The Mathematics of Voting Systems

Arrow's Theorem

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Outline

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Participate in our Mock Election

A survey is available at `https://www.surveymonkey.com/r/62WLLV9` where you can rank some "integers" in order of preference.

Later in the talk, a top secret method will be used to combine everyone's rankings into one overall ranking.

Outline

Clickbait

Some of you may have noticed that the poster advertising this talk is a bit clickbaity.

Betteridge's Law of Headlines

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Any headline that ends in a question mark can be answered by the word "no".

- Betteridge's Law of Headlines may apply!

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Any headline that ends in a question mark can be answered by the word "no".

- Betteridge's Law of Headlines may apply!
- There is a Twitter account dedicated to this premise that has unfortunately been inactive since 2015: @YourTitleSucks.

@YourTitleSucks

@YourTitleSucks

@YourTitleSucks

The Answer Is No @YourTitleSucks · Oct 17, 2015

Replying to @Telegraph

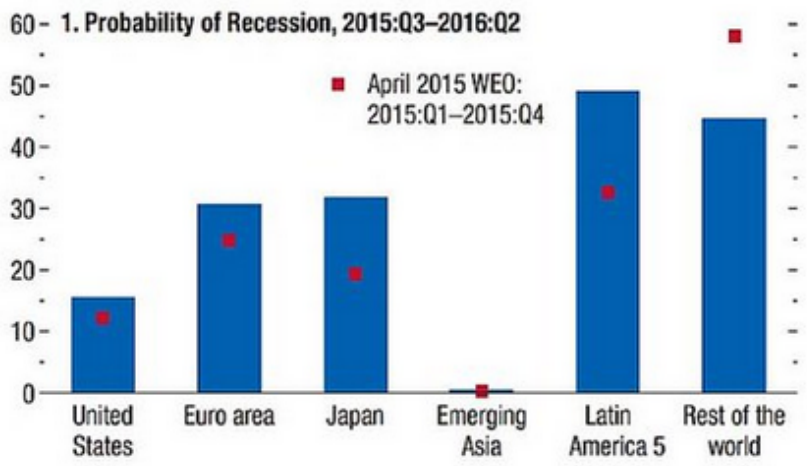
No. RT @telegraph: Does bad grammar stand in the way of true love?
telegraph.co.uk/men/relationsh...

@YourTitleSucks

The Answer Is No @YourTitleSucks · Oct 10, 2015

Replying to @Telegraph

No. RT @telegraph: Is the world heading for its third global financial crisis? telegraph.co.uk/finance/econom...



Outline

Disclaimer

Last time I gave this talk, it was suggested that I not use actual political party names so as not to make the talk too political, but I didn't listen. I'll try to do better this time.

Disclaimer

Political parties referenced in this talk are either the product of the author's imagination or used in a fictitious manner. Any resemblance to actual political parties, local or abroad, or actual events, ideological positions, or policies is purely coincidental .

Outline

Ranked List Voting System

- The systems that we will be considering require the voters to rank all of the candidates in order of preference rather than picking a single candidate to vote for.
- The goal of the system will be to combine these individual rankings into one overall ranking of all of the candidates.

Outline

Determinism

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- If we have two different elections, and every voter gives the same ranking of the candidates in each election, then the output for the two elections should be the same.
- We can't, for example, pick the winner based on what the weather is like.

Universality

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Remark

The properties of Determinism and Universality are really just saying that our aggregation system should be a (surjective) function from the space of possible elections to the space of possible rankings of the candidates.

Unanimity

- If every voter prefers candidate A to candidate B, the the voting system should rank candidate A above candidate B overall.

Independence of Irrelevant Alternatives (IIA)

- Suppose that we have two elections. Suppose that every voter who preferred the Vulture Party to the Hyena Party in the first election still preferred the Vulture Party to the Hyena Party in the second election, and vice versa.

