PAMO Shortlist

2018

1 Algebra

1. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $(f(x+y))^2 = f(x^2) + f(y^2)$ for all $x, y \in \mathbb{Z}$.

2. Find a non-zero polynomial $f(x, y)$ such that $f(\lfloor 3t \rfloor, \lfloor 5t \rfloor) = 0$ for any $t \in \mathbb{R}$.

3. Akello divides a square up into finitely many white and red rectangles, each (rectangle) with sides parallel to the sides of the paren square. Within each white rectangle, she writes down the value of its width divided by its height, while within each red rectangle, she writes down the value of its height divided by its width. Finally, she calculates x, the sum of these numbers. If the total are of the white rectangles equals the total area of the red rectangles, what is the least possible value of x she can get?

4. Let a, b, c and d be non-zero pairwise different real numbers such that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4 \text{ and } ac = bd.$$

   Show that

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} \leq -12$$

   and that -12 is the maximum.

5. Let $g : \mathbb{N} \to \mathbb{N}$ be a function satisfying:

   (a) $g(xy) = g(x)g(y), \forall x, y \in \mathbb{N}$,
   (b) $g(g(x)) = x, \forall x \in \mathbb{N}$,
   (c) $g(x) \neq x$ for $2 \leq x \leq 2018$.

   Find the minimum value of $g(2)$.

6. Let a, b, c be positive real numbers such that $a^3 + b^3 + c^3 = 5abc$.
   Show that

$$\left(\frac{a+b}{c}\right)\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right) \geq 9.$$

7. Let $f(n) = n + \lfloor\sqrt{n}\rfloor$. Prove that for every positive integer m, the integer sequence $m, f(m), f(f(m)), \ldots$ contains at least one square of an integer.

2 Number Theory

1. Does there exist positive integers a, b, c such that $4(ab - a - c^2) = b$.

2. A positive integer is called special if its digits can be arranged to form an integer divisible by 4. How many of the integers from 1 to 2018 are special?

3. For any positive integer x, we set

$$g(x) = \text{the largest odd divisor of } x,$$

$$f(x) = \begin{cases} \frac{x}{2} + \frac{x}{g(x)} & \text{if } x \text{ is even;} \\ 2^{\frac{x+1}{2}} & \text{if } x \text{ is odd.} \end{cases}$$

Consider the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_1 = 1$, $x_{n+1} = f(x_n)$. Show that the integer 2018 appears in this sequence, determine the least integer n such that $x_n = 2018$, and determine whether n is unique or not.

4. Let S be a set of 49-digit numbers n, with the property that each of the digits $1, 2, 3, \ldots, 7$ appears in the decimal expansion of n seven times (and $8, 9$ and 0 do not appear). Show that no two distinct elements of S divide each other.

5. Find all quadruplets (a, b, c, d) of positive integers such that

$$\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right)\left(1 + \frac{1}{d}\right) = 4.$$

6. Prove that there are infinitely many integers n such that both the arithmetic mean of its divisors and the geometric mean of its divisors are integers.

   (Recall that for k positive real numbers, $a_1, a_2, \ldots, a_k$, the arithmetic mean is $\frac{a_1+a_2+\cdots+a_k}{k}$, and the geometric mean is $\sqrt[k]{a_1 a_2 \cdots a_k}$.

7. Find all non-negative integers n for which the equation

$$\left(x^2 + y^2\right)^n = (xy)^{2018}$$

admits positive integral solutions.

3 Geometry

1. In a triangle ABC, let D and E be the midpoints of AB and AC, respectively, and let F be the foot of the altitude through A. Show that the line DE, the angle bisector of $\angle ACB$ and the circumcircle of ACF pass through a common point.

    Alternate version: In a triangle ABC, let D and E be the midpoints of AB and AC, respectively. The line DE and the angle bisector of $\angle ACB$ meet at G. Show that $\angle AGC$ is a right angle.

