Problems Involving Linear Recurrence Relations

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1. **PAMO 2017 Problem 1**

   We consider the real sequence (x_n) defined by $x_0 = 0$, $x_1 = 1$, and $x_{n+2} = 3x_{n+1} - 2x_n$ for $n = 0, 1, 2, \ldots$

   We define the sequence (y_n) by $y_n = {x_n}^2 + 2^{n+2}$ for every non-negative integer n.

   Prove that for every $n > 0$, y_n is the square of an odd integer.

2. **PAMO 2019 Problem 1**

   Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers defined as follows:

   - $a_0 = 3$, $a_1 = 2$, and $a_2 = 12$; and
   - $2a_{n+3} - a_{n+2} - 8a_{n+1} + 4a_n = 0$ for $n \geq 0$.

   Show that a_n is always a strictly positive integer.

3. ***Not* PAMO 2023 Problem 3**

   Consider a sequence of real numbers defined by:

   $$\begin{aligned} x_1 &= c \\ x_{n+1} &= cx_n + \sqrt{c^2 - 1}\sqrt{x_n^2 - 1} \text{ for all } n \geq 1. \end{aligned}$$

   Find a closed form for x_n.

4. **BMO 2012 Round 2 Problem 4**

   Show that there is a positive integer k with the following property: if a, b, c, d, e, and f are integers, and m is a divisor of

   $$a^n + b^n + c^n - d^n - e^n - f^n$$

   for all integers n in the range $1 \leq n \leq k$, then m is a divisor of $a^n + b^n + c^n - d^n - e^n - f^n$ for all positive integers n.

5. Suppose that the sequence (T_n) satisfies the recurrence relation

$$T_{n+2} + aT_{n+1} + bT_n = 0$$

for all $n \geq 0$. Show that

$$T_{n+2}T_n - T_{n+1}^2 = b^n \left(T_2 T_0 - T_1^2\right)$$

for all $n \geq 0$.

6. **Korea 2019 Final Round Problem 3**

Prove that there exist infinitely many positive integers k such that the sequence x_n satisfying $x_1 = 1$, $x_2 = k + 2$, and

$$x_{n+2} - (k+1)x_{n+1} + x_n = 0$$

for all $n \geq 0$ does not contain any prime number.

7. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$f(f(n)) + f(n) = 6n$$

for all $n \in \mathbb{N}_0$.

8. Find a closed form for the sequence T_n satisfying $T_0 = 0$ and

$$T_{n+1} = 2T_n + 2^{n+1} - 2$$

for all integers $n \geq 0$.

9. Let A and B be two fixed real numbers. Find all sequences (T_n) such that

$$T_{n+2} - T_{n+1} - 2T_n = \begin{cases} A & \text{if } n \equiv 0 \pmod 2 \\ B & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

10. Find all sequences T_n such that

$$T_{n+2} + T_{n+1} + T_n = 3\,(n \bmod 3)$$

for all $n \geq 0$. (Here $n \bmod 3$ is the remainder when n is divided by 3)

11. Find all sequences T_n such that

$$T_{n+2} - 3T_{n+1} + 2T_n = n\,(n \bmod 4)$$

for all $n \geq 0$.

12. Show that the n^{th} Fibonacci number F_n is equal to the closest integer to

$$\frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n.$$