Beginner Test 3

Stellenbosch Camp 2018

Time: 4 hours

1. *Let AB be a chord in a circle with centre O, and let C be a point on the larger arc AB. Show that $\angle AOB = 2\angle ACB$.*

   Proven in lectures; refer to lecture notes.

2. *Factorise the following polynomial completely:*

$$(2x+3)^6 - (2x-1)^6.$$

   Using the difference of squares factorisation, we have that

$$(2x+3)^6 - (2x-1)^6 = \left((2x+3)^3 - (2x-1)^3\right)\left((2x+3)^3 + (2x-1)^3\right).$$

   Using the factorisations for differences of cubes, we obtain that

$$\begin{aligned}(2x+3)^3 - (2x-1)^3 &= ((2x+3)-(2x-1))\left((2x+3)^2 + (2x+3)(2x-1) + (2x-1)^2\right)\\ &= 4\left((4x^2+12x+9) + (4x^2+4x-3) + (4x^2-4x+1)\right)\\ &= 4(12x^2+12x+7).\end{aligned}$$

   Similarly using the sum of cubes factorisation, we have that

$$(2x+3)^3 + (2x-1)^3 = (4x+2)(4x^2+4x+13).$$

   The factorisation is thus $8(2x+1)(4x^2+4x+13)(12x^2+12x+7)$.

3. *How many different rearrangments are there of the word TARTAGLIA?*

   There are

$$\frac{9!}{2! \times 3!}$$

   arrangements.

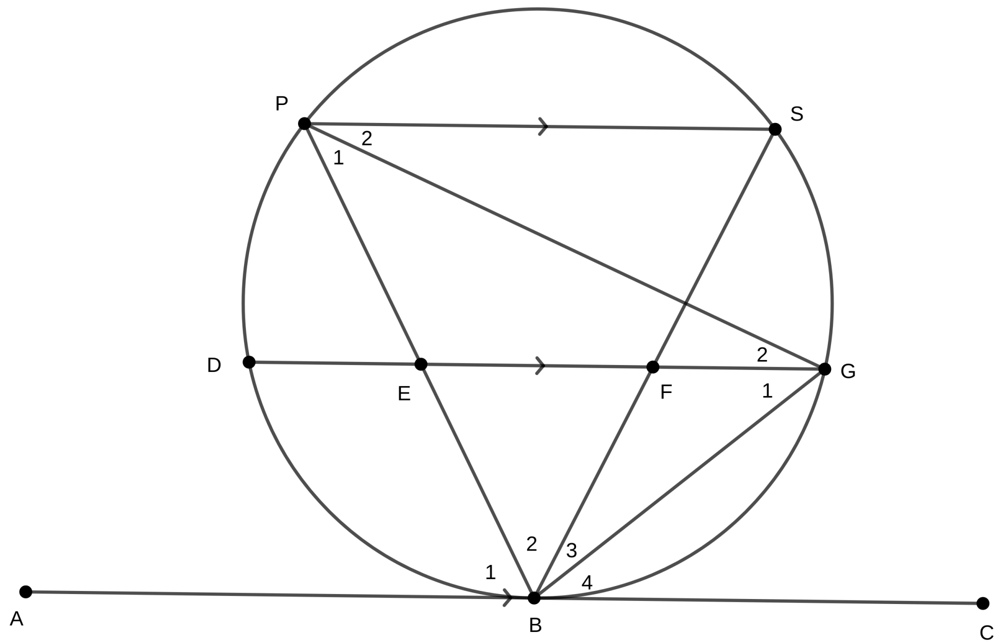
4. *In the figure ABC is a tangent to the circumscribed circle of $\triangle PBG$. PS and DG are both parallel to ABC. Chords BP and BS cut DG at E and F respectively. Prove that:*

   a. $\angle G_1 = \angle P_1$

   b. $\triangle BGP$ *is similar to* $\triangle BEG$

   c. $BG^2 = BP \times BE$

   d. $\frac{BG^2}{BP^2} = \frac{BF}{BS}$



a. We have that $G_1 = B_4$ (alternating angles) $= P_1$ (tan-chord).

b. Since $G_1 = P_1$ and $\angle GBP$ is common to both triangles, the two triangles have two pairs of angles equal and hence are similar.

c. Since $\triangle BGP$ is similar to $\triangle BEG$, we have that

$$\frac{BG}{BE} = \frac{BP}{BG}$$

which is equivalent to the desired result.

d. We know that

$$\frac{BG^2}{BP^2} = \frac{BP \times BE}{BP^2} = \frac{BE}{BP} = \frac{BF}{FS}$$

where the last equality follows because $\triangle BFE$ is similar to $\triangle BSP$.

