Test 3

April Camp 2021

Time: $4\frac{1}{2}$ hours

1. Let p be an odd prime. How many natural numbers n are there such that $0 \leq n < p$ and $n^2 + 1$ is congruent to a perfect square modulo p?

2. Let $\mathcal{A}$ denote the set of all polynomials with three variables x, y, and z with integer coefficients. Let $\mathcal{B}$ denote the subset of $\mathcal{A}$ formed by all polynomials which can be expressed as

$$(x+y+z)P(x,y,z) + (xy+yz+zx)Q(x,y,z) + xyzR(x,y,z)$$

with $P, Q, R \in \mathcal{A}$. Find the smallest nonnegative integer n such that $x^i y^j z^k \in \mathcal{B}$ for all nonnegative integers i, j, and k satisfying $i + j + k \geq n$.

3. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are defined inductively by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Given an integer $m \geq 2$, determine the smallest size of a set S of integers such that for every $k = 2, 3, \ldots, m$ there exist some $x, y \in S$ such that $x - y = F_k$.

- Submit your solutions at `https://forms.gle/uhMSLew7qTQ9Qbqr6`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.