Intermediate Test 3 Solutions

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. *Find the smallest positive integer which only has 1 and 2 as digits and which is divisible by 99.*

   Let this smallest positive integer be N. Note that $99 \mid N \iff 11 \mid N$ and $9 \mid N$.

   Using the divisibility rules, we know that $9 \mid N$ means that the sum of the digits of N will be a multiple of 9 and that $11 \mid N$ means that the sum of every second digit minus the sum of every other digit will be a multiple of 11. Let A be the sum of every second digit and B the sum of every other digit.

   If the sum of N's digits is equal to 9, we will have that $A + B = 9$ and either A will be odd and B even, or vice versa. Since we have that $11 \mid A - B$ and $A + B = 9 < 11 \implies \mid A - B \mid = 0$ which is impossible since A and B have different parity.

   Next we have the case of the sum of N's digits being equal to 18 (the next multiple of 9). To minimise the number of digits we can make as many digits as possible equal to 2. This would make the smallest option $N = 222222222$, however this would not be divisible by 11, as $\mid A - B \mid = 2$. The next smallest possible option would $N = 1122222222$. Note that we place the 1's at the front of number to minimise the number. In this case, the number would be divisible by 11 which would make $N = 1122222222$ the solution.

2. *Γ is a circle and AB and CD are $\perp$ chords that intersect inside Γ. P is on Γ, with Q diametrically opposite to P. Let the feet of the perpendiculars from P to AB and CD be P' and P'' respectively, and the feet of the perpendiculars from Q to AB and CD be Q' and Q'' respectively. Show that:*

   $$(P'P)^2 + (P''P)^2 + (Q'Q)^2 + (Q''Q)^2 < d^2$$

   where d is the diameter of Γ.

   Let AB and CD intersect at X. By Pythagoras $(P'P)^2 + (P''P)^2 = (P'P'')^2$. Notice $PP'XP''$ forms a rectangle. The length of the diagonals of a rectangle are equal, so $(P'P'')^2 = (PX)^2$. Similar arguments give $(Q'Q)^2 + (Q''Q)^2 = (QX)^2$.

   We consider $\triangle PXQ$. Notice $\angle PXQ > 90^\circ$, because X is in the circle and PQ is a diameter. Hence $\triangle PXQ$ is obtuse. We claim it well known that in an obtuse triangle with $\angle PXQ$ being the obtuse angle that $(PX)^2 + (QX)^2 < (PQ)^2$. For a formal proof of this we look at cos-law

   $$(PQ)^2 = (PX)^2 + (QX)^2 - 2(PX)(QX)\cos\angle PXQ$$

   since $\angle PXQ > 90^\circ$, $-2(PX)(QX)\cos\angle PXQ > 0$ and the inequality follows.

   We end by putting everything together

   $$(P'P)^2 + (P''P)^2 + (Q'Q)^2 + (Q''Q)^2 = (PX)^2 + (QX)^2 < (PQ)^2 = d^2$$

3. *Let $q(x) = x^3 - x^2 - 2x + 1$ with roots a, b, and c. Find $a^2 + b^2 + c^2$.*

   Since a, b, c are roots and $q(x)$ has no leading coefficient, we can factorise $q(x)$ into $q(x) = (x - a)(x - b)(x - c)$. Multiplying out gives

   $$q(x) = x^3 + x^2(-a - b - c) + x(ab + bc + ca) - abc$$

which means that $-a-b-c=-1$ and $ab+bc+ca=-2$. The first equation is equivalent to $a+b+c=1$. These results could've been read directly from Viviani's. We finish with the following

$$\begin{aligned}
a^2+b^2+c^2 &= (a+b+c)^2-2ab-2bc-2c \\
&= (a+b+c)^2-2(ab+bc+ca) \\
&= 1^2-2(-2) \\
&= 5
\end{aligned}$$

4. *There is a book with n chapters where chapter i has i pages. The probability of opening the book in the same chapter twice in a row is p. Is it possible for p to be $1/k$ for some integer k?*

   Notice that the book has $\frac{n(n+1)}{2}$ pages $(1+2+3+...+n)$. Now, the number of ways that we can land in the same chapter twice is $1^2+2^2+3^2+...+n^2$, since each chapter has i pages that we could have landed in each time. This can be simplified as: $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$. The total number of ways to open the book twice is just $(\frac{n(n+1)}{2})^2$, since we can land on any page, then any page again. So $p=\frac{n(n+1)(2n+1)}{6}/(\frac{n(n+1)}{2})^2=\frac{(2n+1)}{3}/(\frac{n(n+1)}{2})=\frac{2(2n+1)}{3n(n+1)}$. Since $n(n+1)$ is always divisible by 2, the 2 on the top will cancel. Now, we seek n such that $2n+1$ will cancel i.e. since $2n+1$ is odd, we seek n with $2n+1 \mid 3n(n+1)$.

$$\begin{aligned}
&2n+1 \mid 3n^2+3n \\
&2n+1 \mid 2(3n^2+3n)-3n(2n+1)=3n \\
&2n+1 \mid 2(3n)-3(2n+1) \\
&2n+1 \mid -3
\end{aligned}$$

   Finally, we get $2n+1=1,3,-1,-3$ which gives $n=0,1,-1,-2$, none of which are valid numbers of chapters. So there is no $n>1$ giving $p=\frac{1}{k}$.

5. *Points D, E, and F lie respectively on sides BC, CA, and AB of triangle ABC such that $BDEF$ is a parallelogram. Prove that the area of $BDEF$ is maximal when D, E, and F are the midpoints of the sides.*

   Let $BF=\alpha AB$ and $BD=\beta BC$. Then the area of the parallelogram is $BD.BF\sin\hat{B}=\alpha\beta AB.BC\sin\hat{B}$. Since $AB.BC\sin\hat{B}$ does not depend on D,E,F, we seek to maximise $\alpha\beta$. Note that $DE \parallel BA$ so $\triangle CAB \sim \triangle CED$. Then $\frac{DE}{BA}=\frac{CD}{CB}$, with $DE=BF=\alpha AB$ (parallelogram) and $CD=BC-DB=(1-\beta)BC$, so $\frac{\alpha AB}{AB}=\frac{(1-\beta)BC}{BC}$, giving $\alpha=1-\beta$. Finally, we are left with maximising $\beta(1-\beta)=\beta-\beta^2=\frac{1}{4}-(\beta-\frac{1}{2})^2$, which is clearly maximised by $\beta=\frac{1}{2}$. This also gives $\alpha=\frac{1}{2}$, which means D and F are midpoints, so E is a midpoint. So D,E,F being midpoints gives us $\alpha=\beta=\frac{1}{2}$ and a maximal area.

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