# Real Analysis Royden - Fourth Edition

Notes + Solved Exercises :)

Latex Symbols

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# Contents

# I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

# Preliminaries on Sets, Mappings, and Relations

**Definition.** *A relation $R$ on a set $X$ is called an **equivalence relation** provided:*

*(i) $xRx$ for all $x \in X$ (reflexive),*

*(ii) $xRy$ implies $yRx$ for all $x, y \in X$ (symmetric),*

*(iii) $xRy$ and $yRz$ imply $xRz$ for all $x, y, z \in X$ (transitive).*

**Partial Ordering on a set $X$.** *A relation $R$ on a nonempty set $X$ is called a **partial ordering** provided:*

*(i) $xRx$ for all $x \in X$ (reflexive),*

*(ii) $xRy$ and $yRx$ imply $x = y$ for all $x, y \in X$ (antisymmetric),*

*(iii) $xRy$ and $yRz$ imply $xRz$ for all $x, y, z \in X$ (transitive).*

*A subset $E$ of $X$ is **totally ordered** provided either $xRy$ or $yRx$ for all $x, y \in E$. A member $x$ of $X$ is said to be an **upper bound** for a subset $E$ of $X$ provided that*

$$yRx \text{ for all } y \in E.$$

*A member $x$ of $X$ is said to be **maximal** provided that*

$$xRy \text{ implies that } y = x \text{ for } y \in X.$$

**Strict Partial Ordering on a set $X$.** *A relation $R$ on a nonempty set $X$ is called a **strict partial ordering** provided:*

*(i) not $xRx$ for all $x \in X$ (irreflexive),*

*(ii) $xRy$ implies not $yRx$ for all $x, y \in X$ (asymmetric),*

*(iii) $xRy$ and $yRz$ imply $xRz$ for all $x, y, z \in X$ (transitive).*

*A subset $E$ of $X$ is* ***strictly totally ordered*** *provided either $xRy$ or $yRx$ if $x \neq y$ for all $x, y \in E$.*

**Zorn's Lemma.** *Let $X$ be a partially ordered set for which every totally ordered subset has an upper bound. Then $X$ has a maximal member.*

**Every vector space has a basis.**

*Proof.* Let V be any vector space, and let L be the collection of all linearly independent subsets of V. L is nonempty as the singleton sets are linearly independent. Define a partial order on L in the form $C \subseteq C'$ for $C, C' \in L$. For any chain (a totally ordered subset of a partially ordered set) $\mathcal{C}$ of $L$, where $\mathcal{C}$ consists of the sets $C_1 \subseteq C_2 \subseteq \cdots$, we can construct a linearly independent set $C' = \bigcup_{C \in \mathcal{C}} C$ that is an upper bound of $\mathcal{C}$. By Zorn's Lemma, L has a maximal element, say M. This collection $M$ is a basis for $V$. To show this, suppose by contradiction that there exists a vector $v \in V$ s.t. $v \notin$ Span$\{M\}$. Then $v \cup M$ is linearly independent and $M \subseteq v \cup M$, a contradiction to the fact that $M$ is maximal. $\square$

# Chapter 1

# The Real Numbers: Sets, Sequences, and Functions

## Contents



## 1.1 The Field, Positivity, and Completeness Axioms

**The field axioms**
Consider $a, b, c \in \mathbb{R}$:

1. Closure of Addition: $a + b \in \mathbb{R}$.
2. Associativity of Addition: $(a + b) + c = a + (b + c)$.
3. Additive Identity: $0 + a = a + 0 = a$.
4. Additive Inverse: $(-a) + a = a + (-a) = 0$.
5. Commutativity of Addition: $a + b = b + a$.
6. Closure of Multiplication: $ab \in \mathbb{R}$.
7. Associativity of Multiplication: $(ab)c = a(bc)$.
8. Distributive Property: $a(b + c) = ab + ac$.
9. Commutativity of Multiplication: $ab = ba$.
10. Multiplicative Identity: $1a = a1 = a$.
11. No Zero Divisors: $ab = 0 \implies a = 0$ or $b = 0$.

12. Multiplicative Inverse: $a^{-1}a = aa^{-1} = 1$.

13. Nontriviality: $1 \neq 0$.

**The positivity axioms**
The set of **positive numbers**, $\mathcal{P}$, has the following two properties:

P1 If $a$ and $b$ are positive, then $ab$ and $a + b$ are both positive.

P2 For a real number $a$, exactly one of the three is true: $a$ is positive, $-a$ is positive, $a = 0$.

We call a nonempty set $I$ of real numbers an **interval** provided for any two points in $I$, all the points that lie between these two points also lie in $I$. That is, $\forall x, y \in I, \lambda x + (1 - \lambda)y \in I$ for $\lambda \in [0, 1]$.

**The completeness axiom**
A nonempty set $E$ of real numbers is said to be **bounded above** provided there is a real number $b$ such that $x \leq b$ for all $x \in E$: the number $b$ is called an **upper bound** for $E$. We can similarly define a set being **bounded below** and having a **lower bound**. A set that is bounded above need not have a largest member.

**The Completeness Axiom.** *Let $E$ be a nonempty set of real numbers that is bounded above. The among the set of upper bounds for $E$ there is a smallest, or least, upper bound. (This least upper bound is called the **supremum** of $E$. Also, it can be shown that any nonempty set $E$ that is bounded below has a greatest lower bound, called the **infimum** of $E$).*

**The extended real numbers**
The extended real numbers: $\mathbb{R} \cup \{-\infty, \infty\}$
Every set of real numbers has a supremum and infimum that belongs to the extended real numbers.

**PROBLEMS**

1. For $a \neq 0$ and $a \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

$$
\begin{aligned}
(ab)(ab)^{-1} &= 1 && \text{by multiplicative inverse} \\
a(b(ab)^{-1}) &= 1 && \text{by associativity of multiplication} \\
a^{-1}a(b(ab)^{-1}) &= a^{-1}1 && \\
1(b(ab)^{-1}) &= a^{-1}1 && \text{by multiplicative inverse} \\
b(ab)^{-1} &= a^{-1} && \text{by multiplicative identity} \\
b^{-1}b(ab)^{-1} &= b^{-1}a^{-1} && \\
1(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative inverse} \\
(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative identity} \\
(ab)^{-1} &= a^{-1}b^{-1} && \text{by commutativity of multiplication}
\end{aligned}
$$

2. Verify the following:

(i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.

By positivity axiom P2, since $a \neq 0$, either $a$ is positive or $-a$ is positive.

In the case $a$ is positive, $a^2$ is positive by positivity axiom P1.

In the case $-a$ is positive, $(-a)(-a)$ is positive by P1.

$$\begin{aligned}
(-a)(-a) &= (-a)(-a) + a(0) && \text{by additive identity}\\
&= (-a)(-a) + a(-a + a) && \text{by additive inverse}\\
&= (-a)(-a) + a(-a) + a(a) && \text{by distributive property}\\
&= (-a + a)(-a) + a^2 && \text{by distributive property}\\
&= 0(-a) + a^2 && \text{by additive inverse}\\
&= a^2 && \text{by additive identity}
\end{aligned}$$

Therefore $a^2$ is positive by equality.

(ii) For each positive number $a$, its multiplicative inverse $a^{-1}$ also is positive.

The multiplication of two positive numbers is positive by positivity axiom P1.

The multiplication of two non-positive numbers is positive: by reformulating the previous result from (i), we can see $0 < (-a)(-b) = ab$ for $a, b \neq 0$.

The multiplication of a positive number and a non-positive number is not positive. To see this, suppose $a$ is positive and $b$ is not positive, but $ab$ is positive. By P1 and P2, $a(-b)$ is also positive. By P1, $ab + a(-b)$ is positive. However,

$$ab + a(-b) = a(b - b) = a(0) = 0.$$

This is a contradiction to P2. Therefore $ab$ is not positive.

The result from (i) shows that $1$ is positive. By multiplicative inverse, $aa^{-1} = 1 > 0$. Therefore $a^{-1}$ must be positive because $a$ is positive.

(iii) If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$

Proof that $a(-1) = -a$ for a real number $a$:

$$a + (-1)a = 1a + (-1)a = (1 + -1)a = 0a = 0.$$

$a > b$ implies that $a - b$ is positive.

If $c$ is positive, then $(a - b)c = ac - bc$ is positive, and $ac > bc$.

If $c$ is not positive, then $(a - b)c = ac - bc$ is not positive, and $-(ac - bc) = bc - ac$ is positive, so $bc > ac$.

3. For a nonempty set of real numbers $E$, show that $\inf E = \sup E$ iff $E$ consists of a single point.

( $\Longrightarrow$ ) Suppose $\inf E = \sup E$.

Then $\inf E \leq x \leq \sup E$ for all $x \in E$. But this implies $x = \inf E = \sup E$ for all $x \in E$, so $E$ consists of the single point $x$.

( $\Longleftarrow$ ) Suppose $E = x$ is a singleton set.

Clearly $x$ is an upper bound and a lower bound for $E$, as $x \leq x$. By completeness of the reals, there exists $\sup E$ and $\inf E$ s.t. $x \leq \inf E \leq x \leq \sup E \leq x$, as $\inf E$ is the greatest lower bound, and $\sup E$ is the least upper bound. Therefore $\inf E = \sup E$.

4. Let $a$ and $b$ be real numbers.

(i) Show that if $ab = 0$, then $a = 0$ or $b = 0$.

Contrapositive: Let $a \neq 0$ and $b \neq 0$. In 2.(ii), it was shown that the multiplication of two nonzero numbers is either positive or not positive. Therefore $ab \neq 0$.

(ii) Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

$$\begin{aligned}
(a-b)(a+b) &= (a-b)(a) + (a-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(a) + (a)(b) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b+b)(a) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(b) && \text{by additive inverse}\\
&= a^2 - b^2
\end{aligned}$$

Suppose $a^2 = b^2$. Then $(a-b)(a+b) = a^2 - b^2 = 0$, and by (i), $(a-b) = 0 \implies a = b$ or $(a+b) = 0 \implies a = -b$.

(iii) Let $c$ be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that $E$ is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$.

We can consider $0 \in \mathbb{R}$. $0^2 = 0 < c$, so $0 \in E$ and $E$ is nonempty. Also, $c+1$ is a real number and an upper bound for $E$; thus by the completeness axiom, $E$ has a supremum, say $x_0$. We can see that for any upper bound $b$ of $E$, $x \leq x_0 \leq b$ for all $x \in E$. Then $x^2 \leq x_0^2 \leq b^2$ implies $x_0^2 = c$, else $x_0$ is not the supremum.

Suppose there exists $x_1, x_2 > 0$ such that $x_1^2 = c$ and $x_2^2 = c$. This implies $x_1^2 = x_2^2$, and by part (ii), $x_1 = x_2$ or $x_1 = -x_2$. Because $x_1, x_2$ are positive, $x_1 = x_2$.

5. Let $a, b, c$ be real numbers s.t. $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}.$$

(i) Suppose $b^2 - 4ac > 0$. Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\begin{aligned}
ax^2 + bx + c &= 0\\
4a(ax^2 + bx + c) &= 4a(0)\\
4a^2x^2 + 4abx + 4ac &= 0 && \text{by distributive property}\\
4a^2x^2 + 4abx + 4ac + b^2 - b^2 &= 0 && \text{by additive inverse}\\
4a^2x^2 + 4abx + b^2 &= b^2 - 4ac\\
(2ax + b)^2 &= b^2 - 4ac
\end{aligned}$$

By 4(iii), because $b^2 - 4ac > 0$, there is a unique $y > 0$ for which $y^2 = b^2 - 4ac$. It is denoted by $y = \sqrt{b^2 - 4ac}$.

By 4(ii), $(2ax+b)^2 = b^2 - 4ac = y^2$ implies $(2ax+b) = \sqrt{b^2-4ac} = y$ or $(2ax+b) = -\sqrt{b^2-4ac} = -y$.

$$\begin{aligned} 2ax + b &= \pm\sqrt{b^2-4ac} \\ 2ax &= -b \pm \sqrt{b^2-4ac} \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \end{aligned}$$

6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x \mid x \in E\}.$$

Let $E$ be a set that is bounded below; that is, there exists $l \in \mathbb{R}$ such that $l \le x$ for all $x \in E$. Then $-l \ge -x$ for all $x \in E$, and $-l$ is an upper bound for $-E = \{-x \mid x \in E\}$. Therefore the set $-E$ is bounded above, and by the completeness axiom, there exists a least upper bound $c = \sup(-E)$. Then for any upper bound $u$ of $-E$, $u \ge c \ge -x$ for all $x \in E$. Then $-u$ is a lower bound of $E$, and $-u \le c \le x$ for all $x \in E$, and $c$ is the greatest lower bound and thus infimum of $E$.

7. For real numbers $a$ and $b$, verify the following:

(i) $|ab| = |a||b|$.
We have

$$|ab| = \begin{cases} ab & \text{if } ab \ge 0, \\ -(ab) & \text{if } ab < 0. \end{cases}$$

The case where either $a$ or $b$ are zero is trivial. In problem 2(ii), it was shown that $ab > 0$ if $a, b$ are the same sign, and $ab < 0$ if $a, b$ are opposite signs.
Case $a, b > 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = a$ and $|b| = b$ so $|a||b| = ab$.
Case $a, b < 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = -a$ and $|b| = -b$ so $|a||b| = (-a)(-b) = ab$.
Case $a < 0, b > 0$: Then $ab < 0$ so $|ab| = -(ab) = (-1)ab$, and $|a| = -a = (-1)a$ and $|b| = b$ so $|a||b| = (-1)ab$.

(ii) $|a+b| \le |a| + |b|$.
The case where both $a, b = 0$ is trivial.
Case $a, b > 0$: Then $a + b > 0$, so $|a+b| = a+b$ and $|a| + |b| = a + b$.
Case $a > 0, b = 0$: Then $a + b = a + 0 = a > 0$, so $|a+b| = a$ and $|a| + |b| = a + 0 = a$.
Case $a < 0, b = 0$: Then $a+b = a+0 = a < 0$, so $|a+b| = -a$ and $|a|+|b| = -a+0 = -a$.
Case $a, b < 0$: Then $a + b < 0$, so $|a+b| = -(a+b) = -a-b$ and $|a| + |b| = -a - b$.
That is, equality holds except for the case where $a, b$ are nonzero opposite signs:
Case $a > 0, b < 0$: $|a+b| \in \{a+b, -(a+b)\}$.
$b < 0 < -b \implies a+b < a < a-b$, and $-a < 0 < a \implies -(a+b) = -a-b < -b < a-b$.
$|a| + |b| = a - b$, so $|a+b| < |a| + |b|$.

(iii) For $\epsilon > 0$,

$$|x-a| < \epsilon \text{ iff } a - \epsilon < x < a + \epsilon.$$

We have

$$|x-a| = \begin{cases} x-a & \text{if } x-a \geq 0, \\ -(x-a) & \text{if } x-a < 0. \end{cases}$$

( $\implies$ ) Suppose $|x-a| < \epsilon$.
Then $x-a < \epsilon$ and $a-x < \epsilon$.
Then $x < a+\epsilon$ and $a-\epsilon < x$.
( $\impliedby$ ) Suppose $a-\epsilon < x < a+\epsilon$.
Then

$$\begin{aligned} a-\epsilon-a < x-a < a+\epsilon-a \\ -\epsilon < x-a < \epsilon \end{aligned}$$

So $x-a < \epsilon$ and $-\epsilon < x-a \implies -(x-a) < \epsilon$, so $|x-a| < \epsilon$.

## 1.2 The Natural and Rational Numbers

**Definition.** *A set $E$ of real numbers is said to be **inductive** provided it contains 1 and if the number $x$ belongs to $E$, the number $x+1$ also belongs to $E$.*
The set of **natural numbers**, denoted by $\mathbb{N}$, is defined to be the intersection of all inductive subsets of $\mathbb{R}$.

**Theorem 1.** *Every nonempty set of natural numbers has a smallest member.*

*Proof.* Let $E$ be a nonempty set of natural numbers. Since the set $\{x \in \mathbb{R} \mid x \geq 1\}$ is an inductive set, by definition of intersection, $\mathbb{N} \subseteq \{x \in \mathbb{R} \mid x \geq 1\}$, and the natural numbers are bounded below by 1. Therefore $E$ is bounded below by 1. By the Completeness Axiom, $E$ has an infimum; let $c = \inf E$. Since $c+1$ is not a lower bound for $E$, there is an $m \in E$ for which $m < c+1$. We claim that $m$ is the smallest member of $E$. Otherwise, there is an $n \in E$ for which $n < m$. Since $n \in E$, $c \leq n$. Thus $c \leq n < m < c+1$ and therefore $m-n < 1$. Therefore the natural number $m$ belongs to the interval $(n, n+1)$. However, an induction argument shows that $(n, n+1) \cap \mathbb{N} = \emptyset$ (Problem 8). This is a contradiction to $m \in E$. Therefore $m$ is the smallest member of $E$. □

**Archimedean Property.** *For each pair of positive real numbers $a$ and $b$, there is a natural number $n$ for which $na > b$. This can be reformulated: for each positive real number $\epsilon$, there is a natural number $n$ for which $\frac{1}{n} < \epsilon$.*

The set of **integers**, denoted $\mathbb{Z}$, is defined to be the set of numbers consisting of the natural numbers, their negatives, and zero.

Consider the number 2. From problem 4(iii), there is a unique $x > 0$ for which $x^2 = 2$. It is denoted by $\sqrt{2}$. This number is not rational. Suppose that $x$ is rational: then there exist $p, q \in \mathbb{Z}$ such that $(\frac{p}{q})^2 = 2$. Then $p^2 = 2q^2$. By the unique prime factorizations of $p$ and $q$, $p^2$ is divisible by $2^{2k}$ for some $k \in \mathbb{Z}_{\geq 0}$, while $2q^2$ is divisible by $2 \cdot 2^{2j} = 2^{2j+1}$ for some $j \in \mathbb{Z}_{\geq 0}$. $2^{2k} \neq 2^{2j+1}$ for any combinations of $k, j$ so $p^2 = 2q^2$ is not possible, and $\sqrt{2}$ is not rational.

**Definition.** *A set $E$ of real numbers is said to be **dense** in $\mathbb{R}$ provided that between any two real numbers there lies a member of $E$.*

**Theorem 2.** *The rational numbers are dense in* $\mathbb{R}$.

*Proof.* Let $a, b \in \mathbb{R}$ with $a < b$.
Case $a > 0$:
By the Archimedean Property, for $(b - a) > 0$, there exists $q \in \mathbb{N}$ for which $\frac{1}{q} < b - a$.
Again by the Archimedean Property, for $b, \frac{1}{q} > 0$, there exists $n \in \mathbb{N}$ for which $n(\frac{1}{q}) > b$.
Therefore the set $S = \{n \in \mathbb{N} \mid \frac{n}{q} \geq b\}$ is nonempty. Because $S$ is a set of natural numbers, by Theorem 1, $S$ has a smallest member $p$. Noticing $\frac{1}{q} < b - a < b$, we see that $1 \notin S$ and $p > 1$. Therefore $p - 1$ is a natural number (Problem 9). Because $p$ is the smallest member of $S$, $p - 1 \notin S$ and $\frac{(p-1)}{q} < b$. Also,

$$a = b - (b - a) < \frac{p}{q} - (\frac{1}{q}) = \frac{(p-1)}{q}.$$

Therefore the rational number $\frac{(p-1)}{q}$ lies between $a$ and $b$.
Case $a < 0$:
By the Archimedean Property, for $1, -a > 0$, there exists $n \in \mathbb{N}$ for which $n(1) > -a$, which implies $n + a > 0$, and $b > a$ implies $n + b > n + a > 0$. Then we can use the first case to show that there exists a rational number $r$ such that $n + a < r < n + b$. Therefore the rational number $r - n$ lies between $a$ and $b$. □

**PROBLEMS**

8. Use an induction argument to show that for each natural number $n$, the interval $(n, n + 1)$ fails to contain any natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(n, n + 1) \cap \mathbb{N} = \emptyset$.

   $P(1)$: $(1, 2) = \{x \mid 1 < x < 2\}$. Suppose there exists a natural number $q \in (1, 2)$. Then $q > 1$ and by problem 9, $q - 1$ is a natural number. However, $1 < q < 2 \implies 0 < q - 1 < 1$, which is a contradiction to the fact that the natural numbers are bounded below by 1 (Theorem 1). Therefore there are no natural numbers in $(1, 2)$.

   Suppose $P(k)$ is true for some natural number $k$.

   $P(k+1)$: Suppose there exists a natural number $p \in (k+1, (k+1)+1)$; that is, $k+1 < p < k+2$.

   Case $p = 1$: then $k + 1 < 1 < k + 2$. but $k \in \mathbb{N}$ so $k + 1 > 1$. Thus $p = 1$ is not possible.

   Case $p > 1$: then by problem 9, $p - 1 \in \mathbb{N}$, so $k + 1 < p < k + 2 \implies k < p - 1 < k + 1$. This is a contradiction to $P(k)$, the assumption that there are no natural numbers between $(k, k + 1)$. Therefore $P(k + 1)$ is true.

9. Use an induction argument to show that if $n > 1$ is a natural number, then $n - 1$ also is a natural number. The use another induction argument to show that if $m$ and $n$ are natural numbers with $n > m$, then $n - m$ is a natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $n = 1$ or $n - 1 \in \mathbb{N}$.

   $P(1)$: $1 = 1$, true.

   Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

   $P(k + 1)$: $(k + 1) - 1 = k \in \mathbb{N}$, true.

For $n \in \mathbb{N}$, let $Q(n)$ be the assertion that for all $m \in \mathbb{N}$ such that $n > m$, then $n - m \in \mathbb{N}$.

$Q(1)$: true trivially, because there are no natural numbers less than 1.

Suppose $Q(k)$ is true for some $k \in \mathbb{N}$; that is, for all $m \in \mathbb{N}$ such that $k > m$, then $k - m \in \mathbb{N}$.

$Q(k+1)$: For all the $m$ from $Q(k)$, we have $(k+1) > k > m$.

We want to show that $(k+1) - m \in \mathbb{N}$.

This is clearly true for $m = 1$ because $(k+1) - 1 = k \in \mathbb{N}$. Otherwise, $m > 1$, so by $P(m)$, $m - 1 \in \mathbb{N}$ and $(k+1) - m = k - (m-1)$. $Q(k)$ is true tells us that because $(m-1) \in \mathbb{N}$ and $k > m > m - 1$, then $k - (m-1) \in \mathbb{N}$. Therefore $Q(k+1)$ is true.

10. Show that for any real number $r$, there is exactly one integer in the interval $[r, r+1)$.

    This is trivial if $r \in \mathbb{Z}$.

    Consider the smallest integer $p$ less than $[r, r+1)$. Then $p < r < r+1$ (and $r < p+1$, because $r = p+1 \implies r \in \mathbb{Z}$ and $r > p+1 \implies p$ is not the smallest integer less than $[r, r+1)$), therefore $r < p+1 < r+1$. Because the integers are inductive, $p+1 \in \mathbb{Z}$.

    To show that there is not more than one integer between $[r, r+1)$: let $q$ be a natural number such that $r \leq q < r+1$. Then $q - 1 < r \leq q$ and $q < r+1 \leq q+1$. From problem 8, we see that there are no integers between $(q-1, q)$ and $(q, q+1)$, so there is only one integer in $(q-1, q) \cup q \cup (q, q+1) \supseteq [r, r+1)$.

11. Show that any nonempty set of integers that is bounded above has a largest member.

    Let $E$ be a nonempty set of integers that is bounded above. By the completeness axiom, there exists $c = \sup E$. That is, $x \leq c$ for all $x \in E$. Then $c - 1 < z \leq c$ for some $z \in E$ because $c - 1$ is not an upper bound of $E$. Suppose $c$ is not in $E$. Then $c - 1 < z < c$. This implies that $c - 1 < z < w \leq c$ for some $w \in E$ because $z$ is not an upper bound of E. But then there exists two integers in the interval $(c-1, c]$, which is a contradiction to problem 10. Therefore $c$ is an element of $E$, and it is the largest member.

12. Show that the irrational numbers are dense in $\mathbb{R}$.

    Choose any two real numbers $a, b$ and any irrational number $z$. Then $\frac{a}{z}, \frac{b}{z}$ are real numbers. By density of the rationals in $\mathbb{R}$, there exists a rational $r$ such that $\frac{a}{z} < r < \frac{b}{z}$. This implies $a < rz < b$, where $rz$ is an irrational number.

    Proof that $rz$ is irrational:
    Let $r = \frac{p}{q}$ and suppose that $rz$ is rational; then $rz = \frac{m}{n}$.

$$\begin{aligned} \frac{p}{q} z &= \frac{m}{n} \\ z &= \frac{m}{n} \frac{q}{p} \\ z &= \frac{mq}{np} \end{aligned}$$

    Then $z$ is rational, a contradiction.

13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

Choose any real number $a$. Let $S = \{r \in \mathbb{Q} \mid r \leq a\}$. Then $a$ is an upper bound for this set. To show that $a$ is the supremum, suppose by contradiction that it is not. Then there exists $c \in \mathbb{R}$ such that $r \leq c < a$. However, the rational numbers are dense in $\mathbb{R}$, so there exists a rational between $c$ and $a$, a contradiction to the assumption that $c$ is an upper bound to $S$.

The same argument can easily be shown for the irrational numbers.

14. Show that if $r > 0$, then, for each natural number $n$, $(1+r)^n \geq 1 + n \cdot r$.

    Let $r > 0$.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(1+r)^n \geq 1 + n \cdot r$.

    $P(1)$: $(1+r)^1 = 1 + 1 \cdot r$, true.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $(1+r)^k \geq 1 + k \cdot r$.

    $P(k+1)$:

    $(1+r)^{k+1} = (1+r)^k(1+r) \geq (1+kr)(1+r) = 1+kr+r+kr^2 > 1+kr+r = 1+(k+1)\cdot r.$

15. Use induction arguments to prove that for every natural number $n$,

    (i)
    $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

    $P(1)$: $\sum_{j=1}^{1} j^2 = 1 = \dfrac{1(1+1)(2+1)}{6}$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.
    $P(k+1)$:

    $$\begin{aligned}
    \sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\
    &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
    &= \frac{k(2k^2+k+2k+1)}{6} + \frac{6(k^2+2k+1)}{6} \\
    &= \frac{(2k^3+k^2+2k^2+k)+(6k^2+12k+6)}{6} \\
    &= \frac{2k^3+9k^2+13k+6}{6} \\
    &= \frac{(k+1)(2k^2+7k+6)}{6} \\
    &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}.
    \end{aligned}$$

    (ii)
    $$1^3 + 2^3 + \cdots + n^3 = (1+2+\cdots+n)^2,$$

    $P(1)$: $a^3 = 1 = (1)^3$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.

$P(k+1)$:

$$\begin{aligned}1^3+2^3+\cdots+(k+1)^3 &= 1^3+2^3+\cdots+k^3+(k+1)^3\\ &= (1+2+\cdots+k)^2+(k+1)^3\\ &= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3\\ &= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}\\ &= \frac{k^2(k+1)^2+(4k+4)(k+1)^2}{4}\\ &= \frac{(k^2+4k+4)(k+1)^2}{4}\\ &= \frac{(k+2)^2(k+1)^2}{2^2}\\ &= \left(\frac{(k+2)(k+1)}{2}\right)^2\\ &= \left(\frac{((k+1)+1)(k+1)}{2}\right)^2\\ &= \Big(1+2+\cdots+(k+1)\Big)^2.\end{aligned}$$

(iii)

$$1+r+\cdots+r^n=\frac{1-r^{n+1}}{1-r} \text{ if } r\neq 1.$$

$P(1)$: $1+r^1=\dfrac{(1+r)(1-r)}{1-r}=\dfrac{1-r^2}{1-r}$.
Suppose $P(k)$ is true for $k\in\mathbb{N}$.
$P(k+1)$:

$$\begin{aligned}1+r+\cdots+r^{k+1} &= 1+r+\cdots+r^k+r^{k+1}\\ &= \frac{1-r^{k+1}}{1-r}+r^{k+1}\\ &= \frac{1-r^{k+1}}{1-r}+\frac{(1-r)r^{k+1}}{1-r}\\ &= \frac{1-r^{k+1}+r^{k+1}-r^{(k+1)+1}}{1-r}\\ &= \frac{1-r^{(k+1)+1}}{1-r}.\end{aligned}$$

## 1.3 Countable and Uncountable Sets

Two sets $A$ and $B$ are **equipotent** provided there exists a bijection between them.
A set $E$ is **countable** if it is equipotent to a set of natural numbers.
For a countably infinite set $X$, we say that $\{x_n \mid n\in\mathbb{N}\}$ is an **enumeration** of $X$ provided

$$X=\{x_n \mid n\in\mathbb{N}\} \text{ and } x_n\neq x_m \text{ if } n\neq m.$$

**Theorem 3.** *A subset of a countable set is countable. In particular, every set of natural numbers is countable.*

**Corollary 4.** *The following sets are countably infinite:*

*(i) For each natural numbers $n$, the Cartesian product $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$.*

*(ii) The set of natural numbers $\mathbb{Q}$.*

The rationals are countable: $\mathbb{Q} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\}$.

