Polynomials

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1 What is a polynomial?

A field is a set with well-behaved enough notions of addition, subtraction, multiplication and division, e.g. $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. $\mathbb{Z}$ is not a field since dividing two nonzero integers does not always yield an integer.

Problem: Show that $\mathbb{Z}_p$ (i.e. the integers modulo a prime number p) is a field.

A ring is a set with well-behaved enough notions of addition, subtraction, and multiplication (not necessarily division), e.g. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$. $\mathbb{N}$ is not a ring since we cannot always subtract two natural numbers and expect a positive result.

Problem: Show that $\mathbb{Z}_n$ (i.e. the integers modulo a fixed integer n) is a ring.

Every field is a ring.

A polynomial is an algebraic expression $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots a_1 x + a_0$, where the *coefficients* $a_0, a_1, \ldots$ are in some ring R. The set of all polynomials with coefficients in R and variable x is denoted $R[x]$; this is itself a ring! But never a field...

The largest n such that the coefficient of x^n is nonzero (i.e. $a_n \neq 0$) is called the *degree* of p, sometimes denoted $\deg p$. a_0 is called the *constant term*, $a_{\deg p}$ is called the *leading term*, and the polynomial with all coefficients equal to zero is called the *zero polynomial*.

A number r such that $p(r) = 0$ is called a *root* or *zero* of p.

2 Fundamental Results

3 Euclidean Division Algorithm

4 Fundamental Theorem of Algebra

5 Viète's Formulas

6 Problems

1.