Advanced Test 2

Stellenbosch Camp 2019

Time: $2\frac{1}{2}$ hours

Each question in worth 7 marks.

1. The points E and F lie on sides AB and AD, respectively, of a parallelogram $ABCD$ such that $|AB| = 4|AE|$ and $|AD| = 4|AF|$. Prove that BF, DE, and AC are concurrent.

2. The cells of an 8×8 chessboard are all coloured in white. A move consists of inverting the colours in a horizontal or vertical 1×3 rectangle (the white cells become black and vice versa). Is it possible to make all the cells of the chessboard black in a finite number of moves?

3. Three numbers 2^{100}, 3^{100}, and 5^{100} are written on a long paper strip without any space in between, creating one big number N. Ralpf claims that he can change the last digit of N so that the new number is a power of 13. Is he right?

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(2xy) + f(f(x + y)) = xf(y) + yf(x) + f(x + y)$$

   for all $x, y \in \mathbb{R}$.

5. Let $\triangle ABC$ be acute and let D be the foot of the perpendicular from A onto BC. The circle centred at A passing through D intersects the circumcircle of $\triangle ABC$ at X and Y (with X on the same side as B with respect to to the line AD). Prove that $\angle BXD = \angle CYD$.