Advanced February Monthly Assignment Solutions

1. *Let c and d be positive divisors of a natural number n such that $c > d$. Prove that*

$$c > d + \frac{d^2}{n}.$$

Let $n = kc = md$. The inequality that we want to prove is then equivalent to

$$\frac{n}{k} > \frac{n}{m} + \frac{\frac{n^2}{m^2}}{n} \iff \frac{1}{k} > \frac{1}{m} + \frac{1}{m^2} \iff m^2 > mk + k.$$

Since $c > d$, we have that $k < m$, and so $k \leq m - 1$, and so we have that

$$mk + k = k(m+1) \leq (m-1)(m+1) = m^2 - 1 < m^2$$

which proves the desired result.

2. *Suppose $a, b, c > 0$ and $\sqrt{a-b} + \sqrt{a-c} > \sqrt{b+c}$. Prove that $a > \frac{3}{4}(b+c)$.*

Suppose that $a \leq \frac{3}{4}(b+c)$. We will show that $\sqrt{a-b} + \sqrt{a-c} \leq \sqrt{b+c}$. Since $a \leq \frac{3}{4}(b+c)$, we have that

$$\sqrt{a-b} + \sqrt{a-c} \leq \sqrt{\frac{3}{4}(b+c) - b} + \sqrt{\frac{3}{4}(b+c) - c} = \frac{1}{2}\left(\sqrt{3c-b} + \sqrt{3b-c}\right).$$

We thus wish to show that

$$\frac{1}{2}\left(\sqrt{3c-b} + \sqrt{3b-c}\right) \leq \sqrt{b+c}.$$

By squaring both sides, this is equivalent to

$$(3c-b) + (3b-c) + 2\sqrt{(3c-b)(3b-c)} \leq 4(b+c).$$

We move the terms not under the square-root to the right hand side, divide by 2, and square again to obtain

$$(3c-b)(3b-c) \leq (b+c)^2.$$

Expanding each side of this inequality leads us to want to prove that

$$10bc - 3b^2 - 3c^2 \leq b^2 + 2bc + c^2$$

which is equivalent to

$$4(b-c)^2 \geq 0$$

and so we are done.

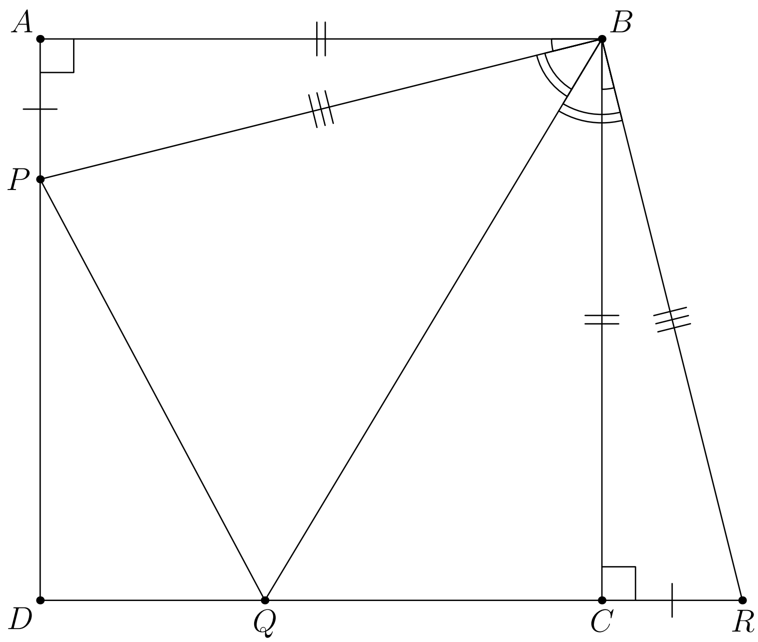

Figure 1: Problem 3

3. *Let $ABCD$ be a square. Points P and Q lie on the segments AD and DC such that $\angle PBQ = 45°$. Prove that BP bisects $\angle APQ$ and BQ bisects $\angle CQP$.*

   Let R be a point on DC extended such that $CR = AP$. Then in triangles BAP and BCR, we have that $AP = CR$, $BA = BC$, and $\angle BAP = \angle BCR = 90°$. It follows that $\triangle BAP \cong \triangle BCR$, and so in particular we have that $BP = BR$ and $\angle ABP = \angle CBR$.

   We note this this implies that $\angle QBR = \angle QBC + \angle CBR = \angle QBC + \angle ABP = 90° - \angle PBQ = 45° = \angle QBP$.

   In triangles QBP and QBR, we thus have that QB is common, $BP = BR$, and $\angle QBP = \angle QBR$. Thus $\triangle QBP \cong \triangle QBR$, and so $\angle RQB = \angle PQB$, as we wished to show. A similar argument shows that $\angle QPB = \angle APB$.

