

## Senior January Monthly Problem Set

Due: 18 January 2019

1.  $P$ ,  $Q$  and  $R$  are any points on  $BC$ ,  $CA$  and  $AB$  respectively of a triangle  $ABC$ . Let the centres of the circumcircles  $AQR$ ,  $BRP$  and  $CPQ$  be  $X$ ,  $Y$  and  $Z$ . Prove that triangles  $XYZ$  and  $ABC$  are similar.
2. Steve determines the geometric mean of two positive integers in the following way:
  - (a) He writes them down in their decimal representation, one below the other, and prepends zeros to the smaller number (if applicable) such that their lengths are equal.
  - (b) He determines the geometric mean of each pair of digits below each other. If the result is not an integer, only the integer part is used.
  - (c) The digits determined by this procedure form the result.

Determine all pairs  $(a, b)$  of positive integers for which Steve's procedure yields the correct result. (For example, one such pair is  $(12; 48)$ .)

3.
  - (a) Prove that if  $p > 10$  is a prime number that divides  $a^4 + a^3 + a^2 + a + 1$  for some integers  $a$ , then  $p$ 's decimal expansion ends in a 1.
  - (b) For any prime  $p$  whose decimal expansion ends in a 1, and any positive integer  $k$ , prove that there exists an integer  $a$  such that  $p^k$  divides  $a^4 + a^3 + a^2 + a + 1$ .
4. The set  $S$  of nonnegative integers has the property that every nonnegative integer  $n$  can be uniquely written as  $n = a + 2b$  where  $a, b \in S$  are not necessarily distinct. How many elements of  $S$  are less than 2018?
5. Jacob has a balance scale and wishes to buy weights from Sipho. Sipho tells Jacob that he sells weights in the following way: Jacob has to specify a sequence of  $n$  integers  $a_1, a_2, \dots, a_n$ , and then Sipho will make 1 weight of mass  $a_1$ , two weights of mass  $a_2$ , etc., and  $n$  weights of mass  $a_n$ .

What is the largest  $k$  for which Jacob can specify some sequence  $(a_1, \dots, a_n)$  and still be able to measure every integral weight from 1 to  $k$ ? (For example, with weights with mass 4 and 7, he can measure a weight of 3 by putting one weight on the one side and the other on the other side of the balance scale.)

6. Does there exist a natural number  $n$  such that

$$1^{2018} + 2^{2018} + \dots + n^{2018}$$

is prime?

7. Fix a natural number  $n \geq 2$ . Find the smallest constant  $C$  such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (3x_i^2 + x_j^2)(x_i^2 + 3x_j^2) \leq C \left( \sum_{i=1}^n x_i \right)^6$$

for all non-negative real numbers  $x_1, x_2, \dots, x_n$ . For this value of  $C$ , when does equality occur?

8. Let  $ABC$  be a triangle circumscribed by  $\Omega$ . Let  $P, Q$  be 2 points not on  $\Omega$  such that the line  $PQ$  passes through the centre of  $\Omega$ . Let  $D, E, F$  be the feet of the perpendiculars from  $P$  to  $BC, CA, AB$ , and let  $X, Y, Z$  be the feet of the perpendiculars from  $Q$  onto the same sides, respectively. Prove that the perpendiculars from  $D, E, F$  to  $YZ, ZX, XY$  are concurrent.

## Email submission guidelines

- Email your solutions to [samf.training.assignments@gmail.com](mailto:samf.training.assignments@gmail.com).
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.