Advanced Test 3

Stellenbosch Camp 2019

Time: $2\frac{1}{2}$ hours

Each question is worth 7 marks.

1. Let a, b, and c be positive real numbers such that $b, c \in [1, 2)$ and
$$\frac{a+b}{b(1+c)} + \frac{a+c}{c(1+b)} = 2.$$
Show that a, b, and c are the lengths of the sides of a triangle.

2. You have 22 points on a plane such that the perimeter of the triangle formed by any three of these points is at most 2. Is it possible to cover all of the points in the plane with a strip of width $2\sqrt{2} - 2$?

3. Find the least positive integer k such that 2050^{2051} can be written as a sum of k 5th powers.

4. Consider an acute triangle ABC with circumcircle Γ. Let the tangents to Γ at B and C intersect at a point T, the line TA intersect Γ again at D, and the point diametrically opposite D with respect to Γ be E. Show that the angle bisector of $\angle BEC$ intersects AT on a circle which is tangent to BT, CT, and Γ.

5. Let P be a polynomial with integer coefficients, and define a sequence (a_n) by $a_0 = 0$ and $a_{n+1} = P(a_n)$ for $n \geq 0$. Show that for nonnegative integers m and n,
$$\gcd(a_m, a_n) = a_{\gcd(m,n)}.$$

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