4. *Let n be a positive integer. Show that there is a positive integer m such that $\varphi(m) = n!$, where φ denotes the Euler phi function.*

   Call a natural number n *sufficiently factorial-like* if whenever $p_1, p_2, \ldots, p_k$ are distinct prime factors of n, we have that $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1) \mid n$. Note that $n!$ is always sufficiently factorial-like.

   We will prove the following result, of which the current problem is a special case:

   Lemma. *Suppose that n is sufficiently factorial-like. Then there is a natural number m such that $\varphi(m) = n$, and such that if p is any prime that divides m, then p divides n.*

   Proof. We proceed by strong induction. We note that 1 is sufficiently factorial-like, and that $\phi(1) = 1$.

   Suppose that n is sufficiently factorial-like, and that for all sufficiently factorial-like natural numbers $a < n$, there is a natural number b such that $\varphi(b) = a$ where all of the prime divisors of b are also prime divisors of a. We will construct a natural number m such that $\varphi(m) = n$ and such that all of the prime divisors of m are also prime divisors of n.

Let p be the largest prime number that divides n, and let p^k be the largest power of p that divides n. Let $a = \dfrac{n}{p^k(p-1)}$. Clearly $a < n$. We claim that a is sufficiently factorial-like.

Suppose that $p_1, p_2, \ldots, p_k$ be distinct prime factors of a. Then $p_1, p_2, \ldots, p_k, p$ are distinct prime factors of n, and so since n is sufficiently factorial-like, we have that

$$(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)(p - 1) \mid n = p^k(p-1)a.$$

Since p was the largest prime factor of n, we have that $p \geq p_i > p_i - 1$ for each i, and so p does not divide $p_i - 1$. It follows that

$$\gcd((p_1 - 1)(p_2 - 1) \cdots (p_k - 1), p^k) = 1,$$

and so we have that $(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)$ divides a by Euclid's Lemma. We see that a is indeed sufficiently factorial-like. By the induction hypothesis, there is a natural number b such that $\varphi(b) = a$, and such that all of the prime factors of b are prime factors of a. This implies that p is not a prime factor of b, and so the numbers p^{k+1} and b are relatively prime. Let $m = p^{k+1}b$. Then

$$\varphi(m) = \varphi\left(p^{k+1}b\right) = \varphi\left(p^{k+1}\right)\varphi(b) = p^k(p-1)a = n.$$

Since all of the prime factors of m are also factors of n (they are either p, which is a factor of n, or a prime factor of b, thus a factor of a, thus a factor of n) we see that the desired result holds true for n as well. By the principle of strong mathematical induction, we have that the lemma is true for all sufficiently factorial-like natural numbers n. □

5. *Define the function $f(x) = x^2 + \sin(x)$ (where x is in radians in this context). Furthermore, let $\{a_n\}$ be a sequence with $a_n \in \mathbb{R}^+$ for all $n \in \mathbb{N}$. Let $a_1 = 1$, and $f(a_n) = a_{n-1}$ for $n \geq 2$. Prove that there exists $n \in \mathbb{N}$ such that*

$$\sum_{k=1}^{n} a_i > 2021.$$

We will show by induction on n that $a_n \geq \dfrac{1}{n}$ for all natural numbers n. This is true for $n = 1$. Suppose that $a_n \geq \dfrac{1}{n}$ for some n. Then we have that

$$a_{n+1}^2 + a_{n+1} \geq a_{n+1}^2 + \sin(a_{n+1}) = f(a_{n+1}) = a_n \geq \frac{1}{n},$$

using the well-known fact that $\sin(x) \leq x$ for all positive real numbers x. To show that this implies that $a_{n+1} \geq \dfrac{1}{n+1}$, we note that the function $x \mapsto x^2 + x$ is increasing over the positive real numbers, and so it is sufficient to show that

$$\left(\frac{1}{n+1}\right)^2 + \frac{1}{n+1} \leq \frac{1}{n}.$$

But this is equivalent to

$$n + n(n+1) \leq (n+1)^2$$

which in turn simplifies to $0 \leq 1$.

We recall that the sum

$$\sum_{k=1}^{n} \frac{1}{k}$$

can be made arbitrarily large by taking n large enough. It follows that the same is true of the sum

$$\sum_{k=1}^{n} a_k.$$

6. *Consider a triangle ABC with points M and N on BC and AB respectively such that $AM \perp BC$ and $CN \perp AB$, and let AC and MN intersect at Y. It is given X is a point inside acute-angled triangle ABC such that $MBNX$ is a parallelogram. Prove that the angle bisectors of $\angle MXN$ and $\angle MYC$ are perpendicular.*

