Senior January Monthly Problem Set

Due: 18 January 2019

1. *P, Q and R are any points on BC, CA and AB respectively of a triangle ABC. Let the centres of the circumcircles AQR, BRP and CPQ be X, Y and Z. Prove that triangles XYZ and PQR are similar.*

2. *Steve determines the geometric mean of two positive integers in the following way:*

   (a) He writes them down in their decimal representation, one below the other, and prepends zeros to the smaller number (if applicable) such that their lengths are equal.

   (b) He determines the geometric mean of each pair of digits below each other. If the result is not an integer, only the integer part is used.

   (c) The digits determined by this procedure form the result.

   Determine all pairs (a, b) of positive integers for which Steve's procedure yields the correct result. (For example, one such pair is $(12; 48)$.)

   Let

$$a = \sum_{k=0}^{n} a_k 10^k \quad \text{and} \quad b = \sum_{k=0}^{n} b_k 10^k$$

   be the decimal expansions of a and b, possibly with leading zeros. Then Steve's procedure for calculating the geometric mean gives us

$$\sum_{k=0}^{n} \left\lfloor \sqrt{a_k b_k} \right\rfloor 10^k.$$

   We thus require that

$$\left(\sum_{k=0}^{n} \left\lfloor \sqrt{a_k b_k} \right\rfloor 10^k \right)^2 = ab = \left(\sum_{k=0}^{n} a_k 10^k \right) \left(\sum_{k=0}^{n} b_k 10^k \right).$$

   But we have that

$$\left(\sum_{k=0}^{n} \left\lfloor \sqrt{a_k b_k} \right\rfloor 10^k \right)^2 \leq \left(\sum_{k=0}^{n} \sqrt{a_k b_k} 10^k \right)^2 \leq \left(\sum_{k=0}^{n} a_k 10^k \right) \left(\sum_{k=0}^{n} b_k 10^k \right)$$

   by the Cauchy-Schwarz inequality. Thus equality occurs if and only if

$$\lfloor \sqrt{a_k b_k} \rfloor = \sqrt{a_k b_k}$$

   for each k, and equality occurs in the application of the Cauchy-Schwarz inequality.

   We thus require that $a_k b_k$ is a square for each k, and that either $a = 0$, or there is a real number λ such that $b_k = \lambda a_k$ for each k. If $a \neq 0$ then we have that $a_k b_k = \lambda a_k^2$ for each k, and so λ is the square of a rational number. Noting that b_k is an integer for each k and that the ration b_k / a_k is the square of a rational number for each k, Table 1 shows the allowed values of λ for each digit that may occur in a.

   Combining the above, we see that all solutions are given by the pairs (a, b) such that one of the following conditions hold:

   - a is any natural number and $b = a$.
   - a contains only the digits 0, 1 and 2 and $b = 4a$.
   - b contains only the digits 0, 1 and 2 and $a = 4b$.

Table 1: Allowed ratio between the digits of b and the digits of a. (Problem 2)

Digit	Ratio	Digit	Ratio	Digit	Ratio
0	Any	1	0, 1, 4, 9	2	0, 1, 4
3	0, 1	4	0, 1/4, 1, 9/4	5	0, 1
6	0, 1	7	0, 1	8	0, 1/4, 1
9	0, 1/9, 4/9, 1				

- a contains only the digits 0 and 1 and $b = 9a$.
- b contains only the digits 0 and 1 and $a = 9b$.
- a contains only the digits 0 and 4 and $b = 9a/4$.
- b contains only the digits 0 and 4 and $a = 9b/4$.

3.

