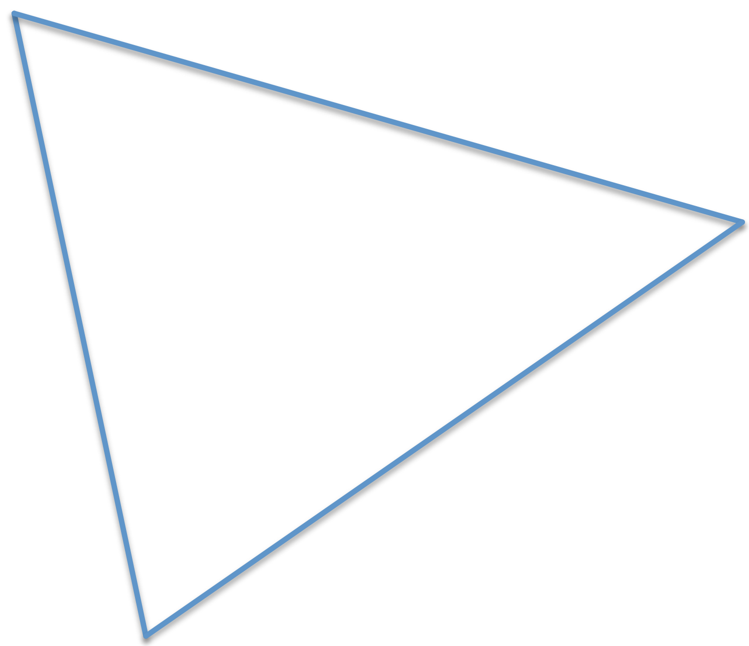
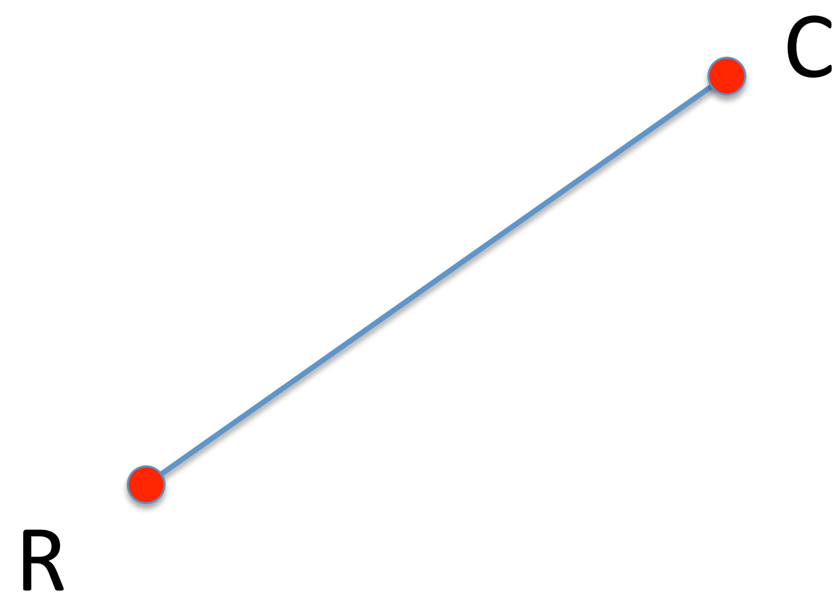
So You Want To Make a Triangle?

