Advanced Test 1 Solutions

January Camp 2021

1. *Find all possible real numbers k such that the values of x satisfying*

$$k(2-k)x^2 - (k+4)x + 6 = 0$$

are positive integers.

Firstly, notice that if $k(2-k) = 0$, then $x = \frac{6}{k+4}$. The only positive integer solution of this is when $k = 2$: $x = 1$. Assuming $k(2-k) \neq 0$, we can use the quadratic formula. We see that the solutions of the equation in x are

$$\begin{aligned} x &= \frac{(k+4) \pm \sqrt{(k+4)^2 - 24k(2-k)}}{2k(2-k)} \\ &= \frac{(k+4) \pm \sqrt{25k^2 - 40k + 16}}{2k(2-k)} \\ &= \frac{(k+4) \pm \sqrt{(5k-4)^2}}{2k(2-k)} \\ &= \frac{(k+4) \pm (5k-4)}{2k(2-k)} \end{aligned}$$

The two solutions are thus $x_1 = \frac{6k}{2k(2-k)} = \frac{3}{2-k}$ and $x_2 = \frac{8-4k}{2k(2-k)} = \frac{2}{k}$. Now we must have that $\frac{2}{k}$ and $\frac{3}{2-k}$ are integers simultaneously. $\frac{2}{k} \in \mathbb{Z} \iff k = \frac{2}{n}$ where $n \in \mathbb{Z}$. Hence we must have the following is an integer

$$\frac{3}{2-k} = \frac{3}{2-\frac{2}{n}} = \frac{3n}{2n-2} = \frac{3n}{2(n-1)}$$

Having $n-1|3$ yields $n \in \{-2, 0, 2, 4\}$. If $n-1 \nmid 3$, we must have $2(n-1)|n$. If $n > 2$, then $2(n-1) > n$. If $n < 0$, then $2(n-1) < n < 0$. Thus we only need to check $n \in \{0, 1, 2\}$. The values of k that we get are $k \in \{-1, 1, \frac{1}{2}, 2\}$. It can then be checked that the only k values that provide positive integers solutions are

$$k \in \{\frac{1}{2}, 1, 2\}$$

2. *Let O be the circumcentre of $\triangle ABC$. Let X, Y and Z be the reflections of O over BC, CA and AB respectively. Prove that $\triangle XYZ$ is congruent to $\triangle ABC$ and the corresponding sides are parallel.*

Let X' be the intersection of OX and BC and Z' be the intersection of OZ and AB. Since X is the reflection of O across BC we have $OX' = X'X$ as well as $OX \perp BC$ and since X' is on OX we also have $OX' \perp BC$. BC is a chord on the circumcircle with centre O and OX' is a perpendicular on BC, thus $BX' = X'C$. By similar arguments we have $OZ' = Z'Z$ and $AZ' = Z'B$.

By midpoint theorem on $\triangle ABC$, we have $X'Z'||CA$ and $2XX' = AC$. By midpoint theorem on $\triangle OZX$, we have $X'Z'||XZ$ and $2X'Z' = XZ$. Therefore, $XZ||X'Z'||CA$ and $XZ = 2X'Z' = AC$. Applying similar arguments to prove $XY = AB$, and $YZ = BC$ gives $\triangle ABC \equiv \triangle XYZ$. The corresponding sides are parallel part follows with similar arguments as well.

3. *Find the smallest non-negative integer which can not be written in the from*

$$\frac{2^a - 2^b}{2^c - 2^d}$$

for some positive integers a, b, c, and d.

We have the following representations for the numbers from 0 to 10:

$$0 = \frac{2^1 - 2^1}{2^2 - 2^1} \qquad 1 = \frac{2^2 - 2^1}{2^2 - 2^1} \qquad 2 = 2 \times 1 = \frac{2^3 - 2^2}{2^2 - 2^1}$$

$$3 = \frac{2^3 - 2^1}{2^2 - 2^1} \qquad 4 = 2 \times 2 = \frac{2^4 - 2^3}{2^2 - 2^1} \qquad 5 = \frac{2^5 - 2^1}{2^3 - 2^1}$$

$$6 = 2 \times 3 = \frac{2^4 - 2^2}{2^2 - 2^1} \qquad 7 = \frac{2^4 - 2^1}{2^2 - 2^1} \qquad 8 = 2 \times 4 = \frac{2^5 - 2^4}{2^2 - 2^1}$$

$$9 = \frac{2^7 - 2^1}{2^4 - 2^1} \qquad 10 = 2 \times 5 = \frac{2^6 - 2^2}{2^3 - 2^1}$$

