Senior Monthly Problem Set

Due: 4 March 2020

1. Let ABC be a triangle with orthocentre H, such that $AB < BC$ and $\angle BAC < 90°$. Let the circle Γ centred at B and passing through A intersect AC again at D. The circumcircle of $\triangle BCD$ intersects Γ again at E. ED and BH intersect at F. Prove that BD is tangent to the circumcircle of $\triangle DHF$.

2. For any 4 points in $\mathbb{R}^3$, does there exist a plane such that the orthogonal projections of the points on the plane make up the vertices of a parallelogram?

3. Find all positive integers n that can be written in the form:
$$n = \left\lfloor m + \sqrt{m} + \frac{1}{2} \right\rfloor$$
where m is also a positive integer.

4. An 8×8 chessboard is divided into several regions by 13 straight lines. Can the lines be placed in such a way that each region contains at most 1 centre of the original 64 squares?

5. Let the midpoints of the sides BC, CA, AB of an equilateral triangle ABC be D, E, and F respectively. Let j, k, and ℓ be lines passing through D, E, and F respectively such that $j \parallel k \parallel \ell$. Define points P, Q, and R by $P = j \cap EF$, $Q = k \cap FD$, and $R = \ell \cap DE$. Prove that the points X, Y, and Z defined by $X = BC \cap QR$, $Y = CA \cap RP$, and $Z = AB \cap PQ$ are collinear.

6. Show that there are only finitely many solutions $(x, y) \in \mathbb{N}^2$ to the equation
$$\sum_{i=1}^{m} (x+i)^n = \sum_{i=1}^{m} (y+i)^{2n}$$
where $m, n \in \mathbb{N} \setminus \{1\}$ are given constants.

7. In the land of Graphopia there are n towns. Between some pairs of towns there are direct roads. The residents of these towns want to have mathematics tournaments. They've decided that in order to have a mathematics tournament they need to have a participating towns. Further they decide that in order to have a tournament the connections between the towns involved must be symmetrical. That is either every pair of towns in a tournament are directly linked by a road or no pair is.

   As mathematics competitions are rightly adored by every member of Graphopia every grouping of a towns that can hold a tournament holds one once a year (some towns may be in multiple groupings). However in some years new roads are built and old ones lost to the elements. If a group of a towns find themselves able to put on a tournament they immediately do. If a tournament finds itself unallowable due to road connections, that tournament is regretfully discontinued. Prove that it is possible that one year poor Graphopia finds itself having less than $\binom{n}{a}2^{1-\binom{a}{2}}$ tournaments.

8. For positive real numbers a, b, and c with $a+b+c=1$ prove that:

$$\frac{a^{n+2}}{a^{n+1}+b^n+c^n}+\frac{b^{n+2}}{a^n+b^{n+1}+c^n}+\frac{c^{n+2}}{a^n+b^n+c^{n+1}}\geq\frac{1}{7}$$

   where $n \in \mathbb{N}$. Where does equality occur?

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.