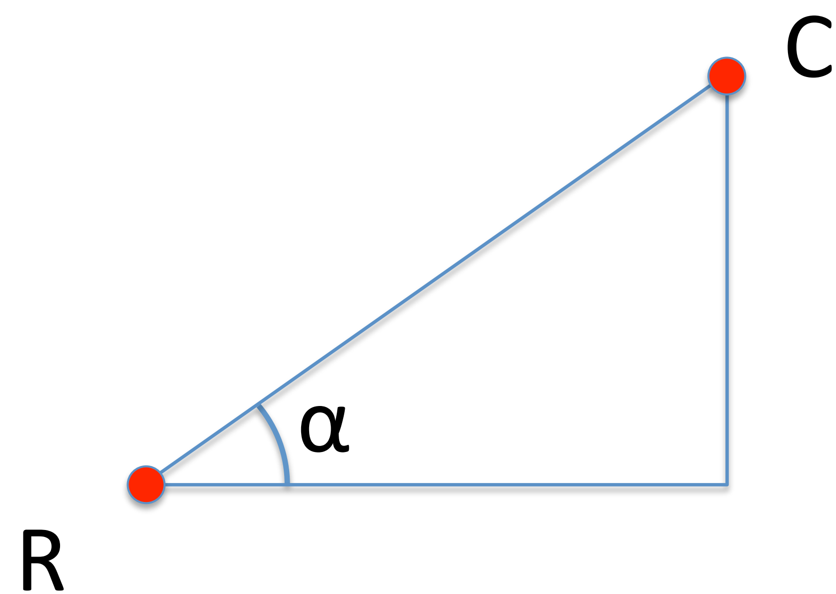


Given two points we can nail down one side of the triangle like so…

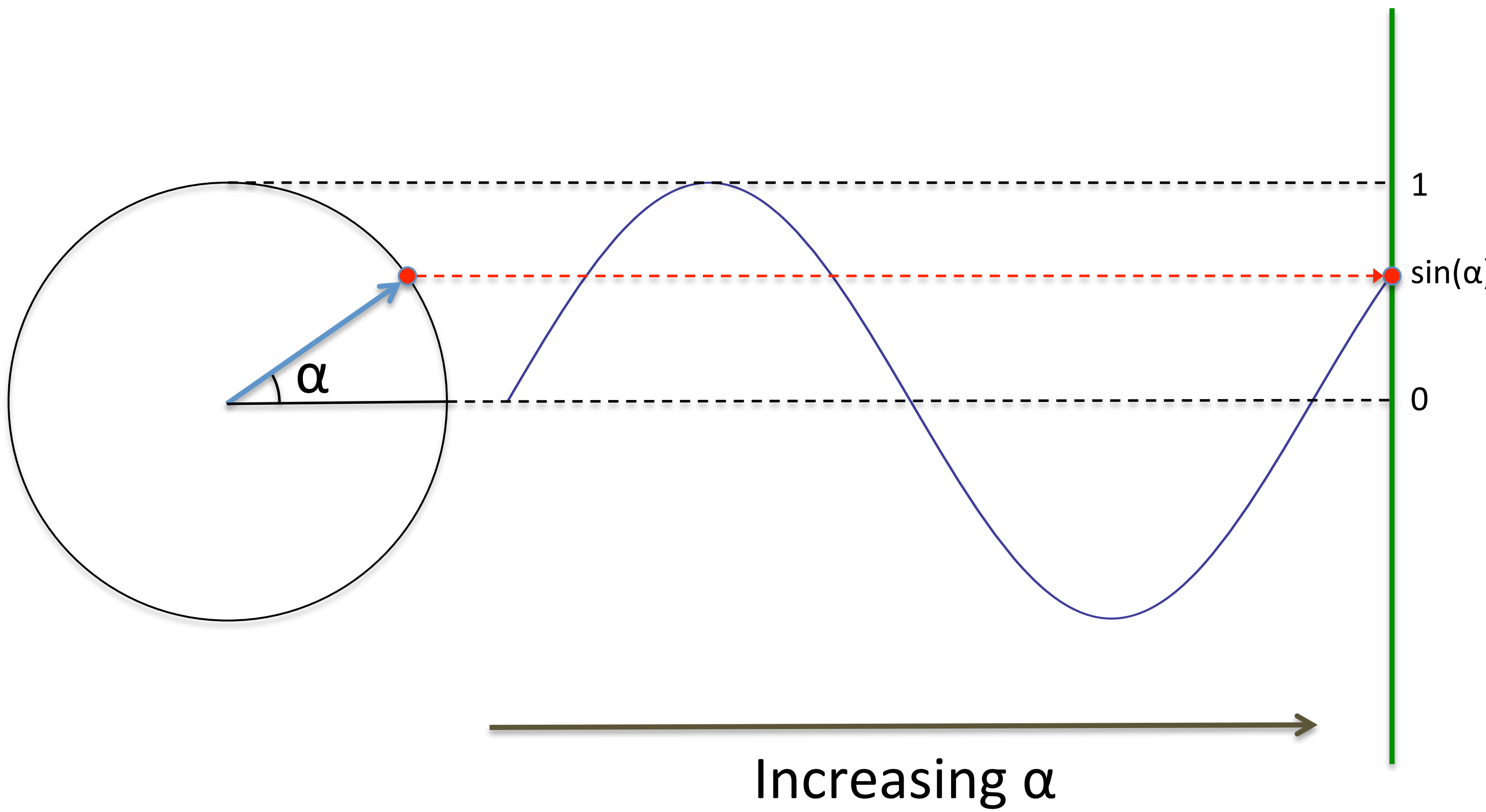
Regardless of where C and R are in space, we can define a vector called RC which points from R to C by RC = C-R.

