Senior Test 4

April Camp 2019

Time: $4\frac{1}{2}$ hours

1. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

2. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n + 1$ squares in a row, numbered from 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses an nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

   Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

   turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

3. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t.$$

   Is it possible that both xy and zt are perfect squares?