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- Then if the voting system ranks the Vulture Party above the Hyena Party in the first election, it should still do so in the second election. Similarly, if the voting system ranks the Vulture Party below the Hyena Party in the first election, then it should still do so in the second election.

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- Then if the voting system ranks the Vulture Party above the Hyena Party in the first election, it should still do so in the second election. Similarly, if the voting system ranks the Vulture Party below the Hyena Party in the first election, then it should still do so in the second election.
- The relative rankings of two candidates should only depend on how each voter ranked those two candidates against to each other.

Outline

Two Candidates

- Suppose that we only have two candidates, the Leech Party, and the Vulture Party.
- We count how many voters rank the Leeches above the Vultures, and vice-versa.
- If more voters prefer the Leeches to the Vultures than vice-versa, we rank the Leeches above the Vultures.
- Otherwise we rank the Vultures above the Leeches.

Two Candidates

Unanimity If every voter prefers the Leeches to the Vultures, then certainly more voters prefer the Leeches to the Vultures than vice-versa, and so we rank the Leeches above the Vultures.

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IIA Suppose we have two elections. If the relative rankings of the Leeches and the Vultures doesn't change between these two elections, then the candidate that is preferred more often also doesn't change, and so the result doesn't change.

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- Every time the candidate is ranked n^{th}, we increase their score by n.
- We then rank the candidates from the lowest score to the highest score.
- To make the system deterministic/universal, we must choose a deterministic method to break ties. e.g. Alphabetically.

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Unanimity

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If every voter prefers the Hyena Party to the Vulture Party, then $H_i < V_i$ for all i.

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If every voter prefers the Hyena Party to the Vulture Party, then $H_i < V_i$ for all i.
Thus

$$\text{Score for Hyena Party} = \sum H_i < \sum V_i = \text{Score for Vulture Party}$$

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Thus

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The system thus ranks the Hyena Party above the Vulture Party.

Positional Voting

IIA

Consider the following elections between the candidates H, V, L, and A:

Voter	Election 1	Election 2
1	H V A L	H A V L
2	L V H A	L H A V

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In the first election, V has 4 points and so is ranked above L which has 5 points. (Regardless of how the tie is broken between H and V)

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In the second election, V has 7 points and so is ranked below L which has 5 points.

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In the second election, V has 7 points and so is ranked below L which has 5 points.
But every voter who preferred H to V in the first election still did in the second election and vice-versa.

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In the second election, V has 7 points and so is ranked below L which has 5 points.
But every voter who preferred H to V in the first election still did in the second election and vice-versa.
Thus *Positional Voting* does not satisfy IIA!

Positional Voting

Copeland's Method

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- For each candidate, we count how many of these virtual elections they win and how many they lose.
- The candidate's score is then the number of victories minus the number of defeats.
- Copeland's Method is guaranteed to find a *Condorcet Winner* if one exists.

Condorcet Winner

Definition

A *Condorcet Winner* is a candidate that would beat any of the other candidates if we were to hold a head-to-head election between the two candidates.

Copeland's Method

Unanimity

- If every voter prefers A to B, then every time B beats a candidate in a sub-election, so does A. Thus the number of times that A wins a sub-election is at least as large as the number of times that B wins.

Copeland's Method

Unanimity

- If every voter prefers A to B, then every time B beats a candidate in a sub-election, so does A. Thus the number of times that A wins a sub-election is at least as large as the number of times that B wins.
- Similarly, A has at most as many defeats as B does.
- In particular, A beats B in the sub-election between A and B, and so in fact A has strictly more victories and strictly fewer defeats than B.
- The score assigned to A is thus larger than that assigned to B, and so Copeland's Method ranks A above B.

Copeland's Method

IIA

Consider the following elections:

Voter	Election 1	Election 2
1	A B C D	A D B C
2	A C B D	A D C B
3, 4	B D A C	D B A C
5, 6	C D A B	D C A B

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In the first election, A is ranked above D, while in the second election, D is ranked above A.
But every voter who preferred A to D in the first election still did in the second election, and vice-versa.

