Senior January Monthly Problem Set Solutions

1. *On the board, we write the integers $1, 2, 3, \ldots, 2019$. At each minute, we pick two numbers on the board a and b, erase them, and write down the number $s(a+b)$ instead where $s(n)$ denotes the sum of the digits of the integer n. Let N be the last number remaining on the board.*

   (a) Is it possible that $N = 19$?

   (b) Is it possible that $N = 15$?

   Since $10^n \equiv 1 \pmod 3$ for all non-negative integers n, we have that $s(n) \equiv n \pmod 3$ and so replacing a and b with $s(a+b)$ does not change the sum of the numbers on the board in mod 3. Now $1 + 2 + \ldots + 2019 = \frac{(2019)(2020)}{2} \equiv 0 \pmod 3$ but $19 \not\equiv 0 \pmod 3$ so we cannot have $N = 19$ as the last number on the board.

   We now show that we can in fact get to 15.

   For each $k \neq \{1010, 906\}$, replace the numbers k and $2020 - k$ with $s(k + 2020 - k) = s(2020) = 4$. So remaining on the board now is $906, 1114, 1010$ and 1008 4s. Now taking the 4s in pairs and applying the procedure gives 504 8s, taking these in pairs gives 252 7s, and taking these 7s in pairs gives 126 5s and again taking these 5s as pairs gives 63 1s. So we have left $906, 1114, 1010$ and 63 1s. Now apply the procedure in 1114 and 1010 to get $s(2125) = 9$ and remain with $9, 906$ and 63 1s. Now from the 63 1s, we can make 7 9s by taking 1s and applying the procedure until we get to a 7 then we stop and take a 1 that has not been used. We then have 8 9s and 906 left on the board. Since $s(9 + 9) = s(18) = 9$, we can apply this procedure to the 9s until just one 9 remains. Finally, we combine this 9 with the 906 to get $s(906 + 9) = s(915) = 15$.

   Remark: *The steps used to get to 15 are not unique.*

2. *For which positive integers n is it possible to divide the set of numbers $\{n, n+1, n+2, \ldots, n+8\}$ into two disjoint sets A and B such that the product of the numbers in A is equal to the product of the numbers in B?*

   Let $S = \{n, n+1, \ldots, n+8\}$. Without loss of generality let A have more elements than B; since we have 9 elements in total, it is impossible for them to have the same number of elements, and $\#A + \#B = 9$ gives that A has at least 5 elements and B has at least 4 elements. The 5 smallest elements of S are n, $n+1$, $n+2$, $n+3$, and $n+4$, while the 4 largest elements of S are $n+5$, $n+6$, $n+7$, and $n+8$.

   Note that for $n \geq 6$ we have that $n^3 \geq 216$ and $9n^3 \geq 1944$, and so

$$
\begin{aligned}
&\prod_{a \in A} a - \prod_{b \in B} b \\
\geq& n(n+1)(n+2)(n+3)(n+4) - (n+5)(n+6)(n+7)(n+8) \\
=& n^5 + 9n^4 + 9n^3 - 201n^2 - 1042n - 1680 \\
=& n^2(n^3 - 201) + n(9n^3 - 1042) + 9n^3 - 1680 \\
>& 0.
\end{aligned}
$$

Thus the products of the elements in A cannot equal those of the elements in B if $n \geq 6$.

However, if $n \leq 5$ then S contains the prime number 7 and no other multiple of 7. Thus either the product of the elements of A will be divisible by 7 and that of B will not, or vice versa; hence it is not possible for $n \leq 5$. Thus there are no positive integers n which satisfy the desired conditions.

3. *Let M be a positive integer, and let S denote the set of finite sequences of positive integers less than or equal to M, including the empty sequence of length zero, which we denote as $\mathbf{0}$. Also, for a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ let $\overline{\mathbf{x}}$ denote the reverse sequence $\overline{\mathbf{x}} = (x_n, x_{n-1}, \ldots, x_1)$.*

   Define a function d from S to the integers as follows:

   - $d(\mathbf{0}) = 0$.
   - *If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a sequence of positive length, let m be the largest integer such that $x_1 + x_2 + \cdots + x_m \leq M$, and let $\mathbf{x}'$ denote the rest of the sequence: $\mathbf{x}' = (x_{m+1}, \ldots, x_n)$. Then $d(\mathbf{x}) = 1 + d(\mathbf{x}')$.*

   Show that $d(\mathbf{x}) = d(\overline{\mathbf{x}})$. Note that d partitions the sequence $\mathbf{x}$ into sub-sequences *s.t.* the sum of each sub-sequence does not exceed M and then counts the total amount of these sub-sequences. Let $s_1, s_2, .., s_{d(\mathbf{x})}$ be the indices of the starts of these sub-sequences.

