

Senior Test 4

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Prove that it is impossible to write a positive integer in every cell of an infinite chessboard, in such a manner that, for all positive integers m, n , the sum of numbers in every $m \times n$ rectangle is divisible by $m + n$.
2. An exam with k questions is presented to n students. A student fails the exam if they get less than half the answers right. We say that a question is easy if more than half of the students get it right. Decide if it is possible that
 - (a) All students fail even though all the questions were easy.
 - (b) No student fails even though no question was easy.
3. Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1, A_5 = A_2$. For $n = 1, 2$ and 3 , suppose that B_n is the midpoint of $A_n A_{n+1}$ and suppose that C_n is the midpoint of $A_n B_n$. Suppose that $A_n C_{n+1}$ and $B_n A_{n+2}$ meet at D_n and that $A_n B_{n+1}$ and $C_n A_{n+2}$ meet at E_n . Calculate the ratio of the area of triangle $\triangle D_1 D_2 D_3$ to the area of triangle $\triangle E_1 E_2 E_3$.
4. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$a^2 + b^2 + c^2 + 3 \geq 2(ab + bc + ca)$$

5. Let $p_1 = 2$ and define a sequence of prime numbers p_1, p_2, p_3, \dots such that, for all positive integers n , p_{n+1} is the least prime factor of $n \cdot p_1^{!} \cdot p_2^{!} \dots p_n^{!} + 1$. Prove that all primes appear in the sequence.

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