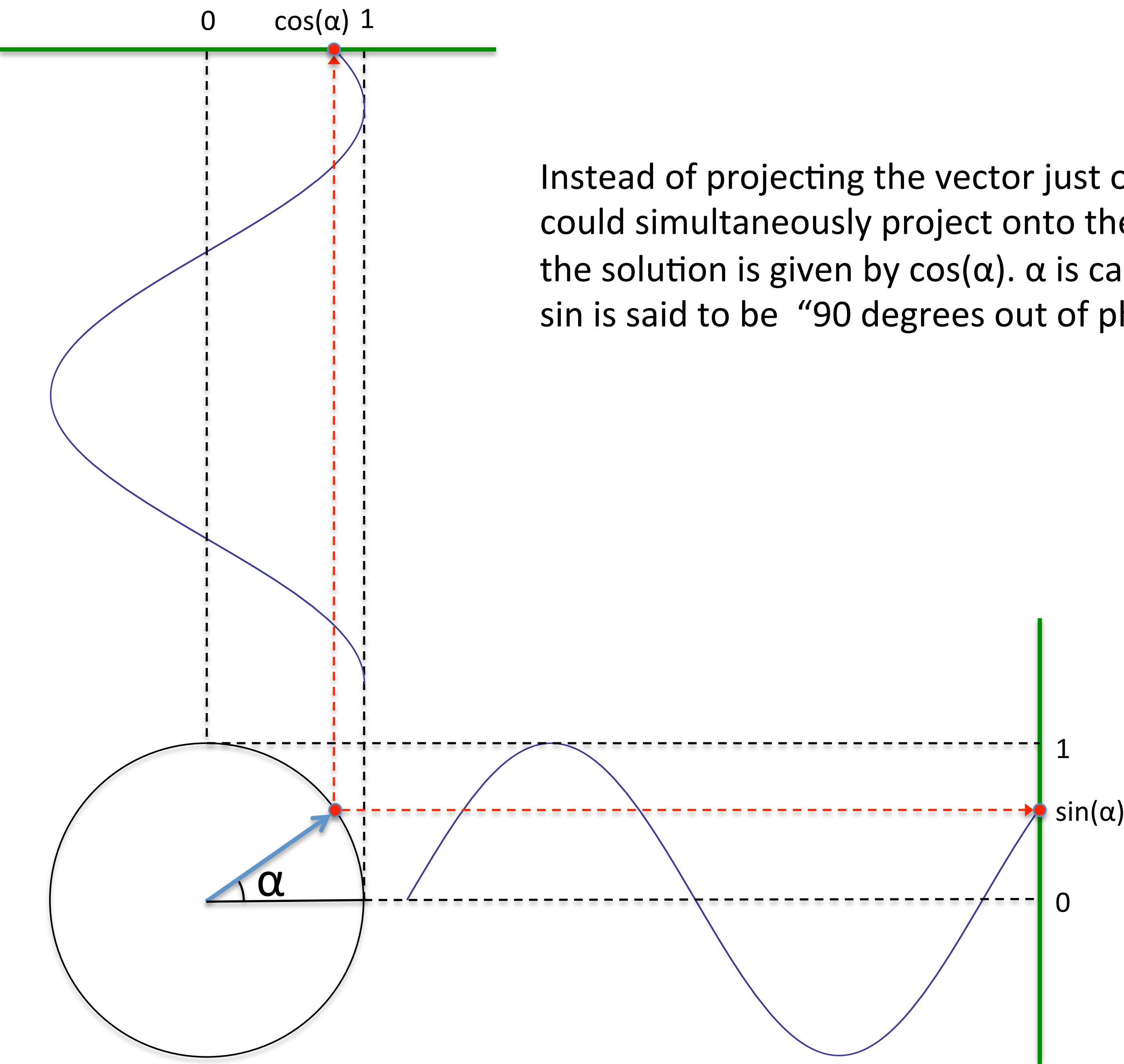


It is useful to figure out the angle between RC and our coordinate system. If we define a right triangle this becomes easier since the properties are well known. In order to determine α we need to use a few tricks…



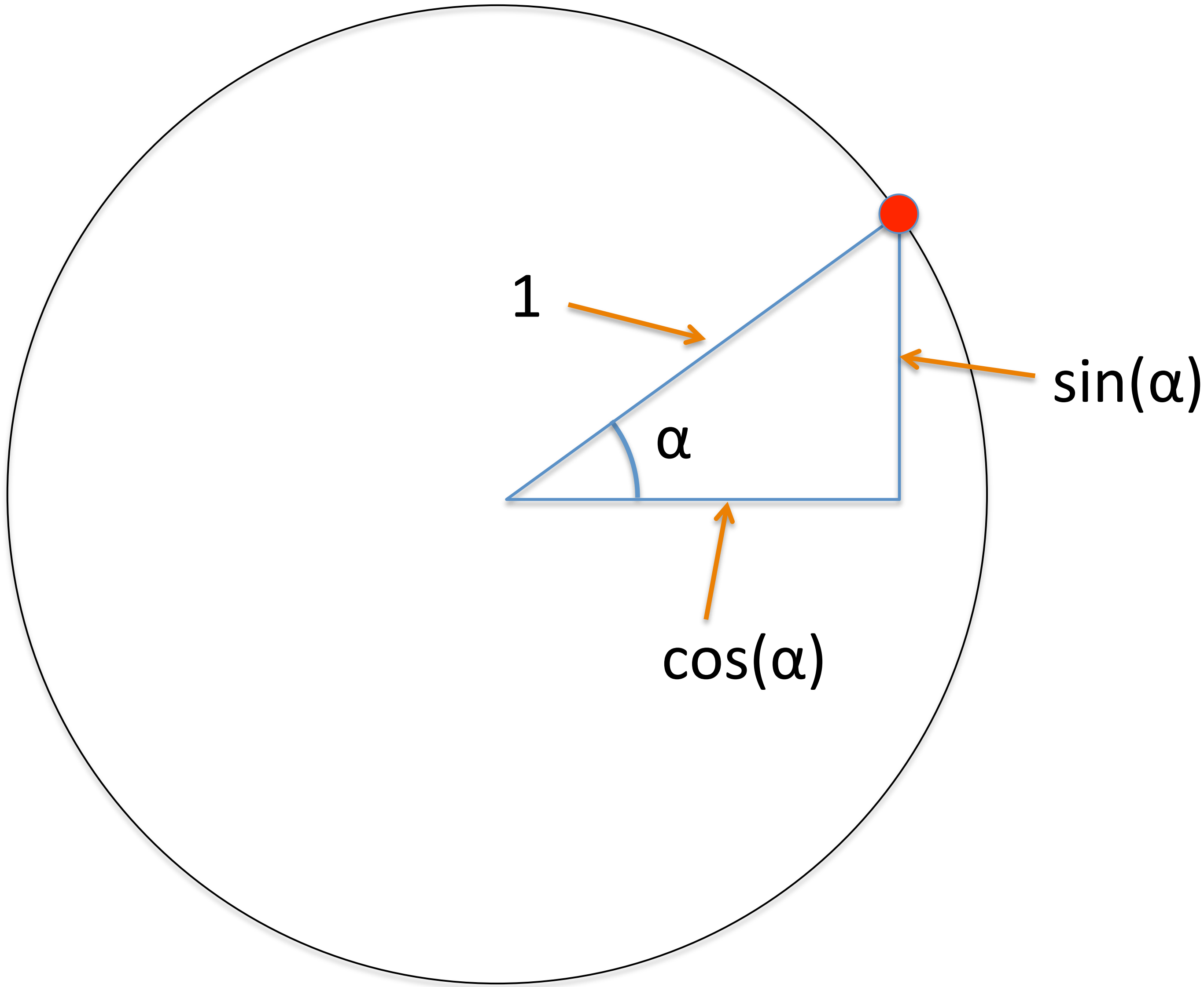
..but first, and tiny bit of background on sine/cosine. There are multiple great/interesting ways to define/understand sin/cos, but this one is pretty straightforward imo. Suppose you Have a unit vector (unit means length = 1) that can rotate around the origin. If we project, or “cast a shadow” of the height of the vector as we increase alpha we trace out a nice sinusoid.



