April Camp 2019
Senior Test 2
Solutions

1. *Let $n \geq 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of the subsets of cardinality m.*

Solution: We show that one of the possible examples is the set

$$S = \{1 \cdot 3^k, 2 \cdot 3^k, k = 1, 2, \ldots, n-1\} \cup \left\{1, \frac{3^n + 9}{2} - 1\right\}$$

It is readily verified that all the numbers listed above are distinct (notice that the last two are not divisible by 3).

The sum of elements in S is

$$\Sigma = 1 + \left(\frac{3^n + 9}{2} - 1\right) + \sum_{k=1}^{n-1}(1 \cdot 3^k + 2 \cdot 3^k) = \frac{3^n + 9}{2} + \sum_{k=1}^{n-1} 3^{k+1} = \frac{3^n + 9}{2} + \frac{3^{n+1} - 9}{2} = 2 \cdot 3^n$$

Hence, in order to show that this set satisfies the problem requirements, it suffices to present, for every $m = 2, 3, \ldots, n$, an m-element subset $A_m \subset S$ whose sum of elements equals 3^n.

Such a subset is

$$A_m = \{2 \cdot 3^k : k = n - m + 1, n - m + 2, \ldots, n - 1\} \cup \{1, 3^{n-m+1}\}.$$

Clearly, $|A_m| = m$. The sum of elements in A_m is

$$3^{n-m+1} + \sum_{k=n-m+1}^{n-1} 2 \cdot 3^k = 3^{n-m+1} + \frac{2 \cdot 3^n - 2 \cdot 3^{n-m+1}}{2} = 3^n,$$

as required. □

Comment: Let us present a more general construction. Let $s_1, s_2, \ldots, s_{2n-1}$ be a sequence of pairwise distinct positive integers satisfying $s_{2i+1} = s_{2i} + s_{2i-1}$ for all $i = 2, 3, \ldots, n-1$. Set $s_{2n} = s_1 + s_2 + \cdots + s_{2n-4}$.

Assume that s_{2n} is distinct from the other terms of the sequence. Then the set $S = \{s_1, s_2, \ldots, s_{2n}\}$ satisfies the problem requirements. Indeed the sum of its elements is

$$\Sigma = \sum_{i=1}^{2n-4} s_i + (s_{2n-3} + s_{2n-2}) + s_{2n-1} + s_{2n} = s_{2n} + s_{2n-1} + s_{2n-1} + s_{2n} = 2s_{2n} + 2s_{2n-1}.$$

Therefore, we have

$$\frac{\Sigma}{2} = s_{2n} + s_{2n-1} = s_{2n} + s_{2n-2} + s_{2n-3} = s_{2n} + s_{2n-2} + s_{2n-4} + s + 2n - 5 = \ldots$$

which shows that the required set A_m can be chosen as

$$a_m = \{s_{2n}, s_{2n-2}, \ldots, s_{2n-2m+4}, s_{2n-2m-3}\}.$$

So, the only condition to be satisfied is $s_{2n} \notin \{s_1, s_2, \ldots, s_{2n-1}\}$, which can be achieved in many different ways (e.g., by choosing properly the number s_1 after specifying $s_2, s_3, \ldots, s_{2n-1}$).

The solution above is an instance of this general construction Another instance, for $n > 3$, is the set

$$\{F_1, F_2, \ldots, F_{2n-1}, F_1 + \cdots + F_{2n-4}\},$$

where $F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}$ is the usual Fibonacci sequence.

2. *A circle ω of radius 1 is given. A collection T of triangles is called* good *if the following conditions both hold:*

   (i) *each triangle from T is inscribed in ω;*

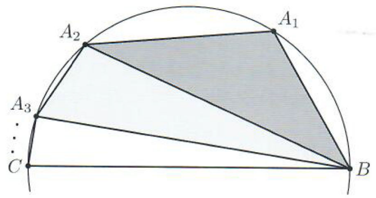
   (ii) *no two triangles from T have a common interior point.*

   Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

Answer: $t \in (0, 4]$.

Solution. First, we show how to construct a good collection of n triangles, each of perimeter greater than 4. This will show that all $t \leq 4$ satisfy the required conditions.

