Real Analysis Royden - Fourth Edition

Notes + Solved Exercises :)
Latex Symbols

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Contents

I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

Preliminaries on Sets, Mappings, and Relations

Definition. *A relation R on a set X is called an **equivalence relation** provided:*

(i) xRx for all $x \in X$ (reflexive),

(ii) xRy implies yRx for all $x, y \in X$ (symmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

Partial Ordering on a set X. *A relation R on a nonempty set X is called a **partial ordering** provided:*

(i) xRx for all $x \in X$ (reflexive),

(ii) xRy and yRx imply $x = y$ for all $x, y \in X$ (antisymmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

*A subset E of X is **totally ordered** provided either xRy or yRx for all $x, y \in E$. A member x of X is said to be an **upper bound** for a subset E of X provided that*

$$yRx \text{ for all } y \in E.$$

*A member x of X is said to be **maximal** provided that*

$$xRy \text{ implies that } y = x \text{ for } y \in X.$$

Strict Partial Ordering on a set X. *A relation R on a nonempty set X is called a **strict partial ordering** provided:*

(i) not xRx for all $x \in X$ (irreflexive),

(ii) xRy implies not yRx for all $x, y \in X$ (asymmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

A subset E of X is ***strictly totally ordered*** *provided either xRy or yRx if $x \neq y$ for all $x, y \in E$.*

Zorn's Lemma. *Let X be a partially ordered set for which every totally ordered subset has an upper bound. Then X has a maximal member.*

Every vector space has a basis.

Proof. Let V be any vector space, and let L be the collection of all linearly independent subsets of V. L is nonempty as the singleton sets are linearly independent. Define a partial order on L in the form $C \subseteq C'$ for $C, C' \in L$. For any chain (a totally ordered subset of a partially ordered set) $\mathcal{C}$ of L, where $\mathcal{C}$ consists of the sets $C_1 \subseteq C_2 \subseteq \cdots$, we can construct a linearly independent set $C' = \bigcup_{C \in \mathcal{C}} C$ that is an upper bound of $\mathcal{C}$. By Zorn's Lemma, L has a maximal element, say M. This collection M is a basis for V. To show this, suppose by contradiction that there exists a vector $v \in V$ s.t. $v \notin$ Span$\{M\}$. Then $v \cup M$ is linearly independent and $M \subseteq v \cup M$, a contradiction to the fact that M is maximal. $\square$

Chapter 1

The Real Numbers: Sets, Sequences, and Functions

Contents

1.1 The Field, Positivity, and Completeness Axioms

The field axioms
Consider $a, b, c \in \mathbb{R}$:

1. Closure of Addition: $a + b \in \mathbb{R}$.
2. Associativity of Addition: $(a + b) + c = a + (b + c)$.
3. Additive Identity: $0 + a = a + 0 = a$.
4. Additive Inverse: $(-a) + a = a + (-a) = 0$.
5. Commutativity of Addition: $a + b = b + a$.
6. Closure of Multiplication: $ab \in \mathbb{R}$.
7. Associativity of Multiplication: $(ab)c = a(bc)$.
8. Distributive Property: $a(b + c) = ab + ac$.
9. Commutativity of Multiplication: $ab = ba$.

10. Multiplicative Identity: $1a = a1 = a$.
11. No Zero Divisors: $ab = 0 \implies a = 0$ or $b = 0$.
12. Multiplicative Inverse: $a^{-1}a = aa^{-1} = 1$.
13. Nontriviality: $1 \neq 0$.

The positivity axioms
The set of **positive numbers**, $\mathcal{P}$, has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are both positive.

P2 For a real number a, exactly one of the three is true: a is positive, $-a$ is positive, $a = 0$.

We call a nonempty set I of real numbers an **interval** provided for any two points in I, all the points that lie between these two points also lie in I. That is, $\forall x, y \in I, \lambda x + (1 - \lambda)y \in I$ for $\lambda \in [0, 1]$.

The completeness axiom
A nonempty set E of real numbers is said to be **bounded above** provided there is a real number b such that $x \leq b$ for all $x \in E$: the number b is called an **upper bound** for E. We can similarly define a set being **bounded below** and having a **lower bound**. A set that is bounded above need not have a largest member.

The Completeness Axiom. *Let E be a nonempty set of real numbers that is bounded above. The among the set of upper bounds for E there is a smallest, or least, upper bound. (This least upper bound is called the **supremum** of E. Also, it can be shown that any nonempty set E that is bounded below has a greatest lower bound, called the **infimum** of E).*

The extended real numbers
The extended real numbers: $\mathbb{R} \cup \{-\infty, \infty\}$
Every set of real numbers has a supremum and infimum that belongs to the extended real numbers.

PROBLEMS

1. For $a \neq 0$ and $a \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

$$
\begin{aligned}
(ab)(ab)^{-1} &= 1 && \text{by multiplicative inverse}\\
a(b(ab)^{-1}) &= 1 && \text{by associativity of multiplication}\\
a^{-1}a(b(ab)^{-1}) &= a^{-1}1 &&\\
1(b(ab)^{-1}) &= a^{-1}1 && \text{by multiplicative inverse}\\
b(ab)^{-1} &= a^{-1} && \text{by multiplicative identity}\\
b^{-1}b(ab)^{-1} &= b^{-1}a^{-1} &&\\
1(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative inverse}\\
(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative identity}\\
(ab)^{-1} &= a^{-1}b^{-1} && \text{by commutativity of multiplication}
\end{aligned}
$$

2. Verify the following:

(i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
By positivity axiom P2, since $a \neq 0$, either a is positive or $-a$ is positive.
In the case a is positive, a^2 is positive by positivity axiom P1.
In the case $-a$ is positive, $(-a)(-a)$ is positive by P1.

$$\begin{aligned}
(-a)(-a) &= (-a)(-a) + a(0) && \text{by additive identity} \\
&= (-a)(-a) + a(-a + a) && \text{by additive inverse} \\
&= (-a)(-a) + a(-a) + a(a) && \text{by distributive property} \\
&= (-a + a)(-a) + a^2 && \text{by distributive property} \\
&= 0(-a) + a^2 && \text{by additive inverse} \\
&= a^2 && \text{by additive identity}
\end{aligned}$$

Therefore a^2 is positive by equality.

(ii) For each positive number a, its multiplicative inverse a^{-1} also is positive.
The multiplication of two positive numbers is positive by positivity axiom P1.
The multiplication of two non-positive numbers is positive: by reformulating the previous result from (i), we can see $0 < (-a)(-b) = ab$ for $a, b \neq 0$.
The multiplication of a positive number and a non-positive number is not positive. To see this, suppose a is positive and b is not positive, but ab is positive. By P1 and P2, $a(-b)$ is also positive. By P1, $ab + a(-b)$ is positive. However,

$$ab + a(-b) = a(b - b) = a(0) = 0.$$

This is a contradiction to P2. Therefore ab is not positive.
The result from (i) shows that 1 is positive. By multiplicative inverse, $aa^{-1} = 1 > 0$. Therefore a^{-1} must be positive because a is positive.

(iii) If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$

Proof that $a(-1) = -a$ for a real number a:

$$a + (-1)a = 1a + (-1)a = (1 + -1)a = 0a = 0.$$

$a > b$ implies that $a - b$ is positive.
If c is positive, then $(a - b)c = ac - bc$ is positive, and $ac > bc$.
If c is not positive, then $(a - b)c = ac - bc$ is not positive, and $-(ac - bc) = bc - ac$ is positive, so $bc > ac$.

3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ iff E consists of a single point.

($\Longrightarrow$) Suppose $\inf E = \sup E$.

Then $\inf E \leq x \leq \sup E$ for all $x \in E$. But this implies $x = \inf E = \sup E$ for all $x \in E$, so E consists of the single point x.

($\Longleftarrow$) Suppose $E = x$ is a singleton set.

Clearly x is an upper bound and a lower bound for E, as $x \leq x$. By completeness of the reals, there exists $\sup E$ and $\inf E$ s.t. $x \leq \inf E \leq x \leq \sup E \leq x$, as $\inf E$ is the greatest lower bound, and $\sup E$ is the least upper bound. Therefore $\inf E = \sup E$.

4. Let a and b be real numbers.

(i) Show that if $ab = 0$, then $a = 0$ or $b = 0$.

Contrapositive: Let $a \neq 0$ and $b \neq 0$. In 2.(ii), it was shown that the multiplication of two nonzero numbers is either positive or not positive. Therefore $ab \neq 0$.

(ii) Verify that $a^2 - b^2 = (a - b)(a + b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

$$\begin{aligned}
(a-b)(a+b) &= (a-b)(a) + (a-b)(b) && \text{by distributive property} \\
&= (a)(a) + (-b)(a) + (a)(b) + (-b)(b) && \text{by distributive property} \\
&= (a)(a) + (-b+b)(a) + (-b)(b) && \text{by distributive property} \\
&= (a)(a) + (-b)(b) && \text{by additive inverse} \\
&= a^2 - b^2
\end{aligned}$$

Suppose $a^2 = b^2$. Then $(a - b)(a + b) = a^2 - b^2 = 0$, and by (i), $(a - b) = 0 \implies a = b$ or $(a + b) = 0 \implies a = -b$.

(iii) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$.

We can consider $0 \in \mathbb{R}$. $0^2 = 0 < c$, so $0 \in E$ and E is nonempty. Also, $c+1$ is a real number and an upper bound for E; thus by the completeness axiom, E has a supremum, say x_0. We can see that for any upper bound b of E, $x \leq x_0 \leq b$ for all $x \in E$. Then $x^2 \leq x_0^2 \leq b^2$ implies $x_0^2 = c$, else x_0 is not the supremum.

Suppose there exists $x_1, x_2 > 0$ such that $x_1^2 = c$ and $x_2^2 = c$. This implies $x_1^2 = x_2^2$, and by part (ii), $x_1 = x_2$ or $x_1 = -x_2$. Because x_1, x_2 are positive, $x_1 = x_2$.

5. Let a, b, c be real numbers s.t. $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}.$$

(i) Suppose $b^2 - 4ac > 0$. Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
4a(ax^2 + bx + c) &= 4a(0) \\
4a^2x^2 + 4abx + 4ac &= 0 && \text{by distributive property} \\
4a^2x^2 + 4abx + 4ac + b^2 - b^2 &= 0 && \text{by additive inverse} \\
4a^2x^2 + 4abx + b^2 &= b^2 - 4ac \\
(2ax + b)^2 &= b^2 - 4ac
\end{aligned}$$

By 4(iii), because $b^2 - 4ac > 0$, there is a unique $y > 0$ for which $y^2 = b^2 - 4ac$. It is denoted by $y = \sqrt{b^2 - 4ac}$.

By 4(ii), $(2ax+b)^2 = b^2 - 4ac = y^2$ implies $(2ax+b) = \sqrt{b^2-4ac} = y$ or $(2ax+b) = -\sqrt{b^2-4ac} = -y$.

$$\begin{aligned} 2ax + b &= \pm\sqrt{b^2-4ac} \\ 2ax &= -b \pm \sqrt{b^2-4ac} \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \end{aligned}$$

6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x \mid x \in E\}.$$

Let E be a set that is bounded below; that is, there exists $l \in \mathbb{R}$ such that $l \leq x$ for all $x \in E$. Then $-l \geq -x$ for all $x \in E$, and $-l$ is an upper bound for $-E = \{-x \mid x \in E\}$. Therefore the set $-E$ is bounded above, and by the completeness axiom, there exists a least upper bound $c = \sup(-E)$. Then for any upper bound u of $-E$, $u \geq c \geq -x$ for all $x \in E$. Then $-u$ is a lower bound of E, and $-u \leq c \leq x$ for all $x \in E$, and c is the greatest lower bound and thus infimum of E.

7. For real numbers a and b, verify the following:

(i) $|ab| = |a||b|$.

We have

$$|ab| = \begin{cases} ab & \text{if } ab \geq 0, \\ -(ab) & \text{if } ab < 0. \end{cases}$$

The case where either a or b are zero is trivial. In problem 2(ii), it was shown that $ab > 0$ if a, b are the same sign, and $ab < 0$ if a, b are opposite signs.

Case $a, b > 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = a$ and $|b| = b$ so $|a||b| = ab$.

Case $a, b < 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = -a$ and $|b| = -b$ so $|a||b| = (-a)(-b) = ab$.

Case $a < 0, b > 0$: Then $ab < 0$ so $|ab| = -(ab) = (-1)ab$, and $|a| = -a = (-1)a$ and $|b| = b$ so $|a||b| = (-1)ab$.

(ii) $|a+b| \leq |a| + |b|$.

The case where both $a, b = 0$ is trivial.

Case $a, b > 0$: Then $a + b > 0$, so $|a+b| = a + b$ and $|a| + |b| = a + b$.

Case $a > 0, b = 0$: Then $a + b = a + 0 = a > 0$, so $|a+b| = a$ and $|a| + |b| = a + 0 = a$.

Case $a < 0, b = 0$: Then $a+b = a+0 = a < 0$, so $|a+b| = -a$ and $|a|+|b| = -a+0 = -a$.

Case $a, b < 0$: Then $a + b < 0$, so $|a+b| = -(a+b) = -a - b$ and $|a| + |b| = -a - b$.

That is, equality holds except for the case where a, b are nonzero opposite signs:

Case $a > 0, b < 0$: $|a+b| \in \{a+b, -(a+b)\}$.

$b < 0 < -b \implies a+b < a < a-b$, and $-a < 0 < a \implies -(a+b) = -a-b < -b < a-b$.

$|a| + |b| = a - b$, so $|a+b| < |a| + |b|$.

(iii) For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ iff } a - \epsilon < x < a + \epsilon.$$

We have

$$|x-a| = \begin{cases} x-a & \text{if } x-a \geq 0, \\ -(x-a) & \text{if } x-a < 0. \end{cases}$$

($\Longrightarrow$) Suppose $|x-a| < \epsilon$.
Then $x-a < \epsilon$ and $a-x < \epsilon$.
Then $x < a+\epsilon$ and $a-\epsilon < x$.
($\Longleftarrow$) Suppose $a-\epsilon < x < a+\epsilon$.
Then

$$\begin{aligned} a-\epsilon-a < x-a < a+\epsilon-a \\ -\epsilon < x-a < \epsilon \end{aligned}$$

So $x-a < \epsilon$ and $-\epsilon < x-a \implies -(x-a) < \epsilon$, so $|x-a| < \epsilon$.

1.2 The Natural and Rational Numbers

Definition. *A set E of real numbers is said to be* ***inductive*** *provided it contains 1 and if the number x belongs to E, the number $x+1$ also belongs to E.*
The set of **natural numbers**, denoted by $\mathbb{N}$, is defined to be the intersection of all inductive subsets of $\mathbb{R}$.

Theorem 1. *Every nonempty set of natural numbers has a smallest member.*

Proof. Let E be a nonempty set of natural numbers. Since the set $\{x \in \mathbb{R} \mid x \geq 1\}$ is an inductive set, by definition of intersection, $\mathbb{N} \subseteq \{x \in \mathbb{R} \mid x \geq 1\}$, and the natural numbers are bounded below by 1. Therefore E is bounded below by 1. By the Completeness Axiom, E has an infimum; let $c = \inf E$. Since $c+1$ is not a lower bound for E, there is an $m \in E$ for which $m < c+1$. We claim that m is the smallest member of E. Otherwise, there is an $n \in E$ for which $n < m$. Since $n \in E$, $c \leq n$. Thus $c \leq n < m < c+1$ and therefore $m-n < 1$. Therefore the natural number m belongs to the interval $(n, n+1)$. However, an induction argument shows that $(n, n+1) \cap \mathbb{N} = \emptyset$ (Problem 8). This is a contradiction to $m \in E$. Therefore m is the smallest member of E. $\square$

Archimedean Property. *For each pair of positive real numbers a and b, there is a natural number n for which $na > b$. This can be reformulated: for each positive real number ϵ, there is a natural number n for which $\frac{1}{n} < \epsilon$.*

The set of **integers**, denoted $\mathbb{Z}$, is defined to be the set of numbers consisting of the natural numbers, their negatives, and zero.

Consider the number 2. From problem 4(iii), there is a unique $x > 0$ for which $x^2 = 2$. It is denoted by $\sqrt{2}$. This number is not rational. Suppose that x is rational: then there exist $p, q \in \mathbb{Z}$ such that $(\frac{p}{q})^2 = 2$. Then $p^2 = 2q^2$. By the unique prime factorizations of p and q, p^2 is divisible by 2^{2k} for some $k \in \mathbb{Z}_{\geq 0}$, while $2q^2$ is divisible by $2 \cdot 2^{2j} = 2^{2j+1}$ for some $j \in \mathbb{Z}_{\geq 0}$. $2^{2k} \neq 2^{2j+1}$ for any combinations of k, j so $p^2 = 2q^2$ is not possible, and $\sqrt{2}$ is not rational.

Definition. *A set E of real numbers is said to be* ***dense*** *in $\mathbb{R}$ provided that between any two real numbers there lies a member of E.*

Theorem 2. *The rational numbers are dense in $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$.
Case $a > 0$:
By the Archimedean Property, for $(b-a) > 0$, there exists $q \in \mathbb{N}$ for which $\frac{1}{q} < b - a$.
Again by the Archimedean Property, for $b, \frac{1}{q} > 0$, there exists $n \in \mathbb{N}$ for which $n(\frac{1}{q}) > b$.
Therefore the set $S = \{n \in \mathbb{N} \mid \frac{n}{q} \geq b\}$ is nonempty. Because S is a set of natural numbers, by Theorem 1, S has a smallest member p. Noticing $\frac{1}{q} < b - a < b$, we see that $1 \notin S$ and $p > 1$. Therefore $p - 1$ is a natural number (Problem 9). Because p is the smallest member of S, $p - 1 \notin S$ and $\frac{(p-1)}{q} < b$. Also,

$$a = b - (b - a) < \frac{p}{q} - (\frac{1}{q}) = \frac{(p-1)}{q}.$$

Therefore the rational number $\frac{(p-1)}{q}$ lies between a and b.
Case $a < 0$:
By the Archimedean Property, for $1, -a > 0$, there exists $n \in \mathbb{N}$ for which $n(1) > -a$, which implies $n + a > 0$, and $b > a$ implies $n + b > n + a > 0$. Then we can use the first case to show that there exists a rational number r such that $n + a < r < n + b$. Therefore the rational number $r - n$ lies between a and b. $\square$

PROBLEMS

8. Use an induction argument to show that for each natural number n, the interval $(n, n+1)$ fails to contain any natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(n, n+1) \cap \mathbb{N} = \emptyset$.

   $P(1)$: $(1, 2) = \{x \mid 1 < x < 2\}$. Suppose there exists a natural number $q \in (1, 2)$. Then $q > 1$ and by problem 9, $q - 1$ is a natural number. However, $1 < q < 2 \implies 0 < q - 1 < 1$, which is a contradiction to the fact that the natural numbers are bounded below by 1 (Theorem 1). Therefore there are no natural numbers in $(1, 2)$.

   Suppose $P(k)$ is true for some natural number k.

   $P(k+1)$: Suppose there exists a natural number $p \in (k+1, (k+1)+1)$; that is, $k+1 < p < k+2$.

   Case $p = 1$: then $k + 1 < 1 < k + 2$. but $k \in \mathbb{N}$ so $k + 1 > 1$. Thus $p = 1$ is not possible.

   Case $p > 1$: then by problem 9, $p - 1 \in \mathbb{N}$, so $k + 1 < p < k + 2 \implies k < p - 1 < k + 1$. This is a contradiction to $P(k)$, the assumption that there are no natural numbers between $(k, k + 1)$. Therefore $P(k + 1)$ is true.

9. Use an induction argument to show that if $n > 1$ is a natural number, then $n - 1$ also is a natural number. The use another induction argument to show that if m and n are natural numbers with $n > m$, then $n - m$ is a natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $n = 1$ or $n - 1 \in \mathbb{N}$.

   $P(1)$: $1 = 1$, true.

   Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

   $P(k + 1)$: $(k + 1) - 1 = k \in \mathbb{N}$, true.

For $n \in \mathbb{N}$, let $Q(n)$ be the assertion that for all $m \in \mathbb{N}$ such that $n > m$, then $n - m \in \mathbb{N}$.

$Q(1)$: true trivially, because there are no natural numbers less than 1.

Suppose $Q(k)$ is true for some $k \in \mathbb{N}$; that is, for all $m \in \mathbb{N}$ such that $k > m$, then $k - m \in \mathbb{N}$.

$Q(k+1)$: For all the m from $Q(k)$, we have $(k+1) > k > m$.

We want to show that $(k+1) - m \in \mathbb{N}$.

This is clearly true for $m = 1$ because $(k+1) - 1 = k \in \mathbb{N}$. Otherwise, $m > 1$, so by $P(m)$, $m - 1 \in \mathbb{N}$ and $(k+1) - m = k - (m-1)$. $Q(k)$ is true tells us that because $(m-1) \in \mathbb{N}$ and $k > m > m - 1$, then $k - (m-1) \in \mathbb{N}$. Therefore $Q(k+1)$ is true.

10. Show that for any real number r, there is exactly one integer in the interval $[r, r+1)$.

    This is trivial if $r \in \mathbb{Z}$.

    Consider the smallest integer p less than $[r, r+1)$. Then $p < r < r+1$ (and $r < p+1$, because $r = p+1 \implies r \in \mathbb{Z}$ and $r > p+1 \implies p$ is not the smallest integer less than $[r, r+1)$), therefore $r < p+1 < r+1$. Because the integers are inductive, $p + 1 \in \mathbb{Z}$.

    To show that there is not more than one integer between $[r, r+1)$: let q be a natural number such that $r \leq q < r+1$. Then $q - 1 < r \leq q$ and $q < r+1 \leq q+1$. From problem 8, we see that there are no integers between $(q-1, q)$ and $(q, q+1)$, so there is only one integer in $(q-1, q) \cup q \cup (q, q+1) \supseteq [r, r+1)$.

11. Show that any nonempty set of integers that is bounded above has a largest member.

    Let E be a nonempty set of integers that is bounded above. By the completeness axiom, there exists $c = \sup E$. That is, $x \leq c$ for all $x \in E$. Then $c - 1 < z \leq c$ for some $z \in E$ because $c - 1$ is not an upper bound of E. Suppose c is not in E. Then $c - 1 < z < c$. This implies that $c - 1 < z < w \leq c$ for some $w \in E$ because z is not an upper bound of E. But then there exists two integers in the interval $(c-1, c]$, which is a contradiction to problem 10. Therefore c is an element of E, and it is the largest member.

12. Show that the irrational numbers are dense in $\mathbb{R}$.

    Choose any two real numbers a, b and any irrational number z. Then $\frac{a}{z}, \frac{b}{z}$ are real numbers. By density of the rationals in $\mathbb{R}$, there exists a rational r such that $\frac{a}{z} < r < \frac{b}{z}$. This implies $a < rz < b$, where rz is an irrational number.

    Proof that rz is irrational:

    Let $r = \frac{p}{q}$ and suppose that rz is rational; then $rz = \frac{m}{n}$.

    $$\begin{aligned} \frac{p}{q} z &= \frac{m}{n} \\ z &= \frac{m}{n} \frac{q}{p} \\ z &= \frac{mq}{np} \end{aligned}$$

    Then z is rational, a contradiction.

13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

Choose any real number a. Let $S = \{r \in \mathbb{Q} \mid r \leq a\}$. Then a is an upper bound for this set. To show that a is the supremum, suppose by contradiction that it is not. Then there exists $c \in \mathbb{R}$ such that $r \leq c < a$. However, the rational numbers are dense in $\mathbb{R}$, so there exists a rational between c and a, a contradiction to the assumption that c is an upper bound to S.

The same argument can easily be shown for the irrational numbers.

14. Show that if $r > 0$, then, for each natural number n, $(1+r)^n \geq 1 + n \cdot r$.

    Let $r > 0$.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(1+r)^n \geq 1 + n \cdot r$.

    $P(1)$: $(1+r)^1 = 1 + 1 \cdot r$, true.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $(1+r)^k \geq 1 + k \cdot r$.

    $P(k+1)$:

    $(1+r)^{k+1} = (1+r)^k(1+r) \geq (1+kr)(1+r) = 1+kr+r+kr^2 > 1+kr+r = 1+(k+1)\cdot r$.

15. Use induction arguments to prove that for every natural number n,

    (i)
    $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

    $P(1)$: $\sum_{j=1}^{1} j^2 = 1 = \dfrac{1(1+1)(2+1)}{6}$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.
    $P(k+1)$:

    $$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(2k^2+k+2k+1)}{6} + \frac{6(k^2+2k+1)}{6} \\ &= \frac{(2k^3+k^2+2k^2+k)+(6k^2+12k+6)}{6} \\ &= \frac{2k^3+9k^2+13k+6}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}.\end{aligned}$$

    (ii)
    $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

    $P(1)$: $a^3 = 1 = (1)^3$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.

$P(k+1)$:

$$\begin{aligned}
1^3+2^3+\cdots+(k+1)^3 &= 1^3+2^3+\cdots+k^3+(k+1)^3\\
&= (1+2+\cdots+k)^2+(k+1)^3\\
&= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3\\
&= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}\\
&= \frac{k^2(k+1)^2+(4k+4)(k+1)^2}{4}\\
&= \frac{(k^2+4k+4)(k+1)^2}{4}\\
&= \frac{(k+2)^2(k+1)^2}{2^2}\\
&= \left(\frac{(k+2)(k+1)}{2}\right)^2\\
&= \left(\frac{((k+1)+1)(k+1)}{2}\right)^2\\
&= \Big(1+2+\cdots+(k+1)\Big)^2.
\end{aligned}$$

(iii)

$$1+r+\cdots+r^n=\frac{1-r^{n+1}}{1-r} \text{ if } r\neq 1.$$

$P(1)$: $1+r^1=\dfrac{(1+r)(1-r)}{1-r}=\dfrac{1-r^2}{1-r}$.

Suppose $P(k)$ is true for $k\in\mathbb{N}$.

$P(k+1)$:

$$\begin{aligned}
1+r+\cdots+r^{k+1} &= 1+r+\cdots+r^k+r^{k+1}\\
&= \frac{1-r^{k+1}}{1-r}+r^{k+1}\\
&= \frac{1-r^{k+1}}{1-r}+\frac{(1-r)r^{k+1}}{1-r}\\
&= \frac{1-r^{k+1}+r^{k+1}-r^{(k+1)+1}}{1-r}\\
&= \frac{1-r^{(k+1)+1}}{1-r}.
\end{aligned}$$

1.3 Countable and Uncountable Sets

Two sets A and B are **equipotent** provided there exists a bijection between them.
A set E is **countable** if it is equipotent to a set of natural numbers.
For a countably infinite set X, we say that $\{x_n \mid n\in\mathbb{N}\}$ is an **enumeration** of X provided

$$X=\{x_n \mid n\in\mathbb{N}\} \text{ and } x_n\neq x_m \text{ if } n\neq m.$$

Theorem 3. *A subset of a countable set is countable. In particular, every set of natural numbers is countable.*

Corollary 4. *The following sets are countably infinite:*

(i) For each natural numbers n, the Cartesian product $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$.

(ii) The set of natural numbers $\mathbb{Q}$.

The rationals are countable: $\mathbb{Q} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\}$.

Corollary 6. *The union of a countable collection of countable sets is countable.*

An interval of real numbers is called degenerate if it is empty or contains a single member.

Theorem 7. *A nondegenerate interval of real numbers is uncountable.*

Proof. Let I be a nondegenerate interval of real numbers. Clearly I is not finite. Suppose I is countably infinite. Let $\{x_n \mid n \in \mathbb{N}\}$ be an enumeration of I. For each $n \in \mathbb{N}$, choose a nondegenerate compact subinterval $[a_n, b_n] \subseteq I$ such that $x_n \notin [a_n, b_n]$. Let the set of such intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ be descending: $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ (That is, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$.) Now, the nonempty set $E = \{a_n \mid n \in \mathbb{N}\}$ is bounded above by b_1. Then the Completeness Axiom implies that E has a supremum, say $x^* = \sup E$. Then for each n, $a_n \leq x^* \leq b_n$ because x^* is the supremum of E and each b_n is an upper bound for E. Therefore x^* belongs to $[a_n, b_n]$ for each n. But then x^* is an element of I and thus has an index $n_0 \in \mathbb{N}$ such that $x^* = x_{n_0}$. But $x^* \in [a_{n_0}, b_{n_0}]$, a contradiction. Therefore I is not countable. $\square$

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16. Show that the set $\mathbb{Z}$ of integers is countable.

    There exists a bijection $\phi : \mathbb{Z} \to \mathbb{N}$ with

$$\phi(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x + 1 & \text{if } x \leq 0. \end{cases}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \cdots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots\}$$

17. Show that a set A is countable iff there is an injective mapping of A to $\mathbb{N}$.

    ($\Longrightarrow$) Suppose A is countable.

    Then either A is equipotent to $\mathbb{N}$, or there is an $n \in \mathbb{N}$ such that A is equipotent to $\{1, 2, \cdots, n\}$. In the case A is countably infinite, we have a bijection with $\mathbb{N}$ and thus an injection. In the case A is finite, we have an injection with a subset of $\mathbb{N}$, and thus an injection with $\mathbb{N}$ (injection: $f(a) = f(b) \implies a = b$ for $a, b \in A$).

    ($\Longleftarrow$) Suppose there is an injective mapping of A to $\mathbb{N}$.

    Then there is a bijection from A to some subset B of $\mathbb{N}$. By Theorem 3, every subset of natural numbers is countable, and because A is equipotent to this countable set B, then A is countable.

18. Use an induction argument to complete the proof of part (i) of Corollary 4.

    (Not an induction argument)

    Consider the function $f : \mathbb{N}^2 \to \mathbb{N}$, where $f(m,n) = 2^m 3^n$. By the Fundamental Theorem of Arithmetic, $2^m 3^n = 2^{m'} 3^{n'} \implies m = m', n = n'$. Then clearly f is an injection. By problem 17, we see that $\mathbb{N}^2$ is countable.

    For any $k \in \mathbb{N}$ we can construct a function $f : \mathbb{N}^k \to \mathbb{N}$, where we have n primes such that $f(m_1, m_2, \cdots, m_k) = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. By the fundamental theorem of arithmetic, this is an injection and thus $\mathbb{N}^k$ is countable.

19. Prove Corollary 6 in the case of a finite family of countable sets.

    Let $\{S_n\}_{n=1}^k$ be a finite family of countable sets. Then each set S_n is countable, and we can enumerate as follows: $S_n = \{s_{nm} \mid m \in \mathbb{N}\}$. Then because there is only a finite number of countable sets, we can construct a function $f : \bigcup_{n=1}^k S_n \to \mathbb{N}$ seeing that

$$\bigcup_{n=1}^{k} S_n = \{s_{11}, s_{21}, s_{31}, \cdots, s_{k1}, s_{12}, s_{22}, s_{32}, \cdots, s_{k2}, s_{13}, \cdots\}.$$

20. Let both $f : A \to B$ and $g : B \to C$ be injective and surjective. Show that the composition $g \circ f : A \to B$ and the inverse $f^{-1} : B \to A$ are also injective and surjective.

    $g \circ f$:

    By surjectivity of g, for all $c \in C$, there exists a $b \in B$ such that $g(b) = c$. Then by surjectivity of f, there exists an $a \in A$ such that $f(a) = b$.

    Therefore for any $c \in C$:

$$\begin{aligned} c &= g(b) && \text{for some } b \in B \\ &= g(f(a)) && \text{for some } a \in A \\ &= g \circ f(a) \end{aligned}$$

    Therefore $g \circ f$ is surjective.

    By injectivity of g, $g(b) = g(b') \implies b = b'$.

    By injectivity of f, $f(a) = f(a') \implies a = a'$.

$$\begin{aligned} g \circ f(a) &= g \circ f(a') \\ g(f(a)) &= g(f(a')) \\ f(a) &= f(a') && \text{by injectivity of } g \\ a &= a' && \text{by injectivity of } f \end{aligned}$$

    Therefore $g \circ f$ is injective.

    f^{-1}:

    Because f is a function from A to B, $f(a) \subseteq B$ is defined for all $a \in A$. That is, for all $a \in A$, there exists a $b \in B$ such that $f^{-1}(b) = a$. Thus f^{-1} is surjective.

    Because f is a function, for each $a \in A$, $f(a) = b$ and $f(a) = b'$ imply $b = b'$. That is, $f^{-1}(b) = f^{-1}(b') \implies b = b'$. Thus f^{-1} is injective.

21. Use an induction argument to establish the pigeonhole principle.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that for any $m \in \mathbb{N}$, the set $\{1, 2, \cdots, n\}$ is not equipotent to the set $\{1, 2, \cdots, n+m\}$.

    $P(1)$: We have the sets $A = \{1\}$ and $B = \{1, 2, \cdots, 1+m\}$, for $m \in \mathbb{N}$. In the case $m = 1$, $B = \{1, 1+1\} = \{1, 2\}$, and clearly A is not equipotent to B. Clearly A is also not equipotent to B for any other natural number $m > 1$.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $\{1, 2, \cdots, k\}$ is not equipotent to the set $\{1, 2, \cdots, k+m\}$, for any $m \in \mathbb{N}$.

    $P(k+1)$: Then clearly $\{1, 2, \cdots, k+1\}$ is not equipotent to the set $\{1, 2, \cdots, (k+1), \cdots, (k+1)+m\}$, for any $m \in \mathbb{N}$.

22. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable.

    (Cantor's Theorem: for a set A, any function $f : A \to \mathcal{P}(A)$ is not surjective.)

    Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any map. Let $E = \{n \in \mathbb{N} \mid n \notin f(n)\}$. Then E is a subset of the naturals that is not in the image of f, so f is not surjective. Therefore there is no bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding theorem to show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.

    In problem 18, we showed that for any $k \in \mathbb{N}$, the set $\mathbb{N}^k = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is countable. It is then trivial to see that the Cartesian product of any finite collection of countable sets $S_1 \times S_2 \times \cdots \times S_k$ is countable.

    Notation:
    $$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}, \cdots$$
    We can let $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ be the set of functions $f : \mathbb{N} \to \{0, 1\}$.

    Then, for any subset $A \subseteq \mathbb{N}$, there exists a function $f \in \{0, 1\}^{\mathbb{N}}$ such that
    $$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$
    and we have a bijection between the elements of $\{0, 1\}^{\mathbb{N}}$ and the subsets of $\mathbb{N}$ ("Two sets that are equipotent are, from a set-theoretic point of view, indistinguishable"). Therefore $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ can be used to represent the collection of subsets of $\mathbb{N}$.

    Now, because the set of functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable, then clearly the set of functions $f : \mathbb{N} \to \mathbb{N} \supseteq \{0, 1\}$ is uncountable (including zero in the naturals for notation convenience).

24. Show that a nondegenerate interval of real numbers fails to be finite.

    Theorem 7 tells us that a nondegenerate interval of real numbers is uncountable, and thus, finite.

25. Show that any two nondegenerate intervals of real numbers are equipotent.

    We can prove this by showing that any interval is equipotent to the interval $(0, 1)$.

    For any bounded interval $(a, b), (a, b], [a, b), [a, b]$, there exists a bijection to $(0, 1), (0, 1], [0, 1), [0, 1]$ respectively, of the form $f(x) = \dfrac{1}{b-a}(x - a)$.

26. Is the set $\mathbb{R} \times \mathbb{R}$ equipotent to $\mathbb{R}$?

    yes (Schröder-Bernstein theorem)

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

Proposition 9. *Every nonempty open set is the union of a countable, disjoint collection of open intervals.*

The Heine-Borel Theorem. *Let F be a closed and bounded set of real numbers. Then every open cover of F has a finite subcover.*

Proof. Let F be the closed, bounded interval $[a,b]$. Let $\mathcal{F}$ be an open cover of $[a,b]$. Define E to be the set of numbers $x \in [a.b]$ with the property that the interval $[a,x]$ can be covered by a finite number of the sets of $\mathcal{F}$. Since $a \in [a,b] \subseteq \mathcal{F}$ implies that a is in one of the sets $\mathcal{O}' \subseteq \mathcal{F}$ by definition of union, $\mathcal{O}'$ is a finite subcover of $[a,a] = \{a\}$, and thus $a \in E$ and E is nonempty. Since $E \subseteq [a,b] = \{x \mid a \leq x \leq b\}$, E is bounded above by b, so by the completeness of $\mathbb{R}$, E has a supremum $c = \sup E$. Because $c \leq b$, clearly c belongs to $[a,b]$, and this implies that there is an $\mathcal{O} \subseteq \mathcal{F}$ that contains c. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$ such that that the interval $(c-\epsilon, c+\epsilon) \subseteq \mathcal{O}$. Now $c-\epsilon$ is not an upper bound for E, and so there must be an $x \in E$ with $c - \epsilon < x$. Because $x \in E$, there exists a finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k\}$ of sets in $\mathcal{F}$ that covers $[a,x]$. Then clearly the finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k, \mathcal{O}\}$ covers the interval $[a, c+\epsilon)$. Therefore $c = b$, otherwise there exists a number $c + \frac{1}{2}\epsilon$ that has a finite subcover and $c < c + \frac{1}{2}\epsilon$ implies that c is not an upper bound for E. Thus $[a,b] \in E$ and $[a,b]$ can be covered by a finite number of sets of $\mathcal{F}$. □

The Heine-Borel Theorem ($\Longleftarrow$). *Let F be a real set such that every open cover of F has a finite subcover. Then F is closed and bounded.*

Proof. Let K be a compact subset of a metric space X. Proving that $X \setminus K$ is open will show that K is closed. Consider any $p \in X \setminus K$. For a $k \in K$, let O_k and I_k be neighborhoods of p and k respectively, with radius less than $\frac{1}{2}d(p,q)$. Because K is compact, there are finitely many points $k_1, \cdots, k_n$ in K such that $K \subseteq I_{k_1} \cup \cdots \cup I_{k_n}$. Let $O = O_{k_1} \cap \cdots \cap O_{k_n}$ so that O is an open neighborhood of p that does not intersect K. Then $O \subseteq X \setminus K$ and $X \setminus K$ is open. Therefore K is closed. □

The Nested Set Theorem. *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof. By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$, and we have an open cover of $\mathbb{R}$ and thus an open cover of $F_1 \subseteq \mathbb{R}$. By the Heine-Borel Theorem, there exists an $N \in \mathbb{N}$ such that $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c$. Because $\{F_n\}$ is descending, $F_n \supseteq F_{n+1}$ for any $n \geq 1$. This implies $F_n^c \subseteq F_{n+1}^c$, and thus $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c = F_N^c = \mathbb{R} \setminus F_N$. This is a contradiction to the assumption that F_N is a nonempty subset of F_1. □

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27. Is the set of rational numbers open or closed?

    The set of rationals is neither open nor closed. The rationals is not open because the irrationals are dense in the rationals; that is, between any two rationals there is an irrational. The rationals is not closed because it does not contain all its limit points; a sequence of rationals can be constructed that converges to an irrational. (Thus we see that the irrationals is neither open nor closed as well.)

28. What are the sets of real numbers that are both open and closed?

    It is clear that $\mathbb{R}$ is open, and $\emptyset$ is open (vacuously). Then because the complement of an open set is closed, $\mathbb{R}$ and $\emptyset$ are both closed as well.

29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

    Let $A = (4, 5)$ and $B = (5, 20)$. Then $(4, 5) \cap (5, 20) = \emptyset$ and $\overline{A} = [4, 5]$ and $\overline{B} = [5, 20]$ so $[4, 5] \cap [5, 20] = \{5\} \neq \emptyset$.

    Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$ and $\overline{A} = \mathbb{R}$ and $\overline{B} = \mathbb{R}$ so $\mathbb{R} \cap \mathbb{R} = \mathbb{R} \neq \emptyset$.

30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \setminus \{x\}$.

    (i) Show that the set E' of accumulation points of E is a closed set.

    Then for $x \in E'$, every open interval that contains x also contains a point in $E \setminus \{x\}$.

    Suppose E' is not closed. Then there exists an element $y \notin E'$ such that every open interval that contains y also contains a point $x \in E'$. Then every open interval that contains x contains a point $z \in E \setminus \{x\}$... It can be shown that $y \in E'$ and so E' contains all its points of closure and is thus closed.

    (ii) Show that $\overline{E} = E \cup E'$.

    E includes all the isolated points not included in E'.

31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.

    Each singleton set $\{x\}$ can be enumerated.

32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E. The set of interior points of E is called the **interior** of E denoted by int E. Show that

    (i) E is open iff $E =$ int E.

    ($\Longrightarrow$) Suppose E is open.

    Then clearly every point of E is an interior point.

    ($\Longleftarrow$) Suppose $E =$ int E.

    Then every point has an open neighborhood contained in E, so E is open.

    (ii) E is dense iff int $(\mathbb{R} \setminus E) = \emptyset$.

33. Show that the nested set theorem is false if F_1 is unbounded.

    The nested set theorem works because the compactness of F_1 allows us to reach a contradiction to the fact that the intersection is empty (see the proof above).

    Consider
    $$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$
    This intersection is empty because for any x, there exists an $n \in \mathbb{N}$ such that $x < n$ and thus $x \notin [n, \infty)$.

34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Heine-Borel Theorem States that Closed and bounded sets are compact; that is, every open cover of a closed and bounded set has a finite subcover. If a set E is bounded, then for any open cover $E \subseteq \mathcal{F}$ there exists a finite open subcover $\mathcal{O} \subseteq \mathcal{F}$. We can consider the intersection of all such $\mathcal{O}$ so that $E \subseteq \bigcap_{\mathcal{O} \subseteq \mathcal{F}} \mathcal{O} \subseteq \mathcal{O}$, and this intersection is the supremum.

    Clearly the descending sets from the nested set theorem are closed and bounded, so the Heine-Borel Theorem discussed above can be used to imply the Completeness Axiom.

35. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

    The Borel sets is defined to be the smallest σ-algebra that contains all the open sets of real numbers. Any sigma-algebra that contains the closed sets contains the open sets by the complement property of a sigma-algebra, so the Borel sets is the smallest sigma-algebra that contains the closed sets as well.

36. Show that the collection of Borel sets is the smallest σ-algebra that contains the intervals of the form $[a, b)$, where $a < b$.

    Any interval $[a, b)$ can be written in the form

$$[a,b) = \bigcup_{n=1}^{\infty} [a, b - \tfrac{1}{n}]$$

37. Show that each open set is an F_σ set.

    Any open set (a, b) can be written in the form

$$(a,b) = \bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b - \tfrac{1}{n}].$$

1.5 Sequences of Real Numbers

Proposition 14. *Let the sequence of real numbers $\{a_n\}$ converge to the real number a. Then the limit is unique, the sequence is bounded, and, for a real number c,*

$$\textit{if } a_n \leq c \textit{ for all } n, \textit{ then } a \leq c.$$

Proof. Suppose there exist a and b such that $\{a_n\} \to a$ and $\{a_n\} \to b$. Then For any $\epsilon > 0$, there exists the index $N = \max\{N_a, N_b\}$ such that for all $n \geq N \geq N_a, N_b$, then $|a - a_n| < \epsilon$ and $|b - a_n| < \epsilon$. By the triangle inequality, $|a - b| \leq |a - a_n| + |a_n - b| < \epsilon + \epsilon = 2\epsilon = \epsilon'$. Therefore $a = b$, and the limit is unique.

Consider $\epsilon = 1$. Then there exists an index $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < 1$. Also, $|a_n| - |a| \leq |a_n - a| < 1 \implies |a_n| < |a| + 1$. Let $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\}$. The maximum exists because this is a finite set of real numbers (totally ordered). Considering any $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - a| < 1 \implies |a_n| < |a| + 1 \leq M$, and if $n < N$, then $|a_n| \leq \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\} = M$, so M is a bound for this sequence.

Suppose that for all n, $a_n \leq c$ but $a > c$. Then $a_n \leq c < a$ for all n, and $0 \leq c - a_n < a - a_n$. Choosing $\epsilon = c - a_n$, there exists an index such that $|a - a_n| < c - a_n$. But this is a contradiction. Therefore $a \leq c$. □

Theorem 15 (the Monotone Convergence Criterion for Real Sequences)**.** *A monotone sequence of real numbers converges iff it is bounded.*

Proof. ($\implies$) Suppose a monotone sequence converges.
By the above proposition, it is bounded.
($\impliedby$) Suppose a monotone sequence $\{a_n\}$ is bounded.
By the Completeness Axiom, there exists a supremum say a such that $a_n \leq a$ for all n. Consider any $\epsilon > 0$. Now, $a - \epsilon$ is not an upper bound, and because the sequence is increasing, there exists an index N for which $a_n \geq a_N > a - \epsilon$ for all $n \geq N$. Then $\epsilon > a - a_n$ and the sequence converges to a. The proof is the same for a decreasing sequence. □

Theorem 16 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let a_n be a bounded sequence of real numbers. Choose $M > 0$ s.t. $|a_n| \leq M$ for all n. Define $E_n = \overline{\{a_j \mid j \geq n\}}$. Then we also have $E_n \subseteq [-M, M]$ and E_n is closed since it is the closure of a set. Therefore $\{E_n\}$ is a descending sequence of nonempty closed bounded subsets of real numbers. The Nested Set Theorem tells us that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$, so there exists $a \in \bigcap_{n=1}^{\infty} E_n$. For each natural number k, a is a point of closure of $\{a_j \mid j \geq k\}$. Thus for infinitely many indices $j \geq n$, a_j belongs to $(a - \frac{1}{k}, a + \frac{1}{k})$. By induction, choose a strictly increasing subsequence of natural numbers n_k such that $|a - a_{n_k}| < \frac{1}{k}$ for all k. From the Archimedean Property of the reals, the subsequence $\{a_{n_k}\}$ converges to a. □

Proposition 19. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.*

(i) $\limsup\{a_n\} = \ell \in \mathbb{R}$ *iff for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > l - \epsilon$ and only finitely many indices n for which $a_n < l - \epsilon$.*

($\Longrightarrow$) *Suppose* $\limsup\{a_n\} = \ell \in \mathbb{R}$.

Then by problem 38, ℓ is a cluster point of the sequence. This means that for all $\epsilon > 0$, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all n_k greater than some index, and thus $\ell - \epsilon < a_{n_k}$ for infinitely many indices n_k.

Suppose by contradiction that for $\epsilon > 0$, there are infinitely many indices n for which $a_n < l - \epsilon$. That is, no matter how large the epsilon we choose, there exists a subsequence $\{a_{n_k}\}$ such that $\epsilon < l - a_{n_k}$ for all n_k after a certain index. This implies that $\{a_n\}$ is not bounded, so by Proposition 14, the sequence does not converge to a real number. This is a contradiction to $\ell \in \mathbb{R}$.

($\Longleftarrow$) *Suppose for $\epsilon > 0$, there are infinitely many indices n for which $a_n > l - \epsilon$ and only finitely many indices n for which $a_n < l - \epsilon$.*

Then choosing specific indices n_k, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all n_k, and this implies the subsequence converges to ℓ. If we suppose that $\ell \neq \limsup\{a_n\}$, then there exists some $\delta > 0$ such that $\ell > \ell - \delta = \limsup\{a_n\}$.

Now, $\ell - \delta = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. That means for any n, $a_k \leq \ell - \delta$ for $k \geq n$. However, this is a contradiction to the fact that there are only finitely many such indices k for which this is true. Therefore $\ell = \limsup\{a_n\}$.

(ii) $\limsup\{a_n\} = \infty$ *iff $\{a_n\}$ is not bounded above.*

($\Longrightarrow$) *Suppose* $\limsup\{a_n\} = \infty$.

This implies that $\infty = \limsup\{a_n\}$ is a cluster point and there exists a subsequence that converges to infinity. Therefore $\{a_n\}$ is not bounded above.

($\Longleftarrow$) *Suppose $\{a_n\}$ is not bounded above.*

By Proposition 4, $\{a_n\}$ does not converge to a real number. Also, $\{a_n\}$ is not bounded above implies that for any real number c, there exists an index such that $a_n > c$. Then the only upper bound of this sequence is ∞ and thus $\limsup\{a_n\} = \infty$.

(iii)

$$\limsup\{a_n\} = -\liminf\{-a_n\}.$$

Definitions of limsup and liminf:

$\limsup\{a_n\} = \lim_{n\to\infty}[\sup\{a_k \mid k \geq n\}] \implies$ *for any $n \in \mathbb{N}$, $\sup\{a_k \mid k \geq n\} \geq a_k$ for $k \geq n$.*

$\liminf\{a_n\} = \lim_{n\to\infty}[\inf\{a_k \mid k \geq n\}]$. $\implies$ *for any* $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k$ *for* $k \geq n$.
Now we have
$\liminf\{-a_n\} = \lim_{n\to\infty}[\inf\{-a_k \mid k \geq n\}]$.
$\implies$ *for any* $n \in \mathbb{N}$, $\inf\{-a_k \mid k \geq n\} \leq -a_k$ *for* $k \geq n$.
$\implies$ *for any* $n \in \mathbb{N}$, $-\inf\{-a_k \mid k \geq n\} \geq a_k$ *for* $k \geq n$*, the definition of limsup.*

(iv) A sequence of real numbers $\{a_n\}$ *converges to an extended real number* a *iff*

$$\liminf\{a_n\} = \limsup\{a_n\} = a.$$

$(\implies)$ *Suppose a sequence of real numbers* $\{a_n\}$ *converges to an extended real number* a.
Clearly $\liminf\{a_n\} \leq a \leq \limsup\{a_n\}$.
If $\liminf\{a_n\} < a < \sup\{a_n\}$*, then we reach a contradiction to the infimum and supremum respectively.*
Therefore $\liminf\{a_n\} = a = \limsup\{a_n\}$.
$(\impliedby)$ *Suppose* $\liminf\{a_n\} = \limsup\{a_n\} = a$.
Then for any $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ *for* $k \geq n$*, which implies*

$$a = \liminf\{a_n\} = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\} \leq \lim_{n\to\infty} a_k \leq \lim_{n\to\infty} \sup\{a_k \mid k \geq n\} = \limsup\{a_n\} = a$$

Clearly $\{a_n\}$ *converges to* a.

(v) If $a_n \leq b_n$ *for all* n*, then*

$$\limsup\{a_n\} \leq \limsup\{b_n\}.$$

For any $n \in \mathbb{N}$, $a_k \leq \sup\{a_k \mid k \geq n\}$ *and* $b_k \leq \sup\{b_k \mid k \geq n\}$ *for all* $k \geq n$.
If we suppose $\limsup\{a_n\} > \limsup\{b_n\}$*, then there exists a natural number* n *such that* $\sup\{a_k \mid k \geq n\} > \sup\{b_k \mid k \geq n\} \geq b_k \geq a_k$ *for all* $k \geq n$*. However, by problem 38, we see that* $\limsup\{a_n\}$ *is a cluster point of* $\{a_n\}$*, and we reach a contradiction. (or contradiction to def of supremum?)*

Proposition 20. *Let* $\{a_n\}$ *be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ *is summable iff for each* $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

$(\implies)$ *Suppose the series* $\sum_{k=1}^{\infty} a_k$ *is summable.*
That is, there exists an s *such that* $\{\sum_{k=1}^{n} a_k\}$ *converges to* s*. Convergent sequences are Cauchy, so for any* $\epsilon > 0$*, there exists and index* N *such that for all* $n + m \geq n - 1 \geq N$,

$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

$$\left|\sum_{k=1}^{n-1} a_k + \sum_{k=n}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon.$$

($\Longleftarrow$) *Suppose that for each* $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

Then

$$\left|\sum_{k=n}^{n+m} a_k + \sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

Without loss of generality, we can suppose that $n - 1 \geq N$*, and because* m *is a natural number,* $n + m > n - 1 \geq N$*. Clearly this describes a Cauchy Sequence, and because the real numbers is complete, this sequence converges and thus the series is summable.*

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ *is summable, then* $\sum_{k=1}^{\infty} a_k$ *is also summable.*

By subadditivity of absolute value, we can show that for each $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| \leq \left|\sum_{k=n}^{n+m} |a_k|\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

(iii) If each term a_k *is nonnegative, then the series* $\sum_{k=1}^{\infty} a_k$ *is summable iff the sequence of partial sums is bounded.*

Let $\{a_k\}$ *be a sequence of nonnegative numbers.*

($\Longrightarrow$) *Suppose the series* $\sum_{k=1}^{\infty} a_k$ *is summable.*

Then the sequence of partial sums converges to a real number. By Proposition 14, the sequence of partial sums is bounded.

($\Longleftarrow$) *Suppose the sequence of partial sums is bounded.*

Because each a_k *is positive, the sequence of partial sums is positive monotonic:*

$$\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} a_k + a_{n+1} = \sum_{k=1}^{n+1} a_k.$$

Therefore by Theorem 15, the sequence of partial sums converges; that is, the series is summable.

PROBLEMS

38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

    Let $s = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. Suppose there exists a subsequence $\{a_{n_k}\}$ that converges to an extended real number a. Fix $\epsilon > 0$. Then there exists an index M such that $|a - a_{n_m}| < \epsilon$ when $n_m \geq M$, and $a_{n_m} \leq \sup\{a_k \mid k \geq M\}$.

    Then $\lim_{M\to\infty} a_{n_m} \leq \lim_{M\to\infty} \sup\{a_k \mid k \geq M\} \implies a \leq s$.

    Therefore $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$. ($\limsup\{a_n\}$ is itself a cluster point else we reach a contradiction to the supremum.) The same method can be used to prove $\liminf\{a_n\}$.

39. Prove proposition 19.

    See above for proof.

40. Show that a sequence $\{a_n\}$ is convergent to an extended real number iff there is exactly one extended real number that is a cluster point of the sequence.

    ($\Longrightarrow$) Suppose $\{a_n\}$ is convergent to an extended real number a.

    By Proposition 19(iv), we have $\liminf\{a_n\} = \limsup\{a_n\} = a$, so clearly any cluster point is equal to a.

    ($\Longleftarrow$) Suppose there is exactly one extended real number a that is a cluster point of $\{a_n\}$.

    Then there exists a subsequence that converges to a. Suppose that $\{a_n\}$ does not converge to a. Then there exists an $\epsilon > 0$ such that there are infinitely many indices n for which $a - a_n > \epsilon$. Collect these indices to construct a subsequence $\{a_{n_k}\}$. In the case that $\{a_{n_k}\}$ is bounded, there exists another subsequence of $\{a_{n_k}\}$ that converges to a real number $b \neq a$. But this is also a subsequence of the original sequence $\{a_n\}$, which implies $\{a_n\}$ has two cluster points a and b, a contradiction. In the case that $\{a_{n_k}\}$ is unbounded, then for any real number c, there exists an index n such that $|a_n| > c$. Then we can construct a subsequence that converges to $+\infty \neq a$ or $-\infty \neq a$, which is again a contradiction to the fact that $\{a_n\}$ has only one cluster point.

41. Show that $\liminf a_n \leq \limsup a_n$.

    For any natural number n, we have $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ for all $k \geq n$. Taking the limit with respect to n clearly proves the statement.

42. Prove that if, for all n, $a_n \geq 0$ and $b_n \geq 0$, then

    $$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

    provided the product on the right is not of the form $0 \cdot \infty$.

    For any natural number n, we can see that

    $$\{a_k \cdot b_k \mid k \geq n\} \subseteq \{a_i \cdot b_j \mid i, j \geq n\}.$$

    Then this clearly implies

    $$\begin{aligned} \sup\{a_k \cdot b_k \mid k \geq n\} &\leq \sup\{a_i \cdot b_j \mid i, j \geq n\} \\ &= \sup\{a_i \mid i \geq n\} \cdot \sup\{b_j \mid j \geq n\}. \end{aligned}$$

    Taking the limit on both sides proves the inequality.

43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

    Let $\{a_n\}$ be any sequence of real numbers. Supposing that there exist no monotone subsequences of $\{a_n\}$, then there are only finitely many indices n for which $a_n \leq a_{n+1}$, and only finitely many indices n for which $a_n \geq a_{n+1}$. Clearly we see a contradiction so there must exist a monotone subsequence.

    Now, in the case that $\{a_n\}$ is bounded, then the monotone subsequence $\{a_{n_k}\}$ is also bounded. By Theorem 15, $\{a_{n_k}\}$ converges. Thus $\{a_n\}$ has a convergent subsequence.

44. Let p be a natural number greater than 1, and x a real number $0 \leq x \leq 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n, $0 < q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal* expansion of x. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

For each $m \in \mathbb{N}$, we can construct a partial sum:

$$\sum_{n=1}^{m} \frac{a_n}{p^n} = \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m}$$

We choose each a_m in the following way:

(The $\sum_{n=1}^{m-1} \frac{a_n}{p^n}$ is a fixed value found from the previous iteration, so for each step, we are simply choosing the best a_m).

Case $\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} = x$ for some $a_m \in \{0, 1, \cdots, p\}$: Then set $a_k = 0$ for all $k \geq m$, and the equality is clear.

Else: Choose $a_m \in \{0, 1, \cdots, p\}$ such that:

$$\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} < x < \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m + 1}{p^m}.$$

In this way we can construct a monotone sequence (of partial sums) that is bounded above by x:

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq \sum_{n=1}^{k+1} \frac{a_n}{p^n} \leq x \text{ for all } k \in \mathbb{N}.$$

By showing that x is the supremum, we can apply Theorem 15 to show that this sequence of partial sums converges to its supremum:

$$\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = x.$$

Suppose that x is not the supremum. Then there exists an $\epsilon > 0$ such that $\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x$ for all k. Now, by the Archimedean Property, there exists a natural number m such that $\frac{1}{m} < \epsilon$; therefore $0 < \epsilon - \frac{1}{m}$. Now, because $p > 1$, there exists a natural number l such that $m < p^l$, so $0 < \frac{1}{p^l} < \frac{1}{m}$ and thus $x - (\epsilon - \frac{1}{p^l}) < x - (\epsilon - \frac{1}{m}) < x$.

Then for all natural numbers k,

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x - \epsilon + \frac{1}{p^l} < x - \epsilon + \frac{1}{m} < x.$$

However, there exists the natural number l such that

$$\sum_{n=1}^{l}\frac{a_n}{p^n}=\sum_{n=1}^{l-1}\frac{a_n}{p^n}+\frac{a_l}{p^l}\leq x-\epsilon<x-\epsilon+\frac{1}{p^l}<x$$
$$\sum_{n=1}^{l-1}\frac{a_n}{p^n}+\frac{a_l+1}{p^l}\leq x-\epsilon+\frac{1}{p^l}<x.$$

This is a contradiction to our choice of a_l so that

$$\sum_{n=1}^{l-1}\frac{a_n}{p^n}+\frac{a_l}{p^l}<x<\sum_{n=1}^{l-1}\frac{a_n}{p^n}+\frac{a_l+1}{p^l}.$$

Therefore x is the supremum, and the series $\sum_{n=1}^{\infty}\frac{a_n}{p^n}$ is summable to x.

In the case that x is of the form q/p^n, the obvious solution would be to set $a_n=q$ (assuming q is an integer), and all other $a_k=0$. The second solution would be to use the method described above.

For the converse, $0\leq a_n\leq p-1$ implies that

$$\sum_{n=1}^{\infty}\frac{a_n}{p^n}\leq\sum_{n=1}^{\infty}\frac{p-1}{p^n}=(p-1)\sum_{n=1}^{\infty}\frac{1}{p^n}$$

Showing that $(p-1)\sum_{n=1}^{\infty}\frac{1}{p^n}<1$ implies that $\sum_{n=1}^{k}\frac{a_n}{p^n}$ is a bounded, monotone sequence of partial sums, and therefore it converges to a number in $[0,1]$.

Ex: $x=.547$; decimal expansion:

$$x=\frac{5}{10^1}+\frac{4}{10^2}+\frac{7}{10^3}+\frac{0}{10^4}+\frac{0}{10^5}+\cdots=.5+.04+.007+0+0+\cdots$$

45. Prove Proposition 20.

    See above.

46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Bolzano-Weierstrass Theorem asserts that every bounded sequence of real numbers has a convergent subsequence.

    The Completeness Axiom asserts that every nonempty set of real numbers that is bounded above has a supremum.

    The Monotone Convergence Theorem asserts that a monotone sequence of real numbers converges iff it is bounded.

1.6 Continuous Real-Valued Functions of a Real Variable

Proposition 21. *A real-valued function f defined on a set E of real numbers is continuous at the point $x_* \in E$ iff whenever a sequence $\{x_n\}$ in E converges to x_*, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.*

Proof. Let f be a real-valued function defined on a set E.
($\implies$) Suppose that f is continuous at the point $x_* \in E$.
Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } x' \in E \text{ and } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.$$

Suppose that a sequence $\{x_n\}$ in E converges to x_*. Then for any $\delta > 0$, there exists an index N such that when $n \geq N$, $|x_* - x_n| < \delta$. Then by continuity of f, $|f(x_*) - f(x_n)| < \epsilon$, and thus the image sequence converges.
($\impliedby$) Suppose that whenever a sequence $\{x_n\}$ in E converges to x_*, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.
That is, for any $\delta > 0$, there exists an index N such that $|x_* - x_n| < \delta$ whenever $n \geq N$, and this implies that for any $\epsilon > 0$, there exists an index M such that $|f(x_*) - f(x_n)| < \epsilon$ whenever $n \geq M$.
Thus continuity is clear. □

Proposition 22. *Let f be a real-valued function defined on a set E of real numbers. Then f is continuous on E iff for each open set $\mathcal{O}$,*

$$f^{-1}(\mathcal{O}) = E \cap \mathcal{U} \text{ where } \mathcal{U} \text{ is an open set.}$$

The Extreme Value Theorem. *A continuous real-valued function on a nonempty, closed, bounded set of real numbers takes a minimum and a maximum value.*

Proof. Let f be a continuous real-valued function on a nonempty, closed, bounded set E of real numbers. Suppose by contradiction that f is not bounded. Then for any $n \in \mathbb{N}$, there exists $x_n \in E$ such that $f(x_n) > n$. With this we can construct a sequence $\{x_n\}$ in E. Because E is bounded, $\{x_n\}$ is bounded, and so by the Bolzano Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$. Because f is continuous, $\{x_{n_k}\}$ is convergent implies $\{f(x_{n_k})\}$ is convergent. However, for each element in the image sequence, $f(x_{n_k}) > n_k$, and $\{f(x_{n_k})\}$ is unbounded, thus it cannot converge, and we reach a contradiction.
Because f is bounded, then it has a supremum s such that $f(x) \leq s$ for all $x \in E$. Suppose that f does not have a maximum. Then there is no $x \in E$ such that $f(x) = s$. Then $f(x) < s \implies f(x) \in (-\infty, s)$ for all $x \in E$. (We can use the fact that $(-\infty, s)$ is open $\implies f^{-1}(-\infty, s)$ is open): Then we reach a contradiction because E is closed. The same proof can be used for the minimum. □

The Intermediate Value Theorem. *Let f be a continuous real-valued function on the closed, bounded interval $[a, b]$ for which $f(a) < c < f(b)$. The there is a point x_0 in (a, b) at which $f(x_0) = c$.*

Theorem 23. *A continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous.*

Proof. Let E be a closed, bounded set of real numbers, and let f be a continuous real-valued function on E.
Fix some $\epsilon > 0$.
By the continuity of f, for all $x \in E$, there exists $\delta_x > 0$ such that for $y \in E$ satisfying $|x - y| < 2\delta_x$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then we can construct an open cover of E consisting of the open balls $\mathbb{B}(x, \delta_x)$ for all $x \in E$.
Because E is compact, there exists a finite subcover $\{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$.

