PAMO Problem Proposals

Dylan Nelson — South Africa

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1. *Find the* 2019^{th} *natural number* n *such that* $\binom{2n}{n}$ *is not divisible by* 5.

   Solution: For a natural number n, let $\nu_5(n)$ denote the exponent of the largest power of 5 which divides n. We recall that

   $$\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!}$$

   and so we require that $\nu_5((2n)!) - 2\nu_5(n!) = 0$. Using Legendre's formula, this is equivalent to

   $$\sum_{k=1}^{\infty} \left(\left\lfloor \frac{2n}{5^k} \right\rfloor - 2 \left\lfloor \frac{n}{5^k} \right\rfloor \right) = 0.$$

   We note that for any real number x, we have that $\lfloor 2x \rfloor \geq 2\lfloor x \rfloor$, with equality if and only if $\{x\} < \frac{1}{2}$, where $\{x\}$ denotes the fractional part of x. We are thus seeking those n such that

   $$\left\{ \frac{n}{5^k} \right\} < \frac{1}{2}$$

   for all k.

   Let

   $$n = \sum_{m=0}^{\infty} d_m \cdot 5^m$$

   be the expansion of n in base 5. We claim that n satisfies the conditions of the problem if and only if $d_m \in \{0, 1, 2\}$ for each m.

   Suppose first that n satisfies the conditions of the problem. Then for each k, we have that

   $$\frac{1}{2} > \left\{ \frac{n}{5^k} \right\} = \sum_{m=0}^{k-1} d_m \cdot 5^{m-k} \geq d_{k-1} \cdot 5^{-1}$$

   and so $d_{k-1} < \frac{5}{2}$.

Conversely, suppose that $d_m \leq 2$ for each m. Then for every k, we have that

$$\left\{\frac{n}{5^k}\right\} = \sum_{m=0}^{k-1} d_m \cdot 5^{m-k} \leq 2\sum_{m=1}^{k} 5^{-m} < 2\sum_{m=1}^{\infty} 5^{-m} = \frac{1}{2}$$

and so n satisfies the conditions of the problem.

We thus wish to find the 2019^{th} natural number n such that the digits of n in base 5 are each $0, 1$ or 2. Now the 2019^{th} string of digits $0, 1, 2$ (at least one of which is not 0) is in fact given by the base 3 expansion of the number 2019, which is equal to 2202210_3, and so the number n which we seek is given by 2202210_5, which is equal to 37805 in base 10.

2. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that*

$$f(x^2) - yf(y) = f(x+y)(f(x) - y)$$

for all real numbers x and y.

Solution: Taking $y = 0$ in the functional equation gives us that $f(x^2) = f(x)^2$ for all real x. In particular, $f(0) = f(0)^2$, and so $f(0) \in \{0, 1\}$.

If $f(0) = 1$, then taking $x = 0$ in the functional equation gives us that $1 - yf(y) = f(y)(1-y) = f(y) - yf(y)$ for all real y, and so $f(y) = 1$ for all real numbers y. We can check that this does indeed satisfy the functional equation.

Now suppose that $f(0) = 0$.

Since $f(-x)^2 = f(x^2) = f(x)^2$ for all x, we know that for each x, either $f(-x) = -f(x)$ or $f(-x) = f(x)$. Now taking $y = -x$ in the functional equation gives us that $f(x^2) + xf(-x) = 0$, and so we derive that $f(x)^2 = xf(x)$ or $f(x)^2 = -xf(x)$ for each x. This shows that $f(x) \in \{0, x, -x\}$ for all real numbers x.

Since $f(x^2) = f(x)^2$, we in fact have that $f(x) \in \{0, x\}$ for all positive real numbers x. Now suppose that there is a positive real number x such that $f(x) = 0$. Then $f(x^2) = 0$, and we find that for all positive y, we have $-yf(y) = f(x+y)(-y)$, and so $f(y) = f(x+y)$. Suppose that $f(y) = y$. Then $f(x+y) \neq 0$, and so $f(x+y) = x+y$. But then $y = x+y$, a contradiction. Thus we must have that $f(y) = 0$ for all positive y, and since $f(-y) = \pm f(y)$ for all y, we have that $f(y) = 0$ for all negative values of y as well. Thus f is identically 0, but this does not satisfy the functional equation.

