ABSTRACT ALGEBRA

DUMMIT, FOOTE

Second Edition

My Own Notes + Exercises

J.B.

July 2025

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Preliminaries

0.1 Basics

The set

$$f(A) = \{b \in B \mid b \in f(a), \text{ for some } a \in A\},$$

is a subset of B, called the *range* or *image* of f. For each subset C of B the set

$$f^{-1}(C) = \{a \in A \mid f(a) \in C\}$$

consisting of the elements of A mapping into C under f is called the *preimage* or *inverse image* of C under f. For each $b \in B$, the preimage of $\{b\}$ under f is called the *fiber* of f over b.

Let $f : A \to B$.

(1) f is *injective* or is an *injection* if whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

(2) f is *surjective* or is an *surjection* if for all $b \in B$ there is some $a \in A$ such that $f(a) = b$; i.e., the image of f is all of B. (The codomain of f is B, while the range/image of f is the subset $f(A) := \{b \in B : b = f(a), \text{ for some } a \in A\}$)

(3) f is *bijective* or is an *bijection* if it is both injective and surjective.

(4) f has a *left inverse* if there is a function $g : B \to A$ such that $g \circ f : A \to A$ is the identity map on A; i.e., $(g \circ f)(a) = a$, for all $a \in A$.

(5) f has a *right inverse* if there is a function $h : B \to A$ such that $f \circ h : B \to B$ is the identity map on B; i.e., $(f \circ h)(b) = b$, for all $b \in B$.

Proposition 1. *Let $f : A \to B$.*

(1) The map f is injective iff f has a left inverse.

(2) The map f is surjective iff f has a right inverse.

(3) The map f is a bijection iff there exists $g : B \to A$ such that $f \circ g$ is the identity map on B and $g \circ f$ is the identity map on A. (The map g is necessarily unique and we say g is the 2-sided inverse of f)

(4) If A and B are finite sets with the same number of elements ($|A| = |B|$), then $f : A \to B$ is bijective iff f is injective iff f is surjective.

Proof. (1) Suppose f is injective. Now, note that by definition of image of f, for all $c \in f(A)$, there exists $a \in A$ s.t. $c = f(a)$. Thus for all such c, we may define the function $g : f(A) \to A$ by $g(f(a)) = g(c) := a$. Note that g is well-defined as a function because each unique $c \in B$ corresponds to a unique $a \in A$ ($c_1 = f(a_1) = f(a_2) = c_2$ implies $g(c_1) = a_1 = a_2 = g(c_2)$). We may extend g to all of B arbitrarily. On the other hand, suppose f has a left inverse. Consider any $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$. Then $a_1 = g(f(a_1)) = g(f(a_2)) = a_2$.

(2) Suppose f is surjective. Then for any $b \in B$, there exists some $a \in A$ such that $f(a) = b$. Thus it is well-defined to define the function $h : B \to A$ such that $h(b) = a$, and we have $f(h(b)) = f(a) = b$. On the other hand, suppose f has a right inverse. Consider any $b \in B$. Then $f(h(b)) = b$, with $a = h(b) \in A$

(3) Suppose f is a bijection. Then by (1) and (2), there exists a left inverse g and a right inverse h. Fix any $b \in B$. Then by surjectivity of f, there exists $a \in A$ such that $b = f(a)$. But then $g(b) = g(f(a)) = a = h(b)$, and $g \equiv h$ is the inverse of f.

(4) Bijective implies injective and surjective by definition. Now suppose f is injective. Suppose that for all $a \in A$ there does not exist $b \in B$ whence $f(a) = b$. But by the pidgeonhole principle there must be (distinct) $a_1 \neq a_2 \in A$ that map to the same element in B; i.e., $f(a_1) = f(a_2)$, and this is a contradiction to the injectivity. On the other hand suppose f is surjective. Suppose that there exists $a_1 \neq a_2 \in A$ but $f(a_1) = f(a_2)$. Again by the pidgeonhole principle there must be a $b \in B$ that is not mapped to, which is a contradiction. □

Let A be a nonempty set.

(1) A binary relation on a set A is a subset R of $A \times A$ and we write $a \sim b$ if $(a, b) \in R$.

(2) The relation $\sim$ on A is said to be:

(a) reflexive if $a \sim a$ for all $a \in A$,

(b) symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in A$,

(c) transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in A$.

A relation is an equivalence relation if it is reflexive, symmetric, and transitive.

(3) If $\sim$ defines an equivalence relation on A, then the equivalence class of $a \in A$ is defined to be $\{x \in A \mid x \sim a\}$. Elements of the equivalence class of a are said to be equivalent to a. If C is an equivalence class, any element of C is called a representative of the class C.

(4) A partition of A is any collection $\{A_i \mid i \in I\}$ of nonempty subsets of A (I some indexing set) such that

(a) $A = \cup_{i \in I} A_i$, and

(b) $A_i \cap A_j = \emptyset$, for all $i, j \in I$ with $i \neq j$.

Preposition 2. *Let A be a nonempty set.*

(1) If $\sim$ defines an equivalence relation on A then the set of equivalence classes of $\sim$ form a partition of A.

