Let line AB meet the circumcircle of PQA at C. Join CP and CQ and extend both to meet Γ_1 and Γ_2 at F and E respectively. Prove that the quadrilateral $PQFE$ is cyclic.

4. Let K be a set of nine different positive integers which only have 2027 and 2029 as prime factors. Show that there are three distinct integers a, b, and c in K such that $\sqrt[3]{abc}$ is an integer.

5. Prove that there are infinitely many $n \in \mathbb{N}$ such that there exists a $d \in \mathbb{N}$ with both d and $d + n$ being factors of $n^2 + 1$.

- Submit your solutions at `https://forms.gle/M1L9KgbwzDxCKEjD9`.

- Submit each question in a single separate PDF file (with multiple pages if necessary).

- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.

- If you have multiple PDF files for a question, combine them using software such as PDFsam.

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