Advanced Test 4

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. Consider a $3 \times 3 \times 3$ 3-dimensional chess cube with some hyperrooks. Hyperrooks can move along any direction parallel to an edge of the cube (like a normal rook, but also up and down). What is the maximum number of hyperrooks you can place in the chess cube without any of them attacking each other?

2. Find all positive integers a, b and c satisfying

$$a + b - c = 14$$

$$a^2 + b^2 - c^2 = 14.$$

3. You are given nine real numbers, a_1, a_2, $\cdots$, a_9 with an average of m. What is the minimum possible number of triples (i, j, k) with $1 \leq i < j < k \leq 9$ and $a_i + a_j + a_k \geq 3m$?

4. Let ABC be an acute-angled triangle with orthocentre H. Let the point H' be the reflection of H over AB. Let N be the intersection of HH' and AB. The circumcircle of $\triangle ANH'$ intersects AC again at M. The circumcircle of $\triangle BNH'$ intersects BC again at P. Show that the points M, N and P are collinear.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = \max\{f(x + y), f(x)f(y)\}$$

for all x, $y \in \mathbb{R}$.

- Submit your solutions at `https://forms.gle/M1L9KgbwzDxCKEjD9`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.