Intermediate Test 2 – Solutions

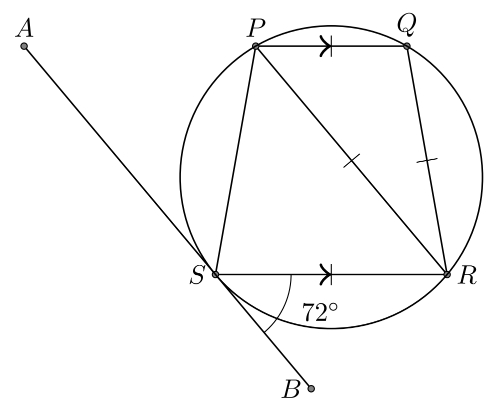
Stellenbosch Camp 2018

1. *For any real number x, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x. Find all real numbers a, b, and c such that*

$$\begin{aligned} \lfloor a \rfloor + \{b\} &= -2.3, \\ \lfloor b \rfloor + \{c\} &= 8.9, \qquad \text{and} \\ \lfloor c \rfloor + \{a\} &= 23.4. \end{aligned}$$

The first equation gives us that $\lfloor a \rfloor = -3$ and $\{b\} = 0.7$. The second equation gives us tht $\lfloor b \rfloor = 8$ and $\{c\} = 0.9$. The last equation implies that $\lfloor c \rfloor = 23$ and $\{a\} = 0.4$. It follows that $a = -2.6$, $b = 8.7$ and $c = 23.9$.

2. *Lets $PQRS$ be a cyclic quadrilateral such that $PQ = QR$, $PR = SR$ and $PQ \parallel SR$. Let ASB be a tangent at S where A lies on PQ. If $\angle BSR = 72°$, find the value of $\angle RPQ$.*



By the tan-chord theorem, we have that $\angle SPR = \angle BSR = 72°$. Since $PR = SR$, we find that $\angle RSP = \angle SPR = 72°$. Since $PQRS$ is a cyclic quadrilateral, we have that $\angle PQR = 180° - \angle RSP = 180° - 72° = 108°$.

Now note that $\angle PQR + \angle RPQ + \angle QRP = 180°$. However, PQR is an isosceles triangle, so $\angle RPQ = \angle QRP$, and so we have that

$$2\angle RPQ + 108° = \angle RPQ + \angle QRP + \angle PQR = 180°,$$

and so $\angle RPQ = 36°$.

3. *Let n be a positive integer. Find the last digit of*

$$n^{2018} + (n+1)^{2018} + \cdots + (n+99)^{2018}.$$

Let

$$S = n^{2018} + (n+1)^{2018} + \cdots + (n+99)^{2018}.$$

Note that among any 10 consecutive integers, we have one integer with each possible remainder modulo 10. We thus have that

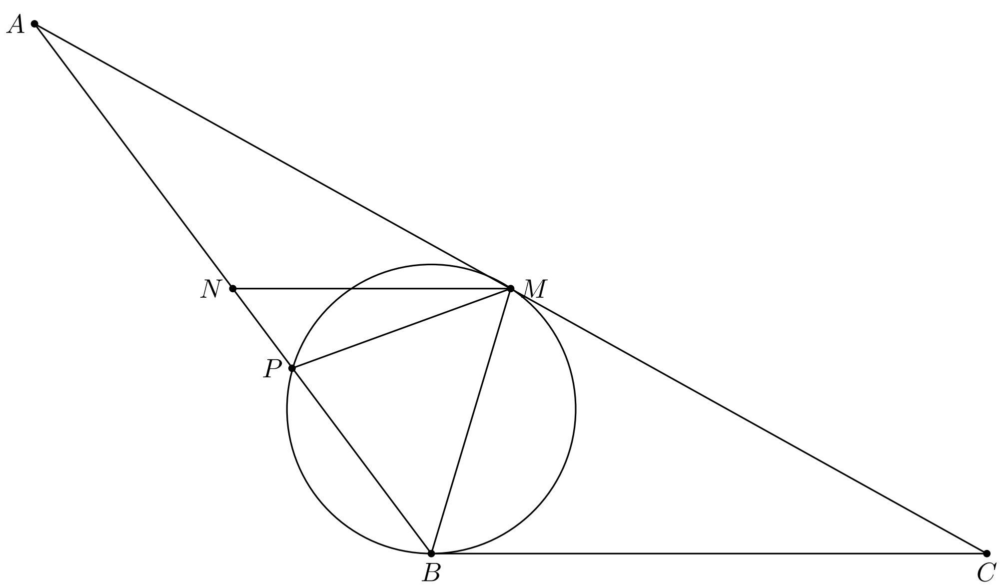
$$k^{2018} + (k+1)^{2018} + \cdots + (k+9)^{2018} \equiv 0^{2018} + 1^{2018} + \cdots + 9^{2018} \pmod{10}.$$

The numbers $n, (n+1), (n+2), \ldots, (n+99)$ consists of 10 groups of 10 consecutive positive integers, and so we have that

$$\begin{aligned} S &\equiv \underbrace{\left(0^{2018} + 1^{2018} + \cdots + 9^{2018}\right) + \left(0^{2018} + 1^{2018} + \cdots + 9^{2018}\right) + \cdots + \left(0^{2018} + 1^{2018} + \cdots + 9^{2018}\right)}_{10 \text{ times}} \\ &\equiv 10\left(0^{2018} + 1^{2018} + \cdots + 9^{2018}\right) \\ &\equiv 0 \pmod{10}, \end{aligned}$$

and so the last digit of S is always a 0.

4. *Let ABC be a triangle, and let the midpoint of AC be M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.*



Construct point N, the midpoint of AB. By midpoint Th$^{\underline{m}}$, $NM || BC$. Using this and Tan-Chord Th$^{\underline{m}}$, we get: $\angle BMN = \angle MBC = \angle MPB$. Since we also have that $\angle PBM = \angle MBN$, we have that $\triangle BMN \sim \triangle BPM$. So

$$\frac{BM}{BP} = \frac{BN}{BM}$$
$$\Rightarrow BM^2 = BN \cdot BP = \left(\frac{1}{2}AB\right) \cdot BP$$

5. *Show that for all positive real numbers x and y,*

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{x+y}{3}.$$

By the Cauchy-Schwarz inequality (in "Engel Form") we have that

$$\frac{x^2}{x+2y} + \frac{y^2}{2x+y} \geq \frac{(x+y)^2}{(x+2y)+(2y+x)} = \frac{x+y}{3}.$$

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