Advanced April Monthly Assignment Solutions

1. *For $x, y, z > 0$, prove that*

$$\frac{x^3}{x+y} + \frac{y^3}{y+z} + \frac{z^3}{z+x} \geq \frac{xy+yz+zx}{2}.$$

We first show that $\dfrac{x^2}{x+y} \geq \dfrac{3x-y}{4}$.

$$\frac{x^2}{x+y} - \frac{3x-y}{4} = \frac{4x^2 - 3x^2 + xy - 3xy + y^2}{4(x+y)} = \frac{(x-y)^2}{4(x+y)} \geq 0.$$

We now have that $\dfrac{x^3}{x+y} \geq \dfrac{3x^2-xy}{4}$. Similarly, $\dfrac{y^3}{y+z} \geq \dfrac{3y^2-yz}{4}$ and $\dfrac{z^3}{z+x} \geq \dfrac{3z^2-zx}{4}$. Adding these three inequalities, we get

$$\frac{x^3}{x+y} + \frac{y^3}{y+z} + \frac{z^3}{z+x} \geq \frac{3x^2 - xy + 3y^2 - yz + 3z^2 - zx}{4}.$$

It is easy to prove that

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

for all positive reals x, y and z. Thus,

$$\frac{3(x^2+y^2+z^2) - xy - yz - zx}{4} \geq \frac{2(xy+yz+zx)}{4} = \frac{xy+yz+zx}{2}$$

and we are done.

2. *Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, $x \neq 0, 1$ we have*

$$f(x) + f\left(\frac{1}{1-x}\right) = x.$$

By replacing successively x with x, $\frac{1}{1-x}$ and $\frac{x-1}{x}$, we obtain the equations

$$f(x) + f\left(\frac{1}{1-x}\right) = x \tag{1}$$

$$f\left(\frac{1}{1-x}\right) + f\left(\frac{x-1}{x}\right) = \frac{1}{x-1} \tag{2}$$

$$f\left(\frac{x-1}{x}\right) + f(x) = \frac{x-1}{x} \tag{3}$$

By taking half of (1)+(3)−(2) we get

$$f(x) = \frac{1}{2}\left(x + \frac{x-1}{x} - \frac{1}{1-x}\right).$$

To check that this solution satisfies the original equation, note that

$$f\left(\frac{1}{1-x}\right)=\frac{1}{2}\left(\frac{1}{1-x}+x-\frac{x-1}{x}\right) \quad \text{and} \quad f\left(\frac{x-1}{x}\right)=\frac{1}{2}\left(\frac{x-1}{x}+\frac{1}{1-x}-x\right),$$

and the check is easy.

3. *Given a convex quadrilateral $ABCD$, $OA=\frac{OB.OD}{OC+OD}$, where O is the intersection point of the diagonals of $ABCD$. The circumcircle of $\triangle ABC$, intersects the line BD in point Q. Prove that CQ bisects $\angle DCA$.*

Let CQ_1, $Q_1 \in BD$ be the angle bisector of $\angle DCA$. From the Angle-Bisector Theorem, we have that $\frac{DQ_1}{Q_1O}=\frac{DC}{CO}$.
Using this, and the fact that $OA=\frac{OB.OD}{OC+DC}$, we have that $OA(OC+DC)=OB.OD \iff OA.OC(1+\frac{DQ_1}{Q_1O})=OB.OD \iff OA.OC(\frac{Q_1O+DQ_1}{Q1O})=OB.OD \iff OA.OC.\frac{DO}{Q_1O}=OB.OD \iff OA.CO=Q_1O.OB$, which proves that quadrilateral $ABCQ_1$ is cyclic. Thus Q_1 is Q.

4. *Find natural numbers x,y,z such that*

$$7^x+13^y=2^z.$$

From the given expression we have that $2^z\equiv(-1)^y \pmod 7$, which is true only if $2\mid y$ and $3\mid z$. Let $z=3k$. Since $7^x\equiv 2^z \pmod{13}$ we have, from Fermat's Little Theorem, that $7^{4x}\equiv 2^{12k}\equiv 1 \pmod{13}$, which holds only if $12\mid 4x$, i.e. $3\mid x$. Let $x=3j$. Now the LHS of our original expression becomes a difference of cubes and we have

$$(2^k-7^j)(2^{2k}+2^k7^j+7^{2j})=13^y.$$

It is straight forward to show that if $13\mid 2^k-7^j$, then $13\not\mid 2^{2k}+2^k7^j+7^{2j}$. Thus, since $2^{2k}+2^k7^j+7^{2j}>1$, we must have that $2^k-7^j=1$. If $k\geq 4$, we must have $7^j\equiv -1 \pmod{16}$, which is impossible. After checking all $k<4$, the only case yielding a solution is $k=3, j=1$, giving $x=3, y=2, z=9$.

