Senior Test 4

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Prove that it is impossible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$.

2.

3. Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$ and 3, suppose that B_n is the midpoint of A_nA_{n+1} and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $\triangle D_1D_2D_3$ to the area of triangle $\triangle E_1E_2E_3$.

4. Lets a, b and c be positive real numbers such that $abc = 1$. Prove that
$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca).$$

5. Let $p_1 = 2$ and define a sequence of prime numbers $p_1, p_2, p_3, \ldots$ such that, for all positive integers n, p_{n+1} is the least prime factor of $n \cdot p_1^{1!} \cdot p_2^{2!} \ldots p_n^{n!} + 1$. Prove that all primes appear in the sequence.

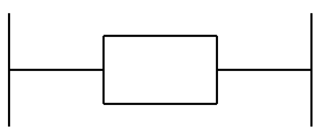

Figure 1: This is **not** real math.