Intermediate Test 2 Solutions

Stellenbosch Camp 2019

1. *In triangle $\triangle ABC$, the angle bisector of $\angle BAC$, the perpendicular bisector of AC and the altitude from C to AB are concurrent. Find the value of $\angle BAC$.*

   Solution:
   Let the given altitude, angle bisector and perpendicular bisector intersect in D, let the foot of the perpendicular from C to AB be E and let the midpoint of AC be F. Since $AF = FC$, $\angle AFE = \angle CFE = 90°$ and FE is a common side, $\triangle EFC$ is congruent to $\triangle EFA$. Therefore $\angle ECF = \angle EAF = \angle EAD$. Furthermore, since $\angle CDA$ is a right angle, we have that $90° = \angle DCA + \angle DAC = 3\angle DAE$, which gives $\angle DAE = 30°$. Thus $\angle BAC = 2\angle DAE = 60°$.

2. *Find all positive integers n such that $\frac{n^2+8n+51}{n+4}$ is also a positive integer.*

   Solution:

$$\frac{n^2+8n+51}{n+4} = n + \frac{n^2+8n+51-n(n+4)}{n+4}$$
$$\implies \frac{n^2+8n+51}{n+4} = n + \frac{4n+51}{n+4}$$
$$\implies \frac{n^2+8n+51}{n+4} = n+4 + \frac{4n+51-4(n+4)}{n+4}$$
$$\implies \frac{n^2+8n+51}{n+4} = n+4 + \frac{35}{n+4}$$

   This shows that if $\frac{n^2+8n+51}{n+4}$ is a positive integer, then $\frac{35}{n+4}$ must be an integer. Since $n > 0$, $n+4 > 4$. The factors of 35 greater than 4 are 5, 7 and 35. This shows that there are three values for n:

   - $n+4 = 5 \implies n = 1$
   - $n+4 = 7 \implies n = 3$
   - $n+4 = 35 \implies n = 31$

   Therefore, the only positive integers, n, such that $\frac{n^2+8n+51}{n+4}$ is a positive integer are $n \in \{1, 3, 31\}$.

3. *Prove that for all real numbers x, y and z,*

$$x^2 + 5y^2 + z^2 \geq 2y(2x+z)$$

   Solution:

$$x^2 + 5y^2 + z^2 \geq 2y(2x+z)$$
$$\iff x^2 + 5y^2 + z^2 \geq 4xy + 2yz$$
$$\iff x^2 + 5y^2 + z^2 - 4xy - 2yz \geq 0$$
$$\iff x^2 - 4xy + 4y^2 + y^2 - 2yz + z^2 \geq 0$$
$$\iff (x-2y)^2 + (y-z)^2 \geq 0$$

   Squares of real numbers are never negative, so the sum of two squares of real numbers is greater than or equal to 0.

4. *The points E and F lie on sides AB and AD, respectively, of a parallelogram $ABCD$ such that $|AB| = 4|AE|$ and $|AD| = 4|AF|$. Prove that BF, DE, and AC are concurrent.*

   Solution: Let DE AND FB intersect at G. Join BD and AG and let AG extended intersect BD at O. By Ceva's Theorem in triangle ABD we have

   $$1 = \frac{DO}{OB} \cdot \frac{BE}{EA} \cdot \frac{AF}{FD} = \frac{DO}{OB} \cdot \frac{3}{1} \cdot \frac{1}{3},$$

   hence $DO = OB$. Since the diagonals of a parallelogram bisect each other, O lies on the diagonal AC; hence BF, DE, and AC are concurrent.

5. *The cells of an 8×8 chessboard are all coloured in white. A move consists in inverting the colours of a 1×3 rectangle, either vertical or horizontal (the white cells become black and the black cells become white). Is it possible to colour all cells of the chessboard in black in a finite number of moves?*

   Solution: Let us label the square in the rth row and cth column as (r, c) where $1 \leq r, c \leq 8$. Now let us colour the chessboard in three repeating diagonal colours red, green, and blue, where a square (r, c) is red if $3 \mid r + c$, green if $3 \mid r + c - 1$, and blue if $3 \mid r + c - 2$. The total number of red squares, which we denote as R, is 22, and analogously $G = 21$ and $B = 21$. We also keep track of the number of the number of white squares of each colour, which we denote by R_o, G_o, and B_o; initially these are also 22, 21, and 21 respectively.

   Note that after each move, the parities of each of R_o, G_o, and B_o changes since one of each category is toggled by each 1×3 rectangle. Thus R_o and G_o always have different parities, and in particular cannot both be zero. Thus we cannot have all the squares of the chessboard be black.