Stellenbosch Camp 2018: Senior Geometry
Lecture 3: Inversion

Definition (Cross Ratio). *Let A, B, C, and D be 4 points in the plane. Then the Cross Ratio on these points is defined as follows:*

$$(A, B : C, D) = \frac{AC}{AD} \bigg/ \frac{BC}{BD} = \frac{AC \cdot BD}{AD \cdot BC}$$

Theorem 1. *Let A, B, C, and D be 4 points lying on a line ℓ, in that order. Let P be a point not lying on ℓ, and let $k \neq \ell$ be a line not passing through P. Let A', B', C', and D' be the intersections of k with PA, PB, PC, and PD respectively. Then:*

$$(A', B' : C', D') = (A, B : C, D)$$

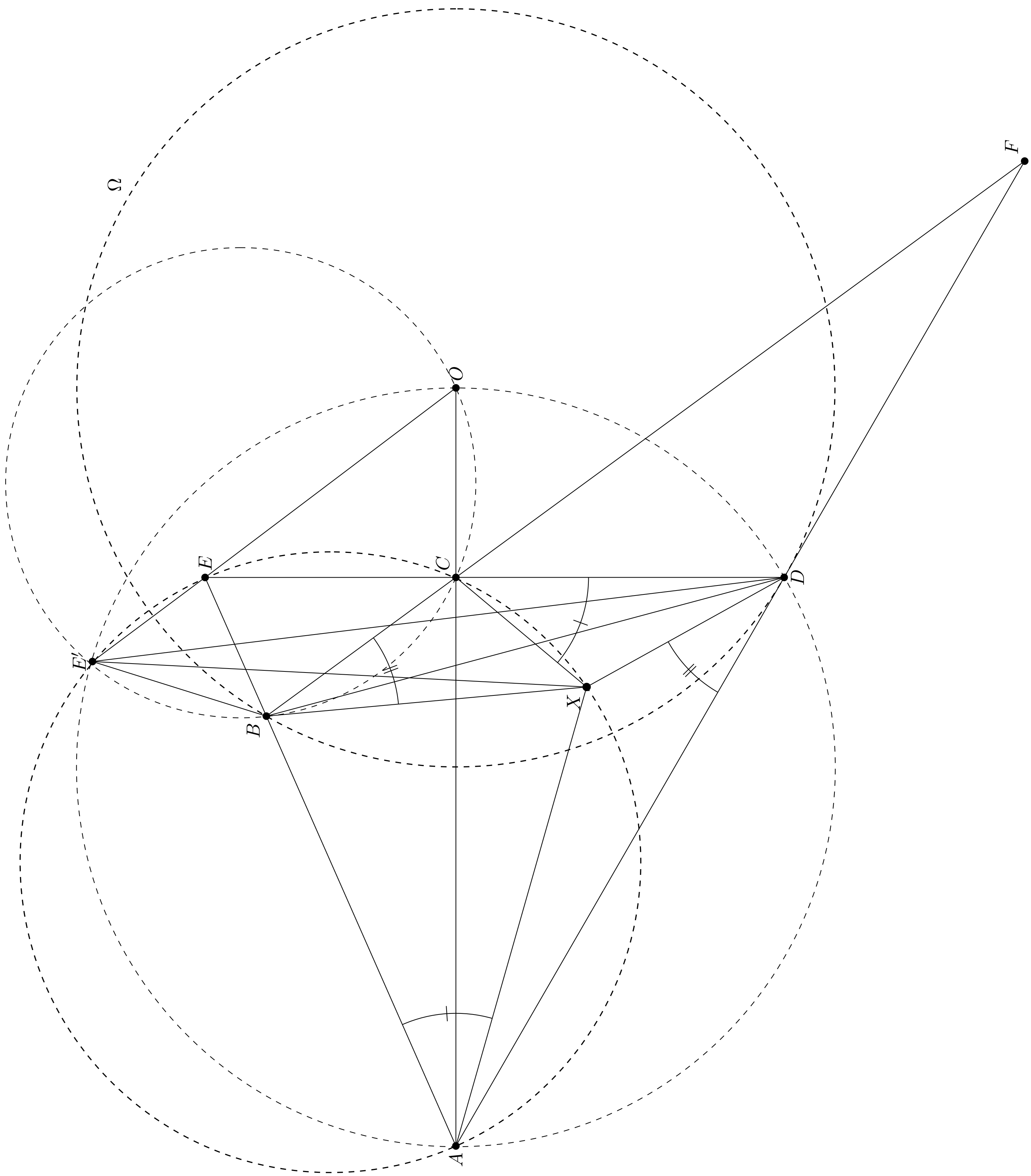
Theorem 2. *Let Ω be a circle with centre O, and let A, B, C, and D be 4 points in the plane, all distinct from O. Let A', B', C', and D' be the images of A, B, C, and D with respect to an inversion $T(\Omega)$, respectively. Then:*

$$(A', B' : C', D') = (A, B : C, D)$$

Theorem 3 (Apollonius' Circle Definition). *Let A and B be 2 points in the plane, and let r be a fixed positive real number. Let P be a point such that $\frac{AP}{BP} = r$. Then the locus of P is a circle.*

Corollary 1 (1). *Let A and B be 2 points in the plane, and let Ω be an Apollonius Circle defined by A, B, and some arbitrary real number r. The inversion $T(\Omega)$ sends point A to point B.*

Corollary 2 (2). *Let A and B be 2 points in the plane, and let Ω be an Apollonius Circle defined by A, B, and some arbitrary real number r. Any circle that passes through both A and B is **orthogonal** to Ω.*


Ω
F
O
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C
D
E'
B
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A