Senior Problem Pool

Algebra

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x^2) - yf(y) = f(x+y)(f(x) - y)$ for all real numbers x and y.

2. Suppose that a, b, and c are real numbers such that
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}.$$
Show that for every odd natural number n, we have that
$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{(a+b+c)^n}.$$

3. Suppose that f is a function from $\mathbb{N}$ to $\mathbb{N}$ such that $f(n) \geq 2$ for all n, and
$$f(n) + f(n+2) = f(n+4)f(n+6) - 1997$$
for all $n \in \mathbb{N}$.
   (a) Find $f(1997)$ and $f(1999)$ if $f(1) = 2$.
   (b) Describe all functions which satisfy the conditions above.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$f(a^3) + f(b^3) + f(c^3) + 3f(a+b)f(b+c)f(c+a) = f(a+b+c)^3$$
for all $a, b, c \in \mathbb{Z}$.

Combinatorics

Geometry

Number Theory

1. Suppose that $2n+1$ and $3n+1$ are both squares. Show that 40 divides n.

2. Does there exist a natural number n such that
$$1^{2018} + 2^{2018} + \cdots + n^{2018}$$
is prime?

3. Suppose that a, b, and N are relatively prime integers. Show that there is an integer m such that a and $b + mN$ are relatively prime.

4. Find all natural numbers n such that there exist n distinct natural numbers $a_1, a_2, \ldots, a_n$ such that $a_i \mid a_{a+1}$ for $i = 1, 2, \ldots, n-1$, and such that $a_n = 2018n$.

Miscellaneous