   For the partitioning of $\bar{\mathbf{x}}$ we conjecture that none of the sub-sequences can have more than one of these x_{s_i}

   If there are exist a sub-sequence with at least two of these s_i we have:

   $$M \geq (\textstyle\sum_{j=s_{i+1}}^{s_i} x_j) = (\sum_{j=s_i}^{s_{i+1}-1} x_j) + x_{s_{i+1}} > M$$

   which is a contradiction $\implies$ each s_i in its own sub-sequence which implies $d(\overline{\mathbf{x}}) \geq d(\mathbf{x})$

   Hence $d(\mathbf{x}) = d(\overline{\overline{\mathbf{x}}}) \geq d(\overline{\mathbf{x}}) \geq d(\mathbf{x}) \implies d(x) = d(\bar{x})$

4. *Let $ABCD$ be a cyclic quadrilateral with its diagonals intersecting at E. Let M be the midpoint of AB. Suppose that ME is perpendicular to CD. Show that either AC is perpendicular to BD or AB is parallel to CD.*

   We will solve this problem using complex numbers; let each point have a corresponding complex number indicated by the corresponding lower case letter (a for A), and the complex conjugate of a complex number x be denoted by $\bar{x}$.

   Without loss of generality, let A, B, C, and D lie on the unit circle, so that $\bar{a} = 1/a$, and similarly for b, c and d. Then $m = (a+b)/2$. E lies on the line AC and on the line BD, so that

   $$\frac{e-a}{c-a} \in \mathbb{R} \iff \frac{e-a}{c-a} = \frac{\bar{e}-\bar{a}}{\bar{c}-\bar{a}} = \frac{\bar{e}-1/a}{1/c-1/a} = \frac{ac\bar{e}-c}{a-c} \iff e-a = c - ac\bar{e}.$$

   Similarly, $e - b = d - bd\bar{e}$, and so

   $$(bd-ac)e = bd(c+a-ac\bar{e}) - ac(b+d-bd\bar{e}) = bcd - cda + dab - abc \implies e = \frac{abc - bcd + cda - dab}{ac - bd}.$$

   Thus $\bar{e} = \frac{1/abc - 1/bcd + 1/cda - 1/dab}{1/ac - 1/bd} = \frac{a-b+c-d}{ac-bd}$.

Now we are given that $ME \perp CD$, in other words

$$\begin{aligned}
&\frac{m-e}{c-d} = -\frac{\bar{m}-\bar{e}}{\bar{c}-\bar{d}} \\
\iff &\frac{a+b-2e}{c-d} = -\frac{1/a+1/b-2\bar{e}}{1/c-1/d} = \frac{2abcd\bar{e}-cd(a+b)}{ab(d-c)} \\
\iff &ab(ac-bd)(a+b) - 2ab(ac-bd)e = -2abcd(ac-bd)\bar{e} + cd(ac-bd)(a+b) \\
\iff &(ab-cd)(ac-bd)(a+b) = -2abcd(a+c-b-d) + 2ab(ac(b+d)-bd(a+c)) \\
\iff &(a-b)(ac+bd)(ab-cd) = 0 \\
\iff &a = b \quad \text{or} \quad ac+bd=0 \quad \text{or} \quad ab=cd.
\end{aligned}$$

Now $a \neq b$ since A and B are distinct points. Moreover, AC is perpendicular to BD if and only if

$$\frac{a-c}{b-d} \in i\mathbb{R} \iff \frac{a-c}{b-d} = -\frac{\bar{a}-\bar{c}}{\bar{b}-\bar{d}} = -\frac{1/a-1/c}{1/b-1/d} = -\frac{bd(a-c)}{ac(b-d)} \iff ac+bd=0,$$

and similarly AB is parallel to CD if and only if $ab = cd$. This completes the solution.

5. *We have done it! We have planted an infinite number of trees on the vertices of an infinite regular grid, one for each vertex. Let T be a positive integer. We define a T-forest as a set of trees such that for any two trees in the T-forest, there exists another tree planted in the grid such that the area of the triangle with these three trees as vertices is T.*

   What is the smallest T such that our T-forest has more than 200 trees?

   Call two points, A and B, T-friends if there exists a point C such that the area of $\triangle ABC$ is T.

   Lemma: (a,b) and (c,d) are T-friends if and only if $\gcd(c-a, d-b) \mid 2T$.

   Proof of Lemma: Note that (a,b) and (c,d) are T-friends if and only if there exists an integer pair (x,y) such that

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & c-a & d-b \\ 1 & x & y \end{vmatrix} = \pm 2T,$$

   or

$$(c-a)y - (d-b)x = \pm 2T.$$

   This is equivalent to $\gcd(c-a, d-b) \mid 2T$ by Bezout's lemma. ■

   We claim that in order to get a T-clique of size at least 200, we must have

$$\operatorname{lcm}(1,2,\ldots,14) \mid 2T.$$

   To see this, note that for each $1 \leq n \leq 14$, we have $n^2+1 \leq 200$, so there exist two points in the clique with matching residue classes mod n for each coordinate, call the points (a,b) and (c,d). Thus, $n \mid \gcd(c-a, d-b) \mid 2T$, so $n \mid 2T$ for each $1 \leq n \leq 14$, as desired.