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In the first election, A is ranked above D, while in the second election, D is ranked above A.
But every voter who preferred A to D in the first election still did in the second election, and vice-versa.
Thus *Copeland's Method* does not satisfy IIA!

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- We continue in this manner until no candidates remain.

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- Any of member of Parliament can nominate any other member of Parliament to stand for President.
- All of the members of Parliament then vote for one of the nominated candidates.
- If any of the candidates have more than 50% of the votes, then they become the President.

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- If no candidate wins the first round of voting, the candidate with the fewest number of votes is eliminated, and every member of Parliament votes again.
- Note that unlike in the version of run-off voting we looked at, the members can change their vote at this point.
- This process continues until one of the candidates has a majority.
- Thus far only one round of voting has ever been necessary because our largest political party has always had a majority in Parliament.

Run-off voting

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The voting system thus ranks A above B.

Run-off voting

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In the first election, the result is A C B, while in the second election, the result is B A C.
But every voter who preferred A to B in the first election still did in the second election, and vice-versa.
Thus *Run-off Voting* does not satisfy IIA!

Run-off Voting

Other Methods

- There are many other methods that exist for the purpose of or can be adapted for the purpose of aggregating a set of rankings into one unified ranking.
  - The Borda Count
  - The Ranked Pairs Voting Method
  - The Kerney-Young Method
  - And many, many more!

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  - The Borda Count
  - The Ranked Pairs Voting Method
  - The Kerney-Young Method
  - And many, many more!
- You might be surprised to hear that *none of them* satisfy the IIA!
- Is there any method that does?

Dictatorship!

- We designate one of the voters as our hopefully-benevolent ~~dictator~~ King or Queen.

Dictatorship!

- We designate one of the voters as our hopefully-benevolent ~~dictator~~ King or Queen.
- The overall ranking is just whatever they decide it is.

Dictatorship!

Unanimity

If every voter prefers the Hyena Party to the Leech Party, then so does the dictator.

Dictatorship!

Unanimity

If every voter prefers the Hyena Party to the Leech Party, then so does the dictator.
A dictatorship would thus rank the Hyena Party above the Leech Party.

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Suppose that we have two elections, and every voter who prefers the Hyena Party to the Leech Party in the first election still does in the second election, and vice-versa.

Dictatorship!

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Suppose that we have two elections, and every voter who prefers the Hyena Party to the Leech Party in the first election still does in the second election, and vice-versa.
If the dictator prefers the Hyena Party to the Leech Party in the first election, then the Hyena Party is ranked above the Leech Party in the first election.

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Thus the Hyena Party is still ranked above the Leech Party in the second election!

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By assumption, every voter who prefers the Hyena Party to the Leech Party in the fist election still does in the second election, and hence so does the dictator.
Thus the Hyena Party is still ranked above the Leech Party in the second election!
Conversely, if the dictator prefers the Leech Party in both elections, then the Leech Party is ranked higher in both elections.

Dictatorship!

IIA

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Thus the Hyena Party is still ranked above the Leech Party in the second election!
Conversely, if the dictator prefers the Leech Party in both elections, then the Leech Party is ranked higher in both elections.
Thus *a Dictatorship* does satisfy IIA!

Outline

Arrow's Theorem

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Suppose that we have an election system for an election that is between 3 or more candidates. If the system satisfies Determinism, Universality, Unanimity, and Independence of Irrelevant Alternatives, then it must be a dictatorship.

Proof of Arrow's Theorem

I do not have slides prepared with the proof of Arrow's Theorem. If there is enough time, and enough people are interested, then I will switch to using another application to provide the sketch.

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- We usually have an opportunity to correct our mistakes.
- An election between "Yes" and "No" only has two candidates :P
- Is IIA even such a desirable property anyway?
- In the real world, people's preferences for parties are correlated.

Outline

Conclusion

Thank you for listening!