**Corollary 6.** *The union of a countable collection of countable sets is countable.*

An interval of real numbers is called degenerate if it is empty or contains a single member.

**Theorem 7.** *A nondegenerate interval of real numbers is uncountable.*

*Proof.* Let $I$ be a nondegenerate interval of real numbers. Clearly $I$ is not finite. Suppose $I$ is countably infinite. Let $\{x_n \mid n \in \mathbb{N}\}$ be an enumeration of $I$. For each $n \in \mathbb{N}$, choose a nondegenerate compact subinterval $[a_n, b_n] \subseteq I$ such that $x_n \notin [a_n, b_n]$. Let the set of such intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ be descending: $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ (That is, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$.) Now, the nonempty set $E = \{a_n \mid n \in \mathbb{N}\}$ is bounded above by $b_1$. Then the Completeness Axiom implies that $E$ has a supremum, say $x^* = \sup E$. Then for each $n$, $a_n \leq x^* \leq b_n$ because $x^*$ is the supremum of $E$ and each $b_n$ is an upper bound for $E$. Therefore $x^*$ belongs to $[a_n, b_n]$ for each $n$. But then $x^*$ is an element of $I$ and thus has an index $n_0 \in \mathbb{N}$ such that $x^* = x_{n_0}$. But $x^* \in [a_{n_0}, b_{n_0}]$, a contradiction. Therefore $I$ is not countable. $\square$

**PROBLEMS**

16. Show that the set $\mathbb{Z}$ of integers is countable.

    There exists a bijection $\phi : \mathbb{Z} \to \mathbb{N}$ with

$$\phi(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x + 1 & \text{if } x \leq 0. \end{cases}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \cdots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots\}$$

17. Show that a set $A$ is countable iff there is an injective mapping of $A$ to $\mathbb{N}$.

    ( $\implies$ ) Suppose $A$ is countable.

    Then either $A$ is equipotent to $\mathbb{N}$, or there is an $n \in \mathbb{N}$ such that $A$ is equipotent to $\{1, 2, \cdots, n\}$. In the case $A$ is countably infinite, we have a bijection with $\mathbb{N}$ and thus an injection. In the case $A$ is finite, we have an injection with a subset of $\mathbb{N}$, and thus an injection with $\mathbb{N}$ (injection: $f(a) = f(b) \implies a = b$ for $a, b \in A$).

    ( $\impliedby$ ) Suppose there is an injective mapping of $A$ to $\mathbb{N}$.

    Then there is a bijection from $A$ to some subset $B$ of $\mathbb{N}$. By Theorem 3, every subset of natural numbers is countable, and because $A$ is equipotent to this countable set $B$, then $A$ is countable.

18. Use an induction argument to complete the proof of part (i) of Corollary 4.

    (Not an induction argument)

    Consider the function $f : \mathbb{N}^2 \to \mathbb{N}$, where $f(m,n) = 2^m 3^n$. By the Fundamental Theorem of Arithmetic, $2^m 3^n = 2^{m'} 3^{n'} \implies m = m', n = n'$. Then clearly $f$ is an injection. By problem 17, we see that $\mathbb{N}^2$ is countable.

    For any $k \in \mathbb{N}$ we can construct a function $f : \mathbb{N}^k \to \mathbb{N}$, where we have $n$ primes such that $f(m_1, m_2, \cdots, m_k) = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. By the fundamental theorem of arithmetic, this is an injection and thus $\mathbb{N}^k$ is countable.

19. Prove Corollary 6 in the case of a finite family of countable sets.

    Let $\{S_n\}_{n=1}^k$ be a finite family of countable sets. Then each set $S_n$ is countable, and we can enumerate as follows: $S_n = \{s_{nm} \mid m \in \mathbb{N}\}$. Then because there is only a finite number of countable sets, we can construct a function $f : \bigcup_{n=1}^k S_n \to \mathbb{N}$ seeing that

$$\bigcup_{n=1}^{k} S_n = \{s_{11}, s_{21}, s_{31}, \cdots, s_{k1}, s_{12}, s_{22}, s_{32}, \cdots, s_{k2}, s_{13}, \cdots\}.$$

20. Let both $f : A \to B$ and $g : B \to C$ be injective and surjective. Show that the composition $g \circ f : A \to B$ and the inverse $f^{-1} : B \to A$ are also injective and surjective.

    $g \circ f$:

    By surjectivity of $g$, for all $c \in C$, there exists a $b \in B$ such that $g(b) = c$. Then by surjectivity of $f$, there exists an $a \in A$ such that $f(a) = b$.

    Therefore for any $c \in C$:

$$\begin{aligned} c &= g(b) && \text{for some } b \in B \\ &= g(f(a)) && \text{for some } a \in A \\ &= g \circ f(a) \end{aligned}$$

    Therefore $g \circ f$ is surjective.

    By injectivity of $g$, $g(b) = g(b') \implies b = b'$.

    By injectivity of $f$, $f(a) = f(a') \implies a = a'$.

$$\begin{aligned} g \circ f(a) &= g \circ f(a') \\ g(f(a)) &= g(f(a')) \\ f(a) &= f(a') && \text{by injectivity of } g \\ a &= a' && \text{by injectivity of } f \end{aligned}$$

    Therefore $g \circ f$ is injective.

    $f^{-1}$:

    Because $f$ is a function from $A$ to $B$, $f(a) \subseteq B$ is defined for all $a \in A$. That is, for all $a \in A$, there exists a $b \in B$ such that $f^{-1}(b) = a$. Thus $f^{-1}$ is surjective.

    Because $f$ is a function, for each $a \in A$, $f(a) = b$ and $f(a) = b'$ imply $b = b'$. That is, $f^{-1}(b) = f^{-1}(b') \implies b = b'$. Thus $f^{-1}$ is injective.

21. Use an induction argument to establish the pigeonhole principle.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that for any $m \in \mathbb{N}$, the set $\{1, 2, \cdots, n\}$ is not equipotent to the set $\{1, 2, \cdots, n + m\}$.

    $P(1)$: We have the sets $A = \{1\}$ and $B = \{1, 2, \cdots, 1 + m\}$, for $m \in \mathbb{N}$. In the case $m = 1$, $B = \{1, 1 + 1\} = \{1, 2\}$, and clearly $A$ is not equipotent to $B$. Clearly $A$ is also not equipotent to $B$ for any other natural number $m > 1$.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $\{1, 2, \cdots, k\}$ is not equipotent to the set $\{1, 2, \cdots, k+m\}$, for any $m \in \mathbb{N}$.

    $P(k + 1)$: Then clearly $\{1, 2, \cdots, k + 1\}$ is not equipotent to the set $\{1, 2, \cdots, (k + 1), \cdots, (k + 1) + m\}$, for any $m \in \mathbb{N}$.

22. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable.

    (Cantor's Theorem: for a set $A$, any function $f : A \to \mathcal{P}(A)$ is not surjective.)

    Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any map. Let $E = \{n \in \mathbb{N} \mid n \notin f(n)\}$. Then $E$ is a subset of the naturals that is not in the image of $f$, so $f$ is not surjective. Therefore there is no bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding theorem to show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.

    In problem 18, we showed that for any $k \in \mathbb{N}$, the set $\mathbb{N}^k = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is countable. It is then trivial to see that the Cartesian product of any finite collection of countable sets $S_1 \times S_2 \times \cdots \times S_k$ is countable.

    Notation:
    $$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}, \cdots$$
    We can let $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ be the set of functions $f : \mathbb{N} \to \{0, 1\}$.

    Then, for any subset $A \subseteq \mathbb{N}$, there exists a function $f \in \{0, 1\}^{\mathbb{N}}$ such that
    $$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$
    and we have a bijection between the elements of $\{0, 1\}^{\mathbb{N}}$ and the subsets of $\mathbb{N}$ ("Two sets that are equipotent are, from a set-theoretic point of view, indistinguishable"). Therefore $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ can be used to represent the collection of subsets of $\mathbb{N}$.

    Now, because the set of functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable, then clearly the set of functions $f : \mathbb{N} \to \mathbb{N} \supseteq \{0, 1\}$ is uncountable (including zero in the naturals for notation convenience).

24. Show that a nondegenerate interval of real numbers fails to be finite.

    Theorem 7 tells us that a nondegenerate interval of real numbers is uncountable, and thus, finite.

25. Show that any two nondegenerate intervals of real numbers are equipotent.

    We can prove this by showing that any interval is equipotent to the interval $(0, 1)$.

    For any bounded interval $(a, b), (a, b], [a, b), [a, b]$, there exists a bijection to $(0, 1), (0, 1], [0, 1), [0, 1]$ respectively, of the form $f(x) = \dfrac{1}{b - a}(x - a)$.

26. Is the set $\mathbb{R} \times \mathbb{R}$ equipotent to $\mathbb{R}$?

    yes (Schröder-Bernstein theorem )

## 1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

**Proposition 9.** *Every nonempty open set is the union of a countable, disjoint collection of open intervals.*

**The Heine-Borel Theorem.** *Let $F$ be a closed and bounded set of real numbers. Then every open cover of $F$ has a finite subcover.*

*Proof.* Let $F$ be the closed, bounded interval $[a, b]$. Let $\mathcal{F}$ be an open cover of $[a, b]$. Define $E$ to be the set of numbers $x \in [a.b]$ with the property that the interval $[a, x]$ can be covered by a finite number of the sets of $\mathcal{F}$. Since $a \in [a, b] \subseteq \mathcal{F}$ implies that $a$ is in one of the sets $\mathcal{O}' \subseteq \mathcal{F}$ by definition of union, $\mathcal{O}'$ is a finite subcover of $[a, a] = \{a\}$, and thus $a \in E$ and $E$ is nonempty. Since $E \subseteq [a, b] = \{x \mid a \leq x \leq b\}$, $E$ is bounded above by $b$, so by the completeness of $\mathbb{R}$, $E$ has a supremum $c = \sup E$. Because $c \leq b$, clearly $c$ belongs to $[a, b]$, and this implies that there is an $\mathcal{O} \subseteq \mathcal{F}$ that contains $c$. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$ such that that the interval $(c-\epsilon, c+\epsilon) \subseteq \mathcal{O}$. Now $c-\epsilon$ is not an upper bound for $E$, and so there must be an $x \in E$ with $c - \epsilon < x$. Because $x \in E$, there exists a finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k\}$ of sets in $\mathcal{F}$ that covers $[a, x]$. Then clearly the finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k, \mathcal{O}\}$ covers the interval $[a, c + \epsilon)$. Therefore $c = b$, otherwise there exists a number $c + \frac{1}{2}\epsilon$ that has a finite subcover and $c < c + \frac{1}{2}\epsilon$ implies that $c$ is not an upper bound for $E$. Thus $[a, b] \in E$ and $[a, b]$ can be covered by a finite number of sets of $\mathcal{F}$. $\square$

**The Heine-Borel Theorem ( $\Longleftarrow$ ).** *Let $F$ be a real set such that every open cover of $F$ has a finite subcover. Then $F$ is closed and bounded.*

*Proof.* Let $K$ be a compact subset of a metric space $X$. Proving that $X \setminus K$ is open will show that $K$ is closed. Consider any $p \in X \setminus K$. For a $k \in K$, let $O_k$ and $I_k$ be neighborhoods of $p$ and $k$ respectively, with radius less than $\frac{1}{2}d(p, q)$. Because $K$ is compact, there are finitely many points $k_1, \cdots, k_n$ in $K$ such that $K \subseteq I_{k_1} \cup \cdots \cup I_{k_n}$. Let $O = O_{k_1} \cap \cdots \cap O_{k_n}$ so that $O$ is an open neighborhood of $p$ that does not intersect $K$. Then $O \subseteq X \setminus K$ and $X \setminus K$ is open. Therefore $K$ is closed. $\square$

**The Nested Set Theorem.** *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which $F_1$ is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

*Proof.* By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$, and we have an open cover of $\mathbb{R}$ and thus an open cover of $F_1 \subseteq \mathbb{R}$. By the Heine-Borel Theorem, there exists an $N \in \mathbb{N}$ such that $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c$. Because $\{F_n\}$ is descending, $F_n \supseteq F_{n+1}$ for any $n \geq 1$. This implies $F_n^c \subseteq F_{n+1}^c$, and thus $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c = F_N^c = \mathbb{R} \setminus F_N$. This is a contradiction to the assumption that $F_N$ is a nonempty subset of $F_1$. $\square$

**PROBLEMS**

27. Is the set of rational numbers open or closed?

    The set of rationals is neither open nor closed. The rationals is not open because the irrationals are dense in the rationals; that is, between any two rationals there is an irrational. The rationals is not closed because it does not contain all its limit points; a sequence of rationals can be constructed that converges to an irrational. (Thus we see that the irrationals is neither open nor closed as well.)

28. What are the sets of real numbers that are both open and closed?

    It is clear that $\mathbb{R}$ is open, and $\emptyset$ is open (vacuously). Then because the complement of an open set is closed, $\mathbb{R}$ and $\emptyset$ are both closed as well.

29. Find two sets $A$ and $B$ such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

    Let $A = (4, 5)$ and $B = (5, 20)$. Then $(4, 5) \cap (5, 20) = \emptyset$ and $\overline{A} = [4, 5]$ and $\overline{B} = [5, 20]$ so $[4, 5] \cap [5, 20] = \{5\} \neq \emptyset$.

    Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$ and $\overline{A} = \mathbb{R}$ and $\overline{B} = \mathbb{R}$ so $\mathbb{R} \cap \mathbb{R} = \mathbb{R} \neq \emptyset$.

30. A point $x$ is called an **accumulation point** of a set $E$ provided it is a point of closure of $E \setminus \{x\}$.

    (i) Show that the set $E'$ of accumulation points of $E$ is a closed set.
    Then for $x \in E'$, every open interval that contains $x$ also contains a point in $E \setminus \{x\}$.
    Suppose $E'$ is not closed. Then there exists an element $y \notin E'$ such that every open interval that contains $y$ also contains a point $x \in E'$. Then every open interval that contains $x$ contains a point $z \in E \setminus \{x\}$... It can be shown that $y \in E'$ and so $E'$ contains all its points of closure and is thus closed.

    (ii) Show that $\overline{E} = E \cup E'$.
    $E$ includes all the isolated points not included in $E'$.

31. A point $x$ is called an **isolated point** of a set $E$ provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set $E$ consists of isolated points, then it is countable.

    Each singleton set $\{x\}$ can be enumerated.

32. A point $x$ is called an **interior point** of a set $E$ if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in $E$. The set of interior points of $E$ is called the **interior** of $E$ denoted by int $E$. Show that

    (i) $E$ is open iff $E =$ int $E$.
    ( $\Longrightarrow$ ) Suppose $E$ is open.
    Then clearly every point of $E$ is an interior point.
    ( $\Longleftarrow$ ) Suppose $E =$ int $E$.
    Then every point has an open neighborhood contained in $E$, so $E$ is open.

    (ii) $E$ is dense iff int $(\mathbb{R} \setminus E) = \emptyset$.

33. Show that the nested set theorem is false if $F_1$ is unbounded.

    The nested set theorem works because the compactness of $F_1$ allows us to reach a contradiction to the fact that the intersection is empty (see the proof above).

    Consider
    $$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$
    This intersection is empty because for any $x$, there exists an $n \in \mathbb{N}$ such that $x < n$ and thus $x \notin [n, \infty)$.

34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Heine-Borel Theorem States that Closed and bounded sets are compact; that is, every open cover of a closed and bounded set has a finite subcover. If a set $E$ is bounded, then for any open cover $E \subseteq \mathcal{F}$ there exists a finite open subcover $\mathcal{O} \subseteq \mathcal{F}$. We can consider the intersection of all such $\mathcal{O}$ so that $E \subseteq \bigcap_{\mathcal{O} \subseteq \mathcal{F}} \mathcal{O} \subseteq \mathcal{O}$, and this intersection is the supremum.

    Clearly the descending sets from the nested set theorem are closed and bounded, so the Heine-Borel Theorem discussed above can be used to imply the Completeness Axiom.

35. Show that the collection of Borel sets is the smallest $\sigma$-algebra that contains the closed sets.

    The Borel sets is defined to be the smallest $\sigma$-algebra that contains all the open sets of real numbers. Any sigma-algebra that contains the closed sets contains the open sets by the complement property of a sigma-algebra, so the Borel sets is the smallest sigma-algebra that contains the closed sets as well.

36. Show that the collection of Borel sets is the smallest $\sigma$-algebra that contains the intervals of the form $[a, b)$, where $a < b$.

    Any interval $[a, b)$ can be written in the form

$$[a, b) = \bigcup_{n=1}^{\infty} [a, b - \tfrac{1}{n}]$$

37. Show that each open set is an $F_\sigma$ set.

    Any open set $(a, b)$ can be written in the form

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b - \tfrac{1}{n}].$$

## 1.5 Sequences of Real Numbers

**Proposition 14.** *Let the sequence of real numbers $\{a_n\}$ converge to the real number $a$. Then the limit is unique, the sequence is bounded, and, for a real number $c$,*

$$\textit{if } a_n \leq c \textit{ for all } n, \textit{ then } a \leq c.$$

*Proof.* Suppose there exist $a$ and $b$ such that $\{a_n\} \to a$ and $\{a_n\} \to b$. Then For any $\epsilon > 0$, there exists the index $N = \max\{N_a, N_b\}$ such that for all $n \geq N \geq N_a, N_b$, then $|a - a_n| < \epsilon$ and $|b - a_n| < \epsilon$. By the triangle inequality, $|a - b| \leq |a - a_n| + |a_n - b| < \epsilon + \epsilon = 2\epsilon = \epsilon'$. Therefore $a = b$, and the limit is unique.

Consider $\epsilon = 1$. Then there exists an index $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < 1$. Also, $|a_n| - |a| \leq |a_n - a| < 1 \implies |a_n| < |a| + 1$. Let $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\}$. The maximum exists because this is a finite set of real numbers (totally ordered). Considering any $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - a| < 1 \implies |a_n| < |a| + 1 \leq M$, and if $n < N$, then $|a_n| \leq \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\} = M$, so $M$ is a bound for this sequence.

Suppose that for all $n$, $a_n \leq c$ but $a > c$. Then $a_n \leq c < a$ for all $n$, and $0 \leq c - a_n < a - a_n$. Choosing $\epsilon = c - a_n$, there exists an index such that $|a - a_n| < c - a_n$. But this is a contradiction. Therefore $a \leq c$. ☐

**Theorem 15** (the Monotone Convergence Criterion for Real Sequences)**.** *A monotone sequence of real numbers converges iff it is bounded.*

*Proof.* ($\implies$) Suppose a monotone sequence converges.
By the above proposition, it is bounded.
($\impliedby$) Suppose a monotone sequence $\{a_n\}$ is bounded.
By the Completeness Axiom, there exists a supremum say $a$ such that $a_n \leq a$ for all $n$. Consider any $\epsilon > 0$. Now, $a - \epsilon$ is not an upper bound, and because the sequence is increasing, there exists an index $N$ for which $a_n \geq a_N > a - \epsilon$ for all $n \geq N$. Then $\epsilon > a - a_n$ and the sequence converges to $a$. The proof is the same for a decreasing sequence. ☐

**Theorem 16** (The Bolzano-Weierstrass Theorem). *Every bounded sequence of real numbers has a convergent subsequence.*

*Proof.* Let $a_n$ be a bounded sequence of real numbers. Choose $M > 0$ s.t. $|a_n| \leq M$ for all $n$. Define $E_n = \overline{\{a_j \mid j \geq n\}}$. Then we also have $E_n \subseteq [-M, M]$ and $E_n$ is closed since it is the closure of a set. Therefore $\{E_n\}$ is a descending sequence of nonempty closed bounded subsets of real numbers. The Nested Set Theorem tells us that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$, so there exists $a \in \bigcap_{n=1}^{\infty} E_n$. For each natural number $k$, $a$ is a point of closure of $\{a_j \mid j \geq k\}$. Thus for infinitely many indices $j \geq n$, $a_j$ belongs to $(a - \frac{1}{k}, a + \frac{1}{k})$. By induction, choose a strictly increasing subsequence of natural numbers $n_k$ such that $|a - a_{n_k}| < \frac{1}{k}$ for all $k$. From the Archimedean Property of the reals, the subsequence $\{a_{n_k}\}$ converges to $a$. □

**Proposition 19.** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.*

*(i)* $\limsup\{a_n\} = \ell \in \mathbb{R}$ *iff for each $\epsilon > 0$, there are infinitely many indices $n$ for which $a_n > l - \epsilon$ and only finitely many indices $n$ for which $a_n < l - \epsilon$.*

( $\implies$ ) *Suppose* $\limsup\{a_n\} = \ell \in \mathbb{R}$.

*Then by problem 38, $\ell$ is a cluster point of the sequence. This means that for all $\epsilon > 0$, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all $n_k$ greater than some index, and thus $\ell - \epsilon < a_{n_k}$ for infinitely many indices $n_k$.*

*Suppose by contradiction that for $\epsilon > 0$, there are infinitely many indices $n$ for which $a_n < l - \epsilon$. That is, no matter how large the epsilon we choose, there exists a subsequence $\{a_{n_k}\}$ such that $\epsilon < l - a_{n_k}$ for all $n_k$ after a certain index. This implies that $\{a_n\}$ is not bounded, so by Proposition 14, the sequence does not converge to a real number. This is a contradiction to $\ell \in \mathbb{R}$.*

( $\impliedby$ ) *Suppose for $\epsilon > 0$, there are infinitely many indices $n$ for which $a_n > l - \epsilon$ and only finitely many indices $n$ for which $a_n < l - \epsilon$.*

*Then choosing specific indices $n_k$, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all $n_k$, and this implies the subsequence converges to $\ell$. If we suppose that $\ell \neq \limsup\{a_n\}$, then there exists some $\delta > 0$ such that $\ell > \ell - \delta = \limsup\{a_n\}$.*

*Now, $\ell - \delta = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. That means for any $n$, $a_k \leq \ell - \delta$ for $k \geq n$. However, this is a contradiction to the fact that there are only finitely many such indices $k$ for which this is true. Therefore $\ell = \limsup\{a_n\}$.*

*(ii)* $\limsup\{a_n\} = \infty$ *iff $\{a_n\}$ is not bounded above.*

( $\implies$ ) *Suppose* $\limsup\{a_n\} = \infty$.

*This implies that $\infty = \limsup\{a_n\}$ is a cluster point and there exists a subsequence that converges to infinity. Therefore $\{a_n\}$ is not bounded above.*

( $\impliedby$ ) *Suppose $\{a_n\}$ is not bounded above.*

*By Proposition 4, $\{a_n\}$ does not converge to a real number. Also, $\{a_n\}$ is not bounded above implies that for any real number c, there exists an index such that $a_n > c$. Then the only upper bound of this sequence is $\infty$ and thus* $\limsup\{a_n\} = \infty$.

*(iii)*

$$\limsup\{a_n\} = -\liminf\{-a_n\}.$$

*Definitions of limsup and liminf:*

$\limsup\{a_n\} = \lim_{n\to\infty}[\sup\{a_k \mid k \geq n\}] \implies$ *for any $n \in \mathbb{N}$, $\sup\{a_k \mid k \geq n\} \geq a_k$ for $k \geq n$.*

$\liminf\{a_n\} = \lim_{n\to\infty}[\inf\{a_k \mid k \geq n\}]$. $\implies$ *for any* $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k$ *for* $k \geq n$.
*Now we have*
$\liminf\{-a_n\} = \lim_{n\to\infty}[\inf\{-a_k \mid k \geq n\}]$.
$\implies$ *for any* $n \in \mathbb{N}$, $\inf\{-a_k \mid k \geq n\} \leq -a_k$ *for* $k \geq n$.
$\implies$ *for any* $n \in \mathbb{N}$, $-\inf\{-a_k \mid k \geq n\} \geq a_k$ *for* $k \geq n$, *the definition of limsup.*

*(iv) A sequence of real numbers* $\{a_n\}$ *converges to an extended real number* $a$ *iff*

$$\liminf\{a_n\} = \limsup\{a_n\} = a.$$

$(\implies)$ *Suppose a sequence of real numbers* $\{a_n\}$ *converges to an extended real number* $a$.
*Clearly* $\liminf\{a_n\} \leq a \leq \limsup\{a_n\}$.
*If* $\liminf\{a_n\} < a < \sup\{a_n\}$, *then we reach a contradiction to the infimum and supremum respectively.*
*Therefore* $\liminf\{a_n\} = a = \limsup\{a_n\}$.
$(\impliedby)$ *Suppose* $\liminf\{a_n\} = \limsup\{a_n\} = a$.
*Then for any* $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ *for* $k \geq n$, *which implies*

$$a = \liminf\{a_n\} = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\} \leq \lim_{n\to\infty} a_k \leq \lim_{n\to\infty} \sup\{a_k \mid k \geq n\} = \limsup\{a_n\} = a$$

*Clearly* $\{a_n\}$ *converges to* $a$.

*(v) If* $a_n \leq b_n$ *for all* $n$, *then*

$$\limsup\{a_n\} \leq \limsup\{b_n\}.$$

*For any* $n \in \mathbb{N}$, $a_k \leq \sup\{a_k \mid k \geq n\}$ *and* $b_k \leq \sup\{b_k \mid k \geq n\}$ *for all* $k \geq n$.
*If we suppose* $\limsup\{a_n\} > \limsup\{b_n\}$, *then there exists a natural number* $n$ *such that* $\sup\{a_k \mid k \geq n\} > \sup\{b_k \mid k \geq n\} \geq b_k \geq a_k$ *for all* $k \geq n$. *However, by problem 38, we see that* $\limsup\{a_n\}$ *is a cluster point of* $\{a_n\}$, *and we reach a contradiction. (or contradiction to def of supremum?)*

**Proposition 20.** *Let* $\{a_n\}$ *be a sequence of real numbers.*

*(i) The series* $\sum_{k=1}^{\infty} a_k$ *is summable iff for each* $\epsilon > 0$, *there is an index* $N$ *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

$(\implies)$ *Suppose the series* $\sum_{k=1}^{\infty} a_k$ *is summable.*
*That is, there exists an* $s$ *such that* $\{\sum_{k=1}^{n} a_k\}$ *converges to* $s$. *Convergent sequences are Cauchy, so for any* $\epsilon > 0$, *there exists and index* $N$ *such that for all* $n+m \geq n-1 \geq N$,

$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

$$\left|\sum_{k=1}^{n-1} a_k + \sum_{k=n}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon.$$

*( $\Longleftarrow$ ) Suppose that for each $\epsilon > 0$, there is an index $N$ for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

*Then*

$$\left|\sum_{k=n}^{n+m} a_k + \sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$
$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

*Without loss of generality, we can suppose that $n-1 \geq N$, and because $m$ is a natural number, $n+m > n-1 \geq N$. Clearly this describes a Cauchy Sequence, and because the real numbers is complete, this sequence converges and thus the series is summable.*

*(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then $\sum_{k=1}^{\infty} a_k$ is also summable.*

*By subadditivity of absolute value, we can show that for each $\epsilon > 0$, there is an index $N$ for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| \leq \left|\sum_{k=n}^{n+m} |a_k|\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

*(iii) If each term $a_k$ is nonnegative, then the series $\sum_{k=1}^{\infty} a_k$ is summable iff the sequence of partial sums is bounded.*

*Let $\{a_k\}$ be a sequence of nonnegative numbers.*

*( $\Longrightarrow$ ) Suppose the series $\sum_{k=1}^{\infty} a_k$ is summable.*

*Then the sequence of partial sums converges to a real number. By Proposition 14, the sequence of partial sums is bounded.*

*( $\Longleftarrow$ ) Suppose the sequence of partial sums is bounded.*

*Because each $a_k$ is positive, the sequence of partial sums is positive monotonic:*

$$\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} a_k + a_{n+1} = \sum_{k=1}^{n+1} a_k.$$

*Therefore by Theorem 15, the sequence of partial sums converges; that is, the series is summable.*

**PROBLEMS**

38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

    Let $s = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. Suppose there exists a subsequence $\{a_{n_k}\}$ that converges to an extended real number $a$. Fix $\epsilon > 0$. Then there exists an index $M$ such that $|a - a_{n_m}| < \epsilon$ when $n_m \geq M$, and $a_{n_m} \leq \sup\{a_k \mid k \geq M\}$.