   We claim that $T = \boxed{\frac{1}{2}\operatorname{lcm}(1,2,\ldots,14)}$ is in fact attainable. To do this, place 225 points in a 15×15 grid formation, and arbitrarily remove 25 points. For each pair of points (a,b) and (c,d), we have $0 \leq |c-a|, |d-b| \leq 14$, so certainly $\gcd(c-a, d-b) \mid \operatorname{lcm}(1,2,\ldots,14)$, as desired.

6. *Given a series $t_1, t_2, \ldots, t_n$ such that*

$$t_{k+1} = \frac{t_k^2+1}{t_{k-1}+1} - 1 \quad \forall\ k \in \{2,3,\ldots,n-1\}.$$

For which $n \in \mathbb{N}$ does there exist a t_1 and t_2 such that $t_i \in \mathbb{N}$ for all $i \in \{1, 2, 3, \ldots, n\}$?

The answers are 1, 2, 3, and 4. There exists a sequence of length 4, as 4, 33, 217, and 1384 form a valid sequence (and hence there exist sequences of length 1, 2, and 3.)

Suppose for the sake of contradiction that there exists a sequence where t_1, t_2, t_3, t_4, t_5 are all integers. Then

$$t_2^2 + 1 = (t_1 + 1)(t_3 + 1)$$
$$t_3^2 + 1 = (t_2 + 1)(t_4 + 1)$$
$$t_4^2 + 1 = (t_3 + 1)(t_5 + 1)$$

This implies that $t_3 + 1 | t_2^2 + 1$ and $t_2 + 1 | t_3^2 + 1$. Suppose for the sake of contradiction that either t_2 or t_3 is odd, WLOG t_2. This means that $t_2 + 1$ is even, so $t_3^2 + 1$ must be even, so t_3 must be odd. This means they must both be odd.

Since $(t_2 + 1)(t_4 + 1) = t_3^2 + 1$, and $t_3^2 + 1 \equiv 2 \pmod 4$, t_4 must be even, so $t_4^2 + 1$ must be odd. But $t_3 + 1 | t_4^2 + 1$, so $t_4^2 + 1$ must be even, which is a contradiction. Hence, $t_3 + 1 | t_2^2 + 1$, $t_2 + 1 | t_3^2 + 1$, and t_2 and t_3 are both even.

We claim that there no positive even integers m and n such that $m+1|n^2+1$ and $n+1|m^2+1$. Suppose there are; take the two that make $m + n$ minimal. WLOG, let $m > n$. Let $(m + 1)(p + 1) = n^2 + 1$. p must be even, otherwise $4|n^2 + 1$. $m = \frac{n^2+1}{p+1} - 1$, so

$$n + 1 | \left(\frac{n^2 + 1}{p + 1} - 1 \right)^2 + 1$$

$$\implies (n + 1)(p + 1)^2 | (n^2 - p)^2 + (p + 1)^2 = (n^4 - 1) - 2(n^2 - 1)p + 2p^2 + 2$$
$$\implies n + 1 | 2(p^2 + 1)$$

Since n is even, $n + 1 | p^2 + 1$ and by construction $p + 1 | n^2 + 1$. Notice that:

$$m \geq n + 1 = \frac{n^2 + 2n + 1}{n + 1} > \frac{n^2 + 1}{n + 1} > \frac{n^2 + 1}{m + 1} = p + 1 > p$$

So we have $n + p < m + n$ which can only happen when $p = 0$ since we assumed $m + n$ was minimal, so $m = n^2$ which has no valid solutions. It follows that there are no sequences with 5 terms, so the proof is complete.

7. *Let AC and BD be two chords of a circle Γ that intersect at X in the interior of Γ. Let Γ_1 and Γ_2 be circles that are mutually tangent at X and are tangent to Γ at P and Q. Let ω be a circle tangent to Γ_1 and Γ_2 at X that intersects the chords AB and CD at M and N respectively. Prove that*

$$\frac{MP}{MQ} = \frac{NP}{NQ} \implies \angle AXM = \angle DXN.$$

8. *Let n be a positive integer greater than 1, and consider a circle of radius 1 in which is inscribed a regular n-gon P with vertices labelled from 1 to n in that order. Consider the set S of positive divisors of n, and the convex polygon G formed by the points of P with labels in S. If the area of G is denoted by $|G|$, show that*

$$|G| < \frac{3}{2}.$$

The largest possible divisors of n are n, $\frac{n}{2}$ and $\frac{n}{3}$. Consider the shape created by joining n and $\frac{n}{2}$ with a straight line, $\frac{n}{2}$ and $\frac{n}{3}$ with a straight line and then joining $\frac{n}{3}$ to n with the arc of the circle. The area of this shape will always be greater than G. Computing the area of the figure, we find:

$$|G| < \frac{\pi}{2} \times \frac{2}{3} + \frac{\sqrt{3}}{4} = 1.4802... < \frac{3}{2}$$