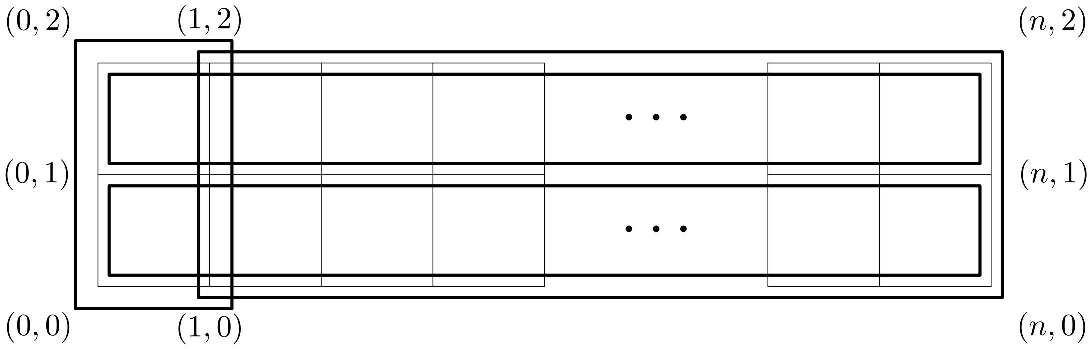
Stellenbosch Camp December 2018
Senior Test 4
Solutions

1. Suppose that it is possible. Let n be a positive integer. The sum of the numbers in the $1 \times n$ rectangle with opposite corners at the coordinates $(0, 0)$ and $(n, 1)$ is divisible by $n + 1$. The sum of the numbers in the $1 \times n$ rectangle with opposite corners $(0, 1)$ and $(n, 2)$ is also divisible by $n + 1$. It follows that the sum of the numbers in the $2 \times n$ rectangle with opposite corners at $(0, 0)$ and $(n, 2)$ is divisible by $n + 1$.

   However, the sum of the numbers in the $2 \times (n - 1)$ rectangle with opposite corners at $(1, 0)$ and $(n, 2)$ is divisible by $n + 1$, and so we see that the sum of the numbers in the rectangle with opposite corners $(0, 0)$ and $(1, 2)$ is divisible by $n + 1$.

   This must be true for every positive integer n. It follows that the sum of the numbers in the rectangle with opposite corners $(0, 0)$ and $(1, 2)$ must be 0. This is a contradiction since we assumed that each square is filled with a positive integer.



□

2. (a) Consider the total number of correct solutions submitted by all of the students. If all of the students fail, then each student submits strictly fewer that $\frac{k}{2}$ correct solutiona. There are therefore strictly fewer than $\frac{nk}{2}$ correct submissions. On the other hand, if every question is easy, then for each question there are strictly more than $\frac{n}{2}$ correct submissions, and there are thus strictly more than $\frac{nk}{2}$ correct submissions. It is thus not possible for every student to fail if all of the questions are easy.

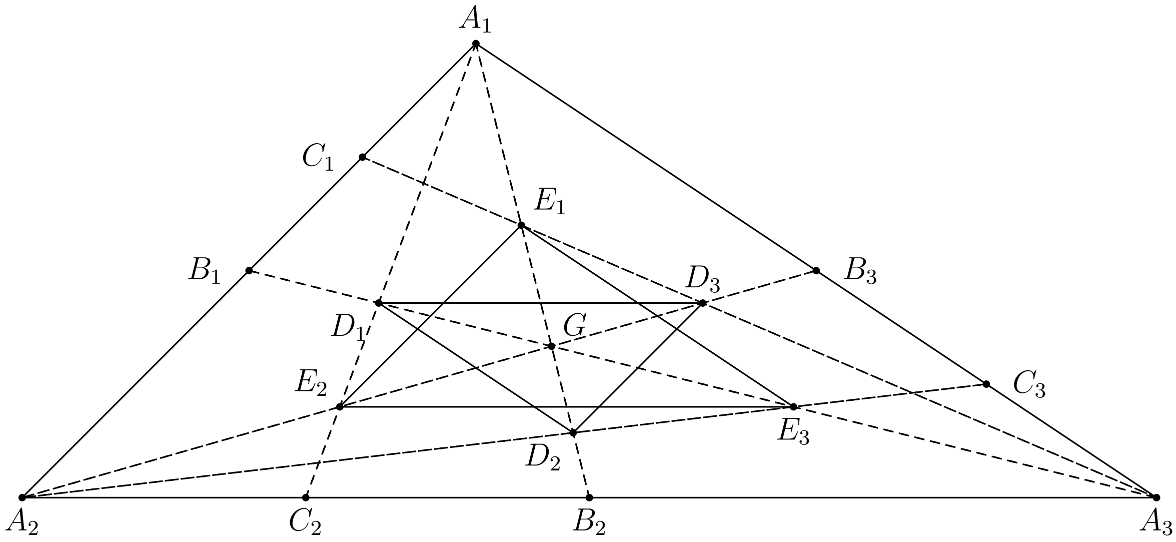
   (b) We first demonstrate that if either n is odd or k is odd then this is not possible. As before, consider the total number of correct submissions from all of the students. If no students fail then each student solves at least $\frac{k}{2}$ questions and so there are at least $\frac{nk}{2}$ correct submissions in total. Note however that each student solves an integer number of questions, and so if k is odd then this inequality is strict. A similar argument shows that if no question is easy then there are at most $\frac{nk}{2}$ correct submissions, and the inequality is strict if n is odd. Thus if either n or k is odd then these considerations imply that $\frac{nk}{2} > \frac{nk}{2}$, which is a contradiction.

