

Test 4

April Camp 2021

Time: $4\frac{1}{2}$ hours

1. We have n boxes labelled B_1, B_2, \dots, B_n . Initially there are n balls in total in these n boxes. (There may be some boxes with more than one ball, and some boxes that are empty.) One each turn, we are allowed to perform one of three operations:

- If there is at least 1 ball in box B_1 , we can remove this ball from box B_1 and add a ball to box B_2 .
- If there are at least 2 balls in box B_i where $1 < i < n$, then we can remove 2 balls from box B_i , and add 1 ball to each of boxes B_{i-1} and B_{i+1} .
- If there is at least 1 ball in box B_n , we can remove this ball from box B_n and add a ball to box B_{n-1} .

Show that no matter how the balls are initially distributed, it is possible to use these operations to obtain the situation where there is exactly one ball in each box.

2. Let ABC be an isosceles triangle with $BC = CA$, and let D be a point on segment AB such that $AD < DB$. Let P and Q be points on segments BC and CA respectively such that $\angle DPB = \angle DQA = 90^\circ$. Let the perpendicular bisector of PQ meet line segment CQ at E , and let the circumcircles of ABC and CPQ meet again at point F , different to C .

Suppose that P , E , and F are collinear. Prove that $\angle ACB = 90^\circ$.

3. For any odd prime p and any integer n , let $d_p(n) \in \{0, 1, 2, \dots, p-1\}$ denote the remainder when n is divided by p . We say that (a_0, a_1, a_2, \dots) is a p -sequence if a_0 is a positive integer coprime to p and $a_{n+1} = a_n + d_p(a_n)$ for $n \geq 0$.
- (a) Do there exist infinitely many primes p for which there exist p -sequences (a_0, a_1, \dots) and (b_0, b_1, \dots) such that $a_n > b_n$ for infinitely many n and $a_n < b_n$ for infinitely many n ?
 - (b) Do there exist infinitely many primes p for which there exist p -sequences (a_0, a_1, \dots) and (b_0, b_1, \dots) such that $a_0 < b_0$ but $a_n > b_n$ for all $n \geq 1$?

- Submit your solutions at <https://forms.gle/uhMSLew7qTQ9Qbqr6>.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

