Advanced Test 2 Solutions

January Camp 2021

1. *A positive integer N has exactly 2021 positive divisors (including 1 and N itself), and it is divisible by 2021. Prove that N is not divisible by 2021^{43}.*

   We recall that if
   $$N = {p_1}^{a_1}{p_2}^{a_2}\cdots{p_k}^{a_k}$$
   is the prime factorisation of N, then the number of divisors of N is given by
   $$(a_1+1)(a_2+1)\cdots(a_k+1).$$
   We thus investigate the solutions to
   $$(a_1+1)(a_2+1)\cdots(a_k+1) = 2021.$$
   We know that $2021 = 43\times 47$, which are both prime, and so the only ways of factorising 2021 as a product of some number of integers are 2021 and 43×47.

   Since N is divisible by 2021, N has at least the two primes factors 43 and 47, and so it is the second factorisation that is relevant: we must have that
   $$a_1+1 = 43 \qquad\text{and}\qquad a_2+1 = 47$$
   or vice versa. Thus the only options for N are $43^{42}\times 47^{46}$, or $43^{46}\times 47^{42}$, neither of which is divisible by $2021^{43} = 43^{43}\times 47^{43}$.

2. *Let a, b, c, x, y and z be positive real numbers with $a+b+c = x+y+z$. Prove that*
   $$\frac{a}{x+y}+\frac{b}{y+z}+\frac{c}{z+x}+\frac{x}{a+b}+\frac{y}{b+c}+\frac{z}{c+a} > 2.$$
   Increasing the value of each of the denominators decreases the value of each fraction, and so
   $$\begin{aligned}&\frac{a}{x+y}+\frac{b}{y+z}+\frac{c}{z+x}+\frac{x}{a+b}+\frac{y}{b+c}+\frac{z}{c+a}\\ &\quad > \frac{a}{x+y+z}+\frac{b}{y+z+x}+\frac{c}{z+x+y}+\frac{x}{a+b+c}+\frac{y}{b+c+a}+\frac{z}{c+a+b}\\ &\quad = \frac{a+b+c}{x+y+z}+\frac{x+y+z}{a+b+c}\\ &\quad = 2.\end{aligned}$$

3. *Let circles Γ_1 and Γ_2 intersect at A and B. One of the tangents to Γ_1 and Γ_2 touches them at P and Q respectively. Let line AB meet the circumcircle of PQA at C. Join CP and CQ and extend both to meet Γ_1 and Γ_2 at F and E respectively. Prove that the quadrilateral $PQFE$ is cyclic.*

   Notice that $PEAB$ is cyclic. Using power of a point with point C, we see that $CP\cdot CE = CB\cdot CA$. Similarly, $QFAB$ is cyclic, so we have $CQ\cdot CF = CB\cdot CA$. Equating these two, we get
   $$CQ\cdot CF = CP\cdot CE$$
   So by the converse of power of a point, we must have that $PQFE$ is cyclic.

4. *Let K be a set of nine different positive integers which only have 2027 and 2029 as prime factors. Show that there are three distinct integers a, b, and c in K such that $\sqrt[3]{abc}$ is an integer.*

Without loss of generality, we may assume that K contains only positive integers. Let

$$K = \left\{2027^{a_i} \cdot 2029^{b_i} \middle| a_i, b_i \in \mathbb{Z}, a_i, a_i \geq 0, 1 \leq i \leq 9\right\}.$$

It suffices to show that there are $1 \leq i_1, i_2, i_3 \leq 9$ such that

$$a_{i_1} + a_{i_2} + a_{i_3} \equiv b_{i_1} + b_{i_2} + b_{i_3} \equiv 0 \pmod 3.$$

Consider a 3×3 grid with rows and columns numbered from 0 to 2. We say that three cells in the grid are *on a line* if all three are in the same row, or all three are in the same column, or if all three are simultaneously in different rows and different columns. In other words, if they are on a physical line in the grid if we consider the grid to wrap around from the bottom to the top, and from the right to the left.

