

Senior February Monthly Problem Set

Due: 22 February 2018

1. Let n be a positive integer. Both n and n^2 only contain the digits 1, 2 and 3 (not necessarily all of them). Determine all possible values of n .
2. Given a (not necessarily convex) quadrilateral $ABCD$, call a point P in the same plane as $ABCD$ an *areal centre* for $ABCD$, if any line through P divides $ABCD$ into two parts of equal area. What are necessary and sufficient conditions on $ABCD$ for it to possess an areal centre?
3. The (English language version of the) game of ScrabbleTM consists of 100 tiles, each containing either a letter from A to Z (some letters occur more than once), except for two blank tiles; see **the relevant Wikipedia page** for the exact distribution of multiplicities of each letter.
In a solo game of Scrabble, the player starts by choosing seven tiles from the 100 available tiles at random. What is the probability that the player does not pick up any vowels?
- 4.
5. Call a positive integer a *triangular* number if it is of the form $1 + 2 + 3 + \cdots + k$ for some positive integer k , and *pentagonal* if it is of the form $1 + 4 + 7 + 10 + 13 + \cdots + (3n - 2)$ for some positive integer n . Prove that there are infinitely many cases where the product of two consecutive pentagonal numbers is equal to the product of two consecutive triangular numbers.
6. Let ABC be a triangle with circumcentre O . Let D be the point of intersection between the bisector of $\angle ABC$ and the perpendicular bisector of AB . Let the circumcircle of ADO be ω . Let $E \neq A$ be the intersection of ω with the segment AB . Let $P \neq E$ be the intersection of the circumcircle of COE with the line AB . Prove that CP is tangent to ω .
7. Call a function $f : \mathbb{N} \rightarrow \mathbb{N}$ *almost linear* if $f(m + n) - f(m) - f(n)$ only takes finitely many values as m and n vary through the natural numbers. Suppose that $f : \mathbb{N} \rightarrow \mathbb{N}$ and $g : \mathbb{N} \rightarrow \mathbb{N}$ are almost linear. Show that $f(g(n)) - g(f(n))$ only takes finitely many values as n varies through the natural numbers.
- 8.

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.