Senior January Monthly Problem Set

Due: 18 January 2019

1.

2.

3. Steve determines the geometric mean of two positive integers in the following way:

   (a) He writes them down in their decimal representation, one below the other, and prepends zeros to the smaller number (if applicable) such that their lengths are equal.

   (b) He determines the geometric mean of each pair of digits below each other. If the result is not an integer, only the integer part is used.

   (c) The digits determined by this procedure form the result.

   Determine all pairs (a, b) of positive integers for which Steve's procedure yields the correct result. (For example, one such pair is $(12; 48)$.)

4.

5.

6. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(mn) = f(\gcd(m, n))f(\operatorname{lcm}(m, n))$$

   for all $m, n \in \mathbb{N}$.

7. Does there exist a natural number n such that

$$1^{2018} + 2^{2018} + \cdots + n^{2018}$$

   is prime?

8. Fix a natural number $n \geq 2$. Find the smallest constant C such that

$$\sum_{1 \leq i < j \leq n} x_i x_j (3x_i^2 + x_j^2)(x_i^2 + 3x_j^2) \leq C \left(\sum_{i=1}^{n} x_i \right)^6$$

   for all non-negative real numbers $x_1, x_2, \ldots, x_n$. For this value of C, when does equality occur?

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as CamScanner to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.