Advanced Test 3

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. There is a book with n chapters where chapter i has i pages. The probability of opening the book in the same chapter twice in a row is p. Is it possible for p to be $1/k$ for some integer k?

2. Points D, E, and F lie respectively on sides BC, CA, and AB of triangle ABC such that $BDEF$ is a parallelogram. Prove that the area of $BDEF$ is maximal when D, E, and F are the midpoints of the sides.

3. Find all polynomials P with real coefficients such that $(x+1)P(x-1)-(x-1)P(x)$ is constant.

4. Does there exist an infinite set A of natural numbers such that any finite sum of distinct elements of A is not a perfect power, where a perfect power is a number of the form a^b with $b > 1$ and $a \in \mathbb{N}$.

5. Let ABC be an acute non-isosceles triangle with altitudes BB_1 and CC_1 intersecting at H. The angle bisectors of $\angle B_1AC_1$ and B_1HC_1 intersect the line B_1C_1 at points L_1 and L_2, respectively. Let P and Q be the second points of intersection of the circumcircles of triangles AHL_1 and AHL_2 with the line B_1C_1 respectively. Prove that the points B, C, P, and Q lie on a circle.

- Submit your solutions at `https://forms.gle/M1L9KgbwzDxCKEjD9`.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.