   Now suppose that both n and k are even. Let there be n students $S_1, S_2, \ldots, S_n$, and k questions $Q_1, Q_2, \ldots, Q_k$. Let students $S_1, S_2, \ldots, S_{n/2}$ solve questions $Q_1, Q_2, \ldots, Q_{k/2}$

and let students $S_{n/2+1}, S_{n/2+2}, \ldots, S_n$ solve $Q_{k/2+1}, Q_{k/2+2}, \ldots, Q_k$. Then each student solves exactly half of the problems and hence no student fails. We also have that every problem is solved by exactly half of the students, and so no question is easy.

$\square$

3. Let G be the centroid of $A_1A_2A_3$.



We use Menelause's Th$^{\underline{\text{m}}}$ on the following sets of points $\{\{A_1E_2C_2\}, \{A_2E_3C_3\}, \{A_3E_1C_1\}\}$ to get that

$$\frac{GE_1}{E_1A_1} = \frac{GE_2}{E_2A_2} = \frac{GE_3}{E_3A_3} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{GE_1}{GA_1} = \frac{GE_2}{GA_2} = \frac{GE_3}{GA_3} = \frac{2}{5}$$

$$\Rightarrow \quad \frac{|\triangle E_1E_2E_3|}{|\triangle A_1A_2A_3|} = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$$

We then use Menelause's Th$^{\underline{\text{m}}}$ on the sets of points $\{\{A_1D_2B_2\}, \{A_2D_3B_3\}, \{A_3D_1B_1\}\}$ and then on the sets of points $\{\{A_1GC_2\}, \{A_2GC_3\}, \{A_3GC_1\}\}$ to get that

$$\frac{D_1C_2}{A_1D_1} = \frac{D_2C_3}{A_2D_2} = \frac{D_3C_1}{A_3D_3} = \frac{3}{4}$$

$$\Rightarrow \quad \frac{A_1C_2}{A_1D_1} = \frac{A_2C_3}{A_2D_2} = \frac{A_3C_1}{A_3D_3} = \frac{7}{4}$$

$$\Rightarrow \quad \frac{D_1G}{GA_3} = \frac{D_2G}{GA_1} = \frac{D_3G}{GA_2} = \frac{2}{7}$$

$$\Rightarrow \quad \frac{|\triangle D_1D_2D_3|}{|\triangle A_1A_2A_3|} = \left(\frac{2}{7}\right)^2 = \frac{4}{49}$$

$$\Rightarrow \quad \frac{|\triangle D_1D_2D_3|}{|\triangle E_1E_2E_3|} = \frac{25}{49}$$

$\square$

4. Let $x = a^{2/3}, y = b^{2/3}$ and $z = c^{2/3}$. By a well-known special case of Schur's inequality, we have

$$x^3 + y^3 + z^2 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$$

By AM-GM, we also have

$$xy(x+y) = x^2y + xy^2 \geq 2\sqrt{x^2y \cdot xy^2} = 2x^{3/2}y^{3/2}$$

and similarly for $yz(y+z)$ and $zx(z+x)$. We therefore obtain

$$\begin{aligned} a^2 + b^2 + c^2 + 3 &= x^3 + y^3 + z^3 + 3xyz \\ &\geq xy(x+y) + yz(y+z) + zx(z+x) \\ &\geq 2x^{3/2}y^{3/2} + 2y^{3/2}z^{3/2} + 2z^{3/2}x^{3/2} \\ &= 2(ab + bc + ca) \end{aligned}$$

which proves the inequality. □

5. Assume for contradiction the problem claim does not hold, and let p be the smallest (odd) prime not appearing in the sequence. We thus have that p is not the least prime factor of

$$1 + n\prod_{i=1}^{n} p_i^{i!}$$

for all i. Let C be the smallest integer such that all primes less than p appear in $p_1, p_2, \ldots, p_C$. Thus, if p divides $1 + n\prod_{i=1}^{n} p_i^{i!}$ for some $n > C$, then it must be the smallest factor.

Therefore, p does not divide $1+n\prod_{i=1}^{n} p_i^{i!}$ for all $i > C$. Now, define $T_m = \prod_{i=1}^{m} p_i^{i!}$. By Fermat's Little Theorem, we have for sufficiently high i and p_i (indeed, for $i > \max(C, p-1)$), $p_i^{i!} \equiv 1 \pmod{p}$. Thus, the residuce class of T_m stays constant for sufficiently high m. Therefore, there exists an m high enough such that $mT_m \equiv -1 \pmod{p}$, noting that T_m is not divisible by p. This proves that p divides $1 + mT_m$, which is a contradiction. □