(a) *Prove that if $p > 10$ is a prime number that divides $a^4 + a^3 + a^2 + a + 1$ for some integers a, then p's decimal expansion ends in a 1.*

(b) *For any prime p whose decimal expansion ends in a 1, and any positive integer k, prove that there exists an integer a such that p^k divides $a^4 + a^3 + a^2 + a + 1$.*

(a) Note that p divides $(a-1)(a^4 + a^3 + \cdots + 1) = a^5 - 1$. Thus the order of a modulo p divides 5, and hence is equal to 1 or 5. If the order of a modulo p is 1, then $a \equiv 1 \pmod{p}$, giving us that $0 \equiv a^4 + a^3 + \cdots + 1 \equiv 1 + 1 + \cdots + 1 \equiv 5 \pmod{p}$, and so $p = 5$, a contradiction. Thus the order of a modulo p is 5. Since $a^{p-1} \equiv 1 \pmod{p}$ by Fermat's little theorem, this implies that $5 \mid p - 1$. We thus have that $p \equiv 1 \pmod{5}$ and $p \equiv 1 \pmod{2}$ (since $p > 10$), and so $p \equiv 1 \pmod{10}$.

(b) Suppose that $p \equiv 1 \pmod{5}$. Then there is a natural number b such that the order of b modulo p is 5. (For example, if g is a primitive root modulo p, then $g^{(p-1)/5}$ is such a number.)
We thus have that $p \mid b^5 - 1$, but $p \nmid b - 1$. Let

$$a = b^{p^{k-1}}.$$

By the Lifting the Exponent Lemma, we have that p^k divides $a^5 - 1$. However, p does not divide $a - 1$, and so p^k divides

$$\frac{a^5 - 1}{a - 1} = a^4 + a^3 + \cdots + 1.$$

4. *The set S of nonnegative integers has the property that every nonnegative integer n can be uniquely written as $n = a + 2b$ where $a, b \in S$ are not necessarily distinct. How many elements of S are less than 2018?*

5. *Jacob has a balance scale and wishes to buy weights from Sipho. Sipho tells Jacob that he sells weights in the following way: Jacob has to specify a sequence of n integers $a_1, a_2, \ldots a_n$, and then Sipho will make 1 weight of mass a_1, two weights of mass a_2, etc., and n weights of mass a_n.*

   What is the largest k for which Jacob can specify some sequence $(a_1, \ldots, a_n)$ and still be able to measure every integral weight from 1 to k? (For example, with weights with mass 4 and 7, he can measure a weight of 3 by putting one weight on the one side and the other on the other side of the balance scale.)

   We claim for any n, the maximum k such that one can specify some sequence $(a_1, a_2, \ldots, a_n)$ and be able to measure every integral weight from 1 to k is

$$k_n = \frac{(2n+1)!! - 1}{2},$$

where $m!!$ denotes the product of the odd integers from 1 to m.

We first show that k_n is an upper bound for k. For some i, suppose that we put s of the weights with mass a_i on the left hand side of the scale, and t of the weights with mass a_i on the right hand side of the scale. If $s > t$, then this is equivalent to placing $s - t$ weights on the left hand side of the scale, and if $t > s$, then this is equivalent to placing $t - s$ of the weights on the right hand side of the scale. We can thus assume that for each i, all of the weights with mass a_i are placed on the same side of the scale.

We now count how many distinct weights we can measure. We note that there are $2i + 1$ options for how we place the weights with mass a_i on the scale: We can put any number from 1 to i on the left hand side of the scale, or we can put any number from 1 to i on the right hand side of the scale, or we can place no weights with mass a_i on the scale.

This implies that there are

$$\prod_{i=1}^{n}(2i+1) = (2n+1)!!$$

possible ways of placing the weights on the scale. However, we have counted each weight that we can measure twice. (Except for 0, which we measure by placing no weights on the scale.) This is because we can swap which side of the scale each of the weights which are on the scale is on. We thus see that we can measure at most

$$\frac{(2n+1)!! - 1}{2}$$

distinct non-zero weights. This number is therefore an upper bound for k, since if we can measure all of the integral weights from 1 to k, then we can measure at least k distinct non-zero weights.

We now show that it is possible to measure each weight from 1 to k_n with a suitable choice of the specified sequence $(a_1, a_2, \dots, a_n)$.

Let Jacob specify the sequence $(a_1, a_2, \dots, a_n)$ where $a_i = (2i - 1)!!$. We proceed by induction on n to show that he can then measure every integral weight from 1 to k_n. This is clearly true for $n = 1$. Suppose that it is true for some n. We show that it is true for $n + 1$ as well.