5. *Consider a game wherein two players Emma and Dylan take turns to take between 1 and 7 stones, inclusive, from a pile which starts with 2018 stones. If Emma plays first, does one of the players have a winning strategy, and if so what is it?*

Emma has a winning strategy. On her first turn, she takes 2 stones. Thereafter, if Dylan takes n stones on his turn then Emma responds by taking $8 - n$ stones on her turn. The number of stones is then always a multiple of 8 after Emma's turn and is never a multiple of 8 after Dylan's turn. In particular, the number of stones can never be 0 after Dylan's turn. Since the game must eventually end, can not end in a draw, and can not be won by Dylan, it must be Emma who wins.

6. *Determine all solutions (x, y) of the system of equations*

$$\frac{4}{x} + \frac{5}{y^2} = 12,$$
$$\frac{3}{x} + \frac{7}{y^2} = 22.$$

Subtracting 3 times the first equation from 4 times the second gives us that

$$\frac{13}{y^2} = 52$$

and so $y^2 = 1/4$. Substituting this back into the equation gives us that $4/x = 12 - 20 = -8$ and so $x = -1/2$. The solutions are thus $(x, y) = (-1/2, -1/2)$ and $(x, y) = (-1/2, 1/2)$.

7. *Suppose k is a positive integer that does not divide 2008. Let $[x]$ denote the greatest integer less than or equal to x. For example, $[11.75] = 11$ and $[\pi] = 3$. What is the maximum possible value of $k \times \left[\frac{2018}{k}\right]$?*

   Let $2018 = kq + r$ where $0 \leq r < k$. Since k does not divide 2018, we have that $0 < r$. Then we have that

   $$\left[\frac{2018}{k}\right] = \left[\frac{kq + r}{k}\right] = \left[q + \frac{r}{k}\right] = q.$$

   We thus want to find the largest possible value of $kq = 2018 - r$. This corresponds to the smallest possible value of r, which is equal to 1, and so the maximum possible value of

   $$k\left[\frac{2018}{k}\right]$$

   is 2017. Equality occurs for any k such that 2018 leaves a remainder of 1 when divided by k, for example $k = 2017$.

8. *The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost 3 cents per piece. Thus, it costs 3 cents to number locker 9 and 6 cents to number locker 42. If it costs R206.91 to label all the lockers, how many lockers are there at the school?*

   We claim there are 2001 lockers. Indeed, the total cost comes to:

   $$3c \cdot (9 - 0) + 6c \cdot (99 - 9) + 9c \cdot (999 - 99) + 12c \cdot (2001 - 999) = 20691c$$

9. *Consider the function $f(x) = \frac{1}{1-x}$ and its iterates f^r defined as*

   $$\begin{aligned} f^1(x) &= f(x) \\ f^2(x) &= f(f(x)) \\ f^3(x) &= f(f(f(x))) \\ f^4(x) &= f(f(f(f(x)))), \end{aligned}$$

   and so on. Calculate the value of $f^{2018}(2018)$.

   Note that

   $$f^1(x) = \frac{1}{1-x},$$

   $$f^2(x) = \frac{1}{1 - \frac{1}{1-x}} = \frac{x-1}{x}$$

   and

   $$f^3(x) = f^2(f(x)) = \frac{\frac{1}{1-x} - 1}{\frac{1}{1-x}} = x$$

We thus have that

$$f^{2018} = f^3(f^{2015}(x)) = f^{2015}(x) = f^3(f^{2012}(x)) = f^{2012}(x) = \cdots = f^2(x) = \frac{x-1}{x}$$

and so

$$f^{2018}(2018) = \frac{2018-1}{2018} = \frac{2017}{2018}.$$

10. *Given the equation* $x^{2018} = y^x$,

   (a) find all pairs (x, y) *of solutions with* x *prime and* y *a positive integer;*

   (b) find all pairs (x, y) *of positive integers satisfying the equation.*

   (a) If x is prime, then the unique prime factorisation of y can only consist of the prime x. Thus $y = x^k$ for some $k \geq 1$. This gives the equation

   $$x^{2018} = x^{kx}$$

   which has solutions when $2018 = kx$. We note the prime factorisation of 2018 is $2018 = 2 \cdot 1009$, which therefore yields two solutions: $(x, y) = (2, 2^{1009})$ and $(x, y) = (1009, 1009^2)$.

   (b) Note that $(x, y) = (1, 1)$ yields a valid solution. Now, assume $x, y \geq 2$. Thus $x = a^m$ and $y = a^n$ where $a \geq 2$ and $m, n \geq 1$. Substituting this into the equation gives:

   $$2018m = a^m \cdot n$$

   Considering the case $m = 1$, yields only one additional solution: $(x, y) = (2018, 2018)$. The case $m = 2$ yields $(x, y) = (2^2, 2^{1009})$. Note that $m \geq 3$ yields no further solutions, as the exponent grows faster than $2018m$. Thus the solutions are

   $$(x, y) \in \{(1, 1), (2, 2^{1009}), (1009, 1009^2), (2018, 2018), (2^2, 2^{1009})\}$$

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          |\_/|            D\___/\
          (0_0)             (0_o)
         ==(Y)==             (V)
----------(u)---(u)----oOo--U--oOo---
__|_______|_______|_______|_______|___
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