    Then $\lim_{M\to\infty} a_{n_m} \leq \lim_{M\to\infty} \sup\{a_k \mid k \geq M\} \implies a \leq s$.

    Therefore $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$. ($\limsup\{a_n\}$ is itself a cluster point else we reach a contradiction to the supremum.) The same method can be used to prove $\liminf\{a_n\}$.

39. Prove proposition 19.

    See above for proof.

40. Show that a sequence $\{a_n\}$ is convergent to an extended real number iff there is exactly one extended real number that is a cluster point of the sequence.

    ( $\Longrightarrow$ ) Suppose $\{a_n\}$ is convergent to an extended real number $a$.

    By Proposition 19(iv), we have $\liminf\{a_n\} = \limsup\{a_n\} = a$, so clearly any cluster point is equal to $a$.

    ( $\Longleftarrow$ ) Suppose there is exactly one extended real number $a$ that is a cluster point of $\{a_n\}$.

    Then there exists a subsequence that converges to $a$. Suppose that $\{a_n\}$ does not converge to $a$. Then there exists an $\epsilon > 0$ such that there are infinitely many indices $n$ for which $a - a_n > \epsilon$. Collect these indices to construct a subsequence $\{a_{n_k}\}$. In the case that $\{a_{n_k}\}$ is bounded, there exists another subsequence of $\{a_{n_k}\}$ that converges to a real number $b \neq a$. But this is also a subsequence of the original sequence $\{a_n\}$, which implies $\{a_n\}$ has two cluster points $a$ and $b$, a contradiction. In the case that $\{a_{n_k}\}$ is unbounded, then for any real number $c$, there exists an index $n$ such that $|a_n| > c$. Then we can construct a subsequence that converges to $+\infty \neq a$ or $-\infty \neq a$, which is again a contradiction to the fact that $\{a_n\}$ has only one cluster point.

41. Show that $\liminf a_n \leq \limsup a_n$.

    For any natural number $n$, we have $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ for all $k \geq n$. Taking the limit with respect to n clearly proves the statement.

42. Prove that if, for all $n$, $a_n \geq 0$ and $b_n \geq 0$, then

    $$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

    provided the product on the right is not of the form $0 \cdot \infty$.

    For any natural number $n$, we can see that

    $$\{a_k \cdot b_k \mid k \geq n\} \subseteq \{a_i \cdot b_j \mid i, j \geq n\}.$$

    Then this clearly implies

    $$\begin{aligned}\sup\{a_k \cdot b_k \mid k \geq n\} &\leq \sup\{a_i \cdot b_j \mid i, j \geq n\} \\ &= \sup\{a_i \mid i \geq n\} \cdot \sup\{b_j \mid j \geq n\}.\end{aligned}$$

    Taking the limit on both sides proves the inequality.

43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

    Let $\{a_n\}$ be any sequence of real numbers. Supposing that there exist no monotone subsequences of $\{a_n\}$, then there are only finitely many indices $n$ for which $a_n \leq a_{n+1}$, and only finitely many indices $n$ for which $a_n \geq a_{n+1}$. Clearly we see a contradiction so there must exist a monotone subsequence.

    Now, in the case that $\{a_n\}$ is bounded, then the monotone subsequence $\{a_{n_k}\}$ is also bounded. By Theorem 15, $\{a_{n_k}\}$ converges. Thus $\{a_n\}$ has a convergent subsequence.

44. Let $p$ be a natural number greater than 1, and $x$ a real number $0 \leq x \leq 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each $n$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when $x$ is of the form $q/p^n$, $0 < q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number $x$ with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal* expansion of $x$. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

For each $m \in \mathbb{N}$, we can construct a partial sum:

$$\sum_{n=1}^{m} \frac{a_n}{p^n} = \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m}$$

We choose each $a_m$ in the following way:

(The $\sum_{n=1}^{m-1} \frac{a_n}{p^n}$ is a fixed value found from the previous iteration, so for each step, we are simply choosing the best $a_m$).

Case $\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} = x$ for some $a_m \in \{0, 1, \cdots, p\}$: Then set $a_k = 0$ for all $k \geq m$, and the equality is clear.

Else: Choose $a_m \in \{0, 1, \cdots, p\}$ such that:

$$\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} < x < \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m + 1}{p^m}.$$

In this way we can construct a monotone sequence (of partial sums) that is bounded above by $x$:

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq \sum_{n=1}^{k+1} \frac{a_n}{p^n} \leq x \text{ for all } k \in \mathbb{N}.$$

By showing that $x$ is the supremum, we can apply Theorem 15 to show that this sequence of partial sums converges to its supremum:

$$\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = x.$$

Suppose that $x$ is not the supremum. Then there exists an $\epsilon > 0$ such that $\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x$ for all $k$. Now, by the Archimedean Property, there exists a natural number $m$ such that $\frac{1}{m} < \epsilon$; therefore $0 < \epsilon - \frac{1}{m}$. Now, because $p > 1$, there exists a natural number $l$ such that $m < p^l$, so $0 < \frac{1}{p^l} < \frac{1}{m}$ and thus $x - (\epsilon - \frac{1}{p^l}) < x - (\epsilon - \frac{1}{m}) < x$.

Then for all natural numbers $k$,

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x - \epsilon + \frac{1}{p^l} < x - \epsilon + \frac{1}{m} < x.$$

However, there exists the natural number $l$ such that

$$\sum_{n=1}^{l} \frac{a_n}{p^n} = \sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l}{p^l} \leq x - \epsilon < x - \epsilon + \frac{1}{p^l} < x$$
$$\sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l + 1}{p^l} \leq x - \epsilon + \frac{1}{p^l} < x.$$

This is a contradiction to our choice of $a_l$ so that

$$\sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l}{p^l} < x < \sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l + 1}{p^l}.$$

Therefore $x$ is the supremum, and the series $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is summable to $x$.

In the case that $x$ is of the form $q/p^n$, the obvious solution would be to set $a_n = q$ (assuming $q$ is an integer), and all other $a_k = 0$. The second solution would be to use the method described above.

For the converse, $0 \leq a_n \leq p - 1$ implies that

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq \sum_{n=1}^{\infty} \frac{p-1}{p^n} = (p-1) \sum_{n=1}^{\infty} \frac{1}{p^n}$$

Showing that $(p-1)\sum_{n=1}^{\infty} \frac{1}{p^n} < 1$ implies that $\sum_{n=1}^{k} \frac{a_n}{p^n}$ is a bounded, monotone sequence of partial sums, and therefore it converges to a number in $[0, 1]$.

Ex: $x = .547$; decimal expansion:

$$x = \frac{5}{10^1} + \frac{4}{10^2} + \frac{7}{10^3} + \frac{0}{10^4} + \frac{0}{10^5} + \cdots = .5 + .04 + .007 + 0 + 0 + \cdots$$

45. Prove Proposition 20.

    See above.

46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Bolzano-Weierstrass Theorem asserts that every bounded sequence of real numbers has a convergent subsequence.

    The Completeness Axiom asserts that every nonempty set of real numbers that is bounded above has a supremum.

    The Monotone Convergence Theorem asserts that a monotone sequence of real numbers converges iff it is bounded.

## 1.6 Continuous Real-Valued Functions of a Real Variable

**Proposition 21.** *A real-valued function $f$ defined on a set $E$ of real numbers is continuous at the point $x_* \in E$ iff whenever a sequence $\{x_n\}$ in $E$ converges to $x_*$, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.*

*Proof.* Let $f$ be a real-valued function defined on a set $E$.
( $\implies$ ) Suppose that $f$ is continuous at the point $x_* \in E$.
Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } x' \in E \text{ and } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.$$

Suppose that a sequence $\{x_n\}$ in $E$ converges to $x_*$. Then for any $\delta > 0$, there exists an index $N$ such that when $n \geq N$, $|x_* - x_n| < \delta$. Then by continuity of $f$, $|f(x_*) - f(x_n)| < \epsilon$, and thus the image sequence converges.
( $\impliedby$ ) Suppose that whenever a sequence $\{x_n\}$ in $E$ converges to $x_*$, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.
That is, for any $\delta > 0$, there exists an index $N$ such that $|x_* - x_n| < \delta$ whenever $n \geq N$, and this implies that for any $\epsilon > 0$, there exists an index $M$ such that $|f(x_*) - f(x_n)| < \epsilon$ whenever $n \geq M$.
Thus continuity is clear. □

**Proposition 22.** *Let $f$ be a real-valued function defined on a set $E$ of real numbers. Then $f$ is continuous on $E$ iff for each open set $\mathcal{O}$,*

$$f^{-1}(\mathcal{O}) = E \cap \mathcal{U} \text{ where } \mathcal{U} \text{ is an open set.}$$

**The Extreme Value Theorem.** *A continuous real-valued function on a nonempty, closed, bounded set of real numbers takes a minimum and a maximum value.*

*Proof.* Let $f$ be a continuous real-valued function on a nonempty, closed, bounded set $E$ of real numbers. Suppose by contradiction that $f$ is not bounded. Then for any $n \in \mathbb{N}$, there exists $x_n \in E$ such that $f(x_n) > n$. With this we can construct a sequence $\{x_n\}$ in $E$. Because $E$ is bounded, $\{x_n\}$ is bounded, and so by the Bolzano Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$. Because $f$ is continuous, $\{x_{n_k}\}$ is convergent implies $\{f(x_{n_k})\}$ is convergent. However, for each element in the image sequence, $f(x_{n_k}) > n_k$, and $\{f(x_{n_k})\}$ is unbounded, thus it cannot converge, and we reach a contradiction.
Because $f$ is bounded, then it has a supremum $s$ such that $f(x) \leq s$ for all $x \in E$. Suppose that $f$ does not have a maximum. Then there is no $x \in E$ such that $f(x) = s$. Then $f(x) < s \implies f(x) \in (-\infty, s)$ for all $x \in E$. (We can use the fact that $(-\infty, s)$ is open $\implies f^{-1}(-\infty, s)$ is open): Then we reach a contradiction because $E$ is closed. The same proof can be used for the minimum. □

**The Intermediate Value Theorem.** *Let $f$ be a continuous real-valued function on the closed, bounded interval $[a, b]$ for which $f(a) < c < f(b)$. The there is a point $x_0$ in $(a, b)$ at which $f(x_0) = c$.*

**Theorem 23.** *A continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous.*

*Proof.* Let $E$ be a closed, bounded set of real numbers, and let $f$ be a continuous real-valued function on $E$.
Fix some $\epsilon > 0$.
By the continuity of $f$, for all $x \in E$, there exists $\delta_x > 0$ such that for $y \in E$ satisfying $|x - y| < 2\delta_x$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then we can construct an open cover of $E$ consisting of the open balls $\mathbb{B}(x, \delta_x)$ for all $x \in E$.
Because $E$ is compact, there exists a finite subcover $\{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$.

Let $\delta_* = \min\{\delta_1, \cdots, \delta_n\}$.
Consider $x, y \in E$ such that $|x - y| < \delta_*$.
Because $y \in E \subseteq \{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$, there exists an index $j \in \{1, \cdots, n\}$ such that $y \in \mathbb{B}(x_j, \delta_j)$; therefore

$$|x_j - y| < \delta_j < 2\delta_j.$$

By continuity of $f$, $|f(x_j) - f(y)| < \frac{\epsilon}{2}$. (A)
By the triangle inequality,

$$|x - x_j| \leq |x - y| + |y - x_j| < \delta_* + \delta_j \leq 2\delta_j.$$

By continuity of $f$, $|f(x) - f(x_j)| < \frac{\epsilon}{2}$. (B)
By the triangle inequality using (A) and (B):

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

**PROBLEMS**

47. Let $E$ be a closed set of real numbers and $f$ a real-valued function that is defined and continuous on $E$. Show that there is a function $g$ defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take $g$ to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)

    Because $E$ is closed, then $\mathbb{R} \setminus E$ is open. In the case that $E = \mathbb{R}$, then $\mathbb{R} \setminus E = \emptyset$ and the conclusion is trivial. Else $\mathbb{R} \setminus E$ is nonempty. By proposition 9, $\mathbb{R} \setminus E$ is the union of a countable, disjoint collection of open intervals.
    In the case that $(-\infty, a)$ [or $(a, \infty)$] is in $\mathbb{R} \setminus E$, then $a \in E$ and $f(a)$ is defined. Simply let $g(x) = f(a)$ be the constant function on $(-\infty, a)$ [or $(a, \infty)$].
    In the case that $(a, b) \in \mathbb{R} \setminus E$, then $a, b \in E$ and $f(a)$,$f(b)$ are defined. Let

    $$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ on } (a, b).$$

    Also let $g(x) = f(x)$ whenever $x \in E$. Then we see that $g$ is continuous.

48. Define the real-valued function $f$ on $\mathbb{R}$ by setting

    $$f(x) = \begin{cases} x & \text{if x irrational} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

    At what points is $f$ continuous?

    See Thomae's Function for something similar.

    $f$ should be discontinuous at each rational number and continuous at each irrational number.

49. Let $f$ and $g$ be continuous real-valued functions with a common domain $E$.

    (i) Show that the sum, $f + g$, and product, $fg$, are also continuous functions.
    Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{f(x_n)\}$ converges to $f(x)$ and $\{g(x_n)\}$ converges to $g(x)$ by continuity of $f, g$.

That is, for any $\epsilon > 0$, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(x_n) - f(x)| < \frac{\epsilon}{2}$ and $|g(x_n) - g(x)| < \frac{\epsilon}{2}$ whenever $|x_n - x| < \delta$. By the triangle inequality,

$$\begin{aligned}
|(f+g)(x_n) - (f+g)(x))| &= |(f(x_n) + g(x_n)) - (f(x) + g(x))| \\
&\leq |f(x_n) - f(x)| + |g(x_n) + g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Fix any $\epsilon > 0$. By continuity of $f, g$, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(y) - f(x)| < \frac{\epsilon}{2|g(x)|}$ and $|g(y) - g(x)| < \frac{\epsilon}{2|f(y)|}$ whenever $|y - x| < \delta$.

$$\begin{aligned}
|fg(y) - fg(x)| &= |f(y)g(y) - f(x)g(x)| \\
&= |f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x)| \\
&= |f(y)(g(y) - g(x)) + g(x)(f(y) - f(x))| \\
&\leq |f(y)(g(y) - g(x))| + |g(x)(f(y) - f(x))| \\
&= |f(y)||g(y) - g(x)| + |g(x)||f(y) - f(x)| \\
&< |f(y)|\frac{\epsilon}{2|f(y)|} + |g(x)|\frac{\epsilon}{2|g(x)|} \\
&= \epsilon
\end{aligned}$$

(This one is a bit janky).

(ii) If $h$ is a continuous function with image contained in $E$, show that the composition $f \circ h$ is continuous.

Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{h(x_n)\} \in E$ converges to $h(x) \in E$ by continuity of $h$. Then $\{f \circ h(x_n)\} = \{f(h(x_n))\}$ converges to $f \circ h(x) = f(h(x))$ by continuity of $f$. Therefore the composition is continuous.

(iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.

Fix $\epsilon > 0$. By continuity of $f, g$, there exists a $0 < \delta \leq \delta_f, \delta_g$ s.t. whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2}$.

We can write

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}.$$

This is by the identity:

$$\begin{aligned}
\max(x, y) + \min(x, y) &= x + y \\
\max(x, y) - \min(x, y) &= |x - y| \\
\max(x, y) &= \frac{1}{2}(x + y + |x - y|) \\
\min(x, y) &= \frac{1}{2}(x + y - |x - y|)
\end{aligned}$$

Now, $|\max\{f(x), g(x)\} - \max\{f(y), g(y)\}|$ is equal to

$$\left|\frac{f(x)+g(x)}{2} + \frac{|f(x)-g(x)|}{2} - (\frac{f(y)+g(y)}{2} + \frac{|f(y)-g(y)|}{2})\right|$$
$$= \left|\frac{f(x)-f(y)+g(x)-g(y)+|f(x)-g(x)|-|f(y)-g(y)|}{2}\right|$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|\,|f(x)-g(x)|-|f(y)-g(y)|\,|}{2}$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-g(x)-f(y)+g(y)|}{2}$$
$$\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-f(y)|+|g(y)+g(x)|}{2}$$
$$< \frac{\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}}{2}$$
$$= \epsilon.$$

(iv) Show that $|f|$ is continuous.
For any $\epsilon > 0$, there exists a delta such that whenever $|x - y| < \delta$, by the reverse triangle inequality:

$$|\,|f(x)| - |f(y)|\,| \leq |f(x) - f(y)| < \epsilon.$$

50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

Lipschitz: there exists $L \geq 0$ s.t. for all $x, x'$:

$$|f(x) - f(x')| \leq L|x - x'|$$

Fixing any $\epsilon > 0$, whenever $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq L|x - x'| < L\delta,$$

so we can set $\delta = \dfrac{\epsilon}{L}$. The $\delta$ is the same for any values of $x$, so $f$ is uniformly continuous.

The function $\sqrt{x}$ is uniformly continuous but not Lipschitz.

51. A continuous function $\phi$ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which $\phi$ is linear on each interval $[x_i, x_{i+1}]$. Let $f$ be a continuous function on $[a, b]$ and $\epsilon$ a positive number. Show that there is a piecewise linear function $\phi$ on $[a, b]$ with $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a, b]$.
Start with $f(x_0)$, and choose $x_1$ so that $f(x_1) = f(x_0) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0) + f(x_0)$ on $[x_0, x_1]$.
Repeat this process to choose each interval:
Start with $f(x_i)$, and choose $x_{i+1}$ so that $f(x_{i+1}) = f(x_i) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}(x - x_i) + f(x_i)$ on $[x_i, x_{i+1}]$.
Then we see that $f$ and $\phi$ are always within $\epsilon$ of each other, and $\phi$ is continuous and piecewise linear.

52. Show that a nonempty set $E$ of real numbers is closed and bounded if and only if every continuous real-valued function on $E$ takes a maximum value.

Let $E$ be a nonempty set of real numbers.
( $\Longrightarrow$ ) Suppose $E$ is closed and bounded.
By the Extreme Value Theorem, every continuous real-valued function on $E$ takes a maximum (and minimum) value.
( $\Longleftarrow$ ) Suppose every continuous real-valued function on $E$ takes a maximum value.
Suppose that $E$ is not closed. The continuous real-valued function $f(x) = \frac{1}{x}$ on the open set $E = (0, 1)$ does not take on a maximum value. Contradiction.
Suppose $E$ is not bounded. The continuous real-valued function $f(x) = x^2$ on the unbounded set $E = [0, \infty)$ does not take on a maximum value. Contradiction.
Therefore $E$ must be closed and bounded. (Not the right way to do this...)

53. Show that a set $E$ of real numbers is closed and bounded iff every open cover of $E$ has a finite subcover.

    Let $E$ be a set of real numbers.

    ( $\Longrightarrow$ ) Suppose $E$ is closed and bounded.

    By the Heine-Borel Theorem, every open cover of $E$ has a finite subcover.

    ( $\Longleftarrow$ ) Suppose every open cover of $E$ has a finite subcover.

    See the proof in 1.4 after the Heine-Borel Theorem.

54. Show that a nonempty set $E$ of real numbers is an interval iff every continuous real-valued function on $E$ has an interval as its image.
    Let $E$ be a nonempty set of real numbers.
    ( $\Longrightarrow$ ) Suppose $E$ is an interval.
    Then for any two points $x, y \in E$, the set $[x, y]$ is in $E$. Let $f$ be a continuous real-valued function on $E$. Now, we have $f(x), f(y)$ in the image of $f$. Suppose, without loss of generality, that $f(x) < f(y)$. By the Intermediate Value Theorem, for any $c$ such that $f(x) < c < f(y)$, there exists $x_0 \in (x, y) \subseteq E$ such that $f(x_0) = c$. That is, for any two points in the image of $f$, every point between them is also in the image of $f$. Therefore the image of $f$ is an interval.
    ( $\Longleftarrow$ ) Suppose every continuous real-valued function on $E$ has an interval as its image.
    Suppose $E$ is not an interval. Then there exist two points $x, y \in E$ such that $x < a < y$ but $a \notin E$. Let $f$ be a continuous real-valued function on $E$, and without loss of generality, let $f$ be monotonically increasing. Because $x, y \in E$, then $f(x)$,$f(y)$ are defined, so $[f(x), f(y)]$ is in the image of $f$.
    Define two disjoint collections of subsets of E: $I_{<a} = \{I \subseteq E \mid x < a \ \forall x \in I\}$ and $I_{>a} = \{I \subseteq E \mid x > a \ \forall x \in I\}$, so that $I_{<a} \cap I_{>a} = \emptyset$. These collections are nonempty because $\{x\} \in I_{<a}$ and $\{y\} \in I_{>a}$. Consider $\bigcup I_{<a} \subseteq E$, the union of all elements of $I_{<a}$, and $\bigcup I_{>a} \subseteq E$, the union of all elements of $I_{>a}$. By monotonicity of $f$, $f(\bigcup I_{<a}) < f(\bigcup I_{>a})$, so $[f(x), f(y)] \not\subseteq f(\bigcup I_{<a}) \bigcup f(\bigcup I_{>a}) = f(E)$, a contradiction.

55. Show that a monotone function on an open interval is continuous iff its image is an interval.
    Let $f$ be a monotone function on an open interval $E = (a, b)$.
    ( $\Longrightarrow$ ) Suppose $f$ is continuous.
    Then by Problem 54, $E$ is an interval implies that the continuous real-valued function $f$ has an interval as its image.
    ( $\Longleftarrow$ ) Suppose the image of $f$ is an interval.
    Let $x_0$ be a point in the open interval $E$, so that $f(x_0)$ is defined. For any sequence $\{x_n\}$ in $E \cap (x_0, \infty)$ that converges to $x_0$, then $\{f(x_n)\}$ converges to $f(x_0^+)$.
    Similarly, for any sequence $\{x_n\}$ in $E \cap (-\infty, x_0)$ that converges to $x_0$, then $\{f(x_n)\}$ converges to$f(x_0^-)$.

Then $f(x_0^-) = f(x_0) = f(x_0^+)$ by monotonicity.
(messy)

56. Let $f$ be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which $f$ is continuous is a $G_\delta$ set.
A $G_\delta$ set is a set that is a countable intersection of open sets.
$f$ is continuous at a point $x$ if for any open set in the image containing $f(x)$, the inverse image is an open set containing $x$.

57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points $x$ at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of $F_\sigma$ sets.
An $F_\sigma$ set is a set that is a countable union of closed sets.

58. Let $f$ be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to $f$ of an open set is open, of a closed set is closed, and of a Borel set is Borel.
The inverse image of an open set is open (See prop 22).
Suppose that the inverse image of a closed set is not closed. That is, let $B$ be a closed set of real numbers and let $f^{-1}(B) = \{x \in \mathbb{R} \mid f(x) \in B\}$ not be closed. Then there exists a sequence $x_n \in f^{-1}(B)$ that converges to $x \notin f^{-1}(B)$. However, by continuity of $f$, $f(x_n) \in B$ converges to $f(x) \notin B$. This implies that $B$ does not contain all its limit points, and thus $B$ is not closed, a contradiction. Therefore $f^{-1}(B)$ must be closed.
Another way:
We have, for any open set $\mathcal{O} \in \mathbb{R}$,
$$\mathbb{R} = dom(f) = f^{-1}(\mathbb{R}) = f^{-1}(\mathcal{O} \cup \mathcal{O}^c) = f^{-1}(\mathcal{O}) \cup f^{-1}(\mathcal{O}^c).$$
Because $f^{-1}(\mathcal{O})$ is open in $\mathbb{R}$, then $f^{-1}(\mathcal{O}^c)$ is closed in $\mathbb{R}$.

59. A sequence $\{f_n\}$ of real-valued functions defined on a set $E$ is said to converge uniformly on $E$ to a function $f$ iff given $\epsilon > 0$, there is an $N$ such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set $E$. Prove that if $\{f_n\}$ converges uniformly to $f$ on $E$, then $f$ is continuous on $E$.
We want to show that uniform convergence preserves continuity.
Fix $\epsilon > 0$.
By uniform convergence of $\{f_n\}$, there exists an index $N$ such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all $x \in E$ and all $n \geq N$.
By continuity of each $f_n$, for all $x \in E$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $|x - y| < \delta$.
Therefore we have:
$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

# Chapter 2

# Lebesgue Measure

### Contents



## 2.1 Introduction

In this chapter we construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure**, denoted by $m$. (A *set function* is a function that associates an extended real number to each set in a collection of sets.) The collection of Lebesgue measurable sets is a $\sigma$-algebra which contains all open sets and all closed sets. The set function $m$ possesses the following three properties:

**The measure of an interval is its length.** *Each nonempty interval $I$ is Lebesgue measurable and*

$$m(I) = \ell(I).$$

**Measure is translation invariant.** *If $E$ is Lebesgue measurable and $y$ is any number then the translate of $E$ by $y$, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and*

$$m(E + y) = m(E).$$

**Measure is countably additive over countable disjoint unions of sets.** *If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then*

$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (See Vitali sets). We first construct a set function called **outer measure**, denoted by $m^*$, such that:

(i) the outer measure of an interval is its length:

$$m^*(I) = \ell(I).$$

(ii) outer measure is translation invariant:

$$m^*(A + y) = m^*(A).$$

(iii) outer measure is countably subadditive:

$$m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k).$$

Outer measure is defined for all sets of real numbers. However, outer measure fails to be countably additive: there exists $A, B$ disjoint s.t. $m^*(A \cup B) < m^*(A) + m^*(B)$.
Then the Lebesgue measure $m$ is the restriction of $m^*$ to the Lebesgue measurable sets.

**PROBLEMS**

In the first three problems, let $m$ be a set function defined for all sets in a $\sigma$-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume $m$ is countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1. Prove that if $A$ and $B$ are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

   $A \subseteq B \implies B = A \cup (B \cap A^c)$, where $A \cap (B \cap A^c) = \emptyset$. The set $(B \cap A^c)$ is measurable because $A^c$ is measurable and countable intersection is measurable, so $m(B) = m(A \cup (B \cap A^c)) = m(A) + m(B \cap A^c)$ by countable additivity, and thus $m(B) \geq m(A)$.

2. Prove that if there is a set $A$ in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

   We have $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

$$\begin{aligned} m(A) &= m(A \cup \emptyset) \\ m(A) &= m(A) + m(\emptyset) && \text{by disjoint additivity} \\ 0 &= m(\emptyset). \end{aligned}$$

3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$. For any two measurable sets $A, B$, we have $A \cup B = (A \setminus B) \cup (B)$. By disjoint additivity,

$$m(A \cup B) = m(A \setminus B) + m(B)$$

   Now, by problem 1, $(A \setminus B) \subseteq A$ implies that $m(A \setminus B) \leq m(A)$. Therefore

$$m(A \cup B) \leq m(A) + m(B).$$

4. A set function $c$, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be $\infty$ if $E$ has infinitely many members and $c(E)$ to be equal to the number of elements in $E$ if $E$ is finite; define $c(\emptyset) = 0$. Show that $c$ is a countably additive and translation invariant set function. This set function is called the **counting measure**.
   Suppose $E = \{x_1, \cdots, x_n\}$.
   Then $m(E) = n$. For any real number $y$, $y + E = \{y + x_1, \cdots, y + x_n\}$, so $m(y + E) = n$.

Suppose $E$ has infinitely many members.
Then $y+E$ has infinitely members as well, so $m(E) = m(y+E) = \infty$.
Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of sets of real numbers. In the case that there exists an $E_k$ with infinitely many members, then the countable additivity is clear.
In the case that all sets $E_k$ are finite, for any two sets $E_i, E_j$:
$E_i = \{x_1, \cdots, x_n\}$
$E_j = \{y_1, \cdots, y_m\}$
Then $E_i \cup E_j = \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ and $m(E_i \cup E_j) = n + m = m(E_i) + m(E_j)$.

