# Advanced March Monthly Assignment

## Due date: 31 March 2021

1. There are $n$ towns in a country. Every road in this country goes from one of these towns to another (different) town. These are not one-way roads. Two independent routes from town $A$ to town $B$ are routes that don't have any intermediate towns in common (i.e. there are no towns that lie between $A$ and $B$ on both routes). If we are told that there are at least two independent routes from any town $A$ to any other town $B$, what is the minimum number of roads that could be in this country?

2. Let $ABC$ be a triangle with centroid $G$ and $AB \neq BC$. $\Gamma$ is the circle with diameter $BG$ and $H$ is its centre. Let $CG$ extended intersect $AB$ at $F$ and $\Gamma$ at $I$ different to $G$. Let $AG$ extended intersect $BC$ at $D$ and $\Gamma$ at $J$ different to $G$. Show that $IJ$ is parallel to $FD$ if and only if $I$, $H$, and $J$ are collinear.

3. Find all monic polynomials $P(x)$ such that the polynomial $P(x)^2 - 1$ is divisible by the polynomial $P(x+1)$.

4. Let $a_0, a_1, \ldots, a_d$ be integers such that $\gcd(a_0, a_1) = 1$. Define the sequence $(u_n)_{n=1}^{\infty}$ by
$$u_n = \sum_{k=0}^{d} a_k \varphi(n+k).$$
Show that 1 is the only positive integer that divides every term of the sequence $(u_n)_{n=1}^{\infty}$.

5. There are some scientists spread throughout a research laboratory in Area 51 which is shaped like a rectangle made up of unit squares. Each unit square is a room, and each edge between two adjacent unit squares is a wall which may or may not have a door in it, but none of the walls on the outside of the laboratory have doors in them (it's a strange building, but what else would you expect from Area 51).

   Each of these scientists has a piece of an alien object, and they all want to bring the pieces together in one room so that the alien object can activate, from which point onward every pet in the world will be completely healthy, never die, and be able to speak. However, these scientists cannot communicate with each other. Instead, there is a mysterious stranger named Anthony Lien outside the building who can see exactly where each scientist is. Anthony Lien has access to the laboratory's intercom system, and one at a time he can announce either north, south, east, or west over the intercom system, at which point each scientist in the building will move one unit in that direction to the next room; if the wall in that direction does not have a door in it, then that scientist will stay where they are.

   Show that if each room can be reached from every other room, then Anthony Lien can always bring all the scientists into the same room.

6. Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{Q}$ we have
$$3f(x+y+z)+f(-x+y+z)+f(x-y+z)+f(x+y-z)+4f(x)+4f(y)+4f(z) = 4f(x+y)+4f(y+z)+4f(z+x).$$

- Submit your solutions at `https://forms.gle/yoD4U3zZxs2yuP4d9`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.