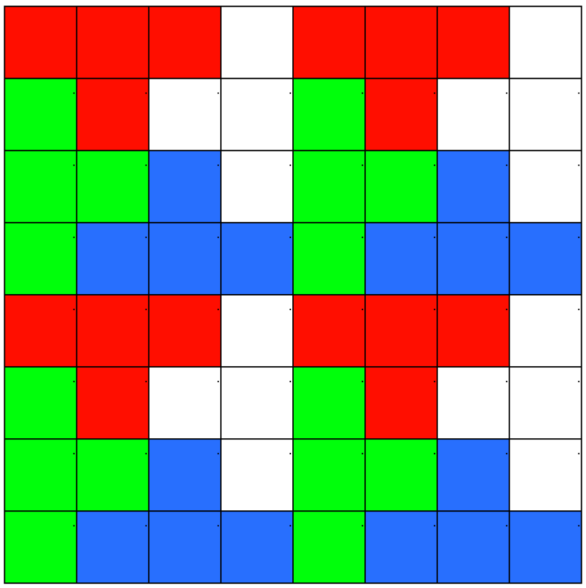
Junior Test 2 Solutions

Stellenbosch Camp 2019

1. *Tile an* 8×8 *chessboard with T-shaped tetrominoes.*

   Solution:



2. *Prove that for all* $a, b > 0$,

$$\frac{a}{b} + \frac{b}{a} \geq 2.$$

   Solution: $a, b > 0$, so $ab > 0$.

$$\begin{aligned} & \frac{a}{b} + \frac{b}{a} \geq 2 \\ \iff \quad & a^2 + b^2 \geq 2ab \\ \iff \quad & a^2 - 2ab + b^2 \geq 0 \\ \iff \quad & (a-b)^2 \geq 0 \end{aligned}$$

   The square of a real number is always non-negative, so it is true.

3. *In* $\triangle ABC$ *let* $\angle ACB = 90°$, $AC = 1$ *and* $AB = 2$. *Let* M *be the midpoint of* AB *and* D *the intersection of the angle bisector of* $\angle CAB$ *and* BC. *Prove that* $AB \perp CM$.

   Solution: Notice that $\triangle ABC$ is a special triangle with angles $90°$, $60°$, and $30°$.

$\therefore DM \perp AB$ and $DM = CD = \frac{\sqrt{3}}{3}$.

$\therefore ACDM$ is a kite.

$\therefore AD \perp MC$.

4. *Find the first number which appears in all 3 the following arithmetic progressions:*

$$21, 34, 57, 70, ...$$
$$33, 37, 41, 45, ...$$
$$42, 75, 108, 141, ...$$

Solution: Let M be the smallest such number

$$\therefore M \equiv_{13} 8$$
$$M \equiv_4 1$$
$$M \equiv_{33} 9$$

$\therefore M = m_1(4 \cdot 33)(8) + m_2(13 \cdot 33)(1) + m_3(13 \cdot 4)(9) + n(4 \cdot 13 \cdot 33)$.

$$\begin{aligned} &\therefore & m_1(4 \cdot 33) &\equiv_{13} 1 & &\implies m_1 = 7 \\ &\therefore & m_2(13 \cdot 33) &\equiv_4 1 & &\implies m_2 = 1 \\ &\therefore & m_3(4 \cdot 13) &\equiv_{33} 1 & &\implies m_3 = 7 \end{aligned}$$

$\therefore M = (7)(4 \cdot 33)(8) + (1)(13 \cdot 33)(1) + (7)(13 \cdot 4)(9) + n(4 \cdot 33 \cdot 13)$.

$\therefore M = 801$.

5. *There are 7 people A, B, C, D, E, F, and G sitting in a row. B wants to sit next to C and E wants to sit next to F. How many different seating arrangements are there?*

   Solution: We can box B and C together considering them as a single entity which can appear in 2! ways (BC or CB). Similarly, we can group E and F into a single entity. This means that the total number of combinations if $5! \cdot 2! \cdot 2! = 480$.