## 2.2 Lebesgue Outer Measure

Let $I$ be a nonempty interval of real numbers. We define its length:

$$\ell(I) = \begin{cases} \infty & \text{if } I \text{ is unbounded} \\ b-a & \text{endpoints } a, b \end{cases}$$

For a set $A$ of real numbers, consider the countable collections $\{I_k\}_{k=1}^{\infty}$ of nonempty open, bounded intervals that cover $A$; that is, collections for which $A \subseteq \bigcup_{k=1}^{\infty} I_k$. For each such collection, consider the sum of the lengths of the intervals in the collection. Since the lengths are positive numbers, each sum is uniquely defined independently of the order of the terms. We define the **outer measure** of $A$, $m^*(A)$, to be the infimum of all such sums, that is

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \ell(I_k) \;\middle|\; A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}.$$

It follows immediately from the definition of outer measure that $m^*(\emptyset) = 0$. Moreover, since any cover of a set $B$ is also a cover of any subset of $B$, outer measure is **monotone** in the sense that

$$A \subseteq B \implies m^*(A) \leq m^*(B).$$

Then because $\emptyset \subseteq A$ for any set $A$, we have $0 = m^*(\emptyset) \leq m^*(A)$.

**Example.** *A countable set $C$ has outer measure zero.*
*Because $C$ is countable, enumerate $C$ such that $C = \{c_k\}_{k=1}^{\infty}$. Fix $\epsilon > 0$. For each $k \in \mathbb{N}$, define an open interval $I_k = (c_k - \frac{\epsilon}{2^{k+1}}, c_k + \frac{\epsilon}{2^{k+1}})$. Then $C \subseteq \bigcup_{k=1}^{\infty} I_k$. Therefore we have, by definition of infimum,*

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} \ell(I_k) = \sum_{k=1}^{\infty} \frac{2\epsilon}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon.$$

*This inequality holds for each $\epsilon > 0$; thus $m^*(C) = 0$.*

**Lemma.** $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$.

*Proof.* To show that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$ (induction).
Let $P(n)$ be the assertion that $\sum_{k=1}^{n} \frac{1}{2^k} = 1 - \frac{1}{2^n}$ for $n \in \mathbb{N}$.
$P(1)$:

$$\sum_{k=1}^{1} \frac{1}{2^k} = \frac{1}{2} = 1 - \frac{1}{2^1}.$$

$P(2)$:

$$\sum_{k=1}^{2} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 1 - \frac{1}{2^2}.$$

Suppose $P(m)$ is true for $m \geq 1$; that is, $\sum_{k=1}^{m} \frac{1}{2^k} = 1 - \frac{1}{2^m}$.
$P(m+1)$:

$$\begin{aligned}
\sum_{k=1}^{m+1} \frac{1}{2^k} &= \sum_{k=1}^{m} \frac{1}{2^k} + \frac{1}{2^{m+1}} \\
&= 1 - \frac{1}{2^m} + \frac{1}{2^{m+1}} \\
&= 1 - \frac{2}{2^{m+1}} + \frac{1}{2^{m+1}} \\
&= 1 - \frac{1}{2^{m+1}}.
\end{aligned}$$

Therefore $P(m)$ is true for all $m \geq 1$.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} (1 - \frac{1}{2^n}) = 1.$$

(An alternate proof would be to see that we have a sequence of partial sums that is monotonic with 1 as the supremum. Then the sequence of partial sums converges to 1 and the series is summable to 1.) □

**PROBLEMS**

5. By using properties of outer measure, prove that the interval $[0,1]$ is not countable.
   Suppose that the interval $[0,1]$ is countable. By an example above, we showed that a countable set has outer measure zero, so $m^*([0,1]) = 0$. Also, the outer measure of an interval is its length. Then $m^*([0,1]) = 1$, and we reach a contradiction.

6. Let $A$ be the set of irrational numbers in the interval $[0,1]$. Prove that $m^*(A) = 1$.
   Let $A = [0,1] \cap \mathbb{Q}^c$.
   Then $A \subseteq [0,1]$, so by monotonicity of outer measure,

$$\begin{aligned}
m^*(A) &\leq m^*([0,1]) \\
m^*(A) &\leq 1.
\end{aligned}$$

   Also, we have

$$\begin{aligned}
[0,1] &= ([0,1] \cap \mathbb{Q}^c) \cup ([0,1] \cap \mathbb{Q}) && \\
[0,1] &= A \cup ([0,1] \cap \mathbb{Q}) && \\
[0,1] &\subseteq A \cup ([0,1] \cap \mathbb{Q}) && A = B \implies A \subseteq B \text{ and } A \supseteq B \\
m^*([0,1]) &\leq m^*(A \cup (m^*([0,1] \cap \mathbb{Q})) && \text{by monotonicity} \\
m^*([0,1]) &\leq m^*(A) + m^*([0,1] \cap \mathbb{Q}) && \text{by countable subadditvity} \\
m^*([0,1]) &\leq m^*(A) + 0 && \text{countable set has outer measure zero} \\
1 &\leq m^*(A). && \text{outer measure of interval is length}
\end{aligned}$$

   Then $m^*(A) \leq 1$ and $1 \leq m^*(A)$ imply that $m^*(A) = 1$.

7. A set of real numbers is said to be a $G_\delta$ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set $E$, there is a $G_\delta$ set $G$ for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

Suppose $E$ is a bounded set of real numbers.
Then there exists a real number $M$ for which $|x| \leq M$ for all $x \in E$; that is, $E \subseteq [-M, M]$. By monotonicity of outer measure, $m^*(E) \leq m^*([-M, M]) = 2M < \infty$, and the outer measure of $E$ is finite.
Now, because outer measure is defined as $m^*(E) = \inf\{\sum_{k=1}^{\infty} \ell(I_k) \mid E \subseteq \bigcup_{k=1}^{\infty} I_k\}$, we have that $m^*(E)$ is the greatest lower bound, so for a natural number $n$, $m^*(E) + \frac{1}{n}$ is not a lower bound.
That is, there exists a countable sequence of open intervals $\{(I_n)_k\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty} (I_n)_k$ and

$$m^*(E) \leq \sum_{k=1}^{\infty} \ell((I_n)_k) < m^*(E) + \frac{1}{n}. \tag{1}$$

Now, for each natural number $n$, we can define the open set

$$\mathcal{O}_n := \bigcup_{k=1}^{\infty} (I_n)_k. \tag{2}$$

Also define the countable intersection of open sets; i.e., a $G_\delta$ set:

$$\mathcal{O} := \bigcap_{n=1}^{\infty} \mathcal{O}_n.$$

Then because we have $E \subseteq \mathcal{O}_n$ for every $n$, this implies $E \subseteq \bigcap_{n=1}^{\infty} \mathcal{O}_n = \mathcal{O}$.

$$\begin{aligned}
m^*(E) &\leq m^*(\mathcal{O}) && \text{by monotonicity of outer measure: } E \subseteq \mathcal{O} \\
&\leq m^*(\mathcal{O}_n) && \text{by monotonicity of outer measure: } \mathcal{O} = \bigcap_{n=1}^{\infty} \mathcal{O}_n \subseteq \mathcal{O}_n \\
&= m^*(\bigcup_{k=1}^{\infty} (I_n)_k) && \text{by (2)} \\
&\leq \sum_{k=1}^{\infty} \ell((I_n)_k) && \text{by countable subadditivity of outer measure} \\
&< m^*(E) + \frac{1}{n}. && \text{by (1)}
\end{aligned}$$

Therefore for any natural number $n$,

$$m^*(E) \leq m^*(\mathcal{O}) < m^*(E) + \frac{1}{n}.$$

Taking the limit as $n \to \infty$, we get that $m^*(E) = m^*(\mathcal{O})$.
Therefore there exists a $G_\delta$ set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m^*(E) = m^*(\mathcal{O})$.

8. Let $B$ be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^{n}$ be a finite collection of open intervals that covers $B$. Prove that $\sum_{k=1}^{n} m^*(I_k) \geq 1$.
The rational numbers are dense in the reals; that is, between any two real numbers, there exists a

rational number. Therefore, the rational numbers are also dense in the real subset $[0,1]$: between any two numbers in $[0,1]$, there exists a rational number.
In the case that $[0,1] \subseteq \bigcup_{k=1}^n I_k$, the inequality is clear by monotonicity and subadditivity:

$$1 = m^*([0,1]) \leq m^*(\bigcup_{k=1}^n I_k) \leq \sum_{k=1}^n m^*(I_k).$$

In the case that $[0,1] \not\subseteq \bigcup_{k=1}^n I_k$, then

$$(\bigcup_{k=1}^n I_k)^c \cap [0,1] = (\bigcap_{k=1}^n I_k^c) \cap [0,1] = \bigcap_{k=1}^n (I_k^c \cap [0,1]) \neq \emptyset.$$

We want to show that $\bigcap_{k=1}^n I_k^c \cap [0,1]$ is countable so that $m^*(\bigcap_{k=1}^n I_k^c \cap [0,1]) = 0$.
Because each $I_k^c \cap [0,1]$ is a closed interval (of irrational numbers), the intersection is also a closed interval (nonempty by assumption); that is, $\bigcap_{k=1}^n (I_k^c \cap [0,1]) = [a,b]$ for some $a \leq b$. Suppose by contradiction that $\bigcap_{k=1}^n (I_k^c \cap [0,1])$ is not countable. Then we have that $a < b$. However, by density of the rationals, there exists a rational between $[a,b]$, leading to a contradiction.
Therefore $\bigcap_{k=1}^n (I_k^c \cap [0,1]) = \{x\}$, where $x \in \mathbb{Q}^c$, and $\bigcap_{k=1}^n (I_k^c \cap [0,1])$ is countable.
Now we can write

$$\begin{aligned}
[0,1] &= (\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1]) && \\
[0,1] &\subseteq (\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1]) && A = B \implies A \subseteq B \text{ and } A \supseteq B \\
m^*([0,1]) &\leq m^*((\bigcup_{k=1}^n I_k \cap [0,1]) \cup (\bigcap_{k=1}^n I_k^c \cap [0,1])) && \text{by monotonicity} \\
m^*([0,1]) &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) + m^*(\bigcap_{k=1}^n I_k^c \cap [0,1]) && \text{by countable subadditivity} \\
m^*([0,1]) &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) + 0 && \text{the outer measure of a countable set is zero} \\
1 &\leq m^*(\bigcup_{k=1}^n I_k \cap [0,1]) && \\
1 &\leq m^*(\bigcup_{k=1}^n I_k) && \text{by monotonicity: } \bigcup_{k=1}^n I_k \cap [0,1] \subseteq [0,1] \\
1 &\leq \sum_{k=1}^n m^*(I_k). && \text{by countable subadditivity}
\end{aligned}$$

9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

$$\begin{aligned}
m^*(A \cup B) &\leq m^*(A) + m^*(B) && \text{by countable subadditivity} \\
m^*(A \cup B) &\leq m^*(B) && \text{because } m^*(A) = 0.
\end{aligned}$$

Also, we have $B \subseteq A \cup B$, so by monotonicity of outer measure,

$$m^*(B) \leq m^*(A \cup B).$$

Then $m^*(A \cup B) \leq m^*(B)$ and $m^*(B) \leq m^*(A \cup B)$ imply that $m^*(A \cup B) = m^*(B)$.

10. Let $A$ and $B$ be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$.
By countable subadditivity of outer measure, $m^*(A \cup B) \leq m^*(A) + m^*(B)$.
We can see that $A$ and $B$ are disjoint: Suppose by contradiction that $A, B$ are not disjoint. Then there exists a real number $x$ such that $x \in A$ and $x \in B$. But then $|x - x| = 0 < \alpha$, a contradiction.
Let $\epsilon$ such that $\alpha/2 > \epsilon > 0$. By definition of outer measure and infimum, there exists a countable sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $(A \cup B) \subseteq \bigcup_{k=1}^{\infty} I_k$ and
$$m^*(A \cup B) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(A \cup B) + \epsilon. \tag{1}$$
Now, each $I_k$ is such that $A \cap I_k \neq \emptyset$ or $B \cap I_k \neq \emptyset$, but not both.
To see this, suppose by contradiction that there exists an $I_k$ such that $A \cap I_k \neq \emptyset$ and $B \cap I_k \neq \emptyset$. Then there exists $a, b \in I_k$ such that $a \in A$ and $b \in B$. Without loss of generality, suppose that these are the closest two elements of $A$ and $B$, and suppose $a < b$. Then the interval $(a, b)$ contains no elements of $A$ or $B$, and $m^*(b - a) \geq \alpha > \alpha/2$. This is a contradiction to the fact that $\sum_{k=1}^{\infty} \ell(I_k)$ is within $\alpha/2$ of $m^*(A \cup B)$.
We can then separate $\{I_k\}_{k=1}^{\infty}$ into two subsequences $\{(I_A)_i\}_{i=1}^{\infty}$ and $\{(I_B)_j\}_{j=1}^{\infty}$ such that $A \subseteq \bigcup_{i=1}^{\infty}(I_A)_i$ and $B \subseteq \bigcup_{j=1}^{\infty}(I_B)_j$. Then because the sum is uniquely defined independently of the order of terms, $\sum_{k=1}^{\infty} \ell(I_k) = \sum_{i=1}^{\infty} \ell((I_A)_i) + \sum_{j=1}^{\infty} \ell((I_B)_j)$.
Therefore we can write
$$\begin{aligned}
m^*(A \cup B) &\leq m^*(A) + m^*(B) && \text{by countable subadditivity of outer measure}\\
&\leq m^*\Big(\bigcup_{i=1}^{\infty}(I_A)_i\Big) + m^*\Big(\bigcup_{j=1}^{\infty}(I_B)_j\Big) && \text{by monotonicity of outer measure}\\
&\leq \sum_{i=1}^{\infty} \ell((I_A)_i) + \sum_{j=1}^{\infty} \ell((I_B)_j) && \text{by countable subadditivity of outer measure}\\
&= \sum_{k=1}^{\infty} \ell(I_k) && \text{rearranging the sum}\\
&< m^*(A \cup B) + \epsilon && \text{by (1)}
\end{aligned}$$
Therefore for any $\epsilon$,
$$m^*(A \cup B) \leq m^*(A) + m^*(B) < m^*(A \cup B) + \epsilon,$$
thus $m^*(A \cup B) = m^*(A) + m^*(B)$.

## 2.3 The $\sigma$-Algebra of Lebesgue Measurable Sets

Outer measure is defined for all sets of real numbers, the outer measure of an interval is its length, outer measure is countably subadditive, and outer measure is translation invariant. However, outer measure fails to be countably additive or even finitely additive. That is, there exists disjoint sets $A, B$ such that
$$m^*(A \cup B) < m^*(A) + m^*(B). \tag{1}$$
We identify a $\sigma$-algebra of sets, called the Lebesgue measurable sets, which contains all intervals and all open sets and has the property that the restriction of the set function outer measure to the collection of Lebesgue measurable sets is countably additive.

**Definition.** *A set $E$ is said to be **measurable** provided for any set $A$,*

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

We see that the strict inequality (1) cannot occur if one of the sets is measurable:
Suppose $A$ is measurable and $B$ is any set disjoint from $A$.

$$\begin{aligned}
m^*(A \cup B) &= m^*([A \cup B] \cap A) + m^*([A \cup B] \cap A^c) && \text{by definition of } A \text{ measurable} \\
&= m^*(A) + m^*([A \cap A^c] \cup [B \cap A^c]) && \text{left: absorbtion, right: distributive property} \\
&= m^*(A) + m^*(\emptyset \cup [B \setminus A]) && \text{complement and def of set difference} \\
&= m^*(A) + m^*(B). && \text{identity of union and set difference of disjoint sets}
\end{aligned}$$

Suppose we want to prove that a set $E$ is measurable.
We already have that for any set $A$,

$$\begin{aligned}
m^*(A) &= m^*([A \cap E] \cup [A \cap E^c]) && \text{by set properties} \\
m^*(A) &\leq m^*(A \cap E) + m^*(A \cap E^c). && \text{by subadditivity of outer measure}
\end{aligned}$$

Therefore to show that $E$ is measurable, it suffices to show the other inequality:

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c). \tag{2}$$

This inequality holds trivially if $m^*(A) = \infty$. Therefore we need only prove (2) for sets $A$ that have finite outer measure.

**Proposition 4.** *Any set of outer measure zero is measurable. In particular, any countable set is measurable.*

*Proof.* Let $E$ be such that $m^*(E) = 0$. Let $A$ be any set.

- $A \cap E \subseteq E$
- $A \cap E^c \subseteq A$

By monotonicity of outer measure,

$$\begin{aligned}
m^*(A \cap E) &\leq m^*(E) = 0 \\
m^*(A \cap E^c) &\leq m^*(A)
\end{aligned}$$

Therefore

$$\begin{aligned}
m^*(A) &\geq m^*(A \cap E^c) + 0 \\
m^*(A) &\geq m^*(A \cap E^c) + m^*(A \cap E).
\end{aligned}$$

□

Every open set is the disjoint union of a countable collection of open intervals. Every interval is measurable, and the countable union of measurable sets is measurable, so all open sets are measurable. By complement, all closed sets are measurable. In the same way, all $G_\delta$ sets and all $F_\sigma$ sets are measurable.

The intersection of all the $\sigma$-algebras of subsets of $\mathbb{R}$ that contain the open sets is a $\sigma$-algebra called the **Borel $\sigma$-algebra**, members of this collection are called **Borel sets**. That is, the Borel sigma-algebra is the sigma-algebra generated by the open sets.

**Lemma 1.** *The set of all subsets of $X$, $\mathcal{P}(X)$ (or $2^X$), is a $\sigma$-algebra of subsets of $X$.*

*Proof.* Let $X$ be any set.

(i) $X \in \mathcal{P}(X)$.

(ii) if $A \in \mathcal{P}(X)$, then $A^c = X \setminus A = \{x \in X \mid x \notin A\} \in \mathcal{P}(X)$.

(iii) if $A_i \in \mathcal{P}(X)$, then $\bigcup_{i=1}^{\infty} A_i = \{x \in X \mid x \in A_i \text{ for some } i\}$.

□

**Lemma 2.** *Given any collection of $\sigma$-algebras $\{\mathcal{F}_\alpha\}_{\alpha \in \mathcal{A}}$ of $X$, the intersection $\bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$ is also a $\sigma$-algebra.*

*Proof.* Let $X$ be any set.

(i) $X \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies X \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

(ii) $A \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha \implies A \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies A^c \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies A^c \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

(iii) $A_i \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha \implies A_i \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A \in \mathcal{F}_\alpha, \forall \alpha \in \mathcal{A} \implies \bigcup_{i=1}^{\infty} A \in \bigcap_{\alpha \in \mathcal{A}} \mathcal{F}_\alpha$.

□

**Theorem.** *Given any collection $\mathcal{C}$ of subsets of $X$, there exists a smallest $\sigma$-algebra containing $\mathcal{C}$. (This is called the $\sigma$-algebra generated by $\mathcal{C}$.)*

*Proof.* Consider $S = \{\mathcal{F} \mid \mathcal{C} \in \mathcal{F}, \mathcal{F} \text{ is a } \sigma\text{-algebra of } X\}$.
Now, $S$ is nonempty because $\mathcal{C} \in \mathcal{P}(X)$ and by Lemma 1, $\mathcal{P}(X)$ is a $\sigma$-algebra of $X$; therefore $\mathcal{P}(X) \in S$.
Consider $\bigcap S$, the intersection of all the elements of $S$.

1. By Lemma 2, $\bigcap S$ is a $\sigma$-algebra,
2. $\mathcal{C} \in \mathcal{F}, \forall \mathcal{F} \in S \implies \mathcal{C} \in \bigcap S$, so $\bigcap S$ is a $\sigma$-algebra that contains $\mathcal{C}$,
3. $\bigcap S \subseteq \mathcal{F}$ for any $\mathcal{F} \in S$ by def of intersection, so $\bigcap S$ is the smallest $\sigma$-algebra containing $\mathcal{C}$.

□

**Proposition 10.** *The translate of a measurable set is measurable.*

*Proof.* Let $E$ be measurable, let $A$ be any set, and let $y$ be any real number.
First we need to see that

$$(A \cap [E+y]) - y = \{x : x \in A, \text{ and } x \in E+y\} - y = \{x : x \in A-y \text{ and } x \in E\} = [A-y] \cap E$$
$$(A \cap [E+y]^c) - y = \{x : x \in A, \text{ and } x \notin E+y\} - y = \{x : x \in A-y \text{ and } x \notin E\} = [A-y] \cap E^c$$

Now, we have

$$\begin{aligned} m^*(A) &= m^*(A-y) && \text{outer measure is translation invariant} \\ &= m^*([A-y] \cap E) + m^*([A-y] \cap E^c) && \text{because } E \text{ is measurable} \\ &= m^*(A \cap [E+y] - y) + m^*(A \cap [E+y]^c - y) && \text{by above} \\ &= m^*(A \cap [E+y]) + m^*(A \cap [E+y]^c). && \text{outer measure is translation invariant} \end{aligned}$$

Therefore $E+y$ is measurable. □

**PROBLEMS**

11. Prove that if a $\sigma$-algebra of subsets of $\mathbb{R}$ contains intervals of the form $(a, \infty)$, then it contains all intervals.
Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of $\mathbb{R}$.
Suppose that for any real number $a$, the interval $(a, \infty) \in \mathcal{M}$.
For any real number $b$, because $\mathcal{M}$ is closed under complements,
$$(b, \infty) \in \mathcal{M} \implies (b, \infty)^c = (-\infty, b] \in \mathcal{M}.$$
For any natural number $n$, because $\mathcal{M}$ is closed under intersections:
$$(a - \frac{1}{n}, \infty), (-\infty, b] \in \mathcal{M} \implies (a - \frac{1}{n}, \infty) \cap (-\infty, b] = (a - \frac{1}{n}, b] \in \mathcal{M},$$
$$(a, \infty), (-\infty, b - \frac{1}{n}] \in \mathcal{M} \implies (a, \infty) \cap (-\infty, b - \frac{1}{n}] = (a, b - \frac{1}{n}] \in \mathcal{M}.$$
Because $\mathcal{M}$ is closed under countable intersections and countable unions:
$$\text{for any } n \in \mathbb{N}, (a - \frac{1}{n}, b] \in \mathcal{M} \implies \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] = [a, b] \in \mathcal{M},$$
$$\text{for any } n \in \mathbb{N}, (a, b - \frac{1}{n}] \in \mathcal{M} \implies \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] = (a, b) \in \mathcal{M}.$$
In short, for any real numbers $a, b$, we have
$$[a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \cap (-\infty, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, \infty) \cap (b, \infty)$$
$$(a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty} (a, \infty) \cap (-\infty, b - \frac{1}{n}] = \bigcup_{n=1}^{\infty} (a, \infty) \cap (b - \frac{1}{n}, \infty)$$
The construction of intervals of the form $(a, b]$ and $[a, b)$ is similar.

12. Show that every interval is a Borel set.
Because any interval of the form $(a, \infty)$ is open, $(a, \infty)$ is a Borel set; i.e., it is a member of the Borel sigma-algebra. By the previous problem 11, any sigma-algebra that contains intervals of the form $(a, \infty)$ contains all intervals. Therefore the Borel sigma-algebra contains all intervals and thus all intervals are Borel sets.

13. Show that

(i) the translate of an $F_\sigma$ set is also $F_\sigma$,
Let $F$ be an $F_\sigma$ set, that is, $F = \bigcup_{n=1}^{\infty} F_n$, with $F_n$ closed.
For any real number $y$,
$$\begin{aligned} F + y &= (\bigcup_{n=1}^{\infty} F_n) + y \\ &= \{x : x \in F_n \text{ for some } n\} + y \\ &= \{x : x \in F_n + y \text{ for some } n\} \\ &= \bigcup_{n=1}^{\infty} (F_n + y) \end{aligned}$$
The translate of any closed set is closed, so this is still an $F_\sigma$ set.

(ii) the translate of a $G_\delta$ set is also $G_\delta$,
Let $\mathcal{O}$ be a $G_\delta$ set, that is, $\mathcal{O} = \bigcap_{n=1}^\infty \mathcal{O}_n$, with $\mathcal{O}_n$ open.
For any real number $y$,

$$\begin{aligned}\mathcal{O} + y &= (\bigcap_{n=1}^\infty \mathcal{O}_n) + y \\ &= \{x : x \in \mathcal{O}_n \text{ for all } n\} + y \\ &= \{x : x \in \mathcal{O}_n + y \text{ for all } n\} \\ &= \bigcap_{n=1}^\infty (\mathcal{O}_n + y)\end{aligned}$$

The translate of any open set is open, so this is still a $G_\delta$ set.

(iii) the translate of a set of measure zero also has measure zero.
Let $E$ be a set of measure zero. That is, $m^*(E) = 0$.
For any $\epsilon > 0$, by definition of infimum, there exists a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ such that

$$m^*(E) \le \sum_{k=1}^\infty \ell(I_k) < m^*(E) + \epsilon,$$

Thus because the outer measure is zero,

$$\sum_{k=1}^\infty \ell(I_k) < \epsilon.$$

Now, for any real number $y$,

$$E + y \subseteq (\bigcup_{k=1}^\infty I_k) + y = \bigcup_{k=1}^\infty (I_k + y).$$

By monotonicity of outer measure,

$$m^*(E + y) \le \sum_{k=1}^\infty \ell(I_k + y) = \sum_{k=1}^\infty \ell(I_k) < \epsilon.$$

Therefore $m^*(E + y) = 0$.

14. Show that if a set $E$ has positive outer measure, then there is a bounded subset of $E$ that also has positive outer measure.
Suppose $E$ has positive outer measure.
If $E$ is bounded, then clearly $E$ itself is a bounded subset of $E$ with positive outer measure.
If $E$ is unbounded:
First, we can partition the real numbers:

$$\mathbb{R} = \bigcup_{n\in\mathbb{Z}} [n, n+1)$$

Then we have that

$$E = E \cap \mathbb{R} = E \cap (\bigcup_{n\in\mathbb{Z}} [n, n+1)) = \bigcup_{n\in\mathbb{Z}} E \cap [n, n+1).$$

By countable subadditivity of outer measure,

$$0 < m^*(E) = m^*(\bigcup_{n\in\mathbb{Z}} E \cap [n, n+1)) \leq \sum_{n\in\mathbb{Z}} m^*(E \cap [n, n+1))$$

Then there exists an $n \in \mathbb{Z}$ such that $m^*(E \cap [n, n+1)) > 0$, else we reach a contradiction. Therefore we have $E \cap [n, n+1) \subseteq E$ that is bounded and has positive outer measure.

15. Show that if $E$ has finite measure and $\epsilon > 0$, then $E$ is the disjoint union of a finite number of measurable sets, each of which has measure at most $\epsilon$.
(We are letting $E$ be a measurable set because we are talking about measure specifically, not outer measure.)
If $E$ is countable, then $E$ has measure zero and $E$ itself is the measurable set whose measure is less than any $\epsilon$: $m(E) = 0 < \epsilon$. In fact, if $E$ has measure zero then the conclusion is trivial.
Suppose $E$ has positive measure.
Fix $\epsilon > 0$.
In the case that $E$ is not bounded, there exists an $M$ such that

$$m(E \setminus [-M, M]) < \epsilon. \ (\star)$$

$(\star)$: To prove this we can partition $\mathbb{R}$:

$$\mathbb{R} = \bigcup_{n=0}^{\infty}\Big([-(n+1), -n) \cup (n, n+1]\Big) = \bigcup_{n=0}^{\infty} I_n.$$

That is, $I_0 = [-1, 1], I_1 = [-2, -1) \cup (1, 2], I_2 = [-3, -2) \cup (2, 3], \cdots$
Therefore $E = E \cap \mathbb{R} = E \cap (\bigcup_{n=0}^{\infty} I_n) = \bigcup_{n=0}^{\infty}(E \cap I_n)$.
By countable additivity of measure, and the fact that $E$ has finite measure,

$$m(E) = m(\bigcup_{n=0}^{\infty}(E \cap I_n)) = \sum_{n=0}^{\infty} m(E \cap I_n) < \infty.$$

Thus we have a sequence of partial sums that converges so there exists an index $M$ such that

$$\sum_{n=M}^{\infty} m(E \cap I_n) = \left|\sum_{n=0}^{\infty} m(E \cap I_n) - \sum_{n=0}^{M-1} m(E \cap I_n)\right| < \epsilon.$$

We see that $m(E \setminus [-M, M]) = m(\bigcup_{n=M}^{\infty}(E \cap I_n)) = \sum_{n=M}^{\infty} m(E \cap I_n) < \epsilon$.
Therefore $E = (E \cap [-M, M]) \cup (E \cap [-M, M]^c))$, a disjoint union, and $m(E \cap [-M, M]^c) < \epsilon$, so we need only worry now about $E \cap [-M, M]$.
Else if $E$ is bounded, then there exists an $M$ such that $E \subseteq [-M, M]$, and $E = E \cap [-M, M]$.
Now, for this $\epsilon$, we can partition the real numbers into a countable collection of disjoint measurable intervals $I_k$ of the form $[x, x+\epsilon)$.
When we choose a natural number $l$ such that $\frac{2M}{\epsilon} < l$, we get $M < -M + l\epsilon$ so that

$$E \cap [-M, M] \subseteq [-M, M] \subseteq \bigcup_{k=1}^{l}[-M + (k-1)\epsilon, -M + k\epsilon) = \bigcup_{k=1}^{l} I_k.$$

Then

$$E \cap [-M, M] = E \cap (\bigcup_{k=1}^{l} I_k) = \bigcup_{k=1}^{l}(E \cap I_k).$$

Thus $E$ is the union of a finite number of disjoint measurable sets, each of which has measure at most $\epsilon$.
(If $E$ is not bounded, $E = (\bigcup_{k=1}^{l}(E \cap I_k)) \cup (E \setminus [-M, M])$, which still satisfies the conclusion.)

