Intermediate Test 4 Solutions

January Camp 2021

Time: $2\frac{1}{2}$ hours

1. *Find all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that for all* $x, y \in \mathbb{R}$ *we have that*

$$xf(y) = yf(x).$$

Let $y = 1$

$$xf(1) = f(x)$$

If we let $m = f(1)$, we see that the solution is just $f(x) = mx$ for $m \in \mathbb{R}$.

Check for $f(x) = mx$:

$$LHS = xf(y) = xmy = mxy$$

$$RHS = yf(x) = ymx = mxy$$

$LHS = RHS$, so the function checks.

2. *Find all positive integers* m *such that* $2^{m^2} - 4$ *is divisible by* 7.

We want to find m such that $2^{m^2} \equiv_7 4$. The possible remainders when 2^k is divided by 7 where k is a natural number will be as follows:

$$\begin{aligned} 2^1 &= 2 \equiv_7 2 \\ 2^2 &= 4 \equiv_7 4 \\ 2^3 &= 8 \equiv_7 1 \\ 2^4 &= 16 \equiv_7 2 \\ 2^5 &= 32 \equiv_7 4 \\ 2^6 &= 64 \equiv_7 1 \end{aligned}$$

This pattern will continue and we will see that 2^k mod 7 will be one of 1, 2, or 4. We also note that $2^k \equiv_7 4$ when $k \equiv_3 2$. If $k = m^2$ we have that $m^2 \equiv_3 2$. The possible remainders for when m^2 is divided by 3 will as follows:

$$\begin{aligned} 1^2 &= 1 \equiv_3 1 \\ 2^2 &= 4 \equiv_3 1 \\ 3^2 &= 9 \equiv_3 0 \\ 4^2 &= 16 \equiv_3 1 \\ 5^2 &= 25 \equiv_3 1 \\ 6^2 &= 36 \equiv_3 0 \end{aligned}$$

This pattern will continue and we will see that m^2 mod 3 will be 0 or 1. This means that a perfect square cannot leave a remainder of 2 when divided by 3, and thus there are no solutions for m.

3. *Consider a triangle ABC with circumcentre O. The angle bisector of $\angle BAC$ meets the opposite side BC at D, and the altitude from B onto AD intersects line AO at E. Show that A, B, D, and E are concyclic.*

   Let the angle bisector of A intersect the perpendicular bisector of BC in point S, which is on the circumcircle of the triangle. Then $\angle EAD = \angle OAS = \angle OSA = \angle OSD = \angle EBD$. Hence $ABDE$ is cyclic.

4. *Consider a $3 \times 3 \times 3$ 3-dimensional chess cube with some hyperrooks. Hyperrooks can move along any direction parallel to an edge of the cube (like a normal rook, but also up and down). What is the maximum number of hyperrooks you can place in the chess cube without any of them attacking each other?*

   First, we prove that 9 hyperrooks is the maximum amount of hyperrooks you can place on a chess cube. Consider the 9 columns (in the up-down direction) of the chess cube. If a column contains 2 hyperrooks, the hyperrooks would attack each other. Therefore, each column has at most 1 hyperrook. Since there are 9 columns, there are at most 9 hyperrooks. We show that placing 9 hyperrooks is possible by construction.

   Construction for 9 rooks:

Top layer

R		
	R	
		R

Middle layer

	R	
		R
R		

Bottom layer

		R
R		
	R	

5. *Find all positive integers a, b and c satisfying*

$$a + b - c = 14$$
$$a^2 + b^2 - c^2 = 14.$$

   Rearranging the first equation, we have $c = a + b - 14$. We can substitute this into the second equation to get

$$\begin{aligned} a^2 + b^2 - c^2 &= 14 \\ \implies a^2 + b^2 - (a+b-14)^2 &= 14 \\ \implies -2ab + 28a + 28b - 196 &= 14 \\ \implies ab - 14a - 14b + 98 &= -7 \\ \implies (a-14)(b-14) - 98 &= -7 \\ \implies (a-14)(b-14) &= 91 = 7 \cdot 13 \end{aligned}$$

   If a and b are integers, then so are $a - 14$ and $b - 14$. So we have that $a - 14$ and $b - 14$ must simply be the paired factors of 91. We can not check every case

$$\begin{aligned} (a-14, b-14) &= (91, 1) \implies (a, b) = (105, 15) \\ &= (13, 7) \implies (a, b) = (27, 21) \\ &= (-13, -7) \implies (a, b) = (1, 7) \\ &= (-91, -1) \implies (a, b) = (-77, 13) \end{aligned}$$

   Notice that swapping a and b will produce more solutions which are just the same as the ones already generated, but also swapped. If we know a and b, we can find c. Hence, using the condition that a, b, $c \in \mathbb{N}$, the only solutions are

$$\begin{aligned} (a, b, c) &= (105, 15, 106) \\ &= (27, 21, 34) \end{aligned}$$

   where a and b may also be swapped in each triple.

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