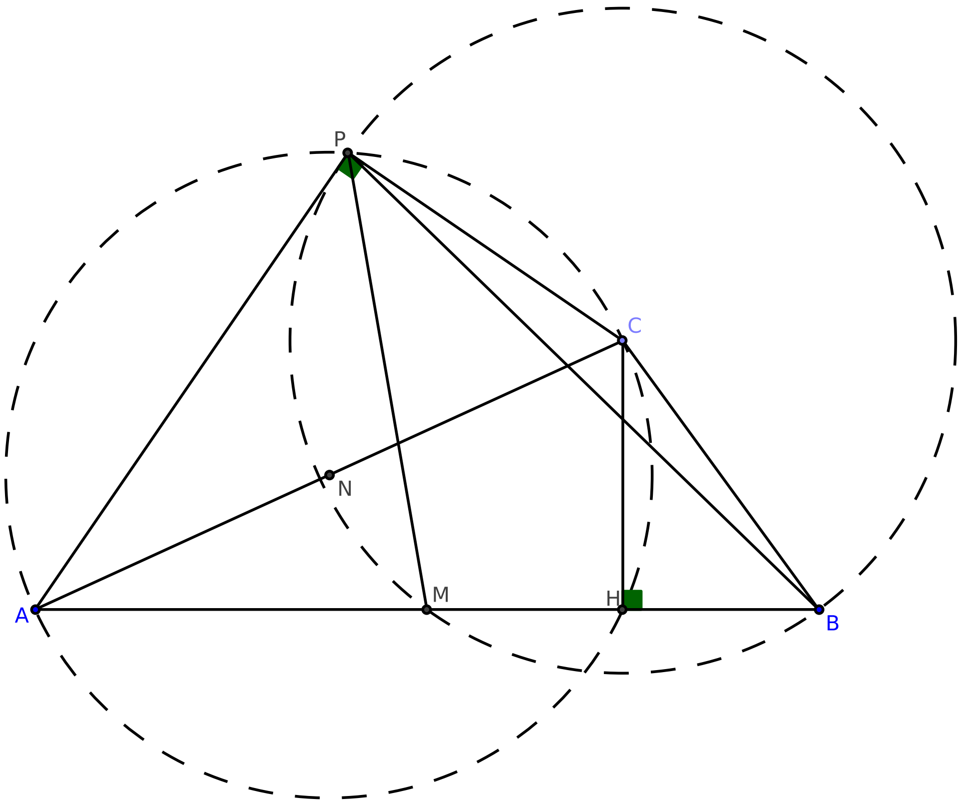
April Camp 2017 Test 1 – Solutions

1. *Determine all positive integers k and n satisfying the equation*

$$k^2 - 2016 = 3^n.$$

Note that if $n \geq 3$, then modulo 27 the equation becomes $k^2 \equiv 18 \pmod{27}$, which is a contradiction since 18 is not a quadratic residue modulo 27. Thus the only possible solutions correspond to $n = 1$ and $n = 2$. For $n = 1$, the equation becomes $k^2 = 2019$, and for $n = 2$, the equation is $k^2 = 2025 = 45^2$. We see that the only solution in positive integers is given by $n = 2$ and $k = 45$.

2. *Let ABC be an acute angled triangle. Let H be the foot of the altitude from C onto AB. Suppose that $|AH| = 3|BH|$. Let M and N be the midpoints of the segments AB and AC respectively. Let P be a point such that $|NP| = |NC|$ and $|CP| = |CB|$ and such that B and P lie on opposite sides of the line AC. Show that $\angle APM = \angle PBA$.*



Note that since $HB = \frac{1}{4}AB = \frac{1}{2}MB$ and $CH \perp MB$, CH is the perpendicular bisector of MB and so $CM = CB = CP$. Thus P, B and M lie on a circle centred at C, which we call Γ. Also, since N is the midpoint of AC and $NC = NP$, P lies on the with diameter AC, centred at N. So $AP \perp PC$, and so AP is tangent to Γ. Thus by the tan-chord theorem, $\angle APM = \angle PBM = \angle PBA$.

3. *Consider a 4×4 grid of unit squares. How many ways are there to write a 0 or 1 in each 1×1 square so that the product of the two numbers written on every neighbouring pair of squares (sharing a common edge) is always 0?*

Suppose that we have a collection of tiles, and in each tile is written either a 0 or a 1. We will call this configuration of tiles *valid* if it satisifies the condition imposed in the problem: whenever two tiles share a border, the product of the numbers written in these two tiles is 0. We wish to count the number of valid 4×4 grids of tiles. We first prove two lemmas about the number of valid assignments of numbers to tiles in certain configurations.