Let $\delta_* = \min\{\delta_1, \cdots, \delta_n\}$.
Consider $x, y \in E$ such that $|x - y| < \delta_*$.
Because $y \in E \subseteq \{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$, there exists an index $j \in \{1, \cdots, n\}$ such that $y \in \mathbb{B}(x_j, \delta_j)$; therefore

$$|x_j - y| < \delta_j < 2\delta_j.$$

By continuity of f, $|f(x_j) - f(y)| < \frac{\epsilon}{2}$. (A)
By the triangle inequality,

$$|x - x_j| \leq |x - y| + |y - x_j| < \delta_* + \delta_j \leq 2\delta_j.$$

By continuity of f, $|f(x) - f(x_j)| < \frac{\epsilon}{2}$. (B)
By the triangle inequality using (A) and (B):

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

PROBLEMS

47. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E. Show that there is a function g defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take g to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)

    Because E is closed, then $\mathbb{R} \setminus E$ is open. In the case that $E = \mathbb{R}$, then $\mathbb{R} \setminus E = \emptyset$ and the conclusion is trivial. Else $\mathbb{R} \setminus E$ is nonempty. By proposition 9, $\mathbb{R} \setminus E$ is the union of a countable, disjoint collection of open intervals.
    In the case that $(-\infty, a)$ [or (a, ∞)] is in $\mathbb{R} \setminus E$, then $a \in E$ and $f(a)$ is defined. Simply let $g(x) = f(a)$ be the constant function on $(-\infty, a)$ [or (a, ∞)].
    In the case that $(a, b) \in \mathbb{R} \setminus E$, then $a, b \in E$ and $f(a)$,$f(b)$ are defined. Let

    $$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ on } (a, b).$$

    Also let $g(x) = f(x)$ whenever $x \in E$. Then we see that g is continuous.

48. Define the real-valued function f on $\mathbb{R}$ by setting

    $$f(x) = \begin{cases} x & \text{if x irrational} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

    At what points is f continuous?

    See Thomae's Function for something similar.

    f should be discontinuous at each rational number and continuous at each irrational number.

49. Let f and g be continuous real-valued functions with a common domain E.

    (i) Show that the sum, $f + g$, and product, fg, are also continuous functions.
    Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{f(x_n)\}$ converges to $f(x)$ and $\{g(x_n)\}$ converges to $g(x)$ by continuity of f, g.

That is, for any $\epsilon > 0$, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(x_n) - f(x)| < \frac{\epsilon}{2}$ and $|g(x_n) - g(x)| < \frac{\epsilon}{2}$ whenever $|x_n - x| < \delta$. By the triangle inequality,

$$\begin{aligned} |(f+g)(x_n) - (f+g)(x))| &= |(f(x_n) + g(x_n)) - (f(x) + g(x))| \\ &\leq |f(x_n) - f(x)| + |g(x_n) + g(x)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Fix any $\epsilon > 0$. By continuity of f, g, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(y) - f(x)| < \frac{\epsilon}{2|g(x)|}$ and $|g(y) - g(x)| < \frac{\epsilon}{2|f(y)|}$ whenever $|y - x| < \delta$.

$$\begin{aligned} |fg(y) - fg(x)| &= |f(y)g(y) - f(x)g(x)| \\ &= |f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x)| \\ &= |f(y)(g(y) - g(x)) + g(x)(f(y) - f(x))| \\ &\leq |f(y)(g(y) - g(x))| + |g(x)(f(y) - f(x))| \\ &= |f(y)||g(y) - g(x)| + |g(x)||f(y) - f(x)| \\ &< |f(y)|\frac{\epsilon}{2|f(y)|} + |g(x)|\frac{\epsilon}{2|g(x)|} \\ &= \epsilon \end{aligned}$$

(This one is a bit janky).

(ii) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.

Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{h(x_n)\} \in E$ converges to $h(x) \in E$ by continuity of h. Then $\{f \circ h(x_n)\} = \{f(h(x_n))\}$ converges to $f \circ h(x) = f(h(x))$ by continuity of f. Therefore the composition is continuous.

(iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.

Fix $\epsilon > 0$. By continuity of f, g, there exists a $0 < \delta \leq \delta_f, \delta_g$ s.t. whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2}$.

We can write

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}.$$

This is by the identity:

$$\begin{aligned} \max(x, y) + \min(x, y) &= x + y \\ \max(x, y) - \min(x, y) &= |x - y| \\ \max(x, y) &= \frac{1}{2}(x + y + |x - y|) \\ \min(x, y) &= \frac{1}{2}(x + y - |x - y|) \end{aligned}$$

Now, $|\max\{f(x), g(x)\} - \max\{f(y), g(y)\}|$ is equal to

$$\left|\frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2} - \left(\frac{f(y)+g(y)}{2} + \frac{|f(y)-g(y)|}{2}\right)\right|$$
$$= \left|\frac{f(x)-f(y)+g(x)-g(y)+|f(x)-g(x)|-|f(y)-g(y)|}{2}\right|$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|\,|f(x)-g(x)|-|f(y)-g(y)|\,|}{2}$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-g(x)-f(y)+g(y)|}{2}$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-f(y)|+|g(y)+g(x)|}{2}$$
$$< \frac{\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}}{2}$$
$$= \epsilon.$$

(iv) Show that $|f|$ is continuous.
For any $\epsilon > 0$, there exists a delta such that whenever $|x-y| < \delta$, by the reverse triangle inequality:

$$|\,|f(x)| - |f(y)|\,| \leq |f(x) - f(y)| < \epsilon.$$

50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

Lipschitz: there exists $L \geq 0$ s.t. for all x, x':

$$|f(x) - f(x')| \leq L|x - x'|$$

Fixing any $\epsilon > 0$, whenever $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq L|x - x'| < L\delta,$$

so we can set $\delta = \dfrac{\epsilon}{L}$. The δ is the same for any values of x, so f is uniformly continuous.

The function $\sqrt{x}$ is uniformly continuous but not Lipschitz.

51. A continuous function ϕ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which ϕ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function ϕ on $[a, b]$ with $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a, b]$.
Start with $f(x_0)$, and choose x_1 so that $f(x_1) = f(x_0) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0) + f(x_0)$ on $[x_0, x_1]$.
Repeat this process to choose each interval:
Start with $f(x_i)$, and choose x_{i+1} so that $f(x_{i+1}) = f(x_i) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}(x - x_i) + f(x_i)$ on $[x_i, x_{i+1}]$.
Then we see that f and ϕ are always within ϵ of each other, and ϕ is continuous and piecewise linear.

52. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

Let E be a nonempty set of real numbers.
($\Longrightarrow$) Suppose E is closed and bounded.
By the Extreme Value Theorem, every continuous real-valued function on E takes a maximum (and minimum) value.
($\Longleftarrow$) Suppose every continuous real-valued function on E takes a maximum value.
Suppose that E is not closed. The continuous real-valued function $f(x) = \frac{1}{x}$ on the open set $E = (0, 1)$ does not take on a maximum value. Contradiction.
Suppose E is not bounded. The continuous real-valued function $f(x) = x^2$ on the unbounded set $E = [0, \infty)$ does not take on a maximum value. Contradiction.
Therefore E must be closed and bounded. (Not the right way to do this...)

53. Show that a set E of real numbers is closed and bounded iff every open cover of E has a finite subcover.

    Let E be a set of real numbers.

    ($\Longrightarrow$) Suppose E is closed and bounded.

    By the Heine-Borel Theorem, every open cover of E has a finite subcover.

    ($\Longleftarrow$) Suppose every open cover of E has a finite subcover.

    See the proof in 1.4 after the Heine-Borel Theorem.

54. Show that a nonempty set E of real numbers is an interval iff every continuous real-valued function on E has an interval as its image.
    Let E be a nonempty set of real numbers.
    ($\Longrightarrow$) Suppose E is an interval.
    Then for any two points $x, y \in E$, the set $[x, y]$ is in E. Let f be a continuous real-valued function on E. Now, we have $f(x), f(y)$ in the image of f. Suppose, without loss of generality, that $f(x) < f(y)$. By the Intermediate Value Theorem, for any c such that $f(x) < c < f(y)$, there exists $x_0 \in (x, y) \subseteq E$ such that $f(x_0) = c$. That is, for any two points in the image of f, every point between them is also in the image of f. Therefore the image of f is an interval.
    ($\Longleftarrow$) Suppose every continuous real-valued function on E has an interval as its image.
    Suppose E is not an interval. Then there exist two points $x, y \in E$ such that $x < a < y$ but $a \notin E$. Let f be a continuous real-valued function on E, and without loss of generality, let f be monotonically increasing. Because $x, y \in E$, then $f(x)$,$f(y)$ are defined, so $[f(x), f(y)]$ is in the image of f.
    Define two disjoint collections of subsets of E: $I_{<a} = \{I \subseteq E \mid x < a \ \forall x \in I\}$ and $I_{>a} = \{I \subseteq E \mid x > a \ \forall x \in I\}$, so that $I_{<a} \cap I_{>a} = \emptyset$. These collections are nonempty because $\{x\} \in I_{<a}$ and $\{y\} \in I_{>a}$. Consider $\bigcup I_{<a} \subseteq E$, the union of all elements of $I_{<a}$, and $\bigcup I_{>a} \subseteq E$, the union of all elements of $I_{>a}$. By monotonicity of f, $f(\bigcup I_{<a}) < f(\bigcup I_{>a})$, so $[f(x), f(y)] \nsubseteq f(\bigcup I_{<a}) \bigcup f(\bigcup I_{>a}) = f(E)$, a contradiction.

55. Show that a monotone function on an open interval is continuous iff its image is an interval.
    Let f be a monotone function on an open interval $E = (a, b)$.
    ($\Longrightarrow$) Suppose f is continuous.
    Then by Problem 54, E is an interval implies that the continuous real-valued function f has an interval as its image.
    ($\Longleftarrow$) Suppose the image of f is an interval.
    Let x_0 be a point in the open interval E, so that $f(x_0)$ is defined. For any sequence $\{x_n\}$ in $E \cap (x_0, \infty)$ that converges to x_0, then $\{f(x_n)\}$ converges to $f(x_0^+)$.
    Similarly, for any sequence $\{x_n\}$ in $E \cap (-\infty, x_0)$ that converges to x_0, then $\{f(x_n)\}$ converges to$f(x_0^-)$.

Then $f(x_0^-) = f(x_0) = f(x_0^+)$ by monotonicity.
(messy)

56. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.
A G_δ set is a set that is a countable intersection of open sets.
f is continuous at a point x if for any open set in the image containing $f(x)$, the inverse image is an open set containing x.

57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.
An F_σ set is a set that is a countable union of closed sets.

58. Let f be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.
The inverse image of an open set is open (See prop 22).
Suppose that the inverse image of a closed set is not closed. That is, let B be a closed set of real numbers and let $f^{-1}(B) = \{x \in \mathbb{R} \mid f(x) \in B\}$ not be closed. Then there exists a sequence $x_n \in f^{-1}(B)$ that converges to $x \notin f^{-1}(B)$. However, by continuity of f, $f(x_n) \in B$ converges to $f(x) \notin B$. This implies that B does not contain all its limit points, and thus B is not closed, a contradiction. Therefore $f^{-1}(B)$ must be closed.
Another way:
We have, for any open set $\mathcal{O} \in \mathbb{R}$,
$$\mathbb{R} = dom(f) = f^{-1}(\mathbb{R}) = f^{-1}(\mathcal{O} \cup \mathcal{O}^c) = f^{-1}(\mathcal{O}) \cup f^{-1}(\mathcal{O}^c).$$
Because $f^{-1}(\mathcal{O})$ is open in $\mathbb{R}$, then $f^{-1}(\mathcal{O}^c)$ is closed in $\mathbb{R}$.

59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f iff given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.
We want to show that uniform convergence preserves continuity.
Fix $\epsilon > 0$.
By uniform convergence of $\{f_n\}$, there exists an index N such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all $x \in E$ and all $n \geq N$.
By continuity of each f_n, for all $x \in E$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $|x - y| < \delta$.
Therefore we have:
$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Chapter 2

Lebesgue Measure

Contents

2.1 Introduction

In this chapter we construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure**, denoted by m. (A *set function* is a function that associates an extended real number to each set in a collection of sets.) The collection of Lebesgue measurable sets is a σ-algebra which contains all open sets and all closed sets. The set function m possesses the following three properties:

The measure of an interval is its length. *Each nonempty interval I is Lebesgue measurable and*

$$m(I) = \ell(I).$$

Measure is translation invariant. *If E is Lebesgue measurable and y is any number then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and*

$$m(E + y) = m(E).$$

Measure is countably additive over countable disjoint unions of sets. *If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then*

$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (See Vitali sets). We first construct a set function called **outer measure**, denoted by m^*, such that:

(i) the outer measure of an interval is its length:

$$m^*(I) = \ell(I).$$

(ii) outer measure is translation invariant:

$$m^*(A + y) = m^*(A).$$

(iii) outer measure is countably subadditive:

$$m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Outer measure is defined for all sets of real numbers. However, outer measure fails to be countably additive: there exists A, B disjoint s.t. $m^*(A \cup B) < m^*(A) + m^*(B)$.
Then the Lebesgue measure m is the restriction of m^* to the Lebesgue measurable sets.

PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

   $A \subseteq B \implies B = A \cup (B \cap A^c)$, where $A \cap (B \cap A^c) = \emptyset$. The set $(B \cap A^c)$ is measurable because A^c is measurable and countable intersection is measurable, so $m(B) = m(A \cup (B \cap A^c)) = m(A) + m(B \cap A^c)$ by countable additivity, and thus $m(B) \geq m(A)$.

2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

   We have $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

   $$\begin{aligned} m(A) &= m(A \cup \emptyset) \\ m(A) &= m(A) + m(\emptyset) && \text{by disjoint additivity} \\ 0 &= m(\emptyset). \end{aligned}$$

3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$. For any two measurable sets A, B, we have $A \cup B = (A \setminus B) \cup (B)$. By disjoint additivity,

   $$m(A \cup B) = m(A \setminus B) + m(B)$$

   Now, by problem 1, $(A \setminus B) \subseteq A$ implies that $m(A \setminus B) \leq m(A)$. Therefore

   $$m(A \cup B) \leq m(A) + m(B).$$

4. A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.
   Suppose $E = \{x_1, \cdots, x_n\}$.
   Then $m(E) = n$. For any real number y, $y + E = \{y + x_1, \cdots, y + x_n\}$, so $m(y + E) = n$.
   Suppose E has infinitely many members.
   Then $y + E$ has infinitely members as well, so $m(E) = m(y + E) = \infty$.
   Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of sets of real numbers. In the case that there exists an E_k with infinitely many members, then the countable additivity is clear.
   In the case that all sets E_k are finite, for any two sets E_i, E_j:
   $E_i = \{x_1, \cdots, x_n\}$
   $E_j = \{y_1, \cdots, y_m\}$
   Then $E_i \cup E_j = \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ and $m(E_i \cup E_j) = n + m = m(E_i) + m(E_j)$.

2.2 Lebesgue Outer Measure

Let I be a nonempty interval of real numbers. We define its length:

$$\ell(I) = \begin{cases} \infty & \text{if } I \text{ is unbounded} \\ b - a & \text{endpoints } a, b \end{cases}$$

For a set A of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover A; that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure** of A, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set B is also a cover of any subset of B, outer measure is **monotone** in the sense that

$$A \subseteq B \implies m^*(A) \leq m^*(B).$$

Then because $\emptyset \subseteq A$ for any set A, we have $0 = m^*(\emptyset) \leq m^*(A)$.

Example. *A countable set C has outer measure zero.*
Because C is countable, enumerate C such that $C = \{c_k\}_{k=1}^{\infty}$. Fix $\epsilon > 0$. For each $k \in \mathbb{N}$, define an open interval $I_k = (c_k - \frac{\epsilon}{2^{k+1}}, c_k + \frac{\epsilon}{2^{k+1}})$. Then $C \subseteq \bigcup_{k=1}^{\infty} I_k$. Therefore we have, by definition of infimum,

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \frac{2\epsilon}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

This inequality holds for each $\epsilon > 0$; thus $m^*(C) = 0$.

Lemma. $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

Proof. To show that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$ (induction).
Let $P(n)$ be the assertion that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$ for $n \in \mathbb{N}$.
$P(1)$:

$$\sum_{k=1}^{1} \frac{1}{2^k} = \frac{1}{2} = 1 - \frac{1}{2^1}.$$

$P(2)$:

$$\sum_{k=1}^{2} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{2^2}.$$

Suppose $P(m)$ is true for $m \geq 1$; that is, $\sum_{k=1}^{m} \frac{1}{2^k} = 1 - \frac{1}{2^m}$.
$P(m+1)$:

$$\begin{aligned}\sum_{k=1}^{m+1} \frac{1}{2^k} &= \sum_{k=1}^{m} \frac{1}{2^k} + \frac{1}{2^{m+1}} \\ &= 1 - \frac{1}{2^m} + \frac{1}{2^{m+1}} \\ &= 1 - \frac{2}{2^{m+1}} + \frac{1}{2^{m+1}} \\ &= 1 - \frac{1}{2^{m+1}}.\end{aligned}$$

Therefore $P(m)$ is true for all $m \geq 1$.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1.$$

(An alternate proof would be to see that we have a sequence of partial sums that is monotonic with 1 as the supremum. Then the sequence of partial sums converges to 1 and the series is summable to 1.) □

PROBLEMS

5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.
   Suppose that the interval $[0, 1]$ is countable. By an example above, we showed that a countable set has outer measure zero, so $m^*([0, 1]) = 0$. Also, the outer measure of an interval is its length. Then $m^*([0, 1]) = 1$, and we reach a contradiction.

6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.
   Let $A = [0, 1] \cap \mathbb{Q}^c$.
   Then $A \subseteq [0, 1]$, so by monotonicity of outer measure,

$$\begin{aligned}m^*(A) &\leq m^*([0, 1]) \\ m^*(A) &\leq 1.\end{aligned}$$

Also, we have

$$
\begin{aligned}
[0,1] &= ([0,1]\cap\mathbb{Q}^c)\cup([0,1]\cap\mathbb{Q}) && \\
[0,1] &= A\cup([0,1]\cap\mathbb{Q}) && \\
[0,1] &\subseteq A\cup([0,1]\cap\mathbb{Q}) && A=B \implies A\subseteq B \text{ and } A\supseteq B \\
m^*([0,1]) &\le m^*(A\cup(m^*([0,1]\cap\mathbb{Q})) && \text{by monotonicity} \\
m^*([0,1]) &\le m^*(A)+m^*([0,1]\cap\mathbb{Q}) && \text{by countable subadditvity} \\
m^*([0,1]) &\le m^*(A)+0 && \text{countable set has outer measure zero} \\
1 &\le m^*(A). && \text{outer measure of interval is length}
\end{aligned}
$$

Then $m^*(A)\le 1$ and $1\le m^*(A)$ imply that $m^*(A)=1$.

7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E\subseteq G \text{ and } m^*(G)=m^*(E).$$

Suppose E is a bounded set of real numbers.
Then there exists a real number M for which $|x|\le M$ for all $x\in E$; that is, $E\subseteq[-M,M]$. By monotonicity of outer measure, $m^*(E)\le m^*([-M,M])=2M<\infty$, and the outer measure of E is finite.
Now, because outer measure is defined as $m^*(E)=\inf\{\sum_{k=1}^\infty \ell(I_k)\mid E\subseteq\bigcup_{k=1}^\infty I_k\}$, we have that $m^*(E)$ is the greatest lower bound, so for a natural number n, $m^*(E)+\frac{1}{n}$ is not a lower bound. That is, there exists a countable sequence of open intervals $\{(I_n)_k\}_{k=1}^\infty$ such that $E\subseteq\bigcup_{k=1}^\infty(I_n)_k$ and

$$m^*(E)\le\sum_{k=1}^\infty \ell((I_n)_k)<m^*(E)+\frac{1}{n}. \tag{1}$$

Now, for each natural number n, we can define the open set

$$\mathcal{O}_n:=\bigcup_{k=1}^\infty (I_n)_k. \tag{2}$$

Also define the countable intersection of open sets; i.e., a G_δ set:

$$\mathcal{O}:=\bigcap_{n=1}^\infty \mathcal{O}_n.$$

Then because we have $E\subseteq\mathcal{O}_n$ for every n, this implies $E\subseteq\bigcap_{n=1}^\infty\mathcal{O}_n=\mathcal{O}$.

$$
\begin{aligned}
m^*(E) &\le m^*(\mathcal{O}) && \text{by monotonicity of outer measure: } E\subseteq\mathcal{O} \\
&\le m^*(\mathcal{O}_n) && \text{by monotonicity of outer measure: } \mathcal{O}=\bigcap_{n=1}^\infty\mathcal{O}_n\subseteq\mathcal{O}_n \\
&= m^*(\bigcup_{k=1}^\infty (I_n)_k) && \text{by (2)} \\
&\le \sum_{k=1}^\infty \ell((I_n)_k) && \text{by countable subadditivity of outer measure} \\
&< m^*(E)+\frac{1}{n}. && \text{by (1)}
\end{aligned}
$$

Therefore for any natural number n,

$$m^*(E) \leq m^*(\mathcal{O}) < m^*(E) + \frac{1}{n}.$$

Taking the limit as $n \to \infty$, we get that $m^*(E) = m^*(\mathcal{O})$.
Therefore there exists a G_δ set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m^*(E) = m^*(\mathcal{O})$.

8. Let B be the set of rational numbers in the interval $[0,1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.
The rational numbers are dense in the reals; that is, between any two real numbers, there exists a rational number. Therefore, the rational numbers are also dense in the real subset $[0,1]$: between any two numbers in $[0,1]$, there exists a rational number.
In the case that $[0,1] \subseteq \bigcup_{k=1}^n I_k$, the inequality is clear by monotonicity and subadditivity:

$$1 = m^*([0,1]) \leq m^*(\bigcup_{k=1}^n I_k) \leq \sum_{k=1}^n m^*(I_k).$$

In the case that $[0,1] \not\subseteq \bigcup_{k=1}^n I_k$, then

$$(\bigcup_{k=1}^n I_k)^c \cap [0,1] = (\bigcap_{k=1}^n I_k^c) \cap [0,1] = \bigcap_{k=1}^n (I_k^c \cap [0,1]) \neq \emptyset.$$

We want to show that $\bigcap_{k=1}^n I_k^c \cap [0,1]$ is countable so that $m^*(\bigcap_{k=1}^n I_k^c \cap [0,1]) = 0$.
Because each $I_k^c \cap [0,1]$ is a closed interval (of irrational numbers), the intersection is also a closed interval (nonempty by assumption); that is, $\bigcap_{k=1}^n (I_k^c \cap [0,1]) = [a,b]$ for some $a \leq b$. Suppose by contradiction that $\bigcap_{k=1}^n (I_k^c \cap [0,1])$ is not countable. Then we have that $a < b$. However, by density of the rationals, there exists a rational between $[a,b]$, leading to a contradiction.
Therefore $\bigcap_{k=1}^n (I_k^c \cap [0,1]) = \{x\}$, where $x \in \mathbb{Q}^c$, and $\bigcap_{k=1}^n (I_k^c \cap [0,1])$ is countable.
Now we can write

$$\begin{aligned}
[0,1] &= (\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1]) \\
[0,1] &\subseteq (\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1]) && A = B \implies A \subseteq B \text{ and } A \supseteq B \\
m^*([0,1]) &\leq m^*((\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1])) && \text{by monotonicity} \\
m^*([0,1]) &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) + m^*(\bigcap_{k=1}^n I_k^c \cap [0,1]) && \text{by countable subadditivity} \\
m^*([0,1]) &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) + 0 && \text{the outer measure of a countable set is zero} \\
1 &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) \\
1 &\leq m^*(\bigcup_{k=1}^n I_k) && \text{by monotonicity: } \bigcup_{k=1}^n I_k \cap [0,1] \subseteq [0,1] \\
1 &\leq \sum_{k=1}^n m^*(I_k). && \text{by countable subadditivity}
\end{aligned}$$

9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

$$\begin{aligned} m^*(A \cup B) &\leq m^*(A) + m^*(B) && \text{by countable subadditivity} \\ m^*(A \cup B) &\leq m^*(B) && \text{because } m^*(A) = 0. \end{aligned}$$

Also, we have $B \subseteq A \cup B$, so by monotonicity of outer measure,

$$m^*(B) \leq m^*(A \cup B).$$

Then $m^*(A \cup B) \leq m^*(B)$ and $m^*(B) \leq m^*(A \cup B)$ imply that $m^*(A \cup B) = m^*(B)$.

10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.
By countable subadditivity of outer measure, $m^*(A \cup B) \leq m^*(A) + m^*(B)$.
We can see that A and B are disjoint: Suppose by contradiction that A, B are not disjoint. Then there exists a real number x such that $x \in A$ and $x \in B$. But then $|x-x| = 0 < \alpha$, a contradiction.
Let ϵ such that $\alpha/2 > \epsilon > 0$. By definition of outer measure and infimum, there exists a countable sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $(A \cup B) \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$m^*(A \cup B) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(A \cup B) + \epsilon. \tag{1}$$

Now, each I_k is such that $A \cap I_k \neq \emptyset$ or $B \cap I_k \neq \emptyset$, but not both.
To see this, suppose by contradiction that there exists an I_k such that $A \cap I_k \neq \emptyset$ and $B \cap I_k \neq \emptyset$. Then there exists $a, b \in I_k$ such that $a \in A$ and $b \in B$. Without loss of generality, suppose that these are the closest two elements of A and B, and suppose $a < b$. Then the interval (a, b) contains no elements of A or B, and $m^*(b-a) \geq \alpha > \alpha/2$. This is a contradiction to the fact that $\sum_{k=1}^{\infty} \ell(I_k)$ is within $\alpha/2$ of $m^*(A \cup B)$.
We can then separate $\{I_k\}_{k=1}^{\infty}$ into two subsequences $\{(I_A)_i\}_{i=1}^{\infty}$ and $\{(I_B)_j\}_{j=1}^{\infty}$ such that $A \subseteq \bigcup_{i=1}^{\infty}(I_A)_i$ and $B \subseteq \bigcup_{j=1}^{\infty}(I_B)_j$. Then because the sum is uniquely defined independently of the order of terms, $\sum_{k=1}^{\infty} \ell(I_k) = \sum_{i=1}^{\infty} \ell((I_A)_i) + \sum_{j=1}^{\infty} \ell((I_B)_j)$.
Therefore we can write

$$\begin{aligned} m^*(A \cup B) &\leq m^*(A) + m^*(B) && \text{by countable subadditivity of outer measure} \\ &\leq m^*(\bigcup_{i=1}^{\infty}(I_A)_i) + m^*(\bigcup_{j=1}^{\infty}(I_B)_j) && \text{by monotonicity of outer measure} \\ &\leq \sum_{i=1}^{\infty} \ell((I_A)_i) + \sum_{j=1}^{\infty} \ell((I_B)_j) && \text{by countable subadditivity of outer measure} \\ &= \sum_{k=1}^{\infty} \ell(I_k) && \text{rearranging the sum} \\ &< m^*(A \cup B) + \epsilon && \text{by (1)} \end{aligned}$$

Therefore for any ϵ,

$$m^*(A \cup B) \leq m^*(A) + m^*(B) < m^*(A \cup B) + \epsilon,$$

thus $m^*(A \cup B) = m^*(A) + m^*(B)$.

2.3 The σ-Algebra of Lebesgue Measurable Sets

Outer measure is defined for all sets of real numbers, the outer measure of an interval is its length, outer measure is countably subadditive, and outer measure is translation invariant. However, outer measure fails to be countably additive or even finitely additive. That is, there exists disjoint sets A, B such that

$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{1}$$

We identify a σ-algebra of sets, called the Lebesgue measurable sets, which contains all intervals and all open sets and has the property that the restriction of the set function outer measure to the collection of Lebesgue measurable sets is countably additive.

Definition. *A set E is said to be **measurable** provided for any set A,*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

We see that the strict inequality (1) cannot occur if one of the sets is measurable:
Suppose A is measurable and B is any set disjoint from A.

$$\begin{aligned}
m^*(A \cup B) &= m^*([A \cup B] \cap A) + m^*([A \cup B] \cap A^c) && \text{by definition of } A \text{ measurable}\\
&= m^*(A) + m^*([A \cap A^c] \cup [B \cap A^c]) && \text{left: absorbtion, right: distributive property}\\
&= m^*(A) + m^*(\emptyset \cup [B \setminus A]) && \text{complement and def of set difference}\\
&= m^*(A) + m^*(B). && \text{identity of union and set difference of disjoint sets}
\end{aligned}$$

Suppose we want to prove that a set E is measurable.
We already have that for any set A,

$$\begin{aligned}
m^*(A) &= m^*([A \cap E] \cup [A \cap E^c]) && \text{by set properties}\\
m^*(A) &\leq m^*(A \cap E) + m^*(A \cap E^c). && \text{by subadditivity of outer measure}
\end{aligned}$$

Therefore to show that E is measurable, it suffices to show the other inequality:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c). \tag{2}$$

This inequality holds trivially if $m^*(A) = \infty$. Therefore we need only prove (2) for sets A that have finite outer measure.

Proposition 4. *Any set of outer measure zero is measurable. In particular, any countable set is measurable.*

Proof. Let E be such that $m^*(E) = 0$. Let A be any set.

- $A \cap E \subseteq E$
- $A \cap E^c \subseteq A$

By monotonicity of outer measure,

$$\begin{aligned}
m^*(A \cap E) &\leq m^*(E) = 0\\
m^*(A \cap E^c) &\leq m^*(A)
\end{aligned}$$

Therefore

$$m^*(A) \geq m^*(A \cap E^c) + 0$$
$$m^*(A) \geq m^*(A \cap E^c) + m^*(A \cap E).$$

□

Every open set is the disjoint union of a countable collection of open intervals. Every interval is measurable, and the countable union of measurable sets is measurable, so all open sets are measurable. By complement, all closed sets are measurable. In the same way, all G_δ sets and all F_σ sets are measurable.

The intersection of all the σ-algebras of subsets of $\mathbb{R}$ that contain the open sets is a σ-algebra called the **Borel σ-algebra**, members of this collection are called **Borel sets**. That is, the Borel sigma-algebra is the sigma-algebra generated by the open sets.

Lemma 1. *The set of all subsets of X, $\mathcal{P}(X)$ (or 2^X), is a σ-algebra of subsets of X.*

Proof. Let X be any set.

(i) $X \in \mathcal{P}(X)$.

(ii) if $A \in \mathcal{P}(X)$, then $A^c = X \setminus A = \{x \in X \mid x \notin A\} \in \mathcal{P}(X)$.

(iii) if $A_i \in \mathcal{P}(X)$, then $\bigcup_{i=1}^{\infty} A_i = \{x \in X \mid x \in A_i \text{ for some } i\}$.

□

Lemma 2. *Given any collection of σ-algebras $\{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{A}}$ of X, the intersection $\bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$ is also a σ-algebra.*

Proof. Let X be any set.

(i) $X \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies X \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

(ii) $A \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha \implies A \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies A^c \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies A^c \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

(iii) $A_i \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha \implies A_i \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

□

Theorem. *Given any collection $\mathcal{C}$ of subsets of X, there exists a smallest σ-algebra containing $\mathcal{C}$. (This is called the σ-algebra generated by $\mathcal{C}$.)*

Proof. Consider $S = \{\mathcal{F} \mid \mathcal{C} \subseteq \mathcal{F}, \mathcal{F} \text{ is a } \sigma\text{-algebra of } X\}$.
Now, S is nonempty because $\mathcal{C} \in \mathcal{P}(X)$ and by Lemma 1, $\mathcal{P}(X)$ is a σ-algebra of X; therefore $\mathcal{P}(X) \in S$.
Consider $\bigcap S$, the intersection of all the elements of S.

1. By Lemma 2, $\bigcap S$ is a σ-algebra,
2. $\mathcal{C} \in \mathcal{F}, \forall \mathcal{F} \in S \implies \mathcal{C} \in \bigcap S$, so $\bigcap S$ is a σ-algebra that contains $\mathcal{C}$,
3. $\bigcap S \subseteq \mathcal{F}$ for any $\mathcal{F} \in S$ by def of intersection, so $\bigcap S$ is the smallest σ-algebra containing $\mathcal{C}$.

□

Proposition 10. *The translate of a measurable set is measurable.*

Proof. Let E be measurable, let A be any set, and let y be any real number.
First we need to see that

$$\begin{aligned}(A \cap [E+y]) - y &= \{x : x \in A, \text{ and } x \in E+y\} - y = \{x : x \in A-y \text{ and } x \in E\} = [A-y] \cap E \\ (A \cap [E+y]^c) - y &= \{x : x \in A, \text{ and } x \notin E+y\} - y = \{x : x \in A-y \text{ and } x \notin E\} = [A-y] \cap E^c\end{aligned}$$

Now, we have

$$\begin{aligned}m^*(A) &= m^*(A-y) && \text{outer measure is translation invariant} \\ &= m^*([A-y] \cap E) + m^*([A-y] \cap E^c) && \text{because } E \text{ is measurable} \\ &= m^*(A \cap [E+y] - y) + m^*(A \cap [E+y]^c - y) && \text{by above} \\ &= m^*(A \cap [E+y]) + m^*(A \cap [E+y]^c). && \text{outer measure is translation invariant}\end{aligned}$$

Therefore $E + y$ is measurable. □

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11. Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a, ∞), then it contains all intervals.
Let $\mathcal{M}$ be a σ-algebra of subsets of $\mathbb{R}$.
Suppose that for any real number a, the interval $(a, \infty) \in \mathcal{M}$.
For any real number b, because $\mathcal{M}$ is closed under complements,

$$(b, \infty) \in \mathcal{M} \implies (b, \infty)^c = (-\infty, b] \in \mathcal{M}.$$

For any natural number n, because $\mathcal{M}$ is closed under intersections:

$$\begin{aligned}(a - \frac{1}{n}, \infty), (-\infty, b] \in \mathcal{M} &\implies (a - \frac{1}{n}, \infty) \cap (-\infty, b] = (a - \frac{1}{n}, b] \in \mathcal{M}, \\ (a, \infty), (-\infty, b - \frac{1}{n}] \in \mathcal{M} &\implies (a, \infty) \cap (-\infty, b - \frac{1}{n}] = (a, b - \frac{1}{n}] \in \mathcal{M}.\end{aligned}$$

Because $\mathcal{M}$ is closed under countable intersections and countable unions:

$$\begin{aligned}\text{for any } n \in \mathbb{N}, (a - \frac{1}{n}, b] \in \mathcal{M} &\implies \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] = [a, b] \in \mathcal{M}, \\ \text{for any } n \in \mathbb{N}, (a, b - \frac{1}{n}] \in \mathcal{M} &\implies \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] = (a, b) \in \mathcal{M}.\end{aligned}$$

In short, for any real numbers a, b, we have

$$\begin{aligned}[a, b] &= \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \cap (-\infty, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \cap (b, \infty) \\ (a, b) &= \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty} (a, \infty) \cap (-\infty, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty} (a, \infty) \cap (b - \frac{1}{n}, \infty)\end{aligned}$$

The construction of intervals of the form $(a, b]$ and $[a, b)$ is similar.

12. Show that every interval is a Borel set.
Because any interval of the form (a, ∞) is open, (a, ∞) is a Borel set; i.e., it is a member of the Borel sigma-algebra. By the previous problem 11, any sigma-algebra that contains intervals of the form (a, ∞) contains all intervals. Therefore the Borel sigma-algebra contains all intervals and thus all intervals are Borel sets.

13. Show that

(i) the translate of an F_σ set is also F_σ,
Let F be an F_σ set, that is, $F = \bigcup_{n=1}^{\infty} F_n$, with F_n closed.
For any real number y,

$$\begin{aligned} F + y &= (\bigcup_{n=1}^{\infty} F_n) + y \\ &= \{x : x \in F_n \text{ for some } n\} + y \\ &= \{x : x \in F_n + y \text{ for some } n\} \\ &= \bigcup_{n=1}^{\infty} (F_n + y) \end{aligned}$$

The translate of any closed set is closed, so this is still an F_σ set.

(ii) the translate of a G_δ set is also G_δ,
Let $\mathcal{O}$ be a G_δ set, that is, $\mathcal{O} = \bigcap_{n=1}^{\infty} \mathcal{O}_n$, with $\mathcal{O}_n$ open.
For any real number y,

$$\begin{aligned} \mathcal{O} + y &= (\bigcap_{n=1}^{\infty} \mathcal{O}_n) + y \\ &= \{x : x \in \mathcal{O}_n \text{ for all } n\} + y \\ &= \{x : x \in \mathcal{O}_n + y \text{ for all } n\} \\ &= \bigcap_{n=1}^{\infty} (\mathcal{O}_n + y) \end{aligned}$$

The translate of any open set is open, so this is still a G_δ set.

(iii) the translate of a set of measure zero also has measure zero.
Let E be a set of measure zero. That is, $m^*(E) = 0$.
For any $\epsilon > 0$, by definition of infimum, there exists a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ such that

$$m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon,$$

Thus because the outer measure is zero,

$$\sum_{k=1}^{\infty} \ell(I_k) < \epsilon.$$

Now, for any real number y,

$$E + y \subseteq (\bigcup_{k=1}^{\infty} I_k) + y = \bigcup_{k=1}^{\infty} (I_k + y).$$

By monotonicity of outer measure,

$$m^*(E+y) \leq \sum_{k=1}^{\infty} \ell(I_k + y) = \sum_{k=1}^{\infty} \ell(I_k) < \epsilon.$$

Therefore $m^*(E+y) = 0$.

14. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.
    Suppose E has positive outer measure.
    If E is bounded, then clearly E itself is a bounded subset of E with positive outer measure.
    If E is unbounded:
    First, we can partition the real numbers:

    $$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1)$$

    Then we have that

    $$E = E \cap \mathbb{R} = E \cap (\bigcup_{n\in\mathbb{Z}} [n, n+1)) = \bigcup_{n\in\mathbb{Z}} E \cap [n, n+1).$$

    By countable subadditivity of outer measure,

    $$0 < m^*(E) = m^*(\bigcup_{n\in\mathbb{Z}} E \cap [n, n+1)) \leq \sum_{n\in\mathbb{Z}} m^*(E \cap [n, n+1))$$

    Then there exists an $n \in \mathbb{Z}$ such that $m^*(E \cap [n, n+1)) > 0$, else we reach a contradiction. Therefore we have $E \cap [n, n+1) \subseteq E$ that is bounded and has positive outer measure.

15. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.
    (We are letting E be a measurable set because we are talking about measure specifically, not outer measure.)
    If E is countable, then E has measure zero and E itself is the measurable set whose measure is less than any ϵ: $m(E) = 0 < \epsilon$. In fact, if E has measure zero then the conclusion is trivial.
    Suppose E has positive measure.
    Fix $\epsilon > 0$.
    In the case that E is not bounded, there exists an M such that

    $$m(E \setminus [-M, M]) < \epsilon. \ (\star)$$

    $(\star)$: To prove this we can partition $\mathbb{R}$:

    $$\mathbb{R} = \bigcup_{n=0}^{\infty} \Big([-(n+1), -n) \cup (n, n+1] \Big) = \bigcup_{n=0}^{\infty} I_n.$$

    That is, $I_0 = [-1, 1], I_1 = [-2, -1) \cup (1, 2], I_2 = [-3, -2) \cup (2, 3], \cdots$
    Therefore $E = E \cap \mathbb{R} = E \cap (\bigcup_{n=0}^{\infty} I_n) = \bigcup_{n=0}^{\infty} (E \cap I_n)$.
    By countable additivity of measure, and the fact that E has finite measure,

    $$m(E) = m(\bigcup_{n=0}^{\infty} (E \cap I_n)) = \sum_{n=0}^{\infty} m(E \cap I_n) < \infty.$$

Thus we have a sequence of partial sums that converges so there exists an index M such that

$$\sum_{n=M}^{\infty} m(E \cap I_n) = \left|\sum_{n=0}^{\infty} m(E \cap I_n) - \sum_{n=0}^{M-1} m(E \cap I_n)\right| < \epsilon.$$

We see that $m(E \setminus [-M, M]) = m(\bigcup_{n=M}^{\infty}(E \cap I_n)) = \sum_{n=M}^{\infty} m(E \cap I_n) < \epsilon$.
Therefore $E = (E \cap [-M, M]) \cup (E \cap [-M, M]^c))$, a disjoint union, and $m(E \cap [-M, M]^c) < \epsilon$, so we need only worry now about $E \cap [-M, M]$.
Else if E is bounded, then there exists an M such that $E \subseteq [-M, M]$, and $E = E \cap [-M, M]$.
Now, for this ϵ, we can partition the real numbers into a countable collection of disjoint measurable intervals I_k of the form $[x, x + \epsilon)$.
When we choose a natural number l such that $\frac{2M}{\epsilon} < l$, we get $M < -M + l\epsilon$ so that

$$E \cap [-M, M] \subseteq [-M, M] \subseteq \bigcup_{k=1}^{l} [-M + (k-1)\epsilon, -M + k\epsilon) = \bigcup_{k=1}^{l} I_k.$$

Then

$$E \cap [-M, M] = E \cap (\bigcup_{k=1}^{l} I_k) = \bigcup_{k=1}^{l} (E \cap I_k).$$

Thus E is the union of a finite number of disjoint measurable sets, each of which has measure at most ϵ.
(If E is not bounded, $E = (\bigcup_{k=1}^{l}(E \cap I_k)) \cup (E \setminus [-M, M])$, which still satisfies the conclusion.)

2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

Measurable sets possess the following **excision property**: If A is a measurable set of finite outer measure that is contained in B, then

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

This holds because

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) = m^*(A) + m^*(B \cap A^c).$$

Theorem 11. *Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E.*
(Outer Approximation by Open Sets and G_δ sets)

(i) For each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \epsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

(Inner Approximation by Closed Sets and F_σ sets)

(iii) For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$.

(iv) There is a F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Proof. (E is measurable $\implies$ (i)):
Assume E is measurable and fix $\epsilon > 0$.
Case: $m^*(E) < \infty$:
By definition of outer measure and infimum, there exists a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon.$$

Defining $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$, we see that $\mathcal{O}$ is an open set containing E. By subadditivity of outer measure,

$$m^*(\mathcal{O}) = m^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon,$$

so that

$$m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

Because E is measurable, has finite outer measure, and is contained in $\mathcal{O}$, we have the excision property:

$$m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

Case: $m^*(E) = \infty$:
Then E may be expressed as the disjoint union of a countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer measure (See Problems 14 and 15 for an example of partitioning $\mathbb{R}$). Now, for each index k, because each E_k is measurable and has finite outer measure, we showed above that there exists an open set $\mathcal{O}_k$ containing E_k for which $m^*(\mathcal{O}_k \setminus E_k) < \epsilon/2^k$. The set $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$ is open, it contains E (because $E = \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k = \mathcal{O}$), and we have $E \supseteq E_k \implies E^c \subseteq E_k^c$, so that

$$\mathcal{O} \setminus E = \mathcal{O} \cap E^c = (\bigcup_{k=1}^{\infty} \mathcal{O}_k) \cap E^c = \bigcup_{k=1}^{\infty} (\mathcal{O}_k \cap E^c) \subseteq \bigcup_{k=1}^{\infty} (\mathcal{O}_k \cap E_k^c) = \bigcup_{k=1}^{\infty} (\mathcal{O}_k \setminus E_k).$$

Therefore by monotonicity and subadditivity of outer measure,

$$m^*(\mathcal{O} \setminus E) \leq m^*(\bigcup_{k=1}^{\infty} (\mathcal{O}_k \setminus E_k)) \leq \sum_{k=1}^{\infty} m^*(\mathcal{O}_k \setminus E_k) < \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

Thus property (i) holds for E.

((i) $\implies$ (ii)):
Now, assume property (i) holds for E. Then for each natural number k, there exists an open set $\mathcal{O}_k$ that contains E for which $m^*(\mathcal{O}_k \setminus E) < 1/k$.
Define $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$ so that $E \subseteq \mathcal{O}_k$ for all $k \implies E \subseteq \bigcap_{k=1}^{\infty} \mathcal{O}_k = G$. Then G is a G_δ set that contains E.
Then because for all k, $G \subseteq \mathcal{O}_k \implies G \setminus E \subseteq \mathcal{O}_k \setminus E$, by monotonicity of outer measure,

$$m^*(G \setminus E) \subseteq m^*(\mathcal{O}_k \setminus E) < 1/k.$$

Thus $m^*(G \setminus E) = 0$, and (ii) holds.

((ii) $\implies$ E is measurable):

Assume property (ii) holds for E.
We can write

$$\begin{aligned} E &= G \cap E \\ &= \emptyset \cup (G \cap E) \\ &= (G \cap G^c) \cup (G \cap E) \\ &= G \cap (G^c \cup E) \\ &= G \cap (G \cap E^c)^c \\ &= G \cap (G \setminus E)^c. \end{aligned}$$

Now, $m^*(G \setminus E) = 0$, and any set of measure zero is measurable, so $G \setminus E$ is measurable and also $(G \setminus E)^c$ is measurable by complement. Also, G is a G_δ set, and all G_δ sets are measurable. Finally, the intersection of measurable sets is measurable so $G \cap (G \setminus E)^c$ is measurable. Thus E is measurable. $\square$

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16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).
(E is measurable $\implies$ (iii)):
Fix $\epsilon > 0$. Suppose E is measurable. Then E^c is also measurable. Also, stating that E^c is measurable is equivalent to property (i), that is, there exists an open set $\mathcal{O}$ containing E^c such that $m^*(\mathcal{O} \setminus E^c) < \epsilon$.
Then because $\mathcal{O}$ is open, we can define the closed set $F = \mathcal{O}^c$, and we have that

$$\mathcal{O} \setminus E^c = \mathcal{O} \cap E = F^c \cap E = E \setminus F,$$

and $E^c \subseteq \mathcal{O} \implies E \supseteq \mathcal{O}^c = F$. Therefore F is a closed set contained in E for which $m^*(E \setminus F) = m^*(\mathcal{O} \setminus E^c) < \epsilon$, and (iii) holds.

((iii) $\implies$ (iv)):
Suppose that property (iii) holds for E. Then for each natural number k, there exists a closed set F_k contained in E for which $m^*(E \setminus F_k) < 1/k$.
Then defining $F = \bigcup_{k=1}^{\infty} F_n$, we have that $F_k \subseteq E, \forall k \implies F = \bigcup_{k=1}^{\infty} F_k \subseteq E$. Then F is an F_σ set that is contained in E. Then $F \supseteq F_k \implies F^c \subseteq F_k^c$ and thus $E \cap F^c \subseteq E \cap F_k^c$ and $E \setminus F \subseteq E \setminus F_k$. By monotonicity of outer measure, for all k, we have

$$m^*(E \setminus F) \leq m^*(E \setminus F_k) < 1/k.$$

Therefore $m^*(E \setminus F) = 0$, and (iv) holds.

((iv) $\implies$ E is measurable):
Suppose that property (iv) holds for E.
We can write

$$\begin{aligned} E &= E \cap \mathbb{R} \\ &= [F \cup E] \cap [F \cup F^c] \\ &= F \cup [E \cap F^c] \\ &= F \cup [E \setminus F]. \end{aligned}$$

Now, $m^*(E \setminus F) = 0$ implies $E \setminus F$ is measurable because all sets of measure zero are measurable. Also, F is an F_σ set, which is measurable. Therefore $F \cup [E \setminus F]$, the intersection of measurable sets, is measurable. Thus E is measurable.

17. Show that a set E is measurable iff for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.
Let E be a set, and let $\epsilon > 0$.
(This case we assuming E has finite measure to assume excision, maybe proof not complete)
($\implies$) Suppose E is measurable.
Then by Theorem 11 (i), (iii), there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \epsilon/2$, and a closed set F contained in E for which $m^*(E \setminus F) < \epsilon/2$. That is, $F \subseteq E \subseteq \mathcal{O}$.
By excision, $m^*(E \setminus F) = m^*(E) - m^*(F)$, and we can write

$$\begin{aligned} m^*(E) - m^*(F) &< \epsilon/2 \\ m^*(E) &< m^*(F) + \epsilon/2 \\ -m^*(E) &> -m^*(F) - \epsilon/2 \end{aligned}$$

Also by excision, we have $m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E)$, and

$$m^*(\mathcal{O}) - m^*(F) - \epsilon/2 < m^*(\mathcal{O}) - m^*(E) < \epsilon/2$$

Therefore $m^*(\mathcal{O} \setminus F) = m^*(\mathcal{O}) - m^*(F) < \epsilon$.
($\impliedby$) Suppose there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.
By excision and monotonicity of outer measure, we have that

$$m^*(E \setminus F) = m^*(E) - m^*(F) \leq m^*(\mathcal{O}) - m^*(F) = m^*(\mathcal{O} \setminus F) < \epsilon.$$

Therefore we have a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$, i.e., proposition (iii), which implies that E is measurable.

18. Let E have finite outer measure. Show that there is a G_δ set $G \supseteq E$ with $m(G) = m^*(E)$. Show that E is measurable iff there is an F_σ set $F \subseteq E$ with $m(F) = m^*(E)$.
Let E be a set with finite outer measure.
Then for each natural number k, by definition of infimum, there exists a countable collection of open intervals $\{(I_k)_n\}_{n=1}^\infty$ whose union contains E for which

$$\sum_{n=1}^\infty \ell((I_k)_n) < m^*(E) + 1/k.$$

Now, $\mathcal{O}_k = \bigcup_{n=1}^\infty (I_k)_n$ is an open set, and we can define $G = \bigcap_{k=1}^\infty \mathcal{O}_k$ so that $E \subseteq \mathcal{O}_k$ for all $k \implies E \subseteq \bigcap_{k=1}^\infty \mathcal{O}_k = G$. Then G is a G_δ set that contains E. Because $G \subseteq \mathcal{O}_k$ for all k, by monotonicity,

$$m^*(G) \leq m^*(\mathcal{O}_k) = m^*(\bigcup_{n=1}^\infty (I_k)_n) \leq \sum_{n=1}^\infty \ell((I_k)_n) < m^*(E) + 1/k.$$

Then we have $m^*(G) < m^*(E) + 1/k$ for any natural number k, which implies $m^*(G) \leq m^*(E)$.
Also, by monotonicity, $E \subseteq G \implies m^*(E) \leq m^*(G)$.
Therefore $m^*(G) = m^*(E)$.

Let E be a set with finite outer measure.
($\implies$) Suppose that E is measurable.
By Theorem 11 (iv), there is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$. Because E has finite outer measure, then F has finite outer measure by monotonicity of outer measure. Then

by excision, we have $m^*(E) - m^*(F) = m^*(E \setminus F) = 0$, which implies $m^*(E) = m^*(F)$.
($\Longleftarrow$) Suppose there is an F_σ set $F \subseteq E$ with $m(F) = m^*(E)$.
Then $0 = m^*(E) - m^*(F)$. Because E has finite outer measure, then F has finite outer measure by monotonicity of outer measure. Therefore by excision we have $0 = m^*(E) - m^*(F) = m^*(E \setminus F)$ and Theorem 11 (iv) holds, which implies that E is measurable.

19. Let E have finite outer measure. Show that if E is not measurable, then there is an open set $\mathcal{O}$ containing E that has finite outer measure and for which

$$m^*(\mathcal{O} \setminus E) > m^*(\mathcal{O}) - m^*(E).$$

Suppose E is not measurable. However, suppose by contradiction that for all open sets $\mathcal{O}$ containing E that have finite outer measure, we have $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O}) - m^*(E)$.
Let $\epsilon > 0$. By definition of outer measure, there exists a countable collection of open intervals $\{I_k\}$ whose union contains E and

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon.$$

We can define $\mathcal{O} := \bigcup_{k=1}^{\infty} I_k$, which is an open set that contains E, and by subadditivity of outer measure, we have that

$$m^*(\mathcal{O}) = m^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon$$

Therefore $m^*(\mathcal{O}) - m^*(E) < \epsilon$, and $\mathcal{O}$ has finite outer measure.
By assumption, we have that $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O}) - m^*(E) < \epsilon$. However, this means that we have an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \epsilon$, Theorem 11 (i), which is equivalent to saying that E is measurable, which is a contradiction.

20. (Lebesgue). Let E have finite outer measure. Show that E is measurable iff for each open, bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

Let E be a set of finite outer measure.
($\Longrightarrow$) Suppose that E is measurable.
Then for any interval (a, b), we have

$$b-a = \ell((a, b)) = m^*((a, b)) = m^*((a, b)\cap E)+m^*((a, b)\cap E^c) = m^*((a, b)\cap E)+m^*((a, b)\setminus E).$$

($\Longleftarrow$) Suppose that for each open, bounded interval (a, b), we have $b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E)$.
Then we have

$$m^*((a, b)) = \ell((a, b)) = b-a = m^*((a, b)\cap E)+m^*((a, b)\setminus E) = m^*((a, b)\cap E)+m^*((a, b)\cap E^c).$$

(This is only proved for any open interval; measurability of E implies this is true for any set)

21. Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).

(ii) Let E be any set of real numbers. Define E to be measurable if there is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

Let A and B be two measurable sets under this definition. Then there exist G_δ sets G_A, G_B containing A, B respectively for which $m^*(G_A \setminus A) = 0$ and $m^*(G_B \setminus B) = 0$.
Now, by definition of G_δ set:

$$\begin{aligned} G_A &= \bigcap_{k=1}^{\infty} \mathcal{O}_k, \text{ for } \mathcal{O}_k \text{ open} \\ G_B &= \bigcap_{n=1}^{\infty} \mathcal{U}_n, \text{ for } \mathcal{U}_n \text{ open} \end{aligned}$$

Therefore

$$\begin{aligned} G_A \cup G_B &= (\bigcap_{k=1}^{\infty} \mathcal{O}_k) \cup (\bigcap_{n=1}^{\infty} \mathcal{U}_n) \\ &= \bigcap_{k=1}^{\infty} (\mathcal{O}_k \cup (\bigcap_{n=1}^{\infty} \mathcal{U}_n) \\ &= \bigcap_{k=1}^{\infty} (\bigcap_{n=1}^{\infty} (\mathcal{O}_k \cup \mathcal{U}_n)) \end{aligned}$$

For each k, n pair, $\mathcal{O}_k \cup \mathcal{U}_n$ is an open set, so $G_A \cup G_B$ is a countable intersection of open sets and thus a G_δ set. Also, $G_A \supseteq A$ and $G_B \supseteq B$ imply that $G_A \cup G_B \supseteq A \cup B$, so $G_A \cup G_B$ is a G_δ set that contains $A \cup B$.
We can write

$$\begin{aligned} (G_A \cup G_B) \setminus (A \cup B) &= (G_A \cup G_B) \cap (A \cup B)^c \\ &= (G_A \cup G_B) \cap (A^c \cap B^c) \\ &= [G_A \cap (A^c \cap B^c)] \cup [G_B \cap (A^c \cap B^c)] \\ &= [G_A \cap A^c \cap B^c] \cup [G_B \cap B^c \cap A^c] \\ &\subseteq [G_A \cap A^c] \cup [G_B \cap B^c] \\ &\subseteq [G_A \setminus A] \cup [G_B \setminus B]. \end{aligned}$$

By monotonicity of outer measure and subadditivity,

$$\begin{aligned} m^*((G_A \cup G_B) \setminus (A \cup B)) &\leq m^*([G_A \setminus A] \cup [G_B \setminus B]) \\ &\leq m^*(G_A \setminus A) + m^*(G_B \setminus B) \\ &= 0. \end{aligned}$$

Therefore $A \cup B$ is measurable.

(iv) Let E be any set of real numbers. Define E to be measurable if there is an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.
Let A and B be two measurable sets under this definition. Then there exist F_σ sets F_A, F_B contained in A, B respectively for which $m^*(A \setminus F_A) = 0$ and $m^*(B \setminus F_B) = 0$.

Now, by definition of F_σ set:

$$F_A = \bigcup_{k=1}^{\infty} I_k, \text{ for } I_k \text{ closed}$$
$$F_B = \bigcup_{n=1}^{\infty} J_n, \text{ for } J_n \text{ closed}$$

Therefore

$$F_A \cup F_B = (\bigcup_{k=1}^{\infty} I_k) \cup (\bigcup_{n=1}^{\infty} J_n),$$

which is clearly a countable union of closed sets, so $F_A \cup F_B$ is an F_σ set. Also, $F_A \subseteq A$ and $F_B \subseteq B$ imply that $F_A \cup F_B \subseteq A \cup B$, so $F_A \cup F_B$ is an F_σ set that is contained in $A \cup B$.
We can write

$$\begin{aligned}
(A \cup B) \setminus (F_A \cup F_B) &= (A \cup B) \cap (F_A \cup F_B)^c \\
&= (A \cup B) \cap (F_A^c \cap F_B^c) \\
&= [A \cap (F_A^c \cap F_B^c)] \cup [B \cap (F_A^c \cap F_B^c)] \\
&= [A \cap F_A^c \cap F_B^c] \cup [B \cap F_B^c \cap F_A^c] \\
&\subseteq [A \cap F_A^c] \cup [B \cap F_B^c] \\
&\subseteq [A \setminus F_A] \cup [B \setminus F_B].
\end{aligned}$$

By monotonicity of outer measure and subadditivity,

$$\begin{aligned}
m^*((A \cup B) \setminus (F_A \cup F_B)) &\leq m^*([A \setminus F_A] \cup [B \setminus F_B]) \\
&\leq m^*(A \setminus F_A) + m^*(B \setminus F_B) \\
&= 0.
\end{aligned}$$

Therefore $A \cup B$ is measurable.

22. For any set A, define $m^{**}(A) \in [0, \infty]$ by

$$m^{**}(A) = \inf\{m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open.}\}$$

How is this set function m^{**} related to outer measure m^*?

Consider any open set $\mathcal{O}$ such that $A \subseteq \mathcal{O}$. By monotonicity of outer measure, $m^*(A) \leq m^*(\mathcal{O})$, and therefore $m^*(A)$ is a lower bound to the set $\{m^*(\mathcal{O}) \mid \mathcal{O} \supseteq A, \mathcal{O} \text{ open.}\}$. Because m^{**} is defined as the greatest lower bound, we get

$$m^*(A) \leq m^{**}(A).$$

Now, if $m^*(A) = \infty$, then trivially we have

$$m^*(A) \geq m^{**}(A),$$

which implies $m^*(A) = m^{**}(A)$.
Thus we consider the case where $m^*(A) < \infty$.
Then for any $\epsilon > 0$, by definition of infimum, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ whose union contains A for which

$$\sum_{n=1}^{\infty} \ell(I_n) < m^*(A) + \epsilon.$$

Now, $\mathcal{O} = \bigcup_{n=1}^{\infty} I_n$ is an open set that contains A, so by definition of m^{**},

$$m^{**}(A) \leq m^*(\mathcal{O}) = m^*(\bigcup_{n=1}^{\infty} I_n) \leq \sum_{n=1}^{\infty} \ell(I_n) < m^*(A) + \epsilon.$$

Then $m^{**}(A) < m^*(A) + \epsilon$ implies $m^{**}(A) \leq m^*(A)$.
Therefore $m^*(A) = m^{**}(A)$.

23. For any set A, define $m^{***}(A) \in [0, \infty]$ by

$$m^{***}(A) = \sup\{m^*(F) \mid F \subseteq A, F \text{ closed.}\}$$

How is this set function m^{***} related to outer measure m^*?

Consider any closed set F such that $F \subseteq A$. By monotonicity of outer measure, $m^*(F) \leq m^*(A)$, and therefore $m^*(A)$ is an upper bound to the set $\{m^*(F) \mid F \subseteq A, F \text{ closed.}\}$. Because m^{**} is defined as the least upper bound, we get

$$m^{***}(A) \leq m^*(A).$$

(In addition, if A is measurable, then $m^{***}(A) = m^*(A)$.)

2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

Theorem 15 (the Continuity of Measure). *Lebesgue measure possesses the following continuity properties:*

(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Proof. Let $\{A_k\}_{k=1}^{\infty}$ be ascending and measurable.
If there exists an index k such that $m(A_k) > \infty$, then by monotonicity of measure, $m(\bigcup_{k=1}^{\infty} A_k) = \infty$. Also, because this collection is ascending, we have $A_k \subseteq A_n$ whenever $k \leq n$; therefore by monotonicity, $\infty = m(A_k) \leq m(A_n)$ for all n such that $k \leq n$, and thus (i) holds.

Therefore it remains to prove the case that $m(A_k) < \infty$ for all k.
Define $A_0 = \emptyset$, and define $C_k = A_k \setminus A_{k-1}$. Then $\{C_k\}_{k=1}^{\infty}$ is disjoint and $\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} A_k$. Now we can write

$$\begin{aligned}
m(\bigcup_{k=1}^{\infty} A_k) &= m(\bigcup_{k=1}^{\infty} C_k) && \\
&= \sum_{k=1}^{\infty} m(C_k) && \text{countable (disjoint) monotonicity} \\
&= \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}) && \\
&= \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})] && \text{by excision: } m(A_{k-1}) < \infty \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] && \\
&= \lim_{n\to\infty} m(A_n) - m(A_0) && \text{by telescoping} \\
&= \lim_{n\to\infty} m(A_n). && \text{because } A_0 = \emptyset
\end{aligned}$$

Let $\{B_k\}_{k=1}^{\infty}$ be descending and measurable.
Define $D_k = B_1 \setminus B_k = B_1 \cap B_k^c$.
Then because $\{B_k\}_{k=1}^{\infty}$ is descending,

$$B_k \supseteq B_{k+1} \implies B_1 \cap B_k^c \subseteq B_1 \cap B_{k+1}^c \implies D_k \subseteq D_{k+1},$$

and $\{D_k\}_{k=1}^{\infty}$ is ascending.
Now we have

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty} [B_1 \cap B_k^c] = B_1 \cap [\bigcup_{k=1}^{\infty} B_k^c] = (B_1 \cap [\bigcap_{k=1}^{\infty} B_k]^c = B_1 \setminus [\bigcap_{k=1}^{\infty} B_k].$$

Then by part (i), we can write

$$\begin{aligned}
m(\bigcup_{k=1}^{\infty} D_k) &= \lim_{k\to\infty} m(D_k) \\
m(B_1 \setminus [\bigcap_{k=1}^{\infty} B_k]) &= \lim_{k\to\infty} m(B_1 \setminus B_k) \\
m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) &= \lim_{k\to\infty} [m(B_1) - m(B_k)] \\
m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) &= m(B_1) - \lim_{k\to\infty} [m(B_k)] \\
m(\bigcap_{k=1}^{\infty} B_k) &= \lim_{k\to\infty} [m(B_k)].
\end{aligned}$$

$\square$

The Borel-Cantelli Lemma. *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k's.*

Proof. By countable subadditivity, for each n,

$$m(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Because $\sum_{k=1}^{\infty} m(E_k) < \infty$, we have a sequence of partial sums such that for any $\epsilon > 0$, there exists an index n for which

$$\sum_{k=n}^{\infty} m(E_k) = |\sum_{k=1}^{\infty} m(E_k) - \sum_{k=1}^{n-1} m(E_k)| < \epsilon.$$

Therefore there exists an n such that $|\sum_{k=n}^{\infty} m(E_k) - 0| < \epsilon$, and $\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$.
By continuity of measure (ii),

$$m(\bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} E_k]) = \lim_{n\to\infty} m(\bigcup_{k=1}^{\infty} E_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore almost all $x \in \mathbb{R}$ fail to belong to $\bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} E_k]$ and therefore belong to at most finitely many E_k's. $\square$

Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of sets that belong to a σ-algebra $\mathcal{A}$. Since $\mathcal{A}$ is closed w.r.t. countable unions and intersections, the following two sets belong to $\mathcal{A}$:

$$\limsup\{A_k\}_{k=1}^{\infty} = \bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} A_k]$$

$$\liminf\{A_k\}_{k=1}^{\infty} = \bigcup_{n=1}^{\infty}[\bigcap_{k=1}^{\infty} A_k]$$

The set $\limsup\{A_k\}_{k=1}^{\infty}$ is the set of points that belong to A_n for countably infinitely many indices n while the set $\liminf\{A_k\}_{k=1}^{\infty}$ is the set of points that belong to A_n except for at most finitely many indices n.

PROBLEMS

24. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

$$\begin{aligned}
m(E_1 \cup E_2) + m(E_1 \cap E_2) &= m([E_1 \cup E_2] \cap E_1) + m([E_1 \cup E_2] \cap E_1^c) + m(E_1 \cap E_2) \\
&= m(E_1) + m([E_1 \cap E_1^c] \cup [E_2 \cap E_1^c]) + m(E_1 \cap E_2) \\
&= m(E_1) + m(\emptyset \cup [E_2 \cap E_1^c]) + m(E_1 \cap E_2) \\
&= m(E_1) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2) \\
&= m(E_1) + m([E_2 \cap E_1^c] \cup [E_1 \cap E_2]) \\
&= m(E_1) + m([E_2 \cup (E_1 \cap E_2)] \cap [E_1^c \cup (E_1 \cap E_2)]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup (E_1 \cap E_2)]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup E_1] \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2 \cap \mathbb{R} \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2).
\end{aligned}$$

25. Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.
In the proof of (ii), we get to the point

$$m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) = m(B_1) - \lim_{k\to\infty}[m(B_k)].$$

If $m(B_1) = \infty$, then we have

$$\infty - m(\bigcap_{k=1}^{\infty} B_k) = \infty - \lim_{k\to\infty}[m(B_k)],$$

and we cannot reach the conclusion we want because $\infty - \infty$ is not defined.

26. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set A,

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

We have by countable subadditivity:

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) = m^*(\bigcup_{k=1}^{\infty}(A \cap E_k)) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Now, for any n, we have $A \cap \bigcup_{k=1}^{\infty} E_k \supseteq A \cap \bigcup_{k=1}^{n} E_k$, so by monotonicity and Proposition 6,

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq m^*(A \cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

The left hand side is independent of n, so taking the limit as $n \to \infty$, we get

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

27. Let $\mathcal{M}'$ be any σ-algebra of subsets of $\mathbb{R}$ and m' a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

(i) Show that m' is finitely additive, monotone, countably monotone, and possesses the excision property.
Countable additivity implies that for any disjoint collection of measurable sets $\{E_k\}_{k=1}^{\infty}$, we have $m'(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m'(E_k)$.
Now, any finite disjoint collection $\{E_k\}_{k=1}^{n}$ can be extended to the infinite disjoint collection $\{E'_k\}_{k=1}^{\infty}$, where $E'_k = E_k$ for $k \in \{1, \cdots, n\}$, and $E'_k = \emptyset$ for $k > n$. Clearly from this we have finite additivity.
In Problem 1 of this chapter, it was shown that a countably additive set function possesses the monotonicity property. Thus m' is monotone. It can clearly be shown that m' is also countably monotone.
To see excision, simply use countable additivity to see that for measurable sets A, B such that $A \subseteq B$, we have

$$m'(B) = m'([B \cap A] \cup [B \cap A^c]) = m'(B \cap A) + m'(B \cap A^c) = m'(A) + m'(B \setminus A).$$

(ii) Show that m' possesses the same continuity properties as Lebesgue measure.
Check Theorem 15 and the Borel-Cantelli Lemma above.

28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.
Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of measurable sets. (if any E_k has infinite measure, countable additivity is clear, so we need only consider sets of finite measure for all E_k.)
Finite additivity implies that for the disjoint collection of measurable sets $\{E_k\}_{k=1}^{n}$, we have $m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k)$.
We can define $F_n = \bigcup_{k=1}^{n} E_k$ so that continuity of measure implies that for the ascending collection $\{F_n\}_{n=1}^{\infty}$ of measurable sets, we have $m(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n)$.
Therefore we can write

$$m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty} m(\bigcup_{k=1}^{n} E_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k) = \sum_{k=1}^{\infty} m(E_k).$$

2.6 Nonmeasurable Sets

Consider the subgroup under addition $\mathbb{Q} \subseteq \mathbb{R}$. Now, $\mathbb{Q}$ is a normal subgroup, and we have the quotient group $\mathbb{R}/\mathbb{Q}$, with the (disjoint) cosets written as $r + \mathbb{Q}$ where $r \in \mathbb{R}$. A Vitali set $V \subseteq [0,1]$ is defined to be a set such that for all $r \in \mathbb{R}$, there exists exactly one unique $v \in V$ such that $v - r \in \mathbb{Q}$. Every Vitali set is uncountable, and $v - u \notin \mathbb{Q}$ for $u, v \in V$, $u \neq v$.

Theorem. *A Vitali set is non-measurable.*

Proof. Suppose by contradiction that a Vitali set V is measurable.
Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[-1,1]$:
recall that $\mathbb{Q}$ looks like

$$\mathbb{Q} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\},$$

therefore

$$\{q_k\}_{k=1}^{\infty} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\}.$$

For each natural number k, let $V_k = V + q_k = \{v + q_k : v \in V\}$.
First we will show the following:

(i) $V_i \cap V_j = \emptyset$ for $i \neq j$

(ii) $[0,1] \subseteq \bigcup_{k=1}^{\infty} V_k \subseteq [-1,2]$

(i) Suppose by contradiction that $V_i \cap V_j = \emptyset$ for some $i \neq j$.
That is, there exists $x \in V_i, y \in V_j$ such that $x = y$.
Also, there exists $v, u \in V$ such that $x = v + q_i$ and $y = u + q_j$.
By equality, we have $v + q_i = u + q_j$.
In the case that $v = u$, we get $q_i = q_j$, a contradiction.
In the case that $v \neq u$, we can write $v - u = q_j - q_i \in \mathbb{Q}$, a contradiction.

(ii) For any real $r \in [0,1]$, there exists a $v \in V \subseteq [0,1]$ such that $r - v \in \mathbb{Q}$.

We can see that

$$\max(r-v) = 1-0 = 1,$$
$$\min(r-v) = 0-1 = -1.$$

which implies $r-v = q_i \in [-1,1] \cap \mathbb{Q}$ for some i, and thus $r = v + q_i \in V_i$.
In short, we can write this as

$$r \in [0,1] \implies r \in V_i \text{ for some } i \implies r \in \bigcup_{k=1}^{\infty} V_k \implies [0,1] \subseteq \bigcup_{k=1}^{\infty} V_k.$$

Now, $V_k = V + q_k, V \subseteq [0,1], q_k \in [-1,1]$, therefore

$$\max(v+q_k) = 1+1 = 2,$$
$$\min(v+q_k) = 0-1 = -1.$$

Therefore $V_k \subseteq [-1,2]$ for all k, and thus $\bigcup_{k=1}^{\infty} V_k \subseteq [-1,2]$.

Then we can write

$$m^*([0,1]) \leq m^*(\bigcup_{k=1}^{\infty} V_k) \leq m^*([-1,2]) \qquad \text{by monotonicity of outer measure}$$
$$1 \leq \sum_{k=1}^{\infty} m^*(V_k) \leq 3 \qquad \text{countable additivity (measurability of } V) \star$$
$$1 \leq \sum_{k=1}^{\infty} m^*(V+q_k) \leq 3$$
$$1 \leq \sum_{k=1}^{\infty} m^*(V) \leq 3 \qquad \text{by translation invariance of outer measure}$$

However, $m^*(V) \geq 0$ is a constant, so $\sum_{k=1}^{\infty} m^*(V) = 0$ or $\sum_{k=1}^{\infty} m^*(V) = \infty$, neither of which is in $[1,3]$, and we reach a contradiction. $\square$

For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to $\mathbb{Q}$. By a **choice set** for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. A choice set $\mathcal{C}_E$ is characterized by the following two properties:

1. the difference of two points in $\mathcal{C}_E$ is not rational;
2. for each point x in E, there is a point c in $\mathcal{C}_E$ for which $x = c + q$, $q \in \mathbb{Q}$.

The first property can be reformulated as

$$\text{For any set } \Lambda \subseteq \mathbb{Q}, \{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda} \text{ is disjoint.}$$

We also have that

$$E \subseteq \bigcup_{\lambda \in \mathbb{Q}} (\lambda + \mathcal{C}_E).$$

PROBLEMS

29. (i) Show that rational equivalence defines an equivalence relation on any set.

    Let X be any set and define $x \sim y$ when $x - y \in \mathbb{Q}$ for $x, y \in X$.

    i. $x - x = 0 \in \mathbb{Q} \iff x \sim x$ for all $x \in X$.
    ii. $x \sim y \iff x - y = q \in \mathbb{Q} \iff y - x = -q \in \mathbb{Q} \iff y \sim x$ for all $x, y \in X$.
    iii. $x \sim y, y \sim z \iff x - y = q \in \mathbb{Q}, y - z = q' \in \mathbb{Q} \iff x - z = x - y + y - z = q + q' \in \mathbb{Q} \iff x \sim z$ for all $x, y, z \in X$.

    (ii) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.

    (For any nonempty set E of real numbers, we define two points in E to be **rationally equivalent** provided their difference belongs to $\mathbb{Q}$. By a **choice set** for the rational equivalence relation on E we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class.) Therefore for the nonempty set $\mathbb{Q}$, we can choose a choice set $\mathcal{C}_{\mathbb{Q}} = \{q\}$ for any $q \in \mathbb{Q}$.

    (iii) Define two numbers to be irrationally equivalent provided their difference is irrational or zero. Is this an equivalence relation on $\mathbb{R}$? Is this an equivalence relation on $\mathbb{Q}$?

    i. $x - x = 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim x$ for all $x \in \mathbb{R}$.
    ii. $x \sim y \iff x - y = q \in \{\mathbb{Q}^c, 0\} \iff y - x = -q \in \{\mathbb{Q}^c, 0\} \iff y \sim x$ for all $x, y \in \mathbb{R}$.
    iii. $2 - \pi \in \{\mathbb{Q}^c, 0\}, \pi - 0 \in \{\mathbb{Q}^c, 0\}$ but $2 - 0 \notin \{\mathbb{Q}^c, 0\}$

    Not an equivalence relation on $\mathbb{R}$.

    i. $x - x = 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim x$ for all $x \in \mathbb{Q}$.
    ii. $x \sim y \iff x - y = 0 \in \{\mathbb{Q}^c, 0\} \iff y - x = 0 \in \{\mathbb{Q}^c, 0\} \iff y \sim x$ for all $x, y \in \mathbb{Q}$.
    iii. $x \sim y, y \sim z \iff x - y = 0 \in \{\mathbb{Q}^c, 0\}, y - z = 0 \in \{\mathbb{Q}^c, 0\} \iff x - z = x - y + y - z = 0 + 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim z$ for all $x, y, z \in \mathbb{Q}$.

    An equivalence relation on $\mathbb{Q}$.

30. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

    Let E be a set of positive outer measure. Suppose there exists a choice set $\mathcal{C}_E$ for the rational equivalence relation on E such that $\mathcal{C}_E$ is countable. All countable sets have outer measure zero, so $m^*(\mathcal{C}_E) = 0$. Because we know $E \subseteq \bigcup_{\lambda \in \mathbb{Q}} (\lambda + \mathcal{C}_E)$, by monotonicity, subadditivity, and translation invariance of outer measure,

    $$m^*(E) \leq m^*(\bigcup_{\lambda \in \mathbb{Q}} (\lambda + \mathcal{C}_E)) \leq \sum_{\lambda \in \mathbb{Q}} m^*(\lambda + \mathcal{C}_E) = \sum_{\lambda \in \mathbb{Q}} m^*(\mathcal{C}_E) = \sum_{\lambda \in \mathbb{Q}} 0 = 0,$$

    and we have a contradiction to the fact that $m^*(E) > 0$.

31. Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that E is bounded.

(Vitali: Any set of real numbers with positive outer measure contains a subset that fails to be measurable.) By Problem 14, we showed that every set of positive outer measure E contains a bounded subset $A \subseteq E$ of positive outer measure. Therefore if there exists a subset $S \subseteq A$ that fails to be measurable, then $S \subseteq A \subseteq E$ is a subset that fails to be measurable.

32. Does Lemma 16 remain true if Λ is allowed to be finite or to be uncountably infinite? Does it remain true if Λ is allowed to be unbounded?

    (Lemma 16: Let E be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers Λ for which the collection of translates of E, $\{\lambda + E\}_{\lambda\in\Lambda}$, is disjoint. Then $m(E) = 0$.)
    Consider the case $\Lambda = \{1, 2\}$ is finite, and $E = (0, 1)$. Then $\{\lambda + E\}_{\lambda\in\Lambda=\{1,2\}} = \{1 + (0, 1), 2 + (0, 1)\} = \{(1, 2), (2, 3)\}$, which is a disjoint collection. However, $m(E) = 1 \neq 0$.
    If Λ is uncountably infinite and satisfies that the translates are disjoint, then we can choose a countable subset of Λ and thus Lemma 16 remains true.
    Consider the case $\Lambda = \{1, 2, 3, \cdots\}$ is unbounded, and $E = (0, 1)$. Then the collection of translates of E, $\{(1, 2), (2, 3), (3, 4), \cdots\}$ is disjoint but $m(E) = 1 \neq 0$.

33. Let E be a nonmeasurable set of finite outer measure. Show that there is a G_δ set G that contains E for which
    $$m^*(E) = m^*(G), \text{ while } m^*(G \setminus E) > 0.$$
    This is a similar construction for the proof from Theorem 11 (i).
    Let E be a nonmeasurable set of finite outer measure.
    By definition of outer measure and infimum, for any natural number n, there exists a countable collection of intervals $\{(I_n)_k\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty}(I_n)_k$ and
    $$m^*(E) \leq \sum_{k=1}^{\infty} \ell((I_n)_k) < m^*(E) + 1/n.$$
    Defining $\mathcal{O}_n = \bigcup_{k=1}^{\infty}(I_n)_k$, we see that $\mathcal{O}_n$ is an open set containing E for each n. Further define $G = \bigcap_{n=1}^{\infty} \mathcal{O}_n$ so that $E \subseteq G \subseteq \mathcal{O}_n$ for any n and G is a G_δ set that contains E.
    By subadditivity of outer measure,
    $$m^*(G) \leq m^*(\mathcal{O}_n) = m^*(\bigcup_{k=1}^{\infty}(I_n)_k) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + 1/n,$$
    so that $m^*(G) < m^*(E) + 1/n \implies m^*(G) \leq m^*(E)$. By subadditivity, $E \subseteq G$ implies we also have $m^*(E) \leq m^*(G)$, and so $m^*(E) = m^*(G)$.
    Now, we know that the outer measure is nonnegative by monotonicity, so we have the inequality $m^*(G \setminus E) \geq 0$.
    By Theorem 11 (ii), $m^*(G \setminus E) = 0 \iff E$ is measurable, so we must have $m^*(G \setminus E) > 0$.

2.7 The Cantor Set and the Cantor-Lebesgue Function

PROBLEMS

34. Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

The function $\psi : [0,1] \to [0,2]$ defined by $\psi(x) = \varphi(x) + x$ maps the Cantor set $C \subseteq [0,1]$ onto a measurable set of positive measure. That is, $m(C) = 0$ and $m(\psi(C)) > 0$. We can consider the inverse function $\psi^{-1} : [0,2] \to [0,1]$ restricted to $[0,1]$: $\psi^{-1}|_{[0,1]} : [0,1] \to [0,1]$. Now consider the set $C' = C \cap [0,1]$. This set C' is a measurable subset of C, a measurable set of measure zero, so by monotonicity of measure, $m(C') = 0$. Then the function has $\psi^{-1}|_{[0,1]}(\psi(C')) = C'$, where $m(\psi(C')) > 0$ and $m(C') = 0$, thus mapping the set $\psi(C')$ of positive measure* onto the set C' of measure zero.
(*We know that $m(\psi(C)) > 0$, but not shown that $m(\psi(C')) > 0$ where $C' \subseteq C$.)

35. Let f be an increasing function on the open interval I. For $x_0 \in I$ show that f is continuous at x_0 iff there are sequences $\{a_n\}$ and $\{b_n\}$ in I such that for each n, $a_n < x_0 < b_n$, and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

Let f be an increasing function on the open interval I and let $x_0 \in I$.
($\Longrightarrow$) Suppose that f is continuous at x_0.
Because I is open, there exists an index N such that for all $n \geq N$, we have that $(x_0 - 1/n, x_0 + 1/n) \subseteq I$. Then for each $n \geq N$ we can choose $a_n \in (x_0 - 1/n, x_0)$ and $b_n \in (x_0, x_0 + 1/n)$, and for $n < N$ let $a_n = a_N$ and $b_n = b_N$, so that $a_n < x_0 < b_n$ for all n. Now,we have

$$x_0 - 1/n < a_n < x_0 \implies x_0 - a_n < 1/n,$$
$$x_0 < b_n < x_0 + 1/n \implies b_n - x_0 < 1/n,$$

therefore $\lim_{n\to\infty} a_n = x_0$ and $\lim_{n\to\infty} b_n = x_0$. Because f is continuous and increasing, for all $\epsilon > 0$, there exists the number $1/n > 0$ such that

$$x_0 - a_n < 1/n \implies f(x_0) - f(a_n) < \epsilon,$$
$$b_n - x_0 < 1/n \implies f(b_n) - f(x_0) < \epsilon.$$

(therefore $\lim_{n\to\infty} f(a_n) = f(x_0)$ and $\lim_{n\to\infty} f(b_n) = f(x_0)$.) We can write

$$[f(b_n) - f(a_n)] = f(x_0) - f(a_n) + f(b_n) - f(x_0) < \epsilon + \epsilon = \epsilon'$$

and so $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

($\Longleftarrow$) Suppose that there exist sequences $\{a_n\}$,$\{b_n\}$ such that $a_n < x_0 < b_n$ and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.
That is, for any $\epsilon > 0$, there exists an index N such that $f(b_n) - f(a_n) < \epsilon$ for all $n \geq N$.
Then $f(b_n) < f(a_n) + \epsilon$ and $f(b_n) - \epsilon < f(a_n)$.
Because f is increasing, we have

$$f(b_n) - \epsilon < f(a_n) < f(x_0) < f(b_n) < f(a_n) + \epsilon.$$

Then $f(x_0) - f(a_n) < \epsilon$ and $f(b_n) - f(x_0) < \epsilon$, which implies $\lim_{n\to\infty} f(a_n) = f(x_0)$ and $\lim_{n\to\infty} f(b_n) = f(x_0)$.
By monotonicity of f, we also have

$$b_n - \epsilon < a_n < x_0 < b_n < a_n + \epsilon.$$

Then $x_0 - a_n < \epsilon$ and $b_n - x_0 < \epsilon$, which implies $\lim_{n\to\infty} a_n = x_0$ and $\lim_{n\to\infty} b_n = x_0$.
Now, clearly we see that for any $\epsilon > 0$, we have $x_0 - a_n < \epsilon \iff f(x_0) - f(a_n) < \epsilon$, and $b_n - x_0 < \epsilon \iff f(b_n) - f(x_0) < \epsilon$, and continuity at x_0 follows.

36. Let f be a continuous function defined on E. Is it true that $f^{-1}(A)$ is always measurable if A is measurable?

    No, the function $\psi : [0,1] \to [0,2]$ defined by $\psi(x) = \varphi(x) + x$ maps a measurable set A, subset of the Cantor set, onto a nonmeasurable set $\psi(A)$. Define $f = \psi^{-1}$ so that $f^{-1}(A) = (\psi^{-1})^{-1}(A)$ is not measurable but A is measurable.

37. Let the function $f : [a,b] \to \mathbb{R}$ be Lipschitz; that is, there is a constant $c \geq 0$ such that for all $u, v \in [a,b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that f maps a set of measure zero onto a set of measure zero. Show that f maps a F_σ set onto an F_σ set. Conclude that f maps a measurable set to a measurable set.

    Let f be a Lipschitz function on the interval I. Clearly f is also continuous.
    Let $E \subseteq I$ be a set of measure zero; that is, $m^*(E) = m(E) = 0$. By definition of infimum, for any $\epsilon > 0$, there exists a countable collection of open intervals $\{I_k\}_{k=1}^\infty$, $I_k = (a_k, b_k)$, such that $E \subseteq \bigcup_{k=1}^\infty I_k$ and
    $$0 \leq \sum_{k=1}^\infty \ell(I_k) < 0 + \frac{\epsilon}{c}.$$
    We also have that $E \subseteq \bigcup_{k=1}^\infty I_k \implies f(E) \subseteq f(\bigcup_{k=1}^\infty I_k) = \bigcup_{k=1}^\infty f(I_k)$.
    Also, by Chapter 1 Problem 54, Because I_k is an interval, the continuous real-valued function f on I_k has an interval as its image; that is, $f(I_k)$ is an interval. Then there exists some $u_k, v_k \in (a,b)$ such that $f(I_k) = (f(u_k), f(v_k))$ and $m(f(I_k)) = f(v_k) - f(u_k)$. Then because f is Lipschitz, $|f(v_k) - f(u_k)| \leq c|v_k - u_k|$ for all k.

    $$\begin{aligned}
    m(f(E)) &\leq m\Big(\bigcup_{k=1}^\infty f(I_k)\Big) && \text{by monotonicity}\\
    &\leq \sum_{k=1}^\infty m(f(I_k)) && \text{by subadditivity}\\
    &= \sum_{k=1}^\infty m(f(v_k) - f(u_k))\\
    &\leq \sum_{k=1}^\infty c|v_k - u_k| && \text{because } f \text{ is Lipschitz}\\
    &\leq \sum_{k=1}^\infty c|b_k - a_k| && \text{because } (u_k, v_k) \subseteq (a,b)\\
    &= \sum_{k=1}^\infty c\ell(I_k)\\
    &< \epsilon.
    \end{aligned}$$

    Therefore $m(f(E)) = 0$.

38. Let F be the subset of $[0,1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the nth deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that F is a closed set, $[0,1] \setminus F$ is dense in $[0,1]$, and $m(F) = 1 - \alpha$. Such a set F is called a generalized Cantor set.

Define F to be constructed in the same manner as the Cantor set, with

$$F = \bigcap_{k=1}^{\infty} F_k,$$

where $\{F_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets, and each F_k is a disjoint union of 2^k closed intervals, each of length $\alpha/3^k$.
It can clearly be seen that F is a closed set because it is an intersection of closed sets.
Now, for any point $x \in [0, 1]$, there exists an index k such that $x \notin F_k$; that is, $x \in F_k^c$, which is an open set. Therefore we can construct sequences in $([0, 1] \setminus F) \setminus \{x\}$ that converge to x.
Each F_k is the disjoint union of 2^k closed intervals each of length $\alpha/3^k$, so at each step we remove 2^{k-1} open intervals of length $\alpha/3^k$:

$$\begin{aligned}
m(F_1) &= 1 - \alpha/3 \\
m(F_2) &= 1 - \alpha/3 - 2\alpha/3^2 \\
m(F_3) &= 1 - \alpha/3 - 2\alpha/3^2 - 2^2\alpha/3^3 \\
&\vdots \\
m(F_n) &= 1 - \sum_{k=1}^{n} 2^{k-1}\alpha/3^k
\end{aligned}$$

Then by the continuity of measure, we have

$$m(\bigcap_{k=1}^{\infty} F_k) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty}(1 - \sum_{k=1}^{n} 2^{k-1}\alpha/3^k).$$

We can see that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} 2^{k-1}\alpha/3^k &= \alpha/3 \lim_{n\to\infty} \sum_{k=1}^{n} (\frac{2}{3})^{k-1} \\
&= \alpha/3 \lim_{n\to\infty} \sum_{k=0}^{n-1} (\frac{2}{3})^{k} \\
&= \alpha/3 \lim_{n\to\infty} \frac{1-(2/3)^n}{1-(2/3)} \\
&= \alpha/3 \frac{1}{1-(2/3)} \\
&= \alpha/3(\frac{1}{1/3}) \\
&= \alpha.
\end{aligned}$$

Therefore $m(F) = m(\bigcap_{k=1}^{\infty} F_k) = 1 - \alpha$.

39. Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: consider the complement of the generalized Cantor set of the preceding problem.)

We have $F \cup (F^c \cap [0, 1]) = [0, 1]$, and $m(F) = 1 - \alpha$, and $m(F^c \cap [0, 1]) = \alpha$...

40. A subset A of $\mathbb{R}$ is said to be **nowhere dense** in $\mathbb{R}$ provided that every open set $\mathcal{O}$ has a non-empty open subset that is disjoint from A. Show that the Cantor set is nowhere dense in $\mathbb{R}$.

The Cantor set $C \subseteq [0,1]$ is defined to be the countable intersection of sets C_k, where C_k is the disjoint union of 2^k closed intervals of length $1/3^k$ each. From Ch1 Proposition 9, we know that every open set is the countable disjoint union of open intervals. Therefore we need only prove Problem 40 for any open interval.
Consider any open interval $(a,b) \in \mathbb{R}$.
In the case that there exists an index k such that $(a,b) \in C_k^c$, then the proof is done:
Ex: $(a,b) = (3/18, 4/18)$. Then for $k = 2$, we have

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1],$$

so that

$$(3/18,4/18) \subseteq C_2^c = (1/9,2/9) \cup (1/3,2/3) \cup (7/9,8/9) = (2/18,4/18) \cup (1/3,2/3) \cup (7/9,8/9).$$

In the case that for all indices k we have that $(a,b) \in C_k$, then simply choose an index far enough so that one of the "open middle third" removal generated from C_k is a subset of (a,b).
Ex: $(a,b) = (6/10,7/10) \ni 2/3$ and $2/3 \in C$ so $(a,b) \cap C \neq \emptyset$. Then for $k = 1$, we have

$$(6/10,2/3) \subseteq (a,b) \text{ and } (6/10,2/3) \notin C_1 = [0,1/3] \cup [2/3,1].$$

Ex: $(a,b) = (2/3, 20/27$. Then for $k = 3$, we have

$$C_3 = [0,\frac{1}{27}] \cup [\frac{2}{27},\frac{1}{9}] \cup [\frac{2}{9},\frac{7}{27}] \cup [\frac{8}{27},\frac{1}{3}] \cup [\frac{2}{3},\frac{19}{27}] \cup [\frac{20}{27},\frac{7}{9}] \cup [\frac{8}{9},\frac{25}{27}] \cup [\frac{26}{27},1].$$

so that

$$(19/27,20/27) \subseteq (a,b) \text{ and } (19/27,20/27) \notin C_3.$$

41. Show that a strictly increasing function that is defined on an interval has a continuous inverse.

Let f be a strictly increasing function on the interval I. Then for $x, y \in I$ such that $x < y$, we have $f(x) < f(y)$.
Then f is injective because

$$x \neq y \implies x < y \text{ or } x > y \implies f(x) < f(y) \text{ or } f(x) > f(y) \implies f(x) \neq f(y).$$

Therefore the inverse $f^{-1} : im(f) \to I$ exists:

$$f^{-1}(x) \neq f^{-1}(y) \implies f^{-1}(f(x)) \neq f^{-1}(f(y)) \implies x \neq y,$$

that is, f^{-1} is a function because $x = y \implies f^{-1}(x) = f^{-1}(y)$ for all $x, y \in im(f)$.
Let $x \in I$ such that $a_n, b_n \in I$ with $a_n < x < b_n$ and $\lim_{n\to\infty} a_n = x$, $\lim_{n\to\infty} b_n = x$. Then clearly, $\lim_{n\to\infty}[b_n - a_n] = 0$. Then because f is strictly increasing, $f(a_n) < f(x) < f(b_n)$.
Now, we have the sequences $f(a_n)$ and $f(b_n)$ in $im(f)$ such that for each n, $f(a_n) < f(x) < f(b_n)$, and $\lim_{n\to\infty}[f^{-1}(f(b_n)) - f^{-1}(f(a_n))] = \lim_{n\to\infty}[b_n - a_n] = 0$. The results from Problem 35 tells us that f^{-1} is continuous at $f(x)$.

42. Let f be a continuous function and B be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: the collection of sets E for which $f^{-1}(E)$ is Borel is a σ-algebra containing the open sets.)

Let $S = \{E \mid f^{-1}(E) \text{ is Borel}\}$.
To show that S is a σ-algebra, know that the Borel sets is a σ-algebra.
Observe that:

(i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in S$.

(ii) $E \in S \implies f^{-1}(E)$ is Borel $\implies f^{-1}(E)^c = f^{-1}(E^c)$ is Borel $\implies E^c \in S$.

(iii) $E_k \in S \implies f^{-1}(E_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} f^{-1}(E_k) = f^{-1}(\bigcup_{k=1}^{\infty} E_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} E_k \in S$.

Also, any open set $\mathcal{O}$ is in S because $f^{-1}(\mathcal{O})$ is open and thus Borel. Thus S is a σ-algebra containing the open sets; that is, the Borel σ-algebra is a subset of S. Therefore for any Borel set B, $B \in S$ and thus $f^{-1}(B)$ is Borel.

43. Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

Let I be an interval and $f : I \to \mathbb{R}$ be a continuous strictly increasing function.
By Problem 41, we showed that $f^{-1} : im(f) \to I$ exists and is continuous.
Let $B \in I$ be any Borel set. By Problem 42, $(f^{-1})^{-1}(B) = f(B)$ is a Borel set.

Chapter 3

Lebesgue Measurable Functions

Contents

3.1 Sums, Products, and Compositions

Proposition 1. *Let the function f have a measurable domain E. Then the following statements are equivalent:*

(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.

(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.

(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.

(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.

Each of these properties implies that for each extended real number c,

the set $\{x \in E \mid f(x) = c\}$ is measurable.

Definition. *An extended real-valued function f defined on E is said to be **Lebesgue measurable**, or simply **measurable**, provided its domain E is measurable and it satisfies one of the four statements of Proposition 1.*

Proposition 2. *Let the real-valued function f be defined on a measurable set E. Then the function f is measurable iff for each open set $\mathcal{O}$, the inverse image of $\mathcal{O}$ under f, $f^{-1}(\mathcal{O}) = \{x \in E \mid f(x) \in \mathcal{O}\}$, is a measurable set.*

Proof. Let $f : E \to \mathbb{R}$, where E is a measurable set.
($\Longrightarrow$) Suppose that f is measurable.
Let $\mathcal{O}$ be open. Then by Chapter 1, Proposition 9, $\mathcal{O}$ can be written as the countable disjoint union of open intervals: $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. We can construct these intervals in the following form:

$$I_k = (a_k, b_k) = (-\infty, b_k) \cap (a_k, \infty)$$

Therefore we see that

$$\begin{aligned} f^{-1}(\mathcal{O}) &= f^{-1}(\bigcup_{k=1}^{\infty} I_k) \\ &= f^{-1}(\bigcup_{k=1}^{\infty} (-\infty, b_k) \cap (a_k, \infty)) \\ &= \bigcup_{k=1}^{\infty} f^{-1}((-\infty, b_k) \cap (a_k, \infty)) \\ &= \bigcup_{k=1}^{\infty} f^{-1}(-\infty, b_k) \cap f^{-1}(a_k, \infty). \end{aligned}$$

Because f is measurable, we see that $f^{-1}((-\infty, b_k))$ and $f^{-1}((a_k, \infty))$ are measurable sets, and countable union and intersection of measurable sets is also a measurable set, so $f^{-1}(\mathcal{O})$ is a measurable set.
($\Longleftarrow$) Suppose that for each open set $\mathcal{O}$, $f^{-1}(\mathcal{O})$ is a measurable set.
Because for any real number c, the interval of the form (c, ∞) is an open set, and therefore we have that the set $f^{-1}((c, \infty)) = \{x \in E \mid f(x) \in (c, \infty)\} = \{x \in E \mid f(x) > c\}$ is measurable, which implies that f is measurable. □

Proposition 5. *Let f be an extended real-valued function on the measurable set E.*

(i) If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.

(ii) For a measurable subset D of E, f is measurable on E iff the restrictions of f to D and $E \setminus D$ are measurable.

Proof. Let f be an extended real-valued function on the measurable set E.

(i) Let f be measurable on E and $f = g$ a.e. on E.
Define $A = \{x \in E \mid f(x) \neq g(x)\} \subseteq E$, so that $f = g$ on $E \setminus A$, and $m(A) = 0$.

$$\begin{aligned} \{x \in E \mid g(x) > c\} &= (\{x \in E \mid g(x) > c\} \cap [E \cap A]) \cup (\{x \in E \mid g(x) > c\} \cap [E \cap A^c]) \\ &= \{x \in A \mid g(x) > c\} \cup (\{x \in E \mid f(x) > c\} \cap [E \setminus A]). \end{aligned}$$

Now, $\{x \in A \mid g(x) > c\} \subseteq A$, and because $m(A) = 0$, $\{x \in A \mid g(x) > c\}$ is measurable and has measure zero. The set $\{x \in E \mid f(x) > c\}$ is measurable because f is measurable, and $E \cap A^c$ is measurable because E and A (and thus A^c) are measurable. Thus $\{x \in E \mid g(x) > c\}$ is measurable because it is the finite union and intersection of measurable sets; therefore g is measurable on E.

(ii) Let D be a measurable subset of E.
($\Longrightarrow$) Suppose f is measurable on E.

Then for any real number c, we see that

$$\{x \in D \mid f|_D(x) > c\} = \{x \in E \mid f(x) > c\} \cap [E \cap D],$$
$$\{x \in E \setminus D \mid f|_{E \setminus D}(x) > c\} = \{x \in E \mid f(x) > c\} \cap [E \setminus D],$$

where both are measurable because they are each intersections of measurable sets. Therefore the restrictions $f|_D$ and $f|_{E\setminus D}$ are measurable.
($\Longleftarrow$) Suppose the restrictions of f to D and $E \setminus D$ are measurable.
Then for any real number c,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f|_D(x) > c\} \cup \{x \in E \setminus D \mid f|_{E\setminus D}(x) > c\},$$

which is measurable because it is a union of measurable sets.

□

PROBLEMS

1. Suppose f and g are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E?

   Let f, g be continuous functions on $[a, b]$, where $f = g$ on $[a, b] \setminus E_0$, where E_0 is a subset of $[a, b]$ and $m(E_0) = 0$.
   Suppose that E_0 is nonempty.
   Consider any point $x_0 \in E_0 \subseteq [a, b]$. For any $\epsilon > 0$, there exists a $c \in (x_0 - \epsilon, x_0 + \epsilon) \cap [a, b]$ such that $f(c) = g(c)$, else we reach a contradiction because $m((x_0 - \epsilon, x_0 + \epsilon) \cap [a, b]) \neq 0$. This means that we can construct a sequence $\{c_i\}_{i=1}^{\infty}$ that converges to x_0, where $f(c_i) = g(c_i)$ is defined for all i. However, because $\{c_i\} \to x_0$, by continuity of f, g, we have $\{f(c_i)\} \to f(x_0)$ and $\{g(c_i)\} \to g(x_0)$, and because $f(c_i) = g(c_i)$ for all i, the limit is unique; that is,

   $$|f(x_0) - g(x_0)| \leq |f(x_0) - f(c_i)| + |f(c_i) - g(x_0)| < \epsilon,$$

   and $f(x_0) = g(x_0)$.
   However, this is a contradiction to $f(x) \neq g(x)$ for all $x \in E_0$, and so $E_0 = \emptyset$.

   In the case that $[a, b]$ is replaced by a general measurable set E, the assertion is not true. Consider the case where $E = \{a\}$, so that f, g are continuous on $\{a\}$, and $f = g$ a.e. on $\{a\}$. This only implies that $f = g$ on E except for a set of measure zero. But E is already of measure zero, so $f(a) \neq g(a)$ is possible, and $f \neq g$ on E.

2. Let D and E be measurable sets and f a function with domain $D \cup E$. We proved that f is measurable on $D \cup E$ iff its restrictions to D and E are measurable. Is the same true if "measurable" is replaced by "continuous"?

   No; consider the function $f : [-1, 1] \to \mathbb{R}$, where $[-1, 1] = [-1, 0) \cup [0, 1]$, and we define

   $$f(x) = \begin{cases} 0 & x \in [-1, 0), \\ 1 & x \in [0, 1]. \end{cases}$$

   Clearly we have a point of discontinuity at $x = 0$, so f is not continuous even though $f|_{[-1,0)}$ and $f|_{[0,1]}$ are continuous.

3. Suppose a function f has a measurable domain and is continuous except at a finite number of points. Is f necessarily measurable?

   Yes; let f be a function on the measurable domain E, and suppose f is continuous on $E \setminus E_0$, where $E_0 = \{x_i\}_{i=1}^n \subseteq E$. Then $m(E_0) = 0$ because countable sets are measurable and have measure zero.
   Now, $f|_{E \setminus E_0}$ is continuous and therefore measurable (Proposition 3), and $f|_{E_0}$ is defined on a set of measure zero, so any subset $\{x \in E_0 \mid f|_{E_0}(x) > c\} \subseteq E_0$ has measure zero and is thus measurable, and therefore $f|_{E_0}$ is a measurable function.
   Recall Proposition 5 to see that for the measurable subset E_0 of E, f is measurable because $f|_{E_0}$ and $f|_{E \setminus E_0}$ are both measurable functions.

4. Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number c. Is f necessarily measurable?

   No; let $V \subseteq [0,1]$ be a Vitali set. Therefore V is nonmeasurable (see Ch 2.6). Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined as
   $$f(x) = \begin{cases} -e^x & x \in V \\ e^x & x \notin V \end{cases}$$
   For any real number c, we have
   $$f^{-1}(c) = \begin{cases} \ln(-c) & c < 0 \\ \ln(c) & c > 0 \\ \emptyset & c = 0 \end{cases}$$
   and so $f^{-1}(c)$ is a singleton set or is the empty set, which are measurable, so $f^{-1}(c)$ is measurable.
   Now, we know that $e^x : \mathbb{R} \to \mathbb{R}_{>0}$ and so $e^x > 0$ for any real number x.
   Therefore $f(x) = -e^x < 0$ only when $x \in V$. However, the set $\{x \in \mathbb{R} \mid f(x) < 0\} = V$ is not measurable, and so f is not a measurable function.

5. Suppose the function f is defined on a measurable set E and $\{x \in E \mid f(x) > c\}$ is a measurable set for each rational number c. Is f necessarily a measurable function?

   Yes. Let $f : E \to \mathbb{R}$ with E a measurable set, and let $\{x \in E \mid f(x) > c\} = \{x \in E \mid f(x) \in (c, \infty)\}$ be measurable for each $c \in \mathbb{Q}$.
   Let a be any real number. Then for any natural number n, there exists a rational number c_n such that $a < c_n < a + \frac{1}{n}$, and therefore $\bigcup_{n=1}^{\infty}(c_n, \infty) = (a, \infty)$. Therefore we have
   $$\begin{aligned} \{x \in E \mid f(x) > a\} &= f^{-1}((a, \infty)) \\ &= f^{-1}\Big(\bigcup_{n=1}^{\infty}(c_n, \infty)\Big) \\ &= \bigcup_{n=1}^{\infty} f^{-1}((c_n, \infty)) \\ &= \bigcup_{n=1}^{\infty} \{x \in E \mid f(x) > c_n\}, \end{aligned}$$
   which is a countable union of measurable sets, and therefore is also measurable.

6. Let f be a function with measurable domain D. Show that f is measurable iff the function g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Let $D \subseteq \mathbb{R}$ be a measurable set, let $f : D \to \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

($\implies$) Suppose that f is measurable.
For any real number c,

$$\{x \in \mathbb{R} \mid g(x) > c\} = \begin{cases} \{x \in D \mid f(x) > c\} & c \geq 0 \\ \{x \in D \mid f(x) > c\} \cup D^c & c < 0 \end{cases}$$

Both of the sets $\{x \in D \mid f(x) > c\}$ and $\{x \in D \mid f(x) > c\} \cap D^c$ are measurable, so $\{x \in \mathbb{R} \mid g(x) > c\}$ is measurable and thus g is a measurable function.

($\impliedby$) Suppose that g is measurable.
Recall Proposition 5 (ii) to see that for the measurable subset D of $\mathbb{R}$, g is measurable on $\mathbb{R}$, which implies that the restrictions $g|_D$ and $g|_{\mathbb{R}\setminus D}$ are measurable. Therefore for any real number c,

$$\{x \in D \mid f(x) > c\} = \{x \in \mathbb{R} \mid g|_D(x) > c\} \cap D \text{ is measurable,}$$

and f is measurable.

7. Let the function f be defined on a measurable set E. Show that f is measurable iff for each borel set A, $f^{-1}(A)$ is measurable. (Hint: the collection of sets A that have the property that $f^{-1}(A)$ is measurable is a σ-algebra.)

Let $f : E \to \mathbb{R}$, where E is a measurable set.
($\implies$) Suppose that f is measurable.
Let $\mathcal{M} = \{A \mid f^{-1}(A) \text{ is measurable}\}$.
To show that $\mathcal{M}$ is a σ-algebra, know that the measurable sets is a σ-algebra.
Observe that:

(i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in \mathcal{M}$.

(ii) $A \in \mathcal{M} \implies f^{-1}(A)$ is measurable $\implies f^{-1}(A)^c = f^{-1}(A^c)$ is measurable $\implies$ $A^c \in \mathcal{M}$.

(iii) $A_k \in \mathcal{M} \implies f^{-1}(A_k)$ is measurable $\implies \bigcup_{k=1}^{\infty} f^{-1}(A_k) = f^{-1}(\bigcup_{k=1}^{\infty} A_k)$ is measurable $\implies$ $\bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

Then because f is measurable, for any real number a, the set $f^{-1}((a,\infty)) = \{x \in E \mid f(x) > a\}$ is measurable. Now, $(a,\infty) \in \mathcal{M}$ because $f^{-1}((a,\infty))$ is measurable. Because (a,∞) is a Borel set, all other Borel sets are in $\mathcal{M}$ because the Borel sets are a σ-algebra.

($\impliedby$) Suppose that for each borel set A, the set $f^{-1}(A) = \{x \in E \mid f(x) \in A\}$ is measurable.
Every interval of the form (a,∞) is a borel set, so we have that for any real number a, the set $f^{-1}((a,\infty)) = \{x \in E \mid f(x) > a\}$ is measurable. This is equivalent to the measurability of f.

8. (Borel measurability) A function f is said to be **Borel measurable** provided its domain E is a Borel set and for each c, the set $\{x \in E \mid f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace "(Lebesgue) measurable set" by "Borel set". Show that:

(i) every Borel measurable function is Lebesgue measurable,

The Borel sets are a subset of the measurable sets. Therefore for a Borel measurable function f, its domain E is a Borel set (and thus a measurable set), and for each c, the set $\{x \in E \mid f(x) > c\}$ is a Borel set (and thus a measurable set). Thus f is a measurable function.

(ii) if f is Borel measurable and B is a Borel set, then $f^{-1}(B)$ is a Borel set,

Let $S = \{B \mid f^{-1}(B) \text{ is Borel}\}$.
To show that S is a σ-algebra, know that the Borel sets is a σ-algebra.
Observe that:

(i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in S$.

(ii) $B \in S \implies f^{-1}(B)$ is Borel $\implies f^{-1}(B)^c = f^{-1}(B^c)$ is Borel $\implies B^c \in S$.

(iii) $B_k \in S \implies f^{-1}(B_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} f^{-1}(B_k) = f^{-1}(\bigcup_{k=1}^{\infty} B_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} B_k \in S$.

Now, because f is Borel measurable, for any real number a, the set $f^{-1}((a,\infty))$ is a Borel set. This implies $(a,\infty) \in S$. Because (a,∞) is a Borel set, all other Borel sets are in S because the Borel sets is a σ-algebra.

(iii) if f and g are Borel measurable, so is $f \circ g$,

Let f, g be Borel measurable, with $g : E \to F$, and $f : F \to \mathbb{R}$, where E, F are Borel sets. Then $f \circ g : E \to \mathbb{R}$ has a Borel set as its domain.
Recall that $\{x \mid f(x) > a\} = \{x \mid f(x) \in (a,\infty)\} = f^{-1}((a,\infty))$.

$$
\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a,\infty)\} \\
&= (f \circ g)^{-1}((a,\infty)) \\
&= (g^{-1} \circ f^{-1})((a,\infty)) \\
&= g^{-1}(f^{-1}((a,\infty))) \\
&= g^{-1}(B) && \text{where } B \text{ is Borel} \\
&= B'. && \text{where } B' \text{ is Borel}
\end{aligned}
$$

Therefore $f \circ g$ is Borel measurable.

(iv) if f is Borel measurable and g is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Let f be Borel measurable, with $f : F \to \mathbb{R}$, and let g be Lebesgue measurable, with $g : E \to F$, where F is a Borel set and E is a measurable set. Then $f \circ g : E \to \mathbb{R}$ has a measurable set as its domain.

$$\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a, \infty)\} \\
&= (f \circ g)^{-1}((a, \infty)) \\
&= (g^{-1} \circ f^{-1})((a, \infty)) \\
&= g^{-1}(f^{-1}((a, \infty))) \\
&= g^{-1}(B) && \text{where } B \text{ is Borel} \\
&= B'. && \text{where } B' \text{ is measurable (Problem 7)}
\end{aligned}$$

Therefore $f \circ g$ is Lebesgue measurable.

9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points of x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable?

   Let $E_0 = \{x \in E \mid \{f_n(x)\}$ converges$\} \subseteq \{x \in E \mid \{f_n(x)\}$ is Cauchy$\}$, because all convergent sequences are Cauchy.
   Therefore $E_0 = \{x \in E \mid \forall k \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{1}{k}$ for all $n, m \geq N\}$.
   This is equivalent to writing

$$E_0 = \bigcap_{k \in \mathbb{N}} \bigcup_{N \in \mathbb{N}} \bigcap_{n,m \geq N} \{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}.$$

   The functions f_n and f_m are measurable, so by Theorem 6, $f_n - f_m$ is also measurable. The absolute value function $|\cdot|$ is continuous, so by Proposition 7, the composition $|\cdot| \circ (f_n - f_m) = |f_n - f_m|$ is a measurable function. Therefore for the real number $\frac{1}{k}$, the set $\{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$ is a measurable set.
   Then E_0 is a countable union and intersection of measurable sets, so E_0 is measurable.

10. Suppose f and g are real-valued functions defined on all of $\mathbb{R}$, f is measurable, and g is continuous. Is the composition $f \circ g$ necessarily measurable?

    No; let f be measurable, with $f : \mathbb{R} \to \mathbb{R}$, and let g be continuous, with $g : \mathbb{R} \to \mathbb{R}$. Then $f \circ g : \mathbb{R} \to \mathbb{R}$.

$$\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a, \infty)\} \\
&= (f \circ g)^{-1}((a, \infty)) \\
&= (g^{-1} \circ f^{-1})((a, \infty)) \\
&= g^{-1}(f^{-1}((a, \infty))) \\
&= g^{-1}(A) && \text{where } A \text{ is measurable}
\end{aligned}$$

    Recall Chapter 2 Problem 36 to see that for a continuous function g, the set $g^{-1}(A)$ is not always measurable when A is measurable.

11. Let f be a measurable function and g be a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable. (Hint: examine Problem 37 in Chapter 2.)

    Let f be measurable, with $f : \mathbb{R} \to \mathbb{R}$, and let g be a bijection from $\mathbb{R}$ to $\mathbb{R}$, where g^{-1} is Lipschitz. From Chapter 2 Problem 37, we have that g^{-1} maps a measurable set to a measurable

set; that is, for the measurable set A, the set $g^{-1}(A)$ is measurable.
We have $f \circ g : \mathbb{R} \to \mathbb{R}$.

$$\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a, \infty)\} \\
&= (f \circ g)^{-1}((a, \infty)) \\
&= (g^{-1} \circ f^{-1})((a, \infty)) \\
&= g^{-1}(f^{-1}((a, \infty))) \\
&= g^{-1}(A) && \text{where } A \text{ is measurable} \\
&= A'. && \text{where } A' \text{ is measurable (Chapter 2 Problem 37)}
\end{aligned}$$

Therefore $f \circ g$ is measurable.

3.2 Sequential Pointwise Limits and Simple Approximation

Definition. *For a sequence $\{f_n\}$ of functions with common domain E, a function f on E, and a subset A of E, we say that*

(i) The sequence $\{f_n\}$ converges to f pointwise on A provided

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in A.$$

a.k.a.

$$\forall x \in A, \forall \epsilon > 0, \exists N_x \in \mathbb{N} : \forall n \geq N_x, |f_n(x) - f(x)| < \epsilon.$$

(ii) The sequence $\{f_n\}$ converges to f pointwise a.e. on A provided it converges to f pointwise on $A \setminus B$, where $m(B) = 0$.

(iii) The sequence $\{f_n\}$ converges to f uniformly on A provided for each $\epsilon > 0$, there is an index N for which

$$|f - f_n| < \epsilon \text{ on } A \text{ for all } n \geq N.$$

a.k.a.

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in A, |f_n(x) - f(x)| < \epsilon.$$

Theorem. *Let $A \subseteq \mathbb{R}^n$ and suppose $\{f_i\}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$ such that*

(i) $\{f_i\} \to f$ uniformly on A

(ii) Each f_i is uniformly continuous on A.

Then $f : A \to \mathbb{R}^m$ is uniformly continuous on A.

Proof. Fix $\epsilon > 0$.
By (i), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in A$, $\|f_n(x) - f(x)\| < \epsilon/3$.
Now, fix N and fix $k \geq N$.
By (ii), there exists a $\delta > 0$ such that for all $x, y \in A$ with $\|x - y\| < \delta$, then $\|f_k(x) - f_k(y)\| < \epsilon/3$.

Therefore we have

$$\begin{aligned}\|f(x)-f(y)\| &= \|f(x)-f_k(x)+f_k(x)-f_k(y)+f_k(y)-f(y)\| \\ &\leq \|f(x)-f_k(x)\|+\|f_k(x)-f_k(y)\|+\|f_k(y)-f(y)\| \\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Therefore for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$ with $\|x-y\| < \delta$, then $\|f(x)-f(y)\| < \epsilon$. Thus f is uniformly continuous on A.
A similar proof can show this for a sequence of continuous functions converging uniformly to a continuous function. ☐

Example. *Consider the sequence of continuous functions $\{f_n\}_{n=2}^{\infty} : [0,1] \to \mathbb{R}$, defined by*

$$f_n(x) = \begin{cases} \frac{n-0}{1/n-0}x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{0-n}{2/n-1/n}(x-\frac{1}{n})+n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases} = \begin{cases} n^2x & \text{if } x \in [0, \frac{1}{n}] \\ -n^2(x-\frac{1}{n})+n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases}$$

(Each f_n is a triangle-shaped function that achieves its max $f(1/n) = n$ and base corners $f(0) = 0$ and $f(2/n) = 0$.)
In addition, consider the continuous function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in [0,1]$.
The sequence $\{f_n\}$ converges to f pointwise but not uniformly on $[0,1]$.
To see this, for any $\epsilon > 0$, and for any $x \in [0,1]$, there exists an index $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, we have $\frac{2}{n} < x$, that is, $x \in (\frac{2}{n}, 1]$, and so $|f_n(x) - f(x)| = 0 - 0 = 0 < \epsilon$.
Therefore the sequence converges pointwise.
To see that the sequence does not converge uniformly, we see that there exists an $\epsilon = 1 > 0$ such that for all indices N, there exists an $n \geq N$ and the point $1/n$ such that $|f_n(1/n) - f(1/n)| = n - 0 = n > 1$.

The pointwise limit of continuous functions may not be continuous.
The pointwise limit of Riemann integrable functions may not be Riemann integrable.

Proposition 9. *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to the function f. Then f is measurable.*

Proof. Let E_0 be a subset of E such that $\{f_n\}$ converges pointwise to f on $E \setminus E_0$, with $m(E_0) = 0$. Because E_0 has measure zero, then it is measurable; by Proposition 5, we have that f is measurable iff the restrictions to E_0 and $E \setminus E_0$ are measurable. By monotonicity of measure, the measure of the set $\{x \in E_0 \mid f(x) < c\}$ is always zero and thus trivially measurable.
We want to show that $\{x \in E \setminus E_0 \mid f(x) < c\}$ is measurable to use Proposition 5.
Now, for any point $x \in E \setminus E_0$, we have that

$$f(x) < c \iff \exists n, k \in \mathbb{N} \text{ s.t. } f_j(x) < c - 1/n \; \forall j \geq k.$$

To see why, suppose that $\forall n, k \in \mathbb{N}, \exists j \geq k$ s.t. $f_j(x) \geq c - 1/n$.
In the case $f_j(x) > c - 1/n$, we have $1/n + f_j(x) \geq c$ for all n, which implies that $f_j(x) \geq c > f$, a contradiction.
In the case $f_j(x) = c - 1/n$, we have that for any n, for any indices k, there exists an index $j \geq k$ such that $c - f_j(x) = 1/n$, but because $f < c$, the convergence $\{f_n\} \to f$ is not possible, a contradiction.
Then we can write

$$\{x \in E \setminus E_0 \mid f(x) < c\} = \bigcup_{1 \leq n, k < \infty} \left[\bigcap_{j=k}^{\infty} \{x \in E \setminus E_0 \mid f_j(x) < c - 1/n\} \right],$$

Which is a countable union and intersection of measurable sets, and thus is measurable. $\square$

If A is any set, the **characteristic function** of A, χ_A, is the function on $\mathbb{R}$ defined by

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

The function χ_A is measurable iff the set A is measurable:
($\Longrightarrow$) Suppose χ_A is measurable.
Then for the real number $c = 0$, the set $\{x \in \mathbb{R} \mid \chi_A(x) > 0\} = A$ is measurable.
($\Longleftarrow$) Suppose the set A is measurable.
Then for any real number c, we have

$$\{x \in \mathbb{R} \mid \chi_A(x) \in (c, \infty)\} = \begin{cases} \emptyset & c \geq 1 \\ A & 1 > c \geq 0 \\ \mathbb{R} & 0 > c \end{cases}$$

Each of the sets $\emptyset$, A, and $\mathbb{R}$ are measurable, so $\{x \in \mathbb{R} \mid \chi_A(x) \in (c, \infty)\}$ is a measurable set and thus χ_A is a measurable function.
Thus the existence of a nonmeasurable set E implies the existence of a nonmeasurable function χ_E.

Definition. *A real-valued function φ defined on a measurable set E is called **simple** provided it is measurable and takes only a finite number of values.*

If φ is simple, has domain E and takes the distinct values $c_1, \cdots, c_n$, then

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\}.$$

(A linear combination of measurable functions)
A simple function is called a **step function** in the special case that the sets E_k are intervals.

The Simple Approximation Lemma. *Let f be a measurable real-valued function on E. Assume f is bounded on E, that is, there is an $M \geq 0$ for which $|f| \leq M$ on E. Then for each $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on E which have the following approximation properties:*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } E.$$

Proof. Because f is bounded, we have that $f(x) \subseteq [-M, M]$ for all $x \in E$.
Let (c, d) be an open, bounded interval such that $f(E) \subseteq [-M, M] \subseteq (c, d)$, so that (c, d) contains $f(E)$, the image of E under f.
Also, choose $\{y_0, \cdots, y_n\}$ to be a partition of the closed, bounded interval $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$:

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

Define

$$I_k = [y_{k-1}, y_k) \text{ and } E_k = f^{-1}(I_k) \text{ for } 1 \leq k \leq n.$$

Because f is a measurable function and each interval $I_k = [y_{k-1}, y_k)$ is Borel, then each set $E_k = f^{-1}(I_k)$ is measurable (see Problem 7). Because E_k is measurable, then χ_{E_k} is a measurable function.

Then we can define the simple functions φ_ϵ and ψ_ϵ on E by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k}$$
$$\psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}$$

Now, for any $x \in E$, because $f(E) \subseteq (c, d)$, and $\{[y_{k-1}, y_k)\}_{k=1}^n$ is a partition that contains (c, d), there exists a unique k, $1 \le k \le n$, for which $f(x) \in [y_{k-1}, y_k)$, and therefore

$$\varphi_\epsilon(x) = y_{k-1} \le f(x) < y_k = \psi_\epsilon(x).$$

Also, for each $x \in E$, $\psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} < \epsilon$ for some k. $\square$

The Simple Approximation Theorem. *An extended real-valued function f on a measurable set E is measurable iff there is a sequence $\{\varphi_n\}$ of simple functions on E which converges pointwise on E to f and has the property that*

$$|\varphi_n| \le |f| \text{ on } E \text{ for all } n.$$

If f is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.

Proof. Let f be an extended real-valued function on a measurable set E.
($\Longleftarrow$) Suppose $\{\varphi_n\}$ is a sequence of simple (and thus measurable) functions on E that converges pointwise on E to f (and has the property that $|\varphi_n| \le |f|$ on E for all n).
Then by Proposition 9, f is measurable.
($\Longrightarrow$) Suppose f is measurable.
We can assume $f \ge 0$ on E (See Problem 23 for the general case, as $f = f^+ - f^-$ on E, a linear combination of two nonnegative measurable functions f^+ and f^-).
For a natural number n, define $E_n = \{x \in E \mid f(x) \le n\}$. Because f is a measurable function, the set $E_n \subseteq E$ is measurable. Then by Proposition 5 (ii), $f|_{E_n}$ is measurable.
Also, $f|_{E_n}$ is bounded because $0 \le f|_{E_n}(x) \le n$ for $x \in E_n$.
Now, recall the Simple Approximation Lemma to see that for the measurable real-valued function $f|_{E_n}$ on E_n, where $f|_{E_n}$ is bounded on E_n, we have that for $\epsilon = 1/n$, there exist simple functions φ_n and ψ_n defined on E_n such that

$$0 \le \varphi_n \le f \le \psi_n \text{ on } E_n \text{ and } 0 \le \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Then we can see that

$$0 \le \varphi_n \le f \text{ and } 0 \le f - \varphi_n \le \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Now, $E = E_n \cup E_n^c = \{x \in E \mid f(x) \le n\} \cup \{x \in E \mid f(x) > n\}$ and φ_n is defined on E_n. We can extend φ_n to all of E by setting $\varphi_n(x) = n$ on $E_n^c = \{x \in E \mid f(x) > n\}$. Then φ_n is a simple function defined on E and $0 \le \varphi_n \le f$ on E. To see that $\{\varphi_n\}$ converges to f pointwise on E:
Consider any $x \in E$.
Case $f(x) < \infty$:
Then there exists a natural number N such that $f(x) \le N$. Then for all $n \ge N$, We have that

$$0 \le f(x) - \varphi_n(x) < 1/n,$$

and thus $\lim_{n\to\infty} \varphi_n(x) = f(x)$.
Case $f(x) = \infty$:
Then $\varphi_n(x) = n$ because $f(x) > n$ for all n, and $\lim_{n\to\infty} \varphi_n(x) = \infty = f(x)$.
By replacing each φ_n with $\max\{\varphi_1, \cdots, \varphi_n\}$, we have $\{\varphi_n\}$ increasing. $\square$

PROBLEMS

12. Let f be a bounded measurable function on E. Show that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing and each of these sequences converges to f uniformly on E.

    Let f be a bounded measurable function on the measurable set E. Because f is bounded, there exists a real number M such that $|f| \leq M$ and thus $-M \leq f \leq M$. Then for all natural numbers $n \geq M$, the set $E_n = \{x \in E \mid f(x) \leq n\} = E$. By the Simple Approximation Lemma, for $\epsilon = 1/n$, there exist the simple function φ_n and ψ_n defined on $E_n = E$ such that

    $$\varphi_n \leq f \leq \psi_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E.$$

    Then for each $n \geq M$, we have the sequences of functions $\{\varphi_n\}$ and $\{\psi_n\}$ such that

    $$\varphi_n \leq f \text{ and } f - \varphi_n \leq \psi_n - \varphi_n < 1/n \text{ on } E,$$
    $$f \leq \psi_n \text{ and } \psi_n - f \leq \psi_n - \varphi_n < 1/n \text{ on } E.$$

    We can replace each φ_n with $\max\{\varphi_1, \cdots, \varphi_n\}$ and each ψ_n with $\min\{\psi_1, \cdots, \psi_n\}$ so that the sequences are increasing and decreasing respectively. The convergence is uniform because we showed that for any $\epsilon = 1/n$, there exists the natural number n such that for all $n' \geq n$, $f(x) - \varphi_{n'}(x) < 1/n$ and $\psi_{n'}(x) - f(x) < 1/n$ for all $x \in E$.

13. A real-valued measurable function is said to be *semisimple* provided it takes only a countable number of values. Let f be any measurable function on E. Show that there is a sequence of semisimple functions $\{f_n\}$ on E that converges to f uniformly on E.

    We can define $f_n(x) = \frac{1}{n}\lfloor nf(x)\rfloor$ (where the floor function $\lfloor x\rfloor$ returns the largest integer less than or equal to x). Then because $f_n = \frac{\lfloor nf(x)\rfloor}{n}$ and $\lfloor nf(x)\rfloor$ and n are integers, we have $f_n(E) \subseteq \mathbb{Q}$ which is countable. Because the floor function rounds down to the nearest integer, we have

    $$|nf(x) - \lfloor nf(x)\rfloor| < 1 \text{ for all } x \in E,$$

    and therefore
    $$|f(x) - \frac{1}{n}\lfloor nf(x)\rfloor| < 1/n \text{ for all } x \in E.$$

14. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set F contained in E such that f is bounded on F and $m(E \backslash F) < \epsilon$.

    For each natural number n, let $F_n = \{x \in E \mid f(x) > n\}$. Because f is measurable, then each F_n is a measurable set. Then $\{F_n\}$ is descending because

    $$F_n = \{x \in E \mid f(x) > n\} \supseteq \{x \in E \mid f(x) > n+1 > n\} = F_{n+1}.$$

    Also, $F_1 \subseteq E$, so by monotonicity of measure, $m(F_1) \leq m(E) < \infty$.
    By the continuity of measure, for the descending collection of measurable sets $\{F_n\}$ for which $m(F_1) < \infty$, we have $m(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n)$.
    Now,
    $$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > n\} = \{x \in E \mid f(x) = \infty\},$$

Because f is finite a.e. on E, we have

$$0 = m(\{x \in E \mid f(x) = \infty\}) = m(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n).$$

Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m(F_n) < \epsilon$.
If we let $F = E \cap F_n^c = E \setminus F_n \subseteq E$, then $F = \{x \in E \mid f(x) \leq n\}$, a measurable set, and

$$E \cap F^c = E \cap [E^c \cup F_n] = [E \cap E^c] \cup [E \cap F_n] = \emptyset \cup F_n = F_n,$$

and thus we have

$$m(E \setminus F) = m(E \cap F^c) = m(F_n) < \epsilon.$$

15. Let f be a measurable function on E that if finite a.e. on E and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set F contained in E and a sequence $\{\varphi_n\}$ of simple functions on E such that $\{\varphi_n\} \to f$ uniformly on F and $m(E \setminus F) < \epsilon$. (Hint: see the preceding problem.)

Because f is finite a.e. on E and $m(E) < \infty$, by the previous problem, for any $\epsilon > 0$, there exists a measurable set $F \subseteq E$ such that f is bounded on F and $m(E \setminus F) < \epsilon$. Furthermore, because f is bounded and measurable on F, by Problem 12, there exists a sequence of simple functions $\{\varphi_n\}$ on F that converges uniformly to f on F. Each φ_n can be extended to E by setting $\varphi_n = n$ on F^c.

16. Let I be a closed, bounded interval and E a measurable subset of I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \chi_E \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: use Theorem 12 of Chapter 2.)