We first show that

$$k_n = 1 \cdot 1!! + 2 \cdot 3!! + \cdots + n \cdot (2n-1)!!$$

for all n. We note that $2i(2i-1)!! = (2i+1)(2i-1)!! - (2i-1)!! = (2i+1)!! - (2i-1)!!$. We thus have that

$$\begin{aligned} 1 + 2\sum_{i=1}^{n} i(2i-1)!! &= 1 + \sum_{i=1}^{n}\left((2i+1)!! - (2i-1)!!\right) \\ &= 1 + (2n+1)!! - 1!! = (2n+1)!! = 1 + 2k_n, \end{aligned}$$

proving our claim. This implies that for our specified sequence, we have that $a_{n+1} = 2k_n + 1$.

We now show that for each m with $0 \le m \le n + 1$, that Jacob can measure all of the positive integers from $ma_{n+1} - k_n$ to $ma_{n+1} + k_n$. For each integer i from $-k_n$ to k_n, the induction hypothesis tells us that Jacob can place the weights with masses $a_1, a_2, \dots, a_n$ on the scale in such a way that the difference between the total weight on the left hand side of the scale and the total weight on the right hand side of the scale is i. Jacob can then measure the weight $ma_{n+1} + i$ by placing m weights with mass a_{n+1} on the left hand side of the scale.

Since $(m+1)a_{n+1} - k_n - (ma_{n+1} + k_n) = a_{n+1} - 2k_n = 1$, this implies that Jacob can measure all of the positive integer weights from $0a_{n+1} - k_n$ to $(n+1)a_{n+1} + k_n$. We are thus finished if we have that $(n+1)a_{n+1} + k_n = k_{n+1}$. But we have that

$$k_{n+1} = \sum_{i=1}^{n+1} i(2i-1)!! = k_n + (n+1)(2n+1)!! = (n+1)a_{n+1} + k_n,$$

and we are done.

6. *Does there exist a natural number n such that*

$$1^{2018} + 2^{2018} + \cdots + n^{2018}$$

is prime?

We will prove that there is no such integer. Let

$$f(n) = 1^{2018} + 2^{2018} + \cdots + n^{2018}.$$

Suppose that p is a prime such that $p \mid n$. For any integer k such that $0 < k < (p-1)$, it is known that

$$1^k + 2^k + \cdots + p^k \equiv 0 \pmod{p}.$$

To show this, let g be a primitive root modulo p. Then we have that

$$1^k + 2^k + \cdots + (p-1)^k \equiv g^k + g^{2k} + \cdots + g^{(p-1)k} \equiv g^k \cdot \frac{g^{(p-1)k} - 1}{g^k - 1}.$$

Since p divides $g^{(p-1)k} - 1$, but p does not divide $g^k - 1$, we see that this is congruent to 0 modulo p.

Now using the division algorithm, we can write $2018 = q(p-1) + r$ where $0 \le r < (p-1)$.

If $r \neq 0$, then we have that

$$\begin{aligned}
&1^{2018} + 2^{2018} + \cdots + n^{2018} \\
&\quad \equiv \underbrace{\left(1^{2018} + 2^{2018} + \cdots + p^{2018}\right) + \cdots + \left(1^{2018} + 2^{2018} + \cdots + p^{2018}\right)}_{n/p \text{ times}} \\
&\quad \equiv \frac{n}{p}\left(1^r + 2^r + \cdots + p^r\right) \\
&\quad \equiv 0 \pmod{p}
\end{aligned}$$

and so $f(n)$ is divisible by p. Since we clearly have that $f(n) > p$, we see that $f(n)$ is not prime.

Thus if $f(n)$ is prime, and $p \mid n$, then we must have that $p - 1 \mid 2018$, and so $p - 1 \in \{1, 2, 1009, 2018\}$. Since p is prime, this implies that $p \in \{2, 3\}$.

