Senior Test 4

Stellenbosch Camp 2018

Time: $2\frac{1}{2}$ hours

1. Assume that $a_1, a_2, a_3, \ldots$ is an infinite strictly increasing sequence of positive integers, and $p_1, p_2, p_3, \ldots$ is a sequence of primes such that $p_n \mid a_n$ for every positive integer n. It appeared that $a_n - a_k = p_n - p_k$ for every positive integers n and k. Prove that all the numbers $a_1, a_2, \ldots$ are primes.

2. A polynomial $P(x)$ is chosen so that each of the polynomials $P(P(x))$ and $P(P(P(x)))$ is strictly monotone on the real axis. Prove that $P(x)$ is also strictly monotone on the real axis.

3. Two arbitrary positive integers a and b are given. Prove that there exists infinitely many positive integers n such that $n^b + 1 \nmid a^n + 1$

4. A positive integer k is given. Initially, N cells are marked on an infinite chequered plane. We say that the *cross* of a cell is the set of all cells lying in the same row or in the same column as A. By a turn, it is allowed to mark an unmarked cell A if the cross of A already contains at least k marked cells. It appears that every cell can be marked in a stretch of turns. Determine the smallest possible value of N

5. Points P and Q are chosen respectively on the sides AB and AC of a triangle ABC so that $PQ||BC$. The segments BQ and CP meet at O. Let A' be the reflection of A with respect to the line BC. The segment $A'O$ meets the circumcircle ω of the triangle APQ at S. Prove that the circumcircle of the triangle BSC is tangent to ω.

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