Theorem 12 of Chapter 2 states:
Let E be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $\mathcal{O} = \bigcup_{k=1}^n I_k$, then

$$m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \epsilon.$$

...
Fix $\epsilon > 0$.
Because $m(E)$ is finite, by definition of infimum, for $\epsilon > 0$, there exists a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ whose union $\mathcal{O} = \bigcup_{k=1}^\infty I_k$ covers E and

$$m(E) \leq \sum_{k=1}^{\infty} \ell(I_k) < m(E) + \epsilon/2,$$

and $\sum_{k=1}^\infty \ell(I_k)$ is finite as well. Because E is a measurable subset of finite outer measure that is contained in $\mathcal{O}$, then we can use the excision property and subadditivity to see that

$$m(\mathcal{O} \setminus E) = m(\mathcal{O}) - m(E) = m(\bigcup_{k=1}^{\infty} I_k) - m(E) \leq \sum_{k=1}^{\infty} \ell(I_k) - m(E) < \epsilon/2, \tag{1}$$

Because the series $\sum_{k=1}^{\infty} \ell(I_k)$ is finite, the sequence of partial sums $\sum_{k=1}^{n} \ell(I_k)$ converges to $\sum_{k=1}^{\infty} \ell(I_k)$, so that for $\epsilon/2$, there exists an index N such that for all $n \geq N$, we have

$$\begin{aligned}
\sum_{k=1}^{\infty} \ell(I_k) - \sum_{k=1}^{n} \ell(I_k) &< \epsilon/2 \\
\sum_{k=n+1}^{\infty} \ell(I_k) + \sum_{k=1}^{n} \ell(I_k) - \sum_{k=1}^{n} \ell(I_k) &< \epsilon/2 \\
\sum_{k=n+1}^{\infty} \ell(I_k) &< \epsilon/2,
\end{aligned}$$

and then we can let $\mathcal{O}' = \bigcup_{k=1}^{N} I_k$ so that, because $E \cap (\mathcal{O} \setminus \mathcal{O}') \subseteq \mathcal{O} \setminus \mathcal{O}'$,

$$m(E \cap (\mathcal{O} \setminus \mathcal{O}')) \leq m(\mathcal{O} \setminus \mathcal{O}') = m(\bigcup_{k=1}^{\infty} I_k \setminus \bigcup_{k=1}^{N} I_k) = m(\bigcup_{k=N+1}^{\infty} I_k) \leq \sum_{k=N+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Because $E \subseteq \mathcal{O}$, then $E \cap \mathcal{O}^c = \emptyset$, and we can derive

$$\begin{aligned}
E \cap \mathcal{O}' &= [\emptyset \cup (E \cap \mathcal{O}')] \\
&= [(E \cap \mathcal{O}^c) \cup (E \cap \mathcal{O}')] \\
&= \emptyset \cup [(E \cap \mathcal{O}^c) \cup (E \cap \mathcal{O}')] \\
&= [E \cap E^c] \cup [E \cap (\mathcal{O}^c \cup \mathcal{O}')] \\
&= E \cap [E^c \cup (\mathcal{O}^c \cup \mathcal{O}')] \\
&= E \cap [E^c \cup (\mathcal{O} \cap \mathcal{O}'^c)^c] \\
&= E \cap [E \cap (\mathcal{O} \setminus \mathcal{O}')]^c \\
&= E \setminus [E \cap (\mathcal{O} \setminus \mathcal{O}')].
\end{aligned}$$

And then we see that by excision,

$$m(E \cap \mathcal{O}') = m(E \setminus [E \cap (\mathcal{O} \setminus \mathcal{O}')]) = m(E) - m(E \cap (\mathcal{O} \setminus \mathcal{O}')) > m(E) - \epsilon/2. \tag{2}$$

We will let $F = (E \cap \mathcal{O}') \cup (I \setminus \mathcal{O})$ so that

$$\begin{aligned}
I \setminus F &= I \cap F^c \\
&= I \cap [(E \cap \mathcal{O}') \cup (I \setminus \mathcal{O})]^c \\
&= I \cap [(I^c \cup \mathcal{O}) \cap (E \cap \mathcal{O}')^c] \\
&= [I \cap (I^c \cup \mathcal{O})] \cap (E \cap \mathcal{O}')^c \\
&= [(I \cap I^c) \cup (I \cap \mathcal{O})] \cap (E \cap \mathcal{O}')^c \\
&= [I \cap \mathcal{O}] \setminus (E \cap \mathcal{O}') \\
&\subseteq \mathcal{O} \setminus [E \cap \mathcal{O}'].
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
m(I \setminus F) &\leq m(\mathcal{O} \setminus [E \cap \mathcal{O}']) && \text{by monotonicity} \\
&= m(\mathcal{O}) - m(E \cap \mathcal{O}') && \text{by excision} \\
&< m(\mathcal{O}) - m(E) + \epsilon/2 && \text{by (2)} \\
&< \epsilon/2 + \epsilon/2 && \text{by (1)} \\
&< \epsilon.
\end{aligned}$$

We can let $h = \sum_{k=1}^{n} \chi_{J_k} = \chi_{\mathcal{O}'}$, with $J_k = I_k \setminus \bigcup_{j=1}^{k-1} I_j$, where each J_k is a finite union of disjoint intervals, and so h is a step function. Then $h = 1$ on $E \cap \mathcal{O}'$ and $h = 0$ on $I \setminus \mathcal{O}$, so that $h = \chi_E$ on F.

17. Let I be a closed, bounded interval and ψ a simple function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

We have

$$\psi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } I, \text{ where } E_k = \{x \in I \mid \psi(x) = c_k\} \text{ measurable.}$$

For each $k = 1, \cdots, n$, we have that I is a closed, bounded interval and E_k a measurable subset of I. For $\epsilon > 0$, by the previous Problem 16, there is a step function h_k on I and a measurable subset F_k of I for which

$$h_k = \chi_{E_k} \text{ on } F_k \text{ and } m(I \setminus F_k) < \epsilon/n.$$

We can let $h = \sum_{k=1}^{n} c_k h_k$ and $F = \bigcap_{k=1}^{n} F_k$ so that $h = \psi$ on F and $I \cap F^c = I \cap \bigcup_{k=1}^{n} F_k^c = \bigcup_{k=1}^{n} (I \cap F_k^c)$ which gives us

$$m(I \setminus F) = m(I \cap F^c) = m(\bigcup_{k=1}^{n} (I \cap F_k^c)) \leq \sum_{k=1}^{n} m(I \cap F_k^c) = \sum_{k=1}^{n} m(I \setminus F_k) < \epsilon.$$

18. Let I be a closed, bounded interval and f a bounded measurable function defined on I. Let $\epsilon > 0$. Show that there is a step function h on I and a measurable subset F of I for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Because f is bounded and measurable on I, by Problem 12, there exists a sequence of simple functions $\{\psi_n\}$ on I that converges uniformly to f on I. Then for any ϵ, we can choose $\psi \in \{\psi_n\}$ such that $|\psi - f| < \epsilon$. By the previous Problem 17, there is a step function h on I and a measurable subset F of I for which

$$h = \psi \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Therefore we have $|h - f| < \epsilon$.

19. Show that the sum and product of two simple functions are simple as are the max and the min.

Consider two simple functions φ and ψ on the measurable set E, with

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\} \text{ measurable.}$$

$$\psi = \sum_{k'=1}^{m} c'_{k'} \cdot \chi_{E'_{k'}} \text{ on } E, \text{ where } E'_{k'} = \{x \in E \mid \psi(x) = c'_{k'}\} \text{ measurable.}$$

Then for any $x \in E$, there exists an $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, m\}$ such that

$$(\varphi + \psi)(x) = c_i + c'_j \text{ on } E_i \cap E'_j,$$

The intersection of measurable sets $E_i \cap E'_j$ is also measurable, so the function $\chi_{E_i \cap E'_j}$ is measurable.
That is, we have the simple function

$$\varphi + \psi = \sum_{k=1}^{n} \sum_{k'=1}^{m} (c_k + c_{k'}) \cdot \chi_{E_k \cap E'_{k'}} \text{ on } E.$$

Similarly for the product, we have

$$\varphi \cdot \psi = \sum_{k=1}^{n} \sum_{k'=1}^{m} (c_k \cdot c_{k'}) \cdot \chi_{E_k \cap E'_{k'}} \text{ on } E.$$

We can recall Chapter 1 Problem 49 to see how we define max and min:

$$\max\{\varphi, \psi\} = \frac{1}{2}(\varphi + \psi + |\varphi - \psi|),$$
$$\min\{\varphi, \psi\} = \frac{1}{2}(\varphi + \psi - |\varphi - \psi|).$$

Clearly scaling a simple function is simple, and the absolute value of a simple function is simple, and we showed that the sum of simple functions is simple, and therefore the max and min of simple functions is simple.

20. Let A, B be any sets. Show that

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$
$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$
$$\chi_{A^c} = 1 - \chi_A$$

We can use DeMorgan's laws to see that

$$\chi_{A\cap B} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \in A^c \cup B^c \end{cases} = \begin{cases} 1 & x \in A \text{ and } x \in B \\ 0 & x \notin A \text{ or } x \notin B \end{cases}$$

Then for any $x \in \mathbb{R}$, we have

$\chi_A(x)$	$\chi_B(x)$	$\chi_{A\cap B}(x)$	$\chi_A(x) \cdot \chi_B(x)$
0	0	0	$0 \cdot 0 = 0$
0	1	0	$0 \cdot 1 = 0$
1	0	0	$1 \cdot 0 = 0$
1	1	1	$1 \cdot 1 = 1$

Similarly see that

$$\chi_{A\cup B} = \begin{cases} 1 & x \in A \cup B \\ 0 & x \notin A \cup B \end{cases} = \begin{cases} 1 & x \in A \cup B \\ 0 & x \in A^c \cap B^c \end{cases} = \begin{cases} 1 & x \in A \text{ or } x \in B \\ 0 & x \notin A \text{ and } x \notin B \end{cases}$$

Then for any $x \in \mathbb{R}$, we have

$\chi_A(x)$	$\chi_B(x)$	$\chi_{A\cup B}(x)$	$\chi_A(x)+\chi_B(x)-\chi_A(x)\cdot\chi_B(x)$
0	0	0	$0+0-0\cdot 0=0$
0	1	1	$0+1-0\cdot 1=1$
1	0	1	$1+0-1\cdot 0=1$
1	1	1	$1+1-1\cdot 1=1$

Finally see that

$$\chi_{A^c}=\begin{cases}1 & x\in A^c\\ 0 & x\notin A^c\end{cases}=\begin{cases}1 & x\notin A\\ 0 & x\in A\end{cases}=\begin{cases}1-0 & x\notin A\\ 1-1 & x\in A\end{cases}=1-\begin{cases}0 & x\notin A\\ 1 & x\in A\end{cases}=1-\chi_A.$$

That is, for any $x\in\mathbb{R}$, we have

$\chi_A(x)$	$\chi_{A^c}(x)$	1-$\chi_A(x)$
1	0	$1-1=0$
0	1	$1-0=1$

21. For a sequence $\{f_n\}$ of measurable functions with common domain E, show that each of the following functions is measurable:
(See Chapter 18 Problem 11)

   - $\inf\{f_n\}$

     We have
     $$\inf\{f_n\}=\{x\in E \mid \inf\{f_n\}>c\}=\bigcap_{n=1}^{\infty}\{x\in E\mid f_n(x)>c\}$$
     or
     $$\inf\{f_n\}=\{x\in E \mid \inf\{f_n\}<c\}=\bigcup_{n=1}^{\infty}\{x\in E\mid f_n(x)<c\}$$

   - $\sup\{f_n\}$

     Similarly,
     $$\sup\{f_n\}=\{x\in E \mid \sup\{f_n\}>c\}=\bigcup_{n=1}^{\infty}\{x\in E\mid f_n(x)>c\}$$
     or
     $$\sup\{f_n\}=\{x\in E \mid \sup\{f_n\}<c\}=\bigcap_{n=1}^{\infty}\{x\in E\mid f_n(x)<c\}$$

   - $\liminf\{f_n\}$
   - $\limsup\{f_n\}$

22. (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a,b]$ which converges pointwise on $[a,b]$ to the continuous function f on $[a,b]$. Show that the convergence is uniform on $[a,b]$. (Hint: let $\epsilon>0$. For each natural number n, define $E_n=\{x\in[a,b]\mid f(x)-f_n(x)<\epsilon\}$. Show that $\{E_n\}$ is an open cover of $[a,b]$ and use the Heine-Borel Theorem.)

Let $\epsilon > 0$. For each natural number n, define

$$\begin{aligned} E_n &= \{x \in [a,b] \mid f(x) - f_n(x) < \epsilon\} \\ &= \{x \in [a,b] \mid f(x) - f_n(x) \in (-\infty, \epsilon)\} \\ &= (f - f_n)^{-1}((-\infty, \epsilon)). \end{aligned}$$

The sum and product of continuous functions is continuous, so the function $f - f_n$ is continuous and therefore $E_n = (f - f_n)^{-1}((-\infty, \epsilon))$ is an open set.
Because $\{f_n\}$ converges pointwise to f on $[a,b]$, for any $\epsilon > 0$, for any $x \in [a,b]$, there exists an index $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, we have $|f(x) - f_n(x)| < \epsilon$.
This means that for any $x \in [a,b]$, there exists an index $N_x \in \mathbb{N}$ such that $x \in \{x \in [a,b] \mid f(x) - f_{N_x}(x) < \epsilon\} = E_{N_x}$, and so $x \in \bigcup_{n\in\mathbb{N}} E_n$, which implies

$$\forall x \in [a,b], \exists N_x \in \mathbb{N} \text{ s.t. } x \in \{x \in [a,b] \mid |f(x) - f_{N_x}(x)| < \epsilon\} = E_{N_x} \implies x \in \bigcup_{n\in\mathbb{N}} E_n,$$

and by definition of subset, $[a,b] \subseteq \bigcup_{n\in\mathbb{N}} E_n$. Now, $\{E_n\}$ is an open cover of $[a,b]$ because it is a union of open sets E_n and it covers $[a,b]$. Because $[a,b]$ is compact, there exists a finite subcover $\{E_{n_k}\}_{k=1}^m \subseteq \{E_n\}$.
This means that for any $x \in [a,b]$, there exists the index $k \in \{1, \cdots, m\}$ such that $x \in E_{n_k} = \{x \in [a,b] \mid |f(x) - f_{n_k}(x)| < \epsilon\}$.
Then we can let $N_0 = \max\{n_1, \cdots, n_m\}$.
Therefore for any $\epsilon > 0$, there exists the index N_0 such that for all $n \geq N_0 \geq n_i$, $i \in \{1, \cdots, m\}$,

$$|f(x) - f_n(x)| < \epsilon \text{ for all } x \in [a,b].$$

Thus we have uniform convergence.

23. Express a measurable function as the difference of nonnegative measurable functions and thereby prove the general Simple Approximation Theorem based on the special case of a nonnegative measurable function.

Let f be a measurable function on E, and we have $f = f^+ - f^-$ on E, a linear combination of the two nonnegative measurable functions $f^+ = \max\{f, 0\} \geq 0$ and $f^- = \max\{-f, 0\} \geq 0$. In our proof for the Simple Approximation Theorem, we proved the case for $f^+ \geq 0$ and $f^- \geq 0$ that there exist sequences $\{\varphi_n^+\}$ and $\{\varphi_n^-\}$ of simple functions on E that converge pointwise on E to f^+ and f^- respectively, and

$$\begin{aligned} 0 \leq \varphi_n^+ \leq f^+ \text{ on } E \text{ for all } n, \\ 0 \leq \varphi_n^- \leq f^- \text{ on } E \text{ for all } n. \end{aligned}$$

then

$$0 \geq -\varphi_n^- \geq -f^- \text{ on } E \text{ for all } n,$$

so that $-f^- \leq -\varphi_n^- \leq f^- \implies -\varphi_n^- \leq |f^-|$.
We have the sets $E^+ = \{x \in E \mid f(x) \geq 0\}$ and $E^- = \{x \in E \mid f(x) \leq 0\}$, so that

$$f(x) = |f(x)| = \begin{cases} 0 & \text{if } x \in E^+ \cap E^- \\ f^+(x) & \text{if } x \in E^+ \text{ only} \\ f^-(x) & \text{if } x \in E^- \text{ only} \end{cases}$$

The function $\varphi_n^+ - \varphi_n^-$ is simple and

$$0 \le \varphi_n^+ \le f^+ = 0, 0 \le \varphi_n^- \le f^- = 0 \text{ on } E^+ \cap E^- \implies \varphi_n^+, \varphi_n^- = 0,$$

and so we have

$$(\varphi_n^+ - \varphi_n^-)(x) = \begin{cases} 0 \le f(x) & \text{if } x \in E^+ \cap E^- \\ \varphi_n^+(x) \le f^+(x) & \text{if } x \in E^+ \text{ only} \\ -\varphi_n^-(x) \le f^-(x) & \text{if } x \in E^- \text{ only} \end{cases}$$

Then clearly $\varphi_n^+ - \varphi_n^-$ converges pointwise to f on E, and therefore we have

$$|\varphi_n^+ - \varphi_n^-| \le |f| \text{ on } E \text{ for all } n.$$

24. Let I be an interval and $f : I \to \mathbb{R}$ be increasing. Show that f is measurable by first showing that, for each natural number n, the strictly increasing function $x \mapsto f(x) + x/n$ is measurable, and then taking pointwise limits.

Let $f_n(x) = f(x) + x/n$.
Then each f_n is strictly increasing; that is, for $x, y \in I$, we have $f_n(x) < f_n(y) \iff x < y$. This also tells us that each f_n is injective because

$$x \neq y \implies x < y \text{ or } y < x \implies f_n(x) < f_n(y) \text{ or } f_n(y) < f_n(x) \implies f_n(x) \neq f_n(y).$$

So for any element $x^* \in dom(f_n)$, we know that $f_n^{-1}(x^*)$ consists of a single element at most.
Now, for any two $x^*, y^* \in (a, \infty)$, by definition of interval, any point z^* between them is also in (a, ∞).
We then have $x^* < z^* < y^* \iff f_n^{-1}(x^*) < f_n^{-1}(z^*) < f_n^{-1}(y^*)$, and so the set $f_n^{-1}((a, \infty)) = \{x \in I \mid f_n(x) \in (a, \infty)\}$ is an interval for all a, and every interval is measurable, so f_n is measurable.
To see that $\{f_n\}$ converges pointwise to f, let $\epsilon > 0$ and consider $x \in I$. Then there exists an index N such that for all $n \ge N$,

$$|f_n(x) - f(x)| = |x/n| < \epsilon.$$

Now, because $\{f_n\}$ is a sequence of measurable functions in I that converges pointwise a.e. on I to f, then by Proposition 9, f is measurable.

3.3 Littlewood's Three Principles, Egoroff's Theorem, and Lusin's Theorem

Egoroff's Theorem. *Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in E for which*

$$\{f_n\} \to f \text{ uniformly on } F \text{ and } m(E \setminus F) < \epsilon.$$

Lemma 10. *Under the assumptions of Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$, there is a measurable subset A of E and an index N for which*

$$|f_n - f| < \eta \text{ on } A \text{ for all } n \ge N \text{ and } m(E \setminus A) < \delta.$$

Proof. First, we can see that because $\{f_n\}$ is a sequence of measurable functions on E that converges pointwise on E to f, by Proposition 9, f is measurable. Then by Theorem 6, the function $f_n - f$ is measurable. Finally by Proposition 7, considering the continuous function $|\cdot|$ and the measurable function $f_n - f$, the composition $|f_n - f|$ is measurable.
This means that the set $\{x \in E \mid |f_n - f| < \eta\}$ is measurable.
Then we see that

$$E_n = \{x \in E \mid |f_k - f| < \eta \text{ for all } k \geq n\} = \bigcap_{k=n}^{\infty} \{x \in E \mid |f_k - f| < \eta\},$$

is also measurable.
Then $\{E_n\}$ is an ascending sequence of measurable sets because

$$E_n = \{x \in E \mid |f_n - f| < \eta\} \cap \left[\bigcap_{k=n+1}^{\infty} \{x \in E \mid |f_k - f| < \eta\}\right] \subseteq \bigcap_{k=n+1}^{\infty} \{x \in E \mid |f_k - f| < \eta\} = E_{n+1}.$$

Also, $E = \bigcup_{n=1}^{\infty} E_n$, because $\{f_n\}$ converges pointwise to f on E.
That is, for $\eta > 0$, for $x \in E$, there exists an index $N \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \eta \text{ for all } k \geq N,$$

and thus $x \in E_N$.
Now, by continuity of measure, we have

$$m(E) = m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} m(E_n).$$

Then because $m(E) < \infty$, there exists an index N_0 for which $m(E) - m(E_{N_0}) < \delta$.
Define $A = E_{N_0}$ so we can use excision to see that

$$m(E \setminus A) = m(E) - m(E_{N_0}) < \delta,$$

and

$$A = \{x \in E \mid |f_k - f| < \eta \text{ for all } k \geq N_0\}.$$

□

Proof. To prove Egoroff's Theorem:
Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f.
For each natural number n, we can let $\eta = 1/n$ and $\delta = \epsilon/2^{n+1}$.
By Lemma 10, there exists a subset A_n of E and an index N_n for which

$$|f_k - f| < 1/n \text{ on } A_n \text{ for all } k \geq N_n \text{ and } m(E \setminus A_n) < \epsilon/2^{n+1}.$$

We define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then we see that

$$\begin{aligned} m(E \setminus A) &= m(E \setminus \left[\bigcap_{n=1}^{\infty} A_n\right]) \\ &= m(E \cap \left[\bigcup_{n=1}^{\infty} A_n^c\right]) \\ &= m(\bigcup_{n=1}^{\infty} [E \cap A_n^c]) \\ &\leq \sum_{n=1}^{\infty} m(E \cap A_n^c) \\ &< \sum_{n=1}^{\infty} \epsilon/2^{n+1} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \epsilon/2^n \\ &= \epsilon/2. \end{aligned}$$

To see that $\{f_n\}$ converges uniformly to f on A:
Let $\epsilon > 0$. Then there exists an index n_0 such that $1/n_0 < \epsilon$ and

$$|f_k - f| < 1/n_0 < \epsilon \text{ on } A \subseteq A_{n_0} \text{ for all } k \geq N_{n_0}.$$

Finally we can use Chapter 2 Theorem 11 to choose a closed set F contained in A for which $m(A \setminus F) < \epsilon/2$. Then we have

$$\begin{aligned} m(E \setminus F) &= m(E) - m(F) && \text{by excision} \\ &= m(E) - m(A) + m(A) - m(F) \\ &= m(E \setminus A) + m(A \setminus F) && \text{by excision} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

Therefore $\{f_n\} \to f$ uniformly on F and $m(E \setminus F) < \epsilon$. $\square$

Proposition 11. *Let f be a simple function defined on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \epsilon.$$

Lusin's Theorem. *Let f be a real-valued measurable function on E. Then for each $\epsilon > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which*

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \epsilon.$$

Proof. Consider the case that $m(E) < \infty$.
The Simple Approximation Theorem tells us that there exists a sequence of simple functions $\{f_n\}$ defined on E that converges to f pointwise on E.

By the previous Proposition 11, for each n, there exists a continuous function g_n on $\mathbb{R}$ and a closed set F_n contained in E for which

$$f_n = g_n \text{ on } F_n \text{ and } m(E \setminus F_n) < \epsilon/2^{n+1}.$$

Also, Egoroff's Theorem states that because E has finite measure and that $\{f_n\}$ is a sequence of simple (measurable) function on E that converges pointwise on E to the real-valued function f, we have that there exists a closed set F_0 contained in E such that

$$\{f_n\} \to f \text{ uniformly on } F_0 \text{ and } m(E \setminus F_0) < \epsilon/2.$$

We define $F = \bigcap_{n=0}^{\infty} F_n$, which is closed because it is an intersection of closed sets. Then we see that

$$\begin{aligned}
m(E \setminus F) &= m(E \cap \left[\bigcup_{n=0}^{\infty} F_n^c \right]) \\
&= m(\bigcup_{n=0}^{\infty} [E \cap F_n^c]) \\
&= m([E \cap F_0^c] \cup \bigcup_{n=1}^{\infty} [E \cap F_n^c]) \\
&= m([E \setminus F_0] \cup \bigcup_{n=1}^{\infty} [E \setminus F_n]) \\
&\leq m([E \setminus F_0]) + \sum_{n=1}^{\infty} m(E \setminus F_n) \\
&< \epsilon/2 + \sum_{n=1}^{\infty} \epsilon/2^{n+1} \\
&= \epsilon.
\end{aligned}$$

Each f_n is continuous on F since $F \subseteq F_n$ and $f_n = g_n$ on F_n.
Also, $\{f_n\}$ converges to f uniformly on F since $F \subseteq F_0$.
Then we can use the fact that because $\{f_n\}$ is a sequence of continuous functions on F that converges uniformly on F to f, then f is continuous on F as well.
Finally see problem 25 to see that there exists a continuous function g that extends f to all of $\mathbb{R}$. Then $f = g$ on F and $m(E \setminus F) < \epsilon$. $\square$

PROBLEMS

25. Suppose f is a function that is continuous on a closed set F of real numbers. Show that f has a continuous extension to all of $\mathbb{R}$. This is a special case of the forthcoming Tietze Extension Theorem. (Hint: express $\mathbb{R} \setminus F$ as the union of a countable disjoint collection of open intervals and define f to be linear on the closure of each of these intervals.)

    (See Chapter 1 Problem 47.)
    Let f be a function that is continuous on the closed set F. Consider the open set F^c. By Chapter 1 Proposition 9, this open F^c is the union of a countable, disjoint collection of intervals.
    In the case that $(-\infty, a)$ [or (a, ∞)] is in F^c, then $a \in F$ and $f(a)$ is defined. Simply let

$f(x) = f(a)$ be the constant function on $(-\infty, a)$ [or (a, ∞)].
In the case that $(a, b) \in F^c$, then $a, b \in F$ and $f(a)$,$f(b)$ are defined. Let

$$f(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ on } (a, b).$$

Then we see that the extension of f is continuous on $\mathbb{R}$.

26. For the function f and the set F in the statement of Lusin's Theorem, show that the restriction of f to F is a continuous function. Must there be any points at which f, considered as a function of E, is continuous?

    See the proof for Lusin's Theorem; because $\{f_n\}$ is a sequence of continuous functions on F that converges uniformly on F to f, then f is continuous on F as well.

27. Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.

    Going back to the proof for Egoroff's Theorem, we see that we used the excision property:

$$\begin{aligned} m(E \setminus F) &= m(E) - m(F) && \text{by excision} \\ &= m(E) - m(A) + m(A) - m(F) \\ &= m(E \setminus A) + m(A \setminus F) && \text{by excision} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

    The excision property requires that

$$\begin{aligned} m(E \setminus F) &= m(E) - m(F) && \text{if } m(F) < \infty \text{ and } F \subseteq E \\ m(E \setminus A) &= m(E) - m(A) && \text{if } m(A) < \infty \text{ and } A \subseteq E \\ m(A \setminus F) &= m(A) - m(F) && \text{if } m(F) < \infty \text{ and } F \subseteq A \end{aligned}$$

    Specifically, we needed that $m(A), m(F) < \infty$. This was only possible because we assumed $m(E) < \infty$.

28. Show that Egoroff's Theorem continues to hold if the convergence is pointwise a.e. and f is finite a.e.

    In Lemma 10 on the way to proving Egoroff's Theorem, we used Proposition 9, which only requires the convergence to be pointwise a.e.
    In Lemma 10 we also used Theorem 6, which only requires f_n and f to be finite a.e.

29. Prove the extension of Lusin's Theorem to the case that E has infinite measure.

    We needed to assume that E had finite measure because we used Egoroff's Theorem in the proof for Lusin's Theorem, which requires finite measure (Problem 27).

30. Prove the extension of Lusin's Theorem to the case that f is not necessarily real-valued, but is finite a.e.

    We needed to assume that f was real valued because we used Egoroff's Theorem in the proof for Lusin's Theorem, which requires f to be real-valued. However, we showed that Egoroff's Theorem continues to hold if f is finite a.e. (Problem 28).

31. Let $\{f_n\}$ be a sequence of measurable functions on E that converges to the real-valued f pointwise on E. Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each index k, E_k is measurable, and $\{f_n\}$ converges uniformly to f on each E_k if $k > 1$, and $m(E_1) = 0$.

    Use Egoroff's Theorem.

Chapter 4

Lebesgue Integration

Contents

4.1 The Riemann Integral

In this chapter the Lebesgue integral is defined in four stages:

0. (define the Riemann integral for bounded functions on a closed, bounded interval):

   For a bounded real-valued function f defined on the closed, bounded interval $[a,b]$, define the Riemann integral of f over $[a,b]$ by

   $$(R)\int_a^b f = \sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ step}, \varphi \leq f \text{ on } [a,b]\right\} = \inf\left\{(R)\int_a^b \psi \mid \psi \text{ step}, \psi \geq f \text{ on } [a,b]\right\},$$

   where the Riemann integral of a step function is defined as

   $$(R)\int_E \psi = \sum_{k=1}^n c_k \cdot (c_k - c_{k-1}).$$

1. define the (Lebesgue) integral for simple functions over a set of finite measure:

   For a (measurable) simple function ψ defined on a set of finite measure E, we define the integral of ψ over E by

   $$\int_E \psi = \sum_{i=1}^n a_i \cdot m(E_i).$$

2. define the (Lebesgue) integral for bounded measurable functions f over a set of finite measure, in terms of integrals of upper and lower approximations of f by simple functions.

   For a bounded measurable function f defined on a set of finite measure E, we define the integral of f over E by

$$\int_E f = \sup\left\{\int_E \varphi \mid \varphi \text{ simple}, \varphi \leq f \text{ on } E\right\} = \inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f \text{ on } E\right\}.$$

3. define the (Lebesgue) integral of a general nonnegative measurable function f over E to be the supremum of the integrals of lower approximations of f by bounded measurable functions that vanish outside a set of finite measure; the integral of such a function is nonnegative, but may be infinite.

   For a nonnegative measurable function f on E, we define the integral of f over E by

$$\int_E f = \sup\left\{\int_E h \mid h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E\right\}.$$

4. define a general measurable function to be (Lebesgue) integrable over E provided $\int_E |f| < \infty$.

The Construction of the Riemann integral:

Let f be a bounded real-valued function defined on the closed, bounded interval $[a, b]$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Define the **lower and upper Darboux sums** for f with respect to P, respectively, by

$$L(f, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}),$$

where, for $1 \leq i \leq n$,

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\} \text{ and } M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

We then define the **lower and upper Riemann integrals** of f over $[a, b]$, respectively, by

$$(R)\underline{\int_a^b} f = \sup\left\{L(f, P) \mid P \text{ a partition of } [a, b]\right\}$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{U(f, P) \mid P \text{ a partition of } [a, b]\right\}.$$

If the upper and lower integrals are equal we say that f is **Riemann integrable** over $[a, b]$ and call this common value the Riemann integral of f over $[a, b]$:

$$(R)\int_a^b f$$

A real-valued function ψ defined on $[a, b]$ is called a **step function** provided there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and numbers $c_1, \cdots, c_n$ such that for $1 \le i \le n$,

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i.$$

Clearly a step function is Riemann integrable:

$$\sum_{i=1}^{n} c_i \cdot (x_i - x_{i-1}) = L(\psi, P) = U(\psi, P) = (R)\int_a^b \psi$$

Then we can reformulate the definition of the lower and upper Riemann integrals:

$$(R)\underline{\int_a^b} f = \sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ a step function and } \varphi \le f \text{ on } [a, b]\right\}$$

and

$$(R)\overline{\int_a^b} f = \inf\left\{(R)\int_a^b \psi \mid \psi \text{ a step function and } \varphi \ge f \text{ on } [a, b]\right\}.$$

Example (Dirichlet's Function) Define $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Let P be any partition of $[0, 1]$. By the density of the rationals and the irrationals, for any open interval (x_{i-1}, x_i) generated by P, there exists both a rational r and an irrational s so that $f(r) = 1$ and $f(s) = 0$ and so $m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\} \le f(r), f(s) \le \sup\{f(x) \mid x_{i-1} < x < x_i\} = M_i$, thus

$$L(f, P) = 0 \text{ and } U(f, P) = 1.$$

Therefore

$$(R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f,$$

so f is not Riemann integrable.

Consider the enumeration of the rationals in $[0, 1]$: $\{q_k\}_{k=1}^{\infty}$. We can define a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ in the following way:

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \cdots, q_n\} \\ 0 & \text{else} \end{cases}$$

Each f_n is a step function (See any partition of the form $0 = q_1 < \cdots < q_n < \cdots < q_2 = 1$) and thus is Riemann integrable, and $\{f_n\}$ is an increasing sequence of Riemann integrable functions on $[0, 1]$,

$$|f_n| \le 1 \text{ on } [0, 1] \text{ for all } n,$$

and

$$\{f_n\} \to f \text{ pointwise on } [0,1].$$

To see this, let $\epsilon > 0$ and let $x \in [0,1]$. Then x is rational or irrational.
If x is irrational, then $|f_n(x) - f(x)| = |0-0| = 0 < \epsilon$.
If x is rational, there exists an index N such that $x = q_N$ and for all $n \geq N$, we have
$|f_n(q_N) - f(q_N)| = |1-1| = 0 < \epsilon$.
Thus we have an increasing sequence of Riemann integrable functions on $[0,1]$ that converges pointwise to a function that is not Riemann integrable.

PROBLEMS

1. Show that, in the above Dirichlet function example, $\{f_n\}$ fails to converge to f uniformly on $[0,1]$.

   Let $\epsilon = 1/2$. Then for any natural number n we choose, there exists $n+1 \geq n$ and $q_{n+1} \in [0,1] \cap \mathbb{Q}$ such that $|f_n(q_{n+1}) - f(q_{n+1})| = |0-1| = 1 > 1/2$. Therefore uniform convergence fails.

2. A partition P' of $[a,b]$ is called a refinement of a partition P provided each partition point of P is also a partition point of P'. For a bounded function f on $[a,b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

   (Ex: the partition $P' = \{a,b,c\}$ is a refinement of $P = \{a,c\}$.)
   Let $P = \{x_0, \cdots, x_n\}$ be any partition. Consider P' to be a refinement of P (and suppose $P' \neq P$). Then for some $k \in \{1, \cdots, n\}$, there exists a point $y \in P'$ such that $x_{k-1} < y < x_k$. Now, we have

$$\{f(x) \mid x_{k-1} < x < x_k\} \supseteq \{f(x) \mid x_{k-1} < x < y\}$$
$$\{f(x) \mid x_{k-1} < x < x_k\} \supseteq \{f(x) \mid y < x < x_k\}$$

   so that

$$m_k = \inf\{f(x) \mid x_{k-1} < x < x_k\} \leq \inf\{f(x) \mid x_{k-1} < x < y\} := m_k^l$$
$$m_k = \inf\{f(x) \mid x_{k-1} < x < x_k\} \leq \inf\{f(x) \mid y < x < x_k\} := m_k^r$$

   and

$$M_k = \sup\{f(x) \mid x_{k-1} < x < x_k\} \geq \sup\{f(x) \mid x_{k-1} < x < y\} := M_k^l$$
$$M_k = \sup\{f(x) \mid x_{k-1} < x < x_k\} \geq \sup\{f(x) \mid x_{k-1} < x < y\} := M_k^r$$

   and finally recall that the lower and upper Darboux sums with respect to P are defined

$$L(f,P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$
$$U(f,P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1})$$

   so that at the index k, we have

$$m_k \cdot (x_k - x_{k-1}) = m_k \cdot (y - x_{k-1}) + m_k \cdot (x_k - y) \leq m_k^l \cdot (y - x_{k-1}) + m_k^r \cdot (x_k - y)$$
$$M_k \cdot (x_k - x_{k-1}) = M_k \cdot (y - x_{k-1}) + M_k \cdot (x_k - y) \geq M_k^l \cdot (y - x_{k-1}) + M_k^r \cdot (x_k - y)$$

and then clearly the lower and upper Darboux sums $L(f,P'), U(f,P')$ with respect to P' are such that

$$L(f,P) \leq L(f,P'),$$
$$U(f,P) \geq U(f,P').$$

That is, the lower Darboux sum of any refinement is an increase, and the upper Darboux sum of any refinement is a decrease.

3. Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.

Let f be a bounded function on a closed, bounded interval $[a,b]$.
Let $P = \{x_0, \cdots, x_n\}$ be any partition of $[a,b]$.
Then for all $k \in \{1, \cdots, n\}$,

$$m_k = \inf\{f(x) \mid x_{k-1} < x < x_k\} \leq \sup\{f(x) \mid x_{k-1} < x < x_k\} = M_k,$$

and therefore

$$L(f,P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) \leq \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) = U(f,P). \tag{1}$$

Then we show that the following holds:

$$(R)\underline{\int_a^b} f = \sup\Big\{L(f,P) \mid P \text{ a partition of } [a,b]\Big\} \leq \inf\Big\{U(f,P) \mid P \text{ a partition of } [a,b]\Big\} = (R)\overline{\int_a^b} f$$

Suppose by contradiction that there exists partitions P, B such that

$$\sup\Big\{L(f,P) \mid P \text{ a partition of } [a,b]\Big\} \geq L(f,P) > U(f,B) \geq \inf\Big\{U(f,P) \mid P \text{ a partition of } [a,b]\Big\}$$

Then $P \cup B$ is a refinement of both P and B, and so by the preceding Problem 2,

$$L(f, P\cup B) \geq L(f,P) > U(f,B) \geq U(f, P\cup B).$$

Furthermore, by (1),

$$U(f,P\cup B) \geq L(f,P\cup B) \geq L(f,P) > U(f,B) \geq U(f,P\cup B),$$

and we reach a contradiction.

4. Suppose the bounded function f on $[a,b]$ is Riemann integrable over $[a,b]$. Show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ for which $\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$.

Because f is Riemann integrable, we have

$$(R)\underline{\int_a^b} f = \sup\Big\{L(f,P) \mid P \text{ a partition of } [a,b]\Big\} = \inf\Big\{U(f,P) \mid P \text{ a partition of } [a,b]\Big\} = (R)\overline{\int_a^b} f.$$

For each natural number n, let $\epsilon = 1/2n$ so that, by definition of supremum and infimum, there exists partitions P_n and B_n such that

$$\left[(R)\underline{\int_a^b} f\right] - 1/2n < L(f, P_n) \leq \left[(R)\underline{\int_a^b} f\right] = (R)\int_a^b f = \left[(R)\overline{\int_a^b} f\right] \leq U(f, B_n) < \left[(R)\overline{\int_a^b} f\right] + 1/2n.$$

Furthermore, because $P_n \cup B_n$ is a refinement of both P_n and B_n, we have

$$\left[(R)\int_a^b f\right] - 1/2n < L(f, P_n) \leq L(f, P_n \cup B_n) \leq (R)\int_a^b f \leq U(f, P_n \cup B_n) \leq U(f, B_n) < \left[(R)\int_a^b f\right] + 1/2n$$

and thus for each n, we have $U(f, P_n \cup B_n) - L(f, P_n \cup B_n) < 1/n$.
Therefore for the sequence $\{P_n \cup B_n\}$ of partitions of $[a, b]$, for any ϵ, there exists an index N such that for all $n \geq N$, then $U(f, P_n \cup B_n) - L(f, P_n \cup B_n) < 1/n \leq 1/N < \epsilon$.

5. Let f be a bounded function on $[a, b]$. Suppose there is a sequence $\{P_n\}$ of partitions of $[a, b]$ for which $\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0$. Show that f is Riemann integrable over $[a, b]$.

   We say that there exists a sequence $\{P_n\}$ of partitions of $[a, b]$ such that for any ϵ, there exists an index N such that for all $n \geq N$, then $U(f, P_n) - L(f, P_n) < \epsilon$.
   In Problem 3 we showed that

   $$(R)\underline{\int_a^b} f = \sup\Big\{L(f, P) \mid P \text{ a partition of } [a, b]\Big\} \leq \inf\Big\{U(f, P) \mid P \text{ a partition of } [a, b]\Big\} = (R)\overline{\int_a^b} f,$$

   and so we have

   $$L(f, P_n) \leq (R)\underline{\int_a^b} f \leq (R)\overline{\int_a^b} f \leq U(f, P_n).$$

   Then for any ϵ, we have that $(R)\overline{\int_a^b} f - (R)\underline{\int_a^b} f < \epsilon$, and thus $(R)\overline{\int_a^b} f \leq (R)\underline{\int_a^b} f$ so that $(R)\underline{\int_a^b} f = (R)\overline{\int_a^b} f$ and f is Riemann integrable.

6. Use the preceding problem to show that since a continuous function f on a closed, bounded interval $[a, b]$ is uniformly continuous on $[a, b]$, it is Riemann integrable over $[a, b]$.

   (Review Chapter 1 Theorem 23 for the proof that a continuous function on a compact set is uniformly continuous.)
   Let f be a continuous function on $[a, b]$. Then f is uniformly continuous and bounded. That means that for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in [a, b]$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. Therefore for each ϵ, we can create a partition $P_\delta = \{x_0, \cdots, x_n\}$ of $[a, b]$ such that for any $k \in \{1, \cdots, n\}$, we have, for the interval (x_{k-1}, x_k), that $x_k - x_{k-1} < \delta$. Then for any $x, y \in (x_{k-1}, x_k)$, we have $|f(x) - f(y)| < \epsilon/n$, and therefore for each k,

   $$M_k - m_k = \sup\{f(x) \mid x_{k-1} < x < x_k\} - \inf\{f(x) \mid x_{k-1} < x < x_k\} < \epsilon/n,$$

   and thus

   $$U(f, P_\delta) - L(f, P_\delta) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) < \epsilon.$$

   This means that, for each natural number m, setting $\epsilon = 1/m$, we can construct a partition P_{δ_m} such that $U(f, P_{\delta_m}) - L(f, P_{\delta_m}) < 1/m$, and therefore $\lim_{m\to\infty}[U(f, P_{\delta_m}) - L(f, P_{\delta_m})] = 0$, and so by the preceding Problem 5, we have that f is Riemann integrable.

7. Let f be an increasing real-valued function on $[0,1]$. For a natural number n, define P_n to be the partition of $[0,1]$ into n subintervals of length $1/n$. Show that $U(f,P_n) - L(f,P_n) \leq 1/n[f(1) - f(0)]$. Use Problem 5 to show that f is Riemann integrable over $[0,1]$.

   Because f is real-valued and increasing on $[0,1]$, we know f is bounded and $0 \leq f(1) - f(0) < \infty$. Because f is increasing, then for each k, we have

   $$f(x_{k-1}) \leq \inf\{f(x) \mid x_{k-1} < x < x_k\} = m_k \leq M_k = \sup\{f(x) \mid x_{k-1} < x < x_k\} \leq f(x_k),$$

   so that

   $$M_k - m_k \leq f(x_k) - f(x_{k-1}).$$

   Then we see that

   $$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}) - \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i) \cdot (x_i - x_{i-1}) \\ &= \sum_{i=1}^{n} (M_i - m_i) \cdot (1/n) \\ &= 1/n \sum_{i=1}^{n} (M_i - m_i) \\ &\leq 1/n \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) \\ &= 1/n[f(x_n) - f(x_0)] \\ &= 1/n[f(1) - f(0)]. \end{aligned}$$

   Then because we just proved that for each natural number n, we have $U(f,P_n) - L(f,P_n) \leq 1/n[f(1) - f(0)]$, then $\lim_{n\to\infty}[U(f,P_n) - L(f,P_n)] = 0$, so by Problem 5, f is Riemann integrable.

8. Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to f on the closed, bounded interval $[a,b]$. If each f_n is Riemann integrable over $[a,b]$, show that f also is Riemann integrable over $[a,b]$. Is it true that

   $$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f?$$

   For any partition $P = \{x_0, \cdots, x_\ell\}$, we have

   $$U(f,P) - L(f,P) = U(f,P) - U(f_n,P) + U(f_n,P) - L(f_n,P) + L(f_n,P) - L(f,P).$$

   Uniform convergence means that for any $\epsilon > 0$, then there exists an index N such that for all $n \geq N$, then $|f(x) - f_n(x)| < \frac{\epsilon}{3\ell}$ for all $x \in [a,b]$.
   We see that for any $x \in [a,b]$,

   $$-\frac{\epsilon}{3\ell} < f(x) - f_n(x) < \frac{\epsilon}{3\ell}.$$

Then for any $k \in \{1, \cdots, \ell\}$, we see that

$$\begin{aligned} M_k = \sup\{f(x) \mid x_{k-1} < x < x_k\} &= \sup\{f(x) - f_n(x) + f_n(x) \mid x_{k-1} < x < x_k\} \\ &< \sup\{\frac{\epsilon}{3\ell} + f_n(x) \mid x_{k-1} < x < x_k\} \\ &= \frac{\epsilon}{3\ell} + \sup\{f_n(x) \mid x_{k-1} < x < x_k\} \\ &= \frac{\epsilon}{3\ell} + (M_n)_k \end{aligned}$$

and

$$\begin{aligned} m_k = \inf\{f(x) \mid x_{k-1} < x < x_k\} &= \inf\{f(x) - f_n(x) + f_n(x) \mid x_{k-1} < x < x_k\} \\ &> \inf\{-\frac{\epsilon}{3\ell} + f_n(x) \mid x_{k-1} < x < x_k\} \\ &= -\frac{\epsilon}{3\ell} + \inf\{f_n(x) \mid x_{k-1} < x < x_k\} \\ &= -\frac{\epsilon}{3\ell} + (m_n)_k \end{aligned}$$

Then the Darboux sums are such that

$$\begin{aligned} U(f,P) = \sum_{k=1}^{\ell} M_k(x_k - x_{k-1}) &< \sum_{k=1}^{\ell} \frac{\epsilon}{3\ell} + \sum_{k=1}^{\ell} (M_n)_k(x_k - x_{k-1}) \\ &= \frac{\epsilon}{3} + U(f_n, P) \end{aligned}$$

and

$$\begin{aligned} L(f,P) = \sum_{k=1}^{\ell} m_k(x_k - x_{k-1}) &> -\sum_{k=1}^{\ell} \frac{\epsilon}{3\ell} + \sum_{k=1}^{\ell} (m_n)_k(x_k - x_{k-1}) \\ &= -\frac{\epsilon}{3} + L(f_n, P) \end{aligned}$$

Therefore we have

$$\begin{aligned} U(f,P) - L(f,P) &= U(f,P) - U(f_n,P) + U(f_n,P) - L(f_n,P) + L(f_n,P) - L(f,P) \\ &= [U(f,P) - U(f_n,P)] + [U(f_n,P) - L(f_n,P)] + [L(f_n,P) - L(f,P)] \\ &< \frac{\epsilon}{3} + [U(f_n,P) - L(f_n,P)] + \frac{\epsilon}{3}. \end{aligned}$$

Because each f_n is bounded and Riemann integrable over $[a,b]$, then by Problem 4, we proved there is a sequence $\{P_m\}_{m=1}^{\infty}$ of partitions of $[a,b]$ s.t. $\lim_{m\to\infty}[U(f_n,P_m) - L(f_n,P_m)] = 0$. Therefore for any $\epsilon > 0$, there exists an index N such that $U(f,P_m) - U(f_n,P_m) < \frac{\epsilon}{3}$ for $m \geq N$.

$$\begin{aligned} U(f,P_m) - L(f,P_m) &< \frac{\epsilon}{3} + [U(f_n,P_m) - L(f_n,P_m)] + \frac{\epsilon}{3} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Thus because there is a sequence $\{P_m\}_{m=1}^{\infty}$ of partitions of $[a,b]$ s.t. $\lim_{m\to\infty}[U(f,P_m) - L(f,P_m)] = 0$, by Problem 5, f is Riemann integrable.
Also, yes, it is true that $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

4.2 The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

Recall the definition of a simple function: If ψ is simple, has domain E and takes the distinct values $a_1, \cdots, a_n$, then

$$\psi = \sum_{k=1}^{n} a_i \cdot \chi_{E_i} \text{ on } E, \text{ where } E_i = \psi^{-1}(a_i) = \{x \in E \mid \psi(x) = a_i\}. \tag{1}$$

The canonical representation is characterized by the E_i's being disjoint and the a_i's being distinct.

Definition. *For a simple function ψ defined on a set of finite measure E, we define the integral of ψ over E by*

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot m(E_i),$$

where ψ has the canonical representation given by (1).

Let f be a bounded real-valued function defined on a set of finite measure E. We define the **lower and upper Lebesgue integral**, respectively, of f over E to begin

$$\sup\left\{\int_E \varphi \mid \varphi \text{ simple and } \varphi \leq f \text{ on } E\right\},$$

and

$$\inf\left\{\int_E \psi \mid \psi \text{ simple and } \varphi \geq f \text{ on } E\right\}.$$

Definition. *A bounded function f on a domain E of finite measure is said to be **Lebesgue measurable** over E provided its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the **Lebesgue integral**, or simply the integral, of f over E and is denoted by $\int_E f$.*

Theorem 3. *Let f be a bounded function defined on the closed, bounded interval $[a, b]$. If f is Riemann integrable over $[a, b]$, then it is Lebesgue integrable over $[a, b]$ and the two integrals are equal.*

Proof. Saying that f is Riemann integrable means that

$$(R)\int_E f = \sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ step and } \varphi \leq f \text{ on } [a,b]\right\} = \inf\left\{(R)\int_a^b \psi \mid \psi \text{ step and } \varphi \geq f \text{ on } [a,b]\right\}.$$

The Riemann integral over a closed, bounded interval of a step function agrees with the Lebesgue integral. Therefore because all step functions are simple functions we have

$$\left\{(R)\int_a^b \varphi \mid \varphi \text{ step and } \varphi \leq f \text{ on } [a,b]\right\} \subseteq \left\{\int_E \varphi \mid \varphi \text{ simple and } \varphi \leq f \text{ on } E\right\}$$
$$\left\{(R)\int_a^b \psi \mid \psi \text{ step and } \varphi \geq f \text{ on } [a,b]\right\} \subseteq \left\{\int_E \psi \mid \psi \text{ simple and } \varphi \geq f \text{ on } E\right\}$$

and thus

$$\sup\left\{(R)\int_a^b \varphi \mid \varphi \text{ step and } \varphi \leq f \text{ on } [a,b]\right\} \leq \sup\left\{\int_E \varphi \mid \varphi \text{ simple and } \varphi \leq f \text{ on } E\right\}$$
$$\inf\left\{(R)\int_a^b \psi \mid \psi \text{ step and } \varphi \geq f \text{ on } [a,b]\right\} \geq \inf\left\{\int_E \psi \mid \psi \text{ simple and } \varphi \geq f \text{ on } E\right\}$$

Thus we can write

$$
\begin{aligned}
(R)\int_E f &= \sup\Big\{(R)\int_a^b \varphi \mid \varphi \text{ step and } \varphi \le f \text{ on } [a,b]\Big\}\\
&\le \sup\Big\{\int_E \varphi \mid \varphi \text{ simple and } \varphi \le f \text{ on } E\Big\}\\
&\le \inf\Big\{\int_E \psi \mid \psi \text{ simple and } \varphi \ge f \text{ on } E\Big\}\\
&\le \inf\Big\{(R)\int_a^b \psi \mid \psi \text{ step and } \varphi \ge f \text{ on } [a,b]\Big\}\\
&= (R)\int_E f,
\end{aligned}
$$

and f is Lebesgue integrable with $\int_E f = (R)\int_E f$.

Example The Dirichlet function is a simple function because

$$
f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}
$$

so that

$$
f(x) = 1 \cdot \chi_{\mathbb{Q}\cap[0,1]} + 0 \cdot \chi_{\mathbb{Q}^c\cap[0,1]} = 1 \cdot \chi_{\mathbb{Q}\cap[0,1]} \text{ on } [0,1],
$$

where

$$
\begin{aligned}
\mathbb{Q}\cap[0,1] &= f^{-1}(1) = \{x \in [0,1] \mid f(x) = 1\},\\
\mathbb{Q}^c\cap[0,1] &= f^{-1}(0) = \{x \in [0,1] \mid f(x) = 0\}.
\end{aligned}
$$

Then f is Lebesgue integrable (but not Riemann integrable) with

$$
\int_{[0,1]} f = 1 \cdot m(\mathbb{Q}\cap[0,1]) + 0 \cdot m(\mathbb{Q}^c\cap[0,1]) = 1\cdot 0 + 0 \cdot 1 = 0.
$$

□

Theorem 4. *Let f be a bounded measurable function on a set of finite measure E. Then f is integrable over E.*

Proof. For each natural number n, by the Simple Approximation Lemma, for $\epsilon = 1/n > 0$, there are two simple functions φ_n and ψ_n on E for which

$$
\varphi_n \le f \le \psi_n \text{ and } 0 \le \psi_n - \varphi_n \le 1/n \text{ on } E.
$$

By the monotonicity of the integral for simple functions,

$$
0 \le \int_E [\psi_n - \varphi_n] \le 1/n \cdot m(E).
$$

Then by linearity of the integral for simple functions,

$$
0 \le \int_E \psi_n - \int_E \varphi_n \le 1/n \cdot m(E).
$$

Then

$$\begin{aligned} 0 &\leq \inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f\right\} - \sup\left\{\int_E \varphi \mid \varphi \text{ simple}, \varphi \leq f\right\} \\ &\leq \int_E \psi_n - \int_E \varphi_n \\ &\leq 1/n \cdot m(E). \end{aligned}$$

Then $\inf\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f\} = \sup\{\int_E \varphi \mid \varphi \text{ simple}, \varphi \geq f\}$ and thus f is integrable over E. □

...

Corollary 6. *Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

Proof. Both $f \cdot \chi_A$ and $f \cdot \chi_B$ are bounded measurable functions on E. Since A and B are disjoint, see Chapter 3 Problem 20 to see that

$$f \cdot \chi_{A\cup B} = f \cdot (\chi_A + \chi_B + \chi_{A\cap B}) = f \cdot (\chi_A + \chi_B + 0) = f \cdot \chi_A + f \cdot \chi_B.$$

Furthermore, for any measurable subset E_1 of E (see Problem 10),

$$\int_{E_1} f = \int_E f \cdot \chi_{E_1}.$$

Therefore, by linearity of integration,

$$\int_{A\cup B} f = \int_E f \cdot \chi_{A\cup B} = \int_E f \cdot \chi_A + \int_E f \cdot \chi_B = \int_A f + \int_B f.$$

□

Corollary 7. *Let f be a bounded measurable function on a set of finite measure E. Then*

$$\left|\int_E f\right| \leq \int_E |f|.$$

Proof. The function $|f|$ is measurable (see Chapter 3 Proposition 7) and bounded: $|f| \leq |f|$ on E, so that

$$-|f| \leq f \leq |f| \text{ on } E.$$

Therefore by linearity and monotonicity of integration,

$$-\int_E |f| \leq \int_E f \leq \int_E |f|,$$

and therefore $|\int_E f| \leq \int_E |f|$. □

Proposition 8. *Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E.*

$$\textit{If } \{f_n\} \to f \textit{ uniformly on } E, \textit{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since the convergence is uniform and each f_n is bounded, then the limit function f is bounded. The function f is measurable since it is the pointwise limit of a sequence of measurable functions (see Chapter 3 Proposition 9).
Let $\epsilon > 0$. By uniform convergence, there exists an index N such that

$$|f - f_n| < \frac{\epsilon}{m(E)} \text{ on } E \text{ for all } n \geq N.$$

Then

$$\begin{aligned}
|\int_E f - \int_E f_n| &= |\int_E [f - f_n]| && \text{linearity of integration} \\
&\leq \int_E |f - f_n| && \text{Corollary 7} \\
&< \int_E [\frac{\epsilon}{m(E)}] \cdot 1 && \text{monotonicity of integration} \\
&= [\frac{\epsilon}{m(E)}] \cdot m(E) \\
&= \epsilon.
\end{aligned}$$

Therefore $\lim_{n\to\infty} \int_E f_n = \int_E f$. □

Recall an example from Chapter 3.2:
Consider the sequence of continuous functions $\{f_n\}_{n=2}^{\infty} : [0,1] \to \mathbb{R}$, defined by

$$f_n(x) = \begin{cases} \frac{n-0}{1/n-0}x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{0-n}{2/n-1/n}(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases} = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ -n^2(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases}$$

(Each f_n is a triangle-shaped function that achieves its max $f(1/n) = n$ and base corners $f(0) = 0$ and $f(2/n) = 0$.)
In addition, consider the continuous function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in [0,1]$.

The sequence $\{f_n\}$ converges to f pointwise (a.e) but not uniformly on $[0,1]$.
Thus we have

$$\{f_n\} \to f \text{ pointwise on } [0,1], \text{ but } \lim_{n\to\infty} \int_0^1 f_n = 1 \neq 0 = \int_0^1 f.$$

Another example: we can define a sequence of nonnegative, measurable, continuous functions on $[0,1]$ by

$$f_n(x) = \begin{cases} -n^2 x + n & x \in [0, \frac{1}{n}] \\ 0 & x \in (\frac{1}{n}, 1] \end{cases}$$

Then $\{f_n\} \to f \equiv 0$ pointwise a.e. on $[0,1]$, but $\int_{[0,1]} f_n = 1/2$ for all n, and so

$$\lim_{n\to\infty} \int_{[0,1]} f_n = 1/2 \neq 0 = \int_{[0,1]} f.$$

The Bounded Convergence Theorem. *Let $\{f_n\}$ be a sequence of measurable functions on a set of finite measure E. Suppose $\{f_n\}$ is uniformly pointwise bounded on E; that is, there is a number $M \geq 0$ for which*

$$|f_n| \leq M \text{ on } E \text{ for all } n.$$

$$\textit{If } \{f_n\} \to f \textit{ pointwise on } E, \textit{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. The pointwise limit of a sequence of measurable functions is measurable (Chapter 3 Proposition 9). Therefore f is measurable. Also, clearly $|f| \leq M$ on E. Let A be any measurable subset of E and n a natural number.
Then

$$\begin{aligned}
\int_E f_n - \int_E f &= \int_E [f_n - f] \\
&= \int_A [f_n - f] + \int_{E \setminus A} [f_n - f] && \text{Corollary 6} \\
&= \int_A [f_n - f] + \int_{E \setminus A} f_n + \int_{E \setminus A} (-f) && \text{Linearity of integration} \\
&\leq \int_A [f_n - f] + \int_{E \setminus A} M + \int_{E \setminus A} M && \text{Monotonicity of integration: } |f_n|, |f| \leq M \\
&= \int_A [f_n - f] + 2M \cdot m(E \setminus A) && \text{Integral of constant functions}
\end{aligned}$$

And by the triangle inequality and Corollary 7,

$$\left| \int_E f_n - \int_E f \right| \leq | \int_A [f_n - f]| + |2M \cdot m(E \setminus A)| \leq \int_A |f_n - f| + 2M \cdot m(E \setminus A)$$

Now let $\epsilon > 0$.
Because E has finite measure and $\{f_n\}$ converges pointwise to f on E, then by Egoroff's Theorem, there is a measurable subset A of E for which $\{f_n\} \to f$ uniformly on A and $m(E \setminus A) < \epsilon/4M$. Then by uniform convergence, there is an index N for which

$$|f_n - f| < \frac{\epsilon}{2 \cdot m(E)} \text{ on } A \text{ for all } n \geq N.$$

Therefore, because $A \subseteq E \implies m(A) \leq m(E) < \infty \implies \frac{m(A)}{m(E)} \leq 1$, using monotonicity of integration,

$$\begin{aligned}
\left| \int_E f_n - \int_E f \right| &\leq \int_A |f_n - f| + 2M \cdot m(E \setminus A) \\
&< \int_A \frac{\epsilon}{2 \cdot m(E)} + 2M \cdot \epsilon/4M \\
&= \frac{\epsilon}{2 \cdot m(E)} m(A) + \epsilon/2 \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Therefore the sequence of integrals $\{\int_E f\}$ converges to $\int_E f$. □

Remark. *Prior to the proof of the Bounded Convergence Theorem, no use was made of the countable additivity of the Lebesgue measure on the real line. Only finite additivity was used, and it was used just once, in the proof of Lemma 1. But for the proof of the Bounded Convergence Theorem we used Egoroff's Theorem. Egoroff's Theorem needed the continuity of Lebesgue measure, a consequence of countable additivity of Lebesgue measure.*

PROBLEMS

9. Let E have measure zero. Show that if f is a bounded function on E, then f is measurable and $\int_E f = 0$.

   First, consider any simple function ψ defined on E, taking the values $a_1, \cdots, a_n$ on the subsets $E_1, \cdots, E_n$ of E.
   For any $i \in \{1, \cdots, n\}$, by monotonicity of Lebesgue measure, $m(E_i) \leq m(E) = 0 \implies m(E_i) = 0$. Therefore the integral of any simple function on E is zero:

$$\int_E \psi = \sum_{i=1}^{n} a_i \cdot m(E_i) = \sum_{i=1}^{n} a_i \cdot 0 = 0$$

   Thus for the bounded function f on the set of finite measure E, we have

$$\sup\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E\right\} = 0 = \inf\left\{\int_E \psi \mid \psi \text{ simple, } \psi \geq f \text{ on } E\right\},$$

   and f is Lebesgue integrable with $\int_E f = 0$.

   To see that f is measurable, consider any sequence of simple (measurable) functions $\{\varphi_n\}$ on E. This sequence trivially converges pointwise $a.e.$ on E to the function f because it converges pointwise on the set $\emptyset = E \setminus E$, where $E \subseteq E$ with $m(E) = 0$. Then by Chapter 3 Proposition 9, f is measurable.

   (A bounded function on a set of finite measure is Lebesgue integrable iff it is measurable;
   ($\implies$) Chapter 5 Theorem 7,
   ($\impliedby$) Chapter 4 Theorem 4)

10. Let f be a bounded measurable function on a set of finite measure E. For a measurable subset A of E, show that $\int_A f = \int_E f \cdot \chi_A$.

   For any simple function ψ on E, we have

$$\psi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k},$$

   Then for the measurable subset A of E, the restriction of ψ to A is measurable and so

$$\psi_{|A} = \sum_{k=1}^{n} c_k \cdot \chi_{E_k \cap A}, \text{ where } E_k \cap A = \{x \in E \mid \psi(x) = c_k\} \cap A = \{x \in A \mid \psi_{|A}(x) = c_k\}$$

   Also, we consider the measurable function χ_A on E, and clearly the product $\psi \cdot \chi_A$ is measurable (and simple) so that

$$\psi \cdot \chi_A = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \cdot \chi_A = \sum_{k=1}^{n} c_k \cdot \chi_{E_k \cap A},$$

   and therefore

$$\int_A \psi = \sum_{k=1}^{n} c_k \cdot m(E_k \cap A) = \int_E \psi \cdot \chi_A.$$

Now, to show $\int_A f = \int_E f \cdot \chi_A$ it is sufficient to see that $\int_A f \leq \int_E f \cdot \chi_A$ and $\int_A f \geq \int_E f \cdot \chi_A$.

For any simple function φ' on E such that $\varphi' \leq f$, we have $\varphi' \cdot \chi_A \leq f \cdot \chi_A$ on E, and so

$$\int_A \varphi' = \int_E \varphi' \cdot \chi_A \leq \sup\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \cdot \chi_A \text{ on } E\right\} = \int_E f \cdot \chi_A$$

Then the supremum of all such φ' shows that

$$\int_A f = \sup\left\{\int_A \varphi' \mid \varphi' \text{ simple, } \varphi' \leq f \text{ on } A\right\} \leq \int_E f \cdot \chi_A \tag{1}$$

Again, for any simple function ψ' on E such that $\psi' \geq f$, we have $\psi' \cdot \chi_A \geq f \cdot \chi_A$ on E, and so

$$\int_A \psi' = \int_E \psi' \cdot \chi_A \geq \inf\left\{\int_E \psi \mid \psi \text{ simple, } \psi \geq f \cdot \chi_A \text{ on } E\right\} = \int_E f \cdot \chi_A$$

Then the infimum of all such ψ' shows that

$$\int_A f = \inf\left\{\int_A \psi' \mid \psi' \text{ simple, } \psi' \geq f \text{ on } A\right\} \geq \int_E f \cdot \chi_A \tag{2}$$

Therefore by (1) and (2) we have $\int_A f = \int_E f \cdot \chi_A$.

11. Does the Bounded Convergence Theorem hold for the Riemann integral?

    No. Recall the Dirichlet function example. The sequence of measurable functions $\{f_n\}$ on the set of finite measure $[0,1]$ is uniformly bounded on $[0,1]$ as each $f_n \in \{0,1\}$ so that

    $$|f_n| \leq 1 \text{ on } [0,1] \text{ for all } n.$$

    Then for the Dirichlet function f,

    $$\{f_n\} \to f \text{ pointwise on } [0,1], \text{ but } (R)\underline{\int_0^1} f = 0 < 1 = (R)\overline{\int_0^1} f,$$

    so that f is not Riemann integrable and so $(R)\int_0^1 f$ is not defined, and we cannot say anything about if $\lim_{n\to\infty}(R)\int_0^1 f_n = (R)\int_0^1 f$.

12. Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f = g$ a.e. on E. Show that $\int_E f = \int_E g$.

    Because $f = g$ a.e. on E, then $f = g$ on $E \setminus E_0$, where $m(E_0) = 0$. Recall from Chapter 3 Proposition 5 (i) that f is measurable on E and $f = g$ a.e. on E implies that g is measurable on E.

Then

$$\begin{aligned}
\int_E f &= \int_{E\setminus E_0} f + \int_{E_0} f && \text{Corollary 6} \\
&= \int_{E\setminus E_0} f && \text{Problem 9: } \int_{E_0} f = 0 \\
&= \int_{E\setminus E_0} g && f = g \text{ on } E \setminus E_0 \\
&= \int_E g - \int_{E_0} g && \text{Corollary 6} \\
&= \int_E g. && \text{Problem 9: } \int_{E_0} g = 0
\end{aligned}$$

13. Does the Bounded Convergence Theorem hold if $m(E) < \infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on E?

No, see the example of the sequence of continuous (and thus measurable) triangular functions $\{f_n\}$ on $E = [0,1]$ so that $m(E) = 1 < \infty$. But $\{f_n\} = \{|f_n|\}$ is not uniformly bounded on E because for every number $M \in \mathbb{N}$ we choose, there always exists the function f_{N+1} with $f_{N+1}(\frac{1}{N+1}) = N + 1 > N$.
Then for the function $f \equiv 0$ on $[0,1]$,

$$\{f_n\} \to f \text{ pointwise on } [0,1], \text{ but } \lim_{n\to\infty} \int_0^1 f_n = 1 \neq 0 = \int_0^1 f,$$

and the Bounded Convergence Theorem does not hold.

14. Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

We see that since the convergence is uniform, then for $\epsilon > 0$, there exists an index N such that for all $n \geq N$,

$$|f| - |f_n| \leq |f - f_n| < \epsilon \text{ on } E. \tag{1}$$

Also, since each f_n is bounded, we have

$$|f_n| \leq M_n \text{ on } E.$$

Therefore at the index N, we have for any $\epsilon > 0$,

$$|f| < |f_N| + \epsilon \leq M_N + \epsilon,$$

and thus f is bounded: $|f| \leq M_N$ on E.

To show that $\{f_n\}$ is uniformly bounded, we can set $\epsilon = 1$ so that there exists an index N' such that for all $n \geq N'$, by (1), we have

$$|f_n| < |f| + 1 \leq M_N + 1 \text{ on } E.$$

Then we have for all $n \in \mathbb{N}$,

$$|f_n| \leq \max\{M_1, \cdots, M_{N'}, M_N + 1\} \text{ on } E,$$

and the sequence $\{f_n\}$ is uniformly bounded on E.

Then Proposition 8 is a special case of the Bounded Convergence Theorem because it requires $\{f_n\}$ to converge uniformly to f.

15. Verify the assertions in the last Remark of this section.

    This is true; we have:

    - Continuity of measure uses countable additivity of measure,
    - Lemma 10 uses continuity of measure,
    - Egoroff's Theorem uses Lemma 10,
    - Bounded Convergence Theorem uses Egoroff's Theorem.

16. Let f be a nonnegative bounded measurable function on a set of finite measure E. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on E.

    We can suppose by contradiction that $f = 0$ on $E \setminus E_0$, but $m(E_0) \neq 0$.
    So $f > 0$ on E_0, or in other words,

$$0 < m(E_0) = m(\{x \in E \mid f(x) > 0\}) = m(\bigcup_{n=1}^{\infty}\{x \in E \mid f(x) \geq 1/n\}) = m(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} m(E_n).$$

Then there must exist an index k for which $m(E_k) \neq 0$, else we have $0 < m(E_0) \leq \sum_{n=1}^{\infty} 0 = 0$ and reach a contradiction.
Thus we have that $f \geq 1/k$ on E_k, with $m(E_k) > 0$.
Therefore it is possible to define the simple function φ_k on E such that $\varphi_n \leq f$ on E:

$$\varphi_k(x) = \begin{cases} 1/k & x \in E_k \\ 0 & x \notin E_k \end{cases}$$

where, because $1/k > 0$ and $m(E_k) > 0$,

$$\int_E \varphi_k = \frac{1}{k} \cdot m(E_k) + 0 \cdot m(E_k^c) = \frac{1}{k} \cdot m(E_k) > 0.$$

But then we have

$$\int_E \varphi_k > 0 = \int_E f = \sup\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E\right\},$$

a contradiction to the supremum.

4.3 The Lebesgue Integral of a Measurable Nonnegative Function

A function f on E is said to be of finite support provided it vanishes outside a set of finite measure, that is, there exists a set E_0 such that $m(E_0) < \infty$ and $f \equiv 0$ on $E \setminus E_0$. Therefore $f = f \cdot \chi_{E_0}$ so that

$$\int_{E_0} f = \int_E f \cdot \chi_{E_0} = \int_E f.$$

Definition. *For a nonnegative measurable function f on E, we define the integral of f over E by*

$$\int_E f = \sup\left\{\int_E h \mid h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E\right\}.$$

Chebyshev's Inequality. *Let f be a nonnegative measurable function on E. Then for any $\lambda > 0$,*

$$m\{x \in E \mid f(x) \geq \lambda\} \leq \frac{1}{\lambda} \cdot \int_E f. \tag{9}$$

Proof. Define $E_\lambda = \{x \in E \mid f(x) \geq \lambda\}$.
First suppose $m(E_\lambda) = \infty$.
Then for a natural number n, define $E_{\lambda,n} = E_\lambda \cap [-n, n]$ and $\psi_n = \lambda \cdot \chi_{E_{\lambda,n}}$. Then ψ_n is a bounded measurable function of finite support,

$$\lambda \cdot m(E_{\lambda,n}) = \int_{E_{\lambda,n}} \lambda \cdot 1 = \int_E \lambda \cdot \chi_{E_{\lambda,n}} = \int_E \psi_n \text{ and } 0 \leq \psi_n \leq f \text{ on } E \text{ for all } n.$$

From the continuity of measure, because $\{E_{\lambda,n}\}_{n=1}^\infty$ is ascending and $E_\lambda = \bigcup_{n=1}^\infty E_{\lambda,n}$,

$$\infty = \lambda \cdot m(E_\lambda) = \lambda \cdot \lim_{n\to\infty} m(E_{\lambda,n}) = \lim_{n\to\infty} \int_E \psi_n \leq \int_E f.$$

Therefore the inequality (9) holds since both sides equal ∞.
Now suppose $m(E_\lambda) < \infty$.
Define $h = \lambda \cdot \chi_{E_\lambda}$. Then h is a bounded measurable function of finite support, and $0 \leq h \leq f$ on E.
Then

$$\lambda \cdot m(E_\lambda) = \int_{E_\lambda} \lambda \cdot 1 = \int_E \lambda \cdot \chi_{E_\lambda} = \int_E h \leq \int_E f.$$

Thus we have $m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$. □

Fatou's Lemma. *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\textit{If } \{f_n\} \to f \textit{ pointwise a.e. on } E, \textit{ then } \int_E f \leq \liminf \int_E f_n.$$

Proof. Let $\{f_n\}$ converge pointwise on $E \setminus E_0$, where $m(E_0) = 0$. We know that sets of measure zero do not contribute to the integral. That is,

$$\int_E f = \int_{E\setminus E_0} f + \int_{E_0} f = \int_{E\setminus E_0} f.$$

Then by excising this set of measure zero, we can assume pointwise convergence on all of E (use E instead of $E \setminus E_0$ for ease of writing).
Then because each f_n is nonnegative, we have $0 \leq f_n,\ \forall n \implies 0 \leq f$. Also, because $\{f_n\}$ is a sequence of measurable functions that converges pointwise to f, then f is also measurable.
Then to verify the inequality of Fatou's Lemma, by the definition of the integral of the nonnegative measurable function f, it is necessary and sufficient to show that if h is any bounded measurable function of finite support for which $0 \leq h \leq f$ on E, then

$$\int_E h \leq \liminf \int_E f_n.$$

This is because $\int_E f = \sup\left\{\int_E h\right\}$; that is, the least upper bound.

Now, consider a function h that is bounded, measurable, finite support, and $0 \leq h \leq f$. Then there exists $M \geq 0$ for which $|h| \leq M$ on E. Let $E_0 = \{x \in E \mid h(x) \neq 0\}$, so because h is of finite support, $m(E_0) < \infty$. For $n \in \mathbb{N}$, define a function h_n on E by

$$h_n = \min\{h, f_n\} \text{ on } E.$$

Then the function h_n is measurable and

$$0 \leq h_n \leq M \text{ on } E_0 \text{ and } h_n \equiv 0 \text{ on } E \setminus E_0 \text{ (finite support).}$$

Also, for each x in E, since $h(x) \leq f(x)$ and $\{f_n(x)\} \to f(x)$, then $\{h_n(x)\} \to h(x)$.

Then we have a sequence of measurable functions $\{h_n\}$ on a set of finite measure E_0. Also, there exists an M such that $|h_n| \leq M$ on E_0 for all n, and $\{h_n\} \to h$ pointwise on E_0. Thus by the Bounded Convergence Theorem, we have $\lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h$, and so

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h.$$

However, for each n, we have $h_n \leq f_n$ on E and by monotonicity, $\int_E h_n \leq \int_E f_n$. Thus

$$\int_E h = \lim_{n\to\infty} \int_E h_n = \liminf \int_E h_n \leq \liminf \int_E f_n.$$

□

The Monotone Convergence Theorem. *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E.*

$$\text{If } \{f_n\} \to f \text{ pointwise a.e. on } E, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. By Fatou's Lemma, we have

$$\int_E f \leq \liminf \int_E f_n.$$

However, for each index n, because $\{f_n\}$ is increasing, we have $f_n \leq f$ a.e. on E, and so $\int_E f_n \leq \int_E f$. Therefore

$$\limsup \int_E f_n \leq \int_E f.$$

Hence, $\limsup \int_E f_n \leq \int_E f \leq \liminf \int_E f_n$ implies that

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

□

Corollary 12. *Let $\{u_n\}$ be a sequence of nonnegative measurable functions on E.*

$$\text{If } f = \sum_{n=1}^{\infty} u_n \text{ pointwise a.e. on } E, \text{ then } \int_E f = \sum_{n=1}^{\infty} \int_E u_n.$$

Proof. Let $f_k = \sum_{n=1}^{k} u_n$ so that $\{f_k\}$ is an increasing sequence of nonnegative measurable functions on E, and

$$f = \sum_{n=1}^{\infty} u_n = \lim_{k\to\infty} \sum_{n=1}^{k} u_n = \lim_{k\to\infty} f_k \text{ pointwise a.e. on } E.$$

Then by the Monotone Convergence Theorem and the linearity of integration, we have

$$\int_E f = \lim_{k\to\infty} \int_E f_k = \lim_{k\to\infty} \int_E \sum_{n=1}^{k} u_n = \lim_{k\to\infty} \sum_{n=1}^{k} \int_E u_n = \sum_{n=1}^{\infty} \int_E u_n.$$

□

Definition. *A nonnegative measurable function f on a measurable set E is said to be* ***integrable*** *over E provided*

$$\int_E f < \infty.$$

PROBLEMS

17. Let E be a set of measure zero and define $f \equiv \infty$ on E. Show that $\int_E f = 0$.

    (Recall Problem 9). If we suppose there exists a simple function that does not have an integral of zero, then there must exist some subset of E_i of E such that $m(E_i) > 0$. But this is a contradiction to the monotonicity of Lebesgue measure: $m(E_i) \leq m(E) = 0$. Therefore it must be that $\sup\left\{\int_E \varphi \mid \varphi \text{ simple}, \varphi \leq f \text{ on } E\right\} = 0 = \inf\left\{\int_E \psi \mid \psi \text{ simple}, \psi \geq f \text{ on } E\right\}$ and $\int_E f = 0$.

18. Show that the integral of a bounded measurable function of finite support is properly defined.

    Let f on E be a bounded measurable function of finite support.
    That is, $m(E_0) = m(\{x \in E \mid f(x) \neq 0\}) < \infty$.
    We want to show that

    $$\int_E f = \int_{E_0} f.$$

    First consider any simple function on E_0; we can write

    $$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ and } \int_{E_0} \varphi = \sum_{k=1}^{n} c_k \cdot m(E_k),$$

    where $E_k = \{x \in E_0 \mid \varphi(x) = c_k\}$.
    Then extending φ to E by setting $\varphi(x) = 0$ for $x \in E \setminus E_0$, see that

    $$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} + 0 \cdot \chi_{E\setminus E_0} \text{ and } \int_E \varphi = \sum_{k=1}^{n} c_k \cdot m(E_k) + 0 \cdot m(E \setminus E_0).$$

    However, $m(E\setminus E_0) = \infty$. We can use the σ-finiteness (see Chapter 17.1) of the Lebesgue measure to partition $E \setminus E_0$ into a countable union of disjoint measurable sets, each of finite measure. That is, $E \setminus E_0 = \bigcup_{i=1}^{\infty} A_i$, where $m(A_i) < \infty$ for all i. We see that

$$\begin{aligned}
\int_E \varphi &= \sum_{k=1}^{n} c_k \cdot m(E_k) + 0 \cdot m(E \setminus E_0) \\
&= \sum_{k=1}^{n} c_k \cdot m(E_k) + 0 \cdot m(\bigcup_{i=1}^{\infty} A_i) \\
&= \sum_{k=1}^{n} c_k \cdot m(E_k) + 0 \cdot \sum_{i=1}^{\infty} m(A_i) \\
&= \sum_{k=1}^{n} c_k \cdot m(E_k) + \sum_{i=1}^{\infty} 0 \cdot m(A_i) \\
&= \sum_{k=1}^{n} c_k \cdot m(E_k) \\
&= \int_{E_0} \varphi.
\end{aligned}$$

and therefore any simple function of finite support has $\int_E \varphi = \int_{E_0} \varphi$.

Now, for any simple functions φ and ψ on on E_0 such that $\varphi \leq f_{|E_0}$ and $\psi \geq f_{|E_0}$, there exists the extension $\varphi(x), \psi(x) = 0$ for $x \in E \setminus E_0$ so that $0 = \varphi(x) \leq f(x) = 0$ and $0 = f(x) \leq \psi(x) = 0$ on $x \in E \setminus E_0$ and $\varphi \leq f$ and $f \leq \psi$ on all of E.
Then because $\int_E \varphi = \int_{E_0} \varphi$ and $\int_E \psi = \int_{E_0} \psi$,

$$\begin{aligned}
\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E\right\} &\supseteq \left\{\int_{E_0} \varphi \mid \varphi \text{ simple, } \varphi \leq f_{|E_0} \text{ on } E_0\right\} \\
\left\{\int_E \psi \mid \psi \text{ simple, } \psi \geq f \text{ on } E\right\} &\supseteq \left\{\int_{E_0} \psi \mid \psi \text{ simple, } \psi \geq f_{|E_0} \text{ on } E_0\right\}
\end{aligned}$$

Then because $\sup\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E\right\} \leq \inf\left\{\int_E \psi \mid \psi \text{ simple, } \psi \geq f \text{ on } E\right\}$ and

$$\begin{aligned}
\sup\left\{\int_E \varphi \mid \varphi \text{ simple, } \varphi \leq f \text{ on } E\right\} &\geq \sup\left\{\int_{E_0} \varphi \mid \varphi \text{ simple, } \varphi \leq f_{|E_0} \text{ on } E_0\right\} = \int_{E_0} f_{|E_0} \\
\inf\left\{\int_E \psi \mid \psi \text{ simple, } \psi \geq f \text{ on } E\right\} &\leq \inf\left\{\int_{E_0} \psi \mid \psi \text{ simple, } \psi \geq f_{|E_0} \text{ on } E_0\right\} = \int_{E_0} f_{|E_0}
\end{aligned}$$

we have $\int_E f = \int_{E_0} f_{|E_0}$.

19. For a number α, define $f(x) = x^{\alpha}$ for $0 < x \leq 1$, and $f(0) = 0$. Compute $\int_0^1 f$.

    Case $0 \leq \alpha$:
    Then the function $f(x) = x^{\alpha}$ is positive monotone on $[0, 1]$, that is,

    $$0 \leq x \leq 1 \implies 0 = 0^{\alpha} \leq x^{\alpha} \leq 1^{\alpha} = 1,$$

    and f is bounded by 1 on the closed bounded interval $[0, 1]$.

Now we see that f is Riemann integrable:
For each natural number m, consider the partition $P_m = \{0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}, 1\} = \{x_0, x_1, \cdots, x_m\}$.
Then because f is increasing, we have $f(x_{k-1}) = \inf\{f(x) \mid x_{k-1} < x < x_k\}$ and $f(x_k) = \sup\{f(x) \mid x_{k-1} < x < x_k\}$, so that

$$\begin{aligned} L(f, P_m) &= \sum_{k=1}^{m} f(x_{k-1}) \cdot \frac{1}{m} \\ &= \sum_{k=1}^{m-1} f(x_k) \cdot \frac{1}{m} + f(x_0) \cdot \frac{1}{m} \\ &= \sum_{k=1}^{m-1} f(x_k) \cdot \frac{1}{m} + 0 \cdot \frac{1}{m} \\ &= \sum_{k=1}^{m-1} f(x_k) \cdot \frac{1}{m} \end{aligned}$$

and

$$\begin{aligned} U(f, P_m) &= \sum_{k=1}^{m} f(x_k) \cdot \frac{1}{m} \\ &= \sum_{k=1}^{m-1} f(x_k) \cdot \frac{1}{m} + f(x_m) \cdot \frac{1}{m} \\ &= \sum_{k=1}^{m-1} f(x_k) \cdot \frac{1}{m} + 1 \cdot \frac{1}{m} \end{aligned}$$

Then clearly we get $\lim_{m\to\infty}[U(f, P_m) - L(f, P_m)] = 0$, with

$$U(f, P_m) = \sum_{k=1}^{m} f(x_k) \cdot \frac{1}{m} = \sum_{k=1}^{m} (\frac{k}{m})^{\alpha} \cdot \frac{1}{m},$$

and $\lim_{m\to\infty} \sum_{k=1}^{m} (\frac{k}{m})^{\alpha} \cdot \frac{1}{m} = \frac{1}{\alpha+1}$. That is, we can see $\lim_{m\to\infty} \frac{\sum_{k=1}^{m} k^{\alpha}}{m^{\alpha+1}} - \frac{1}{\alpha+1} = 0$.
(We can use integration to see:)

$$\int_0^1 x^{\alpha} = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_{x=0}^{x=1} = \left(\frac{1^{\alpha+1}}{\alpha+1} - \frac{0^{\alpha+1}}{\alpha+1}\right) = \left(\frac{1}{\alpha+1}\right).$$

Case $-1 < \alpha < 0$:
For each natural number n, define f_n on $[0, 1]$ such that

$$f_n(x) = \begin{cases} f(x) = x^{\alpha} & \text{if } x \in [1/n, 1] \\ 0 & \text{if } x \in [0, 1/n) \end{cases}$$

Then $\{f_n\}$ is an increasing sequence of nonnegative measurable functions on $[0, 1]$, and $\{f_n\} \to f$ pointwise on $[0, 1]$. By the Monotone Convergence Theorem, $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 f$.

Now, because $f(x) = x^{\alpha} = \frac{1}{x^{-\alpha}}$ is negative monotone on $(0, 1]$, that is, for any $n \in \mathbb{N}$,

$$1/n \leq x \implies (1/n)^{-\alpha} \leq x^{-\alpha} \implies \frac{1}{x^{-\alpha}} \leq \frac{1}{(1/n)^{-\alpha}} \implies x^{\alpha} \leq (1/n)^{\alpha},$$

then f_n is bounded by $(1/n)^\alpha$ on the closed bounded interval $[0,1]$.

Now we see that f is Riemann integrable:
Case $\alpha \leq -1$:
Consider the same sequence $\{f_n\}$ from the previous case. We can use the Monotone Convergence Theorem. See again that f_n is bounded by $(1/n)^\alpha$ on the closed bounded interval $[0,1]$, and is thus Riemann integrable.

20. Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f pointwise on E. Let $M \geq 0$ be such that $\int_E f_n \leq M$ for all n. Show that $\int_E f \leq M$. Verify that this property is equivalent to the statement of Fatou's Lemma.

Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to f pointwise on E.

(Fatou's Lemma $\implies$ Property) Let $M \geq 0$ be such that $\int_E f_n \leq M$ for all n.
Then by Fatou's Lemma,

$$\int_E f \leq \liminf \int_E f_n \leq M.$$

(Property $\implies$ Fatou's Lemma) Suppose that $M \geq 0$ with $\int_E f_n \leq M\ \forall n$ implies $\int_E f \leq M$.
Then, by Chapter 1 Problem 38, we know that $\liminf \left\{\int_E f_n\right\}$ is the smallest cluster point of $\left\{\int_E f_n\right\}$. That is, there exists a subsequence $\left\{\int_E f_{n_k}\right\}$ that converges to $\liminf \left\{\int_E f_n\right\}$.
Fix $\epsilon > 0$.
Then there exists a natural number N such that for $k \geq N$, then $\left|\int_E f_{n_k} - \liminf \left\{\int_E f_n\right\}\right| < \epsilon$.
Then we have that $\{f_{n_k}\}$ is a sequence of measurable functions that converges to f pointwise on E, and $\liminf \left\{\int_E f_n\right\} + \epsilon \geq 0$ with $\int_E f_{n_k} < \liminf \left\{\int_E f_n\right\} + \epsilon$, so the Property implies that $\int_E f \leq \liminf \left\{\int_E f_n\right\} + \epsilon$.
Because this is true for any ϵ, then $\int_E f \leq \liminf \left\{\int_E f_n\right\}$ holds.

21. Let the function f be nonnegative and integrable over E and $\epsilon > 0$. Show there is a simple function η on E that has finite support, $0 \leq \eta \leq f$ on E and $\int_E |f - \eta| < \epsilon$. If E is a closed, bounded interval, show there is a step function h on E that has finite support and $\int_E |f - h| < \epsilon$.

Fix $\epsilon > 0$.
For a nonnegative measurable function f on E, we define the integral of f over E by

$$\int_E f := \sup\left\{\int_E h \mid h \text{ bounded, measurable, of finite support and } 0 \leq h \leq f \text{ on } E\right\},$$

Then by definition of supremum, there exists $h \in \{$h bounded, measurable, of finite support and $0 \leq h \leq f$ on $E\}$ such that

$$\int_E f - \frac{\epsilon}{2} < \int_E h \leq \int_E f.$$

We can write this as

$$\int_E f - \int_E h < \frac{\epsilon}{2} \tag{1}$$

We have the definition of the integral of the bounded measurable function h of finite support:

$$\int_E h := \sup\left\{\int_E \varphi \mid \varphi \text{ simple, } 0 \leq \varphi \leq f \text{ on } E\right\}.$$

Then by definition of supremum, there exists $\eta \in \{\varphi \text{ simple}, 0 \leq \varphi \leq f \text{ on } E\}$ such that

$$\int_E h - \frac{\epsilon}{2} < \int_E \eta \leq \int_E h.$$

We can write this as

$$-\int_E \eta < -\int_E h + \frac{\epsilon}{2}. \tag{2}$$

Then by (1) and (2),

$$\int_E [f - \eta] = \int_E f - \int_E \eta < \int_E f - \int_E h + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

22. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to f and f be integrable over $\mathbb{R}$. Show that

$$\text{if } \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

Let E be a measurable set of real numbers.
By Theorem 11, we have, for each n,

$$\int_{\mathbb{R}} f_n = \int_E f_n + \int_{E^c} f_n. \tag{1}$$

Then because f is integrable, then its integral $\int_{\mathbb{R}} f$ is finite, which by equality to $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$ implies that the sequence $\left\{\int_{\mathbb{R}} f_n\right\}$ converges; that is,

$$\infty > \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n = \liminf \int_{\mathbb{R}} f_n = \limsup \int_{\mathbb{R}} f_n. \tag{2}$$

In particular, we have by (1), (2), and Fatou's Lemma for $\{f_n\}$ on E^c,

$$\int_{\mathbb{R}} f - \limsup \int_E f_n = \liminf \int_{E^c} f_n \geq \int_{E^c} f,$$

so that, rearranging,

$$\int_E f = \int_{\mathbb{R}} f - \int_{E^c} f \geq \limsup \int_E f_n. \tag{a}$$

Then by Fatou's Lemma for $\{f_n\}$ on E,

$$\int_E f \leq \liminf \int_E f_n. \tag{b}$$

Then (a), (b), and the fact that $\liminf \int_E f_n \leq \limsup \int_E f_n$ imply equality:

$$\int_E f = \lim_{n\to\infty} \int_E f_n.$$

23. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function f on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^{\infty} a_n$.

Because each a_n is nonnegative, we can define a sequence of increasing functions $\{f_k\}$ on $[1,\infty)$ by

$$f_k(x) = \begin{cases} a_n & x \in [n, n+1) \text{ for each } n \in \{1, \dots, k\} \\ 0 & \text{else} \end{cases}$$

Then each f_k is a simple function of finite support $\bigcup_{n=1}^{k} [n, n+1)$ so that

$$\int_{[1,\infty)} f_k = \sum_{n=1}^{k} a_n \cdot m([n, n+1)) = \sum_{n=1}^{k} a_n.$$

Also, $\{f_k\} \to f$ pointwise so that, by the Monotone Convergence Theorem,

$$\sum_{n=1}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} \int_{[1,\infty)} f_k = \int_{[1,\infty)} f.$$

24. Let f be a nonnegative measurable function on E.

(i) Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on E, each of finite support, which converges pointwise on E to f.

See the Simple Approximation Theorem from Chapter 3.

(ii) Show that $\int_E f = \sup\{\int_E \varphi \mid \varphi \text{ simple, of finite support and } 0 \le \varphi \le f \text{ on } E\}$.

By the Monotone Convergence Theorem with (i), we have that

$$\sup_n \int_E \varphi_n = \lim_{n\to\infty} \int_E \varphi_n = \int_E f.$$

25. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on E that converges pointwise on E to f. Suppose $f_n \le f$ on E for each n. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

(similar to Monotone Convergence Theorem)
By monotonicity of integration, we get

$$\limsup \int_E f_n \le \int_E f.$$

By Fatou's Lemma,

$$\int_E f \le \liminf \int_E f_n.$$

Then $\liminf \int_E f_n \le \limsup \int_E f_n$ (see Chapter 1 Problem 41) implies equality:

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

26. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

    We can define a decreasing sequence of nonnegative measurable functions $f_n : [0, \infty) \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & x < n \\ 1 & x \geq n \end{cases}$$

    Then $\{f_n\} \to f \equiv 0$ pointwise on $[0, \infty)$, but $\int_{[0,\infty)} f_n = \infty$ (not integrable over $[0, \infty)$) for all n, and so

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n = \infty \neq 0 = \int_{[0,\infty)} f.$$

27. Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on E, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

    We have, for each n,

$$0 \leq \inf_{k\geq n} f_k(x) \leq f_n(x) \text{ for all } x,$$

    and so by monotonicity of integration,

$$\int_E \inf_{k\geq n} f_k(x) \leq \int_E f_n,$$

    and by the Monotone Convergence Theorem for the sequence $\{\inf_{k\geq n} f_k\} \to \lim_n \inf_{k\geq n} f_k$,

$$\int_E \lim_n \inf_{k\geq n} f_k(x) = \lim_n \int_E \inf_{k\geq n} f_k(x) \leq \liminf_n \int_E f_n.$$

4.4 The General Lebesgue Integral

For an extended real-valued function f on E, we have defined the positive part f^+ and the negative part f^- of f, respectively, by

$$f^+(x) = \max\{f(x), 0\}, \text{ and } f^-(x) = \max\{-f(x), 0\} \text{ for all } x \in E.$$

Then f^+ and f^- are nonnegative functions on E,

$$f = f^+ - f^- \text{ on } E,$$

and

$$|f| = f^+ + f^- \text{ on } E.$$

Definition. *A measurable function f on E is said to be **integrable** over E provided $|f|$ is integrable over E. When this is so we define the integral of f over E by*

$$\int_E f = \int_E f^+ - \int_E f^-.$$

Proposition 16 (the Integral Comparison Test)**.** *Let f be a measurable function on E. Suppose there is a nonnegative function g that is integrable over E and dominates f in the sense that*

$$|f| \leq g \text{ on } E.$$

Then f is integrable over E and

$$\left| \int_E f \right| \leq \int_E |f|.$$

Proof. By the monotonicity of integration for nonnegative functions, we have $\int_E |f| \leq \int_E g < \infty$, which implies $|f|$ is integrable, and thus f is integrable.
Then by the triangle inequality and the linearity of integration,

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right| \leq \int_E f^+ + \int_E f^- = \int_E |f|.$$

□

The Lebesgue Dominated Convergence Theorem. *Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n.*

$$\textit{If } \{f_n\} \to f \textit{ pointwise a.e. on } E, \textit{ then } f \textit{ is integrable over } E \textit{ and } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

Proof. Since $|f_n| \leq g$ on E and $|f| \leq g$ a.e. on E and g is integrable over E, by the integral comparison test, f and each f_n also are integrable over E. We infer from Proposition 15 that, by possibly excising from E a countable collection of sets of measure zero and using the countable additivity of Lebesgue measure, we may assume that f and each f_n is finite on E. The function $g - f$ and for each n, the function $g - f_n$, are properly defined, nonnegative, and measurable. Morever, the sequence $\{g - f_n\}$ converges pointwise a.e. on E to $g - f$.
By Fatou's Lemma,

$$\int_E (g - f) \leq \liminf \int_E (g - f_n).$$

Thus, by the linearity of integration for measurable functions and $\liminf(-a_n) = -\limsup(a_n)$ (Chapter 1 Proposition 19 (iii)),

$$\int_E g - \int_E f = \int_E (g - f) \leq \liminf \int_E (g - f_n) = \int_E g + \liminf \left(-\int_E f_n \right) = \int_E g - \limsup \int_E f_n,$$

which tells us that

$$\limsup \int_E f_n \leq \int_E f.$$

Similarly by Fatou's Lemma for the sequence $\{g + f_n\}$,

$$\int_E g + \int_E f = \int_E (g + f) \leq \liminf \int_E (g + f_n) = \int_E g + \liminf \int_E f_n,$$

which tells us that

$$\int_E f \leq \liminf \int_E f_n.$$

□

Theorem 19 (General Lebesgue Dominated Convergence Theorem)**.** *Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise a.e. on E to f. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E that converges pointwise a.e. on E to g and dominates $\{f_n\}$ on E in the sense that*

$$|f_n| \leq g_n \text{ on } E \text{ for all } n.$$

$$\text{If } \lim_{n\to\infty} \int_E g_n = \int_E g < \infty, \text{ then } \lim_{n\to\infty} \int_E f_n = \int_E f.$$

PROBLEMS

28. Let f be integrable over E and let C be a measurable subset of E. Show that $\int_C f = \int_E f \cdot \chi_C$.

    We have

29. For a measurable function f on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number n. Is it true that f is integrable over $[1, \infty)$ iff the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that f is integrable over $[1, \infty)$ iff the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

30. Let g be a nonnegative integrable function over E and suppose $\{f_n\}$ is a sequence of measurable functions on E such that for each n, $|f_n| \leq g$ a.e. on E. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

31. Let f be a measurable function on E which can be expressed as $f = g + h$ on E, where g is finite and integrable over E and h is nonnegative on E. Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function g and nonnegative function h whose sum is f.

32. Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

33. Let $\{f_n\}$ be a sequence of integrable functions on E for which $f_n \to f$ a.e. on E and f is integrable over E. Show that $\int_E |f - f_n| \to 0$ iff $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: use the General Lebesgue Dominated Convergence Theorem.)

34. Let f be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f.$$

    Define, for each n, the function

$$f_n = f \cdot \chi_{[-n,n]},$$

    so that because f is nonnegative and measurable, $\{f_n\}$ is an increasing sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise to f on $\mathbb{R}$.
    Then by the Monotone Convergence Theorem,

$$\lim_{n\to\infty} \int_{[-n,n]} f = \lim_{n\to\infty} \int_E f \cdot \chi_{[-n,n]} = \lim_{n\to\infty} \int_E f_n = \int_E f.$$

35. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y. Suppose for each fixed value of x, $\lim_{y \to 0} f(x, y) = f(x)$ and that for all y, we have $|f(x, y)| \leq g(x)$, where g is integrable over $[0, 1]$. Show that

$$\lim_{y \to 0} \int_0^1 f(x, y) dx = \int_0^1 f(x) dx.$$

Also show that if the function $f(x, y)$ is continuous in y for each x, then

$$h(y) = \int_0^1 f(x, y) dx$$

is a continuous function of y.

36. Let f be a real-valued function of two variables (x, y) that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of x for each fixed value of y. For each $(x, y) \in Q$ let the partial derivative $\partial f / \partial y$ exist. Suppose there is a function g that is integrable over $[0, 1]$ and such that

$$\left| \frac{\partial f}{\partial y}(x, y) \right| \leq g(x) \text{ for all } (x, y) \in Q.$$

Prove that

$$\frac{d}{dy} \left[\int_0^1 f(x, y) dx \right] = \int_0^1 \frac{\partial f}{\partial y}(x, y) dx \text{ for all } y \in [0, 1].$$

4.5 Countable Additivity and Continuity of Integration

Theorem 20 (the Countable Additivity of Integration)**.** *Let f be integrable over E and $\{E_n\}_{n=1}^{\infty}$ a disjoint countable collection of measurable subsets of E whose union is E. Then*

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$$

Proof. Let $n \in \mathbb{N}$, and define

$$f_n := f \cdot \chi_n,$$

where χ_n is the characteristic function of the measurable set $\bigcup_{k=1}^n E_k$. Then f_n is a measurable function on E and

$$|f_n| = |f \cdot \chi_n| \leq |f| \text{ on } E.$$

Observe that $\{f_n\} \to f$ pointwise on E. Thus, by the Lebesgue Dominated Convergence Theorem,

$$\int_E f = \lim_{n \to \infty} \int_E f_n.$$

Then because each set $\{E_k\}_{k=1}^n$ is disjoint, by Corollary 18 (Additivity Over Domains of Integration) we have

$$\int_E f_n = \sum_{k=1}^n \int_{E_k} f.$$

Thus

$$\int_E f = \lim_{n\to\infty}\int_E f_n = \lim_{n\to\infty}\sum_{k=1}^{n}\int_{E_k} f = \sum_{k=1}^{\infty}\int_{E_k} f.$$

□

Theorem 21. *(the Continuity of Integration) Let f be integrable over E.*

(i) If $\{E_n\}_{n=1}^{\infty}$ is an ascending countable collection of measurable subsets of E, then

$$\int_{\bigcup_{n=1}^{\infty} E_n} f = \lim_{n\to\infty}\int_{E_n} f. \tag{24}$$

(ii) If $\{E_n\}_{n=1}^{\infty}$ is a descending countable collection of measurable subsets of E, then

$$\int_{\bigcap_{n=1}^{\infty} E_n} f = \lim_{n\to\infty}\int_{E_n} f. \tag{25}$$

PROBLEMS

37. Let f be an integrable function on E. Show that for each $\epsilon > 0$, there is a natural number N for which if $n \geq N$, then $|int_{E_n} f| < \epsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

38. For each of the two functions f on $[1,\infty)$ defined below, show that $\lim_{n\to\infty}\int_1^n f$ exists while f is not integrable over $[1,\infty)$. Does this contradict the continuity of integration?

    (i) Define $f(x) = \frac{(-1)^n}{n}$, for $n \leq x < n+1$.

    (ii) Define $f(x) = \frac{(\sin x)}{x}$ for $1 \leq x < \infty$.

39. Prove the theorem regarding the continuity of integration.

4.6 Uniform Integrability: The Vitali Convergence Theorem

PROBLEMS

40. Let f be integrable over $\mathbb{R}$. Show that the function F defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbb{R}$$

    is properly defined and continuous. Is it necessarily Lipschitz?

41. Show that Proposition 25 is false if $E = \mathbb{R}$.

42. Show that Theorem 26 is false without the assumption that the h_n's are nonnegative.

43. Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over E. Show that for any α and β, the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable over E.

44. Let f be integrable over $\mathbb{R}$ and let $\epsilon > 0$. Establish the following three approximation properties.

(i) There is a simple function η on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f-\eta| < \epsilon$. (Hint: first verify this if f is nonnegative.)

(ii) There is a step function s on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f-s| < \epsilon$. (Hint: apply part (i) and Problem 18 of Chapter 3.)

(iii) There is a continuous function g on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f-g| < \epsilon$.

45. Let f be integrable over E. Define $\hat{f}$ to be the extension of f to all of $\mathbb{R}$ obtained by setting $\hat{f} \equiv 0$ outside of E. Show that $\hat{f}$ is integrable over $\mathbb{R}$ and $\int_E f = \int_{\mathbb{R}} \hat{f}$. Use this and part (i) and (iii) of the preceding problem to show that for $\epsilon > 0$, there is a simple function η on E and a continuous function g on E for which $\in_E |f-\eta| < \epsilon$ and $\in_E |f-g| < \epsilon$.

46. (Riemann-Lebesgue) Let f be integrable over $(-\infty, \infty)$. Show That

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx dx = 0.$$

(Hint: first show this for f is a step function that vanishes outside a closed, bounded interval and then use the approximation property (ii) of Problem 44.)

47. Let f be integrable over $(-\infty, \infty)$.

(i) Show that for each t,

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x+t)dx.$$

(ii) Let g be a bounded measurable function on $\mathbb{R}$. Show that

$$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x+t)] = 0.$$

(Hint: first show this, using uniform continuity of f on $\mathbb{R}$, if f is continuous and vanishes outside a bounded set. Then use the approximation property (iii) of Problem 44.)

48. Let f be integrable over E and let g be a bounded measurable function on E. Show that $f \cdot g$ is integrable over E.

49. Let f be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:

(i) $f = 0$ a.e. on $\mathbb{R}$.

(ii) $\int_{\mathbb{R}} fg = 0$ for every bounded measurable function g on $\mathbb{R}$.

(iii) $\int_A f = 0$ for every measurable set A.

(iv) $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

50. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \epsilon.$$

51. Let $\mathcal{F}$ be a family of functions, each of which is integrable over E. Show that $\mathcal{F}$ is uniformly integrable over E iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E\cap\mathcal{U}} |f| < \epsilon.$$

Chapter 6

Differentiation and Integration

Contents

For a function f on the closed, bounded interval $[a, b]$, when is

$$\int_a^b f' = f(b) - f(a)? \tag{i}$$

Assume f is continuous. Extend f to take the value $f(b)$ on $(b, b+1]$, and for $0 < h \le 1$, define the divided difference function $\mathrm{Diff}_h f$ and average value function $\mathrm{Av}_h f$ on $[a, b]$ by

$$\mathrm{Diff}_h f(x) := \frac{f(x+h) - f(x)}{h} \text{ and } Av_h f(x) := \frac{1}{h}\int_x^{x+h} f(t)dt \text{ for all } x \in [a, b].$$

A change of variables by translation, and cancellation, provides the discrete formulation of (i) fo the Riemann integral:

$$\int_a^b \mathrm{Diff}_h f = Av_h f(b) - Av_h f(a).$$

The limit of the RHS as $h \to 0^+$ equals $f(b) - f(a)$.

6.1 Continuity of Monotone Functions

Recall from Chapter 1.6 that a real-valued function f defined on a set E of real numbers is said to be increasing provided $f(x) \le f(x')$ whenever $x, x' \in E$ and $x \le x'$, and decreasing provided $f(x) \ge f(x')$ whenever $x, x' \in E$ and $x \le x'$. It is called monotone if it is either increasing or decreasing.

Theorem 1. *Let f be a monotone function on the open interval (a, b). Then f is continuous except possibly at a countable number of points in (a, b).*

Proof. Assume f is increasing.

Case 1: (a, b) is bounded and $f : [a, b] \to \mathbb{R}$:

For each $x_0 \in (a, b)$, f has a limit from the left and from the right:
Because f is increasing, we can define

$$f(x_0^-) = \lim_{x \to x_0^-} f(x) = \sup\{f(x) \mid a < x < x_0\},$$
$$f(x_0^+) = \lim_{x \to x_0^+} f(x) = \sup\{f(x) \mid x_0 < x < b\},$$

and $f(x_0^-) \le f(x_0^+)$. The function f fails to be continuous at x_0 iff $f(x_0^-) < f(x_0^+)$, in which case we define the open "jump" interval $J(x_0)$ by

$$J(x_0) := \{y \mid f(x_0^-) < y < f(x_0^+)\}.$$

Each jump interval is contained in the bounded interval $[f(a), f(b)]$ and the collection of jump intervals is disjoint. Therefore, for each natural number n, the number of jump intervals of length greater than $1/n$ is finite because the number is bounded by $\frac{f(b)-f(a)}{1/n} = n(f(b) - f(a)) < \infty$. Thus the set of points of discontinuity of f is the union of a countable collection of finite sets and therefore is countable.

Case 2: (a, b) is not bounded or $f(a^+)$ or $f(b^-)$ is not finite:

If (a, b) is not bounded, express

$$(a, b) = \bigcup_{n=1}^{\infty} (-n, n),$$

where $(-n, n)$ is bounded and $f : [-n, n] \to \mathbb{R}$ for each n.

If $f(a^+)$ or $f(b^-)$ is not finite, express

$$(a, b) = \bigcup_{n=1}^{\infty} \left(a + \frac{1}{n}, b - \frac{1}{n}\right),$$

where $\left(a + \frac{1}{n}, b - \frac{1}{n}\right)$ is bounded and $f : \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \to \mathbb{R}$ for each n.

Then we can use Case 1 to see that each interval has a countable set of points of discontinuity.
Because (a, b) is the union of an ascending sequence of such intervals, then (a, b) also has a countable set of points of discontinuity. $\square$

Proposition 2. *Let C be a countable subset of the open interval (a, b). Then there is an increasing function on (a, b) that is continuous only at points in $(a, b) \setminus C$.*

Proof. If C is finite the proof is clear. Assume C is countably infinite. Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of C. Define the function f on (a, b) by setting

$$f(x) = \sum_{\{n \mid q_n \le x\}} \frac{1}{2^n} \text{ for all } a < x < b.$$

Then because $\{n \mid q_n \leq x\} \subseteq \{n \mid n \in \mathbb{N}\}$, we have

$$\sum_{\{n \mid q_n \leq x\}} \frac{1}{2^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

We see that the geometric series converges so that f is properly defined. Moreover,

$$\begin{aligned} &\text{if } a < u < v < b, \\ &\text{then } f(v) - f(u) = \sum_{\{n \mid q_n \leq v\}} \frac{1}{2^n} - \sum_{\{n \mid q_n \leq u\}} \frac{1}{2^n} = \sum_{\{n \mid u < q_n \leq v\}} \frac{1}{2^n} \geq 0, \end{aligned} \tag{1}$$

so that f is increasing.

Let $x_0 = q_k$ belong to C.
Then by (1),

$$f(x_0) - f(x) = \sum_{\{n \mid x < q_n \leq x_0 = q_k\}} \frac{1}{2^n} \geq \frac{1}{2^k} \text{ for all } x < x_0.$$

Therefore f fails to be continuous at x_0:
For $\epsilon = \frac{1}{2^{k+1}} > 0$, for all $\delta > 0$, we have $|x_0 - x| < \delta$ but $|f(x_0) - f(x)| \geq \frac{1}{2^k} > \frac{1}{2^{k+1}}$.

Let x_0 belong to $(a, b) \setminus C$.
Fix $\epsilon > 0$.
Then there exists $k \in \mathbb{N}$ for which $\frac{1}{2^k} < \epsilon$, and there is an open interval I containing x_0 for which q_n does not belong to I for $n \in \{1, \dots, k\}$.
Then from (1), for all $x \in I$,

$$\begin{aligned} |f(x_0) - f(x)| &= \sum_{\{n \mid x < q_n \leq x_0, n > k\}} \frac{1}{2^n} \\ &\leq \sum_{n=k+1}^{\infty} \frac{1}{2^n} \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n} - \sum_{n=1}^{k} \frac{1}{2^n} \\ &= 1 - \frac{1}{2} \sum_{n=1}^{k} \left(\frac{1}{2}\right)^{n-1} \\ &= 1 - \frac{1}{2} \frac{1 - (1/2)^k}{1 - (1/2)} \\ &= \frac{1}{2^k} \\ &< \epsilon. \end{aligned}$$

Therefore f is continuous at x_0. □

PROBLEMS

1. Let C be a countable subset of the nondegenerate closed, bounded interval $[a, b]$. Show that there is an increasing function on $[a, b]$ that is continuous only at points in $[a, b] \setminus C$.

Simply take the function f from Proposition 2, where we have

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} \sum_{\{n|q_n\le a\}} \frac{1}{2^n} = 0,$$
$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} \sum_{\{n|q_n\le b\}} \frac{1}{2^n} = 1.$$

If the points a or b are not in C, we can extend f to $[a,b]$ by defining

$$f(a) := 0,$$
$$f(b) := 1,$$

which makes f continuous at a and b.
Else if a or b are in C, then we can extend f to $[a,b]$ by defining

$$f(a) := y, \text{ for any real number } y \neq 0,$$
$$f(b) := y, \text{ for any real number } y \neq 1,$$

which makes f discontinuous at a and b.

2. Show that there is a strictly increasing function on $[0,1]$ that is continuous only at the irrational numbers in $[0,1]$.

Enumerate $\mathbb{Q}\cap[0,1]$ by $\{q_n\}_{n=1}^{\infty}$, and define the function f on $[0,1]$ by setting

$$f(x) = \sum_{\{n|q_n\le x\}} \frac{1}{n^2} \text{ for } 0 < x \le 1.$$

We can see that for any $0 \le a < b \le 1$, by density of the rationals there exists a rational number $q_k \in (a,b)$ so that we have

$$f(b) - f(a) = \sum_{\{n|q_n\le b\}} \frac{1}{n^2} - \sum_{\{n|q_n\le a\}} \frac{1}{n^2} = \sum_{\{n|a<q_n\le b\}} \frac{1}{n^2} \ge \frac{1}{q_k^2} > 0,$$

and therefore f is strictly increasing.

In the case $x_0 \in \mathbb{Q}\cap[0,1]$, $x_0 \neq 0$:
Then there exists an index k such that $x_0 = q_k$.
We can write, for all $x \in [0,1]$ such that $x < x_0$,

$$f(x_0) - f(x) = \sum_{\{n|x<q_n\le x_0=q_k\}} \frac{1}{n^2} \ge \frac{1}{q_k^2}.$$

Then f is discontinuous at x_0:
For $\epsilon = \frac{1}{2\cdot q_k^2} > 0$, for all $\delta > 0$, we have $|x_0 - x| < \delta$ but $|f(x_0) - f(x)| \ge \frac{1}{q_k^2} > \frac{1}{2\cdot q_k^2}$.

In the case $x_0 \in \mathbb{Q}^c \cap [0,1]$:
Fix $\epsilon > 0$.
We have $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, so that the sequence of partial sums $\sum_{n=1}^{k} \frac{1}{n^2}$ converges.

That is, there exists $k \in \mathbb{N}$ for which $\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{k} \frac{1}{n^2} = \sum_{n=k}^{\infty} \frac{1}{n^2} < \epsilon$, and there is an open interval I containing x_0 for which q_n does not belong to I for $n \in \{1, \ldots, k\}$.
Then for all $x \in I$,

$$|f(x_0) - f(x)| = \sum_{\{n|x<q_n \leq x_0, n>k\}} \frac{1}{n^2} \leq \sum_{n=k+1}^{\infty} \frac{1}{n^2} < \epsilon.$$

Therefore f is continuous at x_0.

3. Let f be a monotone function on a subset E of $\mathbb{R}$. Show that f is continuous except possibly at a countable number of points in E.

   a

4. Let E be a subset of $\mathbb{R}$ and let C be a countable subset of E. Is there a monotone function on E that is continuous only at points in $E \setminus C$?

   a

6.2 Differentiability of Monotone Functions: Lebesgue's Theorem

A closed, bounded interval $[c, d]$ is said to be nondegenerate provided $c < d$.

PROBLEMS

5. Show that the Vitali Covering Lemma does not extend to the case in which the covering collection has degenerate closed intervals.

6. Show that the Vitali Covering Lemma does extend to the case in which the covering collection consists of nondegenerate general intervals.

7. let f be continuous on $\mathbb{R}$. Is there an open interval on which f is monotone?

8. Let I and J be closed, bounded intervals and $\gamma > 0$ be such that $\ell(I) > \gamma \cdot \ell(J)$. Assume $I \cap J \neq \emptyset$. Show that if $\gamma \geq 1/2$, then $J \subseteq 5 * I$, where $5 * I$ denotes the interval with the same center as I and five times its length. Is the same true if $0 < \gamma < 1/2$?

9. Show that a set E of real numbers has measure zero iff there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ for which each point in E belongs to infinitely many of the $I_k's$ and $\sum_{k=1}^{\infty} \ell(I_k) < \infty$.

10. (Riesz-Nagy) Let E be a set of measure zero contained in the open interval (a, b). According to the preceding problem, there is a countable collection of open intervals contained in (a, b), $\{(c_k, d_k)\}_{k=1}^{\infty}$, for which each point in E belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty}(d_k - c_k) < \infty$. Define

$$f(x) = \sum_{k=1}^{\infty} \ell((c_k, d_k) \cap (-\infty, x)) \text{ for all } x \text{ in } (a, b).$$

Show that f is increasing and fails to be differentiable at each point in E.

11. For real numbers $\alpha < \beta$ and $\gamma > 0$, show that if g is integrable over $[\alpha + \gamma, \beta + \gamma]$, Then
$$\int_{\alpha}^{\beta} g(t+\gamma)dt = \int_{\alpha+\gamma}^{\beta+\gamma} g(t)dt.$$
Prove this change of variables formula by successively considering simple functions, bounded measurable functions, nonnegative integrable functions, and general integrable functions. Use it to prove (14).

12. Compute the upper and lower derivatives of the characteristic function of the rationals.

13. Let E be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that cover E in the sense of Vitali. Show that there is a countable disjoint collection $\{I_k\}_{k=1}^{\infty}$ of intervals in $\mathcal{F}$ for which
$$m^*\left[E \setminus \bigcup_{k=1}^{\infty} I_k\right] = 0.$$

14. Use the Vitali Covering Lemma to show that the union of any collection (countable or uncountable) of closed, bounded, nondegenerate intervals is measurable.

15. Define f on $\mathbb{R}$ by
$$f(x) = \begin{cases} x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$
Find the upper and lower derivatives of f at $x = 0$.

16. Let g be integrable over $[a, b]$. Define the antiderivative of g of g to be the function f defined on $[a, b]$ by
$$f(x) = \int_a^x g \text{ for all } x \in [a, b].$$
Show that f is differentiable almost everywhere on (a, b).

17. Let f be an increasing bounded function on the open, bounded interval (a, b). Verify (18).

18. Show that if f is defined on (a, b) and $c \in (a, b)$ is a local minimizer for f, then $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$.

19. Let f be continuous on $[a, b]$ with $\underline{D}f \geq 0$ on (a, b). Show that f is increasing on $[a, b]$. (Hint: first show this for a function g for which $\underline{D}g \geq \epsilon > 0$ on (a, b). Apply this to the function $g(x) = f(x) + \epsilon x$.)

20. Let f and g be real-valued functions on (a, b). Show That
$$\underline{D}f + \underline{D}g \leq \underline{D}(f+g) \leq \overline{D}(f+g) \leq \overline{D}(f) + \overline{D}(g) \text{ on } (a, b).$$

21. Let f be defined on $[a, b]$ and g a continuous function on $[\alpha, \beta]$ that is differentiable at $\gamma \in (\alpha, \beta)$ with $g(\gamma) = c \in (a, b)$. Verify the following.
    - (i) If $g'(\gamma) > 0$, then $\overline{D}(f \circ g)(\gamma) = \overline{D}f(c) \cdot g'(\gamma)$.
    - (ii) If $g'(\gamma) = 0$ and the upper and lower derivatives of f at c are finite, then $\overline{D}(f \circ g)(\gamma) = 0$.

22. Show that a strictly increasing function that is defined on an interval is measurable and then use this to show that a monotone function that is defined on an interval is measurable.

23. Show that a continuous function f on $[a, b]$ is Lipschitz if its upper and lower derivatives are bounded on (a, b).

24. Show that for f defined in the last remark of this section, f' is not integrable over $[0, 1]$.

6.3 Functions of Bounded Variation: Jordan's Theorem

PROBLEMS

25. Suppose f is continuous on $[0, 1]$. Must there be a nondegenerate closed subinterval $[a, b]$ of $[0, 1]$ for which the restriction of f to $[a, b]$ is of bounded variation?

26. Let f be the Dirichlet function, the characteristic function of the rationals in $[0, 1]$. Is f of bounded variation on $[0, 1]$?

27. Define $f(x) = \sin x$ on $[0, 2\pi]$. Find two increasing functions h and g for which $f = h - g$ on $[0, 2\pi]$.

28. Let f be a step function on $[a, b]$. Find a formula for its total variation.

29. (i) Define
$$f(x) = \begin{cases} x^2 \cos(1/x^2) & x \neq 0, x \in [-1, 1] \\ 0 & x = 0 \end{cases}$$
Is f of bounded variation on $[-1, 1]$?

(ii) Define
$$g(x) = \begin{cases} x^2 \cos(1/x) & x \neq 0, x \in [-1, 1] \\ 0 & x = 0 \end{cases}$$
Is g of bounded variation on $[-1, 1]$?

30. Show that the linear combination of two functions of bounded variation is also of bounded variation. Is the product of two such functions also of bounded variation?

31. Let P be a partition of $[a, b]$ that is a refinement of the partition P'. For a real-valued function f on $[a, b]$, show that $V(f, P') \leq V(f, P)$.

32. Assume f is of bounded variation on $[a, b]$. Show that there is a sequence of partitions $\{P_n\}$ of $[a, b]$ for which the sequence $\{V(f, P_n)\}$ is increasing and converges to $TV(f)$.

33. Let $\{f_n\}$ be a sequence of real-valued functions on $[a, b]$ that converges pointwise on $[a, b]$ to the real-valued function f. Show that
$$TV(f) \leq \liminf TV(f_n).$$

34. Let f and g be of bounded variation on $[a, b]$. Show that
$$TV(f + g) \leq TV(f) + TV(g) \text{ and } TV(\alpha f) = |\alpha| TV(f).$$

35. For α and β positive numbers, define the function f on $[0,1]$ by

$$f(x) = \begin{cases} x^\alpha \sin(1/x^\beta) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that if $\alpha > \beta$, then f is of bounded variation on $[0,1]$, by showing that f' is integrable over $[0,1]$. Then show that if $\alpha \leq \beta$, then f is not of bounded variation on $[0,1]$.

36. Let f fail to be of bounded variation on $[0,1]$. Show that there is a point x_0 in $[0,1]$ such that there are subintervals of $[0,1]$ that contain x_0 and have arbitrarily small length on which f fails to be of bounded variation.

6.4 Absolutely Continuous Functions

PROBLEMS

37. Let f be a continuous function on $[0,1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$.

    (i) Show that f may not be absolutely continuous on $[0,1]$.

    (ii) Show that f is absolutely continuous on $[0,1]$ if it is increasing.

    (iii) Show that the function f on $[0,1]$, defined by $f(x) = \sqrt(x)$ for $0 \leq x \leq 1$, is absolutely continuous, but not Lipschitz, on $[0,1]$.

38. Show that f is absolutely continuous on $[a,b]$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for every countable disjoint collection $\{(a_k, b_k)\}_{k=1}^\infty$ of open intervals in (a,b),

$$\sum_{k=1}^\infty |f(b_k) - f(a_k)| < \epsilon \text{ if } \sum_{k=1}^\infty [b_k - a_k] < \delta.$$

39. Use the preceding problem to show that if f is continuous and increasing on $[a,b]$, then f is absolutely continuous on $[a,b]$ iff for each ϵ, there is a $\delta > 0$ such that for a measurable subset E of $[a,b]$,

$$m^*(f(E)) < \epsilon \text{ if } m(E) < \delta.$$

40. Use the preceding problem to show that an increasing absolutely continuous function f on $[a,b]$ maps sets of measure zero onto sets of measure zero. Conclude that the Cantor-Lebesgue function φ is not absolutely continuous on $[0,1]$ since the function ψ, defined by $\psi(x) = x + \varphi(x)$ for $0 \leq x \leq 1$, maps the Cantor set to a set of measure 1 (page 52).

41. Let f be an increasing absolutely continuous function on $[a,b]$. Use (i) and (ii) below to conclude that f maps measurable sets to measurable sets.

    (i) Infer from the continuity of f and the compactness of $[a,b]$ that f maps closed sets to closed sets and therefore maps F_σ sets to F_σ sets.

    (ii) The preceding problem tells us that f maps sets of measure zero to sets of measure zero.

42. Show that both the sum and product of absolutely continuous functions are absolutely continuous.

43. Define the functions f and g on $[-1,1]$ by $f(x) = x^{\frac{1}{3}}$ for $-1 \le x \le 1$ and

$$g(x) = \begin{cases} x^2 \cos(\pi/2x) & \text{if } x \neq 0, x \in [-1,1] \\ 0 & \text{if } x = 0 \end{cases}$$

(i) Show that both f and g are absolutely continuous on $[-1,1]$.

(ii) For the partition $P_n = \{-1, 0, 1/2n, 1/[2n-1], \cdots, 1/3, 1/2, 1\}$ of $[-1,1]$, examine $V(f \circ g, P_n)$.

(iii) Show that $f \circ g$ fails to be of bounded variation, and hence also fails to be absolutely continuous, on $[-1,1]$.

44. Let f be Lipschitz on $\mathbb{R}$ and g be absolutely continuous on $[a,b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a,b]$.

45. Let f be absolutely continuous on $\mathbb{R}$ and g be absolutely continuous and strictly monotone on $[a,b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a,b]$.

46. Verify the assertions made in the final remark of this section.

47. Show that a function f is absolutely continuous on $[a,b]$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^{n}$ of open intervals in (a,b),

$$\left|\sum_{k=1}^{n}[f(b_k) - f(a_k)]\right| < \epsilon \text{ if } \sum_{k=1}^{n}[b_k - a_k] < \delta.$$

6.5 Integrating Derivatives: Differentiating Indefinite Integrals

PROBLEMS

48. The Cantor-Lebesgue function φ is continuous and increasing on $[0,1]$. Conclude from Theorem 10 that φ is not absolutely continuous on $[0,1]$. Compare this reasoning with that proposed in Problem 40.

49. Let f be continuous on $[a,b]$ and differentiable almost everywhere on (a,b). Show that

$$\int_a^b f' = f(b) - f(a)$$

if and only if

$$\int_a^b [\lim_{n\to\infty} \text{Diff}_{1/n} f] = \lim_{n\to\infty} [\int_a^b \text{Diff}_{1/n} f].$$

50. Let f be continuous on $[a,b]$ and differentiable almost everywhere on (a,b). Show that if $\{\text{Diff}_{1/n} f\}$ is uniformly integrable over $[a,b]$, then

$$\int_a^b f' = f(b) - f(a).$$

51. Let f be continuous on $[a, b]$ and differentiable almost everywhere on (a, b). Suppose there is a nonnegative function g that is integrable over $[a, b]$ and

$$|\text{Diff}_{1/n} f| \leq g \text{ a.e. on } [a, b] \text{ for all } n.$$

Show that

$$\int_a^b f' = f(b) - f(a).$$

52. Let f and g be absolutely continuous on $[a, b]$. Show that

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

53. Let the function f be absolutely continuous on $[a, b]$. Show that f is Lipschitz on $[a, b]$ iff there is a $c > 0$ for which $|f'| \leq c$ a.e. on $[a, b]$.

54. (i) Let f be a singular increasing function on $[a, b]$. Use the Vitali Covering Lemma to show that f has the following property: Given $\epsilon > 0, \delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in (a, b) for which

$$\sum_{k=1}^n [b_k - a_k] < \delta \text{ and } \sum_{k=1}^n [f(b_k) - f(a_k)] > f(b) - f(a) - \epsilon.$$

(ii) Let f be an increasing function on $[a, b]$ with the property described in part (i). Show that f is singular.

(iii) Let $\{f_n\}$ be a sequence of singular increasing functions on $[a, b]$ for which the series $\sum_{n=1}^\infty f_n(x)$ converges to a finite value for each $x \in [a, b]$. Define

$$f(c) = \sum_{n=1}^\infty f_n(x) \text{ for } x \in [a, b].$$

Show that f is also singular.

55. Let f be of bounded variation on $[a, b]$, and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$.

(i) Show that $|f'| \leq v'$ a.e. on $[a, b]$, and infer from this that

$$\int_a^b |f'| \leq TV(f).$$

(ii) Show that the above is an equality iff f is absolutely continuous on $[a, b]$.

(iii) Compare parts (i) and (ii) with Corollaries 4 and 12, respectively.

56. Let g be strictly increasing and absolutely continuous on $[a, b]$.

(i) Show that for any open subset $\mathcal{O}$ of (a, b),

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'(x)dx.$$

(ii) Show that for any G_δ subset E of (a, b),

$$m(g(E)) = \int_E g'(x)dx.$$

(iii) Show that for any subset E of $[a, b]$ that has measure 0, its image $g(E)$ also has measure 0, so that

$$m(g(E)) = 0 = \int_E g'(x)dx.$$

(iv) Show that for any measurable subset A of $[a, b]$,

$$m(g(A)) = \int_A g'(x)dx.$$

(v) Let $c = g(a)$ and $d = g(b)$. Show that for any simple function φ on $[c, d]$,

$$\int_c^d \varphi(y)dy = \int_a^b \varphi(g(x))g'(x)dx.$$

(vi) Show that for any nonnegative integrable function f over $[c, d]$,

$$\int_c^d f(y)dy = \int_a^b f(g(x))g'(x)dx.$$

(vii) Show that part (i) follows from (vi) in the case that f is the characteristic function of $g(\mathcal{O})$ and the composition is defined.

57. Is the change of variables formula in part (vi) of the preceding problem true if we just assume g is increasing, not necessarily strictly?

58. Construct an absolutely continuous strictly increasing function f on $[0, 1]$ for which $f' = 0$ on a set of positive measure. (Hint: Let E be the relative complement in $[0, 1]$ of a generalized Cantor set of positive measure and f the indefinite integral of χ_E. See Problem 39 of Chapter 2 for the construction of such a Cantor set.)

59. For a nonnegative integrable function f over $[c, d]$, and a strictly increasing absolutely continuous function g on $[a, b]$ such that $g([a, b]) \subseteq [c, d]$, is it possible to justify the change of variables formula

$$\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$$

by showing that

$$\frac{d}{dx}\left[\int_{g(a)}^{g(x)} f(s)ds - \int_a^x f(g(t))g'(t)dt\right] = 0 \text{ for almost all } x \in (a, b)?$$

60. Let f be absolutely continuous and singular on $[a, b]$. Show that f is constant. Also show that the Lebesgue decomposition of a function of bounded variation is unique if the singular function is required to vanish a t $x = a$.

6.6 Convex Functions

PROBLEMS

61. Show that a real-valued function φ on (a, b) is convex iff for points $x_1, \cdots, x_n$ in (a, b) and nonnegative numbers $\lambda_1, \cdots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$,
$$\varphi\left(\sum_{k=1}^n \lambda_k x_k\right) \leq \sum_{k=1}^n \lambda_k \varphi(x_k).$$
Use this to directly prove Jensen's Inequality for f a simple function.

62. Show that a continuous function on (a, b) is convex iff
$$\varphi(\frac{x_1 + x_2}{2}) \leq \frac{\varphi(x_1) + \varphi(x_2)}{2} \text{ for all } x_1, x_2 \in (a, b).$$

63. A function on a general interval I is said to be convex provided it is continuous on I and (38) holds for all $x_1, x_2 \in I$. Is a convex function on a closed, bounded interval $[a, b]$ necessarily Lipschitz on $[a, b]$?

64. Let φ have a second derivative at each point in (a, b). Show that φ is convex iff φ'' is nonnegative.

65. Suppose $a \geq 0$ and $b \geq 0$. Show that the function $\varphi(t) = (a + bt)^p$ is convex on $[0, \infty)$ for $1 \leq p < \infty$.

66. For what functions φ is Jensen's Inequality always an equality?

67. State and prove a version of Jensen's Inequality on a general closed, bounded interval $[a, b]$.

68. Let f be integrable over $[0, 1]$. Show that
$$\exp\left[\int_0^1 f(x)dx\right] \leq \int_0^1 \exp(f(x))dx.$$

69. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers whose sum is 1 and $\{\zeta_n\}$ is a sequence of positive numbers. Show that
$$\prod_{n=1}^{\infty} \zeta_n^{\alpha_n} \leq \sum_{n=1}^{\infty} \alpha_n \zeta_n.$$

70. Let g be a positive measurable function on $[0, 1]$. Show that $\log(\int_0^1 g(x)dx) \geq \int_0^1 \log(g(x))dx$ whenever each side is defined.

71. (Nemytskii) Let φ be a continuous function on $\mathbb{R}$. Show that if there are constants for which (43) holds, then $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is. Then show that if $\varphi \circ f$ is integrable over $[0, 1]$ whenever f is, then there are constants c_1 and c_2 for which (43) holds.

