ABSTRACT ALGEBRA

DUMMIT, FOOTE Second Edition Notes + Exercises

J.B.

July 2025

Contents

Preliminaries	s 1
0.1	Basics 1
0.2	Properties of the Integers 1
0.3	$\mathbb{Z}/n\mathbb{Z}$: The Integers Modulo n 1
	I GROUP THEORY 3
1.1 1.2 1.3 1.4 1.5 1.6 1.7	Introduction to Groups 5 Basic Axioms and Examples 5 Dihedral Groups 5 Symmetric Groups 5 Matrix Groups 5 The Quaternion Group 5 Homomorphisms and Isomorphisms 5 Group Actions 5
2.1 2.2 2.3 2.4 2.5	Subgroups 7 Definition and Examples 7 Centralizers and Normalizers, Stabilizers and Kernels 7 Cyclic Groups and Cyclic Subgroups 7 Subgroups Generated by Subsets of a Group 7 The Lattice of Subgroups of a Group 7
3.1 3.2 3.3 3.4 3.5	Quotient Groups and Homomorphisms 9 Definitions and Examples 9 More on Cosets and Lagrange's Theorem 9 The Isomorphism Theorems 9 Composition Series and the Hölder Program 9 Transpositions and the Alternating Group 9
4.1 4.2 4.3 4.4 4.5 4.6	Group Actions 11 Group Actions and Permutation Representations 11 Groups Acting on Themselves by Left Multiplication—Cayley's Theorem 11 Groups Acting on Themselves by Conjugation—The Class Equation 11 Automorphisms 11 The Sylow Theorems 11 The Simplicity of A_n 11

iv *CONTENTS*

5.1 5.2 5.3 5.4 5.5	Direct and Semidirect Products and Abelian Groups 13 Direct Products 13 The Fundamental Theorem of Finitely Generated Abelian Groups 13 Table of Groups of Small Order 13 Recognizing Direct Products 13 Semidirect Products 13
6.1 6.2 6.3	Further Topics in Group Theory 15 p-groups, Nilpotent Groups, and Solvable Groups 15 Applications in Groups of Medium Order 15 A Word on Free Groups 15
	II RING THEORY 17
7.1 7.2 7.3 7.4 7.5 7.6	Introduction to Rings 19 Basic Definitions and Examples 19 Examples: Polynomial Rings, Matrix Rings, and Group Rings 19 Ring Homomorphisms and Quotient Rings 19 Properties of Ideals 19 Rings of Fractions 19 The Chinese Remainder Theorem 19
Chapter 8	Euclidean Domains, Principal Ideal Domains, and Unique Factorization Domains 21
8.1 8.2 8.3	Euclidean Domains 21 Principal Ideal Domains (P.I.D.s) 21 Unique Factorization Domains (U.F.D.s) 21
9.1 9.2 9.3 9.4 9.5	Polynomial Rings 23 Definitions and Basic Properties 23 Polynomial Rings over Fields I 23 Polynomial Rings that are Unique Factorization Domains 23 Irreducibility Criteria 23 Polynomial Rings over Fields II 23
	III MODULES AND VECTOR SPACES 25
10.1 10.2 10.3 10.4 10.5	Introduction to Module Theory 27 Basic Definitions and Examples 27 Quotient Modules and Module Homomorphisms 27 Generation of Modules, Direct Sums, and Free Modules 27 Tensor Products of Modules 27 Exact Sequences—Projective, Injective, and Flat Modules 27
Chapter 11 11.1 11.2 11.3	Vector Spaces 29 Definitions and Basic Theory 29 The Matrix of a Linear Transformation 29 Dual Vector Spaces 29