2. Let P be a point on the median AM of a triangle ABC. Suppose that the tangents to the circumcircles of ABP and ACP at B and C respectively meet at Q. Show that $\angle PAB = \angle CAQ$.

3. Given a triangle ABC, let D be the intersection of the line through A perpendicular to AB, and the line through B perpendicular to BC. Let P be a point inside the triangle. Show that $DAPB$ is cyclic if and only if $\angle BAP = \angle CBP$.

4. Let ABC be a triangle and Γ be the circle of diameter $[AB]$. The bisectors of $\angle BAC$ and $\angle ABC$ cut the circle Γ again in D and E, respectively. The incircle of the triangle ABC cuts the lines BC and AC in F and G, respectively. Show that the points D, E, F and G lie on the same line.

5. Let ABC be a triangle with $AB \neq AC$. The incircle of ABC touches the sides BC, CA, AB at X, Y, Z respectively. The line through Z and Y intersects BC extended in X'. The lines through B that are parallel to AX and AC intersect AX' in K and L respectively. Prove that $AK = KL$.

6. Let Γ be the circumcircle of an acute triangle ABC. The perpendicular line to AB passing by C cuts AB in D and Γ again in E. The bisector of the angle C cuts AB in F and Γ again in G. The line GD meets again Γ in H and the line HF meets it again in I. Prove that $AI = EB$.

4 Combinatorics

1. A chess tournament is held with the participation of boys and girls. The girls are twice as many as boys. Each pair of players plays only one match. By the end of the tournament, there were no draws and the ratio between girl winnings and boy winnings was $\frac{7}{9}$. How many players took part at the tournament?

2. Adamu and Afaafa choose, each in his turn, positive integers as coefficients of a polynomial of degree n. Adamu wins if the polynomial obtained has an integer root; otherwise, Afaafa wins. Afaafa plays first if n is odd; otherwise Adamu plays first. Prove that:

    i) Adamu has a winning strategy if n is odd.

ii) Afaafa has a winning strategy if n is even.

3. A game is played on an $m \times n$ chessboard. At the beginning, there is a coin on one of the squares. Two players take turns to move the coin to an adjacent square (horizontally or vertically). The coin may never be moved to a square that has been occupied before. If a player cannot move any more, he loses. Prove:

   1. If the size (number of squares) of the board is even, then the player to move first has a winning strategy, regardless of the initial position.

   2. If the size of the board is odd, then the player to move first has a winning strategy if and only if the coin is initially placed on a square whose colour is not the same as the colour of the corners.

4. Seven cyclists follow one another, in a line, on a narrow way. By the end of the training, each cyclist complains about the style of driving of the one in front of him. They agree to rearrange themselves the next day, so that no cyclist would follow the same cyclist he follows the first day. How many such rearrangements are possible?

5. n lines are said to be in *standard form* if no two are parallel and no three are concurrent. Does there exist a value of k such that given any n lines in *standard form*, it is possible to colour the regions bounded by the n lines using k colours in such a way that no two regions of the same colour share a common intersection point of the n lines?

6. A circle is divided into n cells ($n \geq 3$). Each cell can be filled in with either 1 or 0. Choose any cell $\mathcal{C}$ occupied by 0, change it into a 1 and simultaneously change the symbols x, y in the two adjacent cells to $\mathcal{C}$ to their complements $1 - x$, $1 - y$. We repeat this process as long as there exists a zero in some cell. In the initial configuration there is a 0 in one cell and 1s elsewhere. For which values of n we can end this process?
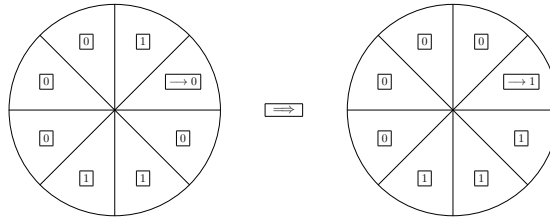

Figure 1: Problem 6