Chapter 7

The L^p Spaces: Completeness and Approximation

Contents

7.1 Normed Linear Spaces

Throughout this chapter E denotes a measurable set of real numbers. Define $\mathcal{F}$ to be the collection of all measurable extended real-valued functions on E that are finite a.e. on E. We can say that two functions $f, g \in \mathcal{F}$ are equivalent, denoted by $f \cong g$, provided

$$f(x) = g(x) \text{ for almost all } x \in E.$$

This is an equivalence relation and induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes, denoted by $\mathcal{F}/\cong$, which is a linear space. There is a natural family $\{L^p(E)\}_{1 \le p \le \infty}$ of subspaces of $\mathcal{F}/\cong$.

For $1 \le p < \infty$, define $L^p(E)$ to be the collection of equivalence class $[f]$ for which

$$\int_E |f|^p < \infty.$$

Then if $f \cong g$, then $\int_E |f|^p = \int_E |g|^p$. Showing that $L^p(E)$ is closed under linear combinations will prove that $L^p(E)$ is a linear subspace. To do this, let $c = \max\{|a|, |b|\}$ so that

$$|a + b| \le |a| + |b| \le 2c,$$

which implies

$$|a+b|^p \leq 2^p c^p \leq 2^p(|a|^p + |b|^p).$$

This inequality, together with the linearity and monotonicity of integration tells us that

$$\int_E |\alpha f + \beta g|^p \leq 2^p(|\alpha|^p \int_E |f|^p + |\beta|^p \int_E |g|^p) < \infty.$$

That is, for $[f], [g] \in L^p(E)$, then $\alpha[f] + \beta[g] \in L^p(E)$.
We call a function $f \in \mathcal{F}$ **essentially bounded** provided there is some $M \geq 0$, called an **essential upper bound** for f, for which

$$|f(x)| \leq M \text{ for almost all } x \in E.$$

Then we can define $L^\infty(E)$ to be the collection of equivalence classes $[f]$ for which f is essentially bounded. Clearly $L^\infty(E)$ is a linear subspace because

$$|\alpha f(x) + \beta g(x)| \leq |\alpha||f(x)| + |\beta||g(x)| \leq |\alpha|M + |\beta|M' = M'' \text{ a.e. on } E$$

To state that a function f in $L^p[a,b]$ is continuous means that there is a continuous function that agrees with f a.e. on $[a,b]$. There is only one such continuous function and it is often convenient to consider this unique continuous function as the representative of $[f]$.
It is useful to consider real-valued functions that have as their domain linear spaces of functions: such functions are called **functionals**.

Definition. *Let X be a linear space. A real-valued functional $\|\cdot\|$ on X is called a **norm** provided for each f and g in X and each real number α,*
(The Triangle Inequality)

$$\|f+g\| \leq \|f\| + \|g\|,$$

(Positive Homogeneity)

$$\|\alpha f\| = |\alpha|\|f\|,$$

(Nonnegativity)

$$\|f\| \geq 0 \textit{ and } \|f\| = 0 \iff f = 0.$$

A **normed linear space** is a linear space together with a norm. If X is a linear space normed by $\|\cdot\|$ we say that a function f in X is a **unit function** provided $\|f\| = 1$. For any $f \in X, f \neq 0$, the function $\frac{f}{\|f\|}$ is a unit function: it is a scalar multiple of f which we call the **normalization** of f.

Example (The Normed Linear Space $L^1(E)$)**.** *For a function f in $L^1(E)$, define*

$$\|f\|_1 = \int_E |f|.$$

Then $\|\cdot\|$ is a norm on $L^1(E)$.
For $f, g \in L^1(E) \subseteq \mathcal{F}$, since f and g are finite a.e. on E, the triangle inequality for real numbers tells us that

$$|f+g| \leq |f| + |g| \textit{ a.e. on } E.$$

Then by the monotonicity and linearity of integration, we have subadditivity:

$$\|f+g\|_1 = \int_E |f+g| \leq \int_E [|f| + |g|] = \int_E |f| + \int_E |g| = \|f\|_1 + \|g\|_1.$$

By the linearity of integration, clearly we have absolute homogeneity:

$$\|\alpha f\|_1 = \int_E |\alpha f| = \int_E |\alpha||f| = |\alpha| \int_E |f| = |\alpha|\|f\|_1.$$

Clearly $\|f\|$ is nonnegative. Finally, if $f \in L^1(E)$ and $\|f\|_1 = 0$, then $f = 0$ a.e. on E. Therefore $[f]$ is the zero element of the linear space $L^1(E) \subseteq \mathcal{F}/\cong$, that is $f = 0$.

Example (The Normed Linear Space $L^\infty(E)$)**.** *For a function f in $L^\infty(E)$, define $\|f\|_\infty$ to be the infimum of the essential upper bounds for f.*

$$\|f\|_\infty = \inf\{M \ : \ |f(x)| \le M \text{ a.e. on } E\}.$$

We call $\|f\|_\infty$ the ***essential supremum*** *of f and claim that $\|\cdot\|_\infty$ is a norm on $L^\infty(E)$.*
Nonnegativity and positive homogeneity are clear.
To show that the triangle inequality holds, we see that for each natural number n, there is a subset E_n of E for which

$$|f| \le \|f\|_\infty + \frac{1}{n} \text{ on } E \setminus E_n \text{ and } m(E_n) = 0.$$

This is true because $\|f\|_\infty$ is the infimum, the greatest lower bound, so $\|f\|_\infty + \frac{1}{n}$ is not a lower bound and thus there exists a real number M in the set of upper bounds a.e. of f for which
$\|f\|_\infty \le M < \|f\|_\infty + \frac{1}{n}$ a.e. on E, and so $|f| \le M < \|f\|_\infty + \frac{1}{n}$ a.e. on E.
Accepting that the union of sets of measure zero is also measure zero, we can let $E_\infty = \bigcup_{n=1}^\infty E_n$, and so

$$|f| \le \|f\|_\infty \text{ on } E \setminus E_\infty \text{ and } m(E_n\infty) = 0.$$

Thus we have that $|f| \le \|f\|_\infty$ a.e. on E; i.e., ess. sup$f$ is the smallest essential upper bound for f.
Now, for $f, g \in L^\infty(E)$,

$$|f + g| \le |f| + |g| \le \|f\|_\infty + \|g\|_\infty \text{ a.e. on } E.$$

Therefore $\|f\|_\infty + \|g\|_\infty$ is an essential bound for $f + g$ and thus the smallest essential upper bound, $\|f + g\|_\infty$, is such that

$$\|f + g\|_\infty \le \|f\|_\infty + \|g\|_\infty.$$

Example (The Normed Linear Spaces ℓ^1 and ℓ^∞)**.** *For $1 \le p < \infty$, define ℓ^p to be the collection of real sequences $a = (a_1, a_2, \cdots)$ for which*

$$\sum_{k=1}^\infty |a_k|^p < \infty.$$

Let $a, b \in \ell^p$, and let α, β be real numbers. Then we have that $\sum_{k=1}^\infty |a_k|^p < \infty$ and $\sum_{k=1}^\infty |b_k|^p < \infty$.
Using the inequality $|a + b|^p \le 2^p(|a|^p + |b|^p)$, we have

$$\begin{aligned}
\sum_{k=1}^\infty |\alpha a_k + \beta b_k|^p &\le \sum_{k=1}^\infty [2^p(|\alpha a_k|^p + |\beta b_k|^p)] \\
&= \sum_{k=1}^\infty 2^p|\alpha|^p|a_k|^p + \sum_{k=1}^\infty 2^p|\beta|^p|b_k|^p \\
&= 2^p|\alpha|^p \sum_{k=1}^\infty |a_k|^p + 2^p|\beta|^p \sum_{k=1}^\infty |b_k|^p \\
&< 2^p|\alpha|^p\infty + 2^p|\beta|^p\infty \\
&= \infty.
\end{aligned}$$

Thus ℓ^p is a linear space.
We define ℓ^∞ to be the linear space of real bounded sequences: that is, for any $\{a_k\}$ in ℓ^∞, there exists a real number M for which $|a_k| \leq M$ for all k. We can define the following norms:
For $\{a_k\} \in \ell^1$:

$$\|\{a_k\}\|_1 = \sum_{k=1}^{\infty} |a_k|$$

For $\{a_k\} \in \ell^\infty$:

$$\|\{a_k\}\|_\infty = \sup_{1\leq k<\infty} |a_k|$$

Example (The Normed Linear Space $C[a,b]$)**.** *Let $[a,b]$ be a closed, bounded interval. The the linear space of continuous real-valued functions on $[a,b]$ is denoted by $C[a,b]$. Since a continuous function on a compact set takes on a maximum value (ch1 problem 52), we can define*

$$\|f\|_{\max} = \max_{x\in[a,b]} |f(x)|.$$

PROBLEMS

1. For f in $C[a,b]$, Define

$$\|f\|_1 = \int_a^b |f|.$$

Show that this is a norm on $C[a,b]$. Also show that there is no number $c \geq 0$ for which

$$\|f\|_{\max} \leq c\|f\|_1 \text{ for all } f \text{ in } C[a,b],$$

but there is a $c \geq 0$ for which

$$\|f\|_1 \leq c\|f\|_{\max} \text{ for all } f \text{ in } C[a,b].$$

Let $f, g \in C[a,b]$. For each $x \in [a.b]$, we have the inequality $|f(x)+g(x)| \leq |f(x)| + |g(x)|$, so by monotonicity and linearity of integration,

$$\|f+g\|_1 = \int_a^b |f(x)+g(x)| \leq \int_a^b [|f(x)|+|g(x)|] = \int_a^b |f(x)| + \int_a^b |g(x)| = \|f\|_1 + \|g\|_1.$$

Therefore subadditivity holds.
Also, by linearity of integration, we have

$$\|\alpha f\|_1 = \int_a^b |\alpha f| = \int_a^b |\alpha||f| = |\alpha| \int_a^b |f| = |\alpha|\|f\|_1.$$

Therefore absolute homogeneity holds.
Finally, by definition of absolute value, $0 \leq |f(x)|$ for all $x \in [a,b]$, and by monotonicity of integration,

$$0 = \int_a^b 0 \leq \int_a^b |f| = \|f\|_1.$$

Clearly $\int_a^b |f| = 0$ iff $f \equiv 0$ on $[a,b]$. Therefore positive definiteness holds.
Thus $\|\cdot\|_1$ is a norm on $C[a,b]$.

Consider the interval $[a, b] = [0, 1]$. For any $c > 0$ we choose, there exists an $n \in \mathbb{N}$ such that $n > c$, with the continuous function $f_n : [0, 1] \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} \frac{n-0}{1/n-0}x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{0-n}{2/n-1/n}(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases} = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ -n^2(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases}$$

(This is a triangle-shaped function that reaches its peak n at $x = \frac{1}{n}$.)
Now, for any n, we have $\|f_n\|_1 = \int_0^1 |f_n| = 1$, and $\|f_n\|_{\max} = n$.
Then $\|f\|_{\max} = n > c = c\|f\|_1$.

Finally, we can see that for any f in $C[a, b]$, by monotonicity of the integral,

$$\begin{aligned} \|f\|_1 &= \int_a^b |f(x)| \\ &\leq \int_a^b \max_{x\in[a,b]} |f(x)| \\ &= \max_{x\in[a,b]} |f(x)| \int_a^b 1 \\ &= \max_{x\in[a,b]} |f(x)| \cdot m([a, b]) \\ &= \|f\|_{\max} \cdot m([a, b]). \end{aligned}$$

Therefore $\|f\|_1 \leq m([a, b])\|f\|_{\max}$ for all $f \in C[a, b]$.

2. Let X be the family of all polynomials with real coefficients defined on $\mathbb{R}$. Show that this is a linear space. For a polynomial p, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Is this a norm?
For any two polynomials $p, q \in X$, there exists natural numbers n, m (suppose without loss of generality that $n \leq m$) such that

$$\begin{aligned} p(x) &= a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n + \cdots + 0x^m \\ q(x) &= b_0 + b_1 x + b_2 x^2 + \cdots + b_{n-1}x^{n-1} + b_n x^n + \cdots + b_m x^m \end{aligned}$$

Now, considering any scalars $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \alpha p(x) + \beta q(x) &= \alpha(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n) \\ &\quad + \beta(b_0 + b_1 x + b_2 x^2 + \cdots + b_{m-1}x^{m-1} + b_m x^m) \\ &= (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_{n-1})x^{n-1} + (\alpha a_n)x^n \\ &\quad + (\beta b_0) + (\beta b_1)x + (\beta b_2)x^2 + \cdots + (\beta b_{n-1})x^{n-1} + (\beta b_n)x^n + \cdots + (\beta b_m)x^m \\ &= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + \cdots + (\alpha a_n + \beta b_n)x^n + \cdots + (\beta b_m)x^m \end{aligned}$$

This is also a polynomial, as for each i, we have $(\alpha a_i + \beta b_i) \in \mathbb{R}$, so X is a linear space.
Now, for any polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

we can define $\|p\| = |a_0| + |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=0}^{n} |a_i|$.
The triangle inequality is clear because

$$\|p+q\| = \sum_{i=0}^{m} |a_i + b_i| \leq \sum_{i=0}^{m} [|a_i| + |b_i|] = \sum_{i=0}^{m} |a_i| + \sum_{i=0}^{m} |b_i| = \|p\| + \|q\|.$$

Absolute homogeneity is clear because

$$\|\alpha p\| = \sum_{i=0}^{n} |\alpha a_i| = \sum_{i=0}^{n} |\alpha||a_i| = |\alpha| \sum_{i=0}^{n} |a_i| = |\alpha|\|p\|.$$

Finally, positive definiteness is clear because

$$0 \leq |a_i| \implies 0 \leq \sum_{i=0}^{n} |a_i| = \|p\|,$$

And $\|p\| = 0$ if and only if $p(x) = 0 + 0x + 0x^2 + \cdots 0x^n = 0$.

3. For f in $L^1[a,b]$, define $\|f\| = \int_a^b x^2 |f(x)| dx$. Show that this is a norm on $L^1[a,b]$.
For $f \in L^1[a,b]$, then f is measurable and finite a.e. on $[a,b]$, and $\int_a^b |f(x)| dx < \infty$.
Let $f, g \in L^1[a,b]$, and let α be a real number.
Because the triangle inequality holds a.e. on $[a,b]$, by monotonicity and linearity of the integral, we have

$$\begin{aligned} \|f+g\| &= \int_a^b x^2 |f(x) + g(x)| dx \\ &\leq \int_a^b x^2 [|f(x)| + |g(x)|] dx \\ &= \int_a^b [x^2 |f(x)| + x^2 |g(x)|] dx \\ &= \int_a^b x^2 |f(x)| dx + \int_a^b x^2 |g(x)| dx \\ &= \|f\| + \|g\|. \end{aligned}$$

Therefore $\|\cdot\|$ is subadditive.
By linearity of the integral, we have

$$\|\alpha f\| = \int_a^b x^2 |\alpha f(x)| dx = \int_a^b x^2 |\alpha||f(x)| dx = |\alpha| \int_a^b x^2 |f(x)| dx = |\alpha|\|f\|.$$

Therefore $\|\cdot\|$ satisfies absolute homogeneity.
We can use the fact that $0 \leq x^2$ and $0 \leq |f(x)|$ implies $0 \leq x^2 |f(x)|$. By monotonicity of the integral, we have

$$0 = \int_a^b 0 dx \leq \int_a^b x^2 |f(x)| dx = \|f\|.$$

Clearly $\|f\| = 0$ if and only if $f = 0$ a.e. on $[a,b]$ because $x^2 \cdot 0 = 0$.
Therefore $\|\cdot\|$ satisfies positive definiteness.

4. For f in $L^\infty[a,b]$, show that

$$\|f\|_\infty = \min\left\{M \,\middle|\, m\{x \in [a,b] \mid |f(x)| > M\} = 0\right\},$$

That is, the sup norm is the smallest real number M such that $|f(x)| > M$ only on a set of measure zero. In an above example, we showed that $\|f\|_\infty$ is the smallest essential upper bound for f. That is, $|f| \leq \|f\|_\infty$ a.e. on E (That is, the inequality is true for $E \setminus E_0$, where $m(E_0) = 0$.)
and if, furthermore, f is continuous on $[a,b]$, that

$$\|f\|_\infty = \|f\|_{\max}.$$

If f is continuous, then there are no jump discontinuities (f is continuous at x_0 iff $f(x_0^-) = f(x_0) = f(x_0^+)$). Then $|f| \leq \|f\|_\infty$ everywhere on E.

5. Show that ℓ^∞ and ℓ^1 are normed linear spaces.
ℓ^∞:
Let $a, b \in \ell^\infty$, and let α, β be real numbers.
Then for some real numbers M, N, we have that $|a_k| \leq M$ and $|b_k| \leq N$ for all k.

$$\begin{aligned}\alpha a + \beta b &= \alpha(a_1, a_2, \cdots) + \beta(b_1, b_2, \cdots)\\ &= (\alpha a_1, \alpha a_2, \cdots) + (\beta b_1, \beta b_2, \cdots)\\ &= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \cdots)\end{aligned}$$

Then $|\alpha a_k + \beta b_k| \leq \alpha M + \beta N$ for all k, and ℓ^∞ is a linear space.
To show that $\|a\|_\infty = \sup_{1\leq k<\infty} |a_k|$ is a norm:

$$\|a+b\|_\infty = \sup_{1\leq k<\infty} |a_k + b_k| \leq \sup_{1\leq i<\infty} |a_i| + \sup_{1\leq j<\infty} |b_j| = \|a\|_\infty + \|b\|_\infty,$$

$$\|\alpha a\|_\infty = \sup_{1\leq k<\infty} |\alpha a_k| = \sup_{1\leq k<\infty} |\alpha||a_k| = |\alpha| \sup_{1\leq k<\infty} |a_k| = |\alpha|\|a\|_\infty,$$

$$0 \leq \sup_{1\leq k<\infty} |a_k| = \|a\|_\infty, \text{ and } \sup_{1\leq k<\infty} |a_k| = 0 \text{ iff } a_k = 0 \text{ for all } k.$$

ℓ^1:
Let $a, b \in \ell^1$, and let α, β be real numbers.
Then we have that $\sum_{k=1}^\infty |a_k| < \infty$ and $\sum_{k=1}^\infty |b_k| < \infty$.
By the triangle inequality for real numbers, we have

$$\sum_{k=1}^\infty |\alpha a_k + \beta b_k| \leq \sum_{k=1}^\infty [|\alpha||a_k| + |\beta||b_k|] = |\alpha| \sum_{k=1}^\infty |a_k| + |\beta| \sum_{k=1}^\infty |b_k| < |\alpha|\infty + |\beta|\infty = \infty.$$

Therefore ℓ^1 is a linear space.
To show that $\|a\|_1 = \sum_{k=1}^\infty |a_k|$ is a norm:

$$\|a+b\|_1 = \sum_{k=1}^\infty |a_k + b_k| \leq \sum_{k=1}^\infty [|a_k| + |b_k|] = \sum_{k=1}^\infty |a_k| + \sum_{k=1}^\infty |b_k| < \infty + \infty = \infty,$$

$$\|\alpha a\|_1 = \sum_{k=1}^\infty |\alpha a_k| = \sum_{k=1}^\infty |\alpha||a_k| = |\alpha| \sum_{k=1}^\infty |a_k| = |\alpha|\|a\|_1,$$

$$0 \leq |a_k| \implies 0 \leq \sum_{k=1}^\infty |a_k| = \|a\|_1, \text{ and } \sum_{k=1}^\infty |a_k| = 0 \text{ iff } a_k = 0 \text{ for all } k.$$

7.2 The Inequalities of Young, Hölder, and Minkowski

PROBLEMS

6. Show that if Hölder's Inequality is true for normalized functions it is true in general.

7. Verify the assertions in the above two examples regarding the membership of the function f in $L^p(E)$.

8. Let f and g belong to $L^2(E)$. From the linearity of integration show that for any number λ,

$$\lambda^2 \int_E f^2 + 2\lambda \int_E f \cdot g + \int_E g^2 = \int_E (\lambda f + g)^2 \geq 0.$$

From this and the quadratic formula directly derive the Cauchy-Schwarz Inequality.

9. Show that in Young's Inequality there is equality iff $a^p = b^q$.

10. Show that in Hölder's Inequality there is equality iff there are constants α, β not both zero, for which

$$\alpha|f|^p = \beta|g|^q \text{ a.e. on } E.$$

For a point $x = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$, define T_x to be the step function on the interval ...

7.3 L^p is Complete: The Riesz-Fischer Theorem

7.4 Approximation and Separability

II ABSTRACT SPACES: METRIC, TOPOLOGICAL, BANACH, AND HILBERT SPACES

Chapter 9

Metric Spaces: General Properties

Contents

9.1 Examples of Metric Spaces

"The object of the present chapter is to study general spaces called metric spaces for which the notion of distance between two points is fundamental."

Definition. *Let X be a nonempty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** provided for all $x, y, z \in X$,*

(i) $\rho(x, y) \geq 0$,

(ii) $\rho(x, y) = 0$ *iff* $x = y$,

(iii) $\rho(x, y) = \rho(y, x)$,

(iv) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

*A nonempty set together with a metric on the set is called a **metric space**, often denoted by* (X, ρ).

An example of a metric space is the set $\mathbb{R}$ of all real numbers with the metric $\rho(x, y) = |x - y|$.

A linear space with a norm is called a normed linear space. A norm $\|\cdot\|$ on a linear space X induces a metric ρ on X by defining

$$\rho(x, y) = \|x - y\| \text{ for all } x, y \in X.$$

To show this, let $x, y \in X$. Because X is a linear space, $x - y \in X$, and $\|x - y\|$ is defined.

(i) $\|x-y\| \geq 0$ by positive definiteness of norm

(ii) $\|x-y\| = 0$ iff $x-y=0 \implies x=y$ by positive definiteness of norm

(iii) $\|x-y\| = \|-1(y-x)\| = |-1|\|y-x\| = \|y-x\|$ by absolute homogeneity of norm

(iv) $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ by subadditivity of norm

Three prominent examples of normed linear spaces: the Euclidean spaces $\mathbb{R}^n$, the $L^p(E)$ spaces, $C[a,b]$. For a natural number n, consider the linear space $\mathbb{R}^n$, whose points are n-tuples of real numbers. For $x = (x_1, \cdots, c_n) \in \mathbb{R}^n$, the Euclidean norm $\|x\|$ is defined by

$$\|x\| = [x_1^2 + \cdots + x_n^2]^{1/2}.$$

The Discrete Metric For any nonempty set X, the discrete metric ρ is defined by setting $\rho(x,y)=0$ if $x=y$ and $\rho(x,y)=1$ if $x \neq y$.

(i) $\rho(x,y) \in \{0,1\} \implies \rho(x,y) \geq 0.$

(ii) $\rho(x,y) = 0 \iff x=y$ by definition.

(iii) By symmetry of the equality relation,

$$\rho(x,y) = \begin{cases} 1 & x=y \\ 0 & x \neq y \end{cases} = \begin{cases} 1 & y=x \\ 0 & y \neq x \end{cases} = \rho(y,x)$$

(iv) In the case $x \neq y$, the triangle inequality is trivial.
In the case $x=y$,
if $x=z$, then by transitivity, $z=y$ and
$\rho(x,y)=0, \rho(x,z)=0, \rho(z,y)=0$, and the triangle inequality is obvious: $0 \leq 0$.
if $x \neq z$, then by transitivity, $z \neq y$ and
$\rho(x,y)=0, \rho(x,z)=1, \rho(z,y)=1 \implies \rho(x,y)=0 \leq 2 = \rho(x,z)+\rho(z,y).$

Metric Subspaces For a metric space (X,ρ), let Y be a nonempty subset of X. Then the restriction of ρ to $Y \times Y$ defines a metric on Y and we call such a metric space a metric **subspace**. Therefore every nonempty subset of Euclidean space, of and $L^p(E)$ space, $1 \leq p \leq \infty$, and of $C[a,b]$ is a metric space.

Metric Products For metric spaces (X_1,ρ_1) and (X_2,ρ_2), we define the **product metric** τ on the Cartesian product $X_1 \times X_2$ by setting, for $(x_1,x_2),(y_1,y_2) \in X_1 \times X_2$,

$$\tau((x_1,x_2),(y_1,y_2)) = \{[\rho(x_1,y_1)]^2 + [\rho(x_2,y_2)]^2\}^{1/2}.$$

To see that this is a metric:

(i) By metric property (i) of ρ_1, ρ_2, we have

$$\begin{aligned} [\rho_1(x_1,y_1)]^2 \geq 0 \text{ and } [\rho_2(x_2,y_2)]^2 \geq 0 &\iff \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2} \geq 0 \\ &\iff \tau((x_1,x_2),(y_1,y_2)) \geq 0 \end{aligned}$$

(ii) By metric property (ii) of ρ_1, ρ_2, we have

$$\begin{aligned} \tau((x_1,x_2),(y_1,y_2)) = \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2} &= 0 \\ \iff [\rho_1(x_1,y_1)]^2 = 0 \text{ and } [\rho_2(x_2,y_2)]^2 &= 0 \\ \iff x_1 = y_1 \text{ and } x_2 &= y_2 \\ \iff (x_1,x_2) &= (y_1,y_2). \end{aligned}$$

(iii) By metric property (iii) (symmetry) of ρ_1, ρ_2, we have

$$\begin{aligned}\tau((x_1,x_2),(y_1,y_2)) &= \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2}\\ &= \{[\rho_1(y_1,x_1)]^2 + [\rho_2(y_2,x_2)]^2\}^{1/2}\\ &= \tau((y_1,y_2),(x_1,x_2)).\end{aligned}$$

(iv) We must first prove an inequality $\star$. Because $x^2 \geq 0$ for any real number x, we have

$$\begin{aligned}0 &\leq [\rho_1(x_1,z_1)\rho_2(z_2,y_2) - \rho_2(x_2,z_2)\rho_1(z_1,y_1)]^2\\ 0 &\leq \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2\\ &\quad - 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2)\\ 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2\end{aligned}$$

Adding $\rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2$ to both sides, we have

$$\begin{aligned}&\rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2 + 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2)\\ &\leq \rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2 + \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2\end{aligned}$$

Therefore we end up with the inequality: $\star$

$$\begin{aligned}[\rho_1(x_1,z_1)\rho_1(z_1,y_1) + \rho_2(x_2,z_2)\rho_2(z_2,y_2)]^2 &\leq [\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2][\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2]\\ \rho_1(x_1,z_1)\rho_1(z_1,y_1) + \rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq \sqrt{[\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2][\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2]}\\ 2\rho_1(x_1,z_1)\rho_1(z_1,y_1) + 2\rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq 2\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2}\sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2} \quad \star\end{aligned}$$

Now, to prove the triangle inequality of the product metric, we use the metric property (iv) (triangle inequality) of ρ_1, ρ_2:

$$\begin{aligned}&\rho(x_1,y_1)^2 + \rho(x_2,y_2)^2\\ &\leq [\rho(x_1,z_1) + \rho(z_1,y_1)]^2 + [\rho(x_2,z_2) + \rho(z_2,y_2)]^2\\ &= \rho(x_1,z_1)^2 + \rho(z_1,y_1)^2 + 2\rho(x_1,z_1)\rho(z_1,y_1) + \rho(x_2,z_2)^2 + \rho(z_2,y_2)^2 + 2\rho(x_2,z_2)\rho(z_2,y_2)\\ &\leq \rho(x_1,z_1)^2 + \rho(z_1,y_1)^2 + \rho(x_2,z_2)^2 + \rho(z_2,y_2)^2 + 2\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2}\sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2} \quad \star\\ &= \left[\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2} + \sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2}\right]^2\end{aligned}$$

Therefore we have

$$\sqrt{\rho(x_1,y_1)^2 + \rho(x_2,y_2)^2} \leq \sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2} + \sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2}$$

and thus

$$\tau((x_1,x_2),(y_1,y_2)) \leq \tau((x_1,x_2),(z_1,z_2)) + \tau((z_1,z_2),(y_1,y_2)).$$

This construction extends to countable products (problem 10).

Definition. *Two metrics ρ and σ on a set X are said to be **equivalent** provided there are positive numbers c_1, c_2 such that for all $x_1, x_2 \in X$,*

$$c_1 \cdot \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \sigma(x_1, x_2).$$

Definition. *A mapping f from a metric space (X, ρ) to a metric space (Y, σ) is said to be an **isometry** provided it maps X onto Y and for all $x_1, x_2 \in X$,*

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2).$$

Two metric spaces are called **isometric** provided there is an isometry from one onto the other. To be isometric is an equivalence relation among metric spaces. From the viewpoint of metric spaces, two isometric metric spaces are exactly the same, an isometry amounting merely to a relabeling of the points.

In the definition of a metric ρ on a set X it is often convenient to relax the condition that $\rho(x, y) = 0$ if and only if $x = y$. When we allow the possibility that $\rho(x, y) = 0$ for some $x \neq y$, we call ρ a **pseudometric** and (X, ρ) a pseudometric space. On such a space, define the relation $x \cong y$ provided $\rho(x, y) = 0$. This is an equivalence relation that separates X into a disjoint collection of equivalence classes $X/\cong$.

PROBLEMS

1. Show that two metrics ρ and τ on the same set X are equivalent iff there is a $c > 0$ such that for all $u, v \in X$,

$$\frac{1}{c}\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

Let ρ and τ be two metrics on the same set X.
($\Longrightarrow$) Suppose ρ and τ are equivalent.
Then there exist $c_1, c_2 > 0$ such that for all $u, v \in X$,

$$c_1 \cdot \tau(u, v) \leq \rho(u, v) \leq c_2 \cdot \tau(u, v).$$

By the Archimedean Property of $\mathbb{R}$, for the positive real number c_1, there exists a natural number n for which $\frac{1}{n} < c_1$.
Let $c = \max\{n, c_2\}$ so that $n \leq c \implies \frac{1}{c} \leq \frac{1}{n} < c_1$ and also $c_2 \leq c$, so we have

$$\frac{1}{c} \cdot \tau(u, v) < c_1 \cdot \tau(u, v) \leq \rho(u, v) \leq c_2 \cdot \tau(u, v) \leq c\tau(u, v).$$

($\Longleftarrow$) Suppose that there is a $c > 0$ such that for all $u, v \in X$,

$$\frac{1}{c}\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

We showed that for a positive number c, its multiplicative inverse $\frac{1}{c}$ is also positive [ch1, 2(ii)], and so we have $\frac{1}{c}, c > 0$ such that for all $u, v \in X$,

$$(\frac{1}{c})\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

Therefore ρ and τ are equivalent.

2. Show that the following define equivalent metrics on $\mathbb{R}^n$:

$$\rho^*(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|;$$
$$\rho^+(x,y) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}.$$

For some $j \in \{1, \cdots, n\}$, we have $|x_j - y_j| = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$.
Then

$$\begin{aligned}\rho^+(x,y) &= |x_j - y_j| \\ &= 0 + \cdots + |x_j - y_j| + \cdots + 0 \\ &\leq |x_1 - y_1| + \cdots |x_j - y_j| + \cdots + |x_n - y_n| \\ &= \rho^*(x,y).\end{aligned}$$

Also,

$$\begin{aligned}\rho^*(x,y) &= |x_1 - y_1| + \cdots |x_j - y_j| + \cdots + |x_n - y_n| \\ &\leq |x_j - y_j| + \cdots |x_j - y_j| + \cdots + |x_j - y_j| \\ &= n|x_j - y_j| \\ &= n\rho^+(x,y).\end{aligned}$$

Therefore we have $1, n > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\rho^+(x,y) \leq \rho^*(x,y) \leq n\rho^+(x,y).$$

3. Find a metric on $\mathbb{R}^n$ that fails to be equivalent to either of those defined in the preceding problem.
Let

$$\rho(x,y) = \begin{cases} \min\{|x-y|, 1\} & \text{if } x, y \text{ are both rational or } x, y \text{ are both irrational} \\ 1 & \text{else} \end{cases}$$

4. For a closed, bounded interval $[a,b]$, consider the set $X = C[a,b]$ of continuous real-valued functions on $[a,b]$. Show that the metric induced by the maximum norm and that induced by the $L^1[a,b]$ norm are not equivalent.
From Chapter 7 Problem 1, we proved that there is no number $c \geq 0$ for which

$$\|f\|_{\max} \leq c\|f\|_1 \text{ for all } f \text{ in } C[a,b].$$

Therefore there exists no $c_1, c_2 > 0$ such that for all f, g in $C[a,b]$,

$$c_1\|f - g\|_1 \leq \|f - g\|_{\max} \leq c_2\|f - g\|_1,$$

and the metrics induced by the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_1$ are not equivalent.

5. *The Nikodym Metric*. Let E be a Lebesgue measurable set of real numbers of finite measure, X the set of measurable subsets of E, and m Lebesgue measure. For $A, B \in X$, define $\rho(A,B) = m(A\Delta B)$, where $A\Delta B = [A \setminus B] \cup [B \setminus A]$, the symmetric difference of A and B. Show that this is a pseudometric on X. Define two measurable sets to be equivalent provided their symmetric difference has measure zero. Show that ρ induces a metric on the collection of equivalence classes. Finally, show that for $A, B \in X$,

$$\rho(A,B) = \int_E |\chi_A - \chi_B|,$$

where χ_A and χ_B are the characteristic functions of A and B, respectively.

6. Show that for $a, b, c \geq 0$,

$$\text{if } a \leq b + c, \text{ then } \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Because $a \leq b + c$, we see that $\frac{1}{1+b+c} \leq \frac{1}{1+a}$, and $-\frac{1}{1+a} \leq -\frac{1}{1+b+c}$ so

$$\frac{a}{1+a} = 1 - \frac{1}{1+a} \leq 1 - \frac{1}{1+b+c} = \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

7. Let E be a Lebesgue measurable set of real numbers that has finite measure and X the set of Lebesgue measurable real-valued functions on E. For $f, g \in X$, define

$$\rho(f, g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Use the preceding problem to show that this is a pseudometric on X. Define two measurable functions to be equivalent provided they are equal a.e. on E. Show that ρ induces a metric on the collection of equivalence classes.

Let $f, g, h \in X$.
Then for all $x \in E$, we have the triangle inequality $|f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$.
By Problem 6 and monotonicity and linearity of integration, we have

$$\int_E \frac{|f-g|}{1+|f-g|} \leq \int_E \frac{|f-h|}{1+|f-h|} + \int_E \frac{|h-g|}{1+|h-g|},$$

Therefore $\rho(f, g) \leq \rho(f, h) + \rho(h, g)$ and (iv) holds.
Clearly $\rho(f, g) \geq 0$ is true so (i) holds.
We also have $\rho(f, g) = \int_E \frac{|f-g|}{1+|f-g|} = \int_E \frac{|g-h|}{1+|g-h|} = \rho(g, f)$ so (iii) holds.
However, (ii) does not hold. We can consider two functions $f, g \in X$ such that $f = g$ on $E \setminus E_0$, where $m(E_0) = 0$. Then $\rho(f, g) = \int_E \frac{|f-g|}{1+|f-g|} = \int_{E \setminus E_0} \frac{|f-f|}{1+|f-f|} = 0$ but $f \neq g$.
If we consider defining the equivalence

$$f \cong g \text{ when } f = g \text{ on } E \setminus E_0 \text{ where } m(E_0) = 0,$$

Then ρ now induces a metric on the collection of equivalence classes $X/\cong$, because we now have $\rho(f, g) = 0 \iff f \cong g$.

8. For $0 < p < 1$, show that
$$(a+b)^p \leq a^p + b^p \text{ for all } a, b \geq 0.$$

9. For E a Lebesgue measurable set of real numbers, $0 < p < 1$, and g, h Lebesgue measurable functions on E that have integrable p^{th} powers, define

$$\rho_p(h, g) = \int_E |g-h|^p.$$

Use the preceding problem to show that this is a pseudometric on the collection of Lebesgue measurable functions on E that have integrable p^{th} powers. Define two such functions to be equivalent provided they are equal a.e. on E. Show that $\rho_p(\cdot, \cdot)$ induces a metric on the collection of equivalence classes.

Let $f, g, h : E \to \mathbb{R}$ such that f, g, h are Lebesgue measurable and have integrable p^{th} powers. For all $x \in E$, Problem 8 tells us that $|g(x) - h(x)|^p \leq |g(x) - f(x)|^p + |f(x) - h(x)|^p$. By monotonicity and linearity of integration, we have

$$\int_E |g - h|^p \leq \int_E |g - f|^p + \int_E |f - h|^p.$$

Therefore $\rho_p(h, g) \leq \rho_p(f, g) + \rho_p(h, f)$ and (iv) holds.
Clearly $\rho_p(h, g) \geq 0$ is true so (i) holds.
We also have $\rho_p(h, g) = \int_E |g - h|^p = \int_E |h - g|^p = \rho_p(g, h)$ so (iii) holds.
However, (ii) does not hold. We can consider two functions h, g that are equivalent a.e. on E but not equal. Then $\rho_p(h, g) = \int_E |g - h|^p = \int_{E \setminus E_0} |g - h|^p = 0$ with $m(E_0) = 0$ but $h \neq g$.
If we define an equivalence when two functions are equal a.e. on E, then ρ_p is a metric on the collection of such equivalence classes.

10. Let $\{(X_n, \rho_n)\}_{n=1}^{\infty}$ be a countable collection of metric spaces. Use problem 6 to show that ρ_* defines a metric on the Cartesian product $\prod_{n=1}^{\infty} X_n$, where for points $x = \{x_n\}$ and $y = \{y_n\}$ in $\prod_{n=1}^{\infty} X_n$,
$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

11. Let (X, ρ) be a metric space and A any set for which there is a one-to-one (injective) mapping f of A onto (surjective?) the set X (bijection?). Show that there is a unique metric on A for which f is an isometry of metric spaces. (This is the sense in which an isometry amounts merely to a relabeling of the points in a space.)

12. Show that the triangle inequality for Euclidean space $\mathbb{R}^n$ follows from the triangle inequality for $L^2[0, 1]$.

9.2 Open Sets, Closed Sets, and Convergent Sequences

9.3 Continuous Mappings Between Metric Spaces

9.4 Complete Metric Spaces

9.5 Compact Metric Spaces

9.6 Separable Metric Spaces

Chapter 13

Continuous Linear Operators Between Banach Spaces

Contents

13.1 Normed Linear Spaces

PROBLEMS

1. Show that a nonempty subset S of a linear space X is a subspace iff $S + S = S$ and $\lambda \cdot S = S$ for each $\lambda \in \mathbb{R}, \lambda \neq 0$.

2. If Y and Z are subspaces of the linear space X, show that $T + Z$ is also a subspace and $Y + Z = \text{span}[Y \cup Z]$.

3. Let S be a subset of a normed linear space X.

   (i) Show that the intersection of a collection of linear subspaces of X is also a linear subspace of X.

   (ii) Show that span$[S]$ is the intersection of all the linear subspaces of X that contain S and therefore is a linear subspace of X.

   (iii) Show that $\overline{\text{span}}[S]$ is the intersection of all the closed linear subspaces of X that contain S and is therefore a closed linear subspace of X.

4. For a normed linear space X, show that the function $\|\cdot\| : X \to \mathbb{R}$ is continuous.

   Fix $\epsilon > 0$.

Then let $\delta := \epsilon > 0$.
Consider any $x, y \in X$ such that $\|x - y\| < \delta$.
Then by the reverse triangle inequality $(*)$,

$$|\|x\| - \|y\|| \leq \|x - y\| < \delta = \epsilon,$$

and $\| \cdot \|$ is continuous.

$(*)$ Proof of reverse triangle inequality:

$$\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|.$$

5. For two normed linear spaces $(X, \| \cdot \|_1)$ and $(Y, \| \cdot \|_2)$, define a linear structure on the Cartesian product $X \times Y$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$ and $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$. Define the product norm $\| \cdot \|$ by $\|(x, y)\| = \|x\|_1 + \|y\|_2$, for $x \in X$ and $y \in Y$. Show that this is a norm with respect to which a sequence converges if and only if each of the two component sequences converges. Furthermore, show that if X and Y are Banach spaces, then so is $X \times Y$.

   Let $(X \times Y, \| \cdot \|)$ be a normed linear space.

   $(\implies)$ Let $\{(x_n, y_n)\}$ be a sequence in $X \times Y$, and suppose that it converges to some $(x, y) \in X \times Y$ with respect to the norm $\| \cdot \|$.

   Fix $\epsilon > 0$.
   Then there exists an index N such that for all $n \geq N$,

$$0 \leq \|x_n - x\|_1 + \|y_n - y\|_2 = \|(x_n - x, y_n - y)\| = \|(x_n, y_n) - (x, y)\| < \epsilon,$$

   which implies that $\{x_n\} \to x$ w.r.t. the norm $\| \cdot \|_1$ and $\{y_n\} \to y$ w.r.t. the norm $\| \cdot \|_2$.

   $(\impliedby)$ Let $\{x_n\}$ be a sequence in X, and $\{y_n\}$ be a sequence in Y, and suppose that there exist $x \in X$, $y \in Y$ such that $\{x_n\} \to x$ w.r.t. the norm $\| \cdot \|_1$ and $\{y_n\} \to y$ w.r.t. the norm $\| \cdot \|_2$.

   Fix $\epsilon > 0$.
   Then there exists an index N_x such that for all $n \geq N_x$,

$$\|x_n - x\|_1 < \frac{\epsilon}{2},$$

   and there also exists an index N_y such that for all $n \geq N_y$,

$$\|y_n - y\|_1 < \frac{\epsilon}{2}.$$

   Thus for all $n \geq \max\{N_x, N_y\}$,

$$\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\|_1 + \|y_n - y\|_2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

   and therefore the sequence $\{(x_n, y_n)\}$ in $X \times Y$ converges to $(x, y) \in X \times Y$ with respect to the norm $\| \cdot \|$.

   Finally, suppose that X and Y are Banach spaces.

Let $\{(x_n, y_n)\}$ be any sequence in $X \times Y$ that is Cauchy.
Then for any ϵ, there exists an index N such that for all $n, m \geq N$, then

$$0 \leq \|x_n - x_m\|_1 + \|y_n - y_m\|_2 = \|(x_n - x_m, y_n - y_m)\| = \|(x_n, y_n) - (x_m, y_m)\| < \epsilon,$$

which implies that the sequences $\{x_n\}$ and $\{y_n\}$ are also Cauchy.
Then because both X and Y are Banach spaces, then $\{x_n\} \to x$ and $\{y_n\} \to y$ for some $x \in X$ and $y \in Y$, and therefore we proved in ($\Longleftarrow$) that $\{(x_n, y_n)\} \to (x, y)$, which implies that $X \times Y$ is a Banach space.

6. Let X be a normed linear space.
   (i) Let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\{x_n\} \to x$ and $\{y_n\} \to y$. Show that for any real numbers α and β, $\{\alpha x_n + \beta y_n\} \to \alpha x + \beta y$.
   (ii) Use (i) to show that if Y is a subspace of X, then its closure $\overline{Y}$ also is a linear subspace of X.
   (iii) Use (i) to show that the vector sum is continuous from $X \times X$ to X and scalar multiplication is continuous from $\mathbb{R} \times X$ to X.
7. Show that the set $\mathcal{P}$ of all polynomials on $[a, b]$ is a linear space. For $\mathcal{P}$ considered as a subset of the normed linear space $C[a, b]$, show that $\mathcal{P}$ fails to be closed. For $\mathcal{P}$ considered as a subset of the normed linear space $L^1[a, b]$, show that $\mathcal{P}$ fails to be closed.
8. A nonnegative real-valued function $\|\cdot\|$ defined on a vector space X is called a **pseudonorm** if $\|x + y\| \leq \|x\| + \|y\|$ and $\|\alpha x\| = |\alpha|\|x\|$. Define $x \cong y$, provided $\|x - y\| = 0$. Show that this is an equivalence relation. Define $X/\cong$ to be the set of equivalence classes of X under $\cong$ and for $x \in X$ define $[x]$ to be the equivalence class of x. Show that $X/\cong$ is a normed vector space if we define $\alpha[x] + \beta[y]$ to be the equivalence class of $\alpha x + \beta y$ and define $\|[x]\| = \|x\|$. Illustrate this procedure with $X = L^p[a, b], 1 \leq p < \infty$.

13.2 Linear Operators

Theorem 1. *A linear operator between normed linear spaces is continuous iff it is bounded.*

Proof. Let $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ be a linear operator.

($\Longrightarrow$) Suppose that T is continuous.
Then for $\epsilon = 1$, there exists a $\delta > 0$ such that, for any $x \in X$ such that $\|x - 0\|_X = \|x\|_X \leq \delta$, then

$$\|T(x) - T(0)\|_Y = \|T(x)\|_Y < 1.$$

(Where $T(0) = 0$ by linearity.)
Therefore consider any $u \in X$, $u \neq 0$.

$$\begin{aligned}
\|T(u)\|_Y &= \left\| T\left(\frac{\delta \cdot \|u\|_X}{\delta \cdot \|u\|_X} u\right) \right\|_Y \\
&= \frac{\|u\|_X}{\delta} \left\| T\left(\frac{\delta}{\|u\|_X} u\right) \right\|_Y && \text{by linearity of } T \text{ and absolute homogeneity of } \|\cdot\|_Y. \\
&< \frac{\|u\|_X}{\delta} \cdot 1, && \text{because } \left\| \frac{\delta}{\|u\|_X} u \right\|_X = \frac{\delta \|u\|_X}{\|u\|_X} = \delta \leq \delta.
\end{aligned}$$

that is, there exists the positive constant $\frac{1}{\delta}$ such that

$$\|T(u)\|_Y \leq \frac{1}{\delta}\|u\|_X \text{ for all } u \in X,$$

which implies that T is bounded.

$(\Longleftarrow)$ Suppose that T is bounded.
Then there exists an $M \geq 0$ such that $\|T(x)\|_Y \leq M\|x\|_X$ for all $x \in X$.
Fix $\epsilon > 0$.
Consider any $x, x' \in X$ such that $\|x - x'\|_X < \frac{\epsilon}{M+1}$.
Then by linearity of T,

$$\|T(x) - T(x')\|_Y = \|T(x - x')\|_Y \leq M\|x - x'\|_X < M\frac{\epsilon}{M+1} < \epsilon,$$

which implies that T is continuous. $\square$

PROBLEMS

9. Let X and Y be normed linear spaces and $T : X \to Y$ be linear.
   (i) Show that T is continuous iff it is continuous at a single point u_0 in X.
   (ii) Show that T is Lipschitz iff it is continuous.
   (iii) Show that neither (i) nor (ii) hold in the absence of the linearity assumption on T.

10. For X and Y normed linear spaces and $T \in \mathcal{L}(X,Y)$, show that $\|T\|$ is the smallest Lipschitz constant for the mapping T; that is, the smallest number $c \geq 0$ for which

    $$\|T(u) - T(v)\| \leq c \cdot \|u - v\| \text{ for all } u, v \in X.$$

    Consider any $x, y \in X$, and consider the vector $(x - y) \in X$.
    Suppose that T is Lipschitz; that is, there exists a $c \geq 0$ such that

    $$\|T(u - v)\|_Y = \|T(u) - T(v)\|_Y \leq L \cdot \|u - v\|_X.$$

    Because T is a Lipschitz function it is thus continuous (previous Problem 9(ii)), and because T is linear, it is also thus bounded (Theorem 1), and so the operator norm of T is well-defined. In particular, $\|T\|$ is the infimum of all such $c \geq 0$.

11. For X and Y normed linear spaces and $Y \in \mathcal{L}(X,Y)$, show that

    $$\|T\| = \sup\{\|T(u)\| \mid u \in X, \|u\| \leq 1\}.$$

12. For X and Y normed linear spaces, let $\{T_n\} \to T$ in $\mathcal{L}(X,Y)$ and $\{u_n\} \to u$ in X. Show that $\{T_n(u_n)\} \to T_u$ in Y.

13. Let X be a Banach space and $T \in \mathcal{L}(X,Y)$ have $\|T\| < 1$.
    (i) Use the Contraction Mapping Principle to show that $I - T \in \mathcal{L}(X,Y)$ is one-to-one and onto.
    (ii) Show that $I - T$ is an isomorphism.

14. (Neumann Series) Let X be a Banach space and $Y \in \mathcal{L}(X,Y)$ have $\|T\| < 1$. Define $T^0 = Id$.

(i) Use the completeness of $\mathcal{L}(X, X)$ to show that $\sum_{n=1}^{\infty} T^n$ converges in $\mathcal{L}(X, X)$.

(ii) Show that $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$

15. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that T is an isomorphism iff there is an operator $S \in \mathcal{L}(Y, X)$ such that for each $u \in X$ and $v \in Y$,

$$S(T(u)) = u \text{ and } T(S(v)) = v.$$

16. For X and Y normed linear spaces and $T \in \mathcal{L}(X, Y)$, show that ker T is a closed subspace of X and that T is one-to-one iff ker $T = \{0\}$.

17. Let (X, ρ) be a metric space containing the point x_0. Define $\text{Lip}_0(X)$ to be the set of real-valued Lipschitz functions f on X that vanish at x_0. Show that $\text{Lip}_0(X)$ is a linear space that is normed by defining, for $f \in \text{Lip}_0(X)$,

$$\|f\| = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}.$$

Show that $\text{Lip}_0(X)$ is a Banach space. For each $x \in X$, define the linear functional F_x on $\text{Lip}_0(X)$ by setting $F_x(f) = f(x)$. Show that F_x belongs to $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$ and that for $x, y \in X$, $\|F_x - F_y\| = \rho(x, y)$. Thus X is isometric to a subset of the Banach space $\mathcal{L}(\text{Lip}_0(X), \mathbb{R})$. Since any closed subset of a complete metric space is complete, this provides another proof of the existence of a completion for any metric space X. It also shows that any metric space is isometric to a subset of a normed linear space.

18. Use the preceding problem to show that every normed linear space is a dense subspace of a Banach space.

19. For X a normed linear space and $T, S \in \mathcal{L}(X, X)$, show that the composition $S \circ T$ also belongs to $\mathcal{L}(X, X)$ and $\|S \circ T\| \leq \|S\| \cdot \|T\|$.

20. Let X be a normed linear space and Y a closed linear subspace of X. Show that $\|x\|_1 = \inf_{y \in Y} \|x - y\|$ defines a pseudonorm on X. The normed linear space induced by the pseudonorm $\|\cdot\|_1$ (see Problem 8) is denoted by X/Y and called the **quotient space** of X modulo Y. Show that the natural map φ of X onto X/Y takes open sets into open sets.

21. Show that if X is a Banach space and Y a closed linear proper subspace of X, then the quotient X/Y also is a Banach space and the natural map $\varphi : X \to X/Y$ has norm 1.

22. Let X and Y be normed linear spaces, $T \in \mathcal{L}(X, Y)$ and ker $T = Z$. Show that there is a unique bounded linear operator S from X/Z onto Y such that $T = S \circ \varphi$ where $\varphi : X \to X/Z$ is the natural map. Moreover, show that $\|T\| = \|S\|$.

13.3 Compactness Lost: Infinite Dimensional Normed Linear Spaces

PROBLEMS

23. Show that a subset of a finite dimensional normed linear space X is compact iff it is closed and bounded.

24. Complete the proof of Riesz's Lemma for $\epsilon \neq 1/2$.

25. Exhibit an open cover of the closed unit ball of $X = \ell^2$ that has not finite subcover. Then do the same for $X = C[0,1]$ and $X = L^2[0,1]$.

26. For normed linear spaces X and Y, let $T : X \to Y$ be linear. If X is finite dimensional, show that T is continuous. If Y is finite dimensional, show that T is continuous iff ker T is closed.

27. (Another proof of Riesz's Theorem) Let X be an infinite dimensional normed linear space, B the closed unit ball in X, and B_0 the unit open ball in X. Suppose B is compact. Then the open cover $\{x + (1/3)B_0\}_{x\in B}$ of B has a finite subcover $\{x_i + (1/3)B_0\}_{1\le i\le n}$. Use Riesz's Lemma with $Y = \text{span}[\{x_1, \ldots, x_n\}]$ to derive a contradiction.

28. Let X be a normed linear space. Show that X is separable iff there is a compact subset K of X for which $\overline{\text{span}}[K] = X$.

13.4 The Open Mapping and Closed Graph Theorems

Hello

PROBLEMS

29. Let X be a finite dimensional normed linear space and Y a normed linear space. Show that every linear operator $T : X \to Y$ is continuous and open.

    Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be the norms on X and Y respectively.
    Because X is finite dimensional, we can choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of X.
    Define
    $$M := \max\{\|T(e_1)\|_Y, \ldots, \|T(e_n)\|_Y\} \ge 0.$$
    By Chapter 13 Theorem 4, any two norms on the finite dimensional linear space X are equivalent; therefore in particular there exists $c \ge 0$ such that
    $$\sum_{i=1}^{n} |x_i| = \|x\|_1 \le c\|x\|_X \text{ for all } x \in X \tag{1}$$
    Now consider any $x \in X$.
    $$\begin{aligned}
    \|T(x)\|_Y &= \|T(\sum_{i=1}^{n} x_i e_i)\|_Y && \text{using the orthonormal basis} \\
    &= \|\sum_{i=1}^{n} x_i T(e_i)\|_Y && \text{by linearity of } T \\
    &\le \sum_{i=1}^{n} \|x_i T(e_i)\|_Y && \text{by subadditivity of the norm } \|\cdot\|_Y \\
    &= \sum_{i=1}^{n} |x_i| \|T(e_i)\|_Y && \text{by absolute homogeneity of the norm } \|\cdot\|_Y \\
    &\le \sum_{i=1}^{n} |x_i| M && \text{by definition of } M \\
    &= \|x\|_1 M && \text{by definition of the 1-norm} \\
    &\le c\|x\|_X M, && \text{by equivalence of norms (1)}
    \end{aligned}$$

and therefore there exists the constant $c \cdot M \geq 0$ such that

$$\|T(x)\|_Y \leq (c \cdot M)\|x\|_X \text{ for all } x \in X,$$

which implies that T is bounded, and thus by Chapter 13 Theorem 1, it is continuous.
(it remains to show that T is open)

30. Let X be a Banach space and $P \in \mathcal{L}(X, X)$ be a projection. Show that P is open.

31. Let $T : X \to Y$ be a continuous linear operator between the Banach spaces X and Y. Show that T is open if the image under T of the open unit ball in X is dense in a neighborhood of the origin in Y.

32. Let $\{u_n\}$ be a sequence in a Banach space X. Suppose that $\sum_{k=1}^{\infty} \|u_k\| < \infty$. Show that there is an $x \in X$ for which
$$\lim_{n\to\infty} \sum_{k=1}^{\infty} u_k = x.$$

33. Let T be a linear operator from a normed linear space X to a finite-dimensional normed linear space Y. Show that T is continuous iff ker T is a closed subspace of X.

34. Let X be a Banach space, the operator $T \in \mathcal{L}(X, X)$ be open and X_0 be a closed subspace of X. The restriction T_0 of T to X_0 is continuous. Is T_0 necessarily open?

35. Let V be a linear subspace of a linear space X. Argue as follows to show that V has a linear complement in X.
    (i) If dim $X < \infty$, let $\{e_i\}_{i=1}^n$ be a basis for V. Extend this basis for V to a basis $\{e_i\}_{i=1}^{n+k}$ for X. Then define $W = \text{span}[\{e_{n+1}, \ldots, e_{n+k}\}]$.
    (ii) If dim $X = \infty$, apply Zorn's Lemma to the collection $\mathcal{F}$ of all subspaces Z of X for which $V \cap Z = \{0\}$, ordered by set inclusion.
    (iii) Verify (15) and (16).
    (iv) Let Y be a normed linear space. Show that Y is a Banach space iff there is a Banach space X and a continuous, open mapping of X onto Y.

13.5 The Uniform Boundedness Principle

PROBLEMS

38. As a consequence of the Baire Category Theorem we showed that a real-valued mapping that is the pointwise limit of a sequence of continuous mapping on a complete metric space must be continuous at all points of a dense subset of its domain. Adapt that proof so that it applies to mapping into any metric space. Use this to prove that the pointwise limit of a sequence of continuous linear operators on a Banach space has a limit that is continuous at some point and hence, by linearity, is continuous.

39. Let $\{f_n\}$ be a sequence in $L^\infty[a, b]$. Suppose that for each $g \in L^1[a, b]$, $\lim_{n\to\infty} \int_a^b g \cdot f_n$ exists. Show that there is a function $f \in L^\infty[a, b]$ such that $\lim_{n\to\infty} \int_a^b g \cdot f_n = \int_a^b g \cdot f$ for all $g \in L^1[a, b]$.

40. Let X be the linear space of all polynomials defined on $\mathbb{R}$. For $p \in X$, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Show that this is a norm on X. For each n, define $\psi_n : X \to \mathbb{R}$ by $\psi_n(p) = p^{(n)}(0)$. Use the properties of the sequence $\{\psi_n\}$ in $\mathcal{L}(X, \mathbb{R})$ to show that X is not a Banach space.

Chapter 16

Continuous Linear Operators on Hilbert Spaces

Contents

16.1 The Inner Product and Orthogonality

Definition. *Let H be a linear space. A function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$ is called an **inner product** on H provided for all $x_1, x_2, x, y \in X$ and real numbers α, β, it satisfies*

(i) linearity in first argument: $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle$.

(ii) symmetry: $\langle x, y \rangle = \langle y, x \rangle$

(iii) positive definiteness: $\langle x, x \rangle > 0$ if $x \neq 0$

*A linear space H together with an inner product on H is called an **inner product space**.*

Property (i) along with property (ii) reveals that the real inner product in linear in both arguments: this is called **bilinearity**.

For two vectors $u = (u_1, \dots u_n)$ ad $u = (v_1, \dots v_n)$ in Euclidean space $\mathbb{R}^n$, the Euclidean inner product, $\langle u, v \rangle$, is defined by

$$\langle u, v \rangle = \sum_{k=1}^{n} u_k v_k.$$

For two sequences $x = \{x_k\}$ ad $y = \{y_k\}$ in ℓ^2, the ℓ^2 inner product, $\langle x, y\rangle$, is defined by

$$\langle x, y\rangle = \sum_{k=1}^{\infty} x_k y_k.$$

For a measurable set $E \subseteq \mathbb{R}$ and two functions f and g in $L^2(E)$, the L^2 inner product, $\langle f, g\rangle$, is defined by

$$\langle f, g\rangle = \int_E f \cdot g.$$

From the Cauchy-Schwarz (Hölder's) inequality, we infer that these inner products are properly defined (finite).

The Cauchy-Schwarz Inequality. *For any two vectors u, v in an inner product space H,*

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

Proposition 1. *For a vector h in an inner product space H, define*

$$\|h\| = \sqrt{\langle h, h\rangle}.$$

Then $\|\cdot\|$ is a norm on H called that norm induced by the inner product $\langle\cdot,\cdot\rangle$.

Proof. Let $h \in H$.

(i) By positive definiteness of the inner product, $\langle h, h\rangle > 0 \implies \sqrt{\langle h, h\rangle} > 0$.
If $h = 0$ then $\langle h, h\rangle = 0 \implies \sqrt{\langle h, h\rangle} = 0$.

(ii) By bilinearity of the inner product, $\|\alpha h\| = \sqrt{\langle \alpha h, \alpha h\rangle} = \sqrt{\alpha^2\langle h, h\rangle} = |\alpha|\sqrt{\langle h, h\rangle} = |\alpha|\|h\|$.

(iii) Use the Cauchy-Schwarz inequality to see that

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle \leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 = (\|u\| + \|v\|)^2.$$

□

The Parallelogram Identity. *For any two vectors u, v in an inner product space H,*

$$\|u-v\|^2 + \|u+v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

To verify this we can simply add the following two equalities:

$$\|u-v\|^2 = \|u\|^2 - 2\langle u, v\rangle + \|v\|^2,$$
$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2.$$

Definition. *An inner product space H is called a **Hilbert space** provided it is a Banach space with respect to the norm induced by the inner product.*

The Riesz-Fischer Theorem tells us that for E a measurable set of real numbers, $L^2(E)$ is a Hilbert space and, as a consequence, so is ℓ^2.

Proposition 2. *Let K be a nonempty, closed, convex subset of a Hilbert space H and h_0 belong to $H \setminus K$. Then there is exactly on vector $h_* \in K$ that is closest to h_0 in the sense that*

$$\|h_0 - h_*\| = \mathit{dist}(h_0, K) = \inf_{h \in K} \|h_0 - h\|.$$

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In the following problems, H is a Hilbert space.

1. Let $[a, b]$ be a closed, bounded interval of real numbers. Show that the $L^2[a, b]$ inner product is also an inner product on $C[a, b]$. Is $C[a, b]$ considered as an inner product space with the $L^2[a, b]$ inner product, a Hilbert space?

   We see that it is true that the $L^2[a, b]$ inner product is also an inner product on $C[a, b]$:

   (i) $\langle \alpha f_1+\beta f_2, g\rangle_{L^2} = \int_{[a,b]}(\alpha f_1+\beta f_2) = \alpha \int_{[a,b]} f_1\cdot g+\beta \int_{[a,b]} f_2\cdot g = \alpha\langle f_1, g\rangle_{L^2}+\beta\langle f_2, g\rangle_{L^2}$

   (ii) $\langle f, g\rangle_{L^2} = \int_{[a,b]} f \cdot g = \int_{[a,b]} g \cdot f = \langle g, f\rangle_{L^2}$

   (iii) $\langle f, f\rangle_{L^2} = \int_{[a,b]} f^2 \geq 0$, with $\langle f, f\rangle_{L^2} = 0 \iff f = 0$

   However, the inner product space $(C[a, b], \langle\cdot, \cdot\rangle_{L^2})$ is not a Hilbert space because it is not complete (not a Banach space) with respect to the norm defined by $\|\cdot\|_{L^2} := \sqrt{\langle\cdot, \cdot\rangle_{L^2}}$.

   To see this, let $t \in (a, b)$ and consider a sequence of continuous functions $\{f_n\}$ defined on $[a, b]$ such that for each n,

$$f_n(x) := \begin{cases} 0 & x \leq t \\ n(x-t) & t < x < t + \frac{1}{n} \\ 1 & x \geq t + \frac{1}{n} \end{cases}$$

   We aim to show that this sequence is Cauchy but does not converge to a continuous function, and therefore the space is not complete.

   First we prove that this sequence is Cauchy:
   Fix $\epsilon > 0$.
   Consider any natural numbers $m, n > \frac{1}{\epsilon^2}$, with $m \geq n$.

$$(f_m-f_n)(x) = \begin{cases} 0-0 & x \leq t \\ m(x-t) - n(x-t) & x \in (t, t + \frac{1}{m}) \\ 1 - n(x-t) & x \in [t + \frac{1}{m}, t + \frac{1}{n}) \\ 1-1 & x \geq t + \frac{1}{n} \end{cases} \leq \begin{cases} 0 & x \leq t \\ 1 & x \in (t, t + \frac{1}{m}) \\ 1 & x \in [t + \frac{1}{m}, t + \frac{1}{n}) \\ 0 & x \geq t + \frac{1}{n} \end{cases} := g(x)$$

   By monotonicity of integration, we get

$$\int_{[a,b]} |f_n - f_m|^2 \leq \int_{[a,b]} |g|^2 = \int_{(t,t+\frac{1}{n})} 1 = m((t, t + \frac{1}{n})) = \frac{1}{n}.$$

   Therefore we see that

$$\|f_n - f_m\|_{L^2} = \left(\int_{[a,b]} |f_n - f_m|^2\right)^{1/2} \leq \left(\frac{1}{n}\right)^{1/2} < \epsilon,$$

   which implies that $\{f_n\}$ is Cauchy.

   Very similarly, we prove that this sequence converges to a discontinuous function:
   Define the discontinuous function $f : [a, b] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & x \leq t \\ 1 & x > t \end{cases}$$

Fix $\epsilon > 0$.
Consider any natural number $n > \frac{1}{\epsilon^2}$.

$$(f - f_n)(x) = \begin{cases} 0 - 0 & x \leq t \\ 1 - n(x-t) & x \in (t, t+\frac{1}{n}) \\ 1 - 1 & x \geq t + \frac{1}{n} \end{cases} \leq \begin{cases} 0 & x \leq t \\ 1 & x \in (t, t+\frac{1}{n}) \\ 0 & x \geq t + \frac{1}{n} \end{cases} := g(x)$$

By monotonicity of integration, we get

$$\int_{[a,b]} |f - f_n|^2 \leq \int_{[a,b]} |g|^2 = \int_{(t, t+\frac{1}{n})} 1 = m((t, t + \frac{1}{n})) = \frac{1}{n}.$$

Therefore we see that

$$\|f - f_n\|_{L^2} = \left(\int_{[a,b]} |f - f_n|^2 \right)^{1/2} \leq \left(\frac{1}{n} \right)^{1/2} < \epsilon,$$

which implies that $\{f_n\}$ converges to f.