(2) If $\{A_i \mid i \in I\}$ is a partition of A then there is an equivalence relation on A whose equivalence classes are precisely the sets $A_i, i \in I$.

EXERCISES

In exercises 1 to 4 let $\mathcal{A}$ be the set of 2×2 matrices with real number entries. Recall that matrix multiplication is defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

and let

$$\mathcal{B} := \{X \in \mathcal{A} \mid MX = XM\}.$$

1. Determine which of the following elements of $\mathcal{A}$ lie in $\mathcal{B}$:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The first is trivially yes. The second is no:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The third is trivially yes. The fourth is no:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

The fifth is yes (identity). The sixth is no:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

2. Prove that if $P, Q \in \mathcal{B}$, then $P + Q \in \mathcal{B}$.

$$(P+Q)M = PM + QM = MP + MQ = M(P+Q)$$

3. Prove that if $P, Q \in \mathcal{B}$, then $P \cdot Q \in \mathcal{B}$.

$$(PQ)M = P(QM) = P(MQ) = (PM)Q = (MP)Q = M(PQ)$$

4. Find conditions on p, q, r, s which determine precisely when $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathcal{B}$.

$$\begin{pmatrix} p & p+q \\ r & r+s \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} p+r & q+s \\ r & s \end{pmatrix}$$

Thus we have

$$\begin{cases} p = p + r \\ r = r \\ p + q = q + s \\ r + s = s \end{cases} \implies \begin{cases} 0 = r \\ p = s \end{cases}$$

5. Determine whether the following functions f are well-defined:

   (a) $f : \mathbb{Q} \to \mathbb{Z}$ defined by $f(a/b) = a$;
   Yes, because the rational numbers are defined to be $\{a/b : a, b \in \mathbb{Z}, b \neq 0\}$.

   (b) $f : \mathbb{Q} \to \mathbb{Q}$ defined by $f(a/b) = a^2/b^2$;
   Yes, because $a, b \in \mathbb{Z} \implies a^2, b^2 \in \mathbb{Z}$, and $b \neq 0 \implies b^2 \neq 0$.

6. Determine whether the function $f : \mathbb{R}^+ \to \mathbb{Z}$ defined by mapping a real number r to the first digit to the right of the decimal point in a decimal expansion of r is well defined.

   False: see $0.0\overline{9} = 0.1$, but $0 = f(0.0\overline{9}) = f(0.1) = 1$, and f is not a function.

7. Let $f : A \to B$ be a surjective map of sets. Prove that the relation

$$a \sim b \iff f(a) = f(b)$$

   is an equivalence relation whose equivalence classes are the fibers of f.

   See that $f(a) = f(a)$, and $f(a) = f(b)$ implies $f(b) = f(a)$, and $f(a) = f(b)$ and $f(b) = f(c)$ implies $f(a) = f(b) = f(c)$. Also see that

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}.$$

0.2 Properties of the Integers

(1) (Well Ordering of $\mathbb{Z}$) If A is any nonempty subset of $\mathbb{Z}^+$, there is some element $m \in A$ such that $m \leq a$, for all $a \in A$ (m is called a *minimal element* of A).

(2) If $a, b \in \mathbb{Z}$ with $a \neq 0$, we say a *divides* b if there is an element $c \in \mathbb{Z}$ such that $b = ac$ (i.e., b/a is an integer). In this case we write $a \mid b$; if a does not divide b we write $a \nmid b$.

(3) If $a, b \in \mathbb{Z} \setminus \{0\}$, there is a unique positive integer d, called the *greatest common divisor* (gcd) of a and b, satisfying:

   (a) $d \mid a$ and $d \mid b$

   (b) if $e \mid a$ and $e \mid b$ then $e \mid d$.

   The gcd of a and b will be denoted (a, b). If $(a, b) = 1$, we say that a and b are *relatively prime*.

(4) If $a, b \in \mathbb{Z} \setminus \{0\}$, there is a unique positive integer l, called the *least common multiple* (lcm) of a and b, satisfying:

   (a) $a \mid l$ and $b \mid l$

   (b) if $a \mid m$ and $b \mid m$ then $l \mid m$.

   The connection between the gcd d and lcm l of any two such a, b is given by $dl = ab$.

(5) *The Division Algorithm*: if $a, b \in \mathbb{Z} \setminus \{0\}$, then there exist unique $q, r \in \mathbb{Z}$ such that

$$a = qb + r, \quad 0 \leq r < |b|,$$

   where q is the *quotient* and r is the *remainder*.

