Intermediate Test 1 Solutions

Stellenbosch Camp 2019

1. *Find the number of paths from A to B, where the only allowed moves are moving down along a diagonal line or moving left or right along a horizontal line, but never crossing the same line twice.*

   Solution: Let us define the diagram in terms of layers, with A being layer 1, the two nodes below A are layer 2, etc. with B as layer 7.

   Note that if you step down a layer, you can go to any of the other nodes on that layer in exactly 1 way using lefts & rights. This implies that the number of paths to nodes in the same layer are the same, and are equal to the number of connecting branches between the previous and current layer multiplied by the number of paths to a node on the previous layer.

   Thus the number of paths from A to B is:

   $$1 \times 2 \times 4 \times 4 \times 4 \times 4 \times 2 = 2^{10}.$$

2. *Two intersecting circles, C_1 and C_2, have a common tangent which touches C_1 at P and C_2 at Q. The two circles intersect at M and N, where N is nearer to PQ than M is. The line PN meets the circle C_2 again at R. Prove that MQ bisects $\angle PMR$.*

   Solution: Draw lines PN, QN and MN. By the tan-chord theorem, we have that $\angle QPN = \angle PMN$ and $\angle PQN = \angle QMN$ and so $\angle PMQ = \angle PMN + \angle QMN = \angle QPN + \angle PQN$. Also $\angle RMQ = \angle RNQ$ since $RQNM$ is cyclic. From triangle PNQ, we obtain that $\angle QPN + \angle PQN = \angle RNQ$ hence $\angle PMQ = \angle RMQ$ and so MQ bisects $\angle PMR$.

3. *Emma and Emile play a game on a 2019×2019 board made up of unit grid squares. Emma plays first by placing a knight on one of the squares and thereafter they take turns to place a knight on a square that does not already contain a knight and is not attacked by one of the already placed knights. The first player who cannot do this loses. Can one of the players always guarantee that they will win? If so, which one?*

   Solution: Emma can guarantee that she always wins.

   Let us assume that the board is coloured like a chessboard. Emma moves first, so she can start by placing a knight on the block in the centre of the board on her first move. Now, whatever move Emile plays, Emma plays the same move, but rotates it 180^o around the centre of the board. If Emile makes a legal move, it will not invalidate the square rotated 180^o from it, since knights attack squares of opposite colour and the squares that are 180^o rotations of each other are the same colour. This guarantees that Emma will always be able to play if Emile can play. Therefore, the first person who will not be able to make a move is Emile, so Emma must win.

4. *Let a, b, and c be positive real numbers such that $b, c \in [1, 2)$ and*

   $$\frac{a+b}{b(1+c)} + \frac{a+c}{c(1+b)} = 2.$$

   Show that a, b, and c are the lengths of the sides of a triangle.

   Solution: By multiplying out and simplifying the given equation, we get that $a = bc$. Thus $a + b > a = bc \geq c$, and similarly $a + c > b$. Finally,

   $$b + c > a = bc \iff 1 > 1 - b - c + bc = (1 - b)(1 - c),$$

   which is true since $0 \leq b - 1, c - 1 < 1$.

5. *Find the least positive integer k such that 2050^{2051} can be written as a sum of k 5th powers.*

Solution: By Fermat's Little Theorem:

$$a^{10} \equiv_{11} 1 \ \forall a \in \mathbb{Z}, \gcd(a, 11) = 1$$

$$\implies (a^5)^2 \equiv_{11} 1$$

$$\implies 11 \mid (a^5)^2 - 1$$

$$\implies 11 \mid (a^5 - 1)(a^5 + 1)$$

Since 11 is prime, we must have $11 \mid a^5 - 1 \implies a^5 \equiv_{11} 1$ or $11 \mid a^5 + 1 \implies a^5 \equiv_{11} -1$. When $11 \mid a$, $a^5 \equiv_{11} 0$. This shows that the only values of powers of 5 mod 11 are -1, 0 and 1. Notice that $2050^{2051} = (2050^{205})^{10} \times 2050 \equiv_{11} 4$ and the smallest way to express 4 as the sum of some values from $\{-1, 0, 1\}$ is $4 = 1 + 1 + 1 + 1$. This shows that 2050^{2051} cannot be expressed as the sum of less than 4 fifth powers, so $k = 4$ is a lower bound.

$$(2050^{410} \times 2)^5 + (2050^{410} \times 2)^5 + (2050^{410} \times 1)^5 + (2050^{410} \times 1)^5 = (2050^{410 \times 5}) \times (2^5 + 2^5 + 1^5 + 1^5) = 2050^{2051}$$

This shows that $k = 4$ is possible, so $k = 4$ is the minimum.