Prime Numbers Containing a Given String of Digits

An Application of the Prime Number Theorem

Dylan Nelson

Stellenbosch University Mathematics Society

8 April 2021

Outline

Reddit Post

- On 4 April 2016, a thread was posted to /r/math on reddit asking for the most surprising examples of divergent series.

Figure: https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/

Reddit Post — Primes with a Prime Number of Digits

- In one example, we consider the set of prime numbers with a prime number of digits.

Reddit Post — Primes with a Prime Number of Digits

- In one example, we consider the set of prime numbers with a prime number of digits.
- It is claimed that the sum of the reciprocals of the elements in this set diverges.

Figure: https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/d1oppgu

Reddit Post — Numbers Without a 9

- In another example, we consider all of the positive integers that *do not* have a 9 *anywhere* in their decimal expansion.

Reddit Post — Numbers Without a 9

- In another example, we consider all of the positive integers that *do not* have a 9 *anywhere* in their decimal expansion.
- In this case, it is claimed that the sum of the reciprocals of these numbers *converges*!

Figure: `https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/d1olh0o`

Combining these Results

- I realised that a combination of (appropriate generalisations) of these two claims implies that there are infinitely many primes which have a prime number of digits, and which contain any given string of decimal digits that you like.

Combining these Results

- I realised that a combination of (appropriate generalisations) of these two claims implies that there are infinitely many primes which have a prime number of digits, and which contain any given string of decimal digits that you like.
- And of course I promptly told everyone I know.

Combining these Results

- I realised that a combination of (appropriate generalisations) of these two claims implies that there are infinitely many primes which have a prime number of digits, and which contain any given string of decimal digits that you like.
- And of course I promptly told everyone I know.
- I even wrote a blog post about it!

Figure: `http://www.mathemafrica.org/?p=12942`

This Talk

- In this talk, we will prove this result following the argument presented in the blog post.

This Talk

- In this talk, we will prove this result following the argument presented in the blog post.
- *BUT*... by considering convergent and divergent series, the blog post is needlessly circuitous.

This Talk

- In this talk, we will prove this result following the argument presented in the blog post.
- *BUT*... by considering convergent and divergent series, the blog post is needlessly circuitous.
- It is possible to give a more direct proof of a stronger result:

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

This Talk

- In this talk, we will prove this result following the argument presented in the blog post.
- *BUT*... by considering convergent and divergent series, the blog post is needlessly circuitous.
- It is possible to give a more direct proof of a stronger result:

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

- I will present a proof of this more general proposition towards the end of the talk.

Outline

The Harmonic Series

- One of the first somewhat surprising examples of a divergent series that students are shown is the *Harmonic Series*

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

- To show that this diverges, we group the terms in blocks of sizes equal to powers of 2, and then approximate each term by the smallest element in its block.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{n=0}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq 1 + \sum_{n=0}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}}$$
$$= 1 + \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = 1 + \sum_{n=0}^{\infty} \frac{1}{2}.$$

The Harmonic Series

- We of course get different behaviour if we sum the reciprocals of some subset of the natural numbers.

The Harmonic Series

- We of course get different behaviour if we sum the reciprocals of some subset of the natural numbers.
- The sum of the reciprocals of the powers of 2 converges:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

The Harmonic Series

- We of course get different behaviour if we sum the reciprocals of some subset of the natural numbers.
- The sum of the reciprocals of the powers of 2 converges:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

- The sum of the reciprocals of the prime numbers diverges.

The Harmonic Series

- We of course get different behaviour if we sum the reciprocals of some subset of the natural numbers.
- The sum of the reciprocals of the powers of 2 converges:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

- The sum of the reciprocals of the prime numbers diverges.
- If we consider only the numbers that do not have a 9 in their decimal expansion, the sum of the reciprocals of these numbers *converges*.