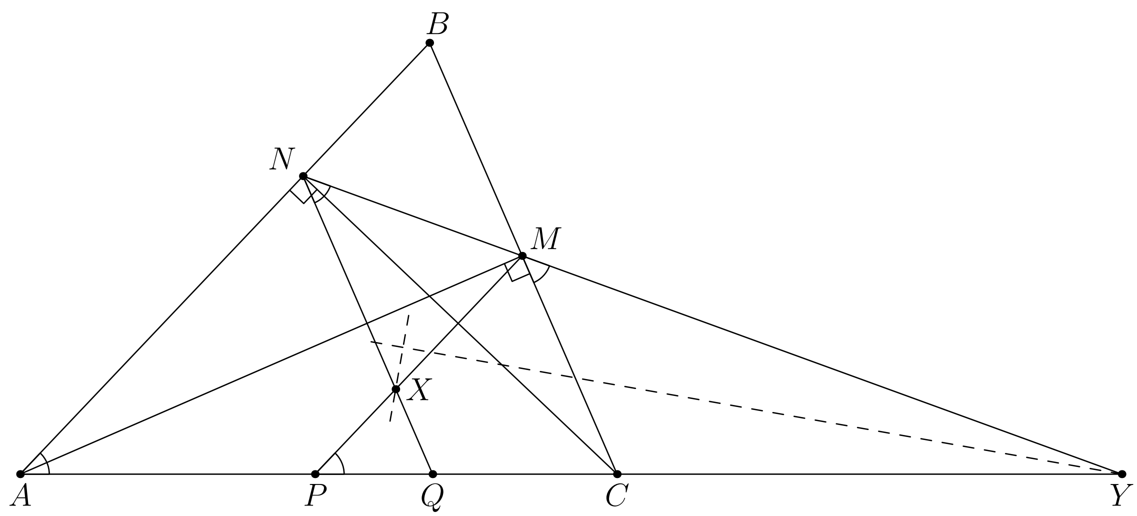

Figure 2: Problem 6

Let P and Q be the points of intersection of the lines MX and NX with the line AC respectively. Consider the arrangement of points in Figure 2. We will show that $\angle QPM = \angle QNM$.

Since $\angle ANC = \angle AMC = 90°$, we have that the points A, N, M, and C are concyclic, implying that $\angle CAN = \angle CMY$. Since $MX \parallel BA$, we have that $\angle QPM = \angle CAN$, and since $NX \parallel BC$, we have that $\angle QNM = \angle CMY$. Hence $\angle QPM = \angle QNM$, as claimed.

Let Z be the intersection of the bisectors of angles $\angle MXN$ and $\angle MYC$. Let T be intersection of PM and YZ (extended), and let S be the intersection of NM and XZ (extended). Let x, α, β, and φ be the angled marked as such in Figure 3.

Then the angles of the triangle XZT are as follows: $\angle TXZ = \beta$, $\angle XZT = 180° - x$, and $\angle XTZ = \varphi - \alpha$ by the exterior angle theorem in triangle TMY. These angles sum to $180°$, and so we have that $\beta + 180° - x + \varphi - \alpha = 180°$, or equivalently that $x = \beta + \varphi - \alpha$.

Finally, we prove that $\beta + \varphi - \alpha = 90°$. The angles in triangle MXN sum to $180°$, and so we have that $\angle QNM = \angle XNM = 180° - 2\beta - \varphi$. We also know that $\angle QPM = \angle YPM = \varphi - 2\alpha$ by the exterior angle theorem in triangle PMY. We proved earlier that $\angle QPM = \angle QNM$. This implies that $180° - 2\beta - \varphi = \varphi - 2\alpha$, and so $\beta + \varphi - \alpha = 90°$, as desired.

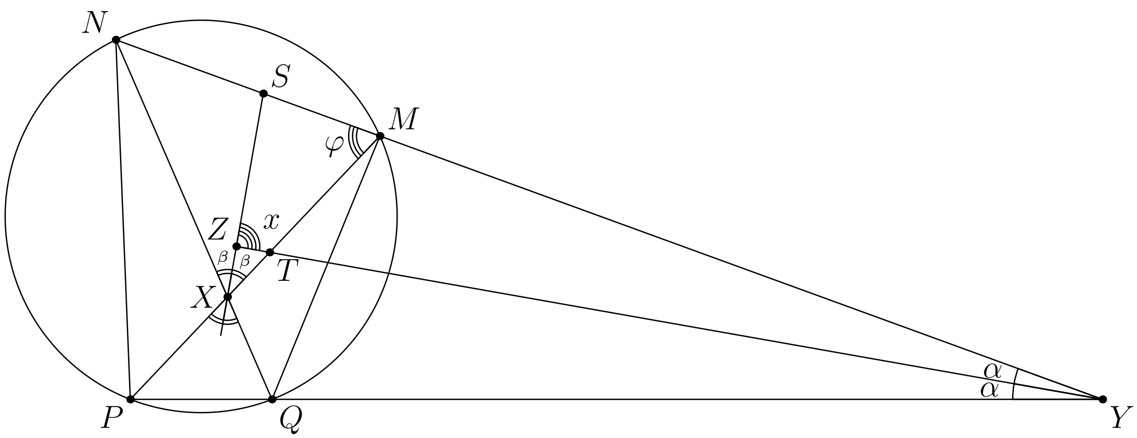

Figure 3: Problem 6