Stellenbosch Camp 2018: Senior Geometry
Weekly Problem Set

9 December 2018 **Due:** 14 December 2018

For the following questions, please provide **full proofs** unless specified otherwise.

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1. Let ABC be a triangle with circumcircle Γ, and Let P be a point interior to $\triangle ABC$. The rays AP, BP, and CP intersect Γ again at D, E, and F, respectively. Prove:

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1$$

2. Prove Ptolemy's Inequality:

   Theorem. *Ptolemy's Inequality*

   Let $ABCD$ be a convex quadrilateral, then:

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$

   With equality if and only if $ABCD$ is a cyclic quadrilateral.

3. Let ABC be a triangle where $\angle BAC > 120°$. Find the Fermat point of this triangle.

4. Let Ω be a circle with circumcentre O,

   (a) Let ℓ be a line not passing through O. Prove that the image of ℓ with respect to Ω is a circle passing through O.
   Thus, prove that if Γ is a circle passing through O, then the image of Γ with respect to Ω is a line not passing through O.

   (b) Let Γ be a circle not passing through O. Prove that the image of Γ with respect to Ω is also a circle not passing through O.

   (c) Let ℓ be a line passing through O. Prove that the image of ℓ with respect to Ω is itself.

   (d) Let Γ be a circle that is orthogonal to Ω. Prove that the image of Γ with respect to Ω is itself.

5. Let Ω be a circle with centre O, and let A, B, C, and D be 4 points in the plane, all distinct from O. Let A', B', C', and D' be the images of A, B, C, and D with respect to an inversion $T(\Omega)$, respectively. Prove that

$$\frac{A'C' \cdot B'D'}{A'D' \cdot B'C'} = \frac{AC \cdot BD}{AD \cdot BC}$$

6. (a) Let P be a point inside triangle ABC. Suppose that P has Barycentric coordinates: $(\alpha : \beta : \gamma)$. Let Q be the isogonal conjugate of P with respect to $\triangle ABC$. Prove that Q has non-homogeneous (*not normalised*) Barycentric coordinates:

$$\left(\frac{a^2}{\alpha} : \frac{b^2}{\beta} : \frac{c^2}{\gamma}\right)$$

   (b) Thus, using the above result, find the Barycentric coordinates of the Incentre

Bonus Question

5. Let ABC be an acute angled triangle. Find the Barycentric coordinates of the first Fermat point of $\triangle ABC$.