Intermediate Test 1 Solutions

Stellenbosch Camp 2019

1. *If $x + \frac{1}{x} = 3$, what is the value of $x^5 + \frac{1}{x^5}$?*

   Solution:

$$3^2 = (x + \frac{1}{x})^2 = x^2 + 2x(\frac{1}{x}) + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2}$$

$$\Rightarrow x^2 + \frac{1}{x^2} = 3^2 - 2 = 7$$

$$7 \times 3 = (x^2 + \frac{1}{x^2})(x + \frac{1}{x}) = x^3 + x + \frac{1}{x} + \frac{1}{x^3} = x^3 + \frac{1}{x^3} + 3$$

$$\Rightarrow x^3 + \frac{1}{x^3} = 21 - 3 = 18$$

$$7^2 = (x^2 + \frac{1}{x^2})^2 = x^4 + 2x^2(\frac{1}{x^2}) + \frac{1}{x^4} = x^4 + 2 + \frac{1}{x^4}$$

$$\Rightarrow x^4 + \frac{1}{x^4} = 7^2 - 2 = 47$$

$$\Rightarrow 47 \times 3 = (x^4 + \frac{1}{x^4})(x + \frac{1}{x}) = x^5 + x^3 + \frac{1}{x^3} + \frac{1}{x^5} = x^5 + \frac{1}{x^5} + 18$$

$$\Rightarrow x^5 + \frac{1}{x^5} = 141 - 18 = 123$$

2. *Given a triangle ABC and two points M and N on sides AB and AC respectively. Let BN and CM intersect at P. It is given that the areas of $\triangle CPN$, $\triangle BPM$ and $\triangle BPC$ are 4, 6 and 5 respectively. Find the area of $\triangle ABC$.*
   (Bonus: if you would like an extra mark, use the areas 20, 19, 2019 instead)

   Solution: Construct line segment AP and denote by X and Y the areas of triangles APM and APN respectively. Now taking MP and PC as bases of triangles APM and APB, we get a common height hence the ratio of their areas is the ratio of their bases, so $\frac{X}{Y+4} = \frac{MP}{PC} = \frac{6}{5}$ and similarly $\frac{Y}{X+6} = \frac{NP}{PB} = \frac{4}{5}$. Then we have $5X = 6Y + 24$ and $5Y = 4X + 24$ and solving for X and Y gives $X = 264$ and $Y = 216$. Hence the total area is $264 + 216 + 4 + 5 + 6 = 495$.

3. *Find all positive integers n where the product of the positive factors of n is n^3.*

   Solution: Notice that $n = 1$ is a trivial solution. Suppose that p is a factor of n, then $\frac{n}{p}$ is an integer and it is also a factor of n. If $p \leq \sqrt{n}$, then $\frac{n}{p} \geq \frac{n}{\sqrt{n}} = \sqrt{n}$. This means that factors come in unique pairs on either side of $\sqrt{n}$ with the product of these factors being n. If n is a perfect square, then there is some k such that $k^2 = n$. This factor will be paired with itself, so it must be dealt with separately. If the product of the factors of n is n^3, then the product of factors excluding k must be $n^{2.5}$. Since the product of pairs of factors is always n, the product of all pairs must be an integer power of n. Thus, it is not possible to get a product of factors being $n^{2.5}$. This shows that n cannot be a perfect square.

   Furthermore, since the product of factors in a pair is always n, the total number of pairs must be 3 so that the product of all factors is n^3. This means that n must have 6 factors in total.

If $n = p_1^{q_1} \times p_2^{q_2} \times \cdots \times p_r^{q_r}$ where all p_i are unique primes and all q_i are positive integers, then the number of factors of n is $(q_1+1) \times (q_2+1) \times \cdots \times (q_r+1)$. Since the factors of 6 are 1, 2, 3 and 6, the possible combinations are:

$$q_1 + 1 = 2, q_2 + 1 = 3 \Rightarrow q_1 = 1, q_2 = 2$$

$$q_1 + 1 = 6 \Rightarrow q_1 = 5$$

This shows that n must either be represented as $n = p^1 \times q^2$ or $n = p^5$ where p and q are unique primes. A simple check of the product of factors shows all numbers of this form work.

$$n = p^1 \times q^2 \Rightarrow 1 \times p \times q \times pq \times q^2 \times pq^2 = p^3 q^6 = n^3$$

$$n = p^5 \Rightarrow 1 \times p \times p^2 \times p^3 \times p^4 \times p^5 = p^{15} = n^3$$

Therefore, the product of factors of n is n^3 if and only if $n = 1$, $n = p \times q^2$ or $n = p^5$ where p and q are unique prime numbers.

4. *A set T of integers is called broken if there are integers $a < b < c$ such that a and c are in T, but b is not in T.*
Find the number of broken subsets of $\{1, 2, \ldots, 2019\}$.

Solution: The number of subsets of $\{1, 2, \ldots, 2019\}$ is 2^{2019}. Let us see how many of these subsets do not have the required property, that is, are not broken. Clearly, neither the empty subset, nor any subset with just one element are broken. So at least

$$\binom{2019}{0} + \binom{2019}{1} = 1 + 2019 = 2020$$

subsets are not broken.
Suppose now that T is a non-broken subset with two or more elements, and let m, M be the smallest and largest elements in T, respectively. If some positive integer k with $m \leq k \leq M$ is not in T, then $a = m$, $b = k$ and $c = M$ satisfy the broken condition, and T would be broken, a contradiction; then all numbers $m, m+1, m+2, \ldots, M$ must be in T, and any non-broken subset with more than two elements must be formed by consecutive numbers of $\{1, 2, \ldots, 2019\}$. The number of such subsets is $\binom{2019}{2}$, since all of them are determined by pairs of the form (m, M).
Then, the number of broken subsets T of $\{1, 2, \ldots, 2019\}$ is

$$2^{2019} - 2020 - \binom{2019}{2}$$

5. *Let ABC denote an equilateral triangle. Let M and N denote the midpoints of AB and BC, respectively. Let P be a point outside ABC such that APC is isosceles and right-angled at P. Lines PM and AN meet at I. Prove that CI is the angle bisector of $\angle ACM$.*

Solution: Since $\triangle BAN \equiv \triangle CAN$ we have $\angle IAB = \angle NAB = \angle CAN = \angle CAI$, and thus AI is an angle bisector of $\angle CAM$. Note that $\triangle AMC \equiv \triangle BMC$ we have $\angle AMC = \angle BMC$ and $\angle AMC + \angle BMC = 180°$, thus $\angle AMC = \angle BMC = 90°$. Since $\angle AMC + \angle CPA = 90° + 90° = 180°$ we have that AMCP is a cyclic quadrilateral. So $\angle AMI = \angle AMP = \angle ACP = \angle CAP = \angle CMP = \angle CMI$. Therefore MI is a bisector of $\angle AMC$. This implies that I is the incentre of $\triangle AMC$ and so CI is the angle bisector of $\angle ACM$ by necessity.