April Camp 2017 Test 1 – Solutions

1. *Determine all positive integers k and n satisfying the equation*

$$k^2 - 2016 = 3^n.$$

2. *Let ABC be an acute angled triangle. Let H be the foot of the altitude from C onto AB. Suppose that $|AH| = 3|BH|$. Let M and N be the midpoints of the segments AB and AC respectively. Let P be a point such that $|NP| = |NC|$ and $|CP| = |CB|$ and such that B and P lie on opposite sides of the line AC. Show that $\angle APM = \angle PBA$.*

3. *Consider a 4×4 grid of unit squares. How many ways are there to write a 0 or 1 in each 1×1 square so that the product of the two numbers written on every neighbouring pair of squares (sharing a common edge) is always 0?*

4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f(xy-1) + f(x)f(y) = 2xy - 1$$

for all $x, y \in \mathbb{R}$.

We call the above equation Rosa. Note that if f is a constant function, then the left-hand side of Rosa is constant as x and y vary, while the right-hand side is obviously not. Hence f is nonconstant. Putting $y = 0$ in Rosa, we see that $f(-1) + f(0)f(x) = -1$ for all $x \in \mathbb{R}$. Since f is nonconstant, we must have that $f(0) = 0$, and consequently $f(-1) = -1$. Also, putting $x = y = 1$ in Rosa we get that $f(1)^2 = 1$, so $f(1) = 1$ or $f(1) = -1$.

Now putting $y = -1$ in Rosa we get that $-2x - 1 = f(-x-1) + f(x)(f-1) = f(-x-1) - f(x)$, putting $y = 1$ in Rosa we get that $f(x-1) + f(1)f(x) = 2x - 1$, and putting $y = x$ in Rosa we get that $f(x^2-1) + f(x)^2 = 2x^2 - 1$.

$$\begin{aligned}\text{Now } f(z)f(y) &= 2zy - 1 - f(zy-1) = 2(-zy)(-1) - 1 - f((-zy)(-1) - 1)\\ &= f(-zy)f(-1) = -f(-zy) \quad \text{for all } z, y \in \mathbb{R}.\end{aligned}$$

Putting $z = x - 1$ and $y = -x - 1$ in this equation, we get that

$$\begin{aligned} &f(x-1)f(-x-1) = -f(x^2-1)\\ \iff\ &[2x - 1 - f(1)f(x)][f(x) - 2x - 1] = -[2x^2 - 1 - f(x)^2] = f(x)^2 - 2x^2 + 1.\end{aligned}$$

Now if $f(1) = 1$, this equation becomes

$$\begin{aligned} f(x)^2 - 2x^2 + 1 &= (1 + 2x - f(x))(1 - 2x + f(x)) \\ &= 1 - (2x - f(x))^2 = 1 - 4x^2 + 4xf(x) - f(x)^2 \\ \iff 0 &= 2x^2 - 4xf(x) - 2f(x)^2 = -2(f(x) - x)^2 \\ f(x) &= x \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

On the other hand, if $f(1) = -1$, the equation becomes

$$\begin{aligned} f(x)^2 - 2x^2 + 1 &= (f(x) - 1 - 2x)(f(x) - 1 + 2x) \\ &= (f(x) - 1)^2 - (2x)^2 = f(x)^2 - 2f(x) + 1 - 4x^2 \\ \iff 2f(x) &= -2x^2 \\ \iff f(x) &= -x^2 \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

So we see that $f(x) = x$ and $f(x) - x^2$ are the only possible solutions. Now it remains to check that both of these are valid solutions by plugging them into Rosa. For $f(x) = x$, $f(xy - 1) + f(x)f(y) = xy - 1 - xy = 2xy - 1$, so the solution checks. For $f(x) = -x^2$, $f(xy - 1) + f(x)f(y) = -(xy - 1)^2 + (-x^2)(-y^2) = -x^2y^2 + 2xy - 1 + x^2y^2 = 2xy - 1$, so this solution also checks. Hence all the possible solutions to this functional equation are $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x^2$ for all $x \in \mathbb{R}$.

5. *Find all infinite sequences $a_1, a_2, a_3 \ldots$ of positive integers such that*

   (a) $a_{mn} = a_m a_n$ for all positive integers m and n, and

   (b) there are infinitely many positive integers n such that

$$\{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}.$$