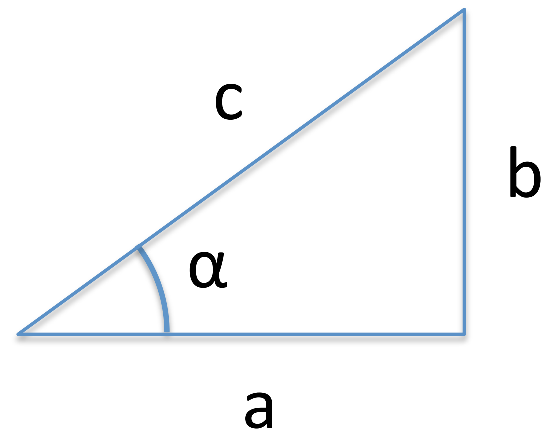


Instead of projecting the vector just onto the the Y axis, we could simultaneously project onto the X axis in which case the solution is given by cos(α). α is called a “phase angle” and sin is said to be “90 degrees out of phase” with cos

Because the two projections were in orthogonal (perpendicular) directions, we can create a right triangle with one point at the origin and its hypotenuse equal to one such that the other legs are equal to sin(α) and cos(α). So, in general the sides of a right triangle can be described with sin/cos!



Up to this point we've been defining our right triangle so that the hypotenuse equals 1. But if the hypotenuse has some arbitrary length c then the cos/sin still work but need to be scaled such that



$a = c*\cos(\alpha)$
$b = c*\sin(\alpha)$

From these we can do some simple manipulation to get

$\cos(\alpha) = a/c$ or $\alpha = \text{acos}(a/c)$
$\sin(\alpha) = b/c$ or $\alpha = \text{asin}(b/c)$

Where asin/acos are the "arcsine" and "arccosine" are the inverse operators of sin/cos which means $\text{asin}(\sin(\alpha)) = \alpha$

Sometimes if you have legs a and b but don't know c you can still figure out α using the tangent function

$\tan(\alpha) = b/a$

There is a helpful little mnemonic for remembering these which is "SOH CAH TOA" (sounds kind of like "soak a toe") and O = opposite, A = adjacent, H = hypotenuse. SOH means Sin = O/H where the "opposite" means, leg opposite the angle in question (in the above case its b). And CAH means Cos = A/H (a/c above) and TOA means Tan = O/A

Now back to our problem, we have two points and can now figure out α pretty easily. But this is really just telling us about one side of our triangle, so we need to add a couple details. Here the right triangle at the bottom has been more or less worked out so we can focus on the new triangle which isn't a right triangle. Generally you can break an arbitrary triangle into two right triangles as indicated by the dashed line. At that point, if the length of the 3 sides is known then the angles can be solved using the "law of cosines".

