PAMO Test 2

2018

Time: $4\frac{1}{2}$ hours

1. Abimbola plays a game with a coin. He tosses the coin a number of times, and records whether each toss was a "heads" or "tails". He stops tossing the coin as soon as he tosses an odd number of heads in a row, followed by a tails. (Note that he stops if the number of heads since the previous time that he tosses tails is odd, and he then tosses another tails. If he has not tossed tails previously, then he stops if the total number of heads is odd, and he then tosses tails.) How many different sequences of coin tosses are there such that he stops after the n^{th} coin toss?

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ for all real numbers x and y.

3. For which prime numbers p can we find three positive integers n, x and y such that $p^n = x^3 + y^3$?