## 2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

Measurable sets possess the following **excision property**: If $A$ is a measurable set of finite outer measure that is contained in $B$, then

$$m^*(B \setminus A) = m^*(B) - m^*(A).$$

This holds because

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c) = m^*(A) + m^*(B \cap A^c).$$

**Theorem 11.** *Let $E$ be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of $E$.*
*(Outer Approximation by Open Sets and $G_\delta$ sets)*

*(i) For each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing $E$ for which $m^*(\mathcal{O} \setminus E) < \epsilon$.*

*(ii) There is a $G_\delta$ set $G$ containing $E$ for which $m^*(G \setminus E) = 0$.*

*(Inner Approximation by Closed Sets and $F_\sigma$ sets)*

*(iii) For each $\epsilon > 0$, there is a closed set $F$ contained in $E$ for which $m^*(E \setminus F) < \epsilon$.*

*(iv) There is a $F_\sigma$ set $F$ contained in $E$ for which $m^*(E \setminus F) = 0$.*

*Proof.* ($E$ is measurable $\implies$ (i)):
Assume $E$ is measurable and fix $\epsilon > 0$.
Case: $m^*(E) < \infty$:
By definition of outer measure and infimum, there exists a countable collection of intervals $\{I_k\}_{k=1}^{\infty}$ such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$m^*(E) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon.$$

Defining $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$, we see that $\mathcal{O}$ is an open set containing $E$. By subadditivity of outer measure,

$$m^*(\mathcal{O}) = m^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon,$$

so that

$$m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

Because $E$ is measurable, has finite outer measure, and is contained in $\mathcal{O}$, we have the excision property:

$$m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E) < \epsilon.$$

Case: $m^*(E) = \infty$:
Then $E$ may be expressed as the disjoint union of a countable collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, each of which has finite outer measure (See Problems 14 and 15 for an example of partitioning $\mathbb{R}$). Now, for each index $k$, because each $E_k$ is measurable and has finite outer measure, we showed above that there exists an open set $\mathcal{O}_k$ containing $E_k$ for which $m^*(\mathcal{O}_k \setminus E_k) < \epsilon/2^k$. The set $\mathcal{O} = \bigcup_{k=1}^{\infty} \mathcal{O}_k$ is open, it contains $E$ (because $E = \bigcup_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} \mathcal{O}_k = \mathcal{O}$), and we have $E \supseteq E_k \implies E^c \subseteq E_k^c$, so that

$$\mathcal{O} \setminus E = \mathcal{O} \cap E^c = (\bigcup_{k=1}^{\infty} \mathcal{O}_k) \cap E^c = \bigcup_{k=1}^{\infty} (\mathcal{O}_k \cap E^c) \subseteq \bigcup_{k=1}^{\infty} (\mathcal{O}_k \cap E_k^c) = \bigcup_{k=1}^{\infty} (\mathcal{O}_k \setminus E_k).$$

Therefore by monotonicity and subadditivity of outer measure,

$$m^*(\mathcal{O} \setminus E) \leq m^*(\bigcup_{k=1}^{\infty}(\mathcal{O}_k \setminus E_k)) \leq \sum_{k=1}^{\infty} m^*(\mathcal{O}_k \setminus E_k) < \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon.$$

Thus property (i) holds for $E$.

((i) $\implies$ (ii)):
Now, assume property (i) holds for $E$. Then for each natural number $k$, there exists an open set $\mathcal{O}_k$ that contains $E$ for which $m^*(\mathcal{O}_k \setminus E) < 1/k$.
Define $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$ so that $E \subseteq \mathcal{O}_k$ for all $k \implies E \subseteq \bigcap_{k=1}^{\infty} \mathcal{O}_k = G$. Then $G$ is a $G_\delta$ set that contains $E$.
Then because for all $k$, $G \subseteq \mathcal{O}_k \implies G \setminus E \subseteq \mathcal{O}_k \setminus E$, by monotonicity of outer measure,

$$m^*(G \setminus E) \subseteq m^*(\mathcal{O}_k \setminus E) < 1/k.$$

Thus $m^*(G \setminus E) = 0$, and (ii) holds.

((ii) $\implies$ $E$ is measurable):
Assume property (ii) holds for $E$.
We can write

$$\begin{aligned} E &= G \cap E \\ &= \emptyset \cup (G \cap E) \\ &= (G \cap G^c) \cup (G \cap E) \\ &= G \cap (G^c \cup E) \\ &= G \cap (G \cap E^c)^c \\ &= G \cap (G \setminus E)^c. \end{aligned}$$

Now, $m^*(G \setminus E) = 0$, and any set of measure zero is measurable, so $G \setminus E$ is measurable and also $(G \setminus E)^c$ is measurable by complement. Also, $G$ is a $G_\delta$ set, and all $G_\delta$ sets are measurable. Finally, the intersection of measurable sets is measurable so $G \cap (G \setminus E)^c$ is measurable. Thus $E$ is measurable. $\square$

**PROBLEMS**

16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).
($E$ is measurable $\implies$ (iii)):
Fix $\epsilon > 0$. Suppose $E$ is measurable. Then $E^c$ is also measurable. Also, stating that $E^c$ is measurable is equivalent to property (i), that is, there exists an open set $\mathcal{O}$ containing $E^c$ such that $m^*(\mathcal{O} \setminus E^c) < \epsilon$.
Then because $\mathcal{O}$ is open, we can define the closed set $F = \mathcal{O}^c$, and we have that

$$\mathcal{O} \setminus E^c = \mathcal{O} \cap E = F^c \cap E = E \setminus F,$$

and $E^c \subseteq \mathcal{O} \implies E \supseteq \mathcal{O}^c = F$. Therefore $F$ is a closed set contained in $E$ for which $m^*(E \setminus F) = m^*(\mathcal{O} \setminus E^c) < \epsilon$, and (iii) holds.

((iii) $\implies$ (iv)):
Suppose that property (iii) holds for $E$. Then for each natural number $k$, there exists a closed set

$F_k$ contained in $E$ for which $m^*(E \setminus F_k) < 1/k$.
Then defining $F = \bigcup_{k=1}^{\infty} F_n$, we have that $F_k \subseteq E, \forall k \implies F = \bigcup_{k=1}^{\infty} F_k \subseteq E$. Then $F$ is an $F_\sigma$ set that is contained in $E$. Then $F \supseteq F_k \implies F^c \subseteq F_k^c$ and thus $E \cap F^c \subseteq E \cap F_k^c$ and $E \setminus F \subseteq E \setminus F_k$. By monotonicity of outer measure, for all $k$, we have

$$m^*(E \setminus F) \leq m^*(E \setminus F_k) < 1/k.$$

Therefore $m^*(E \setminus F) = 0$, and (iv) holds.

((iv) $\implies E$ is measurable):
Suppose that property (iv) holds for $E$.
We can write

$$\begin{aligned} E &= E \cap \mathbb{R} \\ &= [F \cup E] \cap [F \cup F^c] \\ &= F \cup [E \cap F^c] \\ &= F \cup [E \setminus F]. \end{aligned}$$

Now, $m^*(E \setminus F) = 0$ implies $E \setminus F$ is measurable because all sets of measure zero are measurable. Also, $F$ is an $F_\sigma$ set, which is measurable. Therefore $F \cup [E \setminus F]$, the intersection of measurable sets, is measurable. Thus $E$ is measurable.

17. Show that a set $E$ is measurable iff for each $\epsilon > 0$, there is a closed set $F$ and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.
Let $E$ be a set, and let $\epsilon > 0$.
(This case we assuming $E$ has finite measure to assume excision, maybe proof not complete)
($\implies$) Suppose $E$ is measurable.
Then by Theorem 11 (i), (iii), there is an open set $\mathcal{O}$ containing $E$ for which $m^*(\mathcal{O} \setminus E) < \epsilon/2$, and a closed set $F$ contained in $E$ for which $m^*(E \setminus F) < \epsilon/2$. That is, $F \subseteq E \subseteq \mathcal{O}$.
By excision, $m^*(E \setminus F) = m^*(E) - m^*(F)$, and we can write

$$\begin{aligned} m^*(E) - m^*(F) &< \epsilon/2 \\ m^*(E) &< m^*(F) + \epsilon/2 \\ -m^*(E) &> -m^*(F) - \epsilon/2 \end{aligned}$$

Also by excision, we have $m^*(\mathcal{O} \setminus E) = m^*(\mathcal{O}) - m^*(E)$, and

$$m^*(\mathcal{O}) - m^*(F) - \epsilon/2 < m^*(\mathcal{O}) - m^*(E) < \epsilon/2$$

Therefore $m^*(\mathcal{O} \setminus F) = m^*(\mathcal{O}) - m^*(F) < \epsilon$.
($\impliedby$) Suppose there is a closed set $F$ and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.
By excision and monotonicity of outer measure, we have that

$$m^*(E \setminus F) = m^*(E) - m^*(F) \leq m^*(\mathcal{O}) - m^*(F) = m^*(\mathcal{O} \setminus F) < \epsilon.$$

Therefore we have a closed set $F$ contained in $E$ for which $m^*(E \setminus F) < \epsilon$, i.e., proposition (iii), which implies that $E$ is measurable.

18. Let $E$ have finite outer measure. Show that there is a $G_\delta$ set $G \supseteq E$ with $m(G) = m^*(E)$. Show that $E$ is measurable iff there is an $F_\sigma$ set $F \subseteq E$ with $m(F) = m^*(E)$.
Let $E$ be a set with finite outer measure.

Then for each natural number $k$, by definition of infimum, there exists a countable collection of open intervals $\{(I_k)_n\}_{n=1}^{\infty}$ whose union contains $E$ for which

$$\sum_{n=1}^{\infty} \ell((I_k)_n) < m^*(E) + 1/k.$$

Now, $\mathcal{O}_k = \bigcup_{n=1}^{\infty}(I_k)_n$ is an open set, and we can define $G = \bigcap_{k=1}^{\infty} \mathcal{O}_k$ so that $E \subseteq \mathcal{O}_k$ for all $k \implies E \subseteq \bigcap_{k=1}^{\infty} \mathcal{O}_k = G$. Then $G$ is a $G_\delta$ set that contains $E$. Because $G \subseteq \mathcal{O}_k$ for all $k$, by monotonicity,

$$m^*(G) \leq m^*(\mathcal{O}_k) = m^*(\bigcup_{n=1}^{\infty}(I_k)_n) \leq \sum_{n=1}^{\infty} \ell((I_k)_n) < m^*(E) + 1/k.$$

Then we have $m^*(G) < m^*(E) + 1/k$ for any natural number $k$, which implies $m^*(G) \leq m^*(E)$.
Also, by monotonicity, $E \subseteq G \implies m^*(E) \leq m^*(G)$.
Therefore $m^*(G) = m^*(E)$.

Let $E$ be a set with finite outer measure.
( $\implies$ ) Suppose that $E$ is measurable.
By Theorem 11 (iv), there is an $F_\sigma$ set $F$ contained in $E$ for which $m^*(E \setminus F) = 0$. Because $E$ has finite outer measure, then $F$ has finite outer measure by monotonicity of outer measure. Then by excision, we have $m^*(E) - m^*(F) = m^*(E \setminus F) = 0$, which implies $m^*(E) = m^*(F)$.
( $\impliedby$ ) Suppose there is an $F_\sigma$ set $F \subseteq E$ with $m(F) = m^*(E)$.
Then $0 = m^*(E) - m^*(F)$. Because $E$ has finite outer measure, then $F$ has finite outer measure by monotonicity of outer measure. Therefore by excision we have $0 = m^*(E) - m^*(F) = m^*(E \setminus F)$ and Theorem 11 (iv) holds, which implies that $E$ is measurable.

19. Let $E$ have finite outer measure. Show that if $E$ is not measurable, then there is an open set $\mathcal{O}$ containing $E$ that has finite outer measure and for which

$$m^*(\mathcal{O} \setminus E) > m^*(\mathcal{O}) - m^*(E).$$

Suppose $E$ is not measurable. However, suppose by contradiction that for all open sets $\mathcal{O}$ containing $E$ that have finite outer measure, we have $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O}) - m^*(E)$.
Let $\epsilon > 0$. By definition of outer measure, there exists a countable collection of open intervals $\{I_k\}$ whose union contains $E$ and

$$\sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon.$$

We can define $\mathcal{O} := \bigcup_{k=1}^{\infty} I_k$, which is an open set that contains $E$, and by subadditivity of outer measure, we have that

$$m^*(\mathcal{O}) = m^*(\bigcup_{k=1}^{\infty} I_k) \leq \sum_{k=1}^{\infty} \ell(I_k) < m^*(E) + \epsilon$$

Therefore $m^*(\mathcal{O}) - m^*(E) < \epsilon$, and $\mathcal{O}$ has finite outer measure.
By assumption, we have that $m^*(\mathcal{O} \setminus E) \leq m^*(\mathcal{O}) - m^*(E) < \epsilon$. However, this means that we have an open set $\mathcal{O}$ containing $E$ for which $m^*(\mathcal{O} \setminus E) < \epsilon$, Theorem 11 (i), which is equivalent to saying that $E$ is measurable, which is a contradiction.

20. (Lebesgue). Let $E$ have finite outer measure. Show that $E$ is measurable iff for each open, bounded interval $(a, b)$,

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

Let $E$ be a set of finite outer measure.
( $\Longrightarrow$ ) Suppose that $E$ is measurable.
Then for any interval $(a, b)$, we have

$$b-a = \ell((a, b)) = m^*((a, b)) = m^*((a, b) \cap E) + m^*((a, b) \cap E^c) = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

( $\Longleftarrow$ ) Suppose that for each open, bounded interval $(a, b)$, we have $b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E)$.
Then we have

$$m^*((a, b)) = \ell((a, b)) = b-a = m^*((a, b) \cap E) + m^*((a, b) \setminus E) = m^*((a, b) \cap E) + m^*((a, b) \cap E^c).$$

(This is only proved for any open interval; measurability of $E$ implies this is true for any set)

21. Use property (ii) of Theorem 11 as the primitive definition of a measurable set and prove that the union of two measurable sets is measurable. Then do the same for property (iv).

(ii) Let $E$ be any set of real numbers. Define $E$ to be measurable if there is a $G_\delta$ set $G$ containing $E$ for which $m^*(G \setminus E) = 0$.
Let $A$ and $B$ be two measurable sets under this definition. Then there exist $G_\delta$ sets $G_A, G_B$ containing $A, B$ respectively for which $m^*(G_A \setminus A) = 0$ and $m^*(G_B \setminus B) = 0$.
Now, by definition of $G_\delta$ set:

$$G_A = \bigcap_{k=1}^{\infty} \mathcal{O}_k, \text{ for } \mathcal{O}_k \text{ open}$$
$$G_B = \bigcap_{n=1}^{\infty} \mathcal{U}_n, \text{ for } \mathcal{U}_n \text{ open}$$

Therefore

$$\begin{aligned} G_A \cup G_B &= (\bigcap_{k=1}^{\infty} \mathcal{O}_k) \cup (\bigcap_{n=1}^{\infty} \mathcal{U}_n) \\ &= \bigcap_{k=1}^{\infty} (\mathcal{O}_k \cup (\bigcap_{n=1}^{\infty} \mathcal{U}_n) \\ &= \bigcap_{k=1}^{\infty} (\bigcap_{n=1}^{\infty} (\mathcal{O}_k \cup \mathcal{U}_n)) \end{aligned}$$

For each $k, n$ pair, $\mathcal{O}_k \cup \mathcal{U}_n$ is an open set, so $G_A \cup G_B$ is a countable intersection of open sets and thus a $G_\delta$ set. Also, $G_A \supseteq A$ and $G_B \supseteq B$ imply that $G_A \cup G_B \supseteq A \cup B$, so $G_A \cup G_B$ is a $G_\delta$ set that contains $A \cup B$.

We can write

$$\begin{aligned}(G_A \cup G_B) \setminus (A \cup B) &= (G_A \cup G_B) \cap (A \cup B)^c \\ &= (G_A \cup G_B) \cap (A^c \cap B^c) \\ &= [G_A \cap (A^c \cap B^c)] \cup [G_B \cap (A^c \cap B^c)] \\ &= [G_A \cap A^c \cap B^c] \cup [G_B \cap B^c \cap A^c] \\ &\subseteq [G_A \cap A^c] \cup [G_B \cap B^c] \\ &\subseteq [G_A \setminus A] \cup [G_B \setminus B].\end{aligned}$$

By monotonicity of outer measure and subadditivity,

$$\begin{aligned}m^*((G_A \cup G_B) \setminus (A \cup B)) &\leq m^*([G_A \setminus A] \cup [G_B \setminus B]) \\ &\leq m^*(G_A \setminus A) + m^*(G_B \setminus B) \\ &= 0.\end{aligned}$$

Therefore $A \cup B$ is measurable.

(iv) Let $E$ be any set of real numbers. Define $E$ to be measurable if there is an $F_\sigma$ set $F$ contained in $E$ for which $m^*(E \setminus F) = 0$.
Let $A$ and $B$ be two measurable sets under this definition. Then there exist $F_\sigma$ sets $F_A, F_B$ contained in $A, B$ respectively for which $m^*(A \setminus F_A) = 0$ and $m^*(B \setminus F_B) = 0$.
Now, by definition of $F_\sigma$ set:

$$\begin{aligned}F_A &= \bigcup_{k=1}^{\infty} I_k, \text{ for } I_k \text{ closed} \\ F_B &= \bigcup_{n=1}^{\infty} J_n, \text{ for } J_n \text{ closed}\end{aligned}$$

Therefore

$$F_A \cup F_B = (\bigcup_{k=1}^{\infty} I_k) \cup (\bigcup_{n=1}^{\infty} J_n),$$

which is clearly a countable union of closed sets, so $F_A \cup F_B$ is an $F_\sigma$ set. Also, $F_A \subseteq A$ and $F_B \subseteq B$ imply that $F_A \cup F_B \subseteq A \cup B$, so $F_A \cup F_B$ is an $F_\sigma$ set that is contained in $A \cup B$.
We can write

$$\begin{aligned}(A \cup B) \setminus (F_A \cup F_B) &= (A \cup B) \cap (F_A \cup F_B)^c \\ &= (A \cup B) \cap (F_A^c \cap F_B^c) \\ &= [A \cap (F_A^c \cap F_B^c)] \cup [B \cap (F_A^c \cap F_B^c)] \\ &= [A \cap F_A^c \cap F_B^c] \cup [B \cap F_B^c \cap F_A^c] \\ &\subseteq [A \cap F_A^c] \cup [B \cap F_B^c] \\ &\subseteq [A \setminus F_A] \cup [B \setminus F_B].\end{aligned}$$

By monotonicity of outer measure and subadditivity,

$$\begin{aligned} m^*((A\cup B)\setminus(F_A\cup F_B)) &\le m^*([A\setminus F_A]\cup[B\setminus F_B]) \\ &\le m^*(A\setminus F_A)+m^*(B\setminus F_B) \\ &= 0. \end{aligned}$$

Therefore $A\cup B$ is measurable.

22. For any set $A$, define $m^{**}(A)\in[0,\infty]$ by

$$m^{**}(A)=\inf\{m^*(\mathcal{O})\mid \mathcal{O}\supseteq A, \mathcal{O}\text{ open.}\}$$

How is this set function $m^{**}$ related to outer measure $m^*$?

Consider any open set $\mathcal{O}$ such that $A\subseteq\mathcal{O}$. By monotonicity of outer measure, $m^*(A)\le m^*(\mathcal{O})$, and therefore $m^*(A)$ is a lower bound to the set $\{m^*(\mathcal{O})\mid \mathcal{O}\supseteq A,\mathcal{O}\text{ open.}\}$. Because $m^{**}$ is defined as the greatest lower bound, we get

$$m^*(A)\le m^{**}(A).$$

Now, if $m^*(A)=\infty$, then trivially we have

$$m^*(A)\ge m^{**}(A),$$

which implies $m^*(A)=m^{**}(A)$.
Thus we consider the case where $m^*(A)<\infty$.
Then for any $\epsilon>0$, by definition of infimum, there exists a countable collection of open intervals $\{I_n\}_{n=1}^{\infty}$ whose union contains $A$ for which

$$\sum_{n=1}^{\infty}\ell(I_n)<m^*(A)+\epsilon.$$

Now, $\mathcal{O}=\bigcup_{n=1}^{\infty}I_n$ is an open set that contains $A$, so by definition of $m^{**}$,

$$m^{**}(A)\le m^*(\mathcal{O})=m^*(\bigcup_{n=1}^{\infty}I_n)\le\sum_{n=1}^{\infty}\ell(I_n)<m^*(A)+\epsilon.$$

Then $m^{**}(A)<m^*(A)+\epsilon$ implies $m^{**}(A)\le m^*(A)$.
Therefore $m^*(A)=m^{**}(A)$.

23. For any set $A$, define $m^{***}(A)\in[0,\infty]$ by

$$m^{***}(A)=\sup\{m^*(F)\mid F\subseteq A, F\text{ closed.}\}$$

How is this set function $m^{***}$ related to outer measure $m^*$?

Consider any closed set $F$ such that $F\subseteq A$. By monotonicity of outer measure, $m^*(F)\le m^*(A)$, and therefore $m^*(A)$ is an upper bound to the set $\{m^*(F)\mid F\subseteq A,F\text{ closed.}\}$. Because $m^{**}$ is defined as the least upper bound, we get

$$m^{***}(A)\le m^*(A).$$

(In addition, if $A$ is measurable, then $m^{***}(A)=m^*(A)$.)

## 2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

**Theorem 15** (the Continuity of Measure)**.** *Lebesgue measure possesses the following continuity properties:*

*(i) If $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, then*

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

*(ii) If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then*

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

*Proof.* Let $\{A_k\}_{k=1}^{\infty}$ be ascending and measurable.
If there exists an index $k$ such that $m(A_k) > \infty$, then by monotonicity of measure, $m(\bigcup_{k=1}^{\infty} A_k) = \infty$. Also, because this collection is ascending, we have $A_k \subseteq A_n$ whenever $k \leq n$; therefore by monotonicity, $\infty = m(A_k) \leq m(A_n)$ for all $n$ such that $k \leq n$, and thus (i) holds.
Therefore it remains to prove the case that $m(A_k) < \infty$ for all $k$.
Define $A_0 = \emptyset$, and define $C_k = A_k \setminus A_{k-1}$. Then $\{C_k\}_{k=1}^{\infty}$ is disjoint and $\bigcup_{k=1}^{\infty} C_k = \bigcup_{k=1}^{\infty} A_k$. Now we can write

$$\begin{aligned}
m(\bigcup_{k=1}^{\infty} A_k) &= m(\bigcup_{k=1}^{\infty} C_k) \\
&= \sum_{k=1}^{\infty} m(C_k) && \text{countable (disjoint) monotonicity} \\
&= \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}) \\
&= \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})] && \text{by excision: } m(A_{k-1}) < \infty \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} [m(A_k) - m(A_{k-1})] \\
&= \lim_{n\to\infty} m(A_n) - m(A_0) && \text{by telescoping} \\
&= \lim_{n\to\infty} m(A_n). && \text{because } A_0 = \emptyset
\end{aligned}$$

Let $\{B_k\}_{k=1}^{\infty}$ be descending and measurable.
Define $D_k = B_1 \setminus B_k = B_1 \cap B_k^c$.
Then because $\{B_k\}_{k=1}^{\infty}$ is descending,

$$B_k \supseteq B_{k+1} \implies B_1 \cap B_k^c \subseteq B_1 \cap B_{k+1}^c \implies D_k \subseteq D_{k+1},$$

and $\{D_k\}_{k=1}^{\infty}$ is ascending.

Now we have

$$\bigcup_{k=1}^{\infty} D_k = \bigcup_{k=1}^{\infty}[B_1 \cap B_k^c] = B_1 \cap [\bigcup_{k=1}^{\infty} B_k^c] = (B_1 \cap [\bigcap_{k=1}^{\infty} B_k]^c = B_1 \setminus [\bigcap_{k=1}^{\infty} B_k].$$

Then by part (i), we can write

$$\begin{aligned} m(\bigcup_{k=1}^{\infty} D_k) &= \lim_{k\to\infty} m(D_k) \\ m(B_1 \setminus [\bigcap_{k=1}^{\infty} B_k]) &= \lim_{k\to\infty} m(B_1 \setminus B_k) \\ m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) &= \lim_{k\to\infty} [m(B_1) - m(B_k)] \\ m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) &= m(B_1) - \lim_{k\to\infty} [m(B_k)] \\ m(\bigcap_{k=1}^{\infty} B_k) &= \lim_{k\to\infty} [m(B_k)]. \end{aligned}$$

□

**The Borel-Cantelli Lemma.** *Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the $E_k$'s.*

*Proof.* By countable subadditivity, for each $n$,

$$m(\bigcup_{k=n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} m(E_k) < \infty.$$

Because $\sum_{k=1}^{\infty} m(E_k) < \infty$, we have a sequence of partial sums such that for any $\epsilon > 0$, there exists an index n for which

$$\sum_{k=n}^{\infty} m(E_k) = |\sum_{k=1}^{\infty} m(E_k) - \sum_{k=1}^{n-1} m(E_k)| < \epsilon.$$

Therefore there exists an $n$ such that $|\sum_{k=n}^{\infty} m(E_k) - 0| < \epsilon$, and $\lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0$.
By continuity of measure (ii),

$$m(\bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} E_k]) = \lim_{n\to\infty} m(\bigcup_{k=1}^{\infty} E_k) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore almost all $x \in \mathbb{R}$ fail to belong to $\bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} E_k]$ and therefore belong to at most finitely many $E_k$'s. □

Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of sets that belong to a $\sigma$-algebra $\mathcal{A}$. Since $\mathcal{A}$ is closed w.r.t. countable unions and intersections, the following two sets belong to $\mathcal{A}$:

$$\begin{aligned} \limsup\{A_k\}_{k=1}^{\infty} &= \bigcap_{n=1}^{\infty}[\bigcup_{k=1}^{\infty} A_k] \\ \liminf\{A_k\}_{k=1}^{\infty} &= \bigcup_{n=1}^{\infty}[\bigcap_{k=1}^{\infty} A_k] \end{aligned}$$

The set $\limsup\{A_k\}_{k=1}^{\infty}$ is the set of points that belong to $A_n$ for countably infinitely many indices $n$ while the set $\liminf\{A_k\}_{k=1}^{\infty}$ is the set of points that belong to $A_n$ except for at most finitely many indices $n$.

**PROBLEMS**

24. Show that if $E_1$ and $E_2$ are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

$$\begin{aligned}
m(E_1 \cup E_2) + m(E_1 \cap E_2) &= m([E_1 \cup E_2] \cap E_1) + m([E_1 \cup E_2] \cap E_1^c) + m(E_1 \cap E_2) \\
&= m(E_1) + m([E_1 \cap E_1^c] \cup [E_2 \cap E_1^c]) + m(E_1 \cap E_2) \\
&= m(E_1) + m(\emptyset \cup [E_2 \cap E_1^c]) + m(E_1 \cap E_2) \\
&= m(E_1) + m(E_2 \cap E_1^c) + m(E_1 \cap E_2) \\
&= m(E_1) + m([E_2 \cap E_1^c] \cup [E_1 \cap E_2]) \\
&= m(E_1) + m([E_2 \cup (E_1 \cap E_2)] \cap [E_1^c \cup (E_1 \cap E_2)]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup (E_1 \cap E_2)]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup E_1] \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2 \cap \mathbb{R} \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2 \cap [E_1^c \cup E_2]) \\
&= m(E_1) + m(E_2).
\end{aligned}$$

25. Show that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.
In the proof of (ii), we get to the point

$$m(B_1) - m(\bigcap_{k=1}^{\infty} B_k) = m(B_1) - \lim_{k\to\infty}[m(B_k)].$$

If $m(B_1) = \infty$, then we have

$$\infty - m(\bigcap_{k=1}^{\infty} B_k) = \infty - \lim_{k\to\infty}[m(B_k)],$$

and we cannot reach the conclusion we want because $\infty - \infty$ is not defined.

26. Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets. Prove that for any set $A$,

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

We have by countable subadditivity:

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) = m^*(\bigcup_{k=1}^{\infty}(A \cap E_k)) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

Now, for any $n$, we have $A \cap \bigcup_{k=1}^{\infty} E_k \supseteq A \cap \bigcup_{k=1}^{n} E_k$, so by monotonicity and Proposition 6,

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq m^*(A \cap \bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

The left hand side is independent of $n$, so taking the limit as $n \to \infty$, we get

$$m^*(A \cap \bigcup_{k=1}^{\infty} E_k) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

27. Let $\mathcal{M}'$ be any $\sigma$-algebra of subsets of $\mathbb{R}$ and $m'$ a set function on $\mathcal{M}'$ which takes values in $[0, \infty]$, is countably additive, and such that $m'(\emptyset) = 0$.

    (i) Show that $m'$ is finitely additive, monotone, countably monotone, and possesses the excision property.
    Countable additivity implies that for any disjoint collection of measurable sets $\{E_k\}_{k=1}^{\infty}$, we have $m'(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m'(E_k)$.
    Now, any finite disjoint collection $\{E_k\}_{k=1}^{n}$ can be extended to the infinite disjoint collection $\{E_k'\}_{k=1}^{\infty}$, where $E_k' = E_k$ for $k \in \{1, \cdots, n\}$, and $E_k' = \emptyset$ for $k > n$. Clearly from this we have finite additivity.
    In Problem 1 of this chapter, it was shown that a countably additive set function possesses the monotonicity property. Thus $m'$ is monotone. It can clearly be shown that $m'$ is also countably monotone.
    To see excision, simply use countable additivity to see that for measurable sets $A, B$ such that $A \subseteq B$, we have

    $$m'(B) = m'([B \cap A] \cup [B \cap A^c]) = m'(B \cap A) + m'(B \cap A^c) = m'(A) + m'(B \setminus A).$$

    (ii) Show that $m'$ possesses the same continuity properties as Lebesgue measure.
    Check Theorem 15 and the Borel-Cantelli Lemma above.

28. Show that continuity of measure together with finite additivity of measure implies countable additivity of measure.
    Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of measurable sets. (if any $E_k$ has infinite measure, countable additivity is clear, so we need only consider sets of finite measure for all $E_k$.)
    Finite additivity implies that for the disjoint collection of measurable sets $\{E_k\}_{k=1}^{n}$, we have $m(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} m(E_k)$.
    We can define $F_n = \bigcup_{k=1}^{n} E_k$ so that continuity of measure implies that for the ascending collection $\{F_n\}_{n=1}^{\infty}$ of measurable sets, we have $m(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n)$.
    Therefore we can write

    $$m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty} m(\bigcup_{k=1}^{n} E_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(E_k) = \sum_{k=1}^{\infty} m(E_k).$$

## 2.6 Nonmeasurable Sets

Consider the subgroup under addition $\mathbb{Q} \subseteq \mathbb{R}$. Now, $\mathbb{Q}$ is a normal subgroup, and we have the quotient group $\mathbb{R}/\mathbb{Q}$, with the (disjoint) cosets written as $r + \mathbb{Q}$ where $r \in \mathbb{R}$. A Vitali set $V \subseteq [0, 1]$ is defined to be a set such that for all $r \in \mathbb{R}$, there exists exactly one unique $v \in V$ such that $v - r \in \mathbb{Q}$. Every Vitali set is uncountable, and $v - u \notin \mathbb{Q}$ for $u, v \in V$, $u \neq v$.

**Theorem.** *A Vitali set is non-measurable.*

*Proof.* Suppose by contradiction that a Vitali set V is measurable.
Let $\{q_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in $[-1, 1]$:
recall that $\mathbb{Q}$ looks like

$$\mathbb{Q} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\},$$

therefore

$$\{q_k\}_{k=1}^{\infty} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\}.$$

For each natural number $k$, let $V_k = V + q_k = \{v + q_k : v \in V\}$.
First we will show the following:

(i) $V_i \cap V_j = \emptyset$ for $i \neq j$

(ii) $[0, 1] \subseteq \bigcup_{k=1}^{\infty} V_k \subseteq [-1, 2]$

(i) Suppose by contradiction that $V_i \cap V_j = \emptyset$ for some $i \neq j$.
That is, there exists $x \in V_i, y \in V_j$ such that $x = y$.
Also, there exists $v, u \in V$ such that $x = v + q_i$ and $y = u + q_j$.
By equality, we have $v + q_i = u + q_j$.
In the case that $v = u$, we get $q_i = q_j$, a contradiction.
In the case that $v \neq u$, we can write $v - u = q_j - q_i \in \mathbb{Q}$, a contradiction.

(ii) For any real $r \in [0, 1]$, there exists a $v \in V \subseteq [0, 1]$ such that $r - v \in \mathbb{Q}$.
We can see that

$$\max(r - v) = 1 - 0 = 1,$$
$$\min(r - v) = 0 - 1 = -1.$$

which implies $r - v = q_i \in [-1, 1] \cap \mathbb{Q}$ for some $i$, and thus $r = v + q_i \in V_i$.
In short, we can write this as

$$r \in [0, 1] \implies r \in V_i \text{ for some } i \implies r \in \bigcup_{k=1}^{\infty} V_k \implies [0, 1] \subseteq \bigcup_{k=1}^{\infty} V_k.$$

Now, $V_k = V + q_k, V \subseteq [0, 1], q_k \in [-1, 1]$, therefore

$$\max(v + q_k) = 1 + 1 = 2,$$
$$\min(v + q_k) = 0 - 1 = -1.$$

Therefore $V_k \subseteq [-1, 2]$ for all $k$, and thus $\bigcup_{k=1}^{\infty} V_k \subseteq [-1, 2]$.

Then we can write

$$
\begin{aligned}
m^*([0,1]) \le m^*(\bigcup_{k=1}^{\infty} V_k) \le m^*([-1,2]) & \quad \text{by monotonicity of outer measure} \\
1 \le \sum_{k=1}^{\infty} m^*(V_k) \le 3 & \quad \text{countable additivity (measurability of } V\text{)} \star \\
1 \le \sum_{k=1}^{\infty} m^*(V+q_k) \le 3 & \\
1 \le \sum_{k=1}^{\infty} m^*(V) \le 3 & \quad \text{by translation invariance of outer measure}
\end{aligned}
$$

However, $m^*(V) \ge 0$ is a constant, so $\sum_{k=1}^{\infty} m^*(V) = 0$ or $\sum_{k=1}^{\infty} m^*(V) = \infty$, neither of which is in $[1,3]$, and we reach a contradiction. □

For any nonempty set $E$ of real numbers, we define two points in $E$ to be **rationally equivalent** provided their difference belongs to $\mathbb{Q}$. By a **choice set** for the rational equivalence relation on $E$ we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class. A choice set $\mathcal{C}_E$ is characterized by the following two properties:

1. the difference of two points in $\mathcal{C}_E$ is not rational;
2. for each point $x$ in $E$, there is a point $c$ in $\mathcal{C}_E$ for which $x = c + q$, $q \in \mathbb{Q}$.

The first property can be reformulated as

For any set $\Lambda \subseteq \mathbb{Q}$, $\{\lambda + \mathcal{C}_E\}_{\lambda \in \Lambda}$ is disjoint.

We also have that

$$E \subseteq \bigcup_{\lambda \in \mathbb{Q}} (\lambda + \mathcal{C}_E).$$

**PROBLEMS**

29. (i) Show that rational equivalence defines an equivalence relation on any set.

Let $X$ be any set and define $x \sim y$ when $x - y \in \mathbb{Q}$ for $x, y \in X$.

i. $x - x = 0 \in \mathbb{Q} \iff x \sim x$ for all $x \in X$.
ii. $x \sim y \iff x - y = q \in \mathbb{Q} \iff y - x = -q \in \mathbb{Q} \iff y \sim x$ for all $x, y \in X$.
iii. $x \sim y, y \sim z \iff x - y = q \in \mathbb{Q}, y - z = q' \in \mathbb{Q} \iff x - z = x - y + y - z = q + q' \in \mathbb{Q} \iff x \sim z$ for all $x, y, z \in X$.

(ii) Explicitly find a choice set for the rational equivalence relation on $\mathbb{Q}$.

(For any nonempty set $E$ of real numbers, we define two points in $E$ to be **rationally equivalent** provided their difference belongs to $\mathbb{Q}$. By a **choice set** for the rational equivalence relation on $E$ we mean a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class.) Therefore for the nonempty set $\mathbb{Q}$, we can choose a choice set $\mathcal{C}_{\mathbb{Q}} = \{q\}$ for any $q \in \mathbb{Q}$.

(iii) Define two numbers to be irrationally equivalent provided their difference is irrational or zero. Is this an equivalence relation on $\mathbb{R}$? Is this an equivalence relation on $\mathbb{Q}$?

i. $x - x = 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim x$ for all $x \in \mathbb{R}$.
ii. $x \sim y \iff x - y = q \in \{\mathbb{Q}^c, 0\} \iff y - x = -q \in \{\mathbb{Q}^c, 0\} \iff y \sim x$ for all $x, y \in \mathbb{R}$.
iii. $2 - \pi \in \{\mathbb{Q}^c, 0\}, \pi - 0 \in \{\mathbb{Q}^c, 0\}$ but $2 - 0 \notin \{\mathbb{Q}^c, 0\}$

Not an equivalence relation on $\mathbb{R}$.

i. $x - x = 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim x$ for all $x \in \mathbb{Q}$.
ii. $x \sim y \iff x - y = 0 \in \{\mathbb{Q}^c, 0\} \iff y - x = 0 \in \{\mathbb{Q}^c, 0\} \iff y \sim x$ for all $x, y \in \mathbb{Q}$.
iii. $x \sim y, y \sim z \iff x - y = 0 \in \{\mathbb{Q}^c, 0\}, y - z = 0 \in \{\mathbb{Q}^c, 0\} \iff x - z = x - y + y - z = 0 + 0 \in \{\mathbb{Q}^c, 0\} \iff x \sim z$ for all $x, y, z \in \mathbb{Q}$.

An equivalence relation on $\mathbb{Q}$.

30. Show that any choice set for the rational equivalence relation on a set of positive outer measure must be uncountably infinite.

Let $E$ be a set of positive outer measure. Suppose there exists a choice set $\mathcal{C}_E$ for the rational equivalence relation on $E$ such that $\mathcal{C}_E$ is countable. All countable sets have outer measure zero, so $m^*(\mathcal{C}_E) = 0$. Because we know $E \subseteq \bigcup_{\lambda \in \mathbb{Q}}(\lambda + \mathcal{C}_E)$, by monotonicity, subadditivity, and translation invariance of outer measure,

$$m^*(E) \leq m^*(\bigcup_{\lambda \in \mathbb{Q}}(\lambda + \mathcal{C}_E)) \leq \sum_{\lambda \in \mathbb{Q}} m^*(\lambda + \mathcal{C}_E) = \sum_{\lambda \in \mathbb{Q}} m^*(\mathcal{C}_E) = \sum_{\lambda \in \mathbb{Q}} 0 = 0,$$

and we have a contradiction to the fact that $m^*(E) > 0$.

31. Justify the assertion in the proof of Vitali's Theorem that it suffices to consider the case that $E$ is bounded.

(Vitali: Any set of real numbers with positive outer measure contains a subset that fails to be measurable.) By Problem 14, we showed that every set of positive outer measure $E$ contains a bounded subset $A \subseteq E$ of positive outer measure. Therefore if there exists a subset $S \subseteq A$ that fails to be measurable, then $S \subseteq A \subseteq E$ is a subset that fails to be measurable.

32. Does Lemma 16 remain true if $\Lambda$ is allowed to be finite or to be uncountably infinite? Does it remain true if $\Lambda$ is allowed to be unbounded?

(Lemma 16: Let $E$ be a bounded measurable set of real numbers. Suppose there is a bounded, countably infinite set of real numbers $\Lambda$ for which the collection of translates of $E$, $\{\lambda + E\}_{\lambda \in \Lambda}$, is disjoint. Then $m(E) = 0$.)
Consider the case $\Lambda = \{1, 2\}$ is finite, and $E = (0, 1)$. Then $\{\lambda + E\}_{\lambda \in \Lambda = \{1,2\}} = \{1 + (0, 1), 2 + (0, 1)\} = \{(1, 2), (2, 3)\}$, which is a disjoint collection. However, $m(E) = 1 \neq 0$.
If $\Lambda$ is uncountably infinite and satisfies that the translates are disjoint, then we can choose a countable subset of $\Lambda$ and thus Lemma 16 remains true.
Consider the case $\Lambda = \{1, 2, 3, \cdots\}$ is unbounded, and $E = (0, 1)$. Then the collection of translates of $E$, $\{(1, 2), (2, 3), (3, 4), \cdots\}$ is disjoint but $m(E) = 1 \neq 0$.

33. Let $E$ be a nonmeasurable set of finite outer measure. Show that there is a $G_\delta$ set $G$ that contains $E$ for which

$$m^*(E) = m^*(G), \text{ while } m^*(G \setminus E) > 0.$$

This is a similar construction for the proof from Theorem 11 (i).
Let $E$ be a nonmeasurable set of finite outer measure.
By definition of outer measure and infimum, for any natural number $n$, there exists a countable collection of intervals $\{(I_n)_k\}_{k=1}^\infty$ such that $E \subseteq \bigcup_{k=1}^\infty (I_n)_k$ and

$$m^*(E) \leq \sum_{k=1}^\infty \ell((I_n)_k) < m^*(E) + 1/n.$$

Defining $\mathcal{O}_n = \bigcup_{k=1}^\infty (I_n)_k$, we see that $\mathcal{O}_n$ is an open set containing $E$ for each $n$. Further define $G = \bigcap_{n=1}^\infty \mathcal{O}_n$ so that $E \subseteq G \subseteq \mathcal{O}_n$ for any $n$ and $G$ is a $G_\delta$ set that contains $E$.
By subadditivity of outer measure,

$$m^*(G) \leq m^*(\mathcal{O}_n) = m^*(\bigcup_{k=1}^\infty (I_n)_k) \leq \sum_{k=1}^\infty \ell(I_k) < m^*(E) + 1/n,$$

so that $m^*(G) < m^*(E) + 1/n \implies m^*(G) \leq m^*(E)$. By subadditivity, $E \subseteq G$ implies we also have $m^*(E) \leq m^*(G)$, and so $m^*(E) = m^*(G)$.
Now, we know that the outer measure is nonnegative by monotonicity, so we have the inequality $m^*(G \setminus E) \geq 0$.
By Theorem 11 (ii), $m^*(G \setminus E) = 0 \iff E$ is measurable, so we must have $m^*(G \setminus E) > 0$.

## 2.7 The Cantor Set and the Cantor-Lebesgue Function

**PROBLEMS**

34. Show that there is a continuous, strictly increasing function on the interval $[0, 1]$ that maps a set of positive measure onto a set of measure zero.

The function $\psi : [0, 1] \to [0, 2]$ defined by $\psi(x) = \varphi(x) + x$ maps the Cantor set $C \subseteq [0, 1]$ onto a measurable set of positive measure. That is, $m(C) = 0$ and $m(\psi(C)) > 0$. We can consider the inverse function $\psi^{-1} : [0, 2] \to [0, 1]$ restricted to $[0, 1]$: $\psi^{-1}|_{[0,1]} : [0, 1] \to [0, 1]$. Now consider the set $C' = C \cap [0, 1]$. This set $C'$ is a measurable subset of $C$, a measurable set of measure zero, so by monotonicity of measure, $m(C') = 0$. Then the function has $\psi^{-1}|_{[0,1]}(\psi(C')) = C'$, where $m(\psi(C')) > 0$ and $m(C') = 0$, thus mapping the set $\psi(C')$ of positive measure* onto the set $C'$ of measure zero.
(*We know that $m(\psi(C)) > 0$, but not shown that $m(\psi(C')) > 0$ where $C' \subseteq C$.)

35. Let $f$ be an increasing function on the open interval $I$. For $x_0 \in I$ show that $f$ is continuous at $x_0$ iff there are sequences $\{a_n\}$ and $\{b_n\}$ in $I$ such that for each $n$, $a_n < x_0 < b_n$, and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

Let $f$ be an increasing function on the open interval $I$ and let $x_0 \in I$.
( $\implies$ ) Suppose that $f$ is continuous at $x_0$.
Because $I$ is open, there exists an index $N$ such that for all $n \geq N$, we have that $(x_0 - 1/n, x_0 +$

$1/n) \subseteq I$. Then for each $n \geq N$ we can choose $a_n \in (x_0 - 1/n, x_0)$ and $b_n \in (x_0, x_0 + 1/n)$, and for $n < N$ let $a_n = a_N$ and $b_n = b_N$, so that $a_n < x_0 < b_n$ for all $n$. Now,we have

$$x_0 - 1/n < a_n < x_0 \implies x_0 - a_n < 1/n,$$
$$x_0 < b_n < x_0 + 1/n \implies b_n - x_0 < 1/n,$$

therefore $\lim_{n\to\infty} a_n = x_0$ and $\lim_{n\to\infty} b_n = x_0$. Because $f$ is continuous and increasing, for all $\epsilon > 0$, there exists the number $1/n > 0$ such that

$$x_0 - a_n < 1/n \implies f(x_0) - f(a_n) < \epsilon,$$
$$b_n - x_0 < 1/n \implies f(b_n) - f(x_0) < \epsilon.$$

(therefore $\lim_{n\to\infty} f(a_n) = f(x_0)$ and $\lim_{n\to\infty} f(b_n) = f(x_0)$.) We can write

$$[f(b_n) - f(a_n)] = f(x_0) - f(a_n) + f(b_n) - f(x_0) < \epsilon + \epsilon = \epsilon'$$

and so $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.

( $\Longleftarrow$ ) Suppose that there exist sequences $\{a_n\}$,$\{b_n\}$ such that $a_n < x_0 < b_n$ and $\lim_{n\to\infty}[f(b_n) - f(a_n)] = 0$.
That is, for any $\epsilon > 0$, there exists an index $N$ such that $f(b_n) - f(a_n) < \epsilon$ for all $n \geq N$.
Then $f(b_n) < f(a_n) + \epsilon$ and $f(b_n) - \epsilon < f(a_n)$.
Because $f$ is increasing, we have

$$f(b_n) - \epsilon < f(a_n) < f(x_0) < f(b_n) < f(a_n) + \epsilon.$$

Then $f(x_0) - f(a_n) < \epsilon$ and $f(b_n) - f(x_0) < \epsilon$, which implies $\lim_{n\to\infty} f(a_n) = f(x_0)$ and $\lim_{n\to\infty} f(b_n) = f(x_0)$.
By monotonicity of $f$, we also have

$$b_n - \epsilon < a_n < x_0 < b_n < a_n + \epsilon.$$

Then $x_0 - a_n < \epsilon$ and $b_n - x_0 < \epsilon$, which implies $\lim_{n\to\infty} a_n = x_0$ and $\lim_{n\to\infty} b_n = x_0$.
Now, clearly we see that for any $\epsilon > 0$, we have $x_0 - a_n < \epsilon \iff f(x_0) - f(a_n) < \epsilon$, and $b_n - x_0 < \epsilon \iff f(b_n) - f(x_0) < \epsilon$, and continuity at $x_0$ follows.

36. Let $f$ be a continuous function defined on $E$. Is it true that $f^{-1}(A)$ is always measurable if $A$ is measurable?

    No, the function $\psi : [0,1] \to [0,2]$ defined by $\psi(x) = \varphi(x) + x$ maps a measurable set $A$, subset of the Cantor set, onto a nonmeasurable set $\psi(A)$. Define $f = \psi^{-1}$ so that $f^{-1}(A) = (\psi^{-1})^{-1}(A)$ is not measurable but $A$ is measurable.

37. Let the function $f : [a,b] \to \mathbb{R}$ be Lipschitz; that is, there is a constant $c \geq 0$ such that for all $u, v \in [a,b]$, $|f(u) - f(v)| \leq c|u - v|$. Show that $f$ maps a set of measure zero onto a set of measure zero. Show that $f$ maps a $F_\sigma$ set onto an $F_\sigma$ set. Conclude that $f$ maps a measurable set to a measurable set.

    Let $f$ be a Lipschitz function on the interval $I$. Clearly $f$ is also continuous.
    Let $E \subseteq I$ be a set of measure zero; that is, $m^*(E) = m(E) = 0$. By definition of infimum, for

any $\epsilon > 0$, there exists a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$, $I_k = (a_k, b_k)$, such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and

$$0 \leq \sum_{k=1}^{\infty} \ell(I_k) < 0 + \frac{\epsilon}{c}.$$

We also have that $E \subseteq \bigcup_{k=1}^{\infty} I_k \implies f(E) \subseteq f(\bigcup_{k=1}^{\infty} I_k) = \bigcup_{k=1}^{\infty} f(I_k)$.
Also, by Chapter 1 Problem 54, Because $I_k$ is an interval, the continuous real-valued function $f$ on $I_k$ has an interval as its image; that is, $f(I_k)$ is an interval. Then there exists some $u_k, v_k \in (a, b)$ such that $f(I_k) = (f(u_k), f(v_k))$ and $m(f(I_k)) = f(v_k) - f(u_k)$. Then because $f$ is Lipschitz, $|f(v_k) - f(u_k)| \leq c|v_k - u_k|$ for all $k$.

$$\begin{aligned}
m(f(E)) &\leq m(\bigcup_{k=1}^{\infty} f(I_k)) && \text{by monotonicity}\\
&\leq \sum_{k=1}^{\infty} m(f(I_k)) && \text{by subadditivity}\\
&= \sum_{k=1}^{\infty} m(f(v_k) - f(u_k)) &&\\
&\leq \sum_{k=1}^{\infty} c|v_k - u_k| && \text{because } f \text{ is Lipschitz}\\
&\leq \sum_{k=1}^{\infty} c|b_k - a_k| && \text{because } (u_k, v_k) \subseteq (a, b)\\
&= \sum_{k=1}^{\infty} c\ell(I_k) &&\\
&< \epsilon. &&
\end{aligned}$$

Therefore $m(f(E)) = 0$.

38. Let $F$ be the subset of $[0, 1]$ constructed in the same manner as the Cantor set except that each of the intervals removed at the $n$th deletion stage has length $\alpha 3^{-n}$ with $0 < \alpha < 1$. Show that $F$ is a closed set, $[0, 1] \setminus F$ is dense in $[0, 1]$, and $m(F) = 1 - \alpha$. Such a set $F$ is called a generalized Cantor set.

Define $F$ to be constructed in the same manner as the Cantor set, with

$$F = \bigcap_{k=1}^{\infty} F_k,$$

where $\{F_k\}_{k=1}^{\infty}$ is a descending sequence of closed sets, and each $F_k$ is a disjoint union of $2^k$ closed intervals, each of length $\alpha/3^k$.
It can clearly be seen that $F$ is a closed set because it is an intersection of closed sets.
Now, for any point $x \in [0, 1]$, there exists an index $k$ such that $x \notin F_k$; that is, $x \in F_k^c$, which is an open set. Therefore we can construct sequences in $([0, 1] \setminus F) \setminus \{x\}$ that converge to $x$.
Each $F_k$ is the disjoint union of $2^k$ closed intervals each of length $\alpha/3^k$, so at each step we remove

$2^{k-1}$ open intervals of length $\alpha/3^k$:

$$\begin{aligned} m(F_1) &= 1 - \alpha/3 \\ m(F_2) &= 1 - \alpha/3 - 2\alpha/3^2 \\ m(F_3) &= 1 - \alpha/3 - 2\alpha/3^2 - 2^2\alpha/3^3 \\ &\vdots \\ m(F_n) &= 1 - \sum_{k=1}^{n} 2^{k-1}\alpha/3^k \end{aligned}$$

Then by the continuity of measure, we have

$$m(\bigcap_{k=1}^{\infty} F_k) = \lim_{n\to\infty} m(F_n) = \lim_{n\to\infty}(1 - \sum_{k=1}^{n} 2^{k-1}\alpha/3^k).$$

We can see that

$$\begin{aligned} \lim_{n\to\infty}\sum_{k=1}^{n} 2^{k-1}\alpha/3^k &= \alpha/3 \lim_{n\to\infty}\sum_{k=1}^{n}(\frac{2}{3})^{k-1} \\ &= \alpha/3 \lim_{n\to\infty}\sum_{k=0}^{n-1}(\frac{2}{3})^{k} \\ &= \alpha/3 \lim_{n\to\infty}\frac{1-(2/3)^n}{1-(2/3)} \\ &= \alpha/3\frac{1}{1-(2/3)} \\ &= \alpha/3(\frac{1}{1/3}) \\ &= \alpha. \end{aligned}$$

Therefore $m(F) = m(\bigcap_{k=1}^{\infty} F_k) = 1 - \alpha$.

39. Show that there is an open set of real numbers that, contrary to intuition, has a boundary of positive measure. (Hint: consider the complement of the generalized Cantor set of the preceding problem.)

    We have $F \cup (F^c \cap [0,1]) = [0,1]$, and $m(F) = 1 - \alpha$, and $m(F^c \cap [0,1]) = \alpha$...

40. A subset $A$ of $\mathbb{R}$ is said to be **nowhere dense** in $\mathbb{R}$ provided that every open set $\mathcal{O}$ has a non-empty open subset that is disjoint from $A$. Show that the Cantor set is nowhere dense in $\mathbb{R}$.

    The Cantor set $C \subseteq [0,1]$ is defined to be the countable intersection of sets $C_k$, where $C_k$ is the disjoint union of $2^k$ closed intervals of length $1/3^k$ each. From Ch1 Proposition 9, we know that every open set is the countable disjoint union of open intervals. Therefore we need only prove Problem 40 for any open interval.
    Consider any open interval $(a,b) \in \mathbb{R}$.
    In the case that there exists an index $k$ such that $(a,b) \in C_k^c$, then the proof is done:
    Ex: $(a,b) = (3/18, 4/18)$. Then for $k = 2$, we have

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1],$$

so that

$$(3/18, 4/18) \subseteq C_2^c = (1/9, 2/9) \cup (1/3, 2/3) \cup (7/9, 8/9) = (2/18, 4/18) \cup (1/3, 2/3) \cup (7/9, 8/9).$$

In the case that for all indices $k$ we have that $(a, b) \in C_k$, then simply choose an index far enough so that one of the "open middle third" removal generated from $C_k$ is a subset of $(a, b)$.
Ex: $(a, b) = (6/10, 7/10) \ni 2/3$ and $2/3 \in C$ so $(a, b) \cap C \neq \emptyset$. Then for $k = 1$, we have

$$(6/10, 2/3) \subseteq (a, b) \text{ and } (6/10, 2/3) \notin C_1 = [0, 1/3] \cup [2/3, 1].$$

Ex: $(a, b) = (2/3, 20/27$. Then for $k = 3$, we have

$$C_3 = [0, \frac{1}{27}] \cup [\frac{2}{27}, \frac{1}{9}] \cup [\frac{2}{9}, \frac{7}{27}] \cup [\frac{8}{27}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{19}{27}] \cup [\frac{20}{27}, \frac{7}{9}] \cup [\frac{8}{9}, \frac{25}{27}] \cup [\frac{26}{27}, 1].$$

so that

$$(19/27, 20/27) \subseteq (a, b) \text{ and } (19/27, 20/27) \notin C_3.$$

41. Show that a strictly increasing function that is defined on an interval has a continuous inverse.

    Let $f$ be a strictly increasing function on the interval $I$. Then for $x, y \in I$ such that $x < y$, we have $f(x) < f(y)$.
    Then $f$ is injective because

    $$x \neq y \implies x < y \text{ or } x > y \implies f(x) < f(y) \text{ or } f(x) > f(y) \implies f(x) \neq f(y).$$

    Therefore the inverse $f^{-1} : im(f) \to I$ exists:

    $$f^{-1}(x) \neq f^{-1}(y) \implies f^{-1}(f(x)) \neq f^{-1}(f(y)) \implies x \neq y,$$

    that is, $f^{-1}$ is a function because $x = y \implies f^{-1}(x) = f^{-1}(y)$ for all $x, y \in im(f)$.
    Let $x \in I$ such that $a_n, b_n \in I$ with $a_n < x < b_n$ and $\lim_{n\to\infty} a_n = x$, $\lim_{n\to\infty} b_n = x$. Then clearly, $\lim_{n\to\infty}[b_n - a_n] = 0$. Then because $f$ is strictly increasing, $f(a_n) < f(x) < f(b_n)$.
    Now, we have the sequences $f(a_n)$ and $f(b_n)$ in $im(f)$ such that for each $n$, $f(a_n) < f(x) < f(b_n)$, and $\lim_{n\to\infty}[f^{-1}(f(b_n)) - f^{-1}(f(a_n))] = \lim_{n\to\infty}[b_n - a_n] = 0$. The results from Problem 35 tells us that $f^{-1}$ is continuous at $f(x)$.

