Intermediate February Monthly Assignment Solutions

1. *The natural number n can be replaced by ab if $a+b=n$, where a and b are natural numbers. Can the number 2021 be obtained from 22 after a sequence of such replacements?*

   Suppose that the natural number n can be obtained. Then since $(n-1)+1=n$, we have that $(n-1)\times 1 = n-1$ can also be obtained. It follows that if we can obtain any number larger than 2021 after a sequence of such moves, then we can also obtain 2021 after a sequence of such moves.

   Now note that $22 = 11+11$, so we can replace 22 with $11\times 11 = 121$. Then $121 = 100+21$, so we can replace 121 with $100\times 21 = 2100$. This is larger than 2021, so we repeatedly replace the current number that we have with its predecessor until we obtain 2021.

2. *Prove that among the first 30000 positive integers there are at least 22000 composite numbers.*

   Among the first 30000 positive integers, there are $\frac{30000}{2} = 15000$ multiples of 2, 10000 multiples of 3, and 6000 multiples of 5. There are then at most $15000+10000+6000 = 31000$ multiples of 2, 3, or 5. But this counts each multiples of 6, 10, and 15 twice, so we subtract the $5000+3000+2000 = 10000$ multiples of 6, 10, or 15 to arrive at at least $31000-10000 = 21000$ multiples of 2, 3, or 5. But now we have added each multiple of 30 twice, but subtracted them 3 times, so we add $\frac{30000}{30} = 1000$ to account for the multiples of 30.

   We thus see that among the first 30000 positive integers, there are 22000 that are a multiple of 2, 3, or 5. These are all composite except for 2, 3, or 5 themselves, so we need only find 3 more composite numbers below 30000 that are not divisible by any of these. The numbers 49, 77, and 121 will do.

3. *Let a and b be positive real numbers such that $2a^2+2b^2 = 5ab$. If $|x|$ denotes the absolute value of x, calculate*

   $$\left|\frac{a+b}{a-b}\right|.$$

   We rewrite the given equation as

   $$(a+b)^2+(a-b)^2 = \frac{5}{4}\left((a+b)^2-(a-b)^2\right)$$

   we can be rearranged to become

   $$\frac{1}{4}(a+b)^2 = \frac{9}{4}(a-b)^2$$

   or

   $$\left(\frac{a+b}{a-b}\right)^2 = 9.$$

   By taking square-roots, it follows that

   $$\left|\frac{a+b}{a-b}\right| = 3.$$

4. *Triangle ABC is a right angled triangle with $\angle C = 90°$. P is placed randomly inside $\triangle ABC$. What is the probability that the area of $\triangle PBC$ is less than half of the area of $\triangle ABC$?*

   Let the foot of the perpendicular from P onto BC be D. Let M and N be the midpoints of AC and BC respectively. We note that the area of PBC is given by $\frac{1}{2}BC\times PD$, and that the area of triangle ABC is given by $\frac{1}{2}BC\times AC$.

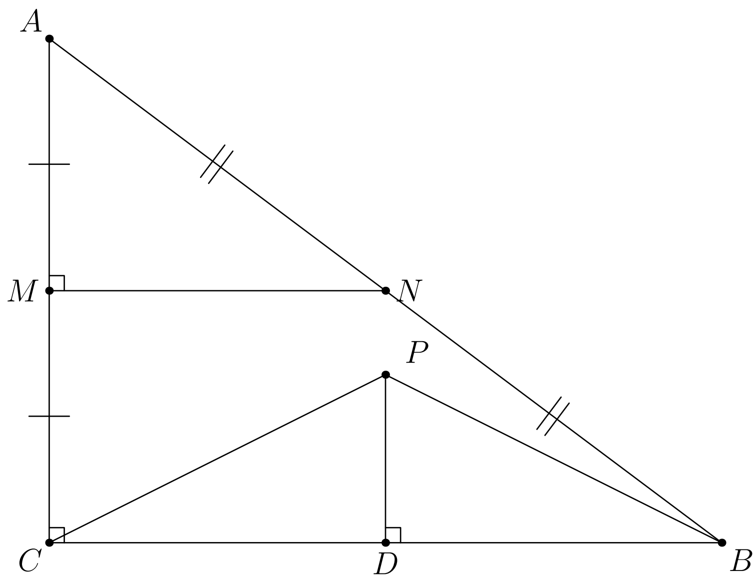

Figure 1: Problem 3

It follows that the area of triangle PBC is less than half of that of ABC if and only if $PD < \frac{1}{2}AC$. We see that this is equivalent to P not lying in the triangle AMN. Now the area of triangle AMN is given by

$$\frac{1}{2}MN \times AM = \frac{1}{2}\left(\frac{1}{2}BC\right) \times \left(\frac{1}{2}AC\right) = \frac{1}{4}|ABC|$$

where $|ABC|$ denotes the area of triangle ABC. It follows that the probability that a randomly chosen point chosen inside triangle ABC lies in AMN is

$$\frac{|AMN|}{|ABC|} = \frac{1}{4},$$

and so the probability that P *does not* lie in triangle AMN is $1 - \frac{1}{4} = \frac{3}{4}$.

5. *Let c and d be positive divisors of a natural number n such that $c > d$. Prove that*

$$c > d + \frac{d^2}{n}.$$

Let $n = kc = md$. The inequality that we want to prove is then equivalent to

$$\frac{n}{k} > \frac{n}{m} + \frac{\frac{n^2}{m^2}}{n} \iff \frac{1}{k} > \frac{1}{m} + \frac{1}{m^2} \iff m^2 > mk + k.$$

Since $c > d$, we have that $k < m$, and so $k \leq m - 1$, and so we have that

$$mk + k = k(m+1) \leq (m-1)(m+1) = m^2 - 1 < m^2$$

which proves the desired result.

6. *Suppose $a, b, c > 0$ and $\sqrt{a-b} + \sqrt{a-c} > \sqrt{b+c}$. Prove that $a > \dfrac{3}{4}(b+c)$.*

Suppose that $a \leq \frac{3}{4}(b+c)$. We will show that $\sqrt{a-b}+\sqrt{a-c} \leq \sqrt{b+c}$. Since $a \leq \frac{3}{4}(b+c)$, we have that

$$\sqrt{a-b}+\sqrt{a-c} \leq \sqrt{\frac{3}{4}(b+c)-b}+\sqrt{\frac{3}{4}(b+c)-c} = \frac{1}{2}\left(\sqrt{3c-b}+\sqrt{3b-c}\right).$$

We thus wish to show that

$$\frac{1}{2}\left(\sqrt{3c-b}+\sqrt{3b-c}\right) \leq \sqrt{b+c}.$$

By squaring both sides, this is equivalent to

$$(3c-b)+(3b-c)+2\sqrt{(3c-b)(3b-c)} \leq 4(b+c).$$

We move the terms not under the square-root to the right hand side, divide by 2, and square again to obtain

$$(3c-b)(3b-c) \leq (b+c)^2.$$

Expanding each side of this inequality leads us to want to prove that

$$10bc-3b^2-3c^2 \leq b^2+2bc+c^2$$

which is equivalent to

$$4(b-c)^2 \geq 0$$

and so we are done.