April Camp 2019
Senior Test 1
Solutions

1. *Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying*
$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y)$$
for all $x, y \in \mathbb{Q}_{>0}$.

Answer: $f(x) = 1$ for all $x \in \mathbb{Q}_{>0}$.

Solution: Take any $a, b \in \mathbb{Q}_{>0}$. By substituting $x = f(a)$, $y = b$, and $x = f(b)$, $y = a$ into the given equation, we get

$$f(f(a))^2 f(b) = f(f(a)^2 f(b)^2) = f(f(b))^2 f(a)$$

which yields
$$\frac{f(f(a))^2}{f(a)} = \frac{f(f(b))^2}{f(b)} \qquad \text{for all } a, b \in \mathbb{Q}_{>0}$$

In other words, this shows that there exists a constant $C \in \mathbb{Q}_{>0}$ such that $f(f(a))^2 = Cf(a)$, or
$$\left(\frac{f(f(a))}{C}\right)^2 = \frac{f(a)}{C} \qquad \text{for all } a \in \mathbb{Q}_{>0} \tag{1}$$

Denote by $f^n(x) = f(f(\ldots(f(x))\ldots))$ the nth iteration of f. Equality (1) yields

$$\frac{f(a)}{C} = \left(\frac{f^2(a)}{C}\right)^2 = \left(\frac{f^3(a)}{C}\right)^4 = \cdots = \left(\frac{f^{n+1}(a)}{C}\right)^{2^n}$$

for all positive integer n. So, $f(a)/C$ is the 2^n-th power of a rational number for all positive integer n. This is impossible unless $f(a)/C = 1$, since otherwise the exponent of some prime in the prime decomposition of $f(a)/C$ is not divisible by sufficiently large powers of 2. Therefore, $f(a) = C$ for all $a \in \mathbb{Q}_{>0}$.

Finally, after substituting $f \equiv C$ into the given condition, we get $C = C^3$, whence $C = 1$. So $f(x) \equiv 1$ is the unique function satisfying the given equation. □

Comment 1. There are several variations of the solution above. For instance, one may start with finding $f(1) = 1$. To do this, let $d = f(1)$. By substituting $x = y = 1$ and $x = d^2$, $y = 1$ into the given, we get $f(d^2) = d^3$ and $f(d^6) = f(d^2)^2 \cdot d = d^7$. By substituting now $x = 1$, $y = d^2$ we obtain $f(d^6) = d^2 \cdot d^3 = d^5$. Therefore, $d^7 = f(d^6) = d^5$, whence $d = 1$.

After that, the rest of the solution simplifies a bit, since we already know that $C = \frac{f(f(1))^2}{f(1)} = 1$. Hence equation (1) becomes merely $f(f(a))^2 = f(a)$, which yields $f(a) = 1$ in a similar manner.

Comment 2 There exist nonconstant functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the given equation for all real $x, y > 0$, for example $f(x) = \sqrt{(}x)$.

2. *Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:*

   (i) *Each number in the table is congruent to 1 modulo n;*

   (ii) *The sum of numbers in any row, as well as the sum of numbers in any column is congruent to n modulo n^2.*

Let R_i be the product of the numbers in the i-th row, and C_j be the product of the numbers in the j-th column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

Proof: Let $A_{i,j}$ be the entry in the ith row and the jth column; let P be the product of all n^2 entries. For convenience, denote $a_{i,j} = A_{i,j} - 1$ and $r_i = R_i - 1$. We show that

$$\sum_{i=1}^{n} R_i \equiv (n-1) + P \pmod{n^4}. \tag{2}$$

Due to symmetry of the problem conditions, the sum of all the C_j is also congruent to $(n-1)+P$ modulo n^4, whence the conclusion.

By condition (i), the number n divides $a_{i,j}$ for all i and j. So, every product of at least two of the $a_{i,j}$ is divisible by n^2, hence

$$R_i = \prod_{j=1}^{n}(1 + a_{i,j}) = 1 + \prod_{j=1}^{n} a_{i,j} + \sum_{1 \le j_1 < j_2 \le n} a_{i,j_1} a_{i,j_2} + \cdots \equiv 1 - n + \sum_{j=1}^{n} A_{i,j} \pmod{n^2}$$

for every index i. Using condition (ii), we obtain $R_i \equiv 1 \pmod{n^2}$, and so $n^2 \mid r_i$.

