Intermediate Test 4 – Solutions

Stellenbosch Camp 2018

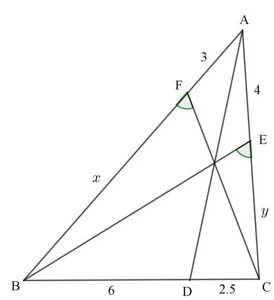
1. *How many numbers from* 1 *to* 2018 *inclusive can be written as the difference of two perfect squares?*

   We say that a number is *nice* if it can be written as the difference of two perfect squares. Any odd number is nice, since $2k+1 = (k+1)^2 - k^2$. Also, any number divisible by 4 is nice since $4k = (k+1)^2 - (k-1)^2$. So looking at remainders modulo 4, any number with remainder 0, 1 or 3 is nice; on the other hand, since all perfect squares have remainder 0 or 1 modulo 4, the only possibilities for the remainder of a difference of two squares are 0, 3 and 1; hence the numbers with these remainders are precisely the nice numbers.

   So to count the nice numbers from 1 to 2018, we take the total amount of numbers (2018) and subtract those with remainder 2; since 2018 has a remainder of 2 modulo 4, the amount of the latter is $\lfloor \frac{2018}{4} \rfloor + 1$, yielding a final answer of

   $$2018 - \left\lfloor \frac{2018}{4} \right\rfloor - 1 = 1513.$$

2. *Solve for lengths x and y in the following diagram:*



   Label the intersection of the lines through A, B and C with sides BC, AC and AB as D, E and F respectively. Using Ceva's theorem we have that:

   $$\frac{3}{x} \times \frac{6}{2.5} \times \frac{y}{4} = 1 \tag{1}$$

   Furthermore, since $\angle BFC = \angle BEC$, $BFEC$ is a cyclic quadrilateral. Point A is a point outside the circle passing through $BFCE$, and the lines through A intersects the circle at points F, B, E and C. From Power of a point we know that

   $$3(3+x) = 4(4+y) \tag{2}$$

   Solving equations (1) and (2) simultaneously, we find that $x = 9$ and $y = 5$.

3. *Find all functions $f : \mathbb{R} -> \mathbb{R}$ such that for all real numbers x,*

   $$2f(x) + 3f(1-x) = x - 4x^3.$$

4. *Prove that it is impossible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$.*

5. *An exam with k questions is presented to n students. A student fails the exam if they get less than half the answers right. We say that a question is easy if more than half of the students get it right. Decide if it is possible that*

   (a) All students fail even though all the questions were easy.

   (b) No student fails even though no question was easy.

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