April Camp 2018: Test 3
PAMO Stream
Time: $4\frac{1}{2}$ hours

1. Consider the sequence generated by $a_0 = 1$, $a_1 = 2$ and

$$2a_{n+2} - 5a_{n+1} + 2a_n = 0$$

   Prove that $a_n \in \mathbb{Z} \forall n \in \mathbb{N}$.

2. Let O be the circumcentre of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q respectively. The altitudes meet at H. Prove that the circumcentre of triangle PQH lies on a median of triangle ABC.

3. Let $p \geq 2$ be a prime number. Lauren and Dylan play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, \ldots, 9\}$. Lauren has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

   Lauren's goal is to make the number M divisible by p, and Dylan's goal is to prevent this.

   Prove that Lauren has a winning strategy.