Junior January Monthly Problem Set

Due: 18 January 2019

1. $$\sqrt[3]{\frac{2+1}{2}\cdot\frac{3+1}{3}\cdot\frac{4+1}{4}\cdots\frac{a+1}{a}}=4.$$

   Find the value of a.

2. In a triangle ABC, let D and E be the midpoints of AB and AC, respectively, and let F be the foot of the altitude through A. Show that the line DE, the angle bisector of $\angle ACB$ and the circumcircle of ACF pass through a common point.

3. Determine the number of triplets (k, l, m) of positive integers such that

$$k+l+m=97 \quad \text{and}$$
$$\frac{4k}{5}+\frac{5l}{6}+\frac{6m}{7}=82.$$

4. A positive integer is called *special* if its digits can be arranged to form an integer divisible by 4. How many of the integers from 1 to 2018 are special?

5. Prove that if $p > 10$ is a prime number that divides $a^4 + a^3 + a^2 + a + 1$ for some integer a, then p's decimal expansion ends in a 1.

6. Steve determines the geometric mean of two positive integers in the following way:

   (a) He writes them down in their decimal representation, one below the other, and prepends zeros to the smaller number (if applicable) such that their lengths are equal.

   (b) He determines the geometric mean of each pair of digits below each other. If the result is not an integer, only the integer part is used.

   (c) The digits determined by this procedure form the result.

   Determine all pairs (a, b) of positive integers for which Steve's procedure yields the correct result. (For example, one such pair is $(12; 48)$.)

7. The set S of nonnegative integers has the property that every nonnegative integer n can be uniquely written as $n = a + 2b$ where $a, b \in S$ are not necessarily distinct. How many elements of S are less than 2018?

8. P, Q and R are any points on BC, CA and AB respectively of a triangle ABC. Let the centres of the circumcircles AQR, BRP and CPQ be X, Y and Z. Prove that triangles XYZ and ABC are similar.

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.