

**Real Analysis Royden - Fourth Edition**  
Notes + Solved Exercises :)  
Latex Symbols

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# **I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE**



# **Preliminaries on Sets, Mappings, and Relations**





# Chapter 1

## The Real Numbers: Sets, Sequences, and Functions

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### 1.1 The Field, Positivity, and Completeness Axioms

#### PROBLEMS

- For  $a \neq 0$  and  $a \neq 0$ , show that  $(ab)^{-1} = a^{-1}b^{-1}$ .
- Verify the following:
  - For each real number  $a \neq 0$ ,  $a^2 > 0$ . In particular,  $1 > 0$  since  $1 \neq 0$  and  $1 = 1^2$ .
  - For each positive number  $a$ , its multiplicative inverse  $a^{-1}$  also is positive.
  - If  $a > b$ , then
$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$
- For a nonempty set of real numbers  $E$ , show that  $\inf E = \sup E$  iff  $E$  consists of a single point.
- Let  $a$  and  $b$  be real numnbers.
  - Show that if  $ab = 0$ , then  $a = 0$  or  $b = 0$ .
  - Verify that  $a^2 - b^2 = (a - b)(a + b)$  and conclude from part (i) that if  $a^2 = b^2$ , then  $a = b$  or  $a = -b$ .
  - Let  $c$  be a positive real number. Define  $E = \{x \in \mathbb{R} \mid x^2 < c\}$ . Verify that  $E$  is nonempty and bounded above. Define  $x_0 = \sup E$ . Show that  $x_0^2 = c$ . Use part (ii) to show that there is a unique  $x > 0$  for which  $x^2 = c$ . It is denoted by  $\sqrt{c}$ .

5. Let  $a, b, c$  be real numbers s.t.  $a \neq 0$  and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}.$$

- (i) Suppose  $b^2 - 4ac > 0$ . Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = \sup\{-x \mid x \in E\}.$$

7. For real numbers  $a$  and  $b$ , verify the following:

- (i)  $|ab| = |a||b|$ .  
(ii)  $|a + b| \leq |a| + |b|$ .  
(iii) For  $\epsilon > 0$ ,

$$|x - a| < \epsilon \text{ iff } a - \epsilon < x < a + \epsilon.$$

## 1.2 The Natural and Rational Numbers

### PROBLEMS

8. Use an induction argument to show that for each natural number  $n$ , the interval  $(n, n + 1)$  fails to contain any natural number.
9. Use an induction argument to show that if  $n > 1$  is a natural number, then  $n - 1$  also is a natural number. Use another induction argument to show that if  $m$  and  $n$  are natural numbers with  $n > m$ , then  $n - m$  is a natural number.
10. Show that for any real number  $r$ , there is exactly one integer in the interval  $[r, r + 1)$ .
11. Show that any nonempty set of integers that is bounded above has a largest member.
12. Show that the irrational numbers are dense in  $\mathbb{R}$ .
13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.
14. Show that if  $r > 0$ , then, for each natural number  $n$ ,  $(1 + r)^n \geq 1 + n \cdot r$ .
15. Use induction arguments to prove that for every natural number  $n$ ,

- (i)

$$\sum_{j=1}^n j^2 = \frac{n(n+1)(2n+1)}{6},$$

- (ii)

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

- (iii)

$$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \text{ if } r \neq 1.$$

## 1.3 Countable and Uncountable Sets

### PROBLEMS

16. Show that the set  $\mathbb{Z}$  of integers is countable.
17. Show that a set  $A$  is countable iff there is an injective mapping of  $A$  to  $\mathbb{N}$ .
18. Use an induction argument to complete the proof of part (i) of Corollary 4.
19. Prove Corollary 6 in the case of a finite family of countable sets.
20. Let both  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be injective and surjective. Show that the composition  $g \circ f : A \rightarrow C$  and the inverse  $f^{-1} : B \rightarrow A$  are also injective and surjective.
21. Use an induction argument to establish the pigeonhole principle.
22. Show that  $2^{\mathbb{N}}$ , the collection of all sets of natural numbers, is uncountable.
23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding theorem to show that  $\mathbb{N}^{\mathbb{N}}$ , the collection of all mappings of  $\mathbb{N}$  into  $\mathbb{N}$ , is not countable.
24. Show that a degenerate interval of real numbers fails to be finite.
25. Show that any two nondegenerate intervals of real numbers are equipotent.
26. Is the set  $\mathbb{R} \times \mathbb{R}$  equipotent to  $\mathbb{R}$ ?

