

Stellenbosch Camp December 2018
Senior Test 1
Solutions

1. Let $d = \gcd(m, n)$, and denote $m = da$ and $n = db$ where a, b are two positive integers. We note that $\text{lcm}(m, n) = mn/d$. Since $a, b \geq 1$, we note

$$\begin{aligned} (a-1)(b-1) &\geq 0 \\ \iff a+b &\leq 1+ab \\ \iff da+db &\leq d+dab \\ \iff m+n &\leq \gcd(m, n) + \text{lcm}(m, n) \end{aligned}$$

which proves the problem statement. Note that equality occurs only if either $a = 1$ or $b = 1$, which occurs iff m divides n or n divides m . \square

2. We first prove injectivity. That is, if $f(a) = f(b)$ for some $a, b \in \mathbb{R}$, then $a = b$. We have

$$\begin{aligned} f(a) &= f(b) \\ \implies f(f(a+0)) &= f(f(b+0)) \\ \implies a+f(0) &= b+f(0) \\ \implies a &= b \end{aligned}$$

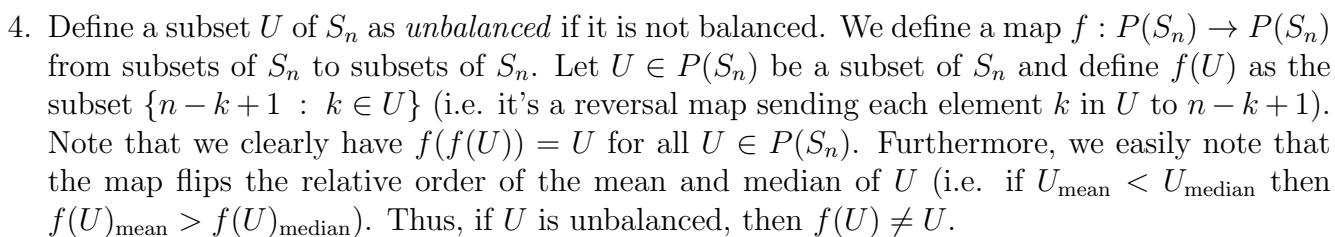
thus proving injectivity. Now substituting $y = 0$ yields $f(f(x)) = f(x)$, which, by injectivity, implies $f(x) = x$ as the only solution. One easily checks that this solution satisfies the functional equation. \square

3. Let E and F be the feet of the perpendiculars from B and C respectively.

Note, since AB and AC are diameters, the angles $\angle BNA = \angle CPA = 90^\circ$. Since we have $\angle NFA = \angle AEP = 90^\circ$, we get that AN is tangent to the circumcircle of $\triangle NFB$, and AP is tangent to the circumcircle of $\triangle PEC$. Lastly, we know that $BFEC$ is a cyclic quadrilateral. So by using power of a point in the circles NFB , PEC , and $BFEC$, we obtain

$$\begin{aligned} AN^2 &= BA \cdot AF = CA \cdot AE = AP^2 \\ AN &= AP \end{aligned}$$

But we already have, $AN = AM$, and $AP = AQ$. Thus, $AN = AM = AP = AQ$. So M , N , P , and Q are all equidistant from A , and thus lie on a common circle centred at A . \square



5. Let $n = 15$. Note that the set $\{1, 2^2, 3^2, 5^2, 7^2, \dots, 43^2\}$ consisting of 1 and the squares of the first 14 primes is a pairwise coprime set not containing any prime. Thus, if n satisfies the problem condition, then $n \geq 16$. We now prove that amongst any 16 pairwise coprime elements, at least one is prime.

Let $A = \{a_1, a_2, \dots, a_n\}$ be a set with $n \geq 16$ pairwise coprime elements, and order it in order of smallest prime dividing each element (if 1 is included, let $a_1 = 1$). Let p_i be the smallest prime dividing a_i . As A is pairwise coprime, we have $p_2 < p_3 < p_4 < \dots < p_n$. Thus, as $p_2 \geq 2$, we have $p_n \geq 47$ as 47 is the 15th smallest prime. Now assuming that a_n is not prime, we have $a_n \geq p_n^2 \geq 2209 > 2018$ which yields a contradiction. Thus a_n is prime, which proves $n = 16$ is the minimal n satisfying the problem condition. \square