Prime Numbers Containing a Given String of Digits

An Application of the Prime Number Theorem

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Outline

Reddit Post

- On 4 April 2016, a thread was posted to /r/math on reddit asking for the most surprising examples of divergent series.

Figure: `https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/`

Reddit Post — Primes with a Prime Number of Digits

- In one example, we consider the set of prime numbers with a prime number of digits.

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- It is claimed that the sum of the reciprocals of the elements in this set diverges.

Figure: https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/d1oppgu

Reddit Post — Numbers Without a 9

- In another example, we consider all of the positive integers that *do not* have a 9 *anywhere* in their decimal expansion.

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- In another example, we consider all of the positive integers that *do not* have a 9 *anywhere* in their decimal expansion.
- In this case, it is claimed that the sum of the reciprocals of these numbers *converges*!

Figure: `https://www.reddit.com/r/math/comments/4d879s/most_surprising_divergent_series/d1olh0o`

Combining these Results

- I realised that a combination of (appropriate generalisations) of these two claims implies that there are infinitely many primes which have a prime number of digits, and which contain any given string of decimal digits that you like.

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- I realised that a combination of (appropriate generalisations) of these two claims implies that there are infinitely many primes which have a prime number of digits, and which contain any given string of decimal digits that you like.
- And of course I promptly told everyone I know.
- I even wrote a blog post about it!

Figure: `http://www.mathemafrica.org/?p=12942`

This Talk

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- It is possible to give a more direct proof of a stronger result:

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

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- *BUT*... by considering convergent and divergent series, the blog post is needlessly circuitous.
- It is possible to give a more direct proof of a stronger result:

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

- I will present a proof of this more general proposition towards the end of the talk.

Outline

The Harmonic Series

- One of the first somewhat surprising examples of a divergent series that students are shown is the *Harmonic Series*

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

- To show that this diverges, we group the terms in blocks of sizes equal to powers of 2, and then approximate each term by the smallest element in its block.

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \sum_{n=0}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} &\geq 1 + \sum_{n=0}^{\infty} \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} \\ &= 1 + \sum_{n=0}^{\infty} \frac{2^n}{2^{n+1}} = 1 + \sum_{n=0}^{\infty} \frac{1}{2}.\end{aligned}$$

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- If we consider only the numbers that do not have a 9 in their decimal expansion, the sum of the reciprocals of these numbers *converges*.

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- The sum of the reciprocals of the prime numbers diverges.
- If we consider only the numbers that do not have a 9 in their decimal expansion, the sum of the reciprocals of these numbers *converges*.
- This feels surprising because it seems like there should be relatively few primes and many, many numbers without a 9 in their decimal expansion, but exactly the opposite is true.

Reciprocals of Numbers Without a Given String of Digits

- Let S be any string of digits. Let $\mathbb{N}_S$ be the set of natural numbers that contain S (contiguously) somewhere in their digits.

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- Let S be any string of digits. Let $\mathbb{N}_S$ be the set of natural numbers that contain S (contiguously) somewhere in their digits.
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  converges.
- The approach will be similar to showing that the harmonic series diverges: we will group the digits in blocks of powers of $10^{\text{length of } S}$ and approximate the summands in each block by the largest element in the block.

Reciprocals of Numbers Without a Given String of Digits

- Let m be the length of S. We group together the numbers with between $km+1$ and $(k+1)m$ digits for some $k \geq 0$.

$$\sum_{n \notin \mathbb{N}_S} \frac{1}{n} = \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{n} \right) \leq \sum_{k=0}^{\infty} \left(\sum_{\substack{10^{km} \leq n < 10^{km+1} \\ n \notin \mathbb{N}_S}} \frac{1}{10^{km}} \right)$$

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- To bound the size of this sum, we need an estimate for how many numbers with between $km+1$ and $(k+1)m$ digits do not contain S.

Estimating the Cardinality

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- There are 10^m possible blocks of m digits. Each block of digits of n can be any one of these possibilities *except* for S.
- There are thus at most

$$(10^m - 1)^{k+1}$$

possible values of n.

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- We see that

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$$\sum_{k=0}^{\infty} \frac{(10^m - 1)^{k+1}}{10^{km}} = (10^m - 1) \sum_{k=0}^{\infty} \left(\frac{10^m - 1}{10^m} \right)^k$$
$$= (10^m - 1) \frac{1}{1 - \frac{10^m - 1}{10^m}} = 10^m (10^m - 1)$$

Outline

The Prime Number Theorem

- In his talk on 1 April 2021, Lourens introduced the *Prime Number Theorem*:

Theorem (The Prime Number Theorem)

Let $\pi(x)$ denote the number of prime numbers that are less than or equal to the real number x. Then

$$\pi(x) \sim \frac{x}{\ln x}.$$

In other words,

$$\lim_{x \to \infty} \pi(x) \Big/ \frac{x}{\ln x} = 1.$$

The Prime Number Theorem

- Formally, this means that for every $\varepsilon > 0$, there exists $N > 0$ such that
$$(1-\varepsilon)\frac{x}{\ln x} \leq \pi(x) \leq (1+\varepsilon)\frac{x}{\ln x}$$
whenever $x > N$.