## 1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

**The Nested Set Theorem.** Let  $\{F_n\}_{n=1}^{\infty}$  be a descending countable collection of nonempty closed sets of real numbers for which  $F_1$  is bounded. Then

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

*Proof.* By contradiction, suppose that  $\bigcap_{n=1}^{\infty} F_n = \emptyset$ . Then  $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$ , and we have an open cover of  $\mathbb{R}$  and thus an open cover of  $F_1 \subseteq \mathbb{R}$ . By the Heine-Borel Theorem, there exists an  $N \in \mathbb{N}$  such that  $F_1 \subseteq \bigcup_{n=1}^N F_n^c$ . Because  $\{F_n\}$  is descending,  $F_n \supseteq F_{n+1}$  for any  $n \geq 1$ . This implies  $F_n^c \subseteq F_{n+1}^c$ , and thus  $F_1 \subseteq \bigcup_{n=1}^N F_n^c = F_N^c = \mathbb{R} \setminus F_N$ . This is a contradiction to the assumption that  $F_N$  is a nonempty subset of  $F_1$ .  $\square$

### PROBLEMS

27. Is the set of rational numbers open or closed?
28. What are the sets of real numbers that are both open and closed?
29. Find two sets  $A$  and  $B$  such that  $A \cap B = \emptyset$  and  $\overline{A} \cap \overline{B} \neq \emptyset$ .
30. A point  $x$  is called an **accumulation point** of a set  $E$  provided it is a point of closure of  $E \setminus \{x\}$ .
  - (i) Show that the set  $E'$  of accumulation points of  $E$  is a closed set.

- (ii) Show that  $\overline{E} = E \cup E'$ .
31. A point  $x$  is called an **isolated point** of a set  $E$  provided there is an  $r > 0$  for which  $(x - r, x + r) \cap E = \{x\}$ . Show that if a set  $E$  consists of isolated points, then it is countable.
32. A point  $x$  is called an **interior point** of a set  $E$  if there is an  $r > 0$  such that the open interval  $(x - r, x + r)$  is contained in  $E$ . The set of interior points of  $E$  is called the **interior** of  $E$  denoted by  $\text{int } E$ . Show that
- (i)  $E$  is open iff  $E = \text{int } E$ .
- (ii)  $E$  is dense iff  $\text{int } (\mathbb{R} \setminus E) = \emptyset$ .
33. Show that the nested set theorem is false if  $F_1$  is unbounded.
34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.
35. Show that the collection of Borel sets is the smallest  $\sigma$ -algebra that contains the closed sets.
36. Show that the collection of Borel sets is the smallest  $\sigma$ -algebra that contains the intervals of the form  $[a, b)$ , where  $a < b$ .
37. Show that each open set is an  $F_\sigma$  set.

## 1.5 Sequences of Real Numbers

### PROBLEMS

38. We call an extended real number a **cluster point** of a sequence  $\{a_n\}$  if a subsequence converges to this extended real number. Show that  $\liminf\{a_n\}$  is the smallest cluster point of  $\{a_n\}$  and  $\limsup\{a_n\}$  is the largest cluster point of  $\{a_n\}$ .
39. Prove proposition 19.
40. Show that a sequence  $\{a_n\}$  is convergent to an extended real number iff there is exactly one extended real number that is a cluster point of the sequence.
41. Show that  $\liminf a_n \leq \limsup a_n$ .
42. Prove that if, for all  $n$ ,  $a_n \geq 0$  and  $b_n \geq 0$ , then

$$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

provided the product on the right is not of the form  $0 \cdot \infty$ .

43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.
44. Let  $p$  be a natural number greater than 1, and  $x$  a real number  $0 \leq x \leq 1$ . Show that there is a sequence  $\{a_n\}$  of integers with  $0 \leq a_n < p$  for each  $n$  such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when  $x$  is of the form  $q/p^n$ ,  $0 < q < p^n$ , in which case there are exactly two such sequences. Show that, conversely, if  $\{a_n\}$  is any sequence of integers with  $0 \leq a_n < p$ , the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number  $x$  with  $0 \leq x \leq 1$ . If  $p = 10$ , this sequence is called the *decimal* expansion of  $x$ . For  $p = 2$  it is called the *binary* expansion; and for  $p = 3$ , the *ternary* expansion.

45. Prove Proposition 20.
46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

## 1.6 Continuous Real-Valued Functions of a Real Variable

### PROBLEMS

47. Let  $E$  be a closed set of real numbers and  $f$  a real-valued function that is defined and continuous on  $E$ . Show that there is a function  $g$  defined and continuous on all of  $\mathbb{R}$  such that  $f(x) = g(x)$  for each  $x \in E$ . (Hint: Take  $g$  to be linear on each of the intervals of which  $\mathbb{R} \setminus E$  is composed.)
48. Define the real-valued function  $f$  on  $\mathbb{R}$  by setting

$$f(x) = \begin{cases} x & \text{if } x \text{ irrational} \\ p \sin \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is  $f$  continuous?

49. Let  $f$  and  $g$  be continuous real-valued functions with a common domain  $E$ .
  - (i) Show that the sum,  $f + g$ , and product,  $fg$ , are also continuous functions.
  - (ii) If  $h$  is a continuous function with image contained in  $E$ , show that the composition  $f \circ h$  is continuous.
  - (iii) Let  $\max\{f, g\}$  be the function defined by  $\max\{f, g\}(x) = \max\{f(x), g(x)\}$ , for  $x \in E$ . Show that  $\max\{f, g\}$  is continuous.
  - (iv) Show that  $|f|$  is continuous.
50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.
51. A continuous function  $\phi$  on  $[a, b]$  is called **piecewise linear** provided there is a partition  $a = x_0 < x_1 < \cdots < x_n = b$  of  $[a, b]$  for which  $\phi$  is linear on each interval  $[x_i, x_{i+1}]$ . Let  $f$  be a continuous function on  $[a, b]$  and  $\epsilon$  a positive number. Show that there is a piecewise linear function  $\phi$  on  $[a, b]$  with  $|f(x) - \phi(x)| < \epsilon$  for all  $x \in [a, b]$ .
52. Show that a nonempty set  $E$  of real numbers is closed and bounded if and only if every continuous real-valued function on  $E$  takes a maximum value.

