2017 Monthlies Pool

February 1, 2021

1. Let n be a positive integer. Determine the minimum number of lines that can be drawn on the plane so that they intersect in exactly n distinct points. (Singapore 2016)

2. In triangle ABC ($AB \neq AC$) the incircle with centre I is tangent to side BC at D. Let M be the midpoint of side BC. Prove that the perpendiculars from points M and D to lines AI and MI, respectively meet on the altitude in $\triangle ABC$ from A or its extension. (Serbia 2016)

3. Given a positive integer n, define $f(0, j) = f(i, 0) = 0$, $f(1, 1) = n$ and
$$f(i, j) = \left\lfloor \frac{f(i-1, j)}{2} \right\rfloor + \left\lfloor \frac{f(i, j-1)}{2} \right\rfloor .$$
for all integers $i, j \geq 0$, $(i, j) \neq (1, 1)$. How many ordered pairs of positive integers (i, j) are there for which $f(i, j)$ is an odd number? (Serbia 2016)

4. A group of tourists get on 10 buses in the outgoing trip. The same group of tourists gets on 8 buses in the return trip. Assuming each bus carries at least 1 tourist, prove that there are at least 3 tourists who have taken a bus in the return trip that has more people than the bus he has taken in the outgoing trip. (Singapore 2016)

5. Let $n \geq 3$ be an integer. Prove that it is impossible to find a set $\{a_1, \dots, a_n\}$ of n integers such that
$$\{a_i + a_j \mid 1 \leq i \leq j \leq n\}$$
leave distinct remainders modulo $n(n+1)/2$. (Singapore 2016)