Advanced Test 2 Solutions

Stellenbosch Camp 2019

1. *The points E and F lie on sides AB and AD, respectively, of a parallelogram $ABCD$ such that $|AB| = 4|AE|$ and $|AD| = 4|AF|$. Prove that BF, DE, and AC are concurrent.*

   Solution: Let DE AND FB intersect at G. Join BD and AG and let AG extended intersect BD at O. By Ceva's Theorem in triangle ABD we have

   $$1 = \frac{DO}{OB} \cdot \frac{BE}{EA} \cdot \frac{AF}{FD} = \frac{DO}{OB} \cdot \frac{3}{1} \cdot \frac{1}{3},$$

   hence $DO = OB$. Since the diagonals of a parallelogram bisect each other, O lies on the diagonal AC; hence BF, DE, and AC are concurrent.

2. *The cells of an 8×8 chessboard are all coloured in white. A move consists in inverting the colours of a 1×3 rectangle, either vertical or horizontal (the white cells become black and the black cells become white). Is it possible to colour all cells of the chessboard in black in a finite number of moves?*

   Solution: Let us label the square in the rth row and cth column as (r, c) where $1 \le r, c \le 8$. Now let us colour the chessboard in three repeating diagonal colours red, green, and blue, where a square (r, c) is red if $3 \mid r + c$, green if $3 \mid r + c - 1$, and blue if $3 \mid r + c - 2$. The total number of red squares, which we denote as R, is 22, and analogously $G = 21$ and $B = 21$. We also keep track of the number of the number of white squares of each colour, which we denote by R_o, G_o, and B_o; initially these are also 22, 21, and 21 respectively.

   Note that after each move, the parities of each of R_o, G_o, and B_o changes since one of each category is toggled by each 1×3 rectangle. Thus R_o and G_o always have different parities, and in particular cannot both be zero. Thus we cannot have all the squares of the chessboard be black.

3. *Three numbers 2^{100}, 3^{100}, and 5^{100} are written on a long paper strip without any space in between, creating one big number N. Ralpf claims that he can change the last digit of N so that the new number is a power of 13. Is he right?*

   Solution: Suppose that the last digit of N is changed from 5 to the digit d, and that the resulting number is 13^k. The last digit of N is the last digit of 5^{100}, which is 5. Also, since $10 \equiv_3 1$, the remainder of N on division by 3 is

   $$N \equiv_3 2^{100} + 3^{100} + 5^{100} \equiv_3 1 + 0 + 1 = 2.$$

   Then since 13^k is odd, d is odd, and so equal to either 1, 3, 5, 7, or 9. Also,

   $$13^k = N - 5 + d \equiv_3 2 - 5 + d \equiv_3 d;$$

   since $3 \nmid 13^k$, we have that $d \neq 3$ and $d \neq 9$. Also, $5 \nmid 13^k$, so $d \neq 5$. Thus $d = 1$ or $d = 7$.

   Since 5^{100} has more than 3 digits, the remainder of N on division by 8 is $N \equiv_8 5^{100} \equiv_8 1$. So if $d = 1$, then $13^k \equiv_8 1 - 5 + 1 \equiv_8 3$. But looking at the last digit of 13^ℓ in general, we see that it is 1 if and only if $4 \mid \ell$; thus k is even, and so $13^k \equiv_8 1$, a contradiction. If on the other hand $d = 7$, looking at the last digit of 13^k we see that $k \equiv_4 3$, so that $13^k \equiv_8 13 \equiv_8 5$. But also $13^k \equiv_8 5^{100} - 5 + 7 \equiv_8 3$, a contradiction.

4. *Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying*

   $$f(2xy) + f(f(x + y)) = xf(y) + yf(x) + f(x + y)$$

for all $x, y \in \mathbb{R}$.

Solution: Substituting $y \leftarrow 0$, we get that

$$f(0) + f(f(x)) = f(0)x + f(x) \implies f(x) - f(f(x)) = f(0) - f(0)x. \tag{1}$$

Now substituting $y \leftarrow \frac{1}{2}$ in the original equation we get that

$$\begin{aligned} f(x) + f\left(f\left(x + \frac{1}{2}\right)\right) &= xf\left(\frac{1}{2}\right) + \frac{1}{2}f(x) + f\left(x + \frac{1}{2}\right) \\ \implies \frac{1}{2}f(x) - f\left(\frac{1}{2}\right)x &= f\left(x + \frac{1}{2}\right) - f\left(f\left(x + \frac{1}{2}\right)\right) \\ &= f(0) - f(0)\left(x + \frac{1}{2}\right) && \text{by (1)} \\ \implies f(x) &= 2\left(\frac{1}{2} - f(0)\right)x + f(0), \end{aligned}$$

so $f(x) = ax + b$ is a linear function.

Substituting $f(x) = ax + b$ into the original equation, we get that

$$\begin{aligned} 2axy + b + a(a(x+y) + b) + b &= x(ay + b) + y(ax + b) + a(x+y) + b \\ \iff a^2(x+y) + ab + b &= (a+b)(x+y). \end{aligned}$$

Since this is true for all $x, y \in \mathbb{R}$, we have that $a^2 = a + b$ and $0 = ab + b = (a+1)b$. From the latter we have that $b = 0$ (in which case $a^2 = a$, so that $a = 0$ or $a = 1$), or that $a = -1$, so that $b = 2$. Thus the possible solutions for f are

- $f(x) = x$,
- $f(x) = 0$, and
- $f(x) = 2 - x$,

all of which satisfy the original equation by the above derivation of the coefficients a and b.

5. *Let $\triangle ABC$ be acute and let D be the foot of the perpendicular from A onto BC. The circle centred at A passing through D intersects the circumcircle of $\triangle ABC$ at X and Y (with X on the same side as B with respect to to the line AD). Prove that $\angle BXD = \angle CYD$.*

   Solution: Let the interior angles of $\triangle ABC$ be α, β, and γ respectively. We are given that $\angle MXA = \angle DAC = \angle MAB$; thus $\triangle MXA$ and $\triangle MAB$ are similar, so that

$$\frac{XM}{AM} = \frac{AM}{BM}.$$

   Let H be the foot of the altitude from A onto BC. We will assume that H lies between B and D; the other case is similar. Since M is the midpoint of the hypotenuse of the right triangle AHD, we have that $AM = HM$. Therefore the previous equation becomes

$$\frac{XM}{HM} = \frac{HM}{BM},$$

   from which it follows that $\triangle XMH$ and $\triangle HMB$ are similar. From this we find that $\angle HXM = \angle BHM$, but

$$\angle BHM = 180^\circ - \angle MHD = 180^\circ - \angle MDH = \beta + \frac{\alpha}{2},$$

   where in the second equation we use the fact that $\triangle MHD$ is isosceles, since $\triangle AHD$ is a right angle and M is the midpoint of the hypotenuse, and in the third we use the fact that the sum of the angles of $\triangle ABD$ is 180°.

   Therefore

$$\angle AXH = \angle MXH + \angle AXM = \left(\beta + \frac{\alpha}{2}\right) + \frac{\alpha}{2} = \beta + \alpha$$

   since we are given that $\angle AXM = \angle DAC = \frac{\alpha}{2}$. Therefore $\angle AXH + \angle ACH = \beta + \alpha + \gamma = 180^\circ$, and so $AXHC$ is cyclic, from which it follows that $\angle AXH + \angle AHC = 90^\circ$.