The Harmonic Series

- We of course get different behaviour if we sum the reciprocals of some subset of the natural numbers.
- The sum of the reciprocals of the powers of 2 converges:

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

- The sum of the reciprocals of the prime numbers diverges.
- If we consider only the numbers that do not have a 9 in their decimal expansion, the sum of the reciprocals of these numbers *converges*.
- This feels surprising because it seems like there should be relatively few primes and many, many numbers without a 9 in their decimal expansion, but exactly the opposite is true.

Reciprocals of Numbers Without a Given String of Digits

- Let S be any string of digits. Let $\mathbb{N}_S$ be the set of natural numbers that contain S (contiguously) somewhere in their digits.

Reciprocals of Numbers Without a Given String of Digits

- Let S be any string of digits. Let $\mathbb{N}_S$ be the set of natural numbers that contain S (contiguously) somewhere in their digits.
- We will show that

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n}$$

converges.

Reciprocals of Numbers Without a Given String of Digits

- Let S be any string of digits. Let $\mathbb{N}_S$ be the set of natural numbers that contain S (contiguously) somewhere in their digits.
- We will show that
$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n}$$
converges.
- The approach will be similar to showing that the harmonic series diverges: we will group the digits in blocks of powers of $10^{\text{length of } S}$ and approximate the summands in each block by the largest element in the block.

Reciprocals of Numbers Without a Given String of Digits

- Let m be the length of S. We group together the numbers with between $km+1$ and $(k+1)m$ digits for some $k \geq 0$.

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} = \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{n} \right) \leq \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{10^{km}} \right)$$

Reciprocals of Numbers Without a Given String of Digits

- Let m be the length of S. We group together the numbers with between $km+1$ and $(k+1)m$ digits for some $k \geq 0$.

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} = \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{n} \right) \leq \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{10^{km}} \right)$$

- To bound the size of this sum, we need an estimate for how many numbers with between $km+1$ and $(k+1)m$ digits do not contain S.

Estimating the Cardinality

- Consider a number n with between $km+1$ and $(k+1)m$ digits, and suppose that n does not contain S in its digits.

Estimating the Cardinality

- Consider a number n with between $km + 1$ and $(k + 1)m$ digits, and suppose that n does not contain S in its digits.
- Break the digits of n up into $k + 1$ consecutive blocks of m digits. (One of the blocks may have fewer than m digits)

Estimating the Cardinality

- Consider a number n with between $km + 1$ and $(k + 1)m$ digits, and suppose that n does not contain S in its digits.
- Break the digits of n up into $k + 1$ consecutive blocks of m digits. (One of the blocks may have fewer than m digits)
- There are 10^m possible blocks of m digits. Each block of digits of n can be any one of these possibilities *except* for S.

Estimating the Cardinality

- Consider a number n with between $km+1$ and $(k+1)m$ digits, and suppose that n does not contain S in its digits.
- Break the digits of n up into $k+1$ consecutive blocks of m digits. (One of the blocks may have fewer than m digits)
- There are 10^m possible blocks of m digits. Each block of digits of n can be any one of these possibilities *except* for S.
- There are thus at most

$$(10^m - 1)^{k+1}$$

possible values of n.

Bounding the Sum

- We see that

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} \leq \sum_{k=0}^{\infty} \frac{(10^m - 1)^{k+1}}{10^{km}}$$

Bounding the Sum

- We see that

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} \leq \sum_{k=0}^{\infty} \frac{(10^m - 1)^{k+1}}{10^{km}}$$

- Since

$$\frac{10^m - 1}{10^m} < 1,$$

this is a geometric series, and converges!

Bounding the Sum

- We see that

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} \leq \sum_{k=0}^{\infty} \frac{(10^m - 1)^{k+1}}{10^{km}}$$

- Since

$$\frac{10^m - 1}{10^m} < 1,$$

this is a geometric series, and converges!