CONTENTS v

11.4 11.5	Determinants 29 Tensor Algebras, Symmetric and Exterior Algebras 29
Chapter 12 12.1 12.2 12.3	Modules over Principal Ideal Domains 31 The Basic Theory 31 The Rational Canonical Form 31 The Jordan Canonical Form 31
	IV FIELD THEORY AND GALOIS THEORY 33
Chapter 13 13.1 13.2 13.3 13.4 13.5 13.6	Field Theory 35 Basic Theory of Field Extensions 35 Algebraic Extensions 35 Classical Straightedge and Compass Constructions 35 Splitting Fields and Algebraic Closures 35 Separable and Inseparable Extensions 35 Cyclotomic Polynomials and Extensions 35
14.1 14.2 14.3 14.4 14.5 14.6 14.7 14.8 14.9	Galois Theory 37 Basic Definitions 37 The Fundamental Theorem of Galois Theory 37 Finite Fields 37 Composite Extensions and Simple Extensions 37 Cyclotomic Extensions and Abelian Extensions over Q 37 Galois Groups of Polynomials 37 Solvable and Radical Extensions: Insolvability of the Quintic 37 Computation of Galois Groups over Q 37 Transcendental Extensions, Inseparable Extensions, Infinite Galois Groups 37
	N INTRODUCTION TO COMMUTATIVE RINGS, ALGEBRAIC BY, AND HOMOLOGICAL ALGEBRA 39
Chapter 15 15.1 15.2 15.3 15.4 15.5	Commutative Rings and Algebraic Geometry Noetherian Rings and Affine Algebraic Sets 41 Radicals and Affine Varieties 41 Integral Extensions and Hilbert's Nullstellensatz 41 Localization 41 The Prime Spectrum of a Ring 41
Chapter 16 16.1 16.2 16.3	Artinian Rings, Discrete Valuation Rings, and Dedekind Domains 43 Artinian Rings 43 Discrete Valuation Rings 43 Dedekind Domains 43
Chapter 17 17.1 17.2 17.3	Introduction to Homological Algebra and Group Cohomology 45 Introduction to Homological Algebra—Ext and Tor $$ 45 The Cohomology of Groups $$ 45 Crossed Homomorphisms and $$ $$ 45

vi *CONTENTS*

17.4 Group Extensions, Factor Sets, and $H^2(G, A)$ 45

VI INTRODUCTION TO THE REPRESENTATION THEORY OF FINITE GROUPS 47

18.1 18.2	Representation Theory and Character Theory 49 Linear Actions and Modules over Group Rings 49 Wedderburn's Theorem and Some Consequences 49
18.3	Character Theory and the Orthogonality Relations 49
19.1 19.2	Examples and Applications of Character Theory 51 Characters of Groups of Small Order 51 Theorems of Burnside and Hall 51

Appendix I: Cartesian Products and Zorn's Lemma 53

Appendix II: Category Theory 55

Preliminaries

- 0.1 Basics
- 0.2 Properties of the Integers
- **0.3** $\mathbb{Z}/n\mathbb{Z}$: The Integers Modulo n

2 CONTENTS

Part I GROUP THEORY

Introduction to Groups

1.1	Basic	Axioms	and	Exami	oles
-----	--------------	---------------	-----	-------	------

EXERCISES

1.2 Dihedral Groups

EXERCISES

1.3 Symmetric Groups

EXERCISES

1.4 Matrix Groups

EXERCISES

1.5 The Quaternion Group

EXERCISES

1.6 Homomorphisms and Isomorphisms

EXERCISES

1.7 Group Actions

EXERCISES

Subgroups

2.1 Definition and Examples

EXERCISES

2.2 Centralizers and Normalizers, Stabilizers and Kernels

EXERCISES

2.3 Cyclic Groups and Cyclic Subgroups

EXERCISES

2.4 Subgroups Generated by Subsets of a Group

EXERCISES

2.5 The Lattice of Subgroups of a Group

EXERCISES

Quotient Groups and Homomorphisms

3.1 Definitions and Examples

EXERCISES

3.2 More on Cosets and Lagrange's Theorem

EXERCISES

3.3 The Isomorphism Theorems

EXERCISES

3.4 Composition Series and the Hölder Program

EXERCISES

3.5 Transpositions and the Alternating Group

EXERCISES

Group Actions

- 4.1 Group Actions and Permutation Representations
- 4.2 Groups Acting on Themselves by Left Multiplication—Cayley's Theorem
- 4.3 Groups Acting on Themselves by Conjugation—The Class Equation
- 4.4 Automorphisms
- 4.5 The Sylow Theorems
- 4.6 The Simplicity of A_n

Direct and Semidirect Products and Abelian Groups

- 5.1 Direct Products
- 5.2 The Fundamental Theorem of Finitely Generated Abelian Groups
- 5.3 Table of Groups of Small Order
- 5.4 Recognizing Direct Products
- 5.5 Semidirect Products

Further Topics in Group Theory

- **6.1** *p*-groups, Nilpotent Groups, and Solvable Groups
- 6.2 Applications in Groups of Medium Order
- 6.3 A Word on Free Groups

Part II RING THEORY

Introduction to Rings

- 7.1 Basic Definitions and Examples
- 7.2 Examples: Polynomial Rings, Matrix Rings, and Group Rings
- 7.3 Ring Homomorphisms and Quotient Rings
- 7.4 Properties of Ideals
- 7.5 Rings of Fractions
- 7.6 The Chinese Remainder Theorem

Euclidean Domains, Principal Ideal Domains, and Unique Factorization Domains

- 8.1 Euclidean Domains
- 8.2 Principal Ideal Domains (P.I.D.s)
- 8.3 Unique Factorization Domains (U.F.D.s)

22CHAPTER 8. EUCLIDEAN DOMAINS, PRINCIPAL IDEAL DOMAINS, AND UNIQUE FACTORIZATION DOM.

