Senior March Monthly Problem Set

Solutions

1. *Prove that the inequality*
$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{x}{z}-\frac{z}{y}-\frac{y}{x}<\frac{1}{4xyz}$$
holds for all real numbers $x, y, z \in (0,1)$.

   Note that
$$\begin{aligned} & \frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{x}{z}-\frac{z}{y}-\frac{y}{x}<\frac{1}{4xyz} \\ \iff \quad & x^2z+y^2x+z^2y-x^2y-z^2x-y^2z<\frac{1}{4} \\ \iff \quad & (x-y)(x-z)(z-y)<\frac{1}{4} \end{aligned}$$

   Note, if $x=y$ or $y=z$ or $z=x$, then the inequality trivially holds. Furthermore, note that the inequality is cyclic. Thus, we may assume wlog that $x>y$ and $x>z$.

   If $y>z$, then we have $(x-y)(x-z)(z-y)<0<\frac{1}{4}$, which trivially proves the inequality. We therefore consider the case where $z>y$. By AM-GM, we have
$$\sqrt{(x-z)(z-y)} \le \frac{(x-z)+(z-y)}{2}=\frac{x-y}{2}<\frac{1}{2}$$
and thus, by squaring, we obtain
$$(x-z)(z-y)<\frac{1}{4}$$
Finally, noting that $x-y<1$, we conclude that
$$(x-y)(x-z)(z-y)<(x-z)(z-y)<\frac{1}{4}$$
which proves the inequality.

2. *In the game Memory you are given* $2n$ *cards, where* n *is a given positive integer. The cards start lying face down in an array on the table. On the face of each card there is a picture. There are* n *different pictures, each occurring on exactly two of the cards. In a turn you may choose two cards and then turn them both face-up. If they have the same picture, you may remove them from the table. Otherwise you turn them face-down again. Your goal is to clear all the cards from the table.*

   What is the least integer k *for which it is always possible to finish the game in at most* k *turns?*

3. *Prove that there are infinitely many integers* n *such that both the arithmetic mean of its divisors and the geometric mean of its divisors are integers.*

   (Recall that for k *positive real numbers* $a_1, a_2, \ldots, a_k$*, the arithmetic mean is* $\frac{a_1+a_2+\cdots+a_k}{k}$ *and the geometric mean is* $\sqrt[k]{a_1a_2\cdots a_k}$*.)*

   Let p be a prime such that $p \equiv 1 \pmod 3$, and let $n=p^2$. Note that there are three divisors of n, which are $1, p, p^2$.

   The geometric mean is therefore $\sqrt[3]{1\cdot p\cdot p^2}=\sqrt[3]{p^3}=p$ which is clearly an integer. The arithmetic mean is $\frac{1+p+p^2}{3}$, however, note that $1+p+p^2 \equiv 1+1+1 \equiv 0 \pmod 3$, thus $\frac{1+p+p^2}{3}$ is also an integer.

   Thus, n satisfies the problem conditions. Finally, note that by Dirichlet's theorem, since gcd(1, 3) = 1, there are infinitely many primes p such that $p \equiv 1 \pmod 3$, and thus infinitely many n satisfying the problem condition.

4. *Let $\mathbb{P}$ be the set of points in the Euclidean plane, and $O \in \mathbb{P}$ be a given point. Let $\mathbb{P}_O = \mathbb{P} \backslash \{O\}$ be the set of points in the Euclidean plane excluding O.*

   Find all functions $f : \mathbb{P}_O \to \mathbb{P}_O$ satisfying both of the following conditions:

   - *If $C \subset \mathbb{P}_O$ is a circle, then $f(C) = \{f(P) \mid P \in C\}$ is also a circle.*
   - *For any point $P \in \mathbb{P}_O$ we have that O, P and $f(P)$ are collinear.*

5. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a strictly monotone function $g : \mathbb{R} \to \mathbb{R}$ which satisfies*

$$f(x)g(y) + g(x) = g(x + y)$$

for all $x, y \in \mathbb{R}$.

We can equate the two following identities

$$f(x)g(y) + g(x) = g(x + y) \quad \text{and} \quad f(y)g(x) + g(y) = g(y + x)$$

to obtain

$$g(x)(f(y) - 1) = g(y)(f(x) - 1) \tag{1}$$

Now, letting $y = 0$ in the original condition yields

$$f(x)g(0) + g(x) = g(x) \implies f(x)g(0) = 0$$

which implies either f identically 0, or $g(0) = 0$. However, f identically 0 would imply that $g(x+y) = g(x)$ which implies g constant, thus contradicting the fact that g is strictly monotone.