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(10^m - 1)^{k+1}}{10^{km}} &= (10^m - 1) \sum_{k=0}^{\infty} \left(\frac{10^m - 1}{10^m} \right)^k \\ &= (10^m - 1) \frac{1}{1 - \frac{10^m - 1}{10^m}} = 10^m (10^m - 1) \end{aligned}$$

Outline

The Prime Number Theorem

- In his talk on 1 April 2021, Lourens introduced the *Prime Number Theorem*:

Theorem (The Prime Number Theorem)

Let $\pi(x)$ *denote the number of prime numbers that are less than or equal to the real number* x*. Then*

$$\pi(x) \sim \frac{x}{\ln x}.$$

In other words,

$$\lim_{x\to\infty} \pi(x) \Big/ \frac{x}{\ln x} = 1.$$

The Prime Number Theorem

- Formally, this means that for every $\varepsilon > 0$, there exists $N > 0$ such that
$$(1-\varepsilon)\frac{x}{\ln x} \leq \pi(x) \leq (1+\varepsilon)\frac{x}{\ln x}$$
whenever $x > N$.

The Prime Number Theorem

- Formally, this means that for every $\varepsilon > 0$, there exists $N > 0$ such that
$$(1-\varepsilon)\frac{x}{\ln x} \leq \pi(x) \leq (1+\varepsilon)\frac{x}{\ln x}$$
whenever $x > N$.
- One consequence of this is that if p_n is the n^{th} prime number, then $p_n \sim n \ln n$.

The Prime Number Theorem

- Formally, this means that for every $\varepsilon > 0$, there exists $N > 0$ such that
$$(1-\varepsilon)\frac{x}{\ln x} \leq \pi(x) \leq (1+\varepsilon)\frac{x}{\ln x}$$
whenever $x > N$.
- One consequence of this is that if p_n is the n^{th} prime number, then $p_n \sim n \ln n$.
- Indeed, since $\pi(p_n) = n$, we have that
$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = \lim_{n\to\infty} \frac{\ln p_n}{\ln n} \times \frac{p_n}{\ln p_n} \Big/ \pi(p_n).$$

The Prime Number Theorem

- Formally, this means that for every $\varepsilon > 0$, there exists $N > 0$ such that
$$(1-\varepsilon)\frac{x}{\ln x} \leq \pi(x) \leq (1+\varepsilon)\frac{x}{\ln x}$$
whenever $x > N$.
- One consequence of this is that if p_n is the n^{th} prime number, then $p_n \sim n \ln n$.
- Indeed, since $\pi(p_n) = n$, we have that
$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = \lim_{n\to\infty} \frac{\ln p_n}{\ln n} \times \frac{p_n}{\ln p_n} \Big/ \pi(p_n).$$
- It is possible to show that $\lim_{n\to\infty} \ln p_n / \ln n = 1$, from which the result follows.

Proving that $\lim_{n\to\infty} \ln p_n / \ln n = 1$

- For all large enough x, we have that

$$\frac{1}{2}\frac{x}{\ln x} \leq \pi(x)$$

and so for large enough n we have that

$$p_n \leq 2n \ln p_n$$

which gives us that

$$\ln p_n \leq \ln 2 + \ln n + \ln(\ln p_n).$$

Proving that $\lim_{n\to\infty} \ln p_n / \ln n = 1$

- For all large enough x, we have that

$$\frac{1}{2}\frac{x}{\ln x} \leq \pi(x)$$

and so for large enough n we have that

$$p_n \leq 2n \ln p_n$$

which gives us that

$$\ln p_n \leq \ln 2 + \ln n + \ln(\ln p_n).$$

- It is thus enough to show that

$$\lim_{n\to\infty} \frac{\ln(\ln p_n)}{\ln n} = 0.$$

Proving that $\lim_{n\to\infty} \ln(\ln p_n)/\ln n = 0$

- Using our earlier estimate, we know that for large n,

$$\begin{aligned}\ln(\ln p_n) &\le \ln\left(\ln 2 + \ln n + \ln(\ln p_n)\right)\\ &= \ln(\ln n) + \ln\left(\frac{\ln 2}{\ln n} + 1 + \frac{\ln(\ln p_n)}{\ln n}\right)\end{aligned}$$

and so it is enough to show that

$$\frac{\ln(\ln p_n)}{\ln n}$$

is bounded.

