Senior Test 4

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Prove that it is impossible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n, the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$.

2. An exam with k questions is presented to n students. A student fails the exam if they get less than half the answers right. We say that a question is easy if more than half of the students get it right. Decide if it is possible that

   (a) All students fail even though all the questions were easy.

   (b) No student fails even though no question was easy.

3. Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$ and 3, suppose that B_n is the midpoint of A_nA_{n+1} and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $\triangle D_1D_2D_3$ to the area of triangle $\triangle E_1E_2E_3$.

4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that
$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca)$$

5. Let $p_1 = 2$ and define a sequence of prime numbers $p_1, p_2, p_3, \ldots$ such that, for all positive integers n, p_{n+1} is the least prime factor of $n \cdot p_1^{1!} \cdot p_2^{2!} \ldots p_n^{n!} + 1$. Prove that all primes appear in the sequence.

```
        |\_/|
        | @ @   Woof!
        |   <>              _
        |  _/\------____ ((| |))
        |               `--' |
    ____|_       ___|   |___.'
   /_/_____/____/_______|
```