Intermediate Test 4 – Solutions

Stellenbosch Camp 2018

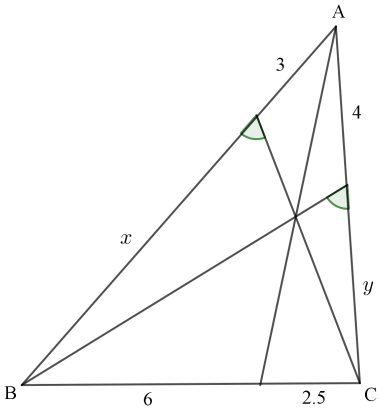
1. *How many numbers from* 1 *to* 2018 *inclusive can be written as the difference of two perfect squares?*

   We say that a number is *nice* if it can be written as the difference of two perfect squares. Any odd number is nice, since $2k+1 = (k+1)^2 - k^2$. Also, any number divisible by 4 is nice since $4k = (k+1)^2 - (k-1)^2$. So looking at remainders modulo 4, any number with remainder 0, 1 or 3 is nice; on the other hand, since all perfect squares have remainder 0 or 1 modulo 4, the only possibilities for the remainder of a difference of two squares are 0, 3 and 1; hence the numbers with these remainders are precisely the nice numbers.

   So to count the nice numbers from 1 to 2018, we take the total amount of numbers (2018) and subtract those with remainder 2; since 2018 has a remainder of 2 modulo 4, the amount of the latter is $\left\lfloor \frac{2018}{4} \right\rfloor + 1$, yielding a final answer of

$$2018 - \left\lfloor \frac{2018}{4} \right\rfloor - 1 = 1513.$$

2. *Solve for lengths x and y in the following diagram:*



3. *Find all functions $f : \mathbb{R} - > \mathbb{R}$ such that for all real numbers x,*

$$2f(x) + 3f(1-x) = x - 4x^3.$$

   We take the original equation: $2f(x) + 3f(1-x) = x - 4x^3$. We also take the original equation with x replaced by $1-x$: $2f(1-x) + 3f(x) = (1-x) - 4(1-x)^3 = -3 + 11x - 12x^2 + 4x^3$. Now we multiply the first equation by 2, the second by 3, and take the difference between the two, yielding

$$4f(x) + 6f(1-x) - 6f(1-x) - 9f(x) = 2x - 8x^3 + 9 - 33x + 36x^2 - 12x^3 = 9 - 31x + 36x^2 - 20x^3.$$

   Plugging this back into the original equation, we see that this function satisfies our condition.

4. *Prove that it is impossible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$.*

5. *An exam with k questions is presented to n students. A student fails the exam if they get less than half the answers right. We say that a question is easy if more than half of the students get it right. Decide if it is possible that*

   (a) All students fail even though all the questions were easy.

   (b) No student fails even though no question was easy.