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whenever $x > N$.
- One consequence of this is that if p_n is the n^{th} prime number, then $p_n \sim n \ln n$.
- Indeed, since $\pi(p_n) = n$, we have that

$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = \lim_{n\to\infty} \frac{\ln p_n}{\ln n} \times \frac{p_n}{\ln p_n} \Big/ \pi(p_n).$$

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$$\lim_{n\to\infty} \frac{p_n}{n \ln n} = \lim_{n\to\infty} \frac{\ln p_n}{\ln n} \times \frac{p_n}{\ln p_n} \Big/ \pi(p_n).$$
- It is possible to show that $\lim_{n\to\infty} \ln p_n / \ln n = 1$, from which the result follows.

Proving that $\lim_{n\to\infty} \ln p_n / \ln n = 1$

- For all large enough x, we have that

$$\frac{1}{2}\frac{x}{\ln x} \leq \pi(x)$$

and so for large enough n we have that

$$p_n \leq 2n \ln p_n$$

which gives us that

$$\ln p_n \leq \ln 2 + \ln n + \ln(\ln p_n).$$

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- It is thus enough to show that

$$\lim_{n\to\infty} \frac{\ln(\ln p_n)}{\ln n} = 0.$$

Proving that $\lim_{n\to\infty} \ln(\ln p_n)/\ln n = 0$

- Using our earlier estimate, we know that for large n,

$$\begin{aligned}\ln(\ln p_n) &\leq \ln\left(\ln 2 + \ln n + \ln(\ln p_n)\right)\\ &= \ln(\ln n) + \ln\left(\frac{\ln 2}{\ln n} + 1 + \frac{\ln(\ln p_n)}{\ln n}\right)\end{aligned}$$

and so it is enough to show that

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is bounded.

Proving that $\ln(\ln p_n)/\ln n$ is bounded

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- Thus

$$\frac{\ln(\ln p_n)}{\ln n} < 1 + \frac{\ln(\ln 4)}{\ln n}$$

which is bounded.

The Sum of the Reciprocals of the Prime Numbers Diverges

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- If one is willing to use the Prime Number Theorem, then by the limit comparison test, this sum converges if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n \ln n}$$

does.

The Sum of the Reciprocals of the Prime Numbers Diverges

- In turn, by the integral test, this sum converges if and only if the integral
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- In turn, by the integral test, this sum converges if and only if the integral
$$\int_2^\infty \frac{1}{x \ln x}\,\mathrm{d}x$$
converges.
- But
$$\int_2^\infty \frac{1}{x \ln x}\,\mathrm{d}x = \ln(\ln x)\Big|_2^\infty \to \infty$$
and so the sum of the reciprocals of the prime numbers diverges.

A Generalisation

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- In fact, we can prove an even more general result:

Proposition

Let S_0 be the set of natural numbers, and for each $n > 0$, let S_n be the set of prime numbers p where the number of digits in the decimal expansion of p is in S_{n-1}. Then

$$\sum_{p \in S_n} \frac{1}{p}$$

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- In particular, the fact that $\sum_{p \in S_2} 1/p$ diverges tells us that there are infinitely many prime numbers with a prime number of digits.

Proof

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$$\sum_{p \in S_n} \frac{1}{p}$$

diverges.

Proof

Let P_k be the set of prime numbers with k digits. Then

$$\sum_{p \in S_{n+1}} \frac{1}{p} = \sum_{k \in S_n} \sum_{p \in P_k} \frac{1}{p} \geq \sum_{k \in S_n} \frac{|P_k|}{10^k}.$$

Proof

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It is thus sufficient to show that there is some constant C such that

$$\frac{|P_k|}{10^k} \geq \frac{C}{k}$$

for all large enough k.

Proof

By the Prime Number Theorem, there is some natural number N such that

$$\frac{1}{2}\frac{m}{\ln m} < \pi(m) < \frac{3}{2}\frac{m}{\ln m}$$

whenever $m > N$.
In particular, if $10^{k-1} > N$, then we have that

$$\pi\left(10^{k-1}\right) < \frac{3}{2}\frac{10^{k-1}}{(k-1)\ln 10}$$

and

$$\pi\left(10^{k}\right) > \frac{1}{2}\frac{10^{k}}{k\ln 10}.$$

Proof

For such a k, we have that

$$\begin{aligned}\frac{|P_k|}{10^k} &= \frac{\pi\left(10^k\right) - \pi\left(10^{k-1}\right)}{10^k} \\ &> \frac{1}{10^k}\left(\frac{1}{2}\frac{10^k}{k\ln 10} - \frac{3}{2}\frac{10^{k-1}}{(k-1)\ln 10}\right) \\ &= \frac{1}{20\ln 10}\frac{10(k-1)-3k}{k(k-1)} = \frac{1}{20\ln 10}\frac{7k-10}{k(k-1)}.\end{aligned}$$

Proof

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For any constant $A < 7$, we have that $7k - 10 > A(k-1)$ provided that k is large enough, and then taking $C = \frac{1}{20\ln 10}$, we have that

$$\frac{|P_k|}{10^k} > \frac{C}{k}$$

for all large enough k.

