Senior February Monthly Problem Set

Solutions

1. *Let n be a positive integer. Both n and n^2 only contain the digits 1, 2 and 3 (not necessarily all of them). Determine all possible values of n.*

   If $n \equiv 2 \text{ or } 3 \pmod{10}$, then we have that $n^2 \equiv 4 \text{ or } 9 \pmod{10}$, and so n is not a solution. We thus require that $n \equiv 1 \pmod{10}$.

   If $n > 10$, then we have that $n \equiv 11 \text{ or } 21 \text{ or } 31 \pmod{100}$. If $n \equiv 21 \pmod{100}$, then $n^2 \equiv 41 \pmod{100}$, and so n is not a solution. Similarly, if $n \equiv 31 \pmod{100}$, then $n^2 \equiv 61 \pmod{100}$, and so n is not a solution. We thus require that $n \equiv 11 \pmod{100}$.

   If $n > 100$, then $n \equiv 111 \text{ or } 211 \text{ or } 311 \pmod{1000}$. If $n \equiv 211 \pmod{1000}$, then we have that $n^2 \equiv 521 \pmod{1000}$, and so n is not a solution. Similarly, if $n \equiv 321 \pmod{1000}$, then $n^2 \equiv 41 \pmod{1000}$, and so n is not a solution. We thus require that $n \equiv 111 \pmod{1000}$.

   Finally, if $n > 1000$, then we must have that $n \equiv 1111 \text{ or } 2111 \text{ or } 3111 \pmod{10000}$. If $n \equiv 1111 \pmod{10000}$, then we have that $n^2 \equiv 4321 \pmod{10000}$, and so n is not a solution. If $n \equiv 2111 \pmod{10000}$, then $n^2 \equiv 6321 \pmod{10000}$, and if $n \equiv 3111 \pmod{10000}$, then $n^2 \equiv 8321 \pmod{10000}$. In each case we find that n is not a solution.

   We see that the only possible values for n are 1, 11, and 111.

2. *Given a (not necessarily convex) quadrilateral $ABCD$, call a point P in the same plane as $ABCD$ an* areal centre *for $ABCD$, if any line through P divides $ABCD$ into two parts of equal area. What are necessary and sufficient conditions on $ABCD$ for it to possess an areal centre?*

   Suppose that P is an areal centre for $ABCD$. Let XX', YY', and ZZ' be three lines passing through P such that X, Y, and Z lie on AB, and X', Y', and Z' lie on CD. Let a, b, x, y, z, and w be the areas of $AZPX'D$, $BXPZ'C$, XPY, YPZ, $X'PY'$, and $Y'PZ'$ respectively. Since P is an areal centre, we find that

   $$a + x + y = b + z + w \qquad a + y + z = b + x + w \qquad \text{and} \qquad a + z + w = b + x + y.$$

   Solving these equations, we find that $a = b$, $x = z$, and $y = w$. These relations then allow us to deduce that

   $$PX \cdot PY = PX' \cdot PY' \qquad PY \cdot PZ = PY' \cdot PZ' \qquad \text{and} \qquad PX \cdot PZ = PX' \cdot PZ',$$

   which upon solving gives us that $PX = PX'$, $PY = PY'$, and $PZ = PZ'$. This gives us that, for example, $PXY \equiv PX'PY'$, and so $\angle PXY = \angle PX'Y'$, giving us that $AB \parallel CD$. A similar argument shows that $BC \parallel DA$, and so $ABCD$ is a parallelogram. Conversely, one can verify that if $ABCD$ is a parallelogram, then the intersection of the diagonals AC and BD is an areal centre for $ABCD$.