Therefore, every product of at least two of the r_i is divisible by n^4. Repeating the same argument, we obtain

$$P = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n}(1 + r_i) \equiv 1 + \sum_{i=1}^{n} r_i \pmod{n^4}$$

whence
$$\sum_{i=1}^{n} R_i = n + \sum_{i=1}^{n} r_i \equiv n + (P - 1) \pmod{n^4}$$

as desired. □

Solution 2: We present a more straightforward (though lengthier) way to establish (2). We also use the notation of $a_{i,j}$.

By condition (i), all the $a_{i,j}$ are divisible by n. Therefore, we have

$$P=\prod_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j})\equiv 1+\sum_{(i,j)}a_{i,j}+\sum_{(i_1,j_1),(i_2,j_2)}a_{(i_1,j_1)}a_{(i_2,j_2)}$$
$$+\sum_{(i_1,j_1),(i_2,j_2),(i_3,j_3)}a_{(i_1,j_1)}a_{(i_2,j_2)}a_{(i_3,j_3)}\pmod{n^4},$$

where the last two sums are taken over all unordered pairs/triples of *pairwise different* pairs (i,j); such conventions are applied through the solution.

Similarly,

$$\sum_{i=1}^{n}R_i=\sum_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j})\equiv n+\sum_i\sum_j a_{i,j}+\sum_i\sum_{j_1,j_2}a_{i,j_1}a_{i,j_2}+\sum_i\sum_{j_1,j_2,j_3}a_{i,j_1}a_{i,j_2}a_{i,j_3}\pmod{n^4}.$$

Therefore,

$$P+(n-1)-\sum_i R_i\equiv\sum_{\substack{(i_1,j_1),(i_2,j_2)\\ i_1\neq i_2}}a_{i_1,j_1}a_{i_2,j_2}+\sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2\neq i_3\neq i_1}}a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3}$$
$$+\sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2=i_3}}a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3}\pmod{n^4}.$$

We show that in fact each of the three sums appearing in the right-hand part of this congruence is divisible by n^4; this yields (2). Denote those three sum sums Σ_1, Σ_2, and Σ_3 in order of appearance. Recall that by condition (i) we have

$$\sum_j a_{i,j}\equiv 0\pmod{n^2}\qquad\text{for all indices } i$$

For every two indices $i_1<i_2$ we have

$$\sum_{j_1}\sum_{j_2}a_{i_1,j_1}a_{i_2,j_2}=\left(\sum_{j_1}a_{i_1,j_1}\right)\cdot\left(\sum_{j_2}a_{i_2,j_2}\right)\equiv 0\pmod{n^4},$$

since each of the two factors is divisible by n^2. Summing over all pairs (i_1,i_2) we obtain $n^4\mid\sum_1$.

Similarly, for every three indices $i_1<i_2<i_3$ we have

$$\sum_{j_1}\sum_{j_2}\sum_{j_3}a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3}=\left(\sum_{j_1}a_{i_1,j_1}\right)\cdot\left(\sum_{j_2}a_{i_2,j_2}\right)\cdot\left(\sum_{j_3}a_{i_3,j_3}\right)$$

which is divisible even by n^6. Hence $n^4\mid\Sigma_2$.

Finally, for every indices $i_1\neq i_2=i_3$ and $j_2<j_3$, we have

$$a_{i_2,j_2}\cdot a_{i_2,j_3}\cdot\sum_{j_1}a_{i_1,j_1}\equiv 0\pmod{n^4},$$

since the three factors are divisible by n, n, and n^2, respectively. Summing over all 4-tuples of indices (i_1,i_2,j_1,j_2) we get $n^4\mid\Sigma_3$. □

3. *A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.*

Proof: By $\sphericalangle(\ell,n)$ we always mean the directed angle of the lines ℓ and n, taken modulo 180°.

Let CC_2 meet Ω again at K (as usual, if CC_2 is tangent to Ω, we set $T=C_2$). We show that the line BB_2 contains K; similarly, AA_2 will also pass through K. For this purpose, it suffices to prove that

$$\sphericalangle(C_2C,C_2A_1)=\sphericalangle(B_2B,B_2A_1)\tag{3}$$

By the problem condition, CB and CA are the perpendicular bisectors of TA_1 and TB_1, respectively. Hence C is the circumcentre of the triangle A_1TB_1. Therefore,

$$\sphericalangle(CA_1,CB)=\sphericalangle(CB,CT)=\sphericalangle(B_1A_1,B_1T)=\sphericalangle(B_1A_1,B_1B_2).$$

