Advanced February Monthly Assignment
Due date: 20 February 2021

1. Let c and d be positive divisors of a natural number n such that $c > d$. Prove that

$$c > d + \frac{d^2}{n}.$$

2. Suppose $a, b, c > 0$ and $\sqrt{a-b} + \sqrt{a-c} > \sqrt{b+c}$. Prove that $a > \frac{3}{4}(b+c)$.

3. Let $ABCD$ be a square. Points P and Q lie on the segments AD and DC such that $\angle PBQ = 45°$. Prove that BP bisects $\angle APQ$ and BQ bisects $\angle CQP$.

4. Let n be a positive integer. Show that there is a positive integer m such that $\varphi(m) = n!$, where φ denotes the Euler phi function.

5. Define the function $f(x) = x^2 + \sin(x)$ (where x is in radians in this context). Furthermore, let $\{a_n\}$ be a sequence with $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$. Let $a_1 = 1$, and $f(a_n) = a_{n-1}$ for $n \geq 2$. Prove that there exists $n \in \mathbb{N}$ such that

$$\sum_{k=1}^{n} a_i > 2021.$$

6. Consider a triangle ABC with points M and N on BC and AB respectively such that $AM \perp BC$ and $CN \perp AB$, and let AC and MN intersect at Y. It is given X is a point inside acute-angled triangle ABC such that $MBNX$ is a parallelogram. Prove that the angle bisectors of $\angle MXN$ and $\angle MYC$ are perpendicular.

- Submit your solutions at `https://forms.gle/yoD4U3zZxs2yuP4d9`.

- Submit each question in a single separate PDF file (with multiple pages if necessary).

- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.

- If you have multiple PDF files for a question, combine them using software such as PDFsam.