

Intermediate Test 2 – Solutions

Stellenbosch Camp 2018

1. For any real number x , let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x , and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x . Find all real numbers a , b , and c such that

$$\begin{aligned}\lfloor a \rfloor + \{b\} &= -2.3, \\ \lfloor b \rfloor + \{c\} &= 8.9, \quad \text{and} \\ \lfloor c \rfloor + \{a\} &= 23.4.\end{aligned}$$

The first equation gives us that $\lfloor a \rfloor = -3$ and $\{b\} = 0.7$. The second equation gives us that $\lfloor b \rfloor = 8$ and $\{c\} = 0.9$. The last equation implies that $\lfloor c \rfloor = 23$ and $\{a\} = 0.4$. It follows that $a = -2.6$, $b = 8.7$ and $c = 23.9$.

2. Let $PQRS$ be a cyclic quadrilateral such that $PQ = QR$, $PR = SR$ and $PQ \parallel SR$. Let ASB be a tangent at S where A lies on PQ . If $\angle BSR = 72^\circ$, find the value of $\angle RPQ$.

By the tan-chord theorem, we have that $\angle SPR = \angle BSR = 72^\circ$. Since $PR = SR$, we find that $\angle RSP = \angle SPR = 72^\circ$. Since $PQRS$ is a cyclic quadrilateral, we have that $\angle PQR = 180^\circ - \angle RSP = 180^\circ - 72^\circ = 108^\circ$.

Now note that $\angle PQR + \angle RPQ + \angle QRP = 180^\circ$. However, PQR is an isosceles triangle, so $\angle RPQ = \angle QRP$, and so we have that

$$2\angle RPQ + 108^\circ = \angle RPQ + \angle QRP + \angle PQR = 180^\circ,$$

and so $\angle RPQ = 36^\circ$.

3. Let n be a positive integer. Find the last digit of

$$n^{2018} + (n+1)^{2018} + \dots + (n+99)^{2018}.$$

Let

$$S = n^{2018} + (n+1)^{2018} + \dots + (n+99)^{2018}.$$

Note that among any 10 consecutive integers, we have one integer with each possible remainder modulo 10. We thus have that

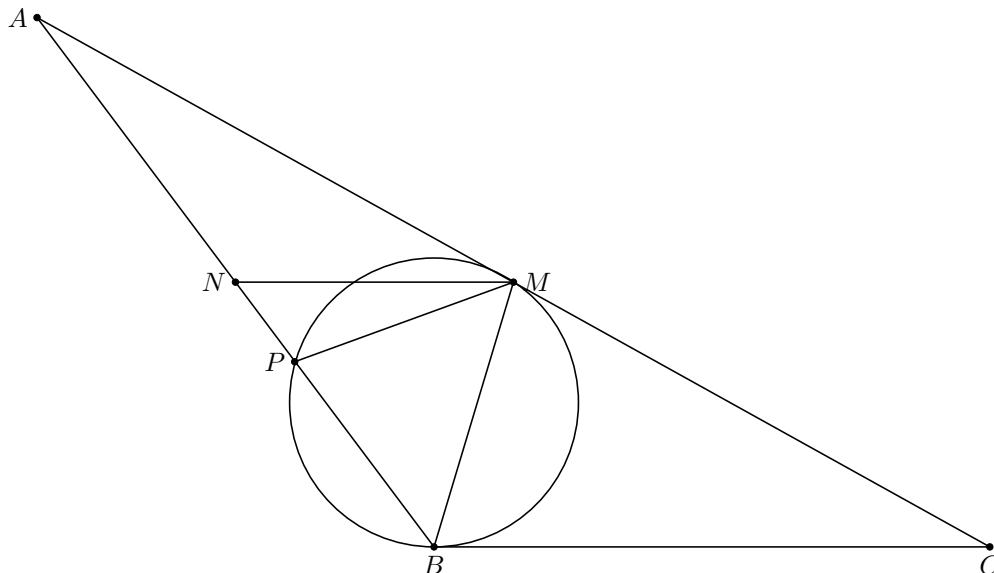
$$k^{2018} + (k+1)^{2018} + \dots + (k+9)^{2018} \equiv 0^{2018} + 1^{2018} + \dots + 9^{2018} \pmod{10}.$$

The numbers $n, (n+1), (n+2), \dots, (n+99)$ consists of 10 groups of 10 consecutive positive integers, and so we have that

$$\begin{aligned}S &\equiv \underbrace{(0^{2018} + 1^{2018} + \dots + 9^{2018}) + (0^{2018} + 1^{2018} + \dots + 9^{2018}) + \dots + (0^{2018} + 1^{2018} + \dots + 9^{2018})}_{10 \text{ times}} \\ &\equiv 10(0^{2018} + 1^{2018} + \dots + 9^{2018}) \\ &\equiv 0 \pmod{10},\end{aligned}$$

and so the last digit of S is always a 0.

4. Let ABC be a triangle, and let the midpoint of AC be M . The circle tangent to BC at B and passing through M meets the line AB again at P . Prove that $AB \times BP = 2BM^2$.



Construct point N , the midpoint of AB . By midpoint Th^{m} , $NM \parallel BC$. Using this and Tan-Chord Th^{m} , we get: $\angle BMN = \angle MBC = \angle MPB$. Since we also have that $\angle PBM = \angle MBN$, we have that $\triangle BMN \sim \triangle BPM$. So

$$\frac{BM}{BP} = \frac{BN}{BM}$$

$$\Rightarrow BM^2 = BN \cdot BP = \left(\frac{1}{2}AB\right) \cdot BP$$

5. Show that for all positive real numbers x and y ,

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{x+y}{3}.$$

By the Cauchy-Schwarz inequality (in “Engel Form”) we have that

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{(x+y)^2}{(x+2y) + (2y+x)} = \frac{x+y}{3}.$$

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