PAMO Stream Test 2

April Camp 2019

Time: $4\frac{1}{2}$ hours

1. *Let Γ be the circumcircle of an acute triangle ABC. The perpendicular line to AB passing by C cuts AB in D and Γ again in E. The bisector of the angle C cuts AB in F and Γ again in G. The line GD meets again Γ in H and the line HF meets it again in I. Prove that $AI = EB$.*

   As CG is the bisector of the angle $\angle ACB$, we have $\angle AHG = \angle ACG = \angle GCB$. We have $\angle HAB = \angle HCB$ as these angles intersect the same arc HB. Considering the triangle ADH, we have

$$\begin{aligned}\angle HDB &= \angle HAB + \angle AHG \\ &= \angle HCB + \angle GCB \\ &= \angle GCH.\end{aligned}$$

   We have $\angle FDH = 180° - \angle HDB = 180° - \angle CGH$. So the quadrilateral $CFDH$ has two of its opposite angles that are supplementary and so the points C, F, D and H are concyclic. Subsequently $\angle GCE = \angle FCD = \angle FHD = \angle IHG = \angle ICG$. Furthermore

$$\begin{aligned}\angle ACI &= \angle ACG - \angle ICG \\ &= \angle GCB - \angle GCE \\ &= \angle ECB.\end{aligned}$$

   Consequently, $AI = BE$.

2. *Find all non-negative integers n for which the equation*

$$\left(x^2 + y^2\right)^n = (xy)^{2018}$$

   admits positive integral solutions.

   Let n, x and y be such that $\left(x^2 + y^2\right)^n = (xy)^{2018}$. According to the AM-GM, we have $x^2 + y^2 \geq 2xy > xy$. So $n < 2018$. Let $d = \gcd(x, y)$ and set $a = \frac{x}{d}$, $b = \frac{y}{d}$. Then

$$\begin{aligned}d^{2n}\left(a^2 + b^2\right)^n &= d^{2\times 2018}(ab)^{2018} \\ \left(a^2 + b^2\right)^n &= d^{2(2018-n)}(ab)^{2018}.\end{aligned}$$

As b divides $(ab)^{2018}$, we have that b divides $(a^2+b^2)^n$. But $\gcd(a,b)=1$ so $\gcd(a^2,b)=1$ and so $\gcd(a^2+b^2,b)=1$. Consequently, $b=1$. The same argument shows that $a=1$. Hence we get

$$2^n = d^{2(2018-n)}.$$

Consequently, $d=2^k$ with $2^n=2^{4036k-2nk}$ and $n=4036k-2nk$. Then $n(2k+1)=4k\cdot 1009$. Since $\gcd(2k+1,4k)=1$, we have that $2k+1$ divides 1009 which is a prime, and so $2k+1=1009$ or $2k+1=1$. Hence $k=504$ or $k=0$, and $n=2016$ or $n=0$, respectively. Conversely, we check that $x=y=2^{504}$ satisfies

$$(2^{1008}+2^{1008})^{2016} = (2^{504}\times 2^{504})^{2018} = 2^{2034144},$$

and so $n=2016$ is a solution. A solution for $n=0$ is provided by $x=y=1$.

3. *Adamu and Afaafa choose, each in his turn, positive integers as coefficients of a polynomial of degree n. Adamu wins if the polynomial obtained has an integer root; otherwise, Afaafa wins. Afaafa plays first if n is odd; otherwise Adamu plays first. Prove that:*

   i) Adamu has a winning strategy if n is odd.

   ii) Afaafa has a winning strategy if n is even.

   i) Assume that n is odd so the polynomial is of the form $a_{2k+1}x^{2k+1}+a_{2k}x^{2k}+\cdots+a_1x+a_0$ for some nonnegative integer k. Afaafa plays first choosing a_i for some $i\in\{0,1,\ldots,2k+1\}$. Next, Adamu chooses a_{2k+1-i} equal to a_i. Using the same process in the next choices we obtain a polynomial having -1 as root so that Adamu wins.

   ii) Assume that n is even then the polynomial is of the form $a_{2k}x^{2k}+a_{2k-1}x^{2k-1}+\cdots+a_1x+a_0$ for some positive integer k. Adamu plays first, if he chooses some a_{2i} or a_{2i-1} (for $i\in\{1,\ldots,k\}$), then Afaafa chooses respectively $a_{2i-1}=a_{2i}$ or $a_{2i}=a_{2i-1}$; if he writes a_0 she writes $a_{2j-1}=1$ for any remaining $j\in\{1,\ldots,k\}$ (the least possible choice). In this way Afaafa is able to get $a_{2i-1}\le a_{2i}\forall i\in\{1,\ldots,k\}$ after her last move. Suppose that the polynomial obtained has an integer root $-\alpha$ (where $\alpha\ge 1$) then

   $$a_0 = \alpha^{2k-1}(a_{2k-1}-a_{2k}\alpha)+\cdots+\alpha(a_1-a_2\alpha)\le 0,$$

   which is a contradiction. So Afaafa wins.