Intermediate Test 4

Stellenbosch Camp 2019

Time: $2\frac{1}{2}$ hours

Each question is worth 7 marks.

1. Prove that for any natural number n, $n^5 - 5n^3 + 4n$ is divisible by 120.

2. Four pair of socks are hung out side by side on a straight washing line. The socks in each pair are identical but the pair themselves are different colours. How many different colour patterns can be made if no sock is allowed to be next to its matching pair?

3. Let $n \geq 3$ be a positive integer. Determine all n such that it is possible to find a pair of diagonals of a regular n-gon which intersect at $90°$.

4. Find the positive integer solutions to the equation
$$\lfloor \sqrt{8n+1} \rfloor + \lfloor \sqrt{8n+2} \rfloor + \cdots + \lfloor \sqrt{8n+7} \rfloor = 2027.$$

5. Let $x_0, x_1, ..., x_n$ be real numbers and define
$$y_k = x_k - x_{n-k}, \quad k = 0, 1, ..., n.$$
Prove that
$$y_0^2 + y_1^2 + ... + y_n^2 \leq 4(x_0^2 + x_1^2 + ... + x_n^2)$$
and determine when equality holds.