Advanced Test 5

Stellenbosch Camp 2019

Time: 4 hours

Each question is worth 7 marks.

1. In the country of Oddland, there are stamps with values 1 cent, 3 cents, 5 cents, etc., one type for each odd number. the rules of Oddland Postal Services stipulate the following: for any two distinct values, the number of stamps of the higher value on an envelope must never exceed the number of stamps of the lower value.

   In the country of Squareland, on the other hand, there are stamps with values 1 cent, 4 cents, 9 cents, etc., one type for each square number. Stamps can be combined in all possible ways in Squareland without additional rules.

   Prove that for every positive integer n: In Oddland and Squareland there are equally many ways to correctly place stamps of a total value n cents on an envelope. Rearranging the stamps on an envelope makes no difference.

2. Find all pairs of positive integers (m, n) satisfying

$$m^2 + n^2 = 2019(m - n).$$

3. Given a set of distinct points (x_1, y_1), (x_2, y_2), $\ldots$, (x_{8074}, y_{8074}) with $x_i, y_i \in \{1, 2, \ldots, 8076\}$. Prove that there exists a positive integer K such that there are 2019 unique pairs $((x_i, y_i), (x_j, y_j))$ with

$$|x_i - x_j| + |y_i - y_j| = K.$$

4. In a triangle ABC, the internal bisector of $\angle A$ meets the side BC at D. The lines through D tangent to the circumcircles of triangles $\triangle ABD$ and $\triangle ACD$ meet the lines AC and AB at points E and F, respectively. Lines BE and CF intersect at G. Prove that $\angle EDG = \angle ADF$.

5. Let n be a positive integer greater than 1, and consider a circle of radius 1 in which is inscribed a regular $2n$-gon P with vertices labelled from 1 to $2n$ in that order. Consider the set S of positive divisors of $2n$, and the convex polygon G formed by the points of P with labels in S. If the area of G is denoted by $|G|$ and the number of elements of S is denoted by $\tau(2n)$, show that
$$\frac{1}{2}\left(\sin\left(\frac{\pi}{n}\right)+\cos\left(\frac{\pi}{n}\right)\right)<|G|<\frac{\tau(2n)}{2}\sin\left(\frac{\pi}{\tau(2n)}\right).$$

6. Do there exist a positive integer k and a nonconstant sequence $a_1, a_2, a_3, \ldots$ of positive integers such that for each positive integer n,
$$a_n = \gcd(a_{n+k}, a_{n+k+1})?$$

```
 "|||||,,,..    ..,,,|||||"
 '||(     |||,|||    )||'
  "||,,,,. "|" .,,,,||"
   .d8888b.    .d8888b.
  o8'  o '8o o8'o   `8o
  o8.    .8o o8.    .8o
   `Y8888P'    `Y8888P'
  ,||''|| \   / ||''||,
 ,||   ||, \ / .||   ||,
 ||     ||  `  ||     ||
,||     '||   ||'     ||,
||      '||   ||'      ||
||       |;   ;|       ||
||      ,|     |,      ||
||,    ,||     ||,    ,||
 ||,  ,|||     |||,  ,||
 '||,,||||,...,||||,,||
   `|||..."|||"...|||'
 %%%%%WWWW%%%%%%WWWW%%%%%
 `````""3$F""""#$F"""""""
        @$.... '$B
       d$$$$$$$$$$:
```
```