

Senior Problem Pool

Algebra

1. Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x^2) - yf(y) = f(x+y)(f(x) - y)$ for all real numbers x and y .
2. Suppose that a , b , and c are real numbers such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a+b+c}.$$

Show that for every odd natural number n , we have that

$$\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n} = \frac{1}{(a+b+c)^n}.$$

3. Suppose that f is a function from \mathbb{N} to \mathbb{N} such that $f(n) \geq 2$ for all n , and

$$f(n) + f(n+2) = f(n+4)f(n+6) - 1997$$

for all $n \in \mathbb{N}$.

- (a) Find $f(1997)$ and $f(1999)$ if $f(1) = 2$.
 - (b) Describe all functions which satisfy the conditions above.
4. Find all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$f(a^3) + f(b^3) + f(c^3) + 3f(a+b)f(b+c)f(c+a) = f(a+b+c)^3$$

for all $a, b, c \in \mathbb{Z}$.

Combinatorics

Geometry

Number Theory

1. Suppose that $2n+1$ and $3n+1$ are both squares. Show that 40 divides n .
2. Suppose that a , b , and N are relatively prime integers. Show that there is an integer m such that a and $b+mN$ are relatively prime.
3. Find all natural numbers n such that there exist n distinct natural numbers a_1, a_2, \dots, a_n such that $a_i \mid a_{a_i+1}$ for $i = 1, 2, \dots, n-1$, and such that $a_n = 2018n$.

Miscellaneous