Now suppose that p^2 divides n. Then a similar argument to earlier shows that

$$f(n) \equiv \frac{n}{p}\left(1^{2018} + 2^{2018} + \cdots + p^{2018}\right) \pmod{p}$$

and since n/p is divisible by p, we have that $f(n)$ is divisible by p and hence is not prime.

Thus we must have that n is square-free, and so $n \in \{1, 2, 3, 6\}$. Checking these values individually, we find that $f(1) = 1$ is not prime, $f(2) = 2^{2018} + 1$ is divisible by 5, $f(3)$ is even, and $f(6)$ is divisible by 7.

7. *Fix a natural number $n \ge 2$. Find the smallest constant C such that*

$$\sum_{1 \le i < j \le n} x_i x_j (3x_i^2 + x_j^2)(x_i^2 + 3x_j^2) \le C \left(\sum_{i=1}^{n} x_i \right)^6$$

for all non-negative real numbers $x_1, x_2, \ldots, x_n$. For this value of C, when does equality occur?

Equivalently, we wish to determine the maxiumum value of the function

$$f(x_1, x_2, \ldots, x_n) = \sum_{1\le i<j\le n} x_i x_j (3x_i^2 + x_j^2)(x_i^2 + 3x_j^2)$$

subject to the constraint $x_1 + x_2 + \cdots + x_n = 1$.

We claim that as long as there are at least three non-zero values among the x_i then it is possible to increase the value of f while maintaining the given constraint.

Suppose WLOG that $0 < x_1 \le x_2 \le x_3$. Consider the values x_i' where $x_1' = x_1 + x_2$, $x_2' = 0$, and $x_i' = x_i$ for $i \ge 3$. We claim that

$$f(x_1, x_2, \ldots, x_n) < f(x_1', x_2', \ldots, x_n')$$

The only terms in the sum defining $f(x_1, x_2, \ldots, x_n)$ which differ from those defining $f(x_1', x_2', \ldots x_n')$ are those for which either i or j is equal to 1 or 2, and so it is enough to prove that

$$\begin{aligned}&\sum_{k=2}^{n} x_1 x_k (3x_1^2 + x_k^2)(x_1^2 + 3x_k^2) + \sum_{k=3}^{n} x_2 x_k (3x_2^2 + x_k^2)(x_2^2 + 3x_k^2)\\ &< \sum_{k=2}^{n} x_1' x_k' (3{x_1'}^2 + {x_k'}^2)({x_1'}^2 + 3{x_k'}^2) + \sum_{k=3}^{n} x_2' x_k' (3{x_2'}^2 + {x_k'}^2)({x_2'}^2 + 3{x_k'}^2)\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\sum_{k=2}^{n} x_1 x_k (3x_1^2 + x_k^2)(x_1^2 + 3x_k^2) + \sum_{k=3}^{n} x_2 x_k (3x_2^2 + x_k^2)(x_2^2 + 3x_k^2)\\ &< \sum_{k=3}^{n} (x_1 + x_2) x_k (3(x_1 + x_2)^2 + x_k^2)((x_1 + x_2)^2 + x_k^2).\end{aligned}$$

For $k > 3$, we have that

$$x_1 x_k (3(x_1 + x_2)^2 + x_k^2)((x_1 + x_2)^2 + x_k^2) > x_1 x_k (3x_1^2 + x_k^2)(x_1^2 + 3x_k^2)$$

and similarly

$$x_2 x_k (3(x_1 + x_2)^2 + x_k^2)((x_1 + x_2)^2 + x_k^2) > x_2 x_k (3x_2^2 + x_k^2)(x_2^2 + 3x_k^2)$$

and thus

$$\begin{aligned}&\sum_{k=4}^{n} (x_1 + x_2) x_k (3(x_1 + x_2)^2 + x_k^2)((x_1 + x_2)^2 + x_k^2)\\ &> \sum_{k=4}^{n} x_1 x_k (3x_1^2 + x_k^2)(x_1^2 + 3x_k^2) + \sum_{k=4}^{n} x_2 x_k (3x_2^2 + x_k^2)(x_2^2 + 3x_k^2).\end{aligned}$$

