# Stellenbosch Camp 2018: Senior Geometry
# Lecture 2: Inversion Basics

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**Definition** (Circle Inversion)**.** *Let $\Omega$ be a circle with circumcentre $O$ and radius $r$. We say that $T$ is an inversion in the circle $\Omega$ if for any point $A \neq O$ in the plane the following is true:*

1. *$T$ is a map that sends $A$ to a point $A'$*
2. *$A'$ is on the ray $OA$*
3. *$OA \cdot OA' = r^2$*

*We say that $A'$ is the image of $A$.*

**Notation.** *When referencing a circle inversion called $T$, in a circle $\Omega$, with centre $O$ and radius $r$. There are a few accepted notations for short:*

- $T(\Omega)$
- $T(O, r)$

**Theorem 1.** *If $A'$ is the image of $A$ with respect to a circle $\Omega$, then $A$ is the image of $A'$ with respect to $\Omega$.*

**Theorem 2.** *Let $A$ be a point on the circumference of a circle $\Omega$. Then $A$ is the image of $A$ with respect to the circle $\Omega$.*

**Theorem 3.** *Let $\Omega$ be a circle, and let $A$,$B$ be 2 points. Let $A'$ and $B'$ be the images of $A$ and $B$, with respect to $\Omega$, respectively. Then $A$, $A'$, $B$, and $B'$ all lie on a common circle.*

**Definition** (Inverting a Set of Points)**.** *Let $S$ be a set of points (E.g. a line, a circle, etc). Then the image of $S$ with respect to a circle $\Omega$, denoted $S'$, is the set containing the image of every point in $S$.*
*In symbols:*

$$S' := \{P' \in \overrightarrow{OP} : OP \cdot OP' = r^2, \forall P \in S\}$$

**Theorem 4.** *Let $\Omega$ be a circle with circumcentre $O$, and let $\ell$ be a line not passing through $O$. Then the image of $\ell$ with respect to $\Omega$ is a circle passing through $O$.*
*Conversely, if $\Gamma$ is a circle passing through $O$, then the image of $\Gamma$ with respect to $\Omega$ is a line not passing through $O$.*

**Theorem 5.** *Let $\Omega$ be a circle with circumcentre $O$, and let $\Gamma$ be a circle not passing through $O$. Then the image of $\Gamma$ with respect to $\Omega$ is also a circle not passing through $O$.*

**Theorem 6.** *Let $\Omega$ be a circle with circumcentre $O$, and let $\ell$ be a line passing through $O$. Then the image of $\ell$ with respect to $\Omega$ is itself.*

**Theorem 7.** *Let $\Omega$ be a circle with circumcentre $O$, and let $\Gamma$ be a circle that is orthogonal to $\Omega$ (intersecting at right angles). Then the image of $\Gamma$ with respect to $\Omega$ is itself.*

**Theorem 8.** *Let $T(O, r)$ be an inversion, and let $\alpha$ and $\beta$ be 2 circles in the plane. Then the angles between the intersections of of the circles is invariant under $T$.*