Test 4

‘April’ Camp 2020

Time: $4\frac{1}{2}$ hours

1. Let x and y be two integers. Prove that if 2013 divides $x^{1433} + y^{1433}$ then 2013 divides $x^7 + y^7$.

2. The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily distinct) integers has the following properties: $0 \leq a_i \leq i$ for all integers $i \geq 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geq 0$.

Prove that all integers $N \geq 0$ occur in the sequence (that is, for all $N \geq 0$, there exists $i \geq 0$ with $a_i = N$).

3. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leq i \leq n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0 & \text{if } n \text{ is even;} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$