Advanced Test 4

Stellenbosch Camp 2019

Time: $2\frac{1}{2}$ hours

Each question is worth 7 marks.

1. Find all positive integer solutions to the equation

$$\lfloor\sqrt{8n+1}\rfloor + \lfloor\sqrt{8n+2}\rfloor + \cdots + \lfloor\sqrt{8n+7}\rfloor = 2027.$$

2. Let $x_0, x_1, ..., x_n$ be real numbers and define

$$y_k = x_k - x_{n-k}, \quad k = 0, 1, ..., n.$$

   Prove that

$$y_0^2 + y_1^2 + ... + y_n^2 \leq 4(x_0^2 + x_1^2 + ... + x_n^2)$$

   and determine when equality holds

3. Let ABC be a triangle, and let M and N be the midpoints of AB and CB respectively. Let T be the intersection of the line through M perpendicular to AC and the line through B perpendicular to BC. Show that TN is equal to the radius of the circumcircle of $\triangle ABC$.

4. Is there a finite set S of positive integers, each of which is greater than 1, such that for every positive integer n greater than 3 we have that $3^3 + 4^3 + \cdots + n^3$ is divisible by one of the values in S?

5. Let n be a positive integer, and let $\mathcal{C}$ be a collection of subsets of $\{1, 2, \ldots, 2n\}$ such that for every two distinct subsets $S_1, S_2 \in \mathcal{C}$ neither of them is a subset of the other. What is the maximal number of sets in $\mathcal{C}$?

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