Beginner Monthly Problem Solutions

Due: Friday, 17 January 2020

1. How many positive square numbers are factors of 1600?

ANS: Factorising yield: $1600 = 2^6 \times 5^2 = (2^2)^3 \times (5^2)^1$ Note that a square number has only even exponents in its factorization, hence is made of square factors. Hence the square factors of $1600 = (3+1)(1+1) = 8$

2. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2882 is a four-digit palindrome and 49194 is a five-digit palindrome. There are pairs of four-digit palindromes whose sum is a five-digit palindrome. One such pair is 2882 and 9339. How many such pairs are there including this pair?

ANS: Let $P_1 = abba, P_2 = cddc$ Note that the first digit, and therefor the last digit, of the 5 digit palindrome, $P_1 + P_2 = xyzyx$, is 1 $\implies x = 1 \implies a + c = 11 \implies y = 1, 2$

If $y = 1 \implies b + d < 10$ and $b + d \equiv_{10} = 0 \implies b + d = 0 \implies b = d = 0$, hence there are $9 - 2 + 1 = 8$ possibilities in this case. If $y = 2 \implies b + d \geq 10$ and $b + d \equiv_{10} = 1 \implies b + d = 11$, hence there are $(9 - 2 + 1) \times (9 - 2 + 1) = 64$ possibilities in this case.

In total there are 72 ordered possibilities $\implies$ 36 pairs.

3. Show all the ways the number 365 can be written as the sum of two or more different perfect square numbers?

ANS: $365 = a^2 + b^2$ W.L.O.G $x > y$ Note we only have check for $y \leq 13$ as $\sqrt{\frac{365}{2}} < 14$ The only cases which work is $y = 2, 13$ which has integer solutions for $x = 19, 14$ respectively.

4. How many squares on an 8×8 chessboard are more black than white?

ANS: Note that even $\times$ even squares have the same amount of black and white squares, hence we will only look at odd $\times$ odd squares. Note that the total amount of odd $\times$ odd squares has the same amount of more white as more black squares by symmetry. Therefor the amount of total odd $\times$ odd squares is twice the amount of more black squares. Total more black $= \frac{2^2+4^2+6^2+8^2}{2} = 60$

5. In triangle ABC, $\widehat{A} = 45°$. Point P is on BC, Q on AB and R on AC such that $BP = QP$ and $CP = RP$. Find $Q\widehat{P}R$.

ANS: $B\hat{Q}P = \hat{B} \implies B\hat{P}Q = 180 - 2\hat{B}$
Similarly $C\hat{P}R = 180 - 2\hat{C}$

$$\implies Q\hat{P}R = 180 - (180 - 2\hat{C}) - (180 - 2\hat{B}) = 180 - 2(180 - \hat{C} - \hat{B}) = 180 - 2\hat{A} = 90$$

6. If X and Y are two digits and the five digit number $30XY5$ can be expressed as the product $225 \times n$, find all possible positive integer values of n.

ANS: $30XY5$ is odd, hence $225 \times n$ is odd $\implies n$ is odd.

$$n = \lceil \tfrac{30XY5}{225} \rceil \geq \lceil \tfrac{30005}{225} \rceil = 134$$
$$n = \lfloor \tfrac{30XY5}{225} \rfloor \leq \lfloor \tfrac{30995}{225} \rfloor = 137$$
$$\implies n = 135, 137$$

7. The symbol $\lfloor x \rfloor$ means the greatest integer less than or equal to x. For example,

$$\lfloor 5.7 \rfloor = 5 \text{ and } \lfloor 4 \rfloor = 4.$$

Calculate the value of the sum

$$\lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \lfloor \sqrt{3} \rfloor + \lfloor \sqrt{4} \rfloor + ... + \lfloor \sqrt{49} \rfloor + \lfloor \sqrt{50} \rfloor$$

ANS: Note $\left\lfloor \sqrt{n^2} \right\rfloor = n$ and $\lfloor \sqrt{n^2 - 1} \rfloor = n - 1$ and $\lfloor \sqrt{x} \rfloor$ is an increasing function.
Hence the sum becomes:
$1 + 1 + 1 + 2 + 2 + ... + 6 + 7 + 7 = 1(2^2 - 1^2) + 2(3^2 - 2^2) + 3(4^2 - 3^2) + ... + 6(7^2 - 6^2) + 7 + 7 = 217$

8. On a page, there are 10 circles. Any pair of them intersect at exactly 2 points but no 3 of them intersect at one common point. Into how many parts do these circles divide the page?

ANS: Let C be one of the circles. C meets each of the other $n - 1$ circles in 2 points. The $2n - 2$ points where C crosses one of the other circles divide the circle C into $2n - 2$ arcs, which we consider as edges of a plane graph. There are $2n - 2$ such arcs on each of the n circles, making a total of $E = n(2n - 2)$ edges in the graph

Now we can use the following formula:

$$F = E - V + 2 = n(2n - 2) - n(n - 1) + 2 = n(n - 1) + 2 = n^2 - n + 2$$

Where F is the amount of parts, V the amount of vertices and E the amount of edges we just calculated.

Subbing in $n = 10$ yields $F = 100 - 10 + 2 = 92$

9. Note that by choosing any two points which aren't diametrically opposite, we can find the other two points which will form a rectangle by taking the two points which are diametrically opposite the two chosen points. Also note that for every rectangle consisting of 4 points, choosing any 2 points (except for 2 which are diametrically opposite) it will define the rectangle.

The total amount of rectangles is: $\frac{\binom{12}{2} - 6}{4} = 15$

10. What is the remainder when $6^{2012} + 8^{2021}$ is divided by 49?

ANS: $6^{2012} + 8^{2021} \equiv_{49} 6^{-4} + 8^5$ by Euler Phi
The multiplicative inverse of 6 is $-8 \pmod{49}$
Hence the above simplifies to $(-8)^4 + 8^5 \equiv_{49} 29 + 36 \equiv_{49} 16$

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).

- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.