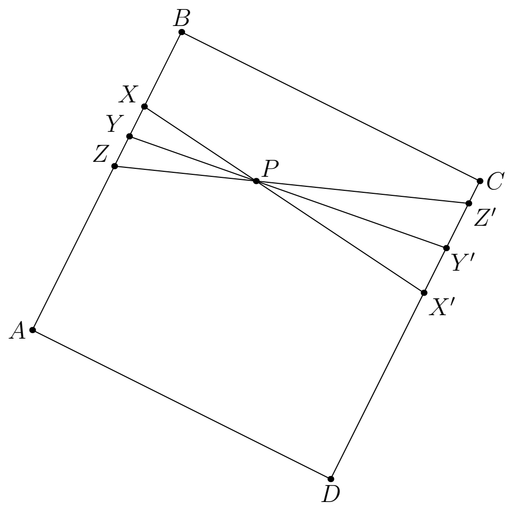


Figure 1: Problem 2

3. *The (English language version of the) game of Scrabble™ consists of 100 tiles, each containing either a letter from A to Z (some letters occur more than once), except for two blank tiles; see* `the relevant Wikipedia page` *for the exact distribution of multiplicities of each letter.*

   In a solo game of Scrabble, the player starts by choosing seven tiles from the 100 available tiles at random. What is the probability that the player does not pick up any vowels?

   Based on the data from the Wikipedia page, 42 of the Scrabble tiles are vowels. (Assuming that we do not count the blank tiles as vowels.)

   Thus the probability that the first tile drawn is not a vowel is given by $\frac{58}{100}$. The probability that the second tile drawn is not a vowel given that the first tile is not a vowel is $\frac{57}{99}$ as there are now only 57 non-vowels remaining among the 99 remaining tiles. Similarly, the probability that the k^{th} tile drawn is not a vowel given that the first $k-1$ tiles were not vowels is given by $\frac{59-k}{101-k}$. The probability that the player draws 7 tiles which are all not vowels is thus

$$\prod_{k=1}^{7} \frac{59-k}{101-k} = \frac{58 \cdot 57 \cdot 56 \cdot 55 \cdot 54 \cdot 53 \cdot 52}{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96 \cdot 95 \cdot 94} = \frac{59943}{3191300}.$$

4. *For each pair of positive integers (a, b), prove that there exists infinitely many positive integers n such that*

$$\frac{a^n + 1}{n^b + 1}$$

   is not an integer.

   We will show that there are infinitely many integers n such that there is a prime p such that $p \mid n^b + 1$ but $p \nmid a^n + 1$.

   We note that for any prime p, there exists a natural number n such that $p \mid n^b + 1$ if and only if -1 is a b^{th} power modulo p. It is sufficient that $2b \mid p - 1$, in which case we have that $g^{(p-1)/2b}$ is a b^{th} root of -1 modulo p, where g is a primitive root modulo p.

   Thus let p be a prime such that $p \equiv 1 \pmod{2b}$ and note that for such a prime p we have $p \neq 2$. (There are infinitely many such primes by Dirichlet's Theorem, but we only need one.)

If $p-1 \mid n$, then we note that by Fermat's Little Theorem, we have $a^n \equiv 0$ or $1 \pmod{p}$, and so $a^n + 1 \equiv 1$ or $2 \pmod{p}$. In particular, since $p \neq 2$, we have that $p \nmid a^n + 1$.

We are thus finished if there are infinitely many n such that $p-1 \mid n$, and $n \equiv g^{(p-1)/2b} \pmod{p}$. Since $\gcd(p, p-1) = 1$, there infinitely many such n by the Chinese Remainder Theorem.

5. *Call a positive integer a* triangular *number if it is of the form $1+2+3+\cdots+k$ for some positive integer k, and* pentagonal *if it is of the form $1+4+7+10+13+\cdots+(3n-2)$ for some positive integer n. Prove that there are infinitely many cases where the product of two consecutive pentagonal numbers is equal to the product of two consecutive triangular numbers.*

   Using the formula for the sum of the terms in an arithmetic progression, one finds that the k^{th} triangular number is given by

   $$\frac{k(k+1)}{2}$$

   and the m^{th} pentagonal number is given by

   $$\frac{m(3m-1)}{2}.$$

   We thus wish to show that there are infinitely many k and m such that

   $$\frac{k(k+1)}{2} \cdot \frac{(k+1)(k+2)}{2} = \frac{m(3m-1)}{2} \cdot \frac{(m+1)(3m+2)}{2},$$