42. Let $f$ be a continuous function and $B$ be a Borel set. Show that $f^{-1}(B)$ is a Borel set. (Hint: the collection of sets $E$ for which $f^{-1}(E)$ is Borel is a $\sigma$-algebra containing the open sets.)

    Let $S = \{E \mid f^{-1}(E) \text{ is Borel}\}$.
    To show that $S$ is a $\sigma$-algebra, know that the Borel sets is a $\sigma$-algebra.
    Observe that:

    (i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in S$.

    (ii) $E \in S \implies f^{-1}(E)$ is Borel $\implies f^{-1}(E)^c = f^{-1}(E^c)$ is Borel $\implies E^c \in S$.

    (iii) $E_k \in S \implies f^{-1}(E_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} f^{-1}(E_k) = f^{-1}(\bigcup_{k=1}^{\infty} E_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} E_k \in S$.

    Also, any open set $\mathcal{O}$ is in $S$ because $f^{-1}(\mathcal{O})$ is open and thus Borel. Thus $S$ is a $\sigma$-algebra containing the open sets; that is, the Borel $\sigma$-algebra is a subset of $S$. Therefore for any Borel set $B$, $B \in S$ and thus $f^{-1}(B)$ is Borel.

43. Use the preceding two problems to show that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

    Let $I$ be an interval and $f : I \to \mathbb{R}$ be a continuous strictly increasing function.
    By Problem 41, we showed that $f^{-1} : im(f) \to I$ exists and is continuous.
    Let $B \in I$ be any Borel set. By Problem 42, $(f^{-1})^{-1}(B) = f(B)$ is a Borel set.

# Chapter 3

# Lebesgue Measurable Functions

## Contents



## 3.1 Sums, Products, and Compositions

**Proposition 1.** *Let the function $f$ have a measurable domain $E$. Then the following statements are equivalent:*

*(i) For each real number c, the set $\{x \in E \mid f(x) > c\}$ is measurable.*

*(ii) For each real number c, the set $\{x \in E \mid f(x) \geq c\}$ is measurable.*

*(iii) For each real number c, the set $\{x \in E \mid f(x) < c\}$ is measurable.*

*(iv) For each real number c, the set $\{x \in E \mid f(x) \leq c\}$ is measurable.*

*Each of these properties implies that for each extended real number c,*

$$\text{the set } \{x \in E \mid f(x) = c\} \text{ is measurable.}$$

**Definition.** *An extended real-valued function $f$ defined on $E$ is said to be* ***Lebesgue measurable****, or simply* ***measurable****, provided its domain $E$ is measurable and it satisfies one of the four statements of Proposition 1.*

**Proposition 2.** *Let the real-valued function $f$ be defined on a measurable set $E$. Then the function $f$ is measurable iff for each open set $\mathcal{O}$, the inverse image of $\mathcal{O}$ under $f$, $f^{-1}(\mathcal{O}) = \{x \in E \mid f(x) \in \mathcal{O}\}$, is a measurable set.*

*Proof.* Let $f : E \to \mathbb{R}$, where $E$ is a measurable set.
($\implies$) Suppose that $f$ is measurable.

Let $\mathcal{O}$ be open. Then by Chapter 1, Proposition 9, $\mathcal{O}$ can be written as the countable disjoint union of open intervals: $\mathcal{O} = \bigcup_{k=1}^{\infty} I_k$. We can construct these intervals in the following form:

$$I_k = (a_k, b_k) = (-\infty, b_k) \cap (a_k, \infty)$$

Therefore we see that

$$\begin{aligned}
f^{-1}(\mathcal{O}) &= f^{-1}(\bigcup_{k=1}^{\infty} I_k) \\
&= f^{-1}(\bigcup_{k=1}^{\infty} (-\infty, b_k) \cap (a_k, \infty)) \\
&= \bigcup_{k=1}^{\infty} f^{-1}((-\infty, b_k) \cap (a_k, \infty)) \\
&= \bigcup_{k=1}^{\infty} f^{-1}(-\infty, b_k) \cap f^{-1}(a_k, \infty).
\end{aligned}$$

Because $f$ is measurable, we see that $f^{-1}((-\infty, b_k))$ and $f^{-1}((a_k, \infty))$ are measurable sets, and countable union and intersection of measurable sets is also a measurable set, so $f^{-1}(\mathcal{O})$ is a measurable set.
( $\Longleftarrow$ ) Suppose that for each open set $\mathcal{O}$, $f^{-1}(\mathcal{O})$ is a measurable set.
Because for any real number $c$, the interval of the form $(c, \infty)$ is an open set, and therefore we have that the set $f^{-1}((c, \infty)) = \{x \in E \mid f(x) \in (c, \infty)\} = \{x \in E \mid f(x) > c\}$ is measurable, which implies that $f$ is measurable. $\square$

**Proposition 5.** *Let $f$ be an extended real-valued function on the measurable set $E$.*

*(i) If $f$ is measurable on $E$ and $f = g$ a.e. on $E$, then $g$ is measurable on $E$.*

*(ii) For a measurable subset $D$ of $E$, $f$ is measurable on $E$ iff the restrictions of $f$ to $D$ and $E \setminus D$ are measurable.*

*Proof.* Let $f$ be an extended real-valued function on the measurable set $E$.

(i) Let $f$ be measurable on $E$ and $f = g$ a.e. on $E$.
Define $A = \{x \in E \mid f(x) \neq g(x)\} \subseteq E$, so that $f = g$ on $E \setminus A$, and $m(A) = 0$.

$$\begin{aligned}
\{x \in E \mid g(x) > c\} &= (\{x \in E \mid g(x) > c\} \cap [E \cap A]) \cup (\{x \in E \mid g(x) > c\} \cap [E \cap A^c]) \\
&= \{x \in A \mid g(x) > c\} \cup (\{x \in E \mid f(x) > c\} \cap [E \setminus A]).
\end{aligned}$$

Now, $\{x \in A \mid g(x) > c\} \subseteq A$, and because $m(A) = 0$, $\{x \in A \mid g(x) > c\}$ is measurable and has measure zero. The set $\{x \in E \mid f(x) > c\}$ is measurable because $f$ is measurable, and $E \cap A^c$ is measurable because $E$ and $A$ (and thus $A^c$) are measurable. Thus $\{x \in E \mid g(x) > c\}$ is measurable because it is the finite union and intersection of measurable sets; therefore $g$ is measurable on $E$.

(ii) Let $D$ be a measurable subset of $E$.
( $\Longrightarrow$ ) Suppose $f$ is measurable on $E$.
Then for any real number $c$, we see that

$$\begin{aligned}
\{x \in D \mid f|_D(x) > c\} &= \{x \in E \mid f(x) > c\} \cap [E \cap D], \\
\{x \in E \setminus D \mid f|_{E \setminus D}(x) > c\} &= \{x \in E \mid f(x) > c\} \cap [E \setminus D],
\end{aligned}$$

where both are measurable because they are each intersections of measurable sets. Therefore the restrictions $f|_D$ and $f|_{E\setminus D}$ are measurable.
( $\Longleftarrow$ ) Suppose the restrictions of $f$ to $D$ and $E \setminus D$ are measurable.
Then for any real number $c$,

$$\{x \in E \mid f(x) > c\} = \{x \in D \mid f|_D(x) > c\} \cup \{x \in E \setminus D \mid f|_{E\setminus D}(x) > c\},$$

which is measurable because it is a union of measurable sets.

□

**PROBLEMS**

1. Suppose $f$ and $g$ are continuous functions on $[a, b]$. Show that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set $E$?

   Let $f, g$ be continuous functions on $[a, b]$, where $f = g$ on $[a, b] \setminus E_0$, where $E_0$ is a subset of $[a, b]$ and $m(E_0) = 0$.
   Suppose that $E_0$ is nonempty.
   Consider any point $x_0 \in E_0 \subseteq [a, b]$. For any $\epsilon > 0$, there exists a $c \in (x_0 - \epsilon, x_0 + \epsilon) \cap [a, b]$ such that $f(c) = g(c)$, else we reach a contradiction because $m((x_0 - \epsilon, x_0 + \epsilon) \cap [a, b]) \neq 0$. This means that we can construct a sequence $\{c_i\}_{i=1}^{\infty}$ that converges to $x_0$, where $f(c_i) = g(c_i)$ is defined for all $i$. However, because $\{c_i\} \to x_0$, by continuity of $f, g$, we have $\{f(c_i)\} \to f(x_0)$ and $\{g(c_i)\} \to g(x_0)$, and because $f(c_i) = g(c_i)$ for all $i$, the limit is unique; that is,

   $$|f(x_0) - g(x_0)| \leq |f(x_0) - f(c_i)| + |f(c_i) - g(x_0)| < \epsilon,$$

   and $f(x_0) = g(x_0)$.
   However, this is a contradiction to $f(x) \neq g(x)$ for all $x \in E_0$, and so $E_0 = \emptyset$.

   In the case that $[a, b]$ is replaced by a general measurable set $E$, the assertion is not true. Consider the case where $E = \{a\}$, so that $f, g$ are continuous on $\{a\}$, and $f = g$ a.e. on $\{a\}$. This only implies that $f = g$ on $E$ except for a set of measure zero. But $E$ is already of measure zero, so $f(a) \neq g(a)$ is possible, and $f \neq g$ on $E$.

2. Let $D$ and $E$ be measurable sets and $f$ a function with domain $D \cup E$. We proved that $f$ is measurable on $D \cup E$ iff its restrictions to $D$ and $E$ are measurable. Is the same true if "measurable" is replaced by "continuous"?

   No; consider the function $f : [-1, 1] \to \mathbb{R}$, where $[-1, 1] = [-1, 0) \cup [0, 1]$, and we define

   $$f(x) = \begin{cases} 0 & x \in [-1, 0), \\ 1 & x \in [0, 1]. \end{cases}$$

   Clearly we have a point of discontinuity at $x = 0$, so $f$ is not continuous even though $f|_{[-1,0)}$ and $f|_{[0,1]}$ are continuous.

3. Suppose a function $f$ has a measurable domain and is continuous except at a finite number of points. Is $f$ necessarily measurable?

   Yes; let $f$ be a function on the measurable domain $E$, and suppose $f$ is continuous on $E \setminus E_0$,

where $E_0 = \{x_i\}_{i=1}^n \subseteq E$. Then $m(E_0) = 0$ because countable sets are measurable and have measure zero.
Now, $f|_{E\setminus E_0}$ is continuous and therefore measurable (Proposition 3), and $f|_{E_0}$ is defined on a set of measure zero, so any subset $\{x \in E_0 \mid f|_{E_0}(x) > c\} \subseteq E_0$ has measure zero and is thus measurable, and therefore $f|_{E_0}$ is a measurable function.
Recall Proposition 5 to see that for the measurable subset $E_0$ of $E$, $f$ is measurable because $f|_{E_0}$ and $f|_{E\setminus E_0}$ are both measurable functions.

4. Suppose $f$ is a real-valued function on $\mathbb{R}$ such that $f^{-1}(c)$ is measurable for each number $c$. Is $f$ necessarily measurable?

   No; let $V \subseteq [0,1]$ be a Vitali set. Therefore $V$ is nonmeasurable (see Ch 2.6). Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined as
$$f(x) = \begin{cases} -e^x & x \in V \\ e^x & x \notin V \end{cases}$$
   For any real number $c$, we have
$$f^{-1}(c) = \begin{cases} \ln(-c) & c < 0 \\ \ln(c) & c > 0 \\ \emptyset & c = 0 \end{cases}$$
   and so $f^{-1}(c)$ is a singleton set or is the empty set, which are measurable, so $f^{-1}(c)$ is measurable.
   Now, we know that $e^x : \mathbb{R} \to \mathbb{R}_{>0}$ and so $e^x > 0$ for any real number $x$.
   Therefore $f(x) = -e^x < 0$ only when $x \in V$. However, the set $\{x \in \mathbb{R} \mid f(x) < 0\} = V$ is not measurable, and so $f$ is not a measurable function.

5. Suppose the function $f$ is defined on a measurable set $E$ and $\{x \in E \mid f(x) > c\}$ is a measurable set for each rational number $c$. Is $f$ necessarily a measurable function?

   Yes. Let $f : E \to \mathbb{R}$ with $E$ a measurable set, and let $\{x \in E \mid f(x) > c\} = \{x \in E \mid f(x) \in (c, \infty)\}$ be measurable for each $c \in \mathbb{Q}$.
   Let $a$ be any real number. Then for any natural number $n$, there exists a rational number $c_n$ such that $a < c_n < a + \frac{1}{n}$, and therefore $\bigcup_{n=1}^{\infty}(c_n, \infty) = (a, \infty)$. Therefore we have
$$\begin{aligned} \{x \in E \mid f(x) > a\} &= f^{-1}((a,\infty)) \\ &= f^{-1}(\bigcup_{n=1}^{\infty}(c_n,\infty)) \\ &= \bigcup_{n=1}^{\infty} f^{-1}((c_n,\infty)) \\ &= \bigcup_{n=1}^{\infty}\{x \in E \mid f(x) > c_n\}, \end{aligned}$$
   which is a countable union of measurable sets, and therefore is also measurable.

6. Let $f$ be a function with measurable domain $D$. Show that $f$ is measurable iff the function $g$ defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Let $D \subseteq \mathbb{R}$ be a measurable set, let $f : D \to \mathbb{R}$, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

( $\implies$ ) Suppose that $f$ is measurable.
For any real number $c$,

$$\{x \in \mathbb{R} \mid g(x) > c\} = \begin{cases} \{x \in D \mid f(x) > c\} & c \geq 0 \\ \{x \in D \mid f(x) > c\} \cup D^c & c < 0 \end{cases}$$

Both of the sets $\{x \in D \mid f(x) > c\}$ and $\{x \in D \mid f(x) > c\} \cap D^c$ are measurable, so $\{x \in \mathbb{R} \mid g(x) > c\}$ is measurable and thus $g$ is a measurable function.

( $\impliedby$ ) Suppose that $g$ is measurable.
Recall Proposition 5 (ii) to see that for the measurable subset $D$ of $\mathbb{R}$, $g$ is measurable on $\mathbb{R}$, which implies that the restrictions $g|_D$ and $g|_{\mathbb{R}\setminus D}$ are measurable. Therefore for any real number $c$,

$$\{x \in D \mid f(x) > c\} = \{x \in \mathbb{R} \mid g|_D(x) > c\} \cap D \text{ is measurable,}$$

and $f$ is measurable.

7. Let the function $f$ be defined on a measurable set $E$. Show that $f$ is measurable iff for each borel set $A$, $f^{-1}(A)$ is measurable. (Hint: the collection of sets $A$ that have the property that $f^{-1}(A)$ is measurable is a $\sigma$-algebra.)

   Let $f : E \to \mathbb{R}$, where $E$ is a measurable set.
   ( $\implies$ ) Suppose that $f$ is measurable.
   Let $\mathcal{M} = \{A \mid f^{-1}(A) \text{ is measurable}\}$.
   To show that $\mathcal{M}$ is a $\sigma$-algebra, know that the measurable sets is a $\sigma$-algebra.
   Observe that:

   (i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in \mathcal{M}$.

   (ii) $A \in \mathcal{M} \implies f^{-1}(A)$ is measurable $\implies f^{-1}(A)^c = f^{-1}(A^c)$ is measurable $\implies A^c \in \mathcal{M}$.

   (iii) $A_k \in \mathcal{M} \implies f^{-1}(A_k)$ is measurable $\implies \bigcup_{k=1}^{\infty} f^{-1}(A_k) = f^{-1}(\bigcup_{k=1}^{\infty} A_k)$ is measurable $\implies \bigcup_{k=1}^{\infty} A_k \in \mathcal{M}$.

   Then because $f$ is measurable, for any real number $a$, the set $f^{-1}((a,\infty)) = \{x \in E \mid f(x) > a\}$ is measurable. Now, $(a,\infty) \in \mathcal{M}$ because $f^{-1}((a,\infty))$ is measurable. Because $(a,\infty)$ is a Borel set, all other Borel sets are in $\mathcal{M}$ because the Borel sets are a $\sigma$-algebra.

   ( $\impliedby$ ) Suppose that for each borel set $A$, the set $f^{-1}(A) = \{x \in E \mid f(x) \in A\}$ is measurable.
   Every interval of the form $(a,\infty)$ is a borel set, so we have that for any real number $a$, the set $f^{-1}((a,\infty)) = \{x \in E \mid f(x) > a\}$ is measurable. This is equivalent to the measurability of $f$.

8. (Borel measurability) A function $f$ is said to be **Borel measurable** provided its domain $E$ is a Borel set and for each $c$, the set $\{x \in E \mid f(x) > c\}$ is a Borel set. Verify that Proposition 1 and Theorem 6 remain valid if we replace "(Lebesgue) measurable set" by "Borel set". Show that:

(i) every Borel measurable function is Lebesgue measurable,

The Borel sets are a subset of the measurable sets. Therefore for a Borel measurable function $f$, its domain $E$ is a Borel set (and thus a measurable set), and for each $c$, the set $\{x \in E \mid f(x) > c\}$ is a Borel set (and thus a measurable set). Thus $f$ is a measurable function.

(ii) if $f$ is Borel measurable and $B$ is a Borel set, then $f^{-1}(B)$ is a Borel set,

Let $S = \{B \mid f^{-1}(B) \text{ is Borel}\}$.
To show that $S$ is a $\sigma$-algebra, know that the Borel sets is a $\sigma$-algebra.
Observe that:

(i) $f^{-1}(\emptyset) = \emptyset \implies \emptyset \in S$.
(ii) $B \in S \implies f^{-1}(B)$ is Borel $\implies f^{-1}(B)^c = f^{-1}(B^c)$ is Borel $\implies B^c \in S$.
(iii) $B_k \in S \implies f^{-1}(B_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} f^{-1}(B_k) = f^{-1}(\bigcup_{k=1}^{\infty} B_k)$ is Borel $\implies \bigcup_{k=1}^{\infty} B_k \in S$.

Now, because $f$ is Borel measurable, for any real number $a$, the set $f^{-1}((a,\infty))$ is a Borel set. This implies $(a,\infty) \in S$. Because $(a,\infty)$ is a Borel set, all other Borel sets are in $S$ because the Borel sets is a $\sigma$-algebra.

(iii) if $f$ and $g$ are Borel measurable, so is $f \circ g$,

Let $f, g$ be Borel measurable, with $g : E \to F$, and $f : F \to \mathbb{R}$, where $E, F$ are Borel sets. Then $f \circ g : E \to \mathbb{R}$ has a Borel set as its domain.
Recall that $\{x \mid f(x) > a\} = \{x \mid f(x) \in (a,\infty)\} = f^{-1}((a,\infty))$.

$$
\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a,\infty)\} \\
&= (f \circ g)^{-1}((a,\infty)) \\
&= (g^{-1} \circ f^{-1})((a,\infty)) \\
&= g^{-1}(f^{-1}((a,\infty))) \\
&= g^{-1}(B) && \text{where } B \text{ is Borel} \\
&= B'. && \text{where } B' \text{ is Borel}
\end{aligned}
$$

Therefore $f \circ g$ is Borel measurable.

(iv) if $f$ is Borel measurable and $g$ is Lebesgue measurable, then $f \circ g$ is Lebesgue measurable.

Let $f$ be Borel measurable, with $f : F \to \mathbb{R}$, and let $g$ be Lebesgue measurable, with $g : E \to F$, where $F$ is a Borel set and $E$ is a measurable set. Then $f \circ g : E \to \mathbb{R}$ has a measurable set as its domain.

$$
\begin{aligned}
\{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a,\infty)\} \\
&= (f \circ g)^{-1}((a,\infty)) \\
&= (g^{-1} \circ f^{-1})((a,\infty)) \\
&= g^{-1}(f^{-1}((a,\infty))) \\
&= g^{-1}(B) && \text{where } B \text{ is Borel} \\
&= B'. && \text{where } B' \text{ is measurable (Problem 7)}
\end{aligned}
$$

Therefore $f \circ g$ is Lebesgue measurable.

9. Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set $E$. Define $E_0$ to be the set of points of $x$ in $E$ at which $\{f_n(x)\}$ converges. Is the set $E_0$ measurable?

   Let $E_0 = \{x \in E \mid \{f_n(x)\}$ converges$\} \subseteq \{x \in E \mid \{f_n(x)\}$ is Cauchy$\}$, because all convergent sequences are Cauchy.
   Therefore $E_0 = \{x \in E \mid \forall k \in \mathbb{N}, \exists N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \frac{1}{k}$ for all $n, m \geq N\}$.
   This is equivalent to writing

   $$E_0 = \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n,m\geq N} \{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}.$$

   The functions $f_n$ and $f_m$ are measurable, so by Theorem 6, $f_n - f_m$ is also measurable. The absolute value function $|\cdot|$ is continuous, so by Proposition 7, the composition $|\cdot| \circ (f_n - f_m) = |f_n - f_m|$ is a measurable function. Therefore for the real number $\frac{1}{k}$, the set $\{x \in E \mid |f_n(x) - f_m(x)| < \frac{1}{k}\}$ is a measurable set.
   Then $E_0$ is a countable union and intersection of measurable sets, so $E_0$ is measurable.

10. Suppose $f$ and $g$ are real-valued functions defined on all of $\mathbb{R}$, $f$ is measurable, and $g$ is continuous. Is the composition $f \circ g$ necessarily measurable?

    No; let $f$ be measurable, with $f : \mathbb{R} \to \mathbb{R}$, and let $g$ be continuous, with $g : \mathbb{R} \to \mathbb{R}$. Then $f \circ g : \mathbb{R} \to \mathbb{R}$.

    $$\begin{aligned} \{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a, \infty)\} \\ &= (f \circ g)^{-1}((a, \infty)) \\ &= (g^{-1} \circ f^{-1})((a, \infty)) \\ &= g^{-1}(f^{-1}((a, \infty))) \\ &= g^{-1}(A) && \text{where } A \text{ is measurable} \end{aligned}$$

    Recall Chapter 2 Problem 36 to see that for a continuous function $g$, the set $g^{-1}(A)$ is not always measurable when $A$ is measurable.

11. Let $f$ be a measurable function and $g$ be a one-to-one function from $\mathbb{R}$ onto $\mathbb{R}$ which has a Lipschitz inverse. Show that the composition $f \circ g$ is measurable. (Hint: examine Problem 37 in Chapter 2.)

    Let $f$ be measurable, with $f : \mathbb{R} \to \mathbb{R}$, and let $g$ be a bijection from $\mathbb{R}$ to $\mathbb{R}$, where $g^{-1}$ is Lipschitz. From Chapter 2 Problem 37, we have that $g^{-1}$ maps a measurable set to a measurable set; that is, for the measurable set $A$, the set $g^{-1}(A)$ is measurable.
    We have $f \circ g : \mathbb{R} \to \mathbb{R}$.

    $$\begin{aligned} \{x \mid (f \circ g)(x) > a\} &= \{x \mid (f \circ g)(x) \in (a, \infty)\} \\ &= (f \circ g)^{-1}((a, \infty)) \\ &= (g^{-1} \circ f^{-1})((a, \infty)) \\ &= g^{-1}(f^{-1}((a, \infty))) \\ &= g^{-1}(A) && \text{where } A \text{ is measurable} \\ &= A'. && \text{where } A' \text{ is measurable (Chapter 2 Problem 37)} \end{aligned}$$

    Therefore $f \circ g$ is measurable.

## 3.2 Sequential Pointwise Limits and Simple Approximation

**Definition.** *For a sequence $\{f_n\}$ of functions with common domain $E$, a function $f$ on $E$, and a subset $A$ of $E$, we say that*

*(i) The sequence $\{f_n\}$ converges to $f$ pointwise on $A$ provided*

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ for all } x \in A.$$

*a.k.a.*

$$\forall x \in A, \forall \epsilon > 0, \exists N_x \in \mathbb{N} : \forall n \geq N_x, |f_n(x) - f(x)| < \epsilon.$$

*(ii) The sequence $\{f_n\}$ converges to $f$ pointwise a.e. on $A$ provided it converges to $f$ pointwise on $A \setminus B$, where $m(B) = 0$.*

*(iii) The sequence $\{f_n\}$ converges to $f$ uniformly on $A$ provided for each $\epsilon > 0$, there is an index $N$ for which*

$$|f - f_n| < \epsilon \text{ on } A \text{ for all } n \geq N.$$

*a.k.a.*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in A, |f_n(x) - f(x)| < \epsilon.$$

**Theorem.** *Let $A \subseteq \mathbb{R}^n$ and suppose $\{f_i\}$ is a sequence of functions $f_i : A \to \mathbb{R}^m$ such that*

*(i) $\{f_i\} \to f$ uniformly on $A$*

*(ii) Each $f_i$ is uniformly continuous on $A$.*

*Then $f : A \to \mathbb{R}^m$ is uniformly continuous on $A$.*

*Proof.* Fix $\epsilon > 0$.
By (i), there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, for all $x \in A$, $\|f_n(x) - f(x)\| < \epsilon/3$.
Now, fix $N$ and fix $k \geq N$.
By (ii), there exists a $\delta > 0$ such that for all $x, y \in A$ with $\|x - y\| < \delta$, then $\|f_k(x) - f_k(y)\| < \epsilon/3$.
Therefore we have

$$\begin{aligned} \|f(x) - f(y)\| &= \|f(x) - f_k(x) + f_k(x) - f_k(y) + f_k(y) - f(y)\| \\ &\leq \|f(x) - f_k(x)\| + \|f_k(x) - f_k(y)\| + \|f_k(y) - f(y)\| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Therefore for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in A$ with $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < \epsilon$. Thus $f$ is uniformly continuous on $A$.
A similar proof can show this for a sequence of continuous functions converging uniformly to a continuous function. $\square$

**Example.** *Consider the sequence of continuous functions $\{f_n\}_{n=2}^{\infty} : [0,1] \to \mathbb{R}$, defined by*

$$f_n(x) = \begin{cases} \frac{n-0}{1/n-0}x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{0-n}{2/n-1/n}(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases} = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ -n^2(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases}$$

*(Each $f_n$ is a triangle-shaped function that achieves its max $f(1/n) = n$ and base corners $f(0) = 0$ and $f(2/n) = 0$.)*
*In addition, consider the continuous function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = 0$ for all $x \in [0, 1]$.*
*The sequence $\{f_n\}$ converges to $f$ pointwise but not uniformly on $[0, 1]$.*
*To see this, for any $\epsilon > 0$, and for any $x \in [0, 1]$, there exists an index $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, we have $\frac{2}{n} < x$, that is, $x \in (\frac{2}{n}, 1]$, and so $|f_n(x) - f(x)| = 0 - 0 = 0 < \epsilon$.*
*Therefore the sequence converges pointwise.*
*To see that the sequence does not converge uniformly, we see that there exists an $\epsilon = 1 > 0$ such that for all indices $N$, there exists an $n \geq N$ and the point $1/n$ such that $|f_n(1/n) - f(1/n)| = n - 0 = n > 1$.*

The pointwise limit of continuous functions may not be continuous.
The pointwise limit of Riemann integrable functions may not be Riemann integrable.