We thus have that there is no positive real number x such that $f(x) = 0$, and so $f(x) = x$ for all $x \geq 0$. We now recall that $f(x^2) + xf(-x)$ for all x. Since $f(x^2) = x^2$, this gives us that $f(-x) = -x$ for all x. We thus find that $f(x) = x$ for all real numbers x, which does indeed satisfy the functional equation. All solutions to the functional equation are thus given by the constant function $f(x) = 1$, and the identity function $f(x) = x$.

3. *ABC is an acute-angled triangle. The bisectors of angles A and B meet BC and AC at D and E respectively. P is a point on DE such that the distances from P to AC and BC are x and y respectively. Show that the distance from P to AB is $x + y$. (Alternatively, give specific values for x and y and ask to calculate the distance from P to AB.)*

   Solution: Let the feet of the perpendiculars from D onto AB and AC respectively be X and X'. Let the feet of the perpendiculars from E onto AB and AC respectively be Y and Y'. We note that since AD and BE are angle bisectors, we have $DX = DX'$ and $EY = EY'$. Let the foot of the perpendicular from P onto AB be Z, so that we are looking for the distance PZ.

   By similarity, we have that

$$x = \frac{PE}{DE} \cdot DX' \quad \text{and} \quad y = \frac{DP}{DE} \cdot EY'.$$

   We thus wish to show that

$$\frac{PE}{DE} \cdot DX + \frac{DP}{DE} \cdot EY = PZ.$$

   The area of the trapezoid $XDEY$ is $\frac{1}{2}XY(DX + EY)$. The area of the trapezoids $XDPZ$ and $ZPEY$ are $\frac{1}{2}XZ(DX+PZ)$ and $\frac{1}{2}ZY(PZ+EY)$ respectively. We thus have that

$$(XZ + ZY)(DX + EY) = XZ(DX + PZ) + ZY(PZ + EY)$$

   and so

$$PZ = \frac{ZY \cdot DX + XZ \cdot EY}{XZ + ZY}.$$

   The result follows by noting that

$$\frac{XZ}{XY} = \frac{DP}{DE} \quad \text{and} \quad \frac{ZY}{XY} = \frac{PE}{DE}.$$

4. *A pawn is a chess piece which attacks the two squares diagonally in front of it. What is the maximum number of pawns which can be placed on an $n \times n$ chessboard such that no two pawns attack each other? (Alternatively, ask for the maximum number of pawns which can be placed on an $a \times b$ chessboard.)*

   Solution: We claim that the maximum number of pawns which can be placed is given by $n \left\lfloor \frac{n+1}{2} \right\rfloor$. Note that this pawns that we place if we fill every second row of the chessboard with pawns, and so this number is attainable.

   First suppose that n is even. In this case, we can divide the board into $\frac{n^2}{4}$ 2×2 squares. In each such square, we can place at most 2 pawns, as otherwise two pawns will fall on the same diagonal of the square and thus one would attack the other. We can thus place at most $2 \cdot \frac{n^2}{4} = n \left\lfloor \frac{n+1}{2} \right\rfloor$ pawns on the chess board, as claimed.

Now suppose that n is odd. Consider one of the diagonals of the chessboard. (Not necessarily one of the main diagonals.) If the diagonal has length $2m$, then we can divide it into m pairs of squares that are diagonally adjacent. In each pair, we can place at most one pawn, and so we can place at most m pawns along this diagonal. Similarly, if the diagonal has length $2m+1$, then we can divide the diagonal into m pairs of diagonally adjacent squares, and one extra square. In this case, we can place at most one pawn in each of the pairs, and at most one pawn in the extra square for a total of $m+1$ pawns.

Summing up over all of the diagonals, we see that we can place at most

$$\begin{aligned}
&\left\lfloor \frac{n+1}{2} \right\rfloor + 2\left(\sum_{1\leq 2m\leq n-1} m\right) + 2\left(\sum_{1\leq 2m+1\leq n-1} (m+1)\right) \\
&= \frac{n+1}{2} + \frac{n-1}{2}\left(\frac{n-1}{2}+1\right) + \frac{n-1}{2}\left(\frac{n-1}{2}+1\right) \\
&= \frac{n+1}{2} + \frac{n^2-1}{2} = n\left\lfloor \frac{n+1}{2} \right\rfloor
\end{aligned}$$

pawns on the chessboard, as claimed.

