2018 Stellenbosch Mathematics Camp
Senior Number Theory Problem Set

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1 Diophantine equations

1. Solve for positive integers m, n:
$$1 + 5 \cdot 2^m = n^2$$

2. Solve for $m, n \in \mathbb{Z}$:
$$9m^2 + 3n = n^2 + 8$$

3. Solve for $a, d \in \mathbb{Z}$:
$$a^4 + 120a = d^2$$

4. Solve for $x, y \in \mathbb{Z}$:
$$7^x + x^4 + 47 = y^2$$

5. Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

6. Find all positive integers a, b such that $a! + b! = a^b + b^a$.

7. Solve for $x, y, z \in \mathbb{Z}$:
$$2(x^2 + y^2 + z^2) = (x - y)^3 + (y - z)^3 + (z - x)^3$$

2 Divisibility

1. Prove that n does not divide $2^n - 1$ for all positive integers $n > 1$.

2. Prove that n does not divide $2^{n-1} + 1$ for all positive integers $n > 1$.

3. Let p be an odd prime. Prove that $(p - 1)^p + 1$ is divisible by p^2, but not divisible by p^3.

4. Prove that for each prime number p and positive integer n, p^n divides
$$\binom{p^n}{p} - p^{n-1}$$

5. If positive integers a, b, c are such that b divides a^3, c divides b^3, and a divides c^3, prove that abc divides $(a + b + c)^{13}$.

3 Sequences

1. Suppose $a_1, a_2, \ldots$ is an infinite strictly increasing sequence of positive integers and $p_1, p_2, \ldots$ is a sequence of distinct primes such that $p_n \mid a_n$ for all $n \geq 1$ and such that $a_n - a_k = p_n - p_k$ for all $n, k \geq 1$. Prove that the sequence (a_n) consists only of prime numbers.

2. Show that there exists an infinite arithmetic progression of natural numbers such that the first term is 16 and the number of positive divisors of each term is divisible by 5. Of all such sequences, find the one with the smallest possible positive common difference.

3. Let k be a positive integer and let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers which satisfies
$$\sum_{d|n} a_d = k^n$$
for all $n \geq 1$. Prove that n divides a_n for all $n \geq 1$.

4. Let $a_0 = a_1 = 1$ and $a_{n+1} = 7a_n - a_{n-1} - 2$ for all positive integers n. Prove that a_n is a perfect square for all n.

5. Let $p_1 = 2$ and define a sequence of prime numbers $p_1, p_2, p_3, \ldots$ such that, for all positive integers n, p_{n+1} is the least prime factor of $n \cdot p_1^{1!} \cdot p_2^{2!} \ldots p_n^{n!} + 1$. Prove that all primes appear in the sequence.

4 Miscellaneous

1. Let
$$E(x, y) = \frac{x}{y} + \frac{x+1}{y+1} + \frac{x+2}{y+2}.$$
   (a) Find all integers $x, y \in \mathbb{Z}$ such that $E(x, y) = 3$.
   (b) Prove that there are infinitely many natural numbers n such that$E(x, y) = n$ has at least one solution in $x, y \in \mathbb{Z}$

2. Prove that $m + n \leq \gcd(m, n) + \operatorname{lcm}(m, n)$ for all positive integers m, n. When does equality occur?

3. Let $a < b$ be natural numbers such that for all prime numbers $p > b$, at least one of a and b divides $p - 1$. Prove that $a \leq 2$.

4. Let $a = 222\ldots2$ where the digit 2 is denoted 2018 times. Prove that there are no positive integers x, y such that $a = xy(x + y)$.

5. If $f : \mathbb{N} \to \mathbb{R}$ is a function such that
$$\prod_{d|n} f(d) = 2^n$$
holds for all $n \in N$, show that f sends $\mathbb{N}$ to $\mathbb{N}$.

6. Find all positive integers $n \geq 2$ such that $n^{n-1} - 1$ is square-free.

7. Let m and n be two integers such that both the quadratic equations $x^2 + mx - n = 0$ and $x^2 - mx + n = 0$ have integer roots. Prove that n is divisible by 6.

8. Show that for any positive integers a and b, $(36a+b)(a+36b)$ cannot be a power of 2.

9. Denote by $a \bmod b$ the remainder of the euclidean division of a by b. Determine all pairs of positive integers (a, p) such that p is prime and

$$a \bmod p + a \bmod 2p + a \bmod 3p + a \bmod 4p = a + p.$$

10. Prove that there exist infinitely many even positive integers k such that for every prime p the number $p^2 + k$ is composite.

11. An integer $n > 1$ and a prime p are such that n divides $p-1$, and p divides n^3-1. Prove that $4p-3$ is a perfect square.

12. Let p be an odd prime. Find all primes p for which the quotient

$$\frac{2^{p-1}-1}{p}$$

is a square.

13. Set $S = \{1, 2, 3, ..., 2018\}$. If among any n pairwise coprime numbers in S there exists at least a prime number, find the minimum of n.

14. Let m and n be given positive integers such that mn divides $m^2 + n^2 + m$. Prove that m is a square of an integer.

15. Let a, b be two positive integers such that $\gcd(a, b) = 1$. Prove that

$$a^{\phi(b)} + b^{\phi(a)} \equiv 1 \pmod{ab}$$

16. Determine all prime numbers p such that p is the sum of all primes less than p.

17. Let p be a prime number. Prove that $2^p + 3^p$ cannot be non-trivial perfect power (i.e. a positive integer of the form a^b where $b > 1$).

18. Let n be a positive integer and define $S_n = \{1, 2, 3, \ldots, n\}$. We denote a non-empty subset T of S_n as *balanced* if the median of T is equal to the average of T. For each $n \geq 1$, prove that the number of balanced subsets of S_n is odd.

19. Let n be a positive integer and let k be an odd positive integer. Moreover, let a, b and c be integers (not necessarily positive) satisfying the equation

$$a^n + kb = b^n + kc = c^n + ka$$

Prove that $a = b = c$.