Senior Test 3

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Let ABC be a triangle, and let the midpoint of AC be M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.

2. Show that for all positive real numbers x and y,

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{x+y}{3}.$$

3. Consider an 8×8 grid of squares. On each square is a lightbulb which is initially switched off. A move consists of choosing a square and either the vertical or horizontal direction, and toggling the lightbulb on that square and it's immediate neighbours in the chosen direction. For clarity this means that usually 3 bulbs are flipped unless the square is on an edge in which case 2 bulbs may be flipped. After some amount of moves a single bulb is switched on (the other 63 are off). Determine which of the 64 bulbs can possibly be on.

4. Determine whether or not there is a positive integer m such that

$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3$$

is a square.

5. In triangle ABC with incentre I, let M_A, M_B and M_C be the midpoints of BC, CA, and AB respectively, and H_A, H_B, and H_C be the feet of altitudes from A, B and C to the respective sides. Denote by l_b the line being tangent to the circumcircle of triangle ABC and passing through B, and denote by l'_b the reflection of l_b in BI. Let P_B be the intersection of M_AM_C and l_b, and let Q_B be the intersection of H_AH_C and l'_b. Define l_c, l'_c, P_C and Q_C analogously. If R is the intersection of P_BQ_B and P_CQ_C, prove that $RB = RC$.

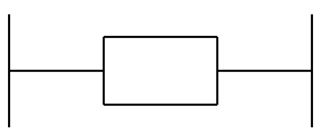

Figure 1: This is **not** real math.