Test 3

April Camp 2017

Time: 4 hours

1. Given 5 positive real numbers, show that there exist two of them, a and b, for which
$$0 \leq \frac{a}{1+a^2} - \frac{b}{1+b^2} \leq \frac{1}{8}.$$

2. An acute-angled triangle ABC is given. Points D, E, and F lie on the sides BC, CA, and AB respectively, and satisfy $\angle FDE = \angle BAC$. and $\angle DEF = \angle ABC$. Prove that the orthocenter of triangle DEF coincides with the circumcenter of triangle ABC.

3. Prove that for every prime p, there exists an integer x such that
$$x^8 \equiv 16 \pmod{p}.$$

4. Let $n > 3$ be an integer. John and Mary play the following game: First John labels the sides of a regular n-gon with the numbers $1, 2, \ldots, n$ in whatever order he wants, using each number exactly once. Then Mary divides this n-gon into triangles by drawing $n-3$ diagonals which do not intersect each other inside the n-gon. All these diagonals are labeled with number 1. Into each of the triangles the product of the numbers on its sides is written. Let S be the sum of those $n-2$ products.

   Determine the value of S if Mary wants the number S to be as small as possible and John wants S to be as large as possible and if they both make the best possible choices.

Each problem is worth 7 points.