Lemma 1. *Let a_n be the number of valid assignments of numbers to n tiles placed in a line. (i.e. a $1 \times n$ strip of tiles.) Then $a_n = F_{n+2}$, where F_n is the n^{th} Fibonacci number.*

Proof. for $n = 0$, there is 1 valid assignment. (The "empty assignment"), and for $n = 1$, there are 2 valid assignments. Thus $a_0 = F_2$ and $a_1 = F_3$. We now show that $a_n = a_{n-1} + a_{n-2}$, which then establishes the claim.

Consider the number placed in the final tile in the strip. If it is a 0, then we can place any numbers in the remaining tiles as long as the remaining $1 \times (n-1)$ strip is valid, and so there are a_{n-1} valid configurations in this case. If the number in the final tile is a 1, then this forces the penultimate tile to contain a 0, and there are then no other restrictions other than that the remaining $1 \times (n-2)$ strip of tiles is valid. There are thus a_{n-2} ways to validly assign the numbers in this case, and so we find that $a_n = a_{n-1} + a_{n-2}$. □

Lemma 2. *Let b_n be the number of valid assignments of numbers to n tiles fixed to a wall in a ring pattern. (i.e. the border of a rectangle.) Then $b_n = a_{n-1} + a_{n-3} = F_{n+1} + F_{n-1}$ for all $n \geq 4$.*

Proof. Consider any tile in the ring. If it contains a 0, then the rest of the ring is a (bent) valid $1 \times (n-1)$ strip of tiles, and so we have that there are a_{n-1} valid configurations in this case. On the other hand, if the chosen tile contains a 1, then the two tiles which border it must both contain a 0, and the remaining $(n-3)$ tiles in the ring is now just any valid $1 \times (n-3)$ strip of tiles, and so there are a_{n-3} valid configurations in this case.

We see that $b_n = a_{n-1} + a_{n-3}$, and by Lemma 1, this is equal to $F_{n+1} + F_{n-1}$. □

We now return to the problem. We consider three cases based on the number of 0's contained in the central 2×2 subgrid of squares.

Case 1: Four 0's:

0	0
0	0

Consider the ring of 12 squares surrounding the central 2×2 subgrid. The board as a whole is valid precisely when these 12 squares are, and so by Lemma 2, we find that there are thus $F_{11} + F_{13} = 89 + 233 = 322$ valid grids in this case.

Case 2: Three 0's: $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline 1 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 0 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 0 \\ \hline\end{array}$

In this case, the squares on the border adjacent to the central square with a 1 in must contain 0's. The square on the border which shares a corner with the square with a 1 in can then be either a 1 or a 0, and in each of these cases, the remaining 9 squares on the border can be any of the $a_9 = F_{11} = 89$ possible 1×9 strips of valid tiles. In each of the 4 configurations for the central 2×2 subgrid, we thus have $2 \times 89 = 178$ possible valid grids, giving us $178 \times 4 = 712$ valid configurations in this case.

Case 3: Two 0's: $\begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$ $\begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array}$

The squares on the border adjecent to the 1's must be filled with 0's. The squares on the border sharing a corner with the 1's can be filled in any any way. For each of the 4 possible ways of filling these squares, the remaining border squares form two independent 1×3 strips of tiles, which can each be filled in $a_3 = F_5 = 5$ ways, and so we see that the number of valid configurations in this case is $2 \times 4 \times 5^2 = 200$.

Combining the above three cases, we find that the total number of valid 4×4 grids is given by $322 + 712 + 200 = 1234$.

4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

$$f(xy - 1) + f(x)f(y) = 2xy - 1$$

for all $x, y \in \mathbb{R}$.

We call the above equation Rosa. Note that if f is a constant function, then the left-hand side of Rosa is constant as x and y vary, while the right-hand side is obviously not. Hence f is nonconstant. Putting $y = 0$ in Rosa, we see that $f(-1) + f(0)f(x) = -1$ for all $x \in \mathbb{R}$. Since f is nonconstant, we must have that $f(0) = 0$, and consequently $f(-1) = -1$. Also, putting $x = y = 1$ in Rosa we get that $f(1)^2 = 1$, so $f(1) = 1$ or $f(1) = -1$.

