Senior January Monthly Problem Set

Due: Friday, 17 January 2020

1. On the board, we write the integers $1, 2, 3, \ldots, 2019$. At each minute, we pick two numbers on the board a and b, erase them, and write down the number $s(a+b)$ instead where $s(n)$ denotes the sum of the digits of the integer n. Let N be the last number remaining on the board.

   (a) Is it possible that $N = 19$?

   (b) Is it possible that $N = 15$?

2. For which positive integers n is it possible to divide the set of numbers $\{n, n+1, n+2, \ldots, n+8\}$ into two disjoint sets A and B such that the product of the numbers in A is equal to the product of the numbers in B?

3. Let M be a positive integer, and let S denote the set of finite sequences of positive integers less than or equal to M, including the empty sequence of length zero, which we denote as $\mathbf{0}$. Also, for a sequence $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ let $\overline{\mathbf{x}}$ denote the reverse sequence $\overline{\mathbf{x}} = (x_n, x_{n-1}, \ldots, x_1)$.

   Define a function d from S to the integers as follows:

   - $d(\mathbf{0}) = 0$.
   - If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a sequence of positive length, let m be the largest integer such that $x_1 + x_2 + \cdots + x_m \leq M$, and let $\mathbf{x}'$ denote the rest of the sequence: $\mathbf{x}' = (x_{m+1}, \ldots, x_n)$. Then $d(\mathbf{x}) = 1 + d(\mathbf{x}')$.

   Show that $d(\mathbf{x}) = d(\overline{\mathbf{x}})$.

4. Let $ABCD$ be a cyclic quadrilateral with its diagonals intersecting at E. Let M be the midpoint of AB. Suppose that ME is perpendicular to CD. Show that either AC is perpendicular to BD or AB is parallel to CD.

5. We have done it! We have planted an infinite number of trees on the vertices of an infinite regular grid, one for each vertex. Let T be a positive integer. We define a *T-forest* as a set of trees such that for any two trees in the T-forest, there exists another tree planted in the grid such that the area of the triangle with these three trees as vertices is T.

   What is the smallest T such that our T-forest has more than 200 trees?

6. Given a series $t_1, t_2, \ldots, t_n$ such that

$$t_{k+1} = \frac{t_k^2 + 1}{t_{k-1} + 1} - 1 \quad \forall\, k \in \{2, 3, \ldots, n-1\}.$$

   For which $n \in \mathbb{N}$ does there exist a t_1 and t_2 such that $t_i \in \mathbb{N}$ for all $i \in \{1, 2, 3, \ldots, n\}$?

7. Let AC and BD be two chords of a circle Γ that intersect at X in the interior of Γ. Let Γ_1 and Γ_2 be circles that are mutually tangent at X and are tangent to Γ at P and Q. Let ω be a circle tangent to Γ_1 and Γ_2 at X that intersects the chords AB and CD at M and N respectively. Prove that

$$\frac{MP}{MQ} = \frac{NP}{NQ} \implies \angle AXM = \angle DXN.$$

8. Let n be a positive integer greater than 1, and consider a circle of radius 1 in which is inscribed a regular n-gon P with vertices labelled from 1 to n in that order. Consider the set S of positive divisors of n, and the convex polygon G formed by the points of P with labels in S. If the area of G is denoted by $|G|$, show that

$$|G| < \frac{3}{2}.$$

Email submission guidelines

- Email your solutions to `samf.training.assignments@gmail.com`.
- In the subject of your email, include your name and the level of the assignment (Beginner, Intermediate or Senior).
- Submit each question in a separate PDF file (with multiple pages if necessary), with your name and the question number written on each page.
- If you take photographs of your work, use a document scanner such as Office Lens to convert to PDF.
- If you have multiple PDF files for a question, combine them using software such as PDFsam.