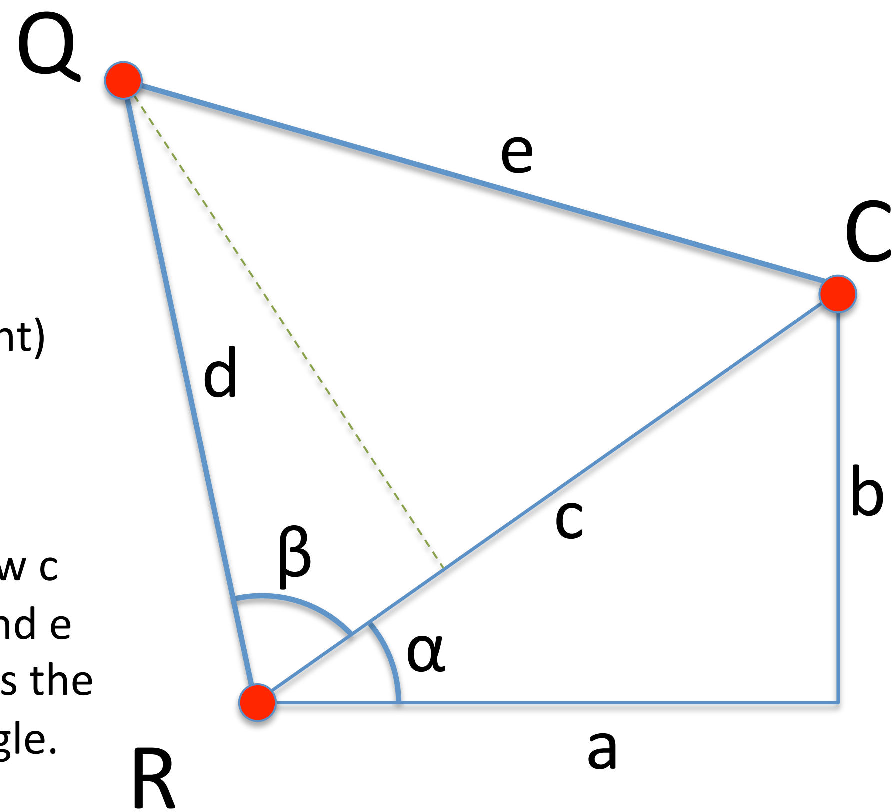
For right triangles the Pythagorean theorem applies and is given by

$$a^2 + b^2 = c^2$$

where c is the hypotenuse. This turns out to be a special case of the law of cosines which is given by (in the context of our triangle at the right)

$$e^2 = d^2 + c^2 - 2*d*c*\cos(\beta)$$

For the case we're interested in, we already know c from sqrt(a^2+b^2) = norm(RC). The lengths d and e are fixed, so we're mainly interested in β which is the most simple angle to work with in the new triangle.