2. Show that the maximum norm on $C[a, b]$ is not induced by an inner product and neither is the usual norm on ℓ^1.

   See Problem 7 to see that for a normed linear space $(X, \|\cdot\|)$, the norm is induced by an inner product iff the parallelogram identity holds. Therefore it is sufficient to show that the parallelogram identity does not hold for the spaces $(C[a, b], \|\cdot\|_{\max})$ and $(\ell^1, \|\cdot\|_1)$.

   $(C[a, b], \|\cdot\|_{\max})$

   Let $f, g \in C[0, 1]$ defined by $f(x) = x^2$, $g(x) = 2$.
   Recall that $\|f\|_{\max} := \max_{x \in [a,b]} |f(x)|$.
   Then $\|f - g\|_{\max} = 2$, $\|f + g\|_{\max} = 3$, $\|f\|_{\max} = 1$, $\|g\|_{\max} = 2$, so that

   $$\|f - g\|_{\max}^2 + \|f + g\|_{\max}^2 = 13 \neq 6 = 2\|f\|_{\max} + 2\|g\|_{\max}.$$

   $(\ell^2, \|\cdot\|_1)$

   Let $x = (1, 0, 0, \dots) \in \ell^1$ and $y = (0, 1, 0, \dots) \in \ell^1$.
   Recall that $\|x\|_1 := \sum_{i=1}^{\infty} |x_i|$.
   Then $\|x - y\|_1 = 2,\ \|x + y\|_1 = 2,\ \|x\|_1 = 1,\ \|y\|_1 = 1$, so that

   $$\|x - y\|_2^2 + \|x + y\|_2^2 = 8 \neq 4 = 2\|x\|_2 + 2\|y\|_2.$$

3. Let H_1 and H_2 be Hilbert spaces. Show that the Cartesian product $H_1 \times H_2$ is also a Hilbert space with an inner product with respect to which $H_1 \times \{0\} = [\{0\} \times H_2]^\perp$.

4. Show that if S is a subset of an inner product space H, then $S^\perp$ is a closed subspace of H.

5. Let S be a subset of H. Show that $S = (S^\perp)^\perp$ iff S is a closed subspace of H.

6. (Polarization Identity) Show that for any two vectors $u, v \in H$,

$$\langle u, v\rangle = \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2].$$

Recall the Parallelogram inequality and instead of adding, simply subtract the first inequality:

$$\begin{aligned} -\|u-v\|^2 &= -\|u\|^2 + 2\langle u, v\rangle - \|v\|^2, \\ \|u+v\|^2 &= \|u\|^2 + 2\langle u, v\rangle + \|v\|^2, \end{aligned}$$

so that

$$\|u+v\|^2 - \|u-v\|^2 = 4\langle u, v\rangle.$$

7. (Jordan-von Neumann) Let X be a linear space normed by $\|\cdot\|$. Use the polarization identity to show that a norm $\|\cdot\|$ is induced by an inner product iff the parallelogram identity holds.

Let $(X, \|\cdot\|)$ be a normed linear space.

($\Longrightarrow$) Suppose that the norm is induced by some inner product $\langle\cdot,\cdot\rangle$ on X.
Then for any two vectors $u, v \in X$,

$$\begin{aligned} \|u-v\|^2 &= \langle u-v, u-v\rangle = \langle u,u\rangle + \langle u,-v\rangle + \langle -v,u\rangle + \langle -v,-v\rangle = \|u\|^2 - 2\langle u,v\rangle + \|v\|^2, \\ \|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle = \|u\|^2 + 2\langle u,v\rangle + \|v\|^2, \end{aligned}$$

and adding the two equalities shows that the parallelogram identity holds.

($\Longleftarrow$) Suppose that the parallelogram identity holds.
Define the function $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{R}$ by

$$\langle u, v\rangle = \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2].$$

The aim is to show that $\sqrt{\langle v, v\rangle} = \|v\|$ for $v \in X$, and that $\langle\cdot,\cdot\rangle$ is an inner product.
First see that

$$\sqrt{\langle v, v\rangle} = \sqrt{\frac{1}{4}[\|v+v\|^2 - \|v-v\|^2]} = \sqrt{\frac{2^2}{4}\|v\|^2} = \|v\|. \tag{1}$$

(i) Bilinearity (additivity):
For any $x, y, v \in X$, by the parallelogram identity,

$$\begin{aligned} \|(u_1+v)+u_2\|^2 &= 2\|u_1+v\|^2 + 2\|u_2\|^2 - \|(u_1+v)-u_2\|^2 \\ \|(u_2+v)+u_1\|^2 &= 2\|u_2+v\|^2 + 2\|u_1\|^2 - \|(u_2+v)-u_1\|^2 \end{aligned}$$

Then

$$\begin{aligned} \|u_1+u_2+v\|^2 &= \frac{1}{2}(\|(u_1+v)+u_2\|^2 + \|(u_2+v)+u_1\|^2) \\ &= \|u_1+v\|^2 + \|u_2+v\|^2 + \|u_1\|^2 + \|u_2\|^2 - \frac{1}{2}\|(u_1+v)-u_2\|^2 - \frac{1}{2}\|(u_2+v)-u_1\|^2 \end{aligned}$$

And so this also holds for $v = -v$:

$$\begin{aligned}\|u_1 + u_2 + (-v)\|^2 &= \|u_1 - v\|^2 + \|u_2 - v\|^2 + \|u_1\|^2 + \|u_2\|^2 - \frac{1}{2}\|(u_1 - v) - u_2\|^2 - \frac{1}{2}\|(u_2 - v) - u_1\|^2 \\ &= \|u_1 - v\|^2 + \|u_2 - v\|^2 + \|u_1\|^2 + \|u_2\|^2 - \frac{1}{2}\|(u_2 + v) - u_1\|^2 - \frac{1}{2}\|(u_1 + v) - u_2\|^2\end{aligned}$$

Therefore we can write

$$\begin{aligned}\langle u_1 + u_2, v\rangle &= \frac{1}{4}[\|u_1 + u_2 + v\|^2 - \|u_1 + u_2 - v\|^2] \\ &= \frac{1}{4}[\|u_1 + v\|^2 - \|u_1 - v\|^2] + \frac{1}{4}[\|u_2 + v\|^2 - \|u_2 - v\|^2] \\ &= \langle u_1, v\rangle + \langle u_2, v\rangle.\end{aligned}$$

(i) Bilinearity (homogeneity):
Let $S = \{\alpha \in \mathbb{R} \mid \alpha\langle u, v\rangle = \langle \alpha u, v\rangle\}$.
Clearly $1, 0 \in S$, and $-1 \in S$ because

$$-1\langle u, v\rangle = \frac{1}{4}[\|u - v\|^2 - \|u + v\|^2] = \frac{1}{4}[\| - u + v\|^2 - \| - u - v\|^2] = \langle -1u, v\rangle.$$

Suppose $\alpha, \beta \in S$. Then

$$\begin{aligned}(\alpha + \beta)\langle u, v\rangle &= \alpha\langle u, v\rangle + \beta\langle u, v\rangle \\ &= \langle \alpha u, v\rangle + \langle \beta u, v\rangle \\ &= \langle \alpha u + \beta u, v\rangle && \text{by bilinearity (additivity)} \\ &= \langle (\alpha + \beta)u, v\rangle,\end{aligned}$$

so that $(\alpha + \beta) \in S$ and S contains all integers.

Suppose $\alpha, \beta \in S$, $\beta \neq 0$. Then

$$\alpha\langle u, v\rangle = \langle \alpha u, v\rangle = \langle \frac{\beta}{\beta}\alpha u, v\rangle = \beta\langle \frac{\alpha}{\beta}u, v\rangle \implies \frac{\alpha}{\beta}\langle u, v\rangle = \langle \frac{\alpha}{\beta}u, v\rangle,$$

so that $\frac{\alpha}{\beta} \in S$ and S contains all rational numbers.

Fix any $x, y \in X$. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by $f(\alpha) = \alpha\langle u, v\rangle$ and $g(\alpha) = \langle \alpha u, v\rangle$. The function f is linear on a finite dimensional space and thus continuous (Chapter 13 Problem 29), and the function g is a composition of $\alpha \mapsto \alpha x$ and $t \mapsto \langle t, y\rangle$, which are both continuous $(*)$. Then we have that f, g are continuous with $f(\alpha) = g(\alpha)$ for all $\alpha \in \mathbb{Q}$, which implies that $f(\alpha) = g(\alpha)$ for all $\alpha \in \mathbb{R}$.
That is, $\alpha\langle u, v\rangle = \langle \alpha u, v\rangle$ for all scalars α.
(ii) Symmetry:

$$\langle u, v\rangle = \frac{1}{4}[\|u + v\|^2 - \|u - v\|^2] = \frac{1}{4}[\|v + u\|^2 - \|v - u\|^2] = \langle v, u\rangle.$$

(iii) Positive Definiteness:
For $v \neq 0$: $\langle v, v\rangle = \|v\|^2 > 0$ by (1) and positive definiteness of norm.
For $v = 0$: $\langle v, v\rangle = \|v\|^2 = 0$ by (1) and positive definiteness of norm.

Therefore $\langle \cdot, \cdot \rangle$ is an inner product.

($*$) For a normed linear space X and $y \in X$, the function $x \mapsto \|x+y\|$ for $x \in X$ is continuous.
(Use this as a simplified version of $x \mapsto \langle x, y\rangle = \frac{1}{4}[\|x+y\|^2 - \|x-y\|^2]$, which will also be continuous as it is the product and sum of continuous functions.)
Fix $\varepsilon > 0$.
Let $\delta = \varepsilon > 0$.
For $x_1, x_2 \in X$ such that $\|x_1 - x_2\| < \delta = \varepsilon$, we use the reverse triangle inequality to see

$$|f(x_1) - f(x_2)| = |\|x_1 + y\| - \|x_2 + y\|| \leq \|x_1 + y - (x_2 + y)\| < \varepsilon.$$

8. Let V be a closed subspace of H and P a projection of H onto V. Show that P is the orthogonal projection of H onto V iff (4) holds.

9. Let T belong to $\mathcal{L}(H)$. Show that T is an isometry iff

$$\langle T(u), T(v)\rangle = \langle u, v\rangle \text{ for all } u, v \in H.$$

($\implies$) Suppose that T is an isometry.
Then $\|Tu\| = \|u\|$ for any $u \in H$.
Let $u, v \in H$ so that we can derive:

$$\begin{aligned}
\langle T(u), T(v)\rangle &= \frac{1}{4}[\|Tu + Tv\|^2 - \|Tu - Tv\|^2] && \text{Polarization identity} \\
&= \frac{1}{4}[\|T(u+v)\|^2 - \|T(u-v)\|^2] && T \text{ is linear} \\
&= \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2] && T \text{ is an isometry} \\
&= \frac{1}{4}[\langle u+v, u+v\rangle - \langle u-v, u-v\rangle] && \text{norm induced by inner product} \\
&= \frac{1}{4}[\|u\|^2 + 2\langle u, v\rangle + \|v\|^2 - \|u\|^2 + 2\langle u, v\rangle - \|v\|^2] && \text{norm induced by inner product} \\
&= \langle u, v\rangle.
\end{aligned}$$

($\impliedby$) Suppose that $\langle T(u), T(v)\rangle = \langle u, v\rangle$ for all $u, v \in H$.
Therefore for any $u \in H$, we have

$$\|Tu\| = \sqrt{\langle Tu, Tu\rangle} = \sqrt{\langle u, u\rangle} = \|u\|.$$

10. Let V be a finite dimensional subspace of H and $\varphi_1, \cdots, \varphi_n$ a basis for V consisting of unit vectors, each pair of which is orthogonal. Show that the orthogonal projection P of H onto V is given by

$$P(h) = \sum_{k=1}^{n} \langle h, \varphi_k\rangle \varphi_k \text{ for all } h \in V.$$

11. For h a vector in H, show that the function $u \mapsto \langle h, u\rangle$ belongs to H^*.

The aim is to show that the function $\varphi : H \to \mathbb{R}$ defined by $\varphi(u) = \langle h, u\rangle$ is bounded and linear (see Chapter 8, Proposition 1).
To see boundedness (continuity), use the Cauchy-Schwarz Inequality so that for any $u \in H$,

$$\varphi(u) = \langle h, u\rangle \leq \|h\| \cdot \|u\| < \infty.$$

(Norms are defined to be **real-valued**; therefore the norm of any element in a linear space is a real number and thus finite: $\|h\|, \|u\| < \infty$)
To see linearity, simply use bilinearity of the inner product:

$$\varphi(\alpha u + \beta v) = \langle h, \alpha u + \beta v\rangle = \alpha\langle h, u\rangle + \beta\langle h, v\rangle = \alpha\varphi(u) + \beta\varphi(v).$$

12. For any vector $h \in H$, show that there is a bounded linear functional $\psi \in H^*$ for which

$$\|\psi\| = 1 \text{ and } \psi(h) = \|h\|.$$

For $h \in H$ $(h \neq 0)$, let $\psi : H \to \mathbb{R}$ be defined by

$$\psi(u) = \langle \frac{h}{\|h\|}, u\rangle \text{ for any } u \in H.$$

By the previous Problem 11, we proved that $\psi \in H^*$.
Then

$$\psi(h) = \langle \frac{h}{\|h\|}, h\rangle = \frac{1}{\|h\|}\langle h, h\rangle = \frac{1}{\|h\|}\|h\|^2 = \|h\|.$$

It remains to show $\|\psi\|_* = 1$.
Recall Chapter 8.1 for the following:
We define the operator norm:

$$\|\psi\|_* := \inf\{M \mid |\psi(u)| \leq M\|u\| \text{ for all } u \in H\},$$

which implies the two:

- $|\psi(u)| \leq \|\psi\|_*\|u\|$ for all $u \in H$
- $\|\psi\|_* = \sup\{\psi(u) \mid u \in H, \|u\| \leq 1\}$

Therefore

$$\|h\| = |\psi(h)| \leq \|\psi\|_*\|h\| \implies 1 \leq \|\psi\|_* \tag{a}$$

$$\|\psi\|_* = \sup_{\substack{u \in H \\ \|u\| \leq 1}} \langle \frac{h}{\|h\|}, u\rangle \leq \|\frac{h}{\|h\|}\| \cdot \|u\| \leq 1 \tag{b}$$

Then (a) and (b) imply $\|\psi\|_* = 1$.

13. Let V be a closed subspace of H and P the orthogonal projection of H onto V. For any normed linear space X and $T \in \mathcal{L}(V, X)$, show that $T \circ P$ belongs to $\mathcal{L}(H, X)$, and is an extension of $T : V \to X$ for which $\|T \circ P\| = \|T\|$.

14. Prove the Hyperplane Separation Theorem for H, considered as a locally convex topological vector space with respect to the strong topology, by directly using Proposition 2.

15. Use Proposition 2 to prove the Krein-Milman Lemma in a Hilbert space.

16.2 The Dual Space and Weak Sequential Convergence

PROBLEMS

In the following problems, H is a Hilbert space.

16. Show that neither $\ell^1, \ell^\infty, L^1[a,b]$ nor $L^\infty[a,b]$ is Hilbertable.

17. Prove Proposition 7.

18. Let H be an inner product space. Show that since H is a dense subset of a Banach space X whose norm restricts to the norm induced by the inner product on H, the inner product on H extends to X and induces the norm on X. Thus inner product spaces have Hilbert space completions.

16.3 Bessel's Inequality and Orthonormal Bases

PROBLEMS

In the following problems, H is a Hilbert space.

19. Show that an orthonormal subset of a separable Hilbert space H must be countable.

20. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space H. Show that $\{\varphi_k\}$ converges weakly to 0 in H.

21. Let $\{\varphi_k\}$ be an orthonormal basis for the separable Hilbert space H. Show that $\{u_n\} \to u$ in H iff $\{u_n\}$ is bounded and, for each k, $\lim_{n\to\infty}\langle u_n, \varphi_k\rangle = \langle u, \varphi_k\rangle$.

22. Show that any two infinite dimensional separable Hilbert spaces are isometrically isomorphic and that any such isomorphism preserves the inner product.

23. Let H be a Hilbert space and V a closed separable subspace of H for which $\{\varphi_k\}$ is an orthonormal basis. Show that the orthogonal projection of H onto V, P, is given By

$$P(h) = \sum_{k=1}^{\infty}\langle\varphi_k, h\rangle\varphi_k \text{ for all } h \in H.$$

24. (Parseval's Indentities) Let $\{\varphi_k\}$ be an orthonormal basis for a Hilbert space H. Verify that

$$\|h\|^2 = \sum_{k=1}^{\infty}\langle\varphi_k, h\rangle^2 \text{ for all } h \in H.$$

Also verify that

$$\langle u, v\rangle = \sum_{k=1}^{\infty} a_k \cdot b_k \text{ for all } u, v \in H,$$

where, for each natural number k, $a_k = \langle u, \varphi_k\rangle$ and $b_k = \langle v, \varphi_k\rangle$.

25. Verify the assertions in the example of the orthonormal basis for $L^2[0, 2\pi]$.

26. Use Proposition 10 and the Stone-Weierstrass Theorem to show that for each $f \in L^2[-\pi, \pi]$,

$$f(x) = a_0/2 + \sum_{k=1}^{\infty} [a_k \cdot \cos kx + b_k \cdot \sin kx],$$

where the convergence is in $L^2[-\pi, \pi]$ and each

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx \text{ and } b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx.$$

16.4 Adjoints and Symmetry for Linear Operators

PROBLEMS

In the following problems, H is a Hilbert space.

27. Verify (12).

28. Let T and S belong to $\mathcal{L}(H)$ and be symmetric. Show that $T = S$ iff $Q_T = Q_S$.

29. Show the symmetric operators are a closed subspace of $\mathcal{L}(H)$. Also show that if T and S are symmetric, then so is the composition $S \circ T$ iff T commutes with S with respect to composition; that is, $S \circ T = T \circ S$.

30. (Helliger-Toplitz) Let H be a Hilbert space and the linear operator $T : H \to H$ have the property that $\langle T(u), v\rangle = \langle u, T(v)\rangle$ for all $u, v \in H$. Show that T belongs to $\mathcal{L}(H)$.

31. Exhibit an operator $T \in \mathcal{L}(\mathbb{R}^2)$ for which $\|T\| > \sup_{\|u\|=1} |\langle T(u), u\rangle|$.

32. Let S and T in $\mathcal{L}(H)$ be symmetric. Assume $S \geq T$ and $T \geq S$. Prove that $T = S$.

33. Let V be a closed nontrivial subspace of a Hilbert space H and P the orthogonal projection of H onto V. Show that $P = P^*$, $P \geq 0$, and $\|P\| = 1$.

34. Let $P \in \mathcal{L}(H)$ be a projection. Show that P is the orthogonal projection of H onto $P(H)$ iff $P = P^*$.

35. Let $\{\varphi_k\}$ be an orthonormal basis for a Hilbert space H and for each natural number n, define P_n to be the orthogonal projection of H onto the linear span of $\{\varphi_1, \ldots, \varphi_n\}$. Show that P_n is symmetric and

$$0 \leq P_n \leq P_{n+1} \leq Id \text{ for all } n.$$

Show that $\{P_n\}$ converges pointwise on H to Id but does not converge uniformly on the unit ball.

36. Show that if $T \in \mathcal{L}(H)$ is invertible, so is $T^* \circ T$ and therefore so is T.

37. (A General Cauchy-Schwarz Inequality) Let $T \in \mathcal{L}(H)$ be symmetric and nonnegative. Show that for all $u, v \in H$,

$$|\langle T(u), v\rangle|^2 \leq \langle T(u), u\rangle \cdot \langle T(v), v\rangle.$$

38. Use the preceding problem to show that if $S, T \in \mathcal{L}(H)$ are symmetric and $S \geq T$, then for each $u \in H$,

$$\|S(u) - T(u)\|^4 = \langle (S-T)(u), (S-T)(u)\rangle^2 \leq |\langle (S-T)(u), u\rangle||\langle (S-T)^2(u), (S-T)(u)\rangle|$$

and thereby conclude that

$$\|S(u) - T(u)\|^4 \leq |\langle S(u), u\rangle - \langle T(u), u\rangle| \cdot \|S-T\|^3 \cdot \|u\|^2.$$

39. (a Monotone Convergence Theorem for Symmetric Operators) A sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ is said to be monotone increasing provided $T_{n+1} \geq T_n$ for each n, and said to be bounded above provided there is a symmetric operator S in $\mathcal{L}(H)$ such that $T_n \leq S$ for all n.

    (i) Use the preceding problem to show that a monotone increasing sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ converges pointwise to a symmetric operator in $\mathcal{L}(H)$ iff it is bounded above.

    (ii) Show that a monotone increasing sequence $\{T_n\}$ of symmetric operators in $\mathcal{L}(H)$ is bounded above iff it is pointwise bounded; that is, for each $h \in H$, the sequence $\{T_n(h)\}$ is bounded.

40. Let $S \in \mathcal{L}(H)$ be a symmetric operator for which $0 \leq S \leq Id$. Define a sequence $\{T_n\}$ in $\mathcal{L}(H)$ by letting $T_1 = 1/2(Id - S)$ and if n is a natural number for which $T_n \in \mathcal{L}(H)$ has been defined, defining $T_{n+1} = 1/2(Id - S + T_n^2)$.

    (i) Show that for each natural umber n, T_n and $T_{n+1} - T_n$ are polynomials in $Id - S$ with nonnegative coefficients.

    (ii) Show that $\{T_n\}$ is a monotone increasing sequence of symmetric operators that is bounded above by Id.

    (iii) Use the preceding problem to show that $\{T_n\}$ converges pointwise to a symmetric operator T for which $0 \leq T \leq Id$ and $T = 1/2(Id - S + T^2)$.

    (iv) Define $A = (Id - T)$. Show that $A^2 = S$.

41. (Square Roots of Nonnegative Symmetric Operators) Let $T \in \mathcal{L}(H)$ be a nonnegative symmetric operator. A nonnegative symmetric operator $A \in \mathcal{L}(H)$ is called a square root of T provided $A^2 = T$. Use the inductive construction in the preceding problem to show that T has a square root A which commutes with each operator in $\mathcal{L}(H)$ that commutes with T. Show that the square root is unique: it is denoted by $\sqrt{T}$. Finally, show that T is invertible iff $\sqrt{T}$ is invertible.

42. An invertible operator $T \in \mathcal{L}(H)$ is said to be **orthogonal** provided $T^{-1} = T^*$. Show that an invertible operator is orthogonal iff it is an isometry.

43. (Polar Decompositions) Let $T \in \mathcal{L}(H)$ be invertible. Show that there is an orthogonal invertible operator $A \in \mathcal{L}(H)$ and a nonnegative symmetric invertible operator $B \in \mathcal{L}(H)$ such that $T = B \circ A$. (Hint: show that TT^* is invertible and symmetric and let $B = \sqrt{T \circ T^*}$.)

16.5 Compact Operators

PROBLEMS

44. Show that if H is infinite dimensional and $T \in \mathcal{L}(H)$ is invertible, then T is not compact.

45. Prove Proposition 18.

46. Let $\mathcal{K}(H)$ denote the set of compact operators in $\mathcal{L}(H)$. Show that $\mathcal{K}(H)$ is a linear subspace of $\mathcal{L}(H)$. Moreover, show that for $K \in \mathcal{K}(H)$ and $T \in \mathcal{L}(H)$, both $K \circ T$ and $T \circ K$ belong to $\mathcal{K}(H)$.

47. Show that a linear operator $T : H \to H$ is continuous iff it maps weakly convergent sequences to weakly convergent sequences.

48. Show that $K \in \mathcal{L}(H)$ is compact iff whenever $\{u_n\} \to u$ in H and $\{v_n\} \to v$ in H, then $\langle K(u_n), v_n\rangle \to \langle K(u), v\rangle$.

49. Let $\{P_n\}$ be a sequence of orthogonal projections in $\mathcal{L}(H)$ with the property that for natural numbers n and m, $P_n(H)$ and $P_m(H)$ are orthogonal finite dimensional subspaces of H. Let $\{\lambda_n\}$ be a bounded sequence of real numbers. Show that

$$K = \sum_{n=1}^{\infty} \lambda_n \cdot P_n$$

is a properly defined symmetric operator in $\mathcal{L}(H)$ that is compact iff $\{\lambda_n\}$ converges to 0.

50. For X a Banach space, define an operator $T \in \mathcal{L}(X)$ to be compact provided $T(B)$ has compact closure. Show that Proposition 18 holds for a general Banach space and Proposition 19 holds for a reflexive Banach space.

16.6 The Hilbert-Schmidt Theorem

PROBLEMS

51. Let H be a Hilbert space and $T \in \mathcal{L}(H)$ be compact and symmetric. Define

$$\alpha = \inf_{\|h\|=1} \langle T(h), h\rangle \text{ and } \beta = \sup_{\|h\|=1} \langle T(h), h\rangle.$$

Show that if $\alpha < 0$, then α is an eigenvalue of T and if $\beta > 0$, then β is an eigenvalue of T. Exhibit an example where $\alpha = 0$ and yet α is not an eigenvalue of T; that is, T is one-to-one (injective).

52. Let H be a Hilbert space and $K \in \mathcal{L}(H)$ be compact and symmetric. Suppose

$$\sup_{\|h\|=1} \langle K(h), h\rangle = \beta > 0.$$

Let $\{h_n\}$ be a sequence of unit vectors for which $\lim_{n\to\infty}\langle K(h_n), h_n\rangle = \beta$. Show that a subsequence of $\{h_n\}$ converges strongly to an eigenvector of T with corresponding eigenvalue β.

16.7 The Riesz-Schauder Theorem: Characterization of Fredholm Operators

PROBLEMS

53. Let $K \in \mathcal{L}(H)$ be compact. Show that $T = K^*K$ is compact and symmetric. Then use the Hilbert-Schmidt Theorem to show that there is an orthonormal sequence $\{\varphi_k\}$ of H such that $T(\varphi_k) - \lambda_k \varphi_k$ for all k and $T(h) = 0$ if h is orthogonal to $\{\varphi_k\}_{k=0}^{\infty}$. Conclude that if h is orthogonal to $\{\varphi_k\}_{k=0}^{\infty}$, then
$$\|K(h)\|^2 \langle K(h), K(h)\rangle = \langle T(h), h\rangle = 0.$$
Define H_0 to be the closed linear span of $\{K^m(\varphi_k) \mid m \geq 0, k \geq 1\}$. Show that H_0 is closed and separable, $K(H_0) \subseteq H_0$ and $K = 0$ on $H_0^{\perp}$.

54. Let $\mathcal{K}(H)$ denote the set of compact operators in $\mathcal{L}(H)$. Show that $\mathcal{K}(H)$ is a closed subspace of $\mathcal{L}(H)$ that has the set of operators of finite rank as a dense subspace. Is $\mathcal{K}(H)$ an open subset of $\mathcal{L}(H)$?

55. Show that the composition in either order of a Fredholm operator of index 0 with an invertible operator is also Fredholm of index 0.

56. Show that the composition of two Fredholm operators of index 0 is also Fredholm of index 0.

57. Show that an operator $T \in \mathcal{L}(H)$ is Fredholm of index 0 iff it is the perturbation of an invertible operator by an operator of finite rank.

58. Argue as follows to show that the collection of invertible operators in $\mathcal{L}(H)$ is an open subset of $\mathcal{L}(H)$.

    (i) For $A \in \mathcal{L}(H)$ with $\|A\| < 1$, use the completeness of $\mathcal{L}(H)$ to show that the so-called Neumann series $\sum_{n=0}^{\infty} A^n$ converges to an operator in $\mathcal{L}(H)$ that is the inverse of $Id - A$.

    (ii) For a invertible operator $S \in \mathcal{L}(H)$ show that for any $T \in \mathcal{L}(H), T = S[Id + S^{-1}(T - S)]$.

    (iii) Use (i) and (ii) to show that if $S \in \mathcal{L}(H)$ is invertible then so is any $T \in \mathcal{L}(H)$ of which $\|S - T\| < 1/\|S^{-1}\|$.

59. Show that the set of operators in $\mathcal{L}(H)$ that are Fredholm of index 0 is an open subset of $\mathcal{L}(H)$.

60. By following the orthogonal approximation sequence method used in the proof of Proposition 22, provide another proof of Proposition 14 in case H is separable.

61. For $T \in \mathcal{L}(H)$, suppose that $\langle T(h), h\rangle \geq \|h\|^2$ for all $h \in H$. Assume that $K \in \mathcal{L}(H)$ is compact and $T + K$ is one-to-one. Show that $T + K$ is onto.

62. Let $K \in \mathcal{L}(H)$ be compact an $\mu \in \mathbb{R}$ have $|\mu| > \|K\|$. Show that $\mu - K$ is invertible.

63. Let $S \in \mathcal{L}(H)$ have $\|S\| < 1$,$K \in \mathcal{L}(H)$ be compact and $(Id + S + K)(H) = H$. Show that $Id + S + K$ is one-to-one.

64. Let $\mathcal{G}L(H)$ denote the set of invertible operators in $\mathcal{L}(H)$.

    (i) Show that under the operation of composition of operators, $\mathcal{G}L(H)$ is a group: it is called the general linear group of H.

    (ii) An operator T in $\mathcal{G}L(H)$ is said to be orthogonal, provided that $T^* = T^{-1}$. Show that the set of orthogonal operators is a subgroup of $\mathcal{G}L(H)$: it is called the orthogonal group.

65. Let H be a Hilbert space, $T \in \mathcal{L}(H)$ be Fredholm of index zero, and $K \in \mathcal{L}(H)$ be compact. Show that $T + K$ is Fredholm of index zero.

66. Let X_0 be a finite codimensional subspace of a Banach space X. Show that all finite dimensional linear complements of X_0 in X have the same dimension.

III MEASURE AND INTEGRATION: GENERAL THEORY

Chapter 17

General Measure Spaces: Their Properties and Construction

Contents

17.1 Measures and Measurable Sets

Definition. *By a **measurable space** we mean a couple $(X, \mathcal{M})$ consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A subset E of X is called **measurable** (or measurable with respect to $\mathcal{M}$) provided E belongs to $\mathcal{M}$.*

Definition. *By a **measure** μ on a measurable space $(X, \mathcal{M})$ we mean an extended real-valued nonnegative set function $\mu : \mathcal{M} \to [0, \infty]$ for which $\mu(\emptyset) = 0$ and which is **countably additive** in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,*

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

Definition. *By a **measure space** $(X, \mathcal{M}, \mu)$ we mean a measurable space $(X, \mathcal{M})$ together with a measure μ defined on $\mathcal{M}$.*

Proposition 1. *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

(Monotonicity) If A and B are measurable sets and $A \subseteq B$, then

$$\mu(A) \leq \mu(B).$$

(Excision) If, moreover, $A \subseteq B$ and $\mu(A) < \infty$, then

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

so that if $\mu(A) = 0$, then

$$\mu(B \setminus A) = \mu(B).$$

(Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^n$ of measurable sets that covers a measurable set E,

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Definition. *Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is called **finite** provided $\mu(X) < \infty$. It is called σ-**finite** provided X is the union of a countable collection of measurable sets, each of which has finite measure. A measurable set E is said to be of **finite measure** provided $\mu(E) < \infty$, and is said to be σ-**finite** provided E is the union of a countable collection of measurable sets, each of which has finite measure.*

Definition. *A measure space $(X, \mathcal{M}, \mu)$ is said to be **complete** provided $\mathcal{M}$ contains all subsets of sets of measure zero, that is, if E belongs to $\mathcal{M}$ and $\mu(E) = 0$, then every subset of E also belongs to $\mathcal{M}$.*

For example, the Lebesgue measure m on the real line is complete. Moreover, in Chapter 2 Proposition 22, we showed that the Cantor set C, a Borel set that has Lebesgue measure zero, contains a Lebesgue measurable set that is not a Borel set. Therefore the Lebesgue measure restricted to the Borel σ-algebra $\mathcal{B}$ is not complete because C belongs to $\mathcal{B}$ and $m(C) = 0$ but there exists a subset $A \subseteq C$ such that $A \notin \mathcal{B}$.

The following proposition tells us that each measure space can be completed.

Proposition 3. *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets E of X of the form $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set E define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a σ-algebra that contains $\mathcal{M}$, μ_0 is a measure that extends μ, and $(X, \mathcal{M}_0, \mu_0)$ (the **completion** of $(X, \mathcal{M}, \mu)$) is a complete measure space.*

PROBLEMS

1. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$, define $\mu(E) = \int_E f$, the Lebesgue integral of f over E. Show that μ is a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

   Because f is nonnegative, by monotonicity of integration, for any Lebesgue measurable set E,

   $$0 \leq f \implies 0 = \int_E 0 \leq \int_E f = \mu(E).$$

   Check Chapter 4 Problem 28 to see that for f Lebesgue integrable over $\mathbb{R}$ and $\emptyset$ a Lebesgue measurable subset of $\mathbb{R}$, we have that

   $$\mu(\emptyset) = \int_\emptyset f = \int_\mathbb{R} f \cdot \chi_\emptyset = \int_\mathbb{R} f \cdot \chi_\emptyset = \int_\mathbb{R} 0 = 0.$$

Even more simply, even if f is not Lebesgue integrable, check Chapter 4 Problem 17 to see that

$$\mu(\emptyset) = \int_\emptyset f = 0.$$

Let $\{E_n\}_{n=1}^\infty$ be a disjoint countable collection of Lebesgue measurable sets so that each $\mu(E_n) = \int_{E_n} f$ is defined. Then $E = \bigcup_{n=1}^\infty E_n$ is Lebesgue measurable, and $\mu(E) = \int_E f$ is defined.
For f Lebesgue integrable over $\mathbb{R}$, by Chapter 4 Theorem 20,

$$\mu(\bigcup_{n=1}^\infty E_n) = \mu(E) = \int_E f = \sum_{n=1}^\infty \int_{E_n} f = \sum_{n=1}^\infty \mu(E_n).$$

If f is not Lebesgue integrable, we can still consider the sequence of functions f_n on E defined by

$$f_n := f \cdot \chi_{A_n}, \text{ where } A_n := \bigcup_{k=1}^n E_k.$$

Then because f is nonnegative, $\{f_n\}$ is an increasing sequence of nonnegative measurable functions on E, where $\{f_n\} \to f$ pointwise on E. By the Monotone Convergence Theorem (Chapter 4), and (finite) Additivity Over Domains of Integration for each f_n (Chapter 4 Theorem 11), we have

$$\infty = \int_E f = \lim_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \sum_{k=1}^n \int_{E_k} f_n = \sum_{k=1}^\infty \int_{E_k} f_n = \sum_{k=1}^\infty \mu(E_k),$$

so that $\mu(\bigcup_{n=1}^\infty E_n) = \mu(E) = \int_E f = \infty = \sum_{n=1}^\infty \mu(E_n)$.
Therefore μ is a measure on the σ-algebra of Lebesgue measurable sets.

2. Let $\mathcal{M}$ be a σ-algebra of subsets of a set X and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive. Prove that μ is a measure iff whenever $\{A_k\}_{k=1}^\infty$ is an ascending sequence of sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^\infty A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

($\Longrightarrow$) Suppose that μ is a measure.
Then by Continuity of Measure, the conclusion follows.

($\Longleftarrow$) Suppose that whenever $\{A_k\}_{k=1}^\infty$ is an ascending sequence of sets in $\mathcal{M}$, then $\mu(\bigcup_{k=1}^\infty A_k) = \lim_{k\to\infty} \mu(A_k)$. (See Chapter 2 Problem 28.)
Finite additivity of μ means that for any finite disjoint collection $\{E_k\}_{k=1}^n$ of measurable sets, we have $\mu(\bigcup_{k=1}^n E_k) = \sum_{k=1}^n \mu(E_k)$.
We define $F_n = \bigcup_{k=1}^n E_k$ so that $\{F_n\}_{n=1}^\infty$ is an ascending sequence of sets in $\mathcal{M}$, and thus $\mu(\bigcup_{n=1}^\infty F_n) = \lim_{n\to\infty} \mu(F_n)$.
Thus we see

$$\mu(\bigcup_{n=1}^\infty E_n) = \mu(\bigcup_{n=1}^\infty F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^n E_k) = \lim_{n\to\infty} \sum_{k=1}^n \mu(E_k) = \sum_{k=1}^\infty \mu(E_k),$$

that is, μ satisfies countable additivity, and thus μ is a measure.

3. Let $\mathcal{M}$ be a σ-algebra of subsets of a set X. Formulate and establish a correspondent of the preceding problem for descending sequences of sets in $\mathcal{M}$.

Let $\mathcal{M}$ be a σ-algebra of subsets of a set X and the set function $\mu : \mathcal{M} \to [0,\infty)$ be finitely additive. Prove that μ is a measure iff whenever $\{A_k\}_{k=1}^{\infty}$ is a descending sequence of sets in $\mathcal{M}$ with $m(A_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

($\Longrightarrow$) Suppose that μ is a measure.
Then by Continuity of Measure, the conclusion follows.

($\Longleftarrow$) Suppose that whenever $\{A_k\}_{k=1}^{\infty}$ is a descending sequence of sets in $\mathcal{M}$ with $\mu(A_1) < \infty$, then $\mu(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$.
Finite additivity of μ means that for any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets, we have $\mu(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu(E_k)$.
We can consider $\{\bigcup_{k=n+1}^{\infty} E_k\}_{n=1}^{\infty}$, a descending sequence of sets in $\mathcal{M}$ with $\mu(\bigcup_{k=2}^{\infty} E_k) < \infty$, and then because $\{E_k\}_{k=1}^{n}$ is disjoint,

$$\mu(\bigcap_{n=1}^{\infty}[\bigcup_{k=n+1}^{\infty} E_k]) = \lim_{n\to\infty} \mu(\bigcup_{k=n+1}^{\infty} E_k) = 0.$$

Thus we see

$$\begin{aligned}
\mu(\bigcup_{k=1}^{\infty} E_k) &= \mu([\bigcup_{k=1}^{n} E_k] \cup [\bigcup_{k=n+1}^{\infty} E_k]) \\
&= \mu(\bigcup_{k=1}^{n} E_k) + \mu(\bigcup_{k=n+1}^{\infty} E_k) && \text{by disjoint additivity} \\
&= \sum_{k=1}^{n} \mu(E_k) + \mu(\bigcup_{k=n+1}^{\infty} E_k) && \text{by disjoint additivity}
\end{aligned}$$

The left hand side is independent of n, so taking the limit, we have

$$\begin{aligned}
\mu(\bigcup_{k=1}^{\infty} E_k) &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) + \lim_{n\to\infty} \mu(\bigcup_{k=n+1}^{\infty} E_k) \\
&= \sum_{k=1}^{\infty} \mu(E_k) + 0 \\
&= \sum_{k=1}^{\infty} \mu(E_k),
\end{aligned}$$

that is, μ satisfies countable additivity, and thus μ is a measure.

4. Let $\{(X_\lambda, \mathcal{M}_\lambda, \mu_\lambda)\}_{\lambda\in\Lambda}$ be a collection of measure spaces parametrized by the set Λ. Assume the collection of sets $\{X_\lambda\}_{\lambda\in\Lambda}$ is disjoint. Then we can form a new measure space (called their union) $(X, \mathcal{B}, \mu)$ by letting $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, letting $\mathcal{B}$ be the collection of subsets B of X such that $B \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$, and defining $\mu(B) = \sum_{\lambda\in\Lambda} \mu_\lambda[B \cap X_\lambda]$ for $B \in \mathcal{B}$.

   (i) Show that $\mathcal{B}$ is a σ-algebra.

   We have:

(i) $X \in \mathcal{B}$ because $X \subseteq X$ such that for any $\lambda' \in \Lambda$,

$$X \cap X_{\lambda'} = \bigcup_{\lambda \in \Lambda} X_\lambda \cap X_{\lambda'} = X_{\lambda'},$$

where $X_{\lambda'} \in \mathcal{M}_{\lambda'}$ because $(X_{\lambda'}, \mathcal{M}_{\lambda'}, \mu_{\lambda'})$ is a measure space.

(ii) if $B \in \mathcal{B}$, then $B \subseteq X$ such that for any $\lambda' \in \Lambda$, $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$.
Then $B^c \subseteq X$ and because $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ and $X_{\lambda'} \in \mathcal{M}_{\lambda'}$,

$$\mathcal{M}_{\lambda'} \ni [B \cap X_{\lambda'}]^c \cap X_{\lambda'} = [B^c \cup X_{\lambda'}^c] \cap X_{\lambda'} = [B^c \cap X_{\lambda'}] \cup [X_{\lambda'}^c \cap X_{\lambda'}] = B^c \cap X_{\lambda'}.$$

Therefore $B^c \in \mathcal{B}$.

(iii) if $B_i \in \mathcal{B}$, then $B_i \in X$ such that for any $\lambda' \in \Lambda$, $B_i \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all i.
Then $\bigcup_{i=1}^\infty B_i \in X$ and $[\bigcup_{i=1}^\infty B_i] \cap X_{\lambda'} = \bigcup_{i=1}^\infty [B_i \cap X_{\lambda'}] \in \mathcal{M}_{\lambda'}$.
Therefore $\bigcup_{i=1}^\infty B_i \in \mathcal{B}$.

(ii) Show that μ is a measure.

For $B \in \mathcal{B}$, we have $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all $\lambda' \in \Lambda$, and so $\mu_{\lambda'}[B \cap X_{\lambda'}]$ is defined.
Then $\mu_{\lambda'}[B \cap X_{\lambda'}] \geq 0$ for all $\lambda' \in \Lambda$, which implies $\mu(B) = \sum_{\lambda \in \Lambda} \mu_\lambda[B \cap X_\lambda] \geq 0$.
Then because $\emptyset = X^c$ is in $\mathcal{B}$, then $\emptyset = \emptyset \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all $\lambda' \in \Lambda$, and then $\mu(\emptyset) = \sum_{\lambda \in \Lambda} \mu_\lambda[\emptyset \cap X_\lambda] = \sum_{\lambda \in \Lambda} 0 = 0$.
Finally, consider any countable disjoint collection $\{B_k\}_{k=1}^\infty$ in $\mathcal{B}$. Then for any $\lambda' \in \Lambda$, the collection $\{B_k \cap X_{\lambda'}\}_{k=1}^\infty$ is disjoint so that

$$\begin{aligned}
\mu(\bigcup_{k=1}^\infty B_k) &= \sum_{\lambda \in \Lambda} \mu_\lambda[(\bigcup_{k=1}^\infty B_k) \cap X_\lambda] \\
&= \sum_{\lambda \in \Lambda} \mu_\lambda[\bigcup_{k=1}^\infty (B_k \cap X_\lambda)] \\
&= \sum_{\lambda \in \Lambda} \sum_{k=1}^\infty \mu_\lambda(B_k \cap X_\lambda) \\
&= \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(B_k \cap X_\lambda) \\
&= \sum_{k=1}^\infty \mu(B_k).
\end{aligned}$$

Therefore μ is a measure.

(iii) Show that μ is σ-finite iff all but a countable number of the measures μ_λ have $\mu(X_\lambda) = 0$ and the remainder are σ-finite.

($\implies$) Suppose μ is σ-finite.
Then X can be written as the countable union of disjoint measurable sets, each of which has finite measure under μ.
That is, we have $X = \bigcup_{k=1}^\infty A_k$, with $A_k \in \mathcal{B}$ s.t. $\mu(A_k) < \infty$ for each k.
So $A_k \in \mathcal{B} \implies A_k \cap X_\lambda \in \mathcal{M}_\lambda$ for each λ, and $\sum_{\lambda \in \Lambda} \mu_\lambda(A_k \cap X_\lambda) = \mu(A_k) < \infty \implies \mu_\lambda(A_k \cap X_\lambda) < \infty$ for each λ.
Then

$$\mu(X) = \mu(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k) = \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(A_k \cap X_\lambda)$$

Then all but a countable number of the measures μ_λ can be nonzero, and the remainder must be σ-finite.

($\Longleftarrow$) Suppose all but a countable number of the measures μ_λ have $\mu(X_\lambda) = 0$ and the remainder are σ-finite.
Let $\Lambda^* \subseteq \Lambda$ be the set of measures μ_λ such that $\mu_\lambda(X_\lambda) = 0$.
Let $\Lambda^{*c} = \{\lambda_k\}_{k=1}^\infty \subseteq \Lambda$ be a countable collection such that each μ_{λ_k} is σ-finite.
By definition of σ-finite, for each k, we have $X_{\lambda_k} = \bigcup_{i=1}^\infty [A_{\lambda_k}]_i$, with $[A_{\lambda_k}]_i \in \mathcal{M}_{\lambda_k}$ s.t. $\mu_{\lambda_k}([A_{\lambda_k}]_i) < \infty$ for each i.
Then because the collection $\{X_\lambda\}_{\lambda\in\Lambda}$ is disjoint, then $[A_{\lambda_k}]_i \cap X_\lambda = ([A_{\lambda_k}]_i \cap X_{\lambda_k}) \cap X_\lambda = \emptyset$ for $\lambda \neq \lambda_k$.
Also $[A_{\lambda_k}]_i = [A_{\lambda_k}]_i \cap X_{\lambda_k} \in \mathcal{M}_{\lambda_k}$ so that $[A_{\lambda_k}]_i \in \mathcal{B}$ and $\mu([A_{\lambda_k}]_i)$ is defined.
Then we have for each i,

$$\begin{aligned}
\mu([A_{\lambda_k}]_i) &= \sum_{\lambda\in\Lambda} \mu_\lambda([A_{\lambda_k}]_i \cap X_\lambda) \\
&= \sum_{\lambda\neq\lambda_k} \mu_\lambda([A_{\lambda_k}]_i \cap X_\lambda) + \mu_{\lambda_k}([A_{\lambda_k}]_i \cap X_\lambda) \\
&= \sum_{\lambda\neq\lambda_k} 0 + \mu_{\lambda_k}([A_{\lambda_k}]_i) \\
&= \mu_{\lambda_k}([A_{\lambda_k}]_i).
\end{aligned}$$

Therefore $\mu_{\lambda_k}([A_{\lambda_k}]_i) = \mu([A_{\lambda_k}]_i) < \infty$.
Then we can write

$$\begin{aligned}
\mu(X) &= \sum_{\lambda\in\Lambda} \mu_\lambda(X_\lambda) \\
&= \sum_{\lambda\in\Lambda*} \mu_\lambda(X_\lambda) + \sum_{\lambda\notin\Lambda^*} \mu_\lambda(X_\lambda) \\
&= \sum_{\lambda\in\Lambda*} 0 + \sum_{k=1}^\infty \mu_{\lambda_k}(X_{\lambda_k}) \\
&= \sum_{k=1}^\infty \mu_{\lambda_k}(\bigcup_{i=1}^\infty [A_{\lambda_k}]_i) \\
&= \sum_{k=1}^\infty \sum_{i=1}^\infty \mu_{\lambda_k}([A_{\lambda_k}]_i) \\
&= \sum_{k=1}^\infty \sum_{i=1}^\infty \mu([A_{\lambda_k}]_i),
\end{aligned}$$

and thus X can be written as a countable disjoint union of measurable sets $[A_{\lambda_k}]_i$, each of which has finite measure under μ.
Therefore μ is σ-finite.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference, $E_1 \Delta E_2$, of two subsets E_1 and E_2 of X is defined by

$$E_1 \Delta E_2 = [E_1 \setminus E_2] \cup [E_2 \setminus E_1].$$

(i) Show that if E_1 and E_2 are measurable and $\mu(E_1 \Delta E_2) = 0$, then $\mu(E_1) = \mu(E_2)$.

We can see that

$$\mu(E_1 \cup E_2) = \mu([E_1 \Delta E_2] \cup [E_1 \cap E_2]) = \mu(E_1 \Delta E_2) + \mu(E_1 \cap E_2) = \mu(E_1 \cap E_2).$$

Then we also know that by monotonicity we have

$$E_1 \cap E_2 \subseteq E_1, E_2 \subseteq E_1 \cup E_2 \implies \mu(E_1 \cap E_2) \leq \mu(E_1), \mu(E_2) \leq \mu(E_1 \cup E_2),$$

and therefore $\mu(E_1) = \mu(E_2)$.

(ii) Show that if μ is complete and $E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ if $\mu(E_1 \Delta E_2) = 0$.

Because $\mu(E_1 \Delta E_2) = 0$, then because μ is complete, the subsets $[E_1 \setminus E_2] \subseteq E_1 \Delta E_2$ and $[E_2 \setminus E_1] \subseteq E_1 \Delta E_2$ are measurable. Therefore the set $[E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c$ is also measurable, and

$$\begin{aligned}
[E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c &= [E_2 \cup E_1] \cap [E_1^c \cup E_1] \cap [E_1^c \cup E_2] \\
&= [E_2 \cup E_1] \cap [E_1^c \cup E_2] \\
&= ([E_2 \cup E_1] \cap E_1^c) \cup ([E_2 \cup E_1] \cap E_2) \\
&= ([E_2 \cap E_1^c] \cup [E_1 \cap E_1^c]) \cup E_2 \\
&= (E_2 \cap E_1^c) \cup E_2 \\
&= E_2,
\end{aligned}$$

therefore $E_2 = [E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c$ is measurable.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and X_0 belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of X_0 and μ_0 the restriction of μ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

We want to show that $(X_0, \mathcal{M}_0)$ is a measurable space (i.e., that $\mathcal{M}_0$ is a σ-algebra of subsets of X_0) and that μ_0 is a measure on $\mathcal{M}_0$.
To see that $\mathcal{M}_0$ is a σ-algebra:

(i) $X_0 \in \mathcal{M}_0$ because $X_0 \in \mathcal{M}$ and $X_0 \subseteq X_0$.

(ii) if $A \in \mathcal{M}_0$, then $A \in \mathcal{M}$ and $A \subseteq X_0$.
Then $X_0 \cap A^c \in \mathcal{M}$ and $X_0 \cap A^c \subseteq X_0$ imply that $X_0 \cap A^c \in \mathcal{M}_0$.

(iii) if $A_i \in \mathcal{M}_0$, then $A_i \in \mathcal{M}$ and $A_i \subseteq X_0$ for all i.
Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ and $\bigcup_{i=1}^{\infty} A_i \subseteq X_0$ imply that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}_0$.

Therefore $(X_0, \mathcal{M}_0)$ is a measurable space.
Clearly μ_0 is a measure on $\mathcal{M}_0$, because it inherits the properties of a measure from μ.
Thus $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

7. Let $(X, \mathcal{M})$ be a measurable space. Verify the following:

(i) If μ and ν are measures defined on $\mathcal{M}$, then set set function λ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure. We denote λ by $\mu + \nu$.

Because $\mu(E) \geq 0$ and $\nu(E) \geq 0$ for any $E \in \mathcal{M}$, then $\lambda(E) = \mu(E) + \nu(E) \geq 0$.

Also, $\mu(\emptyset) = 0$ and $\nu(\emptyset) = 0$ imply that $\lambda(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0$.
Finally, for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \sum_{k=1}^{\infty} \mu(E_k) + \sum_{k=1}^{\infty} \nu(E_k) \\
&= \sum_{k=1}^{\infty} [\mu(E_k) + \nu(E_k)] \\
&= \sum_{k=1}^{\infty} \lambda(E_k).
\end{aligned}$$

Therefore λ is a measure.

(ii) If μ and ν are measures on $\mathcal{M}$ and $\mu \geq \nu$, then there is a measure λ on $\mathcal{M}$ for which $\mu = \nu + \lambda$.

In the case $\mu(E) < \infty$, then we also have $\nu(E) \leq \mu(E) < \infty$, and we can let $\lambda = \mu - \nu$.
We clearly see that $\mu \geq \nu \implies \mu - \nu \geq 0$ so that $\lambda(E) = \mu(E) - \nu(E) \geq 0$ for any $E \in \mathcal{M}$ (of finite measure under μ).
Also, $\mu(\emptyset) = 0$ and $\nu(\emptyset) = 0$ imply that $\lambda(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0$.
Finally, for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets each of finite measure,

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) - \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \sum_{k=1}^{\infty} \mu(E_k) - \sum_{k=1}^{\infty} \nu(E_k) \\
&= \sum_{k=1}^{\infty} [\mu(E_k) - \nu(E_k)] \\
&= \sum_{k=1}^{\infty} \lambda(E_k).
\end{aligned}$$

In the case $\mu(E) = \infty$, we can let $\lambda(E) = \infty$ so that $\nu(E) + \lambda(E) = \mu(E)$.
Then $\lambda(E) = \infty \geq 0$.
For any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, supposing there exists an index j such that $\mu(E_j) = \infty$, then we defined $\lambda(E_j) = \infty$ so that by monotonicity, we have

$$\infty = \lambda(E_j) \leq \lambda\left(\bigcup_{k=1}^{\infty} E_k\right),$$

so $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty = \sum_{k=1}^{\infty} \lambda(E_k)$.

Then we also have $\sum_{k=1}^{\infty} \mu(E_k) = \infty$ and

$$\begin{aligned}\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \lambda\left(\bigcup_{k=1}^{\infty} E_k\right)\\ &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \infty\\ &= \infty.\end{aligned}$$

In conclusion, we have defined

$$\lambda(E) = \begin{cases} \mu(E) - \nu(E) & \text{if } \mu(E) < \infty \\ \infty & \text{if } \mu(E) = \infty, \end{cases}$$

and we have proved that λ is a measure.

(iii) If ν is σ-finite, the measure λ in (ii) is unique.

Because ν is σ-finite, then X is the union of a countable collection of measurable sets (may be taken to be disjoint), each of which has finite measure under ν. That is, $X = \bigcup_{k=1}^{\infty} X_k$, where $\nu(X_k) < \infty$. Then for any $E \in \mathcal{M}$, we have

$$E = E \cap X = E \cap \bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} [E \cap X_k],$$

where by monotonicity of measure we have $\nu(E \cap X_k) \leq \nu(X_k) < \infty$, and thus any measurable set E is also σ-finite when ν is σ-finite.
Now, suppose there exist measures λ_1 and λ_2 such that $\mu = \nu + \lambda_1$ and $\mu = \nu + \lambda_2$. Then $\nu + \lambda_1 = \nu + \lambda_2$ and thus $\nu - \nu = \lambda_2 - \lambda_1$.
For any $E \in \mathcal{M}$ such that $\nu(E) < \infty$, then clearly $\lambda_1(E) = \lambda_2(E)$.
For any $E \in \mathcal{M}$ such that $\nu(E) = \infty$, $\nu(E) - \nu(E) = \infty - \infty$ is not defined.
However, because ν is σ-finite, there exists a countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ such that $E = \bigcup_{k=1}^{\infty} E_k$ and $\nu(E_k) < \infty$ for each k. Then we see that $\nu(E_k) - \nu(E_k)$ is defined for all k, and

$$\nu(E) = \nu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \nu(E_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k).$$

Then we can write

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k) - \lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k) &= \lambda_2(E) - \lambda_1(E)\\ \lim_{n\to\infty} \sum_{k=1}^{n} [\nu(E_k) - \nu(E_k)] &= \lambda_2(E) - \lambda_1(E)\\ \lim_{n\to\infty} \sum_{k=1}^{n} 0 &= \lambda_2(E) - \lambda_1(E)\\ 0 &= \lambda_2(E) - \lambda_1(E).\end{aligned}$$

Therefore $\lambda_1(E) = \lambda_2(E)$, and the measure λ is unique.

(iv) Show that in general the measure λ need not be unique but that there is always a smallest such λ.

Suppose there exists a set $E \in \mathcal{M}$ such that $\mu(E) = \infty$ and $\nu(E) = \infty$. Then regardless of the number $\lambda(E) \in [0, \infty]$ we define λ to be, we always have $\infty = \mu(E) = \nu(E) + \lambda(E)$. Then $\lambda(E) = 0$ is the smallest value that we can set λ to be, and we can define the smallest λ in the following way:

$$\lambda(E) = \begin{cases} \mu(E) - \nu(E) & \text{if } \mu(E) < \infty \text{ (forces } \nu(E) < \infty) \\ \infty & \text{if } \mu(E) = \infty, \nu(E) < \infty \\ 0 & \text{if } \mu(E) = \infty, \nu(E) = \infty \end{cases}$$

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure μ is said to be **semifinite** provided each measurable set of infinite measure contains measurable sets of arbitrarily large finite measure.

(i) Show that each σ-finite measure is semifinite.

If we suppose μ is σ-finite, then we can write any $E \in \mathcal{M}$ as the countable disjoint union of measurable sets of finite measure under μ: $E = \bigcup_{k=1}^{\infty} E_k$ with $\mu(E_k) < \infty$.
Consider any measurable set E such that $\mu(E) = \infty$. Then

$$\mu(E) = \mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) = \infty.$$

Then the sequence of partial sums $\sum_{k=1}^{n} \mu(E_k)$ converges to infinity. That is, for any real number x, there exists an index j such that $\sum_{k=1}^{j} \mu(E_k) > x$. Because each E_k is disjoint and measurable, we have that $E_x := \bigcup_{k=1}^{j} E_k \in \mathcal{M}$, and we can write

$$x < \sum_{k=1}^{j} \mu(E_k) = \mu(\bigcup_{k=1}^{j} E_k) = \mu(E_x) < \infty.$$

That is, for any real number we choose, there exists a measurable set $E_x \subseteq E$ of finite measure that is larger than x.
Therefore μ is semifinite.

(ii) For $E \in \mathcal{M}$, define $\mu_1(E) = \sup\{\mu(F) \mid F \subseteq E, \mu(F) < \infty\}$. Show that μ_1 is a semifinite measure: it is called the semifinite part of μ.

Consider any measurable set E such that $\mu_1(E) = \infty$. Then for any subset F of E such that $\mu(F) < \infty$, we have that $\mu_1(E) \geq \mu(F) = \mu_1(F)$ by definition of supremum. (We have $\mu_1(F) = \mu(F)$ because F is the largest subset of itself). However, because μ_1 is the least upper bound, for any real number x, there exists a subset F_x of E such that $x < \mu_1(F_x) \leq \mu_1(E)$, else we reach a contradiction to the supremum. Therefore for any real number x we choose, there exists a measurable set $F_x \subseteq E$ of finite measure that is larger than x.

(iii) Find a measure μ_2 on $\mathcal{M}$ that only takes the values 0 and ∞ and $\mu = \mu_1 + \mu_2$.

We can define, for any $E \in \mathcal{M}$,

$$\mu_2(E) = \begin{cases} 0 & \text{if } \mu_1(E) < \infty \\ \infty & \text{if } \mu_1(E) = \infty \end{cases}$$

So that we have

$$\mu(E) = \begin{cases} \mu_1(E) + \mu_2(E) = \mu(E) + 0 & \text{if } \mu_1(E) < \infty \\ \mu_1(E) + \mu_2(E) = \mu(E) + \infty & \text{if } \mu_1(E) = \infty \end{cases}$$

9. Prove Proposition 3; that is, show that $\mathcal{M}_0$ is a σ-algebra, μ_0 is properly defined, and $(X, \mathcal{M}_0, \mu_0)$ is complete. In what sense is $\mathcal{M}_0$ minimal?

We can see

(i) $X \in \mathcal{M}_0$ because $X \subseteq X$, and $X = \emptyset \cup X$ with $X \in \mathcal{M}$ and $\emptyset \subseteq \emptyset$ for $\emptyset \in \mathcal{M}$ where $\mu(\emptyset) = 0$.

(ii) If $E \in \mathcal{M}_0$, then $E \subseteq X$, and $E = A \cup B$ with $B \in \mathcal{M}$ and $A \subseteq C$ for $C \in \mathcal{M}$ where $\mu(C) = 0$.
Then $A \subseteq C \implies A^c \supseteq C^c$, and $A^c = [A^c \cap C] \cup [A^c \cap C^c] = [A^c \cap C] \cup C^c$.
Now, $X \cap E^c \subseteq X$.
We can write

$$\begin{aligned} E^c &= A^c \cap B^c \\ &= ([A^c \cap C] \cup C^c) \cap B^c \\ &= ([A^c \cap C] \cap B^c) \cup (C^c \cap B^c), \end{aligned}$$

Where $C^c \cap B^c \in \mathcal{M}$ and $[A^c \cap C] \cap B^c \subseteq C$ for $C \in \mathcal{M}$ where $\mu(C) = 0$.
Therefore $E^c \in \mathcal{M}_0$.

(iii) If $E_k \in \mathcal{M}_0$, then $E_k \subseteq X$, and $E_k = A_k \cup B_k$ with $B_k \in \mathcal{M}$ and $A_k \subseteq C_k$ for $C_k \in \mathcal{M}$ where $\mu(C_k) = 0$ for all k.
Then $\bigcup_{k=1}^{\infty} E_k \subseteq X$, and

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} [A_k \cup B_k] = [\bigcup_{k=1}^{\infty} A_k] \cup [\bigcup_{k=1}^{\infty} B_k],$$

Where $\bigcup_{k=1}^{\infty} B_k \in \mathcal{M}$ and $A_k \subseteq C_k \implies \bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} C_k$ for $\bigcup_{k=1}^{\infty} C_k \in \mathcal{M}$ with $\mu(\bigcup_{k=1}^{\infty} C_k) \leq \sum_{k=1}^{\infty} \mu(C_k) = \sum_{k=1}^{\infty} 0 = 0$.
Thus $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}_0$.

Thus $\mathcal{M}_0$ is a σ-algebra of subsets of X.

To see that μ_0 is a measure on the measurable space $(X, \mathcal{M}_0)$:
For any $E \in \mathcal{M}_0$, we have $E = A \cup B, B \in \mathcal{M}$, so that $\mu_0(E) = \mu(B) \geq 0$.
Then for $\emptyset \in \mathcal{M}_0$, we have $\emptyset = \emptyset \cup \emptyset, \emptyset \in \mathcal{M}$, so that $\mu_0(\emptyset) = \mu(\emptyset) = 0$.
Finally, consider a disjoint collection $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{M}_0$.
See (iii) to see that $\bigcup_{k=1}^{\infty} E_k = [\bigcup_{k=1}^{\infty} A_k] \cup [\bigcup_{k=1}^{\infty} B_k]$, where $\{E_k\}_{k=1}^{\infty}$ disjoint implies $\{B_k\}_{k=1}^{\infty}$ disjoint and we have $\bigcup_{k=1}^{\infty} B_k \in \mathcal{M}$.
Then

$$\mu_0(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \mu(B_k) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Therefore $(X, \mathcal{M}_0, \mu_0)$ is a measure space.

To see that $(X, \mathcal{M}_0, \mu_0)$ is complete, consider any set $E \in \mathcal{M}_0$ such that $\mu_0(E) = 0$.

$$E \in \mathcal{M}_0 \implies E \subseteq X, E = A \cup B, B \in \mathcal{M} \text{ and } A \subseteq C \text{ with } C \in \mathcal{M}, \mu(C) = 0.$$

Then $A \subseteq C \implies A \cup B \subseteq C \cup B$, and $C, B \in \mathcal{M} \implies C \cup B \in \mathcal{M}$. Thus $\mu(C \cup B) \leq \mu(C) + \mu(B) = 0$ is well-defined.
Consider any $D \subseteq E$.

$$D \subseteq E \subseteq X, D = D \cup \emptyset, \emptyset \in \mathcal{M} \text{ and } D \subseteq A \cup B \subseteq C \cup B \text{ with } C \cup B \in \mathcal{M}, \mu(C \cup B) = 0.$$

Therefore $D \in \mathcal{M}_0$ and $(X, \mathcal{M}_0, \mu_0)$ is complete.

10. If $(X, \mathcal{M}, \mu)$ is a measure space, we say that a subset E of X is **locally measurable** provided for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $E \cap B$ belongs to $\mathcal{M}$. The measure μ is called **saturated** provided every locally measurable set is measurable.

(i) Show that each σ-finite measure is saturated.

