Senior Test 2

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Find all positive integers m, n such that

$$1 + 5 \cdot 2^m = n^2$$

2. Show that if we are given 50 segments on the real line, then either there are 8 of them which are pairwise disjoint or 8 of them with a common point.

   A segment is defined as a closed interval $[a, b] = \{x \in \mathbb{R} \ : \ a \leq x \leq b\}$.

3. Let $a_0 = a_1 = 1$ and $a_{n+1} = 7a_n - a_{n-1} - 2$ for all positive integers n. Prove that a_n is a perfect square for all n.

4. Let ABC be triangle with $AB = AC$. A circle Γ lies outside triangle ABC and is tangent to line AC at C. Point D lies on Γ such that the circumcircle of triangle ABD is internally tangent to Γ. Segment AD meets Γ again at E. Prove that BE is tangent to Γ.

5. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

   for all $x, y \in \mathbb{Q}$.

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