

# Intermediate Test 2 – Solutions

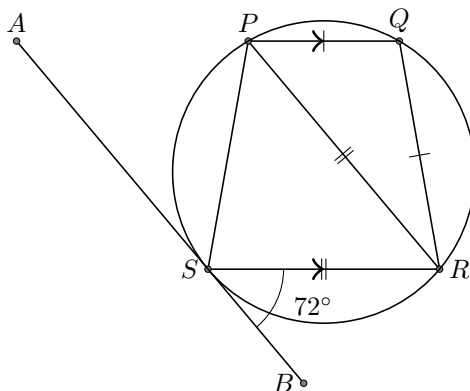
Stellenbosch Camp 2018

1. For any real number  $x$ , let  $\lfloor x \rfloor$  denote the greatest integer less than or equal to  $x$ , and let  $\{x\} = x - \lfloor x \rfloor$  be the fractional part of  $x$ . Find all real numbers  $a$ ,  $b$ , and  $c$  such that

$$\begin{aligned}\lfloor a \rfloor + \{b\} &= -2.3, \\ \lfloor b \rfloor + \{c\} &= 8.9, \quad \text{and} \\ \lfloor c \rfloor + \{a\} &= 23.4.\end{aligned}$$

The first equation gives us that  $\lfloor a \rfloor = -3$  and  $\{b\} = 0.7$ . The second equation gives us that  $\lfloor b \rfloor = 8$  and  $\{c\} = 0.9$ . The last equation implies that  $\lfloor c \rfloor = 23$  and  $\{a\} = 0.4$ . It follows that  $a = -2.6$ ,  $b = 8.7$  and  $c = 23.9$ .

2. Let  $PQRS$  be a cyclic quadrilateral such that  $PQ = QR$ ,  $PR = SR$  and  $PQ \parallel SR$ . Let  $ASB$  be a tangent at  $S$  where  $A$  lies on  $PQ$ . If  $\angle BSR = 72^\circ$ , find the value of  $\angle RPQ$ .



By the tan-chord theorem, we have that  $\angle SPR = \angle BSR = 72^\circ$ . Since  $PR = SR$ , we find that  $\angle RSP = \angle SPR = 72^\circ$ . Since  $PQRS$  is a cyclic quadrilateral, we have that  $\angle PQR = 180^\circ - \angle RSP = 180^\circ - 72^\circ = 108^\circ$ .

Now note that  $\angle PQR + \angle RPQ + \angle QRP = 180^\circ$ . However,  $PQR$  is an isosceles triangle, so  $\angle RPQ = \angle QRP$ , and so we have that

$$2\angle RPQ + 108^\circ = \angle RPQ + \angle QRP + \angle PQR = 180^\circ,$$

and so  $\angle RPQ = 36^\circ$ .

3. Let  $n$  be a positive integer. Find the last digit of

$$n^{2018} + (n+1)^{2018} + \cdots + (n+99)^{2018}.$$

Let

$$S = n^{2018} + (n+1)^{2018} + \cdots + (n+99)^{2018}.$$

Note that among any 10 consecutive integers, we have one integer with each possible remainder modulo 10. We thus have that

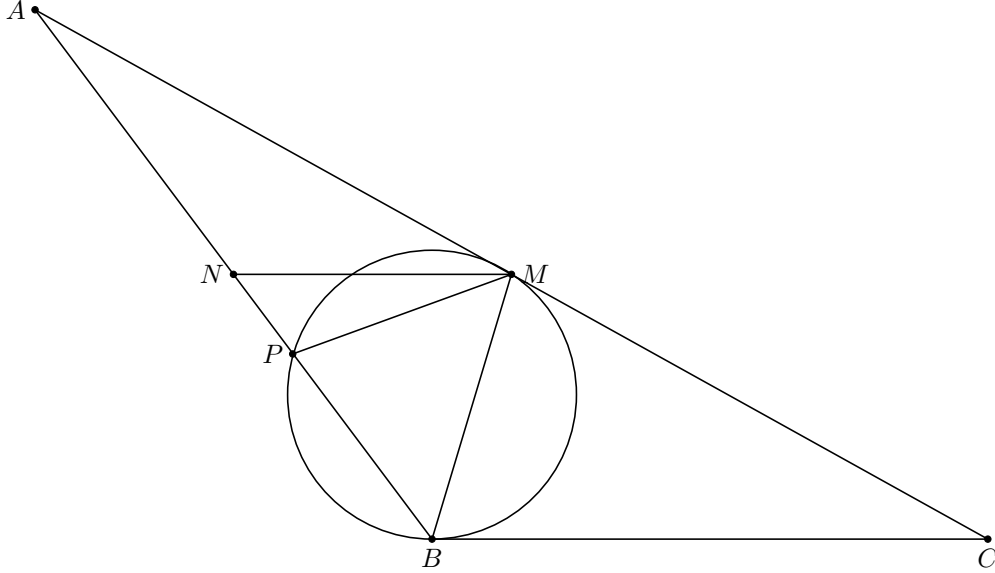
$$k^{2018} + (k+1)^{2018} + \cdots + (k+9)^{2018} \equiv 0^{2018} + 1^{2018} + \cdots + 9^{2018} \pmod{10}.$$

The numbers  $n, (n+1), (n+2), \dots, (n+99)$  consists of 10 groups of 10 consecutive positive integers, and so we have that

$$\begin{aligned} S &\equiv \underbrace{(0^{2018} + 1^{2018} + \cdots + 9^{2018}) + (0^{2018} + 1^{2018} + \cdots + 9^{2018}) + \cdots + (0^{2018} + 1^{2018} + \cdots + 9^{2018})}_{10 \text{ times}} \\ &\equiv 10 (0^{2018} + 1^{2018} + \cdots + 9^{2018}) \\ &\equiv 0 \pmod{10}, \end{aligned}$$

and so the last digit of  $S$  is always a 0.

4. Let  $ABC$  be a triangle, and let the midpoint of  $AC$  be  $M$ . The circle tangent to  $BC$  at  $B$  and passing through  $M$  meets the line  $AB$  again at  $P$ . Prove that  $AB \times BP = 2BM^2$ .



Construct point  $N$ , the midpoint of  $AB$ . By midpoint Th<sup>m</sup>,  $NM \parallel BC$ . Using this and Tan-Chord Th<sup>m</sup>, we get:  $\angle BMN = \angle MBC = \angle MPB$ . Since we also have that  $\angle PBM = \angle MBN$ , we have that  $\triangle BMN \sim \triangle BPM$ . So

$$\begin{aligned} \frac{BM}{BP} &= \frac{BN}{BM} \\ \Rightarrow BM^2 &= BN \cdot BP = \left(\frac{1}{2}AB\right) \cdot BP \end{aligned}$$

5. Show that for all positive real numbers  $x$  and  $y$ ,

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{x+y}{3}.$$

By the Cauchy-Schwarz inequality (in “Engel Form”) we have that

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{(x+y)^2}{(x+2y)+(2y+x)} = \frac{x+y}{3}.$$

:-... ,-----'~~',\_.. ,  
 '\_,,, ,\_ ;'~U'  
 \_,' ,,'\_--; ,--.  
 (\_/'~~ ,',',',(';