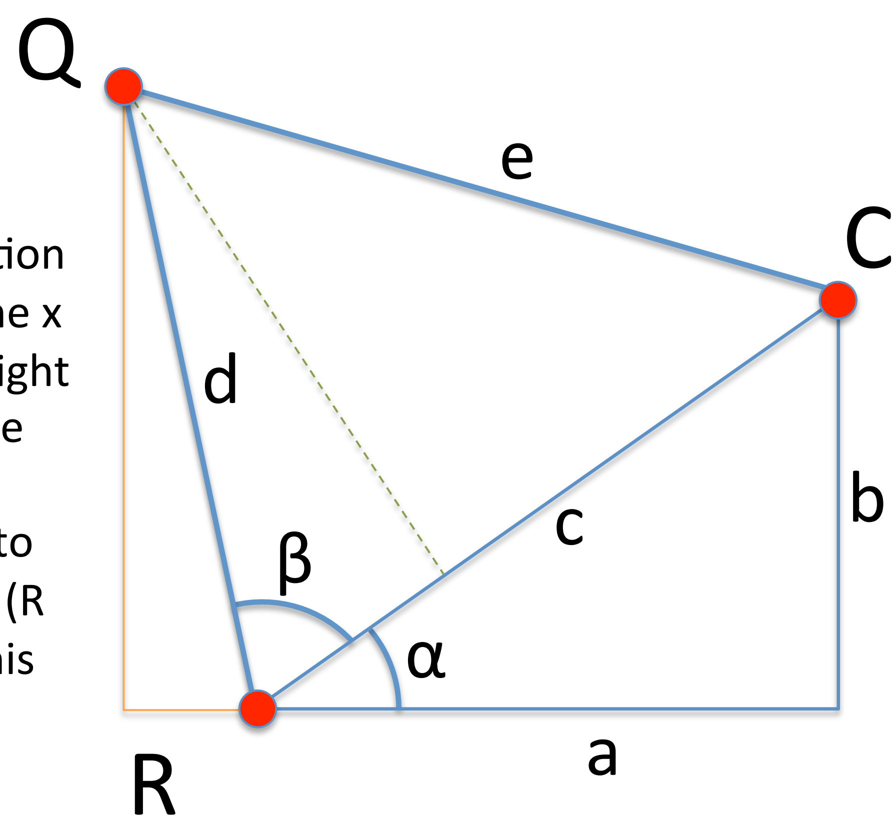


To solve for β we can rearrange the law of cosines to give

$$\cos(\beta) = (-e^2 + d^2 + c^2)/(2*d*c)$$

We could apply an acos to get β directly, but it turns out that a lot of times (as you've already discovered with rotation matrices) having the cosine/sine of an angle is often more useful than the angle itself. And because computers will typically try to calculate the angle to a high precision it's more computationally conservative to throw out actual cos/sin/acos/.. operations all together.

We're interested in finding the coordinates of Q, but we know that Q can be described by a vector [d, 0, 0] which is rotated about the z axis by an angle (α+β). Putting [d,0,0] through a rotation matrix gives [d*cos(α+β), d*sin(α+β),0] where the x and y elements represent 2 legs of yet another right triangle (shown in orange) bringing us back to the circle analogy discussed before. Once we've calculated the RQ vector, we just need to add R to It to put us back in the global coordinate system (R was subtracted to make RC defining R as zero, this allows for the trivial rotation). The only question left to answer is…



…how to calculate $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ without using acos or asin to solve for α and β directly?

The answer is both easy and not so easy. But I'll give you the easy one: go to wikipedia and look up "trig identities" and you should be able to find tables of these things and some look a bit more hairy than others but they are all similar and useful in various situations.
The first one you will find is (note: $\cos^2(\phi)$ is just a prettier way to write $\cos(\phi)^2$)

$$\cos^2(\phi) + \sin^2(\phi) = 1$$

This should be obvious from the circle analogy where cos/sin are the two legs of a triangle with hypotenuse = 1, then pythagorean theorem applies to give this result. This can also be rearranged so

$$\sin(\phi) = \pm \text{sqrt}(1-\cos^2(\phi))$$

and the ± sort of corresponds to the joint bending to the left or to the right. From Wikipedia you'll also find

$$\cos(\alpha+\beta) = \cos(\alpha)*\cos(\beta)-\sin(\alpha)*\sin(\beta)$$
$$\sin(\alpha+\beta) = \cos(\alpha)*\sin(\beta)+\sin(\alpha)*\cos(\beta)$$

These might also seem vaguely familiar, almost as if a vector $[\cos(\alpha),\sin(\alpha)]$ was transformed by a rotation matrix with an angle β.. ;)

From before we already found information about α and β in the form of

$$\cos(\alpha) = a/c$$
$$\sin(\alpha) = b/c$$
$$\cos(\beta) = (-e^2 + d^2 + c^2)/(2*d*c)$$

We can find sin(β) by using

$$\sin(\phi) = \pm \mathrm{sqrt}(1-\cos^2(\phi))$$

Or

$$\sin(\beta) = \pm \mathrm{sqrt}(1-((-e^2 + d^2 + c^2)/(2*d*c))^2)$$

Now we can plug in cos/sines of α and β directly into

$$\cos(\alpha+\beta) = \cos(\alpha)*\cos(\beta)-\sin(\alpha)*\sin(\beta)$$
$$\sin(\alpha+\beta) = \cos(\alpha)*\sin(\beta)+\sin(\alpha)*\cos(\beta)$$

I'll stop at this point, but what you'll notice is that we did a whole bunch of trig with a bunch of sines and cosines but never actually had to calculate the cos/sin of anything and the result will all reduce to simple algebra which will speed up your code significantly (well maybe..)

Anyway, the resulting vector RQ will be [d*cos(α+β), d*sin(α+β),0], and RQ = Q-R, in other words Q = RQ + R. And you're done!