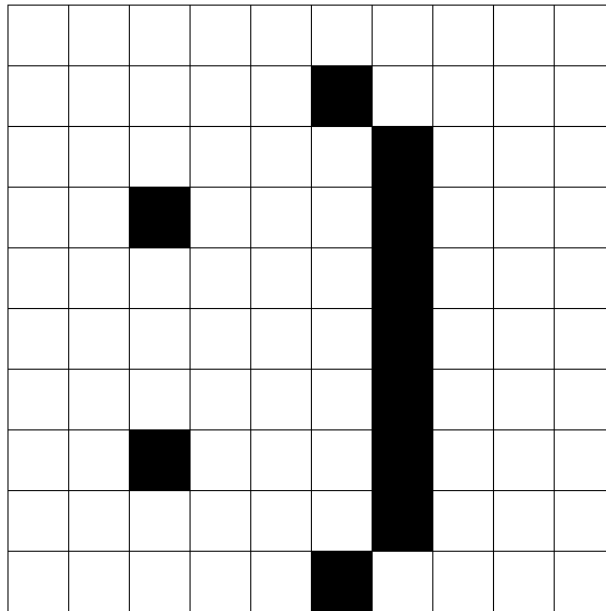# Intermediate Test 5

Stellenbosch Camp 2019

Time: 4 hours

Each question is worth 7 marks.

1. Given the below colouring, is it possible to invert the colours of rows or the colours of columns in some order to achieve a completely white board?

2. Let n be a positive integer greater than 2. Let r_1 be the smallest odd divisor of n greater than 1 and let r_2 be the largest odd divisor of n. Find all n such that

$$n = 5r_1 + 3r_2$$

3. For each positive integer k, define the sequence (a_n) by

$$a_0 = 1$$
$$a_n = kn + (-1)^n a_{n-1} \quad \text{for each } n \geqslant 1.$$

Determine all values of k for which 2000 is a term of the sequence.

4. Let $\triangle XYZ$ be such that $\angle XZY = 30^o$. Let M be a point inside $\triangle XYZ$. Let A and B be points on XZ and YZ respectively such that $\angle ZAM = \angle ZBM = 90^o$. Prove that $ZM = 2 \cdot AB$.

5. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x^2) + f(x + 2y) = (x + 1)f(x) + 2f(y)$$

for all $x, y \in \mathbb{Q}$.

6. In the country of Oddland, there are stamps with values 1 cent, 3 cents, 5 cents, etc., one type for each odd number. the rules of Oddland Postal Services stipulate the following: for any two distinct values, the number of stamps of the higher value on an envelope must never exceed the number of stamps of the lower value.

In the country of Squareland, on the other hand, there are stamps with values 1 cent, 4 cents, 9 cents, etc., one type for each square number. Stamps can be combined in all possible ways in Squareland without additional rules.

Prove that for every positive integer n: In Oddland and Squareland there are equally many ways to correctly place stamps of a total value n cents on an envelope. Rearranging the stamps on an envelope makes no difference.

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