Stellenbosch Camp 2018: Senior Geometry

Lecture 4

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Definition (Barycentric Coordinates)

Let ABC be a triangle with an interior point P. The Barycentric coordinates of P represent the relative "weights" that need to be placed at at the 3 vertices in order for P to become the geometric centroid of the triangle. (I.e centre of gravity).

Notation (Barycentric Coordinates)

*Let ABC be a triangle with an interior point P. Then the Barycentric coordinates of P are written as (α, β, γ). Barycentric coordinates are also normalised so that $\alpha + \beta + \gamma = 1$ (**This is important!**).*

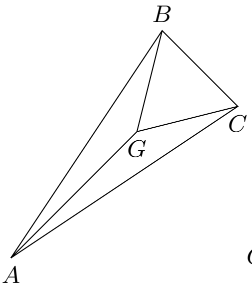
Examples

Barycentric Coordinates Let ABC be a triangle. Here are the Barycentric coordinates of some well known points. For brevity, the normalisation for some of the points have been omitted.

1. Triangle Vertices: $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$
2. Centroid: $G\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$
3. Circumcentre: $O\left(a^2(S_a), b^2(S_b), c^2(S_c)\right)$
4. Orthocentre: $H\left(S_cS_b, S_aS_c, S_bS_a\right)$
5. Incentre: **In Problem Set**
6. Symmedian Point: $K(a^2, b^2, c^2)$

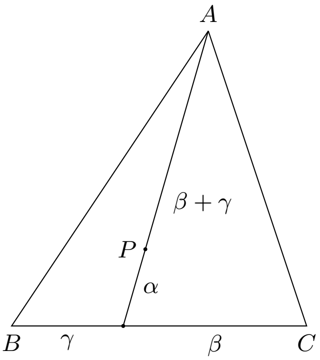
Where $S_A = \frac{b^2+c^2-a^2}{2}$, $S_B = \frac{c^2+a^2-b^2}{2}$, and $S_C = \frac{a^2+b^2-c^2}{2}$

How do you find the barycentric coordinates of a point?



$$G = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

How do you find the barycentric coordinates of a point?
If a point P has Barycentric coordinates (α, β, γ), then the following diagram depicts the ratios that some segments are divided into.



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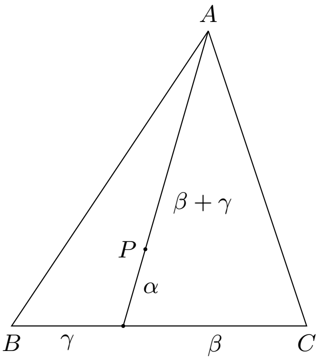


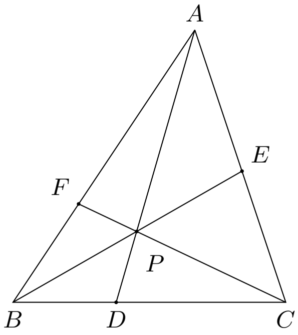
You can claim this as well-known or common knowledge!

Lets do a problem.

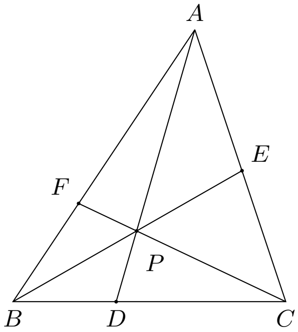
Problem

Let ABC be a triangle with an interior point P. Let D,E, and F be the intersections of the lines AP, BP, and CP, with the lines BC,CA, and AB respectively. Prove:

$$\frac{PD}{DA} + \frac{PE}{EB} + \frac{PF}{FC} = 1$$



Usually with a problem like this, you would work out the ratios as ratios of areas of sub-triangles of ABC, and then add them together to show that the sum is 1.



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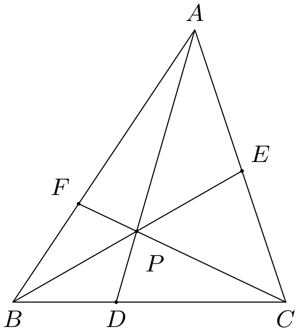
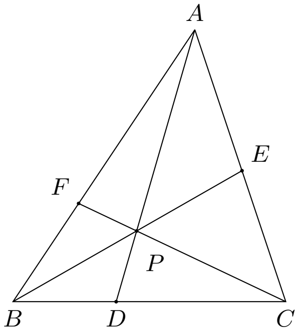
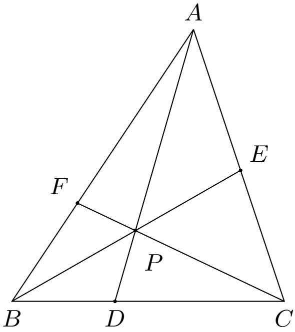


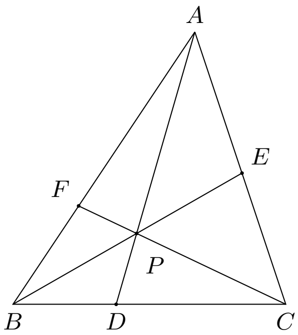
Usually with a problem like this, you would work out the ratios as ratios of areas of sub-triangles of ABC, and then add them together to show that the sum is 1. **Easy but laborious!** So what about Barycentric coordinates? Do they make the problem easier?



