Stellenbosch Camp December 2018
Senior Test 1
Solutions

1. Let $d = \gcd(m, n)$, and denote $m = da$ and $n = db$ where a, b are two positive integers. We note that $\text{lcm}(m, n) = mn/d$. Since $a, b \geq 1$, we note

$$\begin{aligned} & (a-1)(b-1) \geq 0 \\ \iff \quad & a + b \leq 1 + ab \\ \iff \quad & da + db \leq d + dab \\ \iff \quad & m + n \leq \gcd(m, n) + \text{lcm}(m, n) \end{aligned}$$

which proves the problem statement. Note that equality occurs only if either $a = 1$ or $b = 1$, which occurs iff m divides n or n divides m. $\square$

2. We first prove injectivity. That is, if $f(a) = f(b)$ for some $a, b \in \mathbb{R}$, then $a = b$. We have

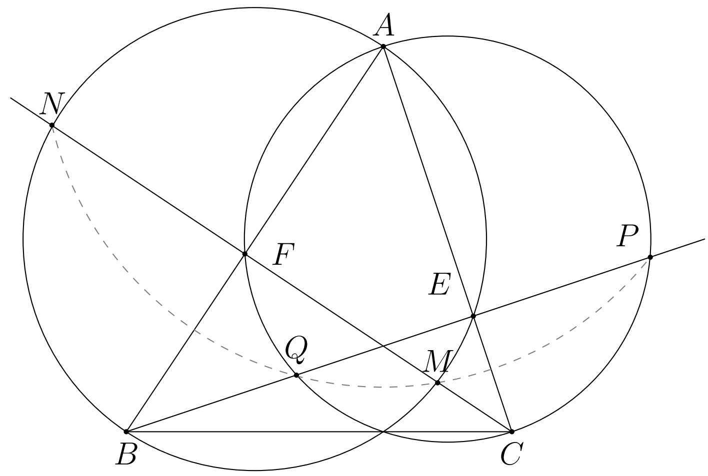
$$\begin{aligned} & f(a) = f(b) \\ \implies \quad & f(f(a+0)) = f(f(b+0)) \\ \implies \quad & a + f(0) = b + f(0) \\ \implies \quad & a = b \end{aligned}$$

thus proving injectivity. Now substituting $y = 0$ yields $f(f(x)) = f(x)$, which, by injectivity, implies $f(x) = x$ as the only solution. One easily checks that this solution satisfies the functional equation. $\square$

3. Let E and F be the feet of the perpendiculars from B and C respectively.
Note, since AB and AC are diameters, the angles $\angle BNA = \angle CPA = 90°$. Since we have $\angle NFA = \angle AEP = 90°$, we get that AN is tangent to the circumcircle of $\triangle NFB$, and AP is tangent to the circumcircle of $\triangle PEC$. Lastly, we know that $BFEC$ is a cyclic quadrilateral. So by using power of a point in the circles NFB, PEC, and $BFEC$, we obtain

$$\begin{aligned} & AN^2 = BA \cdot AF = CA \cdot AE = AP^2 \\ & AN = AP \end{aligned}$$

But we already have, $AN = AM$, and $AP = AQ$. Thus, $AN = AM = AP = AQ$. So M, N, P, and Q are all equidistant from A, and thus lie on a common circle centred at A. $\square$



4. Define a subset U of S_n as *unbalanced* if it is not balanced. We define a map $f : P(S_n) \to P(S_n)$ from subsets of S_n to subsets of S_n. Let $U \in P(S_n)$ be a subset of S_n and define $f(U)$ as the subset $\{n-k+1 \,:\, k \in U\}$ (i.e. it's a reversal map sending each element k in U to $n-k+1$). Note that we clearly have $f(f(U)) = U$ for all $U \in P(S_n)$. Furthermore, we easily note that the map flips the relative order of the mean and median of U (i.e. if $U_{\text{mean}} < U_{\text{median}}$ then $f(U)_{\text{mean}} > f(U)_{\text{median}}$). Thus, if U is unbalanced, then $f(U) \neq U$.

   Therefore, we have paired up each unbalanced set uniquely with another unbalanced set, proving that there are an *even* number of unbalanced sets. As the total number of non-empty subsets is $2^n - 1$, there are thus an *odd* number of balanced subsets. $\square$

5. Let $n = 15$. Note that the set $\{1, 2^2, 3^2, 5^2, 7^2, \ldots, 43^2\}$ consisting of 1 and the squares of the first 14 primes is a pairwise coprime set not containing any prime. Thus, if n satisfies the problem condition, then $n \geq 16$. We now prove that amongst any 16 pairwise coprime elements, at least one is prime.

   Let $A = \{a_1, a_2, \ldots, a_n\}$ be a set with $n \geq 16$ pairwise coprime elements, and order it in order of smallest prime dividing each element (if 1 is included, let $a_1 = 1$). Let p_i be the smallest prime dividing a_i. As A is pairwise coprime, we have $p_2 < p_3 < p_4 < \cdots < p_n$. Thus, as $p_2 \geq 2$, we have $p_n \geq 47$ as 47 is the 15th smallest prime. Now assuming that a_n is not prime, we have $a_n \geq p_n^2 \geq 2209 > 2018$ which yields a contradiction. Thus a_n is prime, which proves $n = 16$ is the minimal n satisfying the problem condition. $\square$