We claim that there is no representation for the number 11. Suppose that

$$2^a - 2^b = 11(2^c - 2^d).$$

We may suppose without loss of generality that $a > b$ and $c > d$. Then the largest power of 2 dividing the left hand side of the equation is 2^b, and the largest power of 2 dividing the right hand side is 2^d. Thus we must have that $b = d$. Letting $x = a - b$ and $y = c - d$, we obtain

$$2^x - 1 = 11(2^y - 1).$$

Consider this equation modulo 5. We obtain that

$$2^x \equiv 2^y \pmod 5 \implies 2^{x-y} \equiv 1 \pmod 5.$$

By trial and error or otherwise, we know that the smallest power of 2 that leaves a remainder of 1 when divided by 5 is 2^4, and so we have that $x - y \geq 4$. But then

$$2^x - 1 \geq 2^{y+4} - 1 > 16(2^y - 1) > 11(2^y - 1)$$

and so it is not possible for $2^x - 1$ to be equal to $11(2^y - 1)$.

4. *Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$f(x + y) = f(x) + f(y) + \frac{1}{2021}$$

Where $\mathbb{R}^+$ is the set of positive real numbers.

Assume that such a function f exists. Consider the function $g : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g(x) = f(x) + \frac{1}{2021}$$

Rewriting the function equation, in terms of g, we get.

$$g(x + y) = g(x) + g(y)$$

Clearly, $g(nx) = ng(x)$ for $n \in \mathbb{Z}$. Letting $x = \frac{1}{n}$, we get $g(1) = ng(\frac{1}{n}) \iff g(\frac{1}{n}) = \frac{g(1)}{n}$. Now no matter what value $g(1)$ takes, we can find a large enough n such that $g(\frac{1}{n}) = \frac{g(1)}{n} < \frac{1}{2021}$. Thus, we must have $f(\frac{1}{n}) + \frac{1}{2021} < \frac{1}{2021} \implies f(\frac{1}{n}) < 0$. Since the codomain is the positive real numbers, we cannot have f returning a negative value. This is thus a contradiction.

Therefore, there are no functions f satisfying the functional equation.

5. *Jon has a collection of weights with different positive integer values. Is it possible that there are exactly* 2020 *ways to choose some of these distinct weights such that their total weight is* 2020*?*

   It is possible. There are several constructions, two of which is given below. Consider a coin purse with weights of values 2, 4, 8, 2014, 2016, 2018, 2020 and every odd number between 503 and 1517. Call such a coin big if its value is between 503 and 1517. Call a coin small if its value is 2, 4 or 8 and huge if its value is 2014, 2016, 2018 or 2020. Suppose some subset of these weights contains no huge weights and sums to 2020. If it contains at least four big weights, then its value must be at least $503 + 505 + 507 + 509 > 2020$. Furthermore since all of the small weights are even in value, if the subset contains exactly one or three big weights, then its value must be odd. Thus the subset must contain exactly two big weights. The eight possible subsets of the small weights have values 0,2, 4, 6, 8, 10, 12, 14. Therefore the ways to make the value 2020 using no huge weights correspond to the pairs of big weights with sums 2006, 2008, 2010, 2012, $2014, 2016$, 2018and 2020. The numbers of such pairs are 250, 251, 251, 252, 252, 253, 253, 254, respectively. Thus there are exactly 2016 subsets of this coin purse with value 2020 using no huge weights. There are exactly four ways to make a value of 2020 using huge weights; these are $\{2020\}$, $\{2, 2018\}$, $\{4, 2016\}$ and $\{2, 4, 2014\}$. Thus there are exactly 2020 ways to make the value 2020. Alternate construction: Take the weights $1, 2, ..., 11, 1954, 1955, ..., 2019$. The only way to get 2020 is a non-empty subset of $1, ..., 11$ and a single large coin. There are 2047 non-empty such subsets of sums between 1 and 66. Thus they each correspond to a unique large coin making 2020, so we have 2047 ways. Thus we only need to remove some large weights, so that we remove exactly 27 small sums. This can be done, for example, by removing weights $2020-n$ for $n = 1, 5, 6, 7, 8, 9$, as these correspond to $1+3+4+5+6+8 = 27$ partitions into distinct numbers that are at most 11.

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