In circle Ω, we have $\sphericalangle(B_1A_1,B_1B_2)=\sphericalangle(C_2A_1,C_2B_2)$. Thus,

$$\sphericalangle(CA_1,CB)=\sphericalangle(B_1A_1,B_1B_2)=\sphericalangle(C_2A_1,C_2B_2).\tag{4}$$

Similarly, we get

$$\sphericalangle(BA_1,BC)=\sphericalangle(C_1A_1,C_1C_2)=\sphericalangle(B_2A_1,B_2C_2).\tag{5}$$

The two obtained relations yield that the triangles A_1BC and $A_1B_2C_2$ are similar and equioriented, hence

$$\frac{A_1B_2}{A_1B}=\frac{A_1C_2}{A_1C}\quad\text{and}\quad\sphericalangle(A_1B,A_1C)=\sphericalangle(A_1B_2,A_1C_2).$$

The second equality may be rewritten as $\sphericalangle(A_1B,A_1B_2)=\sphericalangle(A_1C,A_1C_2)$, so the triangles A_1BB_2 and A_1CC_2 are also similar and equioriented. This establishes (3). □

Comment 1: In fact, the triangle A_1BC is an image of $A_1B_2C_2$ under a spiral similarity centred at A_1; in this case, the triangles ABB_2 and ACC_2 are also spirally similar with the same centre.

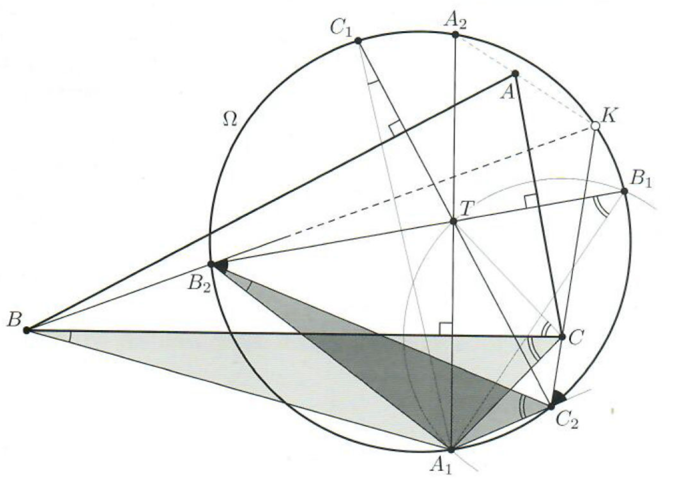
Comment 2: After obtaining (4) and (5), one can finish the solution in different ways.

For instance, introducing the point $X=BC\cap B_2C_2$, one gets from these relations that the 4-tuples (A_1,B,B_2,X) and (A_1,C,C_2,X) are both cyclic. Therefore, K is the Miquel point of the lines BB_2, CC_2, BC, and B_2C_2; this yields that the meeting point of BB_2 and CC_2 lies on Ω.

Yet another way is to show that the points A_1,B,C, and K are concyclic, as

$$\sphericalangle(KC,KA_1)=\sphericalangle(B_2C_2,B_2A_1)=\sphericalangle(BC,BA_1).$$

By symmetry, the second point K' of intersection of BB_2 with Ω is also concyclic to A_1,B, and C, hence $K'=K$.



Comment 3: The requirement that the common point of the lines AA_2, BB_2, and CC_2 should lie on Ω may seem to make the problem easier, since it suggests some approaches. On the other hand, there are also different ways of showing that the lines AA_2, BB_2, and CC_2 are just concurrent.

In particular, the problem conditions yield that the lines A_2T, B_2T, and C_2T are perpendicular to the corresponding sides of the triangle ABC. One may show that the lines AT, BT, and CT are also perpendicular to the corresponding sides of the triangle $A_2B_2C_2$, i.e. the triangles ABC and $A_2B_2C_2$ are *orthologic*, and their orthology centres coincide. It is known that such triangles are also *perspective*, i.e. the lines AA_2, BB_2, and CC_2 are concurrent (in projective sense).

Let A', B' and C' be the projections of T onto the sides of the triangle ABC. Then $A_2T \cdot TA' = B_2T \cdot TB' = C_2T \cdot TC'$, since all three products equal (minus) half the power of T with respect to Ω. This means that A_2, B_2, and C_2 are the poles of the sidelines of the triangle ABC with respect to some circle centred at T and having *pure imaginary* radius (in other words, the reflections of A_2, B_2 and C_2 in T are the poles of those sidelines with respect to some regular circle centred at T). Hence, dually, the vertices of the triangle ABC are also poles of the sidelines of the triangle $A_2B_2C_2$.