Polynomial Rings

- 9.1 Definitions and Basic Properties
- 9.2 Polynomial Rings over Fields I
- 9.3 Polynomial Rings that are Unique Factorization Domains
- 9.4 Irreducibility Criteria
- 9.5 Polynomial Rings over Fields II

Part III MODULES AND VECTOR SPACES

Introduction to Module Theory

- 10.1 Basic Definitions and Examples
- 10.2 Quotient Modules and Module Homomorphisms
- 10.3 Generation of Modules, Direct Sums, and Free Modules
- 10.4 Tensor Products of Modules
- 10.5 Exact Sequences—Projective, Injective, and Flat Modules

Vector Spaces

- 11.1 Definitions and Basic Theory
- 11.2 The Matrix of a Linear Transformation
- 11.3 Dual Vector Spaces
- 11.4 Determinants
- 11.5 Tensor Algebras, Symmetric and Exterior Algebras

Modules over Principal Ideal Domains

- 12.1 The Basic Theory
- 12.2 The Rational Canonical Form
- 12.3 The Jordan Canonical Form

Part IV FIELD THEORY AND GALOIS THEORY

Field Theory

13.1	Basic '	Theory	of Field	Extensions
------	---------	--------	----------	-------------------

- 13.2 Algebraic Extensions
- 13.3 Classical Straightedge and Compass Constructions
- 13.4 Splitting Fields and Algebraic Closures
- 13.5 Separable and Inseparable Extensions
- 13.6 Cyclotomic Polynomials and Extensions

Galois Theory

- 14.1 Basic Definitions
- 14.2 The Fundamental Theorem of Galois Theory
- 14.3 Finite Fields
- 14.4 Composite Extensions and Simple Extensions
- 14.5 Cyclotomic Extensions and Abelian Extensions over Q
- 14.6 Galois Groups of Polynomials
- 14.7 Solvable and Radical Extensions: Insolvability of the Quintic
- 14.8 Computation of Galois Groups over Q
- 14.9 Transcendental Extensions, Inseparable Extensions, Infinite Galois Groups

Part V

AN INTRODUCTION TO COMMUTATIVE RINGS, ALGEBRAIC GEOMETRY, AND HOMOLOGICAL ALGEBRA

Commutative Rings and Algebraic Geometry

- 15.1 Noetherian Rings and Affine Algebraic Sets
- 15.2 Radicals and Affine Varieties
- 15.3 Integral Extensions and Hilbert's Nullstellensatz
- 15.4 Localization
- 15.5 The Prime Spectrum of a Ring

Artinian Rings, Discrete Valuation Rings, and Dedekind Domains

- 16.1 Artinian Rings
- 16.2 Discrete Valuation Rings
- 16.3 Dedekind Domains

44CHAPTER 16. ARTINIAN RINGS, DISCRETE VALUATION RINGS, AND DEDEKIND DOMAINS

Introduction to Homological Algebra and Group Cohomology

- 17.1 Introduction to Homological Algebra—Ext and Tor
- 17.2 The Cohomology of Groups
- **17.3** Crossed Homomorphisms and $H^1(G, A)$
- 17.4 Group Extensions, Factor Sets, and $H^2(G, A)$

46 CHAPTER 17. INTRODUCTION TO HOMOLOGICAL ALGEBRA AND GROUP COHOMOLOGY

Part VI

INTRODUCTION TO THE REPRESENTATION THEORY OF FINITE GROUPS

Representation Theory and Character Theory

- 18.1 Linear Actions and Modules over Group Rings
- 18.2 Wedderburn's Theorem and Some Consequences
- 18.3 Character Theory and the Orthogonality Relations

Examples and Applications of Character Theory

- 19.1 Characters of Groups of Small Order
- 19.2 Theorems of Burnside and Hall
- 19.3 Introduction to the Theory of Induced Characters

Appendix I: Cartesian Products and Zorn's Lemma

Appendix II: Category Theory