Test 2

April Camp 2021

Time: $4\frac{1}{2}$ hours

1. An acute-angled triangle ABC is given. Points D, E, and F lie on the sides BC, CA, and AB respectively and satisfy $\angle FDE = \angle BAC$ and $\angle DEF = \angle ABC$. Prove that the orthocentre of triangle DEF coincides with the circumcentre of triangle ABC.

2. For each prime number p, there is a kingdom of p-Landia consisting of p islands numbered $1, 2, \ldots, p$. Two distinct islands numbered m and n are connected by a bridge if and only if p divides $(m^2 - n + 1)(n^2 - m + 1)$. The bridges may pass over each other, but cannot intersect. Prove that for infinitely many p there are two islands in p-Landia which are not connected by a chain of bridges.

3. A magician intends to perform the following trick, where n is a positive integer. She announces $2n$ real numbers $x_1 < x_2 < \cdots < x_{2n}$ to the audience. A member of the audience then secretly chooses a polynomial $P(x)$ of degree n with real coefficients, computes the $2n$ values $P(x_1), \ldots, P(x_{2n})$, and writes down these $2n$ values on the blackboard in non-decreasing order. After that the magician announces the secret polynomial to the audience.

   Can the magician find a strategy to perform such a trick?

- Submit your solutions at `https://forms.gle/uhMSLew7qTQ9Qbqr6`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.