A similar argument to the above shows that the maximum number of pawns which can be placed on an $a \times b$ chessboard is given by

$$\max\left\{ a\left\lfloor \frac{b+1}{2} \right\rfloor, b\left\lfloor \frac{a+1}{2} \right\rfloor \right\}.$$

5. *A subset S of the set $T_m = \{1, 2, \ldots, m-1\}$ is called* cosy *if for every $x \in S$, either $2x$ or $2x - m$ is also in S. Find the smallest odd number m such that T_m has a cosy subset with exactly* 2009 *members.*

   Solution: Let $C_x = \{2^n \cdot x \mid n \in \mathbb{N}_0\}$. We note that $\{C_x \mid x \in T_m\}$ is a partition of T_m. We also note that if S is a cosy subset of T_m and $x \in S$, then in fact $C_x \subseteq S$. We see that the cosy subsets of T_m are precisely those which are a union of sets of the form C_x, and also that the complement of a cosy subset is also a cosy subset.

   We note that T_{2015} has a cosy subset of size 2009. This cosy subset is the complement of C_{65}, which has 5 elements. We thus need only show that T_{2011} and T_{2013} do not have cosy subsets with 2009 elements.

   If T_{2011} has a cosy subset with 2009 elements, then its complement has 1 element. The complement is thus the cosy subset $\{x\}$ for some x which satisfies $2x \equiv x \pmod{2011}$. But this is equivalent to $x \equiv 0 \pmod{2011}$, which is impossible if $x \in T_m$.

If T_{2013} has a cosy subset with 2009 elements, then it also has a cosy subset with 3 elements. We know that this is a union of sets of the form C_x with $x \in T_m$. The argument above shows that no set of the form C_x has only 1 element, and so this cosy subset must in fact be of the form $\{x, 2x, 4x\}$ for some $x \in T_m$ satisfying $8x \equiv x \pmod{2013}$. But this implies that $7 \mid 2013$, which is a contradiction. Thus $m = 2015$ is indeed the smallest value of m such that T_m has a cosy subset with exactly 2009 elements.

6. *Let $ABCD$ be a parallelogram. The angle bisectors of $\angle BAD$ and $\angle BCD$ intersect BD at E and G respectively. Then angle bisectors of $\angle ABC$ and $\angle ADC$ intersect AC at F and H respectively. Prove that*

$$AE^2 + BH^2 = AG^2 + BF^2.$$

Solution: Let D be the origin. Let $|\vec{A}| = |DA| = a$, and $|\vec{C}| = |DC| = c$. As a consequence of the angle bisector theorem in triangle BCD, we have that

$$\vec{G} = \frac{c}{a+c}\vec{B} = \frac{c}{a+c}(\vec{A} + \vec{C}).$$

Similarly, we have that

$$\vec{E} = \frac{a}{a+c}(\vec{A} + \vec{C}).$$

We also find that

$$\vec{F} = A + \frac{c}{a+c}(\vec{C} - \vec{A}) = \frac{a}{a+c}\vec{A} + \frac{c}{a+c}\vec{C}$$

and similarly,

$$\vec{H} = \frac{c}{a+c}\vec{A} + \frac{a}{a+c}\vec{C}.$$

We thus have that

$$AE^2 = |A - E|^2 = \left|\frac{c}{a+c}\vec{A} - \frac{a}{a+c}\vec{C}\right|^2,$$

and

$$BH^2 = |A + C - H|^2 = \left|\frac{a}{a+c}\vec{A} + \frac{c}{a+c}\vec{C}\right|^2$$

so that

$$AE^2 + BH^2 = \frac{a^2 + c^2}{(a+c)^2}\left(|\vec{A}|^2 + |\vec{C}|^2\right).$$

A similar calculation shows that

$$\begin{aligned} AG^2 + BF^2 &= \left| \frac{a}{a+c}\vec{A} - \frac{c}{a+c}\vec{C} \right|^2 + \left| \frac{c}{a+c}\vec{A} + \frac{a}{a+c}\vec{C} \right|^2 \\ &= \frac{a^2+c^2}{(a+c)^2}\left(|\vec{A}|^2 + |\vec{C}|^2 \right), \end{aligned}$$

and the result follows.