Intermediate Test 5

Stellenbosch Camp 2018

Time: 4 hours

1. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost 3 cents per piece. Thus, it costs 3 cents to number locker 9 and 6 cents to number locker 42. If it costs R206.91 to label all the lockers, how many lockers are there at the school?

2. Given the equation $x^{2018} = y^x$,

   (a) find all pairs (x, y) of solutions with x prime and y a positive integer;

   (b) find all pairs (x, y) of positive integers satisfying the equation.

3. Prove that for any three positive real numbers a, b and c,

$$a^4 + b^4 + c^4 \geq a^2bc + b^2ca + c^2ab.$$

4. Consider two circles Γ_1 and Γ_2 that intersect at points A and B. Let l be a line tangent to circles Γ_1 and Γ_2 at S and T, respectively. Lines AB and ST intersect at point M. Furthermore line BT intersect circle Γ_1 again at point R. Let the intersection of MR and SB be X and the intersection of TX and RS be C. Prove that CB and ST are parallel.

5. Determine the number of ways to choose five numbers from the first eighteen positive integers such that any two chosen numbers differ by at least 2.

6. Show that there exists an infinite arithmetic progression of natural numbers such that the first term is 16 and the number of positive divisors of each term is divisible by 5. Of all such sequences, find the one with the smallest possible positive common difference.