Proving that $\ln(\ln p_n)/\ln n$ is bounded

- If you ask me nicely, I'll prove that $p_n < 4^n$ for all n.

Proving that $\ln(\ln p_n)/\ln n$ is bounded

- If you ask me nicely, I'll prove that $p_n < 4^n$ for all n.
- It follows that

$$\ln(\ln p_n) < \ln\left(\ln\left(4^n\right)\right) = \ln\left(n \ln 4\right) = \ln n + \ln(\ln 4).$$

Proving that $\ln(\ln p_n)/\ln n$ is bounded

- If you ask me nicely, I'll prove that $p_n < 4^n$ for all n.
- It follows that

$$\ln(\ln p_n) < \ln\left(\ln\left(4^n\right)\right) = \ln\left(n \ln 4\right) = \ln n + \ln(\ln 4).$$

- Thus

$$\frac{\ln(\ln p_n)}{\ln n} < 1 + \frac{\ln(\ln 4)}{\ln n}$$

which is bounded.

The Sum of the Reciprocals of the Prime Numbers Diverges

- Consider the series

$$\sum_{p \text{ prime}} \frac{1}{p} = \sum_{n=1}^{\infty} \frac{1}{p_n}.$$

The Sum of the Reciprocals of the Prime Numbers Diverges

- Consider the series

$$\sum_{p \text{ prime}} \frac{1}{p} = \sum_{n=1}^{\infty} \frac{1}{p_n}.$$

- If one is willing to use the Prime Number Theorem, then by the limit comparison test, this sum converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$$

does.

The Sum of the Reciprocals of the Prime Numbers Diverges

- In turn, by the integral test, this sum converges if and only if the integral

$$\int_2^\infty \frac{1}{x \ln x} \, \mathrm{d}x$$

converges.

The Sum of the Reciprocals of the Prime Numbers Diverges

- In turn, by the integral test, this sum converges if and only if the integral
$$\int_2^\infty \frac{1}{x \ln x}\,\mathrm{d}x$$
converges.
- But
$$\int_2^\infty \frac{1}{x \ln x}\,\mathrm{d}x = \ln(\ln x)\Big|_2^\infty \to \infty$$
and so the sum of the reciprocals of the prime numbers diverges.

A Generalisation

- We will show that the sum of the reciprocals of the primes with a prime number of digits diverges.

A Generalisation

- We will show that the sum of the reciprocals of the primes with a prime number of digits diverges.
- In fact, we can prove an even more general result:

Proposition

Let S_0 be the set of natural numbers, and for each $n > 0$, let S_n be the set of prime numbers p where the number of digits in the decimal expansion of p is in S_{n-1}. Then

$$\sum_{p \in S_n} \frac{1}{p}$$

diverges.

A Generalisation

- We will show that the sum of the reciprocals of the primes with a prime number of digits diverges.
- In fact, we can prove an even more general result:

Proposition

Let S_0 be the set of natural numbers, and for each $n > 0$, let S_n be the set of prime numbers p where the number of digits in the decimal expansion of p is in S_{n-1}. Then

$$\sum_{p \in S_n} \frac{1}{p}$$

diverges.

- In particular, the fact that $\sum_{p \in S_2} 1/p$ diverges tells us that there are infinitely many prime numbers with a prime number of digits.

Proof

We prove the result by induction on n.

Proof

We prove the result by induction on n. For $n = 0$, the claim is that the harmonic series diverges, which we have already shown.

Proof

We prove the result by induction on n. For $n = 0$, the claim is that the harmonic series diverges, which we have already shown. Suppose that

$$\sum_{p \in S_n} \frac{1}{p}$$

diverges.

