Beginner Test 3

Stellenbosch Camp 2018

Time: 4 hours

1. Let AB be a chord in a circle with centre O, and let C be a point on the larger arc AB. Show that $\angle AOB = 2\angle ACB$.

2. Factorise the following polynomial completely:

$$(2x+3)^6 - (2x-1)^6.$$

3. How many different rearrangments are there of the word TARTAGLIA?

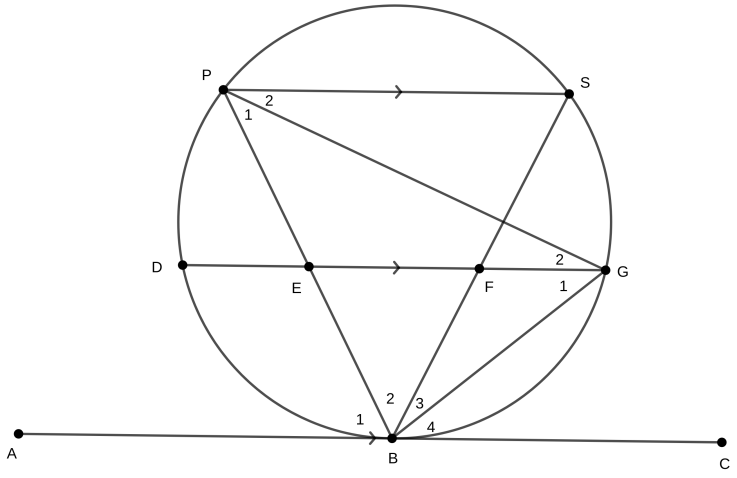
4. In the figure ABC is a tangent to the circumscribed circle of $\triangle PBG$. PS and DG are both parallel to ABC. Chords BP and BS cut DG at E and F respectively. Prove that:

   a. $\angle G_1 = \angle P_1$

   b. $\triangle BGP$ is similar to $\triangle BEG$

   c. $BG^2 = BP \times BE$

   d. $\frac{BG^2}{BP^2} = \frac{BF}{BS}$



5. Consider a game wherein two players Emma and Dylan take turns to take between 1 and 7 stones, inclusive, from a pile which starts with 2018 stones. If Emma plays first, does one of the players have a winning strategy, and if so what is it?

6. Determine all solutions (x, y) of the system of equations

$$\frac{4}{x} + \frac{5}{y^2} = 12,$$
$$\frac{3}{x} + \frac{7}{y^2} = 22.$$

7. Suppose k is a positive integer that does not divide 2008. Let $[x]$ denote the greatest integer less than or equal to x. For example, $[11.75] = 11$ and $[\pi] = 3$. What is the maximum possible value of $k \times \left[\frac{2018}{k}\right]$?

8. The student lockers at Olympic High are numbered consecutively beginning with locker number 1. The plastic digits used to number the lockers cost 3 cents per piece. Thus, it costs 3 cents to number locker 9 and 6 cents to number locker 42. If it costs R206.91 to label all the lockers, how many lockers are there at the school?

9. Consider the function $f(x) = \frac{1}{1-x}$ and its iterates f^r defined as

$$f^1(x) = f(x)$$
$$f^2(x) = f(f(x))$$
$$f^3(x) = f(f(f(x)))$$
$$f^4(x) = f(f(f(f(x)))),$$

and so on. Calculate the value of $f^{2018}(2018)$.

10. Given the equation $x^{2018} = y^x$,
    (a) find all pairs (x, y) of solutions with x prime and y a positive integer;
    (b) find all pairs (x, y) of positive integers satisfying the equation.