Test 2

April Camp 2017

Time: 4 hours

1. For an integer $N > 0$, N boys, no two of them having the same height, are arranged in a circle. A boy in the given arrangement is said to be *tall* if he is taller than both of his neighbours; a boy is said to be *short* if he is shorter than both of his neighbours. Prove that the number of tall boys in the circle is equal to the number of short boys in the circle.

2. Nonzero real numbers a, b, c, d satisfy the equations

$$a + b + c + d = 0, \qquad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{abcd} = 0.$$

   Find all possible values of the product $(ab - cd)(c + d)$.

3. Find all primes p such that $5^p + 4p^4$ is the square of an integer.

4. $ABCD$ is a cyclic quadrilateral. Let the circle Γ_1 pass through A and B and touch CD at E; let the circle Γ_2 pass through B and C and touch DA at F; let the circle Γ_3 pass through C and D and touch AB at G; and let the circle Γ_4 pass through D and A and touch BC at H. Prove that $EG \perp FH$.

5. Given a polynomial P with positive real coefficients, prove that $P(1)P(xy) \geq P(x)P(y)$ for all $x, y \geq 1$.

Each problem is worth 7 points.