Suppose μ is σ-finite, then X can be taken to be the union of a countable collection of measurable sets, each of which has finite measure under μ.
That is, $X = \bigcup_{k=1}^{\infty} X_k$, where $\mu(X_k) < \infty$.
Then for any $E \in X$, we have

$$E = E \cap X = E \cap \bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} [E \cap X_k],$$

In the case that E is locally measurable, then each intersection $E \cap X_k$ is measurable. Then the countable intersection of measurable sets $\bigcup_{k=1}^{\infty} [E \cap X_k] = E$ is measurable.
Thus when μ is σ-finite, every locally measurable set is measurable, and thus μ is saturated.

(ii) Show that the collection $\mathcal{C}$ of locally measurable sets is a σ-algebra.

We have

(i) $X \in \mathcal{C}$ because for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $X \cap B = B \in \mathcal{M}$.

(ii) if $E \in \mathcal{C}$, then for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $E \cap B \in \mathcal{M}$.
Then we have the two measurable sets $E \cap B$ and B so that $[E \cap B]^c \cap B$ is also measurable, and

$$\mathcal{M} \ni [E \cap B]^c \cap B = [E^c \cup B^c] \cap B = [E^c \cap B] \cup [B^c \cap B] = E^c \cap B,$$

and thus $E^c \in \mathcal{C}$.

(iii) if $E_i \in \mathcal{C}$, then for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $E_i \cap B \in \mathcal{M}$ for all i.
Then $[\bigcup_{i=1}^{\infty} E_i] \cap B = \bigcup_{i=1}^{\infty} [E_i \cap B] \in \mathcal{M}$ and thus $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$.

(iii) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{C}$ the σ-algebra of locally measurable sets. For $E \in \mathcal{C}$, define $\overline{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\overline{\mu}(E) = \infty$ if $E \notin \mathcal{M}$. Show that $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

In (ii) we showed that $\mathcal{C}$ is a σ-algebra of subsets of X.
Therefore $(X, \mathcal{C})$ is a measurable space.

We have defined

$$\overline{\mu}(E) = \begin{cases} \mu(E) & \text{if } E \in \mathcal{M} \\ \infty & \text{if } E \notin \mathcal{M} \end{cases}$$

We have $\overline{\mu}(E) \in \{\mu(E), \infty\} \geq 0$ for all $E \in \mathcal{C}$. We have $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$ because $\emptyset \in \mathcal{M}$ and $\emptyset \in \mathcal{C}$. Finally, consider a countable disjoint collection of sets $\{E_k\}_{k=1}^{\infty}$ in $\mathcal{C}$.

(i) If for all k we have $E_k \in \mathcal{M}$, then $\mu(\bigcup_{k=1}^{\infty} E_k)$ is measurable, $\overline{\mu}(E_k) = \mu(E_k)$, and

$$\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \overline{\mu}(E_k).$$

(ii) If there exists an index j such that $E_j \notin \mathcal{M}$, then $\overline{\mu}(E_j) = \infty$ and $\sum_{k=1}^{\infty} \overline{\mu}(E_k) = \infty$.

Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is measurable and $\mu(\bigcup_{k=1}^{\infty} E_k) < \infty$.
Then because for any j, we have $E_j \in \mathcal{C}$, then $E_j = E_j \cap \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.
Then (i) must hold.

Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is measurable and $\mu(\bigcup_{k=1}^{\infty} E_k) = \infty$.
If (i) holds, then $\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \overline{\mu}(E_k)$.
If (ii) holds, then $\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} E_k) = \infty = \sum_{k=1}^{\infty} \overline{\mu}(E_k)$.
Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is not measurable. Then $\mu(\bigcup_{k=1}^{\infty} E_k) = \infty$.

Then (ii) must hold else we reach a contradiction to $\bigcup_{k=1}^{\infty} E_k \notin \mathcal{M}$.
Therefore $(X, \mathcal{C}, \overline{\mu})$ is a measure space.

We can use the definition of $\overline{\mu}$ to see that

$$B \in \mathcal{C} \text{ with } \overline{\mu}(B) < \infty \iff B \in \mathcal{M} \text{ with } \mu(B) < \infty.$$

Consider a set $E \subseteq X$ such that $E \cap B \in \mathcal{C}$ for any such B.
Then by monotonicity, $\overline{\mu}(E \cap B) \leq \overline{\mu}(B) < \infty$.
Because $\overline{\mu}(E \cap B) < \infty$, see the definition of $\overline{\mu}$ to see that $E \cap B \in \mathcal{M}$.
Then we see that $E \in \mathcal{C}$ because for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, we have $E \cap B \in \mathcal{M}$.
Therefore $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

(iv) If μ is semifinite and $E \in \mathcal{C}$, the set $\underline{\mu}(E) = \sup\{\mu(B) \mid B \in \mathcal{M}, B \subseteq E\}$. Show that $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space and that $\underline{\mu}$ is an extension of μ. Give an example to show that $\overline{\mu}$ and $\underline{\mu}$ may be different.

We first want to show that $\underline{\mu}$ is a measure on the measurable space $(X, \mathcal{C})$:
For any $E \in \mathcal{C}$, we have $\underline{\mu}(E) \geq \mu(B) \geq 0$ for $B \in \mathcal{M}, B \subseteq E$.
For $\emptyset \in \mathcal{C}$, we have $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$ because $\{\emptyset\} = \{B \in \mathcal{M} \mid B \subseteq \emptyset\}$.
Finally, for any disjoint collection $\{E_k\}_{k=1}^{\infty}$ in $\mathcal{C}$,

Therefore $(X, \mathcal{C}, \underline{\mu})$ is a measure space.

Consider any $E \subseteq X$ such that $E \cap B \in \mathcal{C}$ whenever $B \in \mathcal{C}$ with $\underline{\mu}(B) < \infty$.
Then $E \cap B \in \mathcal{C}$ implies that $[E \cap B] \cap B' \in \mathcal{M}$ whenever $B' \in \mathcal{M}$ with $\mu(B') < \infty$.

11. Let μ and η be measures on the measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $\nu(E) = \max\{\mu(E), \eta(E)\}$. Is ν a measure on $(X, \mathcal{M})$?

We have $0 \leq \mu(E), \eta(E) \leq \max\{\mu(E), \eta(E)\}$ for any $E \in \mathcal{M}$.
We have $\max\{\mu(E), \eta(E)\} \in [0, \infty]$ for any $E \in \mathcal{M}$.
Counterexample: Let E_1, E_2 be nonempty disjoint measurable (singleton) sets such that

$$\mu(E) = \begin{cases} 1 & E \supseteq E_1 \\ 0 & E \not\supseteq E_1 \end{cases} \text{ and } \eta(E) = \begin{cases} 1 & E \supseteq E_2 \\ 0 & E \not\supseteq E_2 \end{cases}$$

Then $\mu(E) \in \{0,1\} \geq 0$, $\mu(\emptyset) = 0$ because $\emptyset \not\supseteq E_1$, and for any countable disjoint collection $\{A_k\}_{k=1}^{\infty}$ of measurable sets, in the case that for all k, $A_k \not\supseteq E_1$, then $\bigcup_{k=1}^{\infty} A_k \not\supseteq E_1$ and

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = 0.$$

In the case that there exists an index j such that $A_j \supseteq E_1$, then $\bigcup_{k=1}^{\infty} A_k \supseteq E_1$, and because the sets are disjoint, $A_i \not\supseteq E_1$ for $i \neq j$.

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k \in \mathbb{N}\setminus\{j\}} \mu(A_k) + \mu(A_j) = 0 + \mu(A_j) = 1.$$

Then η can also be shown to be a measure in the exact same way.
Then we see that

$$\nu(E_1 \cup E_2) = \max\{\mu(E_1 \cup E_2), \eta(E_1 \cup E_2)\} = \max\{1,1\} = 1,$$
$$\nu(E_1) + \nu(E_2) = \max\{\mu(E_1), \eta(E_1)\} + \max\{\mu(E_2), \eta(E_2)\} = \max\{1,0\} + \max\{0,1\} = 2.$$

Thus ν is not a measure because it does not satisfy countable additivity.

17.2 Signed Measures: The Hahn and Jordan Decompositions

Definition. *By a* ***signed measure*** ν *on the measurable space* $(X, \mathcal{M})$ *we mean an extended real-valued set function* $\nu : \mathcal{M} \to [-\infty, \infty]$ *that possesses the following properties:*

(i) ν *assumes at most one of the values* $+\infty, -\infty$.

(ii) $\nu(\emptyset) = 0$.

(iii) For any countable collection $\{E_k\}_{k=1}^{\infty}$ *of disjoint measurable sets,*

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

where the series $\sum_{k=1}^{\infty} \nu(E_k)$ *converges absolutely if* $\nu(\bigcup_{k=1}^{\infty} E_k)$ *is finite (convergence must hold for any rearrangement).*

A set A is **positive** (with respect to ν) provided that A is measurable and for every measurable subset E of A we have $\nu(E) \geq 0$. The restriction of ν to the measurable subsets of a positive set is a measure. (See Problem 6). Similarly, a set B is called **negative** (with respect to ν) provided that B is measurable and for every measurable subset E of B we have $\nu(E) \leq 0$. The restriction of $-\nu$ to the measurable subsets of a negative set is a measure. A measurable set is called **null** with respect to ν provided every measurable subset of it also has measure zero. (Clearly a null set is both positive and negative.)
Monotonicity for signed measures:

$$A \subseteq B \text{ and } |\nu(B)| < \infty \implies |\nu(A)| < \infty.$$

It is not possible for a signed measure to take on both $\pm\infty$ at the same time.

To see this, suppose there exist two subsets E_1, E_2 of X such that $\nu(E_1) = \infty$ and $\nu(E_2) = -\infty$.
If $-\infty < \nu(E_1 \cap E_2) < \infty$, then

$$\nu(E_1) = \nu(E_1 \cap E_2) + \nu(E_1 \setminus E_2) = \infty \text{ so that } \nu(E_1 \setminus E_2) = \infty,$$
$$\nu(E_2) = \nu(E_2 \cap E_1) + \nu(E_2 \setminus E_1) = -\infty \text{ so that } \nu(E_2 \setminus E_1) = -\infty,$$

and $E_1 \setminus E_2$ and $E_2 \setminus E_1$ are disjoint but $\infty - \infty$ is not defined.
If $\nu(E_1 \cap E_2) = \infty$, then

$$-\infty = (E_2) = \nu(E_2 \cap E_1) + \nu(E_2 \setminus E_1) = \infty + \nu(E_2 \setminus E_1),$$

and we cannot find an $E_2 \setminus E_1$ that satisfies this because $-\infty + \infty$ is not defined.
If $\nu(E_1 \cap E_2) = -\infty$, then

$$\infty = (E_1) = \nu(E_1 \cap E_2) + \nu(E_1 \setminus E_2) = -\infty + \nu(E_1 \setminus E_2),$$

and we cannot find an $E_1 \setminus E_2$ that satisfies this because $\infty - \infty$ is not defined.

The Hahn Decomposition Theorem. *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there is a positive set A for ν and a negative set B for ν for which*

$$X = A \cup B \text{ and } A \cap B = \emptyset.$$

Proof. Without loss of generality assume $+\infty$ is the infinite value omitted by ν. (Otherwise let A be the negative set and B be the positive set. We need this to use Hahn's Lemma: $0 < \nu(E) < \infty$.)
Let $\mathcal{P}$ be the collection of positive subsets of X. Define $\lambda = \sup\{\nu(E) \mid E \in \mathcal{P}\}$, and $\lambda \geq 0$ and it is nonempty because $\emptyset \in \mathcal{P}$. Then by definition of supremum, for each natural number k, there exists an element $A_k \in \lambda$ such that

$$\lambda - \frac{1}{k} < \nu(A_k) \leq \lambda.$$

Then $\{A_k\}_{k=1}^{\infty}$ is a countable collection of positive subsets of X for which $\lambda = \lim_{k\to\infty} \nu(A_k)$. We can let $A = \cup_{k=1}^{\infty} A_k$, and by Proposition 4, the countable union of a positive collection of positive sets is positive, and thus A is a positive subset as well; then $\nu(A) \leq \lambda$. Then for each k, we have $A \cap A_k^c \subseteq A$, and because A is positive, all of its subsets including $A \setminus A_k$ is positive, so $\nu(A \setminus A_k) \geq 0$. Then by countable disjoint additivity,

$$\nu(A) = \nu(A_k) + \nu(A \setminus A_k) \geq \nu(A_k).$$

Therefore $\nu(A) \geq \lim_{k\to\infty} \nu(A_k) = \lambda$ and $\nu(A) \leq \lambda$ implies $\nu(A) = \lambda$. Also, $\lambda < \infty$ because ν does not take on the value ∞.
Let $B = X \setminus A$. Supposing by contradiction that B is not negative, then there exists a subset E of B of nonnegative measure: $\nu(E) \not\leq 0$ and $\nu(E) \neq \infty \implies 0 < \nu(E) < \infty$, and therefore, by Hahn's Lemma, there exists a measurable subset E_0 of E that is both positive and of positive measure. Then $A \cup E_0$ is a positive set and

$$\nu(A \cup E_0) = \nu(A) + \nu(E_0) > \lambda,$$

a contradiction to the choice of λ as the supremum. □

If $\{A, B\}$ is a Hahn decomposition for ν, then we define two measures ν^+ and ν^- (the positive and negative variations of ν) with $\nu = \nu^+ - \nu^-$ by setting

$$\nu^+(E) = \nu(E \cap A) \text{ and } \nu^-(E) = -\nu(E \cap B).$$

(The disjoint sets $E \cap A$ and $E \cap B$ have positive and negative measure, respectively, because A is positive (and thus all subsets have positive measure), and B is negative (and thus all subsets have negative measure).)

Two measures ν_1 and ν_2 on $(X, \mathcal{M})$ are said to be **mutually singular** ($\nu_1 \perp \nu_2$) if there are disjoint measurable sets A, B with $X = A \cup B$ for which $\nu_1(A) = \nu_2(B) = 0$.

The Jordan Decomposition Theorem. *Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ for which $\nu = \nu^+ - \nu^-$. Moreover, there is only one such pair of mutually singular measures.*

Proof. Existence:
By the Hahn Decomposition Theorem, there exists a positive set A and a negative set B for which $X = A \cup B$ and $A \cap B = \emptyset$. Then we can define ν^+ and ν^- on $(X, \mathcal{M})$ such that

$$\nu^+(E) = \nu(A \cap E) \text{ for all } E \in \mathcal{M}$$
$$\nu^-(E) = -\nu(B \cap E) \text{ for all } E \in \mathcal{M}$$

Clearly ν^+ and ν^- are measures because

$$\nu^+(E) \geq 0 \quad \text{because } A \cap E \subseteq A, \text{ and } A \text{ is positive, so for any } C \subseteq A, \text{ then } \nu(C) \geq 0$$
$$\nu^-(E) \geq 0 \quad \text{because } B \cap E \subseteq B, \text{ and } B \text{ is negative, so for any } C \subseteq B, \text{ then } \nu(C) \leq 0 \implies -\nu(C) \geq 0$$

$$\nu^+(\emptyset) = \nu(A \cap \emptyset) = \nu(\emptyset) = 0$$
$$\nu^-(\emptyset) = -\nu(B \cap \emptyset) = -\nu(\emptyset) = 0$$

For disjoint measurable collection $\{E_k\}_{k=1}^{\infty}$,

$$\nu^+(\bigcup_{k=1}^{\infty} E_k) = \nu(A \cap \bigcup_{k=1}^{\infty} E_k) = \nu(\bigcup_{k=1}^{\infty} [A \cap E_k]) = \sum_{k=1}^{\infty} \nu(A \cap E_k) = \sum_{k=1}^{\infty} \nu^+(E_k)$$
$$\nu^-(\bigcup_{k=1}^{\infty} E_k) = -\nu(B \cap \bigcup_{k=1}^{\infty} E_k) = -\nu(\bigcup_{k=1}^{\infty} [B \cap E_k]) = \sum_{k=1}^{\infty} -\nu(B \cap E_k) = \sum_{k=1}^{\infty} \nu^-(E_k)$$

The measures ν^+ and ν^- are mutually singular because $X = A \cup B$, $A \cap B = \emptyset$, and

$$\nu^+(B) = \nu(A \cap B) = \nu(\emptyset) = 0$$
$$\nu^-(A) = -\nu(B \cap A) = -\nu(\emptyset) = 0$$

Then for any measurable set E, we have $E = [E \cap A] \cup [E \cap B]$ so that

$$\nu(E) = \nu([E \cap A] \cup [E \cap B]) = \nu(E \cap A) + \nu(E \cap B) = \nu^+(E) - \nu^-(E).$$

□

We define $|\nu|$ on $\mathcal{M}$ by

$$|\nu|(E) = \nu^+(E) + \nu^-(E) \text{ for all } E \in \mathcal{M}.$$

In Problem 16, prove that

$$|\nu|(X) = \sup \sum_{k=1}^{n} |\nu(E_k)|, \tag{4}$$

where the supremum is taken over all finite disjoint collections $\{E_k\}_{k=1}^n$ of measurable subsets of X. For this reason $|\nu|(X)$ is called the **total variation** of ν and denoted by $\|\nu\|_{var}$.

Example Let $f:\mathbb{R}\to\mathbb{R}$ be a function that is Lebesgue integrable over $\mathbb{R}$. For a Lebesgue measurable set E, define $\nu(E)=\int_E f dm$.
Then ν is a signed measure on the measurable space $(\mathbb{R},\mathcal{L})$:

(i) f is integrable means that we have $\int_{\mathbb{R}}|f|<\infty$. Then because $|f\cdot\chi_E|\le|f|$ on $\mathbb{R}$, by the integral comparison test, $f\cdot\chi_E$ is integrable over $\mathbb{R}$. That is,

$$|\nu(E)|=\left|\int_E f dm\right|=\left|\int_{\mathbb{R}} f\cdot\chi_E dm\right|<\infty.$$

(ii) $\nu(\emptyset)=\int_\emptyset f dm=0$ (See Chapter 4 Problem 17).

(iii) Let $E=\{E_k\}_{k=1}^\infty$ be a countable collection of disjoint measurable sets. Then by Chapter 4 Theorem 20,

$$\nu\left(\bigcup_{k=1}^\infty E_k\right)=\nu(E)=\int_E f dm=\sum_{k=1}^\infty\int_{E_k} f dm=\sum_{k=1}^\infty\nu(E_k).$$

Thus ν is a signed measure.
Define

$$A:=\{x\in\mathbb{R}\mid f(x)\ge0\}$$
$$B:=\{x\in\mathbb{R}\mid f(x)<0\}$$

and define for each Lebesgue measurable set E,

$$\nu^+(E):=\nu(A\cap E)=\int_{A\cap E} f dm$$
$$\nu^-(E):=-\nu(B\cap E)=-\int_{B\cap E} f dm$$

Then $\{A,B\}$ is a Hahn decomposition of $\mathbb{R}$ w.r.t. the signed measure ν. Moreover, $\nu=\nu^+-\nu^-$ is a Jordan decomposition of ν.

PROBLEMS

12. In the above example, let E be a Lebesgue measurable set such that $0<\nu(E)<\infty$. Find a positive set A contained in E for which $\nu(A)>0$.

Consider the set $A'=A\cap E$, which is a positive set because it is a measurable subset of the positive set A.
Then

$$\nu(A')=\nu(A\cap E)=\int_{A\cap A'} f dm-\int_{B\cap A'} f dm=\int_{A\cap E} f dm\ge0.$$

If we suppose that $\nu(A')=0$, then we have

$$\nu(E)=\int_{A\cap E} f dm-\int_{B\cap E} f dm=-\int_{B\cap E} f dm\le0,$$

which is a contradiction to $\nu(E)>0$.
Therefore $\nu(A')>0$.

13. Let μ be a measure and μ_1 and μ_2 be mutually singular measures on a measurable space $(X, \mathcal{M})$ for which $\mu = \mu_1 - \mu_2$. Show that $\mu_2 = 0$. Use this to establish the uniqueness assertion of the Jordan Decomposition Theorem.

Because μ_1 and μ_2 are mutually singular, then there exist disjoint measurable sets A, B with $X = A \cup B$ for which $\mu_1(A) = \mu_2(B) = 0$.
Consider $E \in \mathcal{M}$.
In the case $E \subseteq B$, we have $\mu_2(E) = 0$ by monotonicity of measure.
In the case $E \subseteq A$, we have $\mu_1(E) = 0$, so that

$$\mu(E) = \mu_1(E) - \mu_2(E) = -\mu_2(E) \leq 0,$$

where $\mu(E) \geq 0$ implies $\mu_2(E) = 0$.
Therefore for any measurable set E,

$$\mu_2(E) = \mu_2(E \cap A) + \mu_2(E \cap B) = 0 + 0 = 0.$$

Suppose ν is a signed measure on the measurable space $(X, \mathcal{M})$, and suppose we have any two pairs of mutually singular measures (ν_1^+, ν_1^-) and (ν_2^+, ν_2^-) so that $\nu = \nu_1^+ - \nu_1^- = \nu_2^+ - \nu_2^-$.
By definition of mutually singular, there exist disjoint pairs (A_1, B_1) and (A_2, B_2) with $X = A_1 \cup B_1 = A_2 \cup B_2$ for which $\nu_1^+(A_1) = \nu_1^-(B_1) = 0$ and $\nu_2^+(A_2) = \nu_2^-(B_2) = 0$.
Let $E \in \mathcal{M}$.
In the case $E \subseteq B_1$,

$$\nu_1^+(E) = \nu_2^+(E) - \nu_2^-(E).$$

Restricting ν_1^+ to all measurable subsets of B_1 (see Problem 6), we have that $\nu_2^- = 0$ so that $\nu_1^+ = \nu_2^+$.
In the case $E \subseteq A_1$,

$$\nu_1^-(E) = -\nu_2^+(E) + \nu_2^-(E).$$

Restricting ν_1^- to all measurable subsets of A_1, we have that $\nu_2^+ = 0$ so that $\nu_1^- = \nu_2^-$.

Therefore for any measurable set E,

$$\nu_1^+(E) = \nu_1^+(E \cap B_1) = \nu_2^+(E \cap B_1) = \nu_2^+(E),$$

and Similarly,

$$\nu_1^-(E) = \nu_1^-(E \cap A_1) = \nu_2^-(E \cap A_1) = \nu_2^-(E).$$

14. Show that if E is any measurable set, then

$$-\nu^-(E) \leq \nu(E) \leq \nu^+(E) \text{ and } |\nu(E)| \leq |\nu|(E).$$

We have the Jordan decomposition

$$\nu(E) = \nu^+(E) - \nu^-(E),$$

where ν^+ and ν^- are measures so that they are nonnegative; therefore

$$0 \leq \nu^-(E) \implies \nu(E) \leq \nu^+(E),$$

and

$$\nu^+(E) \geq 0 \implies \nu(E) \geq -\nu^-(E).$$

By the triangle inequality,

$$|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E).$$

15. Show that if ν_1 and ν_2 are any two finite signed measures, then so is $\alpha\nu_1 + \beta\nu_2$, where α and β are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets E.

Let E be any measurable set.

(i) $|\alpha\nu_1(E) + \beta\nu_2(E)| \leq |\alpha||\nu_1(E)| + |\beta||\nu_2(E)| < |\alpha| \cdot \infty + |\beta| \cdot \infty = \infty$

(ii) $\alpha\nu_1(\emptyset) + \beta\nu_2(\emptyset) = \alpha \cdot 0 + \beta \cdot 0 = 0$

(iii) Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable sets.

$$\alpha\nu_1(\bigcup_{k=1}^{\infty} E_k) + \beta\nu_2(\bigcup_{k=1}^{\infty} E_k) = \alpha\sum_{k=1}^{\infty}\nu_1(E_k) + \beta\sum_{k=1}^{\infty}\nu_2(E_k) = \sum_{k=1}^{\infty}\left[\alpha\nu_1(E_k) + \beta\nu_2(E_k)\right].$$

Therefore $\alpha\nu_1 + \beta\nu_2$ is itself a finite signed measure.
Homogeneity and subadditivity of the absolute value come from the properties of the real numbers.

16. Prove (4).

Let $\{E_k\}_{k=1}^{n}$ be a finite, measurable partition of X.
Recall the definition $|\nu| = \nu^+ + \nu^-$.
From problem 14, we have $|\nu(E_k)| \leq |\nu|(E_k)$ for each k.
Then

$$\begin{aligned}
\sum_{k=1}^{n} |\nu(E_k)| &\leq \sum_{k=1}^{n} |\nu|(E_k) \\
&= \sum_{k=1}^{n} \nu^+(E_k) + \sum_{k=1}^{n} \nu^-(E_k) \\
&= \nu^+(\bigcup_{k=1}^{n} E_k) + \nu^-(\bigcup_{k=1}^{n} E_k) \\
&= \nu^+(X) + \nu^-(X) \\
&= |\nu|(X),
\end{aligned}$$

so that

$$\sup \sum_{k=1}^{n} |\nu(E_k)| \leq |\nu|(X). \tag{a}$$

Because ν^+ and ν^- are mutually singular measures, consider the disjoint sets E_1, E_2 such that $X = E_1 \cup E_2$ and $\nu^+(E_1) = \nu^-(E_2) = 0$.

Therefore $\nu(E_1) = \nu^+(E_1) - \nu^-(E_1) = -\nu^-(E_1)$ and $\nu(E_2) = \nu^+(E_2) - \nu^-(E_2) = \nu^+(E_2)$ so that

$$\begin{aligned}
|\nu|(X) &= |\nu|(E_1 \cup E_2) \\
&= \nu^+(E_1 \cup E_2) + \nu^-(E_1 \cup E_2) \\
&= \nu^-(E_1) + \nu^+(E_2) \\
&= |\nu(E_1)| + |\nu(E_2)| \\
&= \sum_{k=1}^{2} |\nu(E_k)| \\
&\leq \sup \sum_{k=1}^{n} |\nu(E_k)|. \qquad \text{(b)}
\end{aligned}$$

Then (a) and (b) imply equality:

$$|\nu|(X) = \sup \sum_{k=1}^{n} |\nu(E_k)|.$$

17. Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ and $\mu \vee \nu = \mu + \nu - \mu \wedge \nu$.

(i) Show that the signed measure $\mu \wedge \nu$ is smaller than μ and ν but larger than any other signed measure that is smaller than μ and ν.

(ii) Show that the signed measure $\mu \vee \nu$ is larger than μ and ν but smaller than any other signed measure that is larger than μ and ν.

(iii) If μ and ν are positive measures, show that they are mutually singular iff $\mu \wedge \nu = 0$.

We have the identities

$$\begin{aligned}
\max(\mu, \nu) + \min(\mu, \nu) &= \mu + \nu \\
\max(\mu, \nu) - \min(\mu, \nu) &= |\mu - \nu| \\
\max(\mu, \nu) &= \frac{1}{2}(\mu + \nu + |\mu - \nu|) \\
\min(\mu, \nu) &= \frac{1}{2}(\mu + \nu - |\mu - \nu|)
\end{aligned}$$

(i) $\mu(E) \wedge \nu(E) = \min(\mu(E), \nu(E)) \leq \mu(E), \nu(E)$.
If $\lambda(E) \leq \mu(E), \nu(E)$, then $\lambda(E) \leq \min\{\mu(E), \nu(E)\} = \mu(E) \wedge \nu(E)$.

(ii) We can see

$$\mu \vee \nu = \frac{1}{2}(2\mu + 2\nu) - \frac{1}{2}(\mu + \nu - |\mu - \nu|) = \frac{1}{2}(\mu + \nu + |\mu - \nu|),$$

So that $\mu(E) \vee \nu(E) = \max(\mu(E), \nu(E)) \geq \mu(E), \nu(E)$.
If $\lambda(E) \geq \mu(E), \nu(E)$, then $\lambda(E) \geq \max\{\mu(E), \nu(E)\} = \mu(E) \vee \nu(E)$.

(iii) Let μ and ν be positive measures.

($\implies$) Suppose that μ and ν are mutually singular.

Then there exist disjoint measurable sets A, B with $X = A \cup B$ s.t. $\mu(A) = \nu(B) = 0$.
Let E be a measurable set.
In the case $E \subseteq A$,

$$(\mu \wedge \nu)(E) = \frac{1}{2}(0 + \nu(E) - |0 - \nu(E)|) = 0.$$

In the case $E \subseteq B$,

$$(\mu \wedge \nu)(E) = \frac{1}{2}(\mu(E) + 0 - |\mu(E) - 0|) = 0.$$

Therefore, for any measurable set E,

$$(\mu \wedge \nu)(E) = (\mu \wedge \nu)(E \cap A) + (\mu \wedge \nu)(E \cap B) = 0 + 0 = 0.$$

($\Longleftarrow$) Suppose that $\mu \wedge \nu = 0$.
This implies that for any measurable set E, at least one of $\mu(E)$ and $\nu(E)$ must equal zero.
Consider the finite signed measure $\lambda = \mu - \nu$.
By the Hahn Decomposition Theorem, there is a positive set P for λ and a negative set N for λ for which

$$X = P \cup N \text{ and } P \cap N = \emptyset.$$

Let E be a measurable set.
In the case $E \subseteq P$,

$$\mu(E) - \nu(E) \geq 0.$$

- If $\mu(E) - \nu(E) > 0$, then $\mu(E) > 0$ and $\nu(E) = 0$.
- If $\mu(E) - \nu(E) = 0$, then $\mu(E) = \nu(E) = 0$.

In the case $E \subseteq N$,

$$\mu(E) - \nu(E) \leq 0.$$

- If $\mu(E) - \nu(E) < 0$, then $\mu(E) = 0$ and $-\nu(E) < 0$.
- If $\mu(E) - \nu(E) = 0$, then $\mu(E) = \nu(E) = 0$.

Then $\mu(N) = \nu(P) = 0$ so that μ and ν are mutually singular.

17.3 The Cathéodory Measure Induced by an Outer Measure

Definition. *A set function $\mu^* : 2^X \to [0, \infty]$ is called an* ***outer measure*** *provided $\mu^*(\emptyset) = 0$ and μ^* is countably monotone in the sense that whenever a set $E \in 2^X$ is covered by a countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in 2^X, then*

$$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k).$$

Clearly an outer measure is finitely monotone, which can be seen by setting $E_k = \emptyset$ for all $k > n$.

Definition. *For an outer measure $\mu^* : 2^X \to [0, \infty]$, we call a subset E of X* ***measurable*** *(with respect to μ^*) provided for every subset A of X,*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Since μ^* is finitely monotone so that $\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$, to show that $E \subseteq X$ is measurable, it is only necessary to prove that

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c) \text{ for all } A \subseteq X \text{ such that } \mu^*(A) < \infty.$$

Theorem 8. *Let μ^* be an outer measure on 2^X. Then the collection $\mathcal{M}$ of sets that are measurable with respect to μ^* is a σ-algebra. If $\overline{\mu}$ is the restriction of μ^* to $\mathcal{M}$, then $(X, \mathcal{M}, \overline{\mu})$ is a complete measure space.*

17.4 The Construction of Outer Measures

Theorem 9. *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Define $\mu^*(\emptyset) = 0$ and for $E \subseteq X$, $E \neq \emptyset$, define*

$$\mu^*(E) = \inf \sum_{k=1}^{\infty} \mu(E_k),$$

*where the infimum is taken over all countable collections $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{S}$ that cover E. Then the set function $\mu^* : 2^X \to [0, \infty]$ is an outer measure called the **outer measure induced by** μ.*

(If a subset E of X cannot be covered by a countable subcollection of $\mathcal{S}$, then it has outer measure equal to ∞.)

Proof. We have $\mu^*(\emptyset) = 0$ by definition so it remains to show countable monotonicity.
Consider any set $E \subseteq X$ that is covered by a countable collection $\{E_k\}_{k=1}^{\infty}$ of sets in X.
We suppose that $\mu^*(E_k) < \infty$ for each k, otherwise the result is trivial.
Fix $\epsilon > 0$.
For each k, by definition of infimum, there exists a countable collection $\{E_{ik}\}_{i=1}^{\infty}$ of sets in $\mathcal{S}$ that covers E_k and

$$\mu^*(E_k) \leq \sum_{i=1}^{\infty} \mu(E_{ik}) < \mu^*(E_k) + \frac{\epsilon}{2^k}. \tag{1}$$

Then $\{E_{ik}\}_{1 \leq k, i < \infty}$ is a countable collection of sets in S that covers $\bigcup_{k=1}^{\infty} E_k$ and therefore covers E. Therefore because μ^* is defined as the infimum of all such collections,

$$\begin{aligned}
\mu^*(E) &\leq \sum_{1 \leq k, i < \infty} \mu(E_{ik}) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \mu(E_{ik}) \\
&< \sum_{k=1}^{\infty} \mu^*(E_k) + \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} && \text{by (1)} \\
&= \sum_{k=1}^{\infty} \mu^*(E_k) + \epsilon,
\end{aligned}$$

and since this holds for all $\epsilon > 0$, then $\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k)$. □

Definition. *Let $\mathcal{S}$ be a collection of subsets of X, let $\mu : \mathcal{S} \to [0, \infty]$ be a set function, and let μ^* be the outer measure induced by μ. The measure $\overline{\mu}$ that is the restriction of μ^* to the σ-algebra $\mathcal{M}$ of μ^*-measurable sets is called the **Carathéodory measure induced by** μ.*

$$\underset{\text{a general set function}}{\mu : \mathcal{S} \to [0,\infty]} \longrightarrow \underset{\text{the induced outer measure}}{\mu^* : 2^X \to [0,\infty]} \longrightarrow \underset{\text{the induced Carathéodory measure}}{\overline{\mu} : \mathcal{M} \to [0,\infty]}$$

For a collection $\mathcal{S}$ of subsets of X, we use $\mathcal{S}_\sigma$ to denote those sets that are countable unions of sets of $\mathcal{S}$ and use $\mathcal{S}_{\sigma\delta}$ to denote those sets that are countable intersections of sets in $\mathcal{S}_\sigma$. Observe that if $\mathcal{S}$ is the collection of open intervals of real numbers, the $\mathcal{S}_\sigma$ is the collection of open subsets of $\mathbb{R}$ (Chapter 1 Proposition 9: Every nonempty open sets is the union of a countable, disjoint collection of open intervals), and $\mathcal{S}_{\sigma\delta}$ is the collection of G_δ subsets of $\mathbb{R}$.

Proposition 10. *Let $\mu : \mathcal{S} \to [0,\infty]$ be a set function defined on a collection $\mathcal{S}$ of subsets of a set X and $\overline{\mu} : \mathcal{M} \to [0,\infty]$ be the Carathéodory measure induced by μ. Let E be a subset of X for which $\mu^*(E) < \infty$. Then there is a subset A of X for which*

$$A \in \mathcal{S}_{\sigma\delta},\ E \subseteq A \text{ and } \mu^*(E) = \mu^*(A).$$

Furthermore, if E and each set in $\mathcal{S}$ is measurable with respect to μ^*, then so is A and

$$\overline{\mu}(A \setminus E) = 0.$$

(See Chapter 2 Theorem 11 (ii))

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18. Let $\mu^* : 2^X \to [0,\infty]$ be an outer measure. Let $A \subseteq X$, $\{E_k\}_{k=1}^\infty$ be a disjoint countable collection of measurable sets and $E = \bigcup_{k=1}^\infty E_k$. Show that

$$\mu^*(A \cap E) = \sum_{k=1}^\infty \mu^*(A \cap E_k).$$

By definition, outer measure is countably monotone so that

$$\mu^*(A \cap E) = \mu^*\left(A \cap \left[\bigcup_{k=1}^\infty E_k\right]\right) = \mu^*\left(\bigcup_{k=1}^\infty [A \cap E_k]\right) \le \sum_{k=1}^\infty \mu^*(A \cap E_k).$$

By monotonicity of outer measure and Propositon 6 (finite additivity of outer measure),

$$\mu^*(A \cap E) = \mu^*\left(A \cap \left[\bigcup_{k=1}^\infty E_k\right]\right) \ge \mu^*\left(A \cap \left[\bigcup_{k=1}^n E_k\right]\right) = \sum_{k=1}^n \mu^*(A \cap E_k).$$

The left-hand side of the inequality is independent of n so that $\mu^*(A \cap E) \ge \sum_{k=1}^\infty \mu^*(A \cap E_k)$.

19. Show that any measure that is induced by an outer measure is complete.

Let X be a set, let μ^* be an outer measure on 2^X, and let $\mathcal{M}$ be the σ-algebra of sets that are measurable w.r.t. μ^*. Then consider the measure $\overline{\mu}$ that is the restriction of μ^* to $\mathcal{M}$. We aim to show that $(X, \mathcal{M}, \overline{\mu})$ is complete; that is, $\mathcal{M}$ contains all subsets of sets of measure zero (under $\overline{\mu}$).

Let $E \in \mathcal{M}$ such that $\mu^*(E) = \overline{\mu}(E) = 0$, and let $A \subseteq X$ be any set.
Consider any subset E' of E.
By monotonicity of outer measure for $A \cap E' \subseteq E' \subseteq E$,

$$\mu^*(A \cap E') \le \mu^*(E') \le \mu^*(E) = 0,$$

and by monotonicity of outer measure for $A \supseteq A \cap E'^c$,

$$\mu^*(A) \geq \mu^*(A \cap E'^c) = \mu^*(A \cap E') + \mu^*(A \cap E'^c).$$

Thus E' is measurable.

20. Let X be any set. Define $\eta : 2^X \to [0, \infty]$ by defining $\eta(\emptyset) = 0$ and for $E \subseteq X, E \neq \emptyset$, defining $\eta(E) = \infty$. Show that η is an outer measure. Also show that the set function that assigns 0 to every subset of X is an outer measure.

We have $\eta(\emptyset) = 0$ by definition so it remains to show countable monotonicity.
Let $E \subseteq X$ and let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets that cover E.
If $E = \emptyset$, then the inequality is trivial, so assume $E \neq \emptyset$.
Then there exists a k such that $E_k \neq \emptyset$, which implies $\eta(E_k) = \infty$ so that we have

$$\eta(E) = \infty \leq \infty = \sum_{k=1}^{\infty} \eta(E_k)$$

The set function μ that assigns 0 to every subset of X is an outer measure trivially because $\mu(\emptyset) = 0$ and

$$\mu(E) = 0 \leq 0 = \sum_{k=1}^{\infty} \mu(E_k)$$

for any set E that is covered by a countable collection $\{E_k\}_{k=1}^{\infty}$.

21. Let X be a set, $\mathcal{S} = \{\emptyset, X\}$, and define $\mu(\emptyset) = 0, \mu(X) = 1$. Determine the outer measure μ^* induced by the set function $\mu : \mathcal{S} \to [0, \infty)$ and the σ-algebra of measurable sets.

Recall that for any $E \subseteq X$, $E \neq \emptyset$, we have the definition of induced outer measure:

$$\mu^*(E) = \inf\left\{\sum_{k=1}^{\infty} \mu(E_k) \mid E_k \in \mathcal{S},\ \bigcup_{k=1}^{\infty} E_k \supseteq E\right\}.$$

For $E \neq \emptyset$, we have that $\mathcal{S} = \{\emptyset, X\}$ implies $\mu(E_k) \in \{0, 1\}$ implies $\sum_{k=1}^{\infty} E_k \in \{1, 2, \dots\}$, so that

$$\mu^*(E) = 1.$$

($\sum_{k=1}^{\infty} E_k \neq 0$ because otherwise we have a contradiction: $\emptyset = \bigcup_{k=1}^{\infty} E_k \supseteq E \neq \emptyset$)
Therefore for any $E \subseteq X$, the induced outer measure μ^* is defined by

$$\mu^*(E) = \begin{cases} 0 & E = \emptyset \\ 1 & E \neq \emptyset \end{cases}$$

We have defined that a subset E of X is measurable provided for every subset A of X,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Consider $A = X$, so that we have

$$1 = \mu^*(X) = \mu^*(E) + \mu^*(E^c),$$

and the only sets E that satisfy this are $E = \emptyset$ and $E = X$.
Therefore the σ-algebra $\mathcal{M}$ of measurable sets (w.r.t. μ^*) is simply

$$\mathcal{M} = \{\emptyset, X\}.$$

22. On the collection $\mathcal{S} = \{\emptyset, [1,2]\}$ of subsets of $\mathbb{R}$, define the set function $\mu : \mathcal{S} \to [0,\infty)$ as follows: $\mu(\emptyset) = 0, \mu([1,2]) = 1$. Determine the outer measure μ^* induced by μ and the σ-algebra of measurable sets.

    Similarly to the previous Problem 21, we deduce that the induced outer measure μ^* will be defined by

    $$\mu^*(E) = \begin{cases} 0 & E = \emptyset \\ 1 & E \subseteq [1,2] \\ \infty & \text{else} \end{cases}$$

    Now let A be any set.
    If $A = \emptyset$, then trivially $\mu^*(A) = 0 \geq 0 = \mu^*(A \cap E) + \mu^*(A \cap E^c)$.
    If $A \not\subseteq [1,2]$, then trivially $\mu^*(A) = \infty \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$.
    Therefore consider the case where $A \subseteq [1,2]$, so that we have

    $$1 = \mu^*([1,2]) = \mu^*([1,2] \cap E) + \mu^*([1,2] \cap E^c).$$

    However, because both $[1,2] \cap E$ and $[1,2] \cap E^c$ are subsets of $[1,2]$, then the sets E that satisfy the inequality are such that $E = \emptyset$ or $E \supseteq [1,2]$.
    Therefore the σ-algebra $\mathcal{M}$ of measurable sets (w.r.t. μ^*) is

    $$\mathcal{M} = \{E \subseteq X \mid E = \emptyset \text{ or } E \supseteq [1,2]\}.$$

23. On the collection $\mathcal{S}$ of all subsets of $\mathbb{R}$, define the set function $\mu : \mathcal{S} \to \mathbb{R}$ by setting $\mu(A)$ to be the number of integers in A. Determine the outer measure μ^* induced by μ and the σ-algebra of measurable sets.

    We can determine the induced outer measure to be

    $$\mu^*(E) = \mu(E) \text{ for any } E \subseteq X,$$

    and the σ-algebra of measurable sets to be

    $$\mathcal{M} = 2^{\mathbb{R}}.$$

24. Let $\mathcal{S}$ be a collection of subsets of X and $\mu : \mathcal{S} \to [0,\infty]$ a set function. Is every set in $\mathcal{S}$ measurable with respect to the outer measure induced by μ?

    Let $\mathcal{S} = 2^{[1,2]}$, the set of all subsets of $[1,2]$.
    Define $\mu : \mathcal{S} \to [0,\infty)$ as follows: $\mu(\emptyset) := 0$, $\mu([1,2]) := 3$, and $\mu(E) := 2, E \in \mathcal{S} \setminus \{\emptyset, [1,2]\}$.
    Then we can define the extension μ^* to be

    $$\mu^*(E) = \begin{cases} 0 & E = \emptyset \\ 2 & E \subset [1,2] \text{ (strict subset, nonempty)} \\ 3 & E = [1,2] \\ \infty & \text{else} \end{cases}$$

    We can see that countable monotonicity holds:

    $$\mu^*(E) \leq \sum_{k=1}^{\infty} \mu^*(E_k),$$

and so μ^* is an outer measure.
Consider $E = [1, 1.5] \in \mathcal{S}$.
Then $[1, 2] \cap E = [1, 1.5] \subset [1, 2]$ and $[1, 2] \cap E^c = (1.5, 2] \subset [1, 2]$ so that for $A = [1, 2]$,

$$\mu^*([1, 2]) = 3 \neq 2 + 2 = \mu^*([1, 2] \cap E) + \mu^*([1, 2] \cap E^c),$$

which implies that E is not measurable.

17.5 The Cathéodory-Hahn Theorem: The Extension of a Premeasure to a Measure

Let $\mu : \mathcal{S} \to [0, \infty]$ be a set function that is defined on a nonempty collection $\mathcal{S}$ of subsets of a set X. We ask the following question: What properties must the collection $\mathcal{S}$ and set function μ possess in order that the Carathéodory measure $\overline{\mu}$ induced by μ be an extension of μ: that is; every set E in $\mathcal{S}$ is measurable w.r.t. the outer measure μ^* induced by μ and, moreover, $\mu(E) = \mu^*(E)$?

In other words: see Problem 24 to see that not every set in $\mathcal{S}$ is measurable w.r.t. the outer measure induced by μ in general. Therefore we want to know what additional properties of $\mathcal{S}$ and μ we must have to guarantee that $\mathcal{S} \subseteq \mathcal{M}$, so that $\overline{\mu}|_{\mathcal{S}} = \mu$.

Proposition 11. *Let $\mathcal{S}$ be a collection of subsets of a set X and let $\mu : \mathcal{S} \to [0, \infty]$ be a set function. In order that the Carathéodory measure induced by μ be an extension of μ it is necessary that μ be both finitely additive and countably monotone and, if $\emptyset$ belongs to $\mathcal{S}$, that $\mu(\emptyset) = 0$.*

Definition. *Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ be a set function. Then μ is called a **premeasure** provided μ is both finitely additive and countably monotone and, if $\emptyset$ belongs to $\mathcal{S}$, then $\mu(\emptyset) = 0$.*

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25. Let X be any set containing more than one point and A a proper nonempty subset of X. Define $\mathcal{S} = \{A, X\}$ and the set function $\mu : \mathcal{S} \to [0, \infty]$ by $\mu(A) = 1$ and $\mu(X) = 2$. Show that $\mu : \mathcal{S} \to [0, \infty]$ is a premeasure. Can μ be extended to a measure? What are the subsets of X that are measurable with respect to the outer measure μ^* induced by μ?

26. Consider the collection $\mathcal{S} = \{\emptyset, [0, 1], [0, 3], [2, 3]\}$ of subsets of $\mathbb{R}$ and define $\mu(\emptyset) = 0, \mu([0, 1]) = 1, \mu([0, 3]) = 1, \mu([2, 3]) = 1$. Show that $\mu : \mathcal{S} \to [0, \infty]$ is a premeasure. Can μ be extended to a measure? What are the subsets of $\mathbb{R}$ that are measurable with respect to the outer measure μ^* induced by μ?

27. Let $\mathcal{S}$ be a collection of subsets of a set X and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Show that μ is countably monotone iff μ^* is an extension of μ.

28. Show that a set function is a premeasure if it has an extension that is a measure.

29. Show that a set function on a σ-algebra is a measure iff it is a premeasure.

30. Let $\mathcal{S}$ be a collection of sets that is closed with respect to the formation of finite unions and finite intersections.

    (i) Show that $\mathcal{S}_\sigma$ is closed with respect to the formation of countable unions and finite intersections.

(ii) Show that each set in $\mathcal{S}_{\sigma\delta}$ is the intersection of a decreasing sequence of $\mathcal{S}_\sigma$ sets.

31. Let $\mathcal{S}$ be a semialgebra of subsets of a set X and $\mathcal{S}'$ the collection of unions of finite disjoint collections of sets in $\mathcal{S}$.

    (i) Show that $\mathcal{S}'$ is an algebra.

    (ii) Show that $\mathcal{S}_\sigma = \mathcal{S}'_\sigma$ and therefore $\mathcal{S}_{\sigma\delta} = \mathcal{S}'_{\sigma\delta}$.

    (iii) Let $\{E_k\}_{k=1}^\infty$ be a collection of sets in $\mathcal{S}'$. Show that we can express

    $$\sum_{k=1}^\infty \mu'(E_k') \geq \sum_{k=1}^\infty \mu(E_k).$$

    (iv) Let A belong to $\mathcal{S}'_{\sigma\delta}$. Show that A is the intersection of a descending sequence $\{A_k\}_{k=1}^\infty$ of sets in $\mathcal{S}_\sigma$.

32. Let $\mathbb{Q}$ be the set of rational numbers and and $\mathcal{S}$ the collection of all finite unions of intervals of the form $(a,b] \cap \mathbb{Q}$, where $a, b \in \mathbb{Q}$ and $a \leq b$. Define $\mu((a,b]) = \infty$ if $a < b$ and $\mu(\emptyset) = 0$. Show that $\mathcal{S}$ is closed with respect to the formation of relative complements and $\mu : \mathcal{S} \to [0,\infty]$ is a premeasure. Then show that the extension of μ to the smallest σ-algebra containing $\mathcal{S}$ is not unique.

33. By a bounded interval of real numbers we mean a set of the form $[a,b]$, $[a,b)$, $(a,b]$, or (a,b) for real numbers $a \leq b$. Thus we consider the empty-set and a set consisting of a single point to be a bounded interval. Show that each of the following three collections of sets $\mathcal{S}$ is a semiring.

    (i) Let $\mathcal{S}$ be the collection of all bounded intervals of real numbers.

    (ii) Let $\mathcal{S}$ be the collection of all subsets of $\mathbb{R} \times \mathbb{R}$ that are products of bounded intervals of real numbers.

    (iii) Let n be a natural number an X be the n-fold Cartesian product of $\mathbb{R}$:

    $$X = \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n.$$

    Let $\mathcal{S}$ be the collection of all subsets of X that are n-fold Cartesian products of bounded intervals of real numbers.

34. If we start with an outer measure μ^* on 2^X and form the induced measure $\overline{\mu}$ on the μ^*-measurable sets, we can view $\overline{\mu}$ as a set function and denote by μ^+ the outer measure induced by $\overline{\mu}$.

    (i) Show that for each set $E \subset X$ we have $\mu^+(E) \geq \mu^*(E)$.

    (ii) For a given set E, show that $\mu^+(E) = \mu^*(E)$ iff there is a μ^*-measurable set $A \supseteq E$ with $\mu^*(A) = \mu^*(E)$.

35. Let $\mathcal{S}$ be a σ-algebra of subsets of X and $\mu : \mathcal{S} \to [0,\infty]$ a measure. Let $\overline{\mu} : \mathcal{M} \to [0,\infty]$ be the measure induced by μ via the Carathéodory construction. Show that $\mathcal{S}$ is a subcollection of $\mathcal{M}$ and it may be a proper subcollection.

36. Let μ be a finite premeasure on an algebra $\mathcal{S}$, and μ^* the induced outer measure. Show that a subset E of X is μ^*-measurable iff for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$, $A \subseteq E$, such that $\mu^*(E \setminus A) < \epsilon$.

Chapter 18

Integration Over General Measure Spaces

Contents

18.1 Measurable Functions

Consider the measurable space $(X, \mathcal{M})$. For an extended real valued function f of X and a measurable subset E of X, the restriction of f to both E and $X \setminus E$ are measurable iff f is measurable on X.

Proposition 3. *Let $(X, \mathcal{M}, \mu)$ be a complete measure space and X_0 be a measurable subset of X for which $\mu(X \setminus X_0) = 0$. Then an extended real valued function f on X is measurable iff its restriction to X_0 is measurable. In particular, if g and h are extended real valued functions on X for which $g = h$ a.e. on X, then g is measurable iff h is measurable.*

Theorem 6. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ a sequence of measurable functions on X for which $\{f_n\} \to f$ pointwise a.e. on X. If either the measure space $(X, \mathcal{M}, \mu)$ is complete or the convergence is pointwise on all of X, then f is measurable.*

Corollary 7. *Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable function on X. Then the following functions are measurable:*

$$\sup\{f_n\},\ \inf\{f_n\},\ \limsup\{f_n\},\ \liminf\{f_n\}.$$

Egoroff's Theorem. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $\{f_n\}$ a sequence of measurable functions on X that converges pointwise a.e. on X to a function f that is finite a.e. on X. Then for each $\epsilon > 0$, there is a measurable subset X_ϵ of X for which*

$$\{f_n\} \to f \text{ uniformly on } X_\epsilon \text{ and } \mu(X \setminus X_\epsilon) < \epsilon.$$

PROBLEMS

In the following problems $(X, \mathcal{M}, \mu)$ is a reference measure space and measurable means with respect to $\mathcal{M}$.

1. Show that an extended real valued function on X is measurable iff $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable and so is $f^{-1}(E)$ for every Borel set of real numbers.

   Let f be an extended real valued function on X.

   ($\Longrightarrow$) Suppose that f is measurable.
   Then the set
   $$f^{-1}\{\infty\} = \{x \in X \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) > n\},$$
   is measurable because it is a countable intersection of measurable sets.
   Similarly the set
   $$f^{-1}\{-\infty\} = \{x \in X \mid f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f(x) < n\},$$
   is measurable because it is a countable intersection of measurable sets.
   By definition of measurable function we have that the set
   $$\{x \in X \mid f(x) > a\} = \{x \in X \mid f(x) \in (a, \infty)\}$$
   is measurable for any real a. From Chapter 2 Problem 11, we have that that if a σ-algebra of subsets of R contains intervals of the form (a, ∞), then it contains all intervals. Therefore by the properties of a σ-algebra, it must contain all open sets (Chapter 1 Proposition 9 - Every nonempty open set is the union of a countable, disjoint collection of open intervals). Then because the collection of Borel sets is the smallest σ-algebra that contains all of the open sets of real numbers, we have
   $$f^{-1}(E) \text{ is measurable for any Borel set } E.$$

   ($\Longleftarrow$) Suppose that the sets $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable, and that $f^{-1}(E)$ is measurable for any Borel set E of real numbers.
   Fix any real number c.
   The collection of Borel sets contain all intervals of the form (c, ∞) so that the set
   $$\{x \in X \mid f(x) > c\} = \{x \in X \mid f(x) \in (c, \infty)\} = f^{-1}((c, \infty))$$
   is measurable.
   Therefore f is a measurable function.

2. Suppose $(X, \mathcal{M}, \mu)$ is not complete. Let E be a subset of a set of measure zero that does not belong to $\mathcal{M}$. Let $f = 0$ on X and $g = \chi_E$. Show that $f = g$ a.e. on X while f is measurable and g is not.

3. Suppose $(X, \mathcal{M}, \mu)$ is not complete. Show that there is a sequence $\{f_n\}$ of measurable functions on X that converges pointwise a.e. on X to a function f that is not measurable.

4. Let E be a measurable subset of X and f an extended real-valued function on X. Show that f is measurable iff its restrictions to E and $X \setminus E$ are measurable.

5. Show that an extended real valued function f on X is measurable iff for each rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set.

   Let f be an extended real valued function on X.

   ($\Longrightarrow$) Suppose that f is measurable.
   Then trivially for any rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set by definition of measurable function.

   ($\Longleftarrow$) Suppose that for each rational number c, $\{x \in X \mid f(x) < c\}$ is a measurable set.
   Let a be any real number.
   Then for each natural number n, by density of the rationals in the reals there exists a rational c_n such that
   $$a_n - \frac{1}{n} < c_n < a_n,$$
   and we have that the set $\{x \in X \mid f(x) < c_n\}$ is measurable.
   Then we have $\bigcup_{n=1}^{\infty}[\infty, c_n) = [\infty, a)$, so that we have the set
   $$\begin{aligned}\bigcup_{n=1}^{\infty}\{x \in X \mid f(x) < c_n\} &= \bigcup_{n=1}^{\infty}\{x \in X \mid f(x) \in [\infty, c_n)\} \\ &= \{x \in X \mid f(x) \in [\infty, a)\}, \\ &= \{x \in X \mid f(x) < a\},\end{aligned}$$
   which is measurable because it is the countable union of measurable sets.
   Therefore f is a measurable function.

6. Consider two extended real valued measurable functions f and g on X that are finite a.e. on X. Define X_0 to be the set of points in X at which both f and g are finite. Show that X_0 is measurable and $\mu(X \setminus X_0) = 0$.

   We have that f and g are finite a.e. on X, which means that there exist (measurable) subsets X_f, X_g of X, both of measure zero, where the property holds on $X \setminus X_f$ and $X \setminus X_g$ respectively.
   Then X_f and X_g are measurable imply that $X \setminus X_f$ and $X \setminus X_g$ are also measurable by the properties of a σ-algebra.
   Therefore we have that the set
   $$X_0 = [X \setminus X_f] \cap [X \setminus X_g] = \{x \in X \mid f(x) \text{ is finite}, g(x) \text{ is finite}\}$$
   is measurable because it is the intersection of measurable sets.
   We see that
   $$X \setminus X_0 = X \cap ([X^c \cup X_f] \cup [X^c \cup X_g]) = X_f \cup X_g,$$
   and we use countable monotonicity to see that
   $$\mu(X \setminus X_0) = \mu(X_f \cup X_g) \leq \mu(X_f) + \mu(X_g) = 0 + 0.$$

7. Let X be a nonempty set. Show that every extended real valued function on X is measurable w.r.t. the measurable space $(X, 2^X)$.

   (i) Let x_0 belong to X and δ_{x_0} be the Dirac measure at x_0 on 2^X. Show that two function on X are equal a.e. $[\delta_{x_0}]$ iff they take the same value at x_0.

(ii) Let η be the counting measure on 2^X. Show that two functions on X are equal a.e. $[\eta]$ iff they take the same value at every point in X.

8. Let X be a topological space and $\mathcal{B}(X)$ be the smallest σ-algebra containing the topology on X. $\mathcal{B}(X)$ is called the Borel σ-algebra associated with the topological space X. Show that any continuous real valued function on X is measurable w.r.t. the Borel measurable space $(X, \mathcal{B}(X))$.

9. If a real valued function on $\mathbb{R}$ is measurable w.r.t. the σ-algebra of Lebesgue measurable sets, is it necessarily measurable w.r.t. the Borel measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$?

10. Check that the proofs of Proposition 1 and Theorem 4 follow from the proofs of the corresponding results in the case of Lebesgue measure on the real line.

11. Prove Corollary 7.

Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of measurable function on X.

(i) $f(x) := \sup_{n\in\mathbb{N}}\{f_n(x)\}$
Fix any real number c.
(1) Let $y \in \{x \in X \mid f(x) > c\}$. Then $f(y) > c$.
By definition of supremum, there exists an index k such that

$$f(y) \geq f_k(y) > c,$$

and therefore $y \in \{x \in X \mid f_k(x) > c\}$ for some k, which implies

$$\{x \in X \mid f(x) > c\} \subseteq \bigcup_{n=1}^{\infty}\{x \in X \mid f_n(x) > c\}. \tag{1}$$

(2) Let $y' \in \bigcup_{n=1}^{\infty}\{x \in X \mid f_n(x) > c\}$.
Then there exists an index k such that $y' \in \{x \in X \mid f_k(x) > c\}$.
By definition of supremum, we have

$$f(y') \geq f_k(y') > c,$$

and therefore $y' \in \{x \in X \mid f(x) > c\}$, which implies

$$\{x \in X \mid f(x) > c\} \supseteq \bigcup_{n=1}^{\infty}\{x \in X \mid f_n(x) > c\} \tag{2}$$

Then by (1) and (2),

$$\{x \in X \mid f(x) > c\} = \bigcup_{n=1}^{\infty}\{x \in X \mid f_n(x) > c\},$$

which is measurable because it is the countable union of measurable sets.

(ii) $f(x) := \inf_{n\in\mathbb{N}}\{f_n(x)\}$
Fix any real number c.
(1) Let $y \in \{x \in X \mid f(x) \geq c\}$. Then $f(y) \geq c$.
By definition of infimum, for all indices n, we have that

$$c \leq f(y) \leq f_n(y),$$

and therefore $y \in \{x \in X \mid f_n(x) \geq c\}$ for all n, which implies

$$\{x \in X \mid f(x) \geq c\} \subseteq \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geq c\}. \tag{1}$$

(2) Let $y' \in \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geq c\}$.
Then for all indices n, we have $y' \in \{x \in X \mid f_n(x) \geq c\}$ so that c is a lower bound to the set $\{f_n(y')\}_{n \in \mathbb{N}}$.
Then by definition of infimum we have

$$c \leq f(y'),$$

and therefore $y' \in \{x \in X \mid f(x) \geq c\}$, which implies

$$\{x \in X \mid f(x) \geq c\} \supseteq \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geq c\} \tag{2}$$

Then by (1) and (2),

$$\{x \in X \mid f(x) \geq c\} = \bigcap_{n=1}^{\infty} \{x \in X \mid f_n(x) \geq c\},$$

which is measurable because it is the countable intersection of measurable sets.

(iii) $f(x) := \limsup_n \{f_n(x)\} = \lim_{n\to\infty} \sup_{k \geq n} \{f_n(x)\} = \inf_{n \in \mathbb{N}} \sup_{k \geq n} \{f_n(x)\}$
Fix any real number c.
Pulling from (i), with a small modification, we have that the function $g_n(x) := \sup_{k \geq n} \{f_k(x)\}$ is measurable for each $n \in \mathbb{N}$.
That is, the set

$$\{x \in X \mid g_n(x) > c\} = \bigcup_{k=n}^{\infty} \{x \in X \mid f_k(x) > c\}$$

is measurable, and thus each function g_n is measurable.
Then using the same process as (ii), the set

$$\{x \in X \mid f(x) \geq c\} = \bigcap_{n=1}^{\infty} \{x \in X \mid g_n(x) \geq c\}$$

is measurable because it is a countable intersection of measurable sets.

(iv) $f(x) := \liminf_n \{f_n(x)\} = \lim_{n\to\infty} \inf_{k \geq n} \{f_n(x)\} = \sup_{n \in \mathbb{N}} \inf_{k \geq n} \{f_n(x)\}$
Fix any real number c.
Pulling from (ii), with modification, we have that the function $g_n(x) := \inf_{k \geq n} \{f_k(x)\}$ is measurable for each $n \in \mathbb{N}$.
That is, the set

$$\{x \in X \mid g_n(x) \geq c\} = \bigcap_{k=n}^{\infty} \{x \in X \mid f_k(x) \geq c\}$$

is measurable, and thus each function g_n is measurable.
Then using the same process as (i), the set

$$\{x \in X \mid f(x) > c\} = \bigcup_{n=1}^{\infty} \{x \in X \mid g_n(x) > c\}$$

is measurable because it is a countable union of measurable sets.

12. Prove Egoroff's Theorem. Is Egoroff's Theorem true in the absence of the assumption that the limit function is finite a.e.?

13. Let $\{f_n\}$ be a sequence of real valued measurable functions on X such that, for each natural number n, $\mu\{x \in X \mid |f_n(x) - f_{n+1}(x)| > 1/2^n\} < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent a.e. on X. (Hint: Use the Borel-Cantelli Lemma.)

14. Under the assumptions of Egoroff's Theorem, show that $X = \bigcup_{k=0}^{\infty} X_k$, where each X_k is measurable, $\mu(X_0) = 0$ and, for $k \geq 1$, $\{f_n\}$ converges uniformly to f on X_k.

15. A sequence $\langle f_n \rangle$ of measurable real-valued functions on X is said to **converge in measure** to a measurable function f provided that for each $\eta > 0$,

$$\lim_{n\to\infty} \mu\{x \in X \mid |f_n(x) - f(x) > \eta|\} = 0.$$

A sequence $\langle f_n \rangle$ of measurable functions is said to **Cauchy in measure** provided that for each $\epsilon > 0$ and $\eta > 0$, there is an index N such that for each $m, n \geq N$,

$$\mu\{x \in X \mid |f_n(x) - f_m(x) > \eta|\} < \epsilon.$$

   (i) Show that if $\mu(X) < \infty$ and $\{f_n\}$ converges pointwise a.e. on X to a measurable function f, then $\{f_n\}$ converges to f in measure. (Hint: Use Egoroff's Theorem.)

   (ii) Show that if $\{f_n\}$ converges to f in measure, then there is a subsequence of $\{f_n\}$ that converges pointwise a.e. on X to f. (Hint: Use the Borel-Cantelli Lemma.)

   (iii) Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function f to which $\{f_n\}$ converges in measure.

16. Assume $\mu(X) < \infty$. Show that $\{f_n\} \to f$ in measure iff each subsequence of $\{f_n\}$ has a further subsequence that converges pointwise a.e. on X to f. Use this to show that for two sequences that converge in measure, the product sequence also converges in measure to the product of the limits.

18.2 Integration of Nonnegative Measurable Functions

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18.3 Integration of General Measurable Functions

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18.4 The Radon-Nikodym Theorem

Let $(X, \mathcal{M})$ be a measurable space. For μ a measure on $(X, \mathcal{M})$ and f a nonnegative function on X that is measurable w.r.t. $\mathcal{M}$, define the set function ν on $\mathcal{M}$ by

$$\nu(E) = \int_E f d\mu \text{ for all } E \in \mathcal{M}.$$

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18.5 The Nikodym Metric Space: The Vitali-Hahn-Saks Theorem

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