Now putting $y = -1$ in Rosa we get that $-2x - 1 = f(-x - 1) + f(x)(f - 1) = f(-x - 1) - f(x)$, putting $y = 1$ in Rosa we get that $f(x - 1) + f(1)f(x) = 2x - 1$, and putting $y = x$ in Rosa we get that $f(x^2 - 1) + f(x)^2 = 2x^2 - 1$.

$$\begin{aligned}\text{Now } f(z)f(y) &= 2zy - 1 - f(zy - 1) = 2(-zy)(-1) - 1 - f((-zy)(-1) - 1) \\ &= f(-zy)f(-1) = -f(-zy) \quad \text{for all } z, y \in \mathbb{R}.\end{aligned}$$

Putting $z = x - 1$ and $y = -x - 1$ in this equation, we get that

$$\begin{aligned}&f(x - 1)f(-x - 1) = -f(x^2 - 1) \\ \iff &[2x - 1 - f(1)f(x)][f(x) - 2x - 1] = -[2x^2 - 1 - f(x)^2] = f(x)^2 - 2x^2 + 1.\end{aligned}$$

Now if $f(1) = 1$, this equation becomes

$$\begin{aligned} f(x)^2 - 2x^2 + 1 &= (1 + 2x - f(x))(1 - 2x + f(x)) \\ &= 1 - (2x - f(x))^2 = 1 - 4x^2 + 4xf(x) - f(x)^2 \\ \iff 0 &= 2x^2 - 4xf(x) - 2f(x)^2 = -2(f(x) - x)^2 \\ f(x) &= x \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

On the other hand, if $f(1) = -1$, the equation becomes

$$\begin{aligned} f(x)^2 - 2x^2 + 1 &= (f(x) - 1 - 2x)(f(x) - 1 + 2x) \\ &= (f(x) - 1)^2 - (2x)^2 = f(x)^2 - 2f(x) + 1 - 4x^2 \\ \iff 2f(x) &= -2x^2 \\ \iff f(x) &= -x^2 \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

So we see that $f(x) = x$ and $f(x) - x^2$ are the only possible solutions. Now it remains to check that both of these are valid solutions by plugging them into Rosa. For $f(x) = x$, $f(xy - 1) + f(x)f(y) = xy - 1 - xy = 2xy - 1$, so the solution checks. For $f(x) = -x^2$, $f(xy - 1) + f(x)f(y) = -(xy - 1)^2 + (-x^2)(-y^2) = -x^2y^2 + 2xy - 1 + x^2y^2 = 2xy - 1$, so this solution also checks. Hence all the possible solutions to this functional equation are $f(x) = x$ for all $x \in \mathbb{R}$ and $f(x) = -x^2$ for all $x \in \mathbb{R}$.

5. *Find all infinite sequences $a_1, a_2, a_3 \ldots$ of positive integers such that*

   (a) $a_{mn} = a_m a_n$ for all positive integers m and n, and

   (b) there are infinitely many positive integers n such that

$$\{1, 2, \ldots, n\} = \{a_1, a_2, \ldots, a_n\}.$$

Note that the sequence (a_n) defined by $a_n = n$ for all positive natural numbers n satisfies the conditions posed in the problem. We will show that it is the only such sequence.

For convenience, define the function $f : \mathbb{N} \longrightarrow \mathbb{N}$ by $f(n) = a_n$. The following observations are easy to prove:

(a) $f(1) = 1$.

(b) The function f is bijective.

(c) A positive integer n is prime if and only if $f(n)$ is prime.

(d) If

$$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$$

is the unique prime factorisation of n, then

$$f(n) = f(p_1)^{a_1} f(p_2)^{a_2} \ldots f(p_k)^{a_k}$$

is the unique prime factorisation of $f(n)$.

(e) For any natural numbers m and n, $f(m) \mid f(n)$ if and only if $m \mid n$.

We now show that for every prime number p, that $f(p) = p$. The result then follows by the above observations.

Consider any prime number p. Then there is a unique prime number q such that $a_q = p$. For any natural number N, we have that there are

$$\left\lfloor \frac{N}{p} \right\rfloor$$

multiples of p among the natural numbers $\{1, 2, \ldots, N\}$.

On the other hand, the multiples of p among the numbers $\{a_1, a_2, \ldots, a_N\}$ are precisely the numbers divisible by a_q, which are precisely the numbers a_m where $1 \leq m \leq N$ and $q \mid m$. Thus there are

$$\left\lfloor \frac{N}{q} \right\rfloor$$

multiples of p among the natural numbers $\{a_1, a_2, \ldots, a_N\}$.

Since there are infinitely many natural numbers N such that

$$\{a_1, a_2, \ldots, a_N\} = \{1, 2, \ldots, N\}$$

it follows that there are infinitely many natural numbers N such that

$$\left\lfloor \frac{N}{p} \right\rfloor = \left\lfloor \frac{N}{q} \right\rfloor.$$

But this is only possible if $p = q$. Suppose, for example, that $p < q$. Then for

$$N > \frac{pq}{q - p},$$

we have that $Nq > Np + pq$, and so

$$\left\lfloor \frac{N}{p} \right\rfloor \geq \frac{N}{p} > \frac{N}{q} + 1 > \left\lfloor \frac{N}{q} \right\rfloor.$$

A similar proof shows that it is not possible to have $p > q$. We find that $f(p) = p$ for all primes p, and so $f(n) = n$ for all positive natural numbers n.