Advanced Test 1

Stellenbosch Camp 2019

Time: $2\frac{1}{2}$ hours

1. A set T of integers is called *broken* if there are integers $a < b < c$ such that a and c are in T but b is not in T.

   Find the number of broken subsets of $\{1, 2, \ldots, 2019\}$.

2. Let ABC denote an equilateral triangle. Let M and N denote the midpoints of AB and BC, respectively. Let P be a point outside ABC such that APC is isosceles and right-angled at P. Lines PM and AN meet at I. Prove that CI is the angle bisector of $\angle ACM$.

3. A number written in base 10 is a string of 3^{2019} digit 3s. No other digit appears. Find the highest power of 3 which divides this number.

4. There are 6 eagles, 17 snakes and 55 mice in Wonderland. An eagle can eat a snake or a mouse, but not another eagle. A snake can eat a mouse, but not an eagle or another snake. A mouse cannot eat an eagle, a snake or another mouse.

   Whenever an eagle eats a snake, it turns into a mouse, and when it eats a mouse it turns into a snake. When a snake eats a mouse, it turns into an eagle.

   After some time a situation is reached in which no animal can eat another animal. What is the maximal possible number of animals alive in this situation?

5. For each integer $n \geq 2$, determine, with proof, which of the two positive real numbers a and b satisfying

$$a^n = a + 1, \qquad b^{2n} = b + 3a$$

   is larger.