# Day 1

## Marking Scheme for Problem 1 Day 1

- Prove that $f(0) = 0$ or $2$ $\boxed{\textbf{1 mark}}$

Case $f(0) = 0$.

- Prove that $f(x^2) = 0$ **or** $(f(x))^2 = x(f(1))^2$ $\boxed{\textbf{1 mark}}$
- Prove that $f(x) = 0 \forall x$. $\boxed{\textbf{1 mark}}$

Case $f(0) = 2$.

- Prove that $f(x^2) = 2$. $\boxed{\textbf{1 mark}}$
- Prove that $f(x) = \pm 2$ $\boxed{\textbf{1 mark}}$
- Prove that $f(x) = \begin{cases} 2 & \text{if x is a square} \\ \pm 2 & \text{otherwise} \end{cases}$ $\boxed{\textbf{1 mark}}$
- Check or prove all solutions. $\boxed{\textbf{1 mark}}$

**Remark:** If the contestant finds that there is a Cauchy functional equation either in one or both cases, he gets 1 mark.

## Marking Scheme for Problem 2 Day 1

- Prove that the total number of matches if $\binom{3n}{2}$ $\boxed{\textbf{1 mark}}$
- Prove that boys winnings is $\frac{7}{16}\binom{3n}{2}$ $\boxed{\textbf{1 mark}}$
- Prove that girls winnings between them is $\binom{2n}{2} = n(2n-1)$ $\boxed{\textbf{1 mark}}$
- Prove that $n \leq 11$ $\boxed{\textbf{1 mark}}$
- Prove that $n = 11$ either by
  - Checking that the only possible value is 11 $\boxed{\textbf{1 mark}}$
  - Justifying that the number of boys winnings is an integer $\boxed{\textbf{1 mark}}$

  or by

  - Checking that $32 \mid 21n(3n-1)$ $\boxed{\textbf{1 mark}}$
  - Finding that $n \geq 11$. $\boxed{\textbf{1 mark}}$

  The last mark is not attained if the conclusion $n = 11$ is not explicitly written.

- Conclusion: 33 players **1 mark**

**Alternatives** We set $m$ to be the number of times a girl wins against a boy

- Prove that girls winnings are $n(2n-1)+m$ **1 mark**
- Prove that boys winnings are $\binom{n}{2}+2n^2-m$ **1 mark**
- Find the equation $n(2n-1)+m=\frac{7}{9}\left(\frac{n(n-1)}{2}+2n^2-m\right)$ **1 mark**
- Conclude that $11n \geq n^2$ **1 mark**
- Prove that $n=11$ as above **2 marks**
- Conclusion: 33 players **1 mark**

## Marking Scheme for Problem 3 Day 1

- Remark that if $\exists r \in \mathbb{N}$ such that $x_{m_r}=2^r$ then

  $$x_{m_r+i}=2^{r-i}(2i+1) \quad i \in \{1,\dots,r\}$$ **1 mark**

  and $x_{m_r+r+1}=2^{r+1}$ **1 mark**
- Conclude that every natural number appears at least once **1 mark**
- Prove that $m_{r+1}=m_r+r+1$ **1 mark**
- Prove that $x_{(r+1)(r+2)}2=2r+1$ or something like **1 mark**
- Find the value of $n$ **1 mark**
- Prove somewhere that $n$ is unique **1 mark**

# Day 2

## Marking Scheme for Problem 4 Day 2

- Considering both configurations or directed angles, etc... **1 mark**
- $\Longrightarrow$ one direction **3 marks**
- no $\Longrightarrow$ or $\Longleftrightarrow$ but works both ways **5 marks**
- with $\Longleftrightarrow$ or `iff` or `ssi` **6 marks**
- for each relevant angle equality **1 mark** maximum **2 marks**

**Alternatives**

- $DAPB$ cyclic $\implies DB$ diameter **1 mark**
- $BC$ perpendicular to radius $\implies BC$ tangent **1 mark**
- $\angle CBP = \angle BAP$ (tan-chord) **1 mark**
- Conversely if $\angle CBP = \angle BAP \implies BC$ is tangent to circumcircle of $ABP$ **1 mark**
- $\implies DB$ passes through centre ($\perp$ to tangent) **1 mark**
- Since $\angle DAB = 90°$, $D$ is diametrically opposite $B$ so $D$ is on the circle **2 marks**

## Marking Scheme for Problem 5 Day 2

- Prove that $(a+c)(b+d) = 4ac$ **or** $\frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}}{4} \geq \sqrt[4]{1}$ **1 mark**
- Prove that $(a+c)(b+d) > 4ac = (a+c)(b+d)$ which is a contradiction **or** checking that equality occurs iff all terms are equal meaning that $b = d$, contradiction again **1 mark**
- Prove that $A = \frac{(a+c)^2+(b+d)^2}{ac} - 4$ **2 marks**
- Prove that $(a+c)^2 + (b+d)^2 \geq 2|a+c||b+d|$ **1 mark**
- Prove that $\frac{(a+c)^2+(b+d)^2}{ac} \leq \frac{2|a+c||b+d|}{ac} = -8$ **1 mark**
- Providing the example of equality **1 mark**

**Alternatives**

- Show that $4 = \frac{d}{c} + \frac{a}{d} + \frac{b}{a} + \frac{c}{b}$. **1 mark**
- State that $x_i$ are pairwise different **1 mark**
- Show that $x_1, x_2, x_3, x_4$ are roots of $x^4 - 4x^3 + (A+2)x^2 - 4x + 1$ **1 mark**
- Show that $x_i + \frac{1}{x_i}$ are roots of $t^2 - 4t + A$ **1 mark**
- Show that $x_i + \frac{1}{x_i} = 2 \pm \sqrt{4-A}$, negative root occurs, and $\sqrt{4-A} > 0$ **1 mark**
- Show that $A \leq -12$ **1 mark**
- Providing the example of equality **1 mark**

## Marking Scheme for Problem 6 Day 2

- Remark that the process ends once we have 1s everywhere **1 mark**
- Case $3 \mid n$: set $s_w, s_r, s_b$ and check that $s_w = \frac{n}{3} - 1$, $s_r = \frac{n}{3}$, $s_b = \frac{n}{3}$ **1 mark**
- Conclude a contradiction **1 mark**

Case $n = 3k + 1$:

- Transform configuration $110111\ldots1111$ to $000000\ldots0001$ **1 mark**
- Divide into blocks of three and perform $k$ steps **1 mark**

Case $n = 3k + 2$:

- Transform configuration $110111\ldots1111$ to $000000\ldots0110$ **1 mark**
- Divide into blocks of three and perform $k$ steps **1 mark**