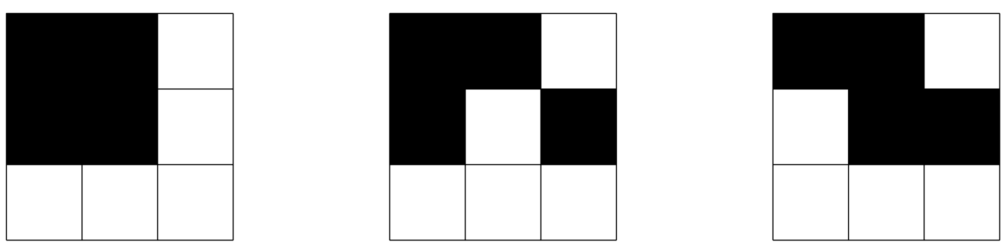
For $n = 2027^a \cdot 2029^b \in K$, place n in row (a mod 3) and column (b mod 3).

We see that the product of three numbers in K is a cube if and only if they are on a line in the grid, or are in the same cell in the grid.

If three numbers are in the same cell, we are done. Suppose that every cell contains at most two numbers. Then by the pigeonhole principle, there are at least five non-empty cells in the grid. (If each number were in one of at most four cells, and there are 9 numbers, then the pigeonhole principle would give us three numbers in the same cell.)

If three of these non-empty cells are in the same row, then we are done. So suppose that each row contains at most two non-empty cells. Then we see that there are at least two rows containing precisely two non-empty cells.

If three cells are on a line in the grid, then we see that they are still on a line if we swap two of the rows, or if we swap two of the columns. We can thus swap the rows so that the top two rows contain two non-empty cells, and the leftmost two cells in the top row are non-empty. Without loss of generality, we are therefore in one of the following three cases:



In each case, we see that every way of placing a fifth non-empty square in the grid results in three squares that are on a line.

5. *Prove that there are infinitely many $n \in \mathbb{N}$ such that there exists a $d \in \mathbb{N}$ with both d and $d + n$ being factors of $n^2 + 1$.*

Solution 1 Note that $d = n = 1$ provides a solution. Suppose that k and m are such that $k \mid m^2 + 1$ and $k + m \mid m^2 + 1$. Let

$$d = k + m \qquad \text{and} \qquad n = m + \frac{m^2 + 1}{k}.$$

We claim that $d \mid n^2 + 1$ and $d + n \mid n^2 + 1$. We first show that $d \mid n^2 + 1$. Since $\gcd(d, k) = \gcd(m, k) = 1$, it is enough to show that

$$k^2(n^2 + 1) = (m^2 + km + 1)^2 + k^2$$

is divisible by $d = k + m$. We note that

$$(m^2 + km + 1)^2 + k^2 \equiv k^2 + 1 \equiv (-m)^2 + 1 \equiv 0 \pmod{k + m}$$

since $k + m \mid m^2 + 1$ by assumption. We now show that $n + d \mid n^2 + 1$. We know that $n^2 \equiv (-d)^2 + 1 \pmod{n + d}$, so we can instead show that $n + d \mid d^2 + 1$. We note that

$$k(n + d) = (km + m^2 + 1) + k(k + m) = k^2 + 2km + m^2 + 1 = (k + m)^2 + 1 = d^2 + 1.$$

and so $d^2 + 1$ is divisible by $n + d$, as claimed.

We see that if k and m are such that $k \mid m^2 + 1$ and $k + m \mid m^2 + 1$, then the values of d and n given above also provide a solution to the problem. Since $n > m$, we obtain infinitely many solutions in this way.

Solution 2 We recall *Cassini's Identity*

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

where F_k is the k^{th} Fibonacci number. Considering the even terms gives us

$$F_{2n}^2 + 1 = F_{2n+1}F_{2n-1} = F_{2n-1}(F_{2n} + F_{2n-1})$$

and so $n = F_{2n}$ and $d = F_{2n-1}$ provides infinitely many solutions to the problem.

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