

Intermediate Test 5

January Camp 2021

Time: 4 hours

1. In the acute-angled triangle ABC , the foot of the perpendicular from B to CA is E . Let l be the tangent to the circumcircle of $\triangle ABC$ at B . The foot of the perpendicular from C to l is F . Prove that EF is parallel to AB .
2. Let x and y be distinct real numbers such that

$$x + 4 = (y - 2)^2 \quad \text{and} \quad y + 4 = (x - 2)^2.$$

Find $x^2 + y^2$.

3. There is a group of 6 people that are very suspicious of each other. It is very important to these people which of the rest of the group consider them to be friends. Now, because of the nature of this group, friendship is not a two way street! Person A may consider himself friends with Person B while Person B does not consider the same about Person A. A true friendship is a pair of people who both consider each other to be friends. If we are told that each of these 6 people considers exactly 3 others to be their friend, what is the minimum and maximum number of true friendships that can exist among these 6 people?
4. Consider relatively prime integers a and b and a prime number p . Show that

$$\gcd(ab, a^2 + pb^2) = \begin{cases} 1 & \text{if } p \nmid a \\ p & \text{if } p \mid a \end{cases}.$$

5. A square-based pyramid has all of its edges the same length. A cube is placed inside so that one of its faces lies on the base of the pyramid, and the opposite face has an edge along each side of the pyramid (think – the natural way to put a cube in a pyramid). Show that the sum of any of the cube's edge lengths with any of the cube's face diagonal lengths is the same as the edge length of the pyramid.
6. There is a soccer competition with five teams where each team plays exactly one match against each other team. If one team wins against another team, that team gets 5 points whereas the losing team gets 0 points. If two teams draw, they each get 1 point if neither team scored a goal and they each get 2 points if they scored at least one goal. At the end of the competition it is found that the total points for the teams form five consecutive nonnegative integers. What is the minimum number of goals scored in this competition?

- Submit your solutions at <https://forms.gle/M1L9KgbwzDxCKEjD9>.
- Submit each question in a single separate PDF file (with multiple pages if necessary).
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.