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- Suppose that $S_n \cap \mathbb{N}_S$ is finite. Then there is some natural number N such that if $p > N$ and $p \in S_n$, we have that $p \notin \mathbb{N}_S$.

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- It follows that

$$\sum_{\substack{p>N \\ p\in S_n}} \frac{1}{p} \leq \sum_{\substack{p>N \\ p\notin \mathbb{N}_S}} \frac{1}{p}.$$

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- It follows that

$$\sum_{\substack{p>N \\ p\in S_n}} \frac{1}{p} \leq \sum_{\substack{p>N \\ p\notin \mathbb{N}_S}} \frac{1}{p}.$$

- But the sum on the left diverges, while the sum on the right converges. A contradiction!

Outline

A More General Result

- The fact that there are infinitely many prime numbers that have a prime number of digits and that contain your phone number somewhere among their digits is also a consequence of the following more general result.

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

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- The fact that there are infinitely many prime numbers that have a prime number of digits and that contain your phone number somewhere among their digits is also a consequence of the following more general result.

Proposition

Given a string of digits S, there is some natural number N, such that for all $n > N$, there is a prime with n digits that starts with S. (Or by some non-zero digit followed by S.)

- This shows that having a prime number of digits isn't special. There is an appropriate prime number with almost every number of digits.

Proof

- Let m be the natural number whose decimal representation is S. If S starts with a 0, instead let m be the number whose decimal representation is 1 followed by S.

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- Equivalently, we want to show that

$$\pi\left(10^{n+1} m - 1\right) - \pi\left(10^n m - 1\right) > 0$$

for all large enough n.

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- Equivalently, we want to show that

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  for all large enough n.
- Since $10^n m$ is never prime, this is the same as proving that

$$\pi\left(10^{n+1} m\right) - \pi\left(10^n m\right) > 0.$$

Proof

- For all large enough x, we know that
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- It follows that for all large enough n, we have that
$$\pi\left(10^{n+1}m\right) - \pi\left(10^{n}m\right) > \frac{1}{2}\frac{10^{n+1}m}{\ln\left(10^{n+1}m\right)} - \frac{3}{2}\frac{10^{n}m}{\ln\left(10^{n}m\right)}.$$

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- It follows that for all large enough n, we have that
$$\pi\left(10^{n+1}m\right) - \pi\left(10^{n}m\right) > \frac{1}{2}\frac{10^{n+1}m}{\ln\left(10^{n+1}m\right)} - \frac{3}{2}\frac{10^{n}m}{\ln\left(10^{n}m\right)}.$$
- We wish to show that this is positive for large enough n. Since
$$\frac{10^{n}m}{2\ln\left(10^{n+1}m\right)\ln\left(10^{n}m\right)}$$
is positive, this is equivalent to showing that
$$10\ln\left(10^{n}m\right) - 3\ln\left(10^{n+1}m\right)$$
is positive for large n.

Proof

- We have that

$$
\begin{aligned}
10 \ln\left(10^n m\right) - 3 \ln\left(10^{n+1} m\right) \\
&= 10\left(n \ln 10 + \ln m\right) - 3\left((n+1)\ln 10 + \ln m\right) \\
&= 7n \ln 10 + 7 \ln m - 3 \ln 10
\end{aligned}
$$

which is positive for all positive integers n.

Proof

- We have that

$$\begin{aligned}10\ln\left(10^n m\right) - 3\ln\left(10^{n+1}m\right) &= 10\left(n\ln 10 + \ln m\right) - 3\left((n+1)\ln 10 + \ln m\right)\\ &= 7n\ln 10 + 7\ln m - 3\ln 10\end{aligned}$$

  which is positive for all positive integers n.
- It follows that as long as n is large enough that $x = 10^n$ satisfies the bound

$$\frac{1}{2}\frac{x}{\ln x} < \pi(x) < \frac{3}{2}\frac{x}{\ln x},$$

  we have that there is a prime with $n + \text{length}(S)$ (possibly $+1$) digits that starts either with S, or with 1 followed by S.

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- They also happen to show up in the Honours course in Analytic Number Theory.

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- You can try it out here:

Figure: `https://dlnnlsn.github.io/prime-strings`