   or equivalently such that

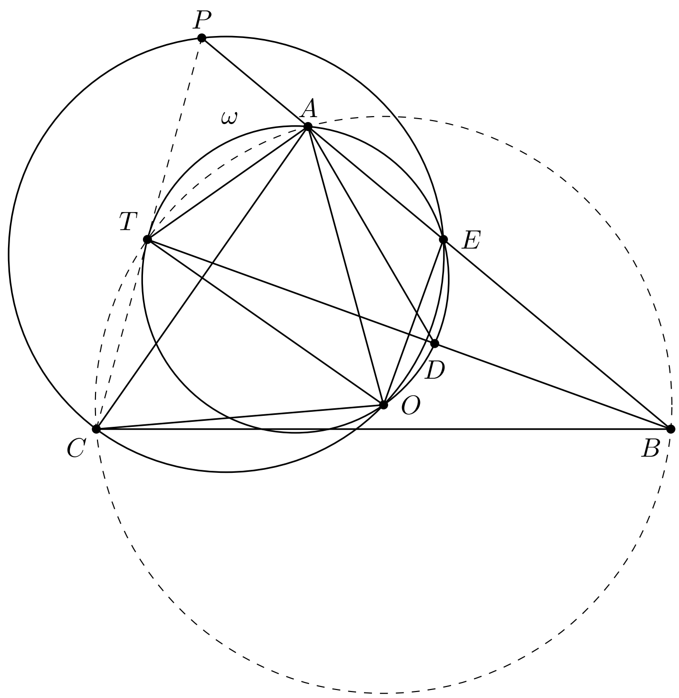
   $$(k^2+2k+1)(k^2+2k) = (3m^2+2m)(3m^2+2m-1).$$

   We see that it is sufficient to show that there are infinitely many k and m such that $k^2+2k+1 = 3m^2+2m$, or equivalently such that $(3m+1)^2 - 3(k+1)^2 = 1$.

   We thus wish to show that the Pell equation $x^2 - 3y^2 = 1$ has infinitely many solutions where $x \equiv 1 \pmod{3}$.

   We note that one solution to the Pell equation is given by $x = 7$ and $y = 4$. Moreover, if (x, y) is a solution such that $x \equiv 1 \pmod{3}$, then $(7x+12y, 4x+7y)$ is another solution and satisfies $7x+12y \equiv 1 \pmod{3}$. There are thus infinitely many solutions such that $x \equiv 1 \pmod{3}$, and we are done.

6. *Let ABC be a triangle with circumcentre O. Let D be the point of intersection between the bisector of $\angle ABC$ and the perpendicular bisector of AB. Let the circumcircle of ADO be ω. Let $E \neq A$ be the intersection of ω with the segment AB. Let $P \neq E$ be the intersection of the circumcircle of COE with the line AB. Prove that CP is tangent to ω.*



Let T be the midpoint of the arc AC not containing B. We will show that T lies on ω. Note:

$$\angle ADT = \angle ABD + \angle DAB = 2\angle ABD = 2\angle ABT = \angle AOT$$

Therefore T lies on ω. Also note that since $\triangle OCT \equiv \triangle OAT$:

$$\angle CTO = \angle TCO = \angle TAO$$

So CT is tangent to ω. Let $P' = CT \cap AB$. We will show that $P' \equiv P$. Note that it is sufficient to show that $P'COE$ is a cyclic quad as this would require $P' \equiv P$. But note that:

$$\angle OEB = \angle OTA = \angle OAT = \angle OCT = \angle OCP$$

So $P'COE$ is a cyclic quad, and so $P' \equiv P$ and thus CP is tangent to ω at T.

7. *Call a function $f : \mathbb{N} \to \mathbb{N}$* almost linear *if $f(m+n) - f(m) - f(n)$ only takes finitely many values as m and n vary through the natural numbers. Suppose that $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ are almost linear. Show that $f(g(n)) - g(f(n))$ only takes finitely many values as n varies through the natural numbers.*

Let M be a natural number such that

$$-M + f(x) + f(y) \leq f(x+y) \leq f(x) + f(y) + M$$

for all natural numbers x and y.