Construct inductively an $(n+2)$-gon $BA_1A_2 \ldots A_nC$ inscribed in ω such that BC is a diameter, and BA_1A_2, $BA_2A_3, \ldots, BA_{n-1}A_n, BA_nC$ is a good collection of n triangles. For $n = 1$, take any triangle BA_1C inscribed in ω such that BC is a diameter; its perimeter s greater than $2BC = 4$. To perform the inductive step, assume that the $(n+2)$-gon $BA_1A_2 \ldots A_nC$ is already constructed. Since $A_nB + A_nC + BC > 4$, one can choose a point A_{n+1} on the small arc CA_n, close enough to C, so that $A_nB + A_nA_{n+1} + BA_{n+1}$ is still greater than 4. Thus each of these new triangles BA_nA_{n+1} and $BA_{n+1}C$ has perimeter greater than 4, which completes the induction step.



We proceed by showing that no $t > 4$ satisfies the conditions of the problem. To this end, we assume that there exists a good collection T of n triangles, each of perimeter greater than t, and then bound n from above.

Take $\epsilon > 0$ such that $t = 4 + 2\epsilon$.

Claim: There exists a positive constant $\sigma = \sigma(\epsilon)$ such that any triangle Δ with perimeter $2s \geq 4 + 2\epsilon$; inscribed in ω; has area $S(\Delta)$ at least σ.

Proof: Let a, b, c be the side lengths of Δ. Since Δ is inscribed in ω, each side has length at most 2. Therefore, $s-a \geq (2+\epsilon)-2 = \epsilon$. Similarly, $s-b \geq \epsilon$ and $s-c \geq \epsilon$. By Heron's formula, $S(\Delta) = \sqrt{s(s-a)(s-b)(s-c)} \geq \sqrt{(2+\epsilon)\epsilon^3}$. Thus, we can set $\sigma(\epsilon) = \sqrt{(2+\epsilon)\epsilon^3}$. $\square$

Now we see that the total area S of all triangles from T is at least $n\sigma(\epsilon)$. On the other hand, S does not exceed the area of the disk bounded by ω. Thus $n\sigma(\epsilon) \leq \pi$, which means that n is bounded from above.

Comment 1. One may prove the Claim using the formula $S = \frac{abc}{4R}$ instead of Heron's formula.

Comment 2. In the statement of the problem condition (i) could be replaced by a weaker one; each triangle from T lies within ω. This does not affect the solution above, but reduces the number of ways to prove the Claim.

3. *Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying*

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right) \tag{1}$$

for all $x, y > 0$.

Answer: $f(x) = C_1 x + \frac{C_2}{x}$ with arbitrary constants C_1 and C_2.

Solution 1: Fix a real number $a > 1$, and take a new variable t. For the values $f(t), f(t^2)$, $f(at)$, and $f(a^2t^2)$, the given relation provides a system of linear equations:

$$x = y = 1: \qquad \left(t + \frac{1}{t}\right) f(t) = f(t^2) + f(1) \tag{2}$$

$$x = \frac{t}{a}, y = at: \qquad \left(\frac{t}{a} + \frac{a}{t}\right) f(at) = f(t^2) + f(a^2) \tag{3}$$

$$x = a^2t, y = t: \qquad \left(a^2t + \frac{1}{a^2t}\right) f(t) = f(a^2t^2) + f\left(\frac{1}{a^2}\right) \tag{4}$$

$$x = y = at: \qquad \left(at + \frac{1}{at}\right) f(at) = f(a^2t^2) + f(1) \tag{5}$$

In order to eliminate $f(t^2)$, take the difference of (2) and (3); from (4) and (5) eliminate $f(a^2t^2)$; then by taking a linear combination, eliminate $f(at)$ as well:

$$\left(t + \frac{1}{t}\right) f(t) - \left(\frac{t}{a} + \frac{a}{t}\right) f(at) = f(1) - f(a^2) \quad \text{and}$$
$$\left(a^2t + \frac{1}{a^2t}\right) f(t) - \left(at + \frac{1}{at}\right) f(at) = f(1/a^2) - f(1), \quad \text{so}$$

$$\left(\left(at + \frac{1}{at}\right)\left(t + \frac{1}{t}\right) - \left(\frac{t}{a} + \frac{a}{t}\right)\left(a^2t + \frac{1}{a^2t}\right)\right) f(t)$$
$$= \left(at + \frac{1}{at}\right)(f(1) - f(a^2)) - \left(\frac{t}{a} + \frac{a}{t}\right)(f(1/a^2) - f(1)).$$

Notice that on the left-hand side, the coefficient of $f(t)$ is nonzero and does not depend on t:

$$\left(at + \frac{1}{at}\right)\left(t + \frac{1}{t}\right) - \left(\frac{t}{a} + \frac{a}{t}\right)\left(a^2t + \frac{1}{a^2t}\right) = a + \frac{1}{a} - \left(a^3 + \frac{1}{a^3}\right) < 0$$

After dividing by this fixed number, we get

$$f(t) = C_1 t + \frac{C_2}{t} \tag{6}$$

where the numbers C_1 and C_2 are expressed in terms of a, $f(1)$, $f(a^2)$, and $f(1/a^2)$, and they do not depend on t.