Therefore $g(0) = 0$, and since g strictly monotone, this implies $g(x) \neq 0$ for all $x \neq 0$. Therefore, from (1), we obtain

$$\frac{f(x) - 1}{g(x)} = \frac{f(y) - 1}{g(y)} \quad \text{for all } x, y \neq 0$$

Therefore, there exists a $c \in \mathbb{R}$ such that $f(x) - 1 = cg(x)$ for all $x \neq 0$. Furthermore, substituting $x = 0$, $y = 1$ into (1), we obtain $0 = g(1)(f(0) - 1)$, and since $g(1) \neq 0$, this implies $f(0) = 1$.

Therefore, we have that $f(x) - 1 = cg(x)$ for all $x \in \mathbb{R}$. Substituting this into the original equation, we obtain

$$\begin{aligned}
& f(x)g(y) + g(x) = g(x + y) \\
\implies \quad & (1 + cg(x))g(y) + g(x) = g(x + y) \\
\implies \quad & 1 + c(1 + cg(x))g(y) + cg(x) = 1 + cg(x + y) \\
\implies \quad & (1 + cg(x))(1 + cg(y)) = 1 + cg(x + y) \\
\implies \quad & f(x)f(y) = f(x + y)
\end{aligned}$$

First, let's consider the case $c = 0$. This implies $f(x) = 1$, which fulfils the problem condition where $g(x) = x$.

Now, assume $c \neq 0$. As g strictly monotonic, this implies f strictly monotonic. Furthermore, noting that $f(x) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f^2\left(\frac{x}{2}\right)$, we have $f(x) \geq 0$ for all $x \in \mathbb{R}$. But since f strictly monotonic, this implies $f(x) > 0$ for all $x \in \mathbb{R}$.

We may therefore define the function $h : \mathbb{R} \to \mathbb{R}$ where $h(x) = \ln f(x)$ for all $x \in \mathbb{R}$. Note that

$$h(x) + h(y) = \ln f(x) + \ln f(y) = \ln f(x) \cdot f(y) = \ln f(x + y) = h(x + y)$$

Therefore, h satisfies the Cauchy condition $h(x+y) = h(x) + h(y)$. Since h is monotonic, this implies that h must be linear. Thus, there exists $k \in \mathbb{R}$ such that $h(x) = kx$ for all $x \in \mathbb{R}$. Thus $f(x) = e^{kx}$ and $g(x) = \frac{e^{kx}-1}{c}$.

One can easily check that this satisfies the given condition. In conclusion, the set of valid functions for f are $f(x) = a^x$ where $a > 0$.

6. *We put a number in each field of an $n \times n$ table T such that no number appears twice in the same row. Prove that it is possible to rearrange the numbers in T in such a way that each row of the rearranged table T^* contains the same numbers that the corresponding row of T contained, and moreover no number appears twice in the same column of T^*.*

7. *Find all positive integers m, n such that*

$$m^{2019} - m! = n^{2019} - n!.$$

Firstly, we note that $m = n$ clearly yields a solution. Now, let $f : \mathbb{N} \to \mathbb{Z}$ be a function such that $f(n) = n^{2019} - n!$ for all $n \in \mathbb{N}$. We shall prove that f is injective, which proves that there exist no further solutions.

Assume for contradiction that there exists $m, n \in \mathbb{N}$ such that $f(m) = f(n)$ and $m > n$. Note that clearly $n \neq 1$, thus we may assume $n > 1$. Let p be a prime divisor of n. Since $m > n$, we have p divides $m!$. Thus, since $m! - n! = m^{2019} - n^{2019}$, we have p divides m

8. *Let ABC be an acute angled triangle with incentre I. Let AI and CI have midpoints M and N respectively and intersect BC and BA at A' and C' respectively. Let K and L be points inside triangles $AC'I$ and $A'CI$ respectively such that $\angle AKI = \angle AIC = \angle CLI$, $\angle AKM = \angle ICA$ and $\angle IAC = \angle CLN$. Show that the radii of the circumcircles of LIK and ABC are equal.*