Senior Test 1

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Prove that $m + n \leq \gcd(m, n) + \operatorname{lcm}(m, n)$ for all positive integers m, n. When does equality occur?

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(f(x + y)) = x + f(y)$$
for all $x, y \in \mathbb{R}$.

3. Let ABC be an acute angled triangle. The circle with diameter AB intersects the altitude from C at M and N. The circle with diameter AC intersects the altitude from B at P and Q. Prove M, N, P, and Q all lie on a common circle.

4. Let n be a positive integer and define $S_n = \{1, 2, 3, \ldots, n\}$. We denote a non-empty subset T of S_n as *balanced* if the median of T is equal to the average of T. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.

   The median of a subset T is defined as follows: Let $T = \{a_1, a_2, \ldots, a_k\}$ with the elements listed in increasing order $a_1 < a_2 < \cdots < a_k$. If k is odd, then the median of T is the element $a_{(k+1)/2}$. Otherwise, if k is even, then the median is the average of $a_{k/2}$ and $a_{k/2+1}$.

5. Set $S = \{1, 2, 3, ..., 2018\}$. Let n be a positive integer such that, among any n pairwise coprime numbers in S, there exists at least a prime number. Find the minimum value of n.

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