(6) The *Euclidean Algorithm* is an important procedure which produces a greatest common divisor of two integers a and b by iterating the Division Algorithm: if $a, b \in \mathbb{Z} \setminus \{0\}$, then we obtain a sequence of quotients and remainders

$$\begin{aligned}
a &= q_0 b + r_0 && (0)\\
b &= q_1 r_0 + r_1 && (1)\\
r_0 &= q_2 r_1 + r_2 && (2)\\
r_1 &= q_3 r_2 + r_3 && (3)\\
&\vdots\\
r_{n-2} &= q_n r_{n-1} + r_n && (n)\\
r_{n-1} &= q_{n+1} r_n \ (r_{n+1} = 0) && (n+1)
\end{aligned}$$

where r_n is the last nonzero remainder. Such an r_n exists since $|b| > |r_0| > |r_1| > \cdots > |r_n|$ is a decreasing sequence of strictly positive integers if the remainders are nonzero and such a sequence cannot continue indefinitely. Then r_n is the gcd (a, b) of a and b.

(7) One consequence of the Euclidean Algorithm: if $a, b \in \mathbb{Z} \setminus \{0\}$, then there exist $x, y \in \mathbb{Z}$ such that

$$(a, b) = ax + by.$$

(8) An element p of $\mathbb{Z}^+$ is called a *prime* if $p > 1$ and the only positive divisors of p are 1 and p. Elements of $\mathbb{Z}^+$ that are not prime are called *composite*.

(9) The *Fundamental Theorem of Arithmetic says*: if $n \in \mathbb{Z}_{>1}$, then n can be factored uniquely into the product of primes; i.e., there are distinct primes $p_1, p_2, \dots, p_n$ and positive integers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}.$$

This factorization is unique. Suppose we have two positive integers a and b with the prime factorizations

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}, \quad b = p_1^{\beta_1} p_2^{\beta_2} \dots p_n^{\beta_n},$$

and the p_i are unique and allow the $\alpha_i, \beta_i \geq 0$. Then the greatest common divisor of a and b is

$$(a, b) = p_1^{\min(\alpha_1,\beta_1)} p_2^{\min(\alpha_2,\beta_2)} \dots p_n^{\min(\alpha_n,\beta_n)}.$$

Then similarly the least common multiple is obtained by taking each maximum instead of the minimum.

(10) The *Euler φ-function* is defined as follows: for $n \in \mathbb{Z}^+$, let $\varphi(n)$ be the number of positive integers $a \leq n$ with a relatively prime to n; i.e., $(a, n) = 1$. For primes p we have $\varphi(p) = p - 1$, and more generally for all $a \geq 1$ we have the formula

$$\varphi(p^a) = p^a - p^{a-1} = p^{a-1}(p - 1).$$

The function φ is *multiplicative* in the sense that

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \text{if } (a, b) = 1.$$

EXERCISES

1. For each of the following pairs of integers a and b, determine:
   - their greatest common divisor ($\gcd(a, b)$),
   - their least common multiple (lcm(a, b)),
   - and write their greatest common divisor in the form $ax + by$ for some integers x and y.

   (a) $a = 20, \quad b = 13$
   gcd: 1, lcm: 260

   (b) $a = 69, \quad b = 372$
   gcd: 3, lcm: 8556

   (c) $a = 792, \quad b = 275$
   gcd: 11, lcm: 19800

   (d) $a = 11391, \quad b = 5673$
   gcd: 3, lcm: 21540381

   (e) $a = 1761, \quad b = 1567$
   gcd: 1, lcm: 2759487

   (f) $a = 507885, \quad b = 60808$
   gcd: 691, lcm: 44693880

2. Prove that if the integer k divides the integers a and b, then k divides $as + bt$ for every pair of integers s and t.

3. Prove that if n is composite then there are integers a and b such that $n \mid ab$ but $n \nmid a$ and $n \nmid b$.

4. Let a, b, and N be fixed integers with a and b nonzero, and let $d = (a, b)$ be the greatest common divisor of a and b. Suppose x_0 and y_0 are particular solutions to $ax + by = N$ (i.e., $ax_0 + by_0 = N$). Prove that for any integer t, the integers

$$x = x_0 + \frac{b}{d}t \quad \text{and} \quad y = y_0 - \frac{a}{d}t$$

   are also solutions to $ax + by = N$ (this is in fact the general solution).

5. Determine the value $\varphi(n)$ for each integer $n \leq 30$ where φ denotes the Euler φ function.

6. Prove the Well-Ordering Property of $\mathbb{Z}$ by induction and prove that the minimal element is unique.

7. If p is a prime, prove that there do not exist nonzero integers a and b such that $a^2 = pb^2$ (i.e., $\sqrt{p}$ is not a rational number).

8. Let p be a prime and $n \in \mathbb{Z}^+$. Find a formula for the largest power of p which divides $n! = n(n-1)(n-2)\cdots 2 \cdot 1$ (it involves the greatest integer function).

9. Write a computer program to determine the greatest common divisor (a, b) of two integers a and b and to express (a, b) in the form $ax + by$ for some integers x and y.

10. Prove that for any given positive integer N there exist only finitely many integers n with $\varphi(n) = N$ where φ denotes the Euler φ function. Conclude in particular that $\varphi(n)$ tends to infinity as n tends to infinity.

11. Prove that if d divides n then $\varphi(d)$ divides $\varphi(n)$ where φ denotes the Euler φ function.

0.3 $\mathbb{Z}/n\mathbb{Z}$: The Integers Modulo n

abcdefg

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