The functions of the form (6) satisfy the equation:

$$\left(x + \frac{1}{x}\right) f(y) = \left(x + \frac{1}{x}\right)\left(C_1 y + \frac{C_2}{y}\right) = \left(C_1 xy + \frac{C_2}{xy}\right) + \left(C_1 \frac{y}{x} + C_2 \frac{x}{y}\right) = f(xy) + f\left(\frac{y}{x}\right)$$

Solution 2: We start with an observation. If we substitute $x = a \neq 1$ and $y = a^n$ in the given, we obtain

$$f(a^{n+1}) - \left(a + \frac{1}{a}\right) f(a^n) + f(a^{n-1}) = 0$$

For the sequence $z_n = a^n$, this is a homogeneous linear recurrence of the second order, and its characteristic polynomial is $t^2 - (a + \frac{1}{a})t + 1 = (t-a)(t - \frac{1}{a})$ with two distinct nonzero roots, namely a and $1/a$. As is well-known, the general solution is $z_n = C_1 a^n + C_2 (1/a)^n$ where the index n can be as well positive as negative. Of course, the numbers C_1 and C_2 may depend of the choice of a, so in fact we have two functions, C_1 and C_2, such that

$$f(a^n) = C_1(a) \cdot a^n + \frac{C_2(a)}{a^n} \quad \text{for every } a \neq 1 \text{ and every integer } n \tag{7}$$

The relation (7) can be easily extended to rational values of n, so we may conjecture that C_1 and C_2 are constants, and whence $f(t) = C_1 t + \frac{C_2}{t}$. As it was seen in the previous solution, such functions indeed satisfy the given.

The equation (1) is linear in f; so if some functions f_1 and f_2 satisfy (1) and c_1, c_2 are real numbers, then $c_1f_1(x)+c_2f_2(x)$ is also a solution of (1). In order to make our formulas simpler, define $f_0(x) = f(x) - f(1)\cdot x$.

This function is another one satisfying (1) and the extra constraint $f_0(1) = 0$. Repeating the same argument on linear recurrences, we can write $f_0(a) = K(a)a^n + \frac{L(a)}{a^n}$ with some functions K and L. By substituting $n = 0$, we can see that $K(a) + L(a) = f_0(1) = 0$ for every a. Hence,

$$f_0(a^n) = K(a)\left(a^n - \frac{1}{a^n}\right)$$

Now take two numbers $a > b > 1$ arbitrarily and substitute $x = (a/b)^n$ and $y = (ab)^n$ in (1):

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) f_0((ab)^n) = f_0(a^{2n}) + f_0(b^{2n}), \quad \text{so}$$
$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) K(ab)\left((ab)^n + \frac{1}{(ab)^n}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right), \quad \text{or equivalently}$$
$$K(ab)\left(a^{2n} - \frac{1}{a^{2n}} + b^{2n} - \frac{1}{b^{2n}}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right). \tag{8}$$

By dividing (8) by a^{2n} and then taking limit with $n \to +\infty$, we get $K(ab) = K(a)$. Then (8) reduced to $K(a) = K(b)$. Hence, $K(a) = K(b)$ for all $a > b > 1$.

Fix $a > 1$. For every $x > 0$, there is some b and an integer n such that $1 < b < a$ and $x = b^n$. Then

$$f_0(x) = f_0(b^n) = K(b)\left(b^n - \frac{1}{b^n}\right) = K(a)\left(x - \frac{1}{x}\right).$$

Hence, we have $f(x) = f_0(x) + f(1)x = C_1x + \frac{C_2}{x}$ with $C_1 = K(a) + f(1)$ and $C_2 = -K(a)$.

Comment: After establishing (8), there are several variants of finishing the solution. For example, instead of taking a limit, we can obtain a system of linear equations for $K(a)$, $K(b)$, and $K(ab)$ by substituting two positive integers n in (8), say $n = 1$ and $n = 2$. This approach leads to a similar ending as in the first solution.

Optionally, we define another function $f_1(x) = f_0(x) - C(x - \frac{1}{x})$ and prescribe $K(c) = 0$ for another fixed c. Then we can choose $ab = c$ and decrease the number of terms in (8).