5. *Triangle ABC has an area of 7. M and N are points on the sides AB and AC respectively, such that $AN=BM$. Let O be the intersection point of BN and CM. The area of triangle BOC is 2.*

   (a) Prove that $MB:AB=1:3$ and $MB:AB=2:3$.

   (b) Find the size of $\angle AOB$.

(a) Let $\frac{MB}{AB}=x$. We then have $|ABN|=7x=|BMC|$. Thus, $|BOM|=7x-2$ and $|AMON|=|BOC|=2$. We now have $|CON|=7-2-2-(7x-2)=5-7x$,
$|ANO|=\frac{x}{1-x}.|CNO|=\frac{x(5-7x)}{1-x}$,
$|AMO|=\frac{1-x}{x}.|BOM|=\frac{(1-x)(7x-2)}{x}$.
From the fact that $|AMON|=|ANO|+|AMO|$ we have that $2=\frac{x(5-7x)}{1-x}+\frac{(1-x)(7x-2)}{x}$, which simplifies to $9x^2-9x+2=0$ which has roots $x_1=\frac{1}{3}$ and $x_2=\frac{2}{3}$.

(b) $\Delta ABN\cong\Delta BMC$ and so $\angle BOM=\angle BCM+\angle CBO=\angle MBO+\angle CBO=60°$.
Since $\angle MAN+\angle MON=180°$, $AMON$ is a cyclic quadrilateral. Let $\frac{MB}{AB}=\frac{1}{3}$, i.e. $AM=2BM=2AN$. Let Q be the midpoint of the line segment AM. Triangle AQN is then isosceles,

and $\angle NAQ = 60^\circ$, thus it is also equilateral. Q is then the centre of the circle $ANOM$ and $\angle AOM = \angle ANM = 90^\circ$.
Thus, $\angle AOB = 150^\circ$. Similarly, when $\frac{MB}{AB} = \frac{2}{3}$, i.e. $2AM = MB = AN$, we have $\angle AMN = \angle AON = 90^\circ$, i.e. $\angle AOB = 90^\circ$.

6. *n points are given in the plane ($n > 4$), such that no three of them are collinear. The points are used as vertices to form at least n triangles. Show that there exist two triangles which have exactly one vertex in common.*

Assume for contradiction that for some n ($n > 4$), no two triangles have exactly one vertex in common and let k be the smallest n such that this is true. Since from k points, we form $k+1$ triangles, by the Pigeon-hole principle, there exists a point, say A, which is a common vertex for at least 4 triangles. Let the first triangle be ABC. The second triangle has, besides point A, either point B or point C as a vertex. So let the second triangle be ABD. If the third triangle is of the form ACX, then $X = D$ since otherwise, ACX and ABD have exactly one vertex in common. We thus have the three triangles ABC, ABD and ACD. Then the fourth triangle with A as a vertex, needs to also have either B or C as a vertex too. It is easy to show that in both cases we end up with two triangles sharing exactly one common vertex, which is a contradiction. Thus, the third triangle needs to have both A and B as vertices, and so does the fourth.
Extending this idea, let A be a vertex in t triangles, $t \geq 4$. Then these triangles are of the form $ABA_1, ABA_2, ABA_3, \ldots, ABA_t$, where points $A_1, A_2, \ldots, A_t$ are all distinct. It is easy to verify that we cannot have a triangle of the form BXY, where X and Y are points different from $A_1, A_2, \ldots A_t$, nor a triangle of the form BA_iA_j; nor of the form $A_iA_jA_m$ - in each of these cases we end up with a pair of triangles sharing exactly one common vertex, which is a contradiction. Hence, the points $A, B, A_1, A_2, \ldots, A_t$ are vertices only of triangles $ABA_1, ABA_2, ABA_3, \ldots, ABA_t$. In this way we used $t+2$ points to form t triangles. We cannot have $t+2 = k$, since the number of triangles are $t < k$. Thus we need $k_0 = k - t - 2$ additional points and at least $k + 1 - t$ additional triangles, no two of which have exactly one vertex in common, to satisfy the conditions of the question. Since the number of triangles is more than the number of points, $k + 1 - t > k_0$, we have that $k_0 > 4$. Thus, we have found $k_0 < k$ satisfying the conditions of the question, which contradicts with the choice of k.