6. *Given $\triangle ABC$, with $AB < AC$, let D be the point where the angle bisector of angleBAC intersects the circumcircle of $\triangle ABC$. Let P and Q be the altitudes dropped onto the extensions of AB and AC. Prove that $PB = QC$.*

   Solution: Construct lines DB and DC, note that $DB = DC$ as they subtend the same angle.

   Additionally, $PD = DQ$ since $\triangle APD \equiv \triangle AQD$.

   $\therefore \triangle BPD \equiv \triangle CQD$ (RHS).

   $\therefore BP = QC$.

7. *What are the last two digits of $7^{7^{7^7}}$?*

Solution: Note $7^4 \equiv_{100} 1$.

$\therefore 7^{7^{7^7}} \equiv_{100} (7^4)^k \cdot 7^r$

where $r \equiv_4 77^7$.

$\therefore r \equiv_4 (-1)^{7^7} \equiv_4 -1 \equiv_4 3$

$$\begin{aligned}
\therefore 7^{7^{7^7}} &\equiv_{100} (7^4)^k \cdot 7^3 \\
&\equiv_{100} (1)^k \cdot 7^3 \\
&\equiv_{100} 43
\end{aligned}$$

8. *Prove that for all $a, b, c, d > 0$,*

$$(a+b+c+d)^4 \geq abcd \times 4^4.$$

Solution: Note that for x, $y > 0$ we have

$$\begin{aligned}
& & (x-y)^2 &\geq 0 \\
&\implies & x^2 - 2xy + y^2 &\geq 0 \\
&\implies & x^2 + y^2 &\geq 2xy \\
&\implies & \frac{x^2+y^2}{2} &\geq xy
\end{aligned}$$

Now, we let $x^2 = \frac{a+b}{2}$ and $y^2 = \frac{c+d}{2}$.

$$\begin{aligned}
\frac{\frac{a+b}{2} + \frac{c+d}{2}}{2} &\geq \sqrt{(\frac{a+b}{2})(\frac{c+d}{2})} \\
\frac{a+b+c+d}{4} &\geq \sqrt{(\frac{a+b}{2})(\frac{c+d}{2})} \\
&\geq \sqrt{\sqrt{ab}\sqrt{cd}} \\
&\geq \sqrt[4]{abcd} \\
\therefore (a+b+c+d) &\geq 4\sqrt[4]{abcd} \\
\therefore (a+b+c+d)^4 &\geq abcd \times 4^4
\end{aligned}$$

9. *Given the smiley face colouring, can the board be made completely white through some order of inverting rows.*

Solution: Let every white squre be denoted by a 1, and every black square by a -1. Let k be the product of all of the values in the grid. Note that an inversion of a row or column would never change the value of k, as you are multiplying each item in the row by (-1) when you invert. Hence, k gets multiplied by $(-1)^8 = 1$.

Note that in the original diagram, $k = -1$ and a completely white board will have $k = 1$. Since inversions do not change the value of k for the board, it must be impossible.

10. *Let n be a positive integer greater than 2. Let r_1 be the smallest odd divisor of n greater than 1 and let r_2 be the largest odd divisor of n. Find all n such that*

$$n = 5r_1 + 3r_2$$

Solution: First notice that r_1 must be prime otherwise there is a smaller odd number that divides n. Let 2^k be the highest power of 2 that divides n and so we may write the given condition as

$$2^k r_1 p = 5r_1 + 3r_1 p$$

for some odd number p.

If $p = 1$ then the equation becomes $2^k r_1 = 8r_1$ so $k = 3$, therefore $n = 8r_1$ satisfies this for any odd prime r_1.

If $p \geq 3$, $r_2 = pr_1$ and so we have

$$\begin{aligned} & 2^k r_1 p = 5r_1 + 3pr_1 \\ \Longrightarrow \quad & 2^k p = 5 + 3p \\ \Longrightarrow \quad & p(2^k - 3) = 5 \end{aligned}$$

and since $p \geq 3$, p must be 5 and k must be 2. So $n = 20r_1$ for some odd prime not greater than 3 and so $n = 60$ and $n = 100$ are valid.