Intermediate Test 4 – Solutions

Stellenbosch Camp 2018

1. We claim there are 2001 lockers. Indeed, the total cost comes to:

$$3c \cdot (9 - 0) + 6c \cdot (99 - 9) + 9c \cdot (999 - 99) + 12c \cdot (2001 - 999) = 20691c$$

2. (a) If x is prime, then the unique prime factorisation of y can only consist of the prime x. Thus $y = x^k$ for some $k \geq 1$. This gives the equation

$$x^{2018} = x^{kx}$$

which has solutions when $2018 = kx$. We note the prime factorisation of 2018 is $2018 = 2 \cdot 1009$, which therefore yields two solutions: $(x, y) = (2, 2^{1009})$ and $(x, y) = (1009, 1009^2)$.

   (b) Note that $(x, y) = (1, 1)$ yields a valid solution. Now, assume $x, y \geq 2$. Thus $x = a^m$ and $y = a^n$ where $a \geq 2$ and $m, n \geq 1$. Substituting this into the equation gives:

$$2018m = a^m \cdot n$$

Considering the case $m = 1$, yields only one additional solution: $(x, y) = (2018, 2018)$. The case $m = 2$ yields $(x, y) = (2^2, 2^{1009})$. Note that $m \geq 3$ yields no further solutions, as the exponent grows faster than $2018m$. Thus the solutions are

$$(x, y) \in \{(1, 1), (2, 2^{1009}), (1009, 1009^2), (2018, 2018), (2^2, 2^{1009})\}$$

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2. *Consider two circles Γ_1 and Γ_2 that intersect at points A and B. Let l be a line tangent to circles Γ_1 and Γ_2 at S and T, respectively. Lines AB and ST intersect at point M. Furthermore line BT intersect circle Γ_1 again at point R. Let the intersection of MR and SB be X and the intersection of TX and RS be C. Prove that CB and ST are parallel.*

   First notice that from Power of a Point theorem with M as point with respect to circle Γ_1 we have that $MS^2 = MB \times MA$. Similarly, $MT^2 = MB \times MA$. Thus we have that $MS = MT$. Furthermore $\triangle RST$ has three concurrent cevians RM, SB and TC intersecting at X. Thus from Ceva's Theorem we have that

$$\frac{RC}{CS} \times \frac{SM}{MT} \times \frac{TB}{BR} = 1$$

   Since $\frac{SM}{MT} = 1$, this simplifies to $\frac{RC}{CS} = \frac{BR}{BT}$. From the triangle proportionality theorem, it is clear that CB is parallel to ST.

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