Senior Monthly Problem Set

Due: 31 March 2019

1. Prove that the inequality

$$\frac{x}{y}+\frac{y}{z}+\frac{z}{x}-\frac{x}{z}-\frac{z}{y}-\frac{y}{x}<\frac{1}{4xyz}$$

   holds for all real numbers $x, y, z \in (0,1)$.

2. In the game Memory you are given $2n$ cards, where n is a given positive integer. The cards start lying face down in an array on the table. On the face of each card there is a picture. There are n different pictures, each occurring on exactly two of the cards. In a turn you may choose two cards and then turn them both face-up. If they have the same picture, you may remove them from the table. Otherwise you turn them face-down again. Your goal is to clear all the cards from the table.

   What is the least integer k for which it is always possible to finish the game in at most k turns?

3. Prove that there are infinitely many integers n such that both the arithmetic mean of its divisors and the geometric mean of its divisors are integers.

   (Recall that for k positive real numbers $a_1, a_2, \ldots, a_k$, the arithmetic mean is $\frac{a_1+a_2+\cdots+a_k}{k}$ and the geometric mean is $\sqrt[k]{a_1 a_2 \cdots a_k}$.)

4. Let $\mathbb{P}$ be the set of points in the Euclidean plane, and $O \in \mathbb{P}$ be a given point. Let $\mathbb{P}_O = \mathbb{P} \backslash \{O\}$ be the set of points in the Euclidean plane excluding O.

   Find all functions $f : \mathbb{P}_O \to \mathbb{P}_O$ satisfying both of the following conditions:

   - If $C \subset \mathbb{P}_O$ is a circle, then $f(C) = \{f(P) \mid P \in C\}$ is also a circle.
   - For any point $P \in \mathbb{P}_O$ we have that O, P and $f(P)$ are collinear.

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that there exists a strictly monotone function $g : \mathbb{R} \to \mathbb{R}$ which satisfies

$$f(x)g(y) + g(x) = g(x + y)$$

for all $x, y \in \mathbb{R}$.

6. We put a number in each field of an $n \times n$ table T such that no number appears twice in the same row. Prove that it is possible to rearrange the numbers in T in such a way that each row of the rearranged table T^* contains the same numbers that the corresponding row of T contained, and moreover no number appears twice in the same column of T^*.

7. Find all positive integers m, n such that

$$m^{2019} - m! = n^{2019} - n!.$$

8. Let ABC be an acute angled triangle with incentre I. Let AI and CI have midpoints M and N respectively and intersect BC and BA at A' and C' respectively. Let K and L be points inside triangles $AC'I$ and $A'CI$ respectively such that $\angle AKI = \angle AIC = \angle CLI$, $\angle AKM = \angle ICA$ and $\angle IAC = \angle CLN$. Show that the radii of the circumcircles of LIK and ABC are equal.

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.