One can show by induction on m that for any natural numbers m and n, we have that

$$-mM + mf(n) \leq f(mn) \leq mf(n) + mM.$$

We note that this is still true even if f is allowed to take values in the integers, rather than in the natural numbers. The only condition that f needs to satisfy is that $f(m+n) - f(m) - f(n)$ is bounded.

We will now show that the sequence $\frac{1}{n}f(n)$ is Cauchy. We note that

$$\left|\frac{f(n)}{n}-\frac{f(m)}{m}\right| \le \left|\frac{f(n)}{n}-\frac{f(mn)}{mn}\right| + \left|\frac{f(mn)}{mn}-\frac{f(m)}{m}\right| \le \frac{M}{n}+\frac{M}{m}$$

which can be made arbitrarily small by taking m and n large enough. We thus have that

$$\lim_{n\to\infty}\frac{f(n)}{n}$$

exists. Let this limit equal α. We now note that the function $h(n) = f(n) - \lfloor \alpha n \rfloor$ is a function such that $h(m+n) - h(m) - h(n)$ is bounded, and such that $\lim_{n\to\infty} \frac{1}{n}h(n) = 0$. We claim that $h(n)$ is bounded. Suppose that $h(n)$ is not bounded. Let K be a natural number such that

$$-K + h(x) + h(y) \le h(x+y) \le h(x) + h(y) + K$$

for all natural numbers x and y. Since h is not bounded, there is a natural number n such that $h(n)-K > 0$.

Note that by a earlier result, we have that $h(mn) \ge -mK + mh(n) = m(h(n) - K)$ for all m. We thus have that

$$\frac{h(mn)}{mn} > \frac{h(n)-K}{n}$$

for all m, contradicting that fact that

$$\lim_{m\to\infty}\frac{h(mn)}{mn} = 0.$$

We thus have that h is bounded. Noting that $\lfloor \alpha n \rfloor - \alpha n$ is bounded, we find that there is a bounded function $b_f : \mathbb{N} \to \mathbb{R}$ such that $f(n) = \alpha n + b_f(n)$ for all n. Similarly, there is a real number β and a bounded function $b_g : \mathbb{N} \to \mathbb{R}$ such that $g(n) = \beta n + b_g(n)$ for all n.

We thus have that

$$\begin{aligned} f(g(n)) - g(f(n)) &= \alpha g(n) + b_f(g(n)) - \beta f(n) - b_g(f(n)) \\ &= \alpha\beta n + \alpha b_g(n) + b_f(g(n)) - \alpha\beta n - \beta b_f(n) - b_g(f(n)). \end{aligned}$$

Since

$$\alpha b_g(n) + b_f(g(n)) - \beta b_f(n) - b_g(f(n))$$

is bounded (each term is bounded), we have that $f(g(n)) - g(f(n))$ is bounded. Since it is always a natural number, it follows that it takes only finitely many values.

8. *Two people, Alf and Bob, wash up on a desert island. They are greeted by a fearsome monster (Maurice), who challenges them to a game. If they win the monster will show them the way off the island, but if they lose the monster will eat them — with a nice nutmeg sauce. Seeing little choice they agree to the game (refusing gets them eaten with an avocado sauce). The monster explains the rules. First Maurice will show Alf a standard $8 \times x$ chessboard, on each square of which is a coin, showing either heads or tails. Maurice will then point to a square (so that Alf can see but Bob cannot). No changes are made to the board at this time. Then Alf will play, choosing a single square he will toggle the coin on that square (i.e. change it from heads to tails or from tails to heads). Bob is then shown the adjusted board (the first time Bob gazes on it's wondrousness). Bob must then state which square Maurice chose. If Bob manages to choose the correct square both Alf and Bob go free, but if not... nibbles. Before the game commences Alf and Bob have a chance to discuss their strategy; what is their optimal strategy and how likely are they to escape?*

First Alf and Bob number the squares from 0 to 63; further they define a function f from board configurations to the integers by taking the xor-sum of the squares with heads-up coins. If Alf receives a board R and a marked square m then he computes $y = f(R) \oplus m$, with $\oplus$ denoting xor-summation, and flips square y. Call this new configuration S. Bob will point to $f(S)$ and they will always win.