Intermediate January Monthly Problem Set

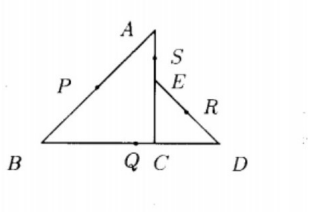
Due: Friday, 17 January 2020

1. Find all integers m and n such that $5m - 7n = 1$.

2. Prove that, if a, b and c are positive numbers such that $a < b + c$, then

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c}.$$

3. In the figure, $BC = CA$, $EC = CD$ and $\angle BCA = \angle DCE = 90°$. If P, Q, R and S are the midpoints of AB, BD, DE and EA respectively, prove, in two different ways, that $PQRS$ is a square.



4. Prove that among any six integers there will be a pair whose sum or difference is divisible by 9.

5. Prove that, if a, b and c are positive numbers such that $abc = 1$, then

$$\frac{a}{ab+a+1} + \frac{b}{bc+b+1} + \frac{c}{ca+c+1} = 1.$$

6. $\triangle ABC$ and $\triangle DBE$ are isosceles with $\angle ABC = \angle DBE = 36°$ and $AB = BC$ and $DB = DE$. Find the angle between lines AD and CE.

7. On the board, we write the integers $1, 2, 3, \ldots, 2019$. At each minute, we pick two numbers on the board a and b, erase them, and write down the number $s(a + b)$ instead where $s(n)$ denotes the sum of the digits of the integer n. Let N be the last number remaining on the board.

   (a) Is it possible that $N = 19$?

   (b) Is it possible that $N = 15$?

8. For which positive integers n is it possible to divide the set of numbers $\{n, n + 1, n + 2, \ldots, n + 8\}$ into two disjoint sets A and B such that the product of the numbers in A is equal to the product of the numbers in B?

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).
- Submit each question in a single separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.