# PAMO Stream Test 3

April Camp 2019

Time: $4\frac{1}{2}$ hours

1. *Find a non-zero polynomial $f(x, y)$ such that $f(\lfloor 3t \rfloor, \lfloor 5t \rfloor) = 0$ for every $t \in \mathbb{R}$.*

   Let $x = \lfloor 3t \rfloor$ and $y = \lfloor 5t \rfloor$, where $t \in \mathbb{R}$ and define $g(x, y) = 5x - 3y$. We claim that $g(x, y) = 0, \pm 1, \pm 2, -3, -4$. For any integer $k$, we have $5\lfloor 3(t+k) \rfloor - 3\lfloor 5(t+k) \rfloor = 5(\lfloor 3t \rfloor + 3k) - 3(\lfloor 5t \rfloor + 5k) = 5\lfloor 3t \rfloor - 3\lfloor 5t \rfloor$. So $g(\lfloor 3t \rfloor, \lfloor 5t \rfloor) = 5\lfloor 3t \rfloor - 3\lfloor 5t \rfloor$ is periodic function with period 1. It suffices to consider $t \in [0, 1]$. We now partition $[0, 1)$ as follows: $[0, 1) = [0, \frac{1}{5}) \cup [\frac{1}{5}, \frac{1}{3}) \cup [\frac{1}{3}, \frac{2}{5}) \cup [\frac{2}{5}, \frac{3}{5}) \cup [\frac{3}{5}, \frac{2}{3}) \cup [\frac{2}{3}, \frac{4}{5}) \cup [\frac{4}{5}, 1)$. So,

$$g(x, y) = 5x - 3y = 5\lfloor 3t \rfloor - 3\lfloor 5t \rfloor = \begin{cases} 0, & t \in \left[0, \frac{1}{5}\right), \\ -3, & t \in \left[\frac{1}{5}, \frac{1}{3}\right), \\ 2, & t \in \left[\frac{1}{3}, \frac{2}{5}\right), \\ -1, & t \in \left[\frac{2}{5}, \frac{3}{5}\right), \\ -4, & t \in \left[\frac{3}{5}, \frac{2}{3}\right), \\ 1, & t \in \left[\frac{2}{3}, \frac{4}{5}\right), \\ -2, & t \in \left[\frac{4}{5}, 1\right]. \end{cases}$$

   So we can take $f(x, y) = (5x - 3y)((5x - 3y)^2 - 1)((5x - 3y)^2 - 4)(5x - 3y + 3)(5x - 3y + 4)$.

2. *Let $ABC$ be a triangle and $\Gamma$ be the circle of diameter $[AB]$. The bisectors of $\angle BAC$ and $\angle ABC$ cut the circle $\Gamma$ again in $D$ and $E$, respectively. The incicrcle of the triangle $ABC$ cuts the lines $BC$ and $AC$ in $F$ and $G$, respectively. Show that the points $D, E, F$ and $G$ lie on the same line.*

3. *A positive integer is called special if its digits can be arranged to form an integer divisible by 4. How many of the integers from 1 to 2018 are special?*

   We characterise the integers which are not special. Recall that a natural number $n$ is divisible by 4 if and only if the number formed by the last two digits of $n$ is divisible by 4.

   Suppose that $n$ is not special. First we consider the case where all of the digits of $n$ are even. If $n$ contains a digit, say $a$, which is divisible by 4, then if $b$ is any other digit of $n$, we know that $10b + a \equiv a \equiv 0 \pmod 4$, and so the digits of $n$ can be rearranged to form a number divisible by 4. (i.e. The one that ends in $ba$.) We see that there are no non-special numbers all of whose digits are even and which contain a $0, 4$, or 8. (If $n$ does not contain another digit other than $a$, then $n$ itself is equal to 0, 4, or 8, and so is special.)

Thus if $n$ contains only even digits, then $n$ consists only of the digits 2 and 6. Conversely, since none of 22, 26, 62, and 66 are divisible by 4, we see that any natural number containing only the digits 2 and 6 is not special. It follows that the number of $m$ digit numbers which are not special and which contain only even digits is equal to $2^m$.

We now consider the case where $m$ contains an odd digit. Let this odd digit be $a$. Note that $10a \equiv 2 \pmod 4$, and so if $n$ contains a 2 or a 6, then the digits of $n$ can be rearranged to end in $a2$ or $a6$ both of which are divisible by 4, and so $n$ would be special. We see that if $n$ is not special, then the only even digits which $n$ can contain are $0, 4$, and 8.

Now suppose that $n$ contains two even digits. Then the digits of $n$ can be rearranged to form the number ending in those two digits. This number is divisible by 4, and so $n$ is special. It follows that if $n$ is not special, then $n$ contains either only odd digits, or exactly one even digit which has to be a $0, 4$, or 8. Conversely, we see that any such number is not special.

The number of $m$ digit numbers containing only odd digits is equal to $5^m$. The number of $m$ digit numbers containing exactly one $0, 4$, or 8 and having the rest of its digits odd is equal to $2 \cdot 5^{m-1} + 3(m-1) \cdot 5^{m-1}$. This is because if the number starts with $0, 4$, or 8, then it starts with a 4, or 8, and so there are 2 options for the first digit, and 5 options for each of the remaining digits. Otherwise, there are $(m-1)$ options for which digit is equal to $0, 4$, or 8, and 3 options for what that digit is equal to. There are then again 5 options for each of the remaining digits. We also note that this formula requires $m > 1$, since a one-digit number can not contain a $0, 4$ or 8, and also contain an odd digit.

For $m > 1$, we thus have that there are $2^m + 5^m + 2 \cdot 5^{m-1} + 3(m-1) \cdot 5^{m-1}$ non-special numbers.

Using the above, we see that there are 7 one-digit non-special numbers, 54 two-digit non-special numbers, and 333 three-digit non-special numbers. This gives a total of 394 non-special numbers below 1000.

We now consider the non-special numbers $n$ such that $1000 \leq n < 2000$. We note that these all start with a 1, and so are a non-special number containing an odd digit. As before, we see that the remaining digits are either all odd, or exactly one of them is equal to a $0, 4$, or 8. There are $5^3$ numbers where the remaining digits are all odd, and $3 \cdot 3 \cdot 5^2$ numbers which contain a $0, 4$, or 8. There are thus 350 non-special numbers between 1000 and 2000.

Finally, note that there are no non-special numbers from 2001 to 2018 since the digits of these numbers can all be rearranged to form a number ending in 20. The number 2000 is also special since it is already a multiple of 4.

We thus see that the number of positive integers less than or equal 2018 which are not special is equal to $394 + 350 = 744$. There are thus $2018 - 744 = 1274$ special natural numbers from 1 to 2018.