It is thus enough to prove that

$$\begin{aligned}&(x_1 + x_2) x_3 (3(x_1 + x_2)^2 + x_3^2)((x_1 + x_2)^2 + 3x_3^2)\\ &> x_1 x_2 (3x_1^2 + x_2^2)(x_1^2 + 3x_2^2) + x_1 x_3 (3x_1^2 + x_3^2)(x_1^2 + 3x_3^2)\\ &+ x_2 x_3 (3x_2^2 + x_3^2)(x_2^2 + 3x_3^2)\end{aligned}$$

which simplifies to

$$\begin{aligned}&x_1 x_3 (3(x_1 + x_2)^4 + 10x_3^2 (x_1 + x_2)^2 + 3x_3^4 - 3x_1^4 - 10x_1^2 x_3^2 - 3x_3^4)\\ &+ x_2 x_3 (3(x_1 + x_2)^4 + 10x_3^2 (x_1 + x_2)^2 + 3x_3^4 - 3x_2^4 - 10x_2^2 x_3^2 - 3x_3^4)\\ &> x_1 x_2 (3x_1^4 + 10x_1^2 x_2^2 + 3x_2^4)\end{aligned}$$

Note that
$$3(x_1+x_2)^4 - 3x_1^4 > 3x_2^4$$
and
$$10x_3^2(x_1+x_2)^2 - 10x_1^2x_3^2 > 10x_2^2x_3^2 \geq 10x_1^2x_2^2$$
where in the last inequality we use that $x_2 \leq x_3$.

We thus have that
$$\begin{aligned}&x_1x_3(3(x_1+x_2)^4 + 10x_3^2(x_1+x_2)^2 + 3x_3^4 - 3x_1^4 - 10x_1^2x_3^2 - 3x_3^4)\\ &> x_1x_3(3x_2^4 + 10x_1^2x_2^2) \geq x_1x_2(3x_2^4 + 10x_1^2x_2^2).\end{aligned}$$

Similarly
$$\begin{aligned}&x_2x_3(3(x_1+x_2)^4 + 10x_3^2(x_1+x_2)^2 + 3x_3^4 - 3x_2^4 - 10x_2^2x_3^2 - 3x_3^4)\\ &> x_1x_2(3x_1^4 + 10x_1^2x_2^2)\end{aligned}$$
and the result follows.

Now for any set of values $x_1, x_2, \ldots, x_n$ such that $x_1 + \cdots + x_n = 1$, we can apply the above procedure to increase the value of $f(x_1, x_2, \ldots, x_n)$. The procedure increases the number of the x_i which are equal to 0 on each step, and so after a finite number of applications of the procedure, we obtain new values for the x_i such that at most two of the x_i are non-zero, and such that f evaluated at the new values of x_i is strictly larger than the original value of f.

It is thus enough to consider the case where at most two of the x_i are non-zero. We thus wish to find the smallest constant C such that
$$xy(3x^2+y)(x^2+3y^2) < C(x+y)^6$$
for all non-negative reals x and y. We claim that the best such value for C is $1/4$. Notice that if $x = y$, then we have equality, so we need only show that
$$(x+y)^6 \geq 4xy(3x^2+y^2)(x^2+3y^2)$$
and determine when equality holds.

But
$$(x+y)^6 \geq 4xy(3x^2+y^2)(x^2+3y^2)$$
is equivalent to
$$(x-y)^6 \geq 0$$
and so we see that the inequality does hold for $C = 1/4$, and that equality occurs if and only if $x = y$. Since equality does occur, $C = 1/4$ is the best possible constant. In terms of the x_i, equality holds if and only if two of the x_i are equal, and the rest are 0.

8. *Let ABC be a triangle with circumcircle Ω. Let P, Q be 2 points not on Ω such that the line PQ passes through the centre of Ω. Let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, and let X, Y, Z be the feet of the perpendiculars from Q onto the same sides, respectively. Prove that the perpendiculars from D, E, F to YZ, ZX, XY are concurrent.*

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.