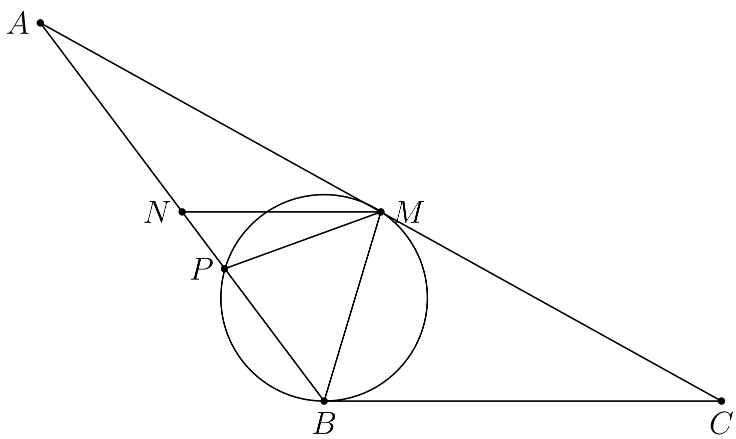
Stellenbosch Camp December 2018
Senior Test 3
Solutions

1. Construct point N, the midpoint of AB.



By midpoint Th$^{\underline{\text{m}}}$, $NM||BC$. Using this and Tan-Chord Th$^{\underline{\text{m}}}$, we get: $\angle BMN = \angle MBC = \angle MPB$. Since we also have that $\angle PBM = \angle MBN$, we have that $\triangle BMN \sim \triangle BPM$. So

$$\frac{BM}{BP} = \frac{BN}{BM}$$
$$\implies \quad BM^2 = BN \cdot BP = \left(\frac{1}{2}AB\right) \cdot BP$$

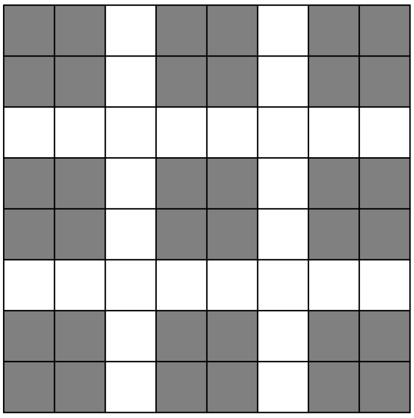
$\square$

2. By the Cauchy-Schwarz inequality (in "Engel Form") we have that

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{(x+y)^2}{(x+2y)+(2y+x)} = \frac{x+y}{3}.$$

$\square$

3. Consider the following colouring:



Note that when we do any change, we are changing an even number of lamps in the coloured squares, so the number of lamps turned on in those squares remains even. Thus, if only one lamp remains on, it must be in a white square. To see that these can be changed it is sufficient to notice that they are at distance 2 from the edge. If X is the square at distance 2 from the edge and S, T are the squares separating it from the edge, we have the following situation:

S	T	X

We can use T and change the state of S, T, X, and then use S and change the state of S, T. With this only X is turned on. When we used this pair of movements we have not affected any other lamp. As we wanted, X is the only lamp turned on in the whole board. $\square$

4. It is well known that the sum of the first n cubes is given by

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

and thus we wish to determine whether

$$\frac{(2m)^2(2m+1)^2}{4} - \frac{m^2(m+1)^2}{4}$$

can be a square.

Suppose that this quantity is a square. Then we have that

$$(4m^2 + 2m)^2 - (m^2 + m)^2 = (3m^2 + m)(5m^2 + 3m) = m^2(3m+1)(5m+3)$$

is a square. It follows that $(3m+1)(5m+3)$ must be a square. Since the greatest common divisor of $3m+1$ and $5m+3$ is a divisor of $3(5m+3) - 5(3m+1) = 4$, this implies that either $(3m+1)$ and $(5m+3)$ are both squares, or are both twice a square.

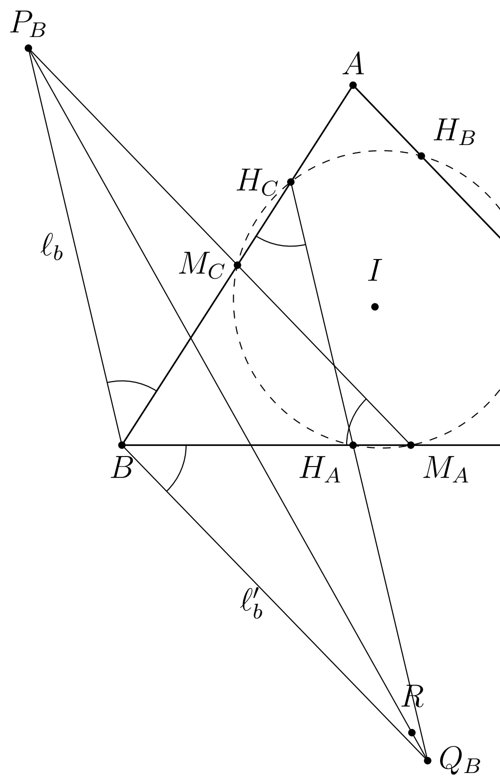
If $3m+1 = x^2$ and $5m+3 = y^2$, then we have that $4 = 3y^2 - 5x^2$. Modulo 5 this becomes $y^2 \equiv 3 \pmod 5$, which is a contradiction.

Similarly, if $3m+1 = 2x^2$ and $5m+3 = 2y^2$, then we have that $3y^2 - 5x^2 = 2$. Modulo 3 we get $x^2 \equiv 2 \pmod 3$, which is again a contradiction.

Thus $(m+1)^3 + \cdots + (2m)^3$ is never a square for positive integers m. $\square$

5. By using Midpoint Th$^{\underline{\text{m}}}$, Tan-Chord Th$^{\underline{\text{m}}}$, properties of reflection and the Nine Point Circle hence forth denoted as ω_9, we get that the following angles are equal.

$$\angle ACB = \angle ABP_B = \angle BH_CH_A = \angle M_CM_AB = \angle Q_BBC$$



So ℓ_b is tangent to the circumcircle of M_CBM_A and So ℓ_b' is tangent to the circumcircle of H_ABH_C. So, by power of a point:

$$BP_B^2 = M_AP_B \cdot M_CP_B \text{ and } BQ_B^2 = H_AQ_B \cdot H_CQ_B$$

So the power of P_B with respect to the point circle at B is equal to the power of P_B with respect to Ω_9. Similarly the power of Q_B with respect to the point circle at B is equal to the power of Q_B with respect to Ω_9. So P_BQ_B is the radical axis with respect to the point circle at B and Ω_9. Similarly we get that P_CQ_C is the radical axis with respect to the point circle at C and Ω_9. Thus R is the radical centre of the 3 circles, and so it lies on the radical axis of the 2 point circles at B and C. Since both circles have the same radius (0) we have that R is equidistant from B and C, as required.