**Proposition 9.** *Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise a.e. on $E$ to the function $f$. Then $f$ is measurable.*

*Proof.* Let $E_0$ be a subset of $E$ such that $\{f_n\}$ converges pointwise to $f$ on $E \setminus E_0$, with $m(E_0) = 0$. Because $E_0$ has measure zero, then it is measurable; by Proposition 5, we have that $f$ is measurable iff the restrictions to $E_0$ and $E \setminus E_0$ are measurable. By monotonicity of measure, the measure of the set $\{x \in E_0 \mid f(x) < c\}$ is always zero and thus trivially measurable.
We want to show that $\{x \in E \setminus E_0 \mid f(x) < c\}$ is measurable to use Proposition 5.
Now, for any point $x \in E \setminus E_0$, we have that

$$f(x) < c \iff \exists n, k \in \mathbb{N} \text{ s.t. } f_j(x) < c - 1/n \, \forall j \geq k.$$

To see why, suppose that $\forall n, k \in \mathbb{N}, \exists j \geq k$ s.t. $f_j(x) \geq c - 1/n$.
In the case $f_j(x) > c - 1/n$, we have $1/n + f_j(x) \geq c$ for all $n$, which implies that $f_j(x) \geq c > f$, a contradiction.
In the case $f_j(x) = c - 1/n$, we have that for any $n$, for any indices $k$, there exists an index $j \geq k$ such that $c - f_j(x) = 1/n$, but because $f < c$, the convergence $\{f_n\} \to f$ is not possible, a contradiction.
Then we can write

$$\{x \in E \setminus E_0 \mid f(x) < c\} = \bigcup_{1 \leq n, k < \infty} \left[ \bigcap_{j=k}^{\infty} \{x \in E \setminus E_0 \mid f_j(x) < c - 1/n\} \right],$$

Which is a countable union and intersection of measurable sets, and thus is measurable. □

If $A$ is any set, the **characteristic function** of $A$, $\chi_A$, is the function on $\mathbb{R}$ defined by

$$\chi_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

The function $\chi_A$ is measurable iff the set $A$ is measurable:
( $\Longrightarrow$ ) Suppose $\chi_A$ is measurable.
Then for the real number $c = 0$, the set $\{x \in \mathbb{R} \mid \chi_A(x) > 0\} = A$ is measurable.
( $\Longleftarrow$ ) Suppose the set $A$ is measurable.
Then for any real number $c$, we have

$$\{x \in \mathbb{R} \mid \chi_A(x) \in (c, \infty)\} = \begin{cases} \emptyset & c \geq 1 \\ A & 1 > c \geq 0 \\ \mathbb{R} & 0 > c \end{cases}$$

Each of the sets $\emptyset$, $A$, and $\mathbb{R}$ are measurable, so $\{x \in \mathbb{R} \mid \chi_A(x) \in (c, \infty)\}$ is a measurable set and thus $\chi_A$ is a measurable function.
Thus the existence of a nonmeasurable set $E$ implies the existence of a nonmeasurable function $\chi_E$.

**Definition.** *A real-valued function $\varphi$ defined on a measurable set $E$ is called* ***simple*** *provided it is measurable and takes only a finite number of values.*

If $\varphi$ is simple, has domain $E$ and takes the distinct values $c_1, \cdots, c_n$, then

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\}.$$

(A linear combination of measurable functions)
A simple function is called a **step function** in the special case that the sets $E_k$ are intervals.

**The Simple Approximation Lemma.** *Let $f$ be a measurable real-valued function on $E$. Assume $f$ is bounded on $E$, that is, there is an $M \geq 0$ for which $|f| \leq M$ on $E$. Then for each $\epsilon > 0$, there are simple functions $\varphi_\epsilon$ and $\psi_\epsilon$ defined on $E$ which have the following approximation properties:*

$$\varphi_\epsilon \leq f \leq \psi_\epsilon \text{ and } 0 \leq \psi_\epsilon - \varphi_\epsilon < \epsilon \text{ on } E.$$

*Proof.* Because $f$ is bounded, we have that $f(x) \subseteq [-M, M]$ for all $x \in E$.
Let $(c, d)$ be an open, bounded interval such that $f(E) \subseteq [-M, M] \subseteq (c, d)$, so that $(c, d)$ contains $f(E)$, the image of $E$ under $f$.
Also, choose $\{y_0, \cdots, y_n\}$ to be a partition of the closed, bounded interval $[c, d]$ such that $y_k - y_{k-1} < \epsilon$ for $1 \leq k \leq n$:

$$c = y_0 < y_1 < \cdots < y_{n-1} < y_n = d$$

Define

$$I_k = [y_{k-1}, y_k) \text{ and } E_k = f^{-1}(I_k) \text{ for } 1 \leq k \leq n.$$

Because $f$ is a measurable function and each interval $I_k = [y_{k-1}, y_k)$ is Borel, then each set $E_k = f^{-1}(I_k)$ is measurable (see Problem 7). Because $E_k$ is measurable, then $\chi_{E_k}$ is a measurable function.
Then we can define the simple functions $\varphi_\epsilon$ and $\psi_\epsilon$ on $E$ by

$$\varphi_\epsilon = \sum_{k=1}^{n} y_{k-1} \cdot \chi_{E_k}$$
$$\psi_\epsilon = \sum_{k=1}^{n} y_k \cdot \chi_{E_k}$$

Now, for any $x \in E$, because $f(E) \subseteq (c, d)$, and $\{[y_{k-1}, y_k)\}_{k=1}^{n}$ is a partition that contains $(c, d)$, there exists a unique $k$, $1 \leq k \leq n$, for which $f(x) \in [y_{k-1}, y_k)$, and therefore

$$\varphi_\epsilon(x) = y_{k-1} \leq f(x) < y_k = \psi_\epsilon(x).$$

Also, for each $x \in E$, $\psi_\epsilon(x) - \varphi_\epsilon(x) = y_k - y_{k-1} < \epsilon$ for some $k$. □

**The Simple Approximation Theorem.** *An extended real-valued function $f$ on a measurable set $E$ is measurable iff there is a sequence $\{\varphi_n\}$ of simple functions on $E$ which converges pointwise on $E$ to $f$ and has the property that*

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n.$$

*If $f$ is nonnegative, we may choose $\{\varphi_n\}$ to be increasing.*

*Proof.* Let $f$ be an extended real-valued function on a measurable set $E$.
( $\Longleftarrow$ ) Suppose $\{\varphi_n\}$ is a sequence of simple (and thus measurable) functions on $E$ that converges pointwise on $E$ to $f$ (and has the property that $|\varphi_n| \leq |f|$ on $E$ for all $n$).
Then by Proposition 9, $f$ is measurable.
( $\Longrightarrow$ ) Suppose $f$ is measurable.
We can assume $f \geq 0$ on $E$ (See Problem 23 for the general case, as $f = f^+ - f^-$ on $E$, a linear combination of two nonnegative measurable functions $f^+$ and $f^-$).
For a natural number $n$, define $E_n = \{x \in E \mid f(x) \leq n\}$. Because $f$ is a measurable function, the set $E_n \subseteq E$ is measurable. Then by Proposition 5 (ii), $f|_{E_n}$ is measurable.
Also, $f|_{E_n}$ is bounded because $0 \leq f|_{E_n}(x) \leq n$ for $x \in E_n$.
Now, recall the Simple Approximation Lemma to see that for the measurable real-valued function $f|_{E_n}$ on $E_n$, where $f|_{E_n}$ is bounded on $E_n$, we have that for $\epsilon = 1/n$, there exist simple functions $\varphi_n$ and $\psi_n$ defined on $E_n$ such that

$$0 \leq \varphi_n \leq f \leq \psi_n \text{ on } E_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Then we can see that

$$0 \leq \varphi_n \leq f \text{ and } 0 \leq f - \varphi_n \leq \psi_n - \varphi_n < 1/n \text{ on } E_n.$$

Now, $E = E_n \cup E_n^c = \{x \in E \mid f(x) \leq n\} \cup \{x \in E \mid f(x) > n\}$ and $\varphi_n$ is defined on $E_n$. We can extend $\varphi_n$ to all of $E$ by setting $\varphi_n(x) = n$ on $E_n^c = \{x \in E \mid f(x) > n\}$. Then $\varphi_n$ is a simple function defined on $E$ and $0 \leq \varphi_n \leq f$ on $E$. To see that $\{\varphi_n\}$ converges to $f$ pointwise on $E$:
Consider any $x \in E$.
Case $f(x) < \infty$:
Then there exists a natural number $N$ such that $f(x) \leq N$. Then for all $n \geq N$, We have that

$$0 \leq f(x) - \varphi_n(x) < 1/n,$$

and thus $\lim_{n\to\infty} \varphi_n(x) = f(x)$.
Case $f(x) = \infty$:
Then $\varphi_n(x) = n$ because $f(x) > n$ for all $n$, and $\lim_{n\to\infty} \varphi_n(x) = \infty = f(x)$.
By replacing each $\varphi_n$ with $\max\{\varphi_1, \cdots, \varphi_n\}$, we have $\{\varphi_n\}$ increasing. □

**PROBLEMS**

12. Let $f$ be a bounded measurable function on $E$. Show that there are sequences of simple functions on $E$, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing and each of these sequences converges to $f$ uniformly on $E$.

    Let $f$ be a bounded measurable function on the measurable set $E$. Because $f$ is bounded, there exists a real number $M$ such that $|f| \leq M$ and thus $-M \leq f \leq M$. Then for all natural numbers $n \geq M$, the set $E_n = \{x \in E \mid f(x) \leq n\} = E$. By the Simple Approximation Lemma, for $\epsilon = 1/n$, there exist the simple function $\varphi_n$ and $\psi_n$ defined on $E_n = E$ such that

    $$\varphi_n \leq f \leq \psi_n \text{ and } 0 \leq \psi_n - \varphi_n < 1/n \text{ on } E.$$

    Then for each $n \geq M$, we have the sequences of functions $\{\varphi_n\}$ and $\{\psi_n\}$ such that

    $$\varphi_n \leq f \text{ and } f - \varphi_n \leq \psi_n - \varphi_n < 1/n \text{ on } E,$$
    $$f \leq \psi_n \text{ and } \psi_n - f \leq \psi_n - \varphi_n < 1/n \text{ on } E.$$

We can replace each $\varphi_n$ with $\max\{\varphi_1, \cdots, \varphi_n\}$ and each $\psi_n$ with $\min\{\psi_1, \cdots, \psi_n\}$ so that the sequences are increasing and decreasing respectively. The convergence is uniform because we showed that for any $\epsilon = 1/n$, there exists the natural number $n$ such that for all $n' \geq n$, $f(x) - \varphi_{n'}(x) < 1/n$ and $\psi_{n'}(x) - f(x) < 1/n$ for all $x \in E$.

13. A real-valued measurable function is said to be *semisimple* provided it takes only a countable number of values. Let $f$ be any measurable function on $E$. Show that there is a sequence of semisimple functions $\{f_n\}$ on $E$ that converges to $f$ uniformly on $E$.

We can define $f_n(x) = \frac{1}{n}\lfloor nf(x)\rfloor$ (where the floor function $\lfloor x\rfloor$ returns the largest integer less than or equal to $x$). Then because $f_n = \frac{\lfloor nf(x)\rfloor}{n}$ and $\lfloor nf(x)\rfloor$ and $n$ are integers, we have $f_n(E) \subseteq \mathbb{Q}$ which is countable. Because the floor function rounds down to the nearest integer, we have

$$|nf(x) - \lfloor nf(x)\rfloor| < 1 \text{ for all } x \in E,$$

and therefore

$$|f(x) - \frac{1}{n}\lfloor nf(x)\rfloor| < 1/n \text{ for all } x \in E.$$

14. Let $f$ be a measurable function on $E$ that is finite a.e. on $E$ and $m(E) < \infty$. For each $\epsilon > 0$, show that there is a measurable set $F$ contained in $E$ such that $f$ is bounded on $F$ and $m(E\backslash F) < \epsilon$.

For each natural number $n$, let $F_n = \{x \in E \mid f(x) > n\}$. Because $f$ is measurable, then each $F_n$ is a measurable set. Then $\{F_n\}$ is descending because

$$F_n = \{x \in E \mid f(x) > n\} \supseteq \{x \in E \mid f(x) > n+1 > n\} = F_{n+1}.$$

Also, $F_1 \subseteq E$, so by monotonicity of measure, $m(F_1) \leq m(E) < \infty$.
By the continuity of measure, for the descending collection of measurable sets $\{F_n\}$ for which $m(F_1) < \infty$, we have $m(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n)$.
Now,

$$\bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} \{x \in E \mid f(x) > n\} = \{x \in E \mid f(x) = \infty\},$$

Because $f$ is finite a.e. on $E$, we have

$$0 = m(\{x \in E \mid f(x) = \infty\}) = m(\bigcap_{n=1}^{\infty} F_n) = \lim_{n\to\infty} m(F_n).$$

Then for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m(F_n) < \epsilon$.
If we let $F = E \cap F_n^c = E \setminus F_n \subseteq E$, then $F = \{x \in E \mid f(x) \leq n\}$, a measurable set, and

$$E \cap F^c = E \cap [E^c \cup F_n] = [E \cap E^c] \cup [E \cap F_n] = \emptyset \cup F_n = F_n,$$

and thus we have

$$m(E \setminus F) = m(E \cap F^c) = m(F_n) < \epsilon.$$

15. Let $f$ be a measurable function on $E$ that if finite a.e. on $E$ and $m(E) < \infty$. Show that for each $\epsilon > 0$, there is a measurable set $F$ contained in $E$ and a sequence $\{\varphi_n\}$ of simple functions on $E$ such that $\{\varphi_n\} \to f$ uniformly on $F$ and $m(E \setminus F) < \epsilon$. (Hint: see the preceding problem.)

Because $f$ is finite a.e. on $E$ and $m(E) < \infty$, by the previous problem, for any $\epsilon > 0$, there exists a measurable set $F \subseteq E$ such that $f$ is bounded on $F$ and $m(E \setminus F) < \epsilon$. Furthermore, because $f$ is bounded and measurable on $F$, by Problem 12, there exists a sequence of simple functions $\{\varphi_n\}$ on $F$ that converges uniformly to $f$ on $F$. Each $\varphi_n$ can be extended to $E$ by setting $\varphi_n = n$ on $F^c$.

16. Let $I$ be a closed, bounded interval and $E$ a measurable subset of $I$. Let $\epsilon > 0$. Show that there is a step function $h$ on $I$ and a measurable subset $F$ of $I$ for which

$$h = \chi_E \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: use Theorem 12 of Chapter 2.)

Theorem 12 of Chapter 2 states:
Let $E$ be a measurable set of finite outer measure. Then for each $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $\mathcal{O} = \bigcup_{k=1}^n I_k$, then

$$m^*(E \setminus \mathcal{O}) + m^*(\mathcal{O} \setminus E) < \epsilon.$$

...
Fix $\epsilon > 0$.
Because $m(E)$ is finite, by definition of infimum, for $\epsilon > 0$, there exists a countable collection of open intervals $\{I_k\}_{k=1}^\infty$ whose union $\mathcal{O} = \bigcup_{k=1}^\infty I_k$ covers $E$ and

$$m(E) \leq \sum_{k=1}^{\infty} \ell(I_k) < m(E) + \epsilon/2,$$

and $\sum_{k=1}^\infty \ell(I_k)$ is finite as well. Because $E$ is a measurable subset of finite outer measure that is contained in $\mathcal{O}$, then we can use the excision property and subadditivity to see that

$$m(\mathcal{O} \setminus E) = m(\mathcal{O}) - m(E) = m(\bigcup_{k=1}^{\infty} I_k) - m(E) \leq \sum_{k=1}^{\infty} \ell(I_k) - m(E) < \epsilon/2, \qquad (1)$$

Because the series $\sum_{k=1}^\infty \ell(I_k)$ is finite, the sequence of partial sums $\sum_{k=1}^n \ell(I_k)$ converges to $\sum_{k=1}^\infty \ell(I_k)$, so that for $\epsilon/2$, there exists an index $N$ such that for all $n \geq N$, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \ell(I_k) - \sum_{k=1}^{n} \ell(I_k) &< \epsilon/2 \\ \sum_{k=n+1}^{\infty} \ell(I_k) + \sum_{k=1}^{n} \ell(I_k) - \sum_{k=1}^{n} \ell(I_k) &< \epsilon/2 \\ \sum_{k=n+1}^{\infty} \ell(I_k) &< \epsilon/2, \end{aligned}$$

and then we can let $\mathcal{O}' = \bigcup_{k=1}^N I_k$ so that, because $E \cap (\mathcal{O} \setminus \mathcal{O}') \subseteq \mathcal{O} \setminus \mathcal{O}'$,

$$m(E \cap (\mathcal{O} \setminus \mathcal{O}')) \leq m(\mathcal{O} \setminus \mathcal{O}') = m(\bigcup_{k=1}^{\infty} I_k \setminus \bigcup_{k=1}^{N} I_k) = m(\bigcup_{k=N+1}^{\infty} I_k) \leq \sum_{k=N+1}^{\infty} \ell(I_k) < \epsilon/2.$$

Because $E \subseteq \mathcal{O}$, then $E \cap \mathcal{O}^c = \emptyset$, and we can derive

$$\begin{aligned}
E \cap \mathcal{O}' &= [\emptyset \cup (E \cap \mathcal{O}')] \\
&= [(E \cap \mathcal{O}^c) \cup (E \cap \mathcal{O}')] \\
&= \emptyset \cup [(E \cap \mathcal{O}^c) \cup (E \cap \mathcal{O}')] \\
&= [E \cap E^c] \cup [E \cap (\mathcal{O}^c \cup \mathcal{O}')] \\
&= E \cap [E^c \cup (\mathcal{O}^c \cup \mathcal{O}')] \\
&= E \cap [E^c \cup (\mathcal{O} \cap \mathcal{O}'^c)^c] \\
&= E \cap [E \cap (\mathcal{O} \setminus \mathcal{O}')]^c \\
&= E \setminus [E \cap (\mathcal{O} \setminus \mathcal{O}')].
\end{aligned}$$

And then we see that by excision,

$$m(E \cap \mathcal{O}') = m(E \setminus [E \cap (\mathcal{O} \setminus \mathcal{O}')]) = m(E) - m(E \cap (\mathcal{O} \setminus \mathcal{O}')) > m(E) - \epsilon/2. \quad (2)$$

We will let $F = (E \cap \mathcal{O}') \cup (I \setminus \mathcal{O})$ so that

$$\begin{aligned}
I \setminus F &= I \cap F^c \\
&= I \cap [(E \cap \mathcal{O}') \cup (I \setminus \mathcal{O})]^c \\
&= I \cap [(I^c \cup \mathcal{O}) \cap (E \cap \mathcal{O}')^c] \\
&= [I \cap (I^c \cup \mathcal{O})] \cap (E \cap \mathcal{O}')^c \\
&= [(I \cap I^c) \cup (I \cap \mathcal{O})] \cap (E \cap \mathcal{O}')^c \\
&= [I \cap \mathcal{O}] \setminus (E \cap \mathcal{O}') \\
&\subseteq \mathcal{O} \setminus [E \cap \mathcal{O}'].
\end{aligned}$$

Therefore we can write

$$\begin{aligned}
m(I \setminus F) &\leq m(\mathcal{O} \setminus [E \cap \mathcal{O}']) && \text{by monotonicity} \\
&= m(\mathcal{O}) - m(E \cap \mathcal{O}') && \text{by excision} \\
&< m(\mathcal{O}) - m(E) + \epsilon/2 && \text{by (2)} \\
&< \epsilon/2 + \epsilon/2 && \text{by (1)} \\
&< \epsilon.
\end{aligned}$$

We can let $h = \sum_{k=1}^{n} \chi_{J_k} = \chi_{\mathcal{O}'}$, with $J_k = I_k \setminus \bigcup_{j=1}^{k-1} I_j$, where each $J_k$ is a finite union of disjoint intervals, and so $h$ is a step function. Then $h = 1$ on $E \cap \mathcal{O}'$ and $h = 0$ on $I \setminus \mathcal{O}$, so that $h = \chi_E$ on $F$.

17. Let $I$ be a closed, bounded interval and $\psi$ a simple function defined on $I$. Let $\epsilon > 0$. Show that there is a step function $h$ on $I$ and a measurable subset $F$ of $I$ for which

$$h = \psi \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

(Hint: use the fact that a simple function is a linear combination of characteristic functions and the preceding problem.)

We have

$$\psi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } I, \text{ where } E_k = \{x \in I \mid \psi(x) = c_k\} \text{ measurable.}$$

For each $k = 1, \cdots, n$, we have that $I$ is a closed, bounded interval and $E_k$ a measurable subset of $I$. For $\epsilon > 0$, by the previous Problem 16, there is a step function $h_k$ on $I$ and a measurable subset $F_k$ of $I$ for which

$$h_k = \chi_{E_k} \text{ on } F_k \text{ and } m(I \setminus F_k) < \epsilon/n.$$

We can let $h = \sum_{k=1}^{n} c_k h_k$ and $F = \bigcap_{k=1}^{n} F_k$ so that $h = \psi$ on $F$ and $I \cap F^c = I \cap \bigcup_{k=1}^{n} F_k^c = \bigcup_{k=1}^{n}(I \cap F_k^c)$ which gives us

$$m(I \setminus F) = m(I \cap F^c) = m(\bigcup_{k=1}^{n}(I \cap F_k^c)) \leq \sum_{k=1}^{n} m(I \cap F_k^c) = \sum_{k=1}^{n} m(I \setminus F_k) < \epsilon.$$

18. Let $I$ be a closed, bounded interval and $f$ a bounded measurable function defined on $I$. Let $\epsilon > 0$. Show that there is a step function $h$ on $I$ and a measurable subset $F$ of $I$ for which

$$|h - f| < \epsilon \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Because $f$ is bounded and measurable on $I$, by Problem 12, there exists a sequence of simple functions $\{\psi_n\}$ on $I$ that converges uniformly to $f$ on $I$. Then for any $\epsilon$, we can choose $\psi \in \{\psi_n\}$ such that $|\psi - f| < \epsilon$. By the previous Problem 17, there is a step function $h$ on $I$ and a measurable subset $F$ of $I$ for which

$$h = \psi \text{ on } F \text{ and } m(I \setminus F) < \epsilon.$$

Therefore we have $|h - f| < \epsilon$.

19. Show that the sum and product of two simple functions are simple as are the max and the min.

Consider two simple functions $\varphi$ and $\psi$ on the measurable set $E$, with

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \text{ on } E, \text{ where } E_k = \{x \in E \mid \varphi(x) = c_k\} \text{ measurable.}$$

$$\psi = \sum_{k'=1}^{m} c'_{k'} \cdot \chi_{E'_{k'}} \text{ on } E, \text{ where } E'_{k'} = \{x \in E \mid \psi(x) = c'_{k'}\} \text{ measurable.}$$

Then for any $x \in E$, there exists an $i \in \{1, \cdots, n\}$ and $j \in \{1, \cdots, m\}$ such that

$$(\varphi + \psi)(x) = c_i + c'_j \text{ on } E_i \cap E'_j,$$

The intersection of measurable sets $E_i \cap E'_j$ is also measurable, so the function $\chi_{E_i \cap E'_j}$ is measurable.
That is, we have the simple function

$$\varphi + \psi = \sum_{k=1}^{n} \sum_{k'=1}^{m} (c_k + c_{k'}) \cdot \chi_{E_k \cap E'_{k'}} \text{ on } E.$$

Similarly for the product, we have

$$\varphi \cdot \psi = \sum_{k=1}^{n} \sum_{k'=1}^{m} (c_k \cdot c_{k'}) \cdot \chi_{E_k \cap E'_{k'}} \text{ on } E.$$

We can recall Chapter 1 Problem 49 to see how we define max and min:

$$\max\{\varphi, \psi\} = \frac{1}{2}(\varphi + \psi + |\varphi - \psi|),$$
$$\min\{\varphi, \psi\} = \frac{1}{2}(\varphi + \psi - |\varphi - \psi|).$$

Clearly scaling a simple function is simple, and the absolute value of a simple function is simple, and we showed that the sum of simple functions is simple, and therefore the max and min of simple functions is simple.

20. Let $A, B$ be any sets. Show that

$$\chi_{A\cap B} = \chi_A \cdot \chi_B$$
$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$$
$$\chi_{A^c} = 1 - \chi_A$$

We can use DeMorgan's laws to see that

$$\chi_{A\cap B} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases} = \begin{cases} 1 & x \in A \cap B \\ 0 & x \in A^c \cup B^c \end{cases} = \begin{cases} 1 & x \in A \text{ and } x \in B \\ 0 & x \notin A \text{ or } x \notin B \end{cases}$$

Then for any $x \in \mathbb{R}$, we have

| $\chi_A(x)$ | $\chi_B(x)$ | $\chi_{A\cap B}(x)$ | $\chi_A(x) \cdot \chi_B(x)$ |
|---|---|---|---|
| 0 | 0 | 0 | $0 \cdot 0 = 0$ |
| 0 | 1 | 0 | $0 \cdot 1 = 0$ |
| 1 | 0 | 0 | $1 \cdot 0 = 0$ |
| 1 | 1 | 1 | $1 \cdot 1 = 1$ |

Similarly see that

$$\chi_{A\cup B} = \begin{cases} 1 & x \in A \cup B \\ 0 & x \notin A \cup B \end{cases} = \begin{cases} 1 & x \in A \cup B \\ 0 & x \in A^c \cap B^c \end{cases} = \begin{cases} 1 & x \in A \text{ or } x \in B \\ 0 & x \notin A \text{ and } x \notin B \end{cases}$$

Then for any $x \in \mathbb{R}$, we have

| $\chi_A(x)$ | $\chi_B(x)$ | $\chi_{A\cup B}(x)$ | $\chi_A(x) + \chi_B(x) - \chi_A(x) \cdot \chi_B(x)$ |
|---|---|---|---|
| 0 | 0 | 0 | $0 + 0 - 0 \cdot 0 = 0$ |
| 0 | 1 | 1 | $0 + 1 - 0 \cdot 1 = 1$ |
| 1 | 0 | 1 | $1 + 0 - 1 \cdot 0 = 1$ |
| 1 | 1 | 1 | $1 + 1 - 1 \cdot 1 = 1$ |

Finally see that

$$\chi_{A^c} = \begin{cases} 1 & x \in A^c \\ 0 & x \notin A^c \end{cases} = \begin{cases} 1 & x \notin A \\ 0 & x \in A \end{cases} = \begin{cases} 1-0 & x \notin A \\ 1-1 & x \in A \end{cases} = 1 - \begin{cases} 0 & x \notin A \\ 1 & x \in A \end{cases} = 1 - \chi_A.$$

That is, for any $x \in \mathbb{R}$, we have

| $\chi_A(x)$ | $\chi_{A^c}(x)$ | 1-$\chi_A(x)$ |
|---|---|---|
| 1 | 0 | $1-1=0$ |
| 0 | 1 | $1-0=1$ |

21. For a sequence $\{f_n\}$ of measurable functions with common domain $E$, show that each of the following functions is measurable:

- $\inf\{f_n\}$

  We have
  $$\inf\{f_n\} = \{x \in E \mid \inf\{f_n\} > c\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f_n(x) > c\}$$
  or
  $$\inf\{f_n\} = \{x \in E \mid \inf\{f_n\} < c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) < c\}$$

- $\sup\{f_n\}$

  Similarly,
  $$\sup\{f_n\} = \{x \in E \mid \sup\{f_n\} > c\} = \bigcup_{n=1}^{\infty} \{x \in E \mid f_n(x) > c\}$$
  or
  $$\sup\{f_n\} = \{x \in E \mid \sup\{f_n\} < c\} = \bigcap_{n=1}^{\infty} \{x \in E \mid f_n(x) < c\}$$

- $\lim\inf\{f_n\}$
- $\lim\sup\{f_n\}$

22. (Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions on $[a, b]$ which converges pointwise on $[a, b]$ to the continuous function $f$ on $[a, b]$. Show that the convergence is uniform on $[a, b]$. (Hint: let $\epsilon > 0$. For each natural number $n$, define $E_n = \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\}$. Show that $\{E_n\}$ is an open cover of $[a, b]$ and use the Heine-Borel Theorem.)

Let $\epsilon > 0$. For each natural number $n$, define
$$\begin{aligned} E_n &= \{x \in [a, b] \mid f(x) - f_n(x) < \epsilon\} \\ &= \{x \in [a, b] \mid f(x) - f_n(x) \in (-\infty, \epsilon)\} \\ &= (f - f_n)^{-1}((-\infty, \epsilon)). \end{aligned}$$

The sum and product of continuous functions is continuous, so the function $f - f_n$ is continuous and therefore $E_n = (f - f_n)^{-1