Proof

Let P_k be the set of prime numbers with k digits. Then

$$\sum_{p \in S_{n+1}} \frac{1}{p} = \sum_{k \in S_n} \sum_{p \in P_k} \frac{1}{p} \geq \sum_{k \in S_n} \frac{|P_k|}{10^k}.$$

Proof

Let P_k be the set of prime numbers with k digits. Then

$$\sum_{p \in S_{n+1}} \frac{1}{p} = \sum_{k \in S_n} \sum_{p \in P_k} \frac{1}{p} \geq \sum_{k \in S_n} \frac{|P_k|}{10^k}.$$

It is thus sufficient to show that there is some constant C such that

$$\frac{|P_k|}{10^k} \geq \frac{C}{k}$$

for all large enough k.

Proof

By the Prime Number Theorem, there is some natural number N such that

$$\frac{1}{2}\frac{m}{\ln m} < \pi(m) < \frac{3}{2}\frac{m}{\ln m}$$

whenever $m > N$.
In particular, if $10^{k-1} > N$, then we have that

$$\pi\left(10^{k-1}\right) < \frac{3}{2}\frac{10^{k-1}}{(k-1)\ln 10}$$

and

$$\pi\left(10^{k}\right) > \frac{1}{2}\frac{10^{k}}{k\ln 10}.$$

Proof

For such a k, we have that

$$\begin{aligned}\frac{|P_k|}{10^k} &= \frac{\pi\left(10^k\right) - \pi\left(10^{k-1}\right)}{10^k} \\ &> \frac{1}{10^k}\left(\frac{1}{2}\frac{10^k}{k\ln 10} - \frac{3}{2}\frac{10^{k-1}}{(k-1)\ln 10}\right) \\ &= \frac{1}{20\ln 10}\frac{10(k-1)-3k}{k(k-1)} = \frac{1}{20\ln 10}\frac{7k-10}{k(k-1)}.\end{aligned}$$

Proof

For such a k, we have that

$$\begin{aligned}\frac{|P_k|}{10^k} &= \frac{\pi\left(10^k\right)-\pi\left(10^{k-1}\right)}{10^k}\\ &> \frac{1}{10^k}\left(\frac{1}{2}\frac{10^k}{k\ln 10}-\frac{3}{2}\frac{10^{k-1}}{(k-1)\ln 10}\right)\\ &= \frac{1}{20\ln 10}\frac{10(k-1)-3k}{k(k-1)} = \frac{1}{20\ln 10}\frac{7k-10}{k(k-1)}.\end{aligned}$$

For any constant $A<7$, we have that $7k-10>A(k-1)$ provided that k is large enough, and then taking $C=\frac{1}{20\ln 10}$, we have that

$$\frac{|P_k|}{10^k} > \frac{C}{k}$$

for all large enough k.

Outline

Combining these Results

- Let S be some string of decimal digits.

Combining these Results

- Let S be some string of decimal digits.
- As before, S_n is the set of primes where the number of digits is prime, the number of digits of the number of digits is prime, and so on.

Combining these Results

- Let S be some string of decimal digits.
- As before, S_n is the set of primes where the number of digits is prime, the number of digits of the number of digits is prime, and so on.
- Let $\mathbb{N}_S$ be the set of natural numbers that contain S somewhere in their decimal expansion.

Combining these Results

- Let S be some string of decimal digits.
- As before, S_n is the set of primes where the number of digits is prime, the number of digits of the number of digits is prime, and so on.
- Let $\mathbb{N}_S$ be the set of natural numbers that contain S somewhere in their decimal expansion.
- Suppose that $S_n \cap \mathbb{N}_S$ is finite. Then there is some natural number N such that if $p > N$ and $p \in S_n$, we have that $p \notin \mathbb{N}_S$.