Using the result I showed you earlier you can immediately write down the following:

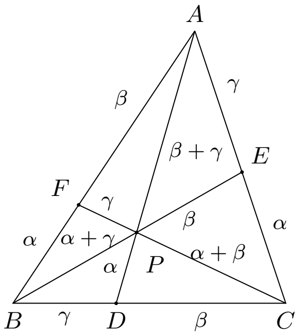


Using the result I showed you earlier you can immediately write down the following:
Suppose P has Barycentric coordinates (α, β, γ).



Using the result I showed you earlier you can immediately write down the following:
Suppose P has Barycentric coordinates (α, β, γ).Then:

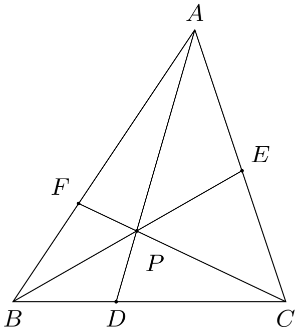
$$\frac{PD}{DA} + \frac{PE}{EB} + \frac{PF}{FC} = \frac{\alpha}{\alpha+\beta+\gamma} + \frac{\beta}{\alpha+\beta+\gamma} + \frac{\gamma}{\alpha+\beta+\gamma} = 1$$



Theorem (Ceva's Theorem)

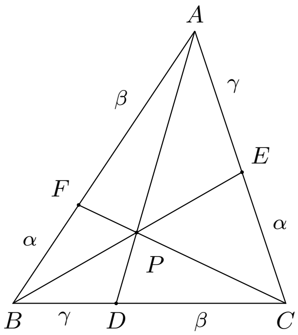
Let ABC be a triangle with points D, E, F on sides BC, CA, and AB. The lines AD, BE, and CF are concurrent if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = 1$$

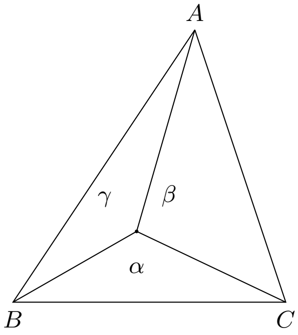


Suppose P has Barycentric coordinates (α, β, γ)

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CF}{FA} = \frac{\beta}{\alpha} \cdot \frac{\gamma}{\beta} \cdot \frac{\alpha}{\gamma} = 1$$



Alternatively, another way to determine the barycentric coordinates is to use the ratio of areas of the opposite *sub-triangle*.



Definition (Proportionality Theorem)

Let $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ be 2 points in triangle ABC. If point X that is on the line PQ, then

$$X = \frac{\overrightarrow{QX} \cdot (\alpha_1, \beta_1, \gamma_1) + \overrightarrow{XP} \cdot (\alpha_2, \beta_2, \gamma_2)}{PQ} \tag{1}$$

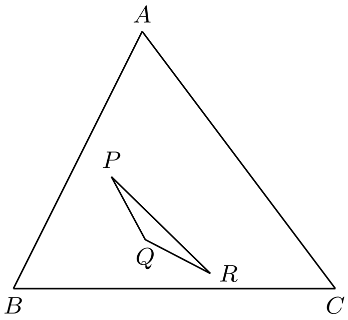
Coordinates are added component wise. Arrow overhead is to indicate directed edges.

Theorem

Let $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ and $R(\alpha_3, \beta_3, \gamma_3)$ be points inside $\triangle ABC$. Then the area of $\triangle PQR$ is given by:

$$|\triangle PQR| = \left|\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}\right| |\triangle ABC|$$

(Note that these coordinates are normalised.)



Corollary (1)

Let $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ *and* $R(\alpha_3, \beta_3, \gamma_3)$ *be points inside* $\triangle ABC$*. Then* P, Q *and* R *are collinear if and only if*

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0$$

Definition (3 × 3 Determinant)

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1$$

Problem (Asiatic Pacific Maths Olympiad, 1989)

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1, 2$ and 3, suppose that B_n is the midpoint of A_nA_{n+1} and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $\triangle D_1D_2D_3$ to the area of triangle $\triangle E_1E_2E_3$.