53. Show that a set  $E$  of real numbers is closed and bounded iff every open cover of  $E$  has a finite subcover.
54. Show that a nonempty set  $E$  of real numbers is an interval iff every continuous real-valued function on  $E$  has an interval as its image.
55. Show that a monotone function on an open interval is continuous iff its image is an interval.
56. Let  $f$  be a real-valued function defined on  $\mathbb{R}$ . Show that the set of points at which  $f$  is continuous is a  $G_\delta$  set.
57. Let  $\{f_n\}$  be a sequence of continuous functions defined on  $\mathbb{R}$ . Show that the set of points  $x$  at which the sequence  $\{f_n(x)\}$  converges to a real number is the intersection of a countable collection of  $F_\sigma$  sets.
58. Let  $f$  be a continuous real-valued function on  $\mathbb{R}$ . Show that the inverse image with respect to  $f$  of an open set is open, of a closed set is closed, and of a Borel set is Borel.
59. A sequence  $\{f_n\}$  of real-valued functions defined on a set  $E$  is said to converge uniformly on  $E$  to a function  $f$  iff given  $\epsilon > 0$ , there is an  $N$  such that for all  $x \in E$  and all  $n \geq N$ , we have  $|f_n(x) - f(x)| < \epsilon$ . Let  $\{f_n\}$  be a sequence of continuous functions defined on a set  $E$ . Prove that if  $\{f_n\}$  converges uniformly to  $f$  on  $E$ , then  $f$  is continuous on  $E$ .

# Chapter 2

## Lebesgue Measure

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### 2.1 Introduction

In this chapter we construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure**, denoted by  $m$ . The collection of Lebesgue measurable sets is a  $\sigma$ -algebra which contains all open sets and all closed sets. The set function  $m$  possesses the following three properties:

**The measure of an interval is its length.** *Each nonempty interval  $I$  is Lebesgue measurable and*

$$m(I) = \ell(I).$$

**Measure is translation invariant.** *If  $E$  is Lebesgue measurable and  $y$  is any number then the translate of  $E$  by  $y$ ,  $E + y = \{x + y \mid x \in E\}$ , also is Lebesgue measurable and*

$$m(E + y) = m(E).$$

**Measure is countably additive over countable disjoint unions of sets.** *If  $\{E_k\}_{k=1}^{\infty}$  is a countable disjoint collection of Lebesgue measurable sets, then*

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (See Vitali sets). We first construct a set function called **outer measure**, denoted by  $m^*$ , such that:

- (i) the outer measure of an interval is its length,
- (ii) outer measure is translation invariant,
- (iii) outer measure is countably subadditive.

Then the Lebesgue measure  $m$  is the restriction of  $m^*$  to the Lebesgue measurable sets.

### PROBLEMS

In the first three problems, let  $m$  be a set function defined for all sets in a  $\sigma$ -algebra  $\mathcal{A}$  with values in  $[0, \infty]$ . Assume  $m$  is countably additive over countable disjoint collections of sets in  $\mathcal{A}$ .

1. Prove that if  $A$  and  $B$  are two sets in  $\mathcal{A}$  with  $A \subseteq B$ , then  $m(A) \leq m(B)$ . This property is called *monotonicity*.
2. Prove that if there is a set  $A$  in the collection  $\mathcal{A}$  for which  $m(A) < \infty$ , then  $m(\emptyset) = 0$ .
3. Let  $\{E_k\}_{k=1}^{\infty}$  be a countable collection of sets in  $\mathcal{A}$ . Prove that  $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$ .
4. A set function  $c$ , defined on all subsets of  $\mathbb{R}$ , is defined as follows. Define  $c(E)$  to be  $\infty$  if  $E$  has infinitely many members and  $c(E)$  to be equal to the number of elements in  $E$  if  $E$  is finite; define  $c(\emptyset) = 0$ . Show that  $c$  is a countably additive and translation invariant set function. This set function is called the **counting measure**.

## 2.2 Lebesgue Outer Measure

## 2.3 The $\sigma$ -Algebra of Lebesgue Measurable Sets

## 2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

## 2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

## 2.6 Nonmeasurable Sets

## 2.7 The Cantor Set and the Cantor-Lebesgue Function



## Chapter 3

# Lebesgue Measurable Functions

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## Chapter 4

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## **Chapter 5**

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## Chapter 8

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## **II ABSTRACT SPACES: METRIC, TOPO- LOGICAL, BANACH, AND HILBERT SPACES**



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# **III MEASURE AND INTEGRATION: GENERAL THEORY**





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