Advanced Test 4 Solutions

Stellenbosch Camp 2019

1. *Find the positive integer solutions to the equation*

$$\lfloor\sqrt{8n+1}\rfloor + \lfloor\sqrt{8n+2}\rfloor + \cdots + \lfloor\sqrt{8n+7}\rfloor = 2027.$$

Solution: Notice that for $n > 1$, $\lfloor\sqrt{8n+1}\rfloor$ and $\lfloor\sqrt{8n+7}\rfloor$ differ by at most 1. This means that all values $\lfloor\sqrt{8n+1}\rfloor$, $\lfloor\sqrt{8n+2}\rfloor$, $\ldots$, $\lfloor\sqrt{8n+7}\rfloor$ are either the same or differ by 1. Suppose that there are a values that differ by 1 and let $\lfloor\sqrt{8n+1}\rfloor = k$; then

$$7k + a = 2027.$$

Notice that $2027 \equiv_7 4 \implies a = 4 \implies k = 289$. From this, we see that $\lfloor\sqrt{8n+1}\rfloor = 289$ and specifically $\sqrt{8n+4} = 290 \implies 8n+4 = 290^2$. Therefore, the only value that works is $n = \frac{290^2-4}{8} = \frac{145^2-1}{2}$ which is an integer.

2. *Let $x_0, x_1, ..., x_n$ be real numbers and define*

$$y_k = x_k - x_{n-k}, \quad k = 0, 1, ..., n.$$

Prove that

$$y_0^2 + y_1^2 + ... + y_n^2 \leq 4(x_0^2 + x_1^2 + ... + x_n^2)$$

and determine when equality holds.

Solution: Substitute the definition of all y_i into the inequality:

$$\begin{aligned}
& \sum_{i=0}^{n} y_i^2 \leq 4\sum_{i=0}^{n} x_i^2 \\
\iff \quad & \sum_{i=0}^{n}(x_i - x_{n-i})^2 \leq 4\sum_{i=0}^{n} x_i^2 \\
\iff \quad & 2\sum_{i=0}^{n} x_i^2 - 2\sum_{i=0}^{n} x_i x_{n-i} \leq 4\sum_{i=0}^{n} x_i^2 \\
\iff \quad & 0 \leq 2\sum_{i=0}^{n} x_i^2 + 2\sum_{i=0}^{n} x_i x_{n-i} = \sum_{i=0}^{n}(x_i + x_{n-i})^2
\end{aligned}$$

Since the square of a real number is always non-negative, the sum on the right must also be non-negative. Equality only holds when

$$x_i + x_{n-i} = 0 \quad \text{for all choices of } i \text{ with } 0 \leq i \leq n.$$

3. *Let ABC be a triangle, and let M and N be the midpoints of AB and CB respectively. Let T be the intersection of the line through M perpendicular to AC and the line through B perpendicular to BC. Show that TN is equal to the radius of the circumcircle of $\triangle ABC$.*

Solution:

Let the line perpendicular to AC at A intersect TB at F. By the midpoint theorem in $\triangle ABF$, we get $TB = TF$ since $AF \parallel MT$ and $MB = MA$. Now by the midpoint theorem in $\triangle BFC$, we have that $CF = 2NT$. We also have that CF is the diameter of the circumcircle of $\triangle BAC$ since $\angle CAF = 90°$. It therefore follows that NT is equal to the radius of the circumcircle of $\triangle ABC$ since it is half the diameter.

4. *Is there a finite set S of positive integers, each of which is greater than 1, such that for every positive integer n greater than 3 we have that $3^3 + 4^3 + \cdots + n^3$ is divisible by one of the values in S?*

   Solution: Let $M = 3^3 + 4^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 - 9$. Assume there exists some set S such that the condition holds. Note that if a composite number in S divides M, its prime factors also divide M. This means that we can consider S to be a set of prime numbers.

   Let the primes in S be $p_1,\ p_2,\ \ldots,\ p_s$ and let $m = \prod_{p_i \neq 3} p_i$. There exists some value k such that $km \equiv_3 2$. Consider what happens when $n = 2km$.

   For every value of p_i:

$$\begin{aligned} & M \equiv_{p_i} (km(2km+1))^2 - 9 \\ \implies \quad & M \equiv_{p_i} 0 - 9 \equiv_{p_i} -9 \\ \implies \quad & p_i \nmid M \end{aligned}$$

   This shows that only possible value in the set that can divide M is 3. However, since n was constructed to have $n \equiv_3 1$:

$$\begin{aligned} & M \equiv_3 (km(2km+1))^2 - 9 \\ \implies \quad & M \equiv_3 (2 \times (1+1))^2 - 9 \equiv_3 1 \\ \implies \quad & 3 \nmid M \end{aligned}$$

   This shows that for any finite set S, a value of n can be constructed such that $3^3 + 4^3 + \cdots + n^3$ is not divisible by any value in the set. Therefore, no such set exists.

5. *Let n be a positive integer, and let $\mathcal{C}$ be a collection of subsets of $\{1, 2, \ldots, 2n\}$ such that for every two distinct subsets $S_1, S_2 \in \mathcal{C}$ neither of them is a subset of the other. What is the maximal number of sets in $\mathcal{C}$?*

   Solution:

   The unique solution is to use the $\binom{2n}{n}$ subsets of size n.

   To see that you can't do better begin by considering the $(2n)!$ permutations of $\{1, 2, \ldots, 2n\}$. Associate a subset of $\{1, 2, \ldots, 2n\}$ to a permutation of $\{1, 2, \ldots, 2n\}$ if the subset is exactly the first k elements of the permutations. Notice that no permutation can be associated to more than one chosen set (or else one would be a subset of the other, violating our condition). Notice further that a size k subset is associated to $k!(2n-k)!$ permutations. This is minimized at $n = k$. Thus each set is associated to at least $(n!)^2$ permutations, and so there are at most $(2n)!/(n!)^2 = \binom{2n}{n}$ subsets. Further as we can choose all n element subsets of $\{1, 2, \ldots, 2n\}$ we are done.