April Camp 2018: Test 3
IMO Stream
Time: $4\frac{1}{2}$ hours

1. Let O be the circumcentre of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q respectively. The altitudes meet at H. Prove that the circumcentre of triangle PQH lies on a median of triangle ABC.

2. Let $p \geq 2$ be a prime number. Lauren and Dylan play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, \ldots, 9\}$. Lauren has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:
$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$
Lauren's goal is to make the number M divisible by p, and Dylan's goal is to prevent this.

   Prove that Lauren has a winning strategy.

3. A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation
$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all } n > 2017.$$
Prove that this sequence is bounded, i.e., there is a constant M such that $|a_n| \leq M$ for all positive integers n.