Super Secret Test

Time: 4 hours

January 21, 2020

1. Find all triples of integers (x, y, p) such that p is a prime, and
$$y(x^2 + p) - x(y^2 + p) = p.$$

2. Thandi is filling the cells of a 2019×2019 table alternately with crosses and circles (she starts with a cross). When the table is filled in completely, she determines her score as $X_r + X_c - O_r - O_c$ where X_r is the sum of the squares of the number of crosses in each row, X_c is the sum of the squares of the number of crosses in each column, O_r is the sum of the squares of the numbers of circles in each row, and O_c is the sum of the squares of the number of circles in each column. Find all possible values of the score.

3. Parallelograms $ABCD$ and $EFGH$ are such that points A, B, E, and F are collinear and points C, D, G, and H are collinear. Lines AD and EH meet at P, lines BC and FG meet at Q, and lines AC and EG meet at R.

   Prove that P, Q and R are collinear.

4. Let a, b, and c be positive integers that are the side-lengths of a non-degenerate triangle, and satisfy $\gcd(a, b, c) = 1$. Suppose that the fractions
$$\frac{a^2 + b^2 - c^2}{a + b - c}, \qquad \frac{b^2 + c^2 - a^2}{b + c - a}, \qquad \frac{c^2 + a^2 - b^2}{c + a - b}$$
are all integers. Prove that the product of the denominators of the three fractions is either a square or twice the square of an integer.

5. Given are 51 positive integers written in a row, with sum equal to 100. An integer is called *representable* if it can be expressed as the sum of several consecutive numbers in the given row. Prove that for each $k \in \{1, 2, \ldots, 100\}$ at least one of the numbers k and $100 - k$ is representable.

6. In a triangle ABC, the interior angle bisector of A intersects the excircle of ABC opposite A at points D and E such that D lies on segment AE. Show that
$$\frac{AD}{AE} \leq \frac{BC^2}{DE^2}.$$