Combining these Results

- Let S be some string of decimal digits.
- As before, S_n is the set of primes where the number of digits is prime, the number of digits of the number of digits is prime, and so on.
- Let $\mathbb{N}_S$ be the set of natural numbers that contain S somewhere in their decimal expansion.
- Suppose that $S_n \cap \mathbb{N}_S$ is finite. Then there is some natural number N such that if $p > N$ and $p \in S_n$, we have that $p \notin \mathbb{N}_S$.
- It follows that

$$\sum_{\substack{p>N \\ p\in S_n}} \frac{1}{p} \leq \sum_{\substack{p>N \\ p\notin \mathbb{N}_S}} \frac{1}{p}.$$

Combining these Results

- Let S be some string of decimal digits.
- As before, S_n is the set of primes where the number of digits is prime, the number of digits of the number of digits is prime, and so on.
- Let $\mathbb{N}_S$ be the set of natural numbers that contain S somewhere in their decimal expansion.
- Suppose that $S_n \cap \mathbb{N}_S$ is finite. Then there is some natural number N such that if $p > N$ and $p \in S_n$, we have that $p \notin \mathbb{N}_S$.
- It follows that

$$\sum_{\substack{p>N \\ p\in S_n}} \frac{1}{p} \leq \sum_{\substack{p>N \\ p\notin \mathbb{N}_S}} \frac{1}{p}.$$

- But the sum on the left diverges, while the sum on the right converges. A contradiction!

Outline

A More General Result

- The fact that there are infinitely many prime numbers that have a prime number of digits and that contain your phone number somewhere among their digits is also a consequence of the following more general result.

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

A More General Result

- The fact that there are infinitely many prime numbers that have a prime number of digits and that contain your phone number somewhere among their digits is also a consequence of the following more general result.

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

- This shows that having a prime number of digits isn't special. There is an appropriate prime number with almost every number of digits.

Proof

- Let m be the natural number whose decimal representation is S. If S starts with a 0, instead let m be the number whose decimal representation is 1 followed by S.

Proof

- Let m be the natural number whose decimal representation is S. If S starts with a 0, instead let m be the number whose decimal representation is 1 followed by S.
- We want to show that for all large enough natural numbers n, there is a prime between $10^n m$ and $10^n(m+1)-1$.

Proof

- Let m be the natural number whose decimal representation is S. If S starts with a 0, instead let m be the number whose decimal representation is 1 followed by S.
- We want to show that for all large enough natural numbers n, there is a prime between $10^n m$ and $10^n(m+1)-1$.
- Equivalently, we want to show that

$$\pi\left(10^n(m+1)-1\right)-\pi\left(10^n m-1\right)>0$$

for all large enough n.

Proof

- Let m be the natural number whose decimal representation is S. If S starts with a 0, instead let m be the number whose decimal representation is 1 followed by S.
- We want to show that for all large enough natural numbers n, there is a prime between $10^n m$ and $10^n(m+1)-1$.
- Equivalently, we want to show that

$$\pi\left(10^n(m+1)-1\right)-\pi\left(10^n m-1\right)>0$$

  for all large enough n.
- Since $10^n m$ is never prime, this is the same as proving that

$$\pi\left(10^n(m+1)\right)-\pi\left(10^n m\right)>0.$$

Proof

- The Prime Number Theorem tells us that for any constants $a < 1$ and $b > 1$ (we'll choose them later) we have that

$$a\frac{x}{\ln x} < \pi(x) < b\frac{x}{\ln x}.$$

for all large enough x.

Proof

- The Prime Number Theorem tells us that for any constants $a < 1$ and $b > 1$ (we'll choose them later) we have that

$$a\frac{x}{\ln x} < \pi(x) < b\frac{x}{\ln x}.$$

for all large enough x.

- It follows that for all large enough n, we have that

$$\pi\left(10^n(m+1)\right) - \pi\left(10^n m\right) > a\frac{10^n(m+1)}{\ln\left(10^n(m+1)\right)} - b\frac{10^n m}{\ln\left(10^n m\right)}.$$

Proof

- We wish to show that this is positive for large enough n. Since

$$\frac{10^n}{\ln\left(10^n(m+1)\right)\ln\left(10^n m\right)}$$

is positive, this is equivalent to showing that

$$a(m+1)\ln\left(10^n m\right) - bm\ln\left(10^n(m+1)\right)$$

is positive for large n.

Proof

- We wish to show that this is positive for large enough n. Since

$$\frac{10^n}{\ln\left(10^n(m+1)\right)\ln\left(10^n m\right)}$$

is positive, this is equivalent to showing that

$$a(m+1)\ln\left(10^n m\right) - bm\ln\left(10^n(m+1)\right)$$

is positive for large n.

- Equivalently, we wish to show that

$$(a(m+1) - bm)(\ln 10)n \\ + \text{ something that only depends on } m, a, \text{ and } b \text{ but not } n$$

is positive for all large enough n.

Proof

- It is sufficient to show that we can choose $a < 1$ and $b > 1$ such that

$$a(m+1) - bm > 0.$$

Proof

- It is sufficient to show that we can choose $a < 1$ and $b > 1$ such that

$$a(m+1) - bm > 0.$$

- Taking

$$a = \frac{2m+1}{2m+2} \qquad \text{and} \qquad b = \frac{4m+1}{4m}$$

gives us

$$a(m+1) - bm = \frac{2m+1}{2} - \frac{4m+1}{4} = \frac{1}{4} > 0.$$

Proof

- It follows that as long as n is large enough that $x = 10^n m$ satisfies the bound
$$\frac{2m+1}{2m+1}\frac{x}{\ln x} < \pi(x) < \frac{4m+1}{4m}\frac{x}{\ln x}$$
and n is large enough that
$$\frac{1}{4}n\ln 10 + \left(m+\frac{1}{2}\right)\ln m - \left(m+\frac{1}{4}\right)\ln(m+1) > 0$$
that there is guaranteed to be a prime between $10^n m$ and $10^n(m+1)-1$.

Outline

Did We Actually Need the Prime Number Theorem?

- For some of the results in this talk, the only consequence of the Prime Number Theorem that we have used is that for large x, we have that

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x}.$$

Did We Actually Need the Prime Number Theorem?

- For some of the results in this talk, the only consequence of the Prime Number Theorem that we have used is that for large x, we have that

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x}.$$

- In fact, even weaker bounds would probably have been sufficient. We did not actually need the full power of the Prime Number Theorem.

Did We Actually Need the Prime Number Theorem?

- For some of the results in this talk, the only consequence of the Prime Number Theorem that we have used is that for large x, we have that

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x}.$$

- In fact, even weaker bounds would probably have been sufficient. We did not actually need the full power of the Prime Number Theorem.
- For some of these bounds, much more elementary proofs are known.

Did We Actually Need the Prime Number Theorem?

- For some of the results in this talk, the only consequence of the Prime Number Theorem that we have used is that for large x, we have that

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x}.$$

- In fact, even weaker bounds would probably have been sufficient. We did not actually need the full power of the Prime Number Theorem.
- For some of these bounds, much more elementary proofs are known.
- Perhaps they can be explored in an upcoming talk *"It's Prime Time ∞: Elementary Proofs of Prime Number Theorem-like Results."*

Did We Actually Need the Prime Number Theorem?

- For some of the results in this talk, the only consequence of the Prime Number Theorem that we have used is that for large x, we have that

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x}.$$

- In fact, even weaker bounds would probably have been sufficient. We did not actually need the full power of the Prime Number Theorem.
- For some of these bounds, much more elementary proofs are known.
- Perhaps they can be explored in an upcoming talk *"It's Prime Time ∞: Elementary Proofs of Prime Number Theorem-like Results."*
- They also happen to show up in the Honours course in Analytic Number Theory.

Outline

Website

- I took some time to implement the Bailie-PSW pseudo-primality test in JavaScript.

Website

- I took some time to implement the Bailie-PSW pseudo-primality test in JavaScript.
- This allowed me to create a site where you can enter a string of digits, and it will find up to 100 prime numbers with a given number of digits containing the given string of digits.

Website

- I took some time to implement the Bailie-PSW pseudo-primality test in JavaScript.
- This allowed me to create a site where you can enter a string of digits, and it will find up to 100 prime numbers with a given number of digits containing the given string of digits.
- You can try it out here:

Figure: `https://dlnnlsn.github.io/prime-strings`