PAMO Stream Test 1

April Camp 2019

Time: $4\frac{1}{2}$ hours

1. In a triangle ABC, let D and E be the midpoints of AB and AC, respectively, and let F be the foot of the altitude through A. Show that the line DE, the angle bisector of $\angle ACB$ and the circumcircle of ACF pass through a common point.

2. Let $f(n) = n + \lfloor \sqrt{n} \rfloor$. Prove that for every positive integer m, the integer sequence m, $f(m)$, $f(f(m))$, $\ldots$ contains at least one square of an integer.

3. A game is played on an $m \times n$ chessboard. At the beginning, there is a coin on one of the squares. Two players take turns to move the coin to an adjacent square (horizontally or vertically). The coin may never be moved to a square that has been occupied before. If a player cannot move any more, he loses. Prove:

   (a) If the size (number of squares) of the board is even, then the player to move first has a winning strategy, regardless of the initial position.

   (b) If the size of the board is odd, then the player to move first has a winning strategy if and only if the coin is initially placed on a square whose colour is not the same as the colour of the corners.