Test 3

'April' Camp 2020

Time: $4\frac{1}{2}$ hours

1. ABC is an acute-angled triangle. The bisectors of angles A and B meet BC and AC at D and E respectively. P is a point on DE such that the distances from P to AC and BC are x and y respectively. Show that the distance from P to AB is $x + y$.

2. Let $n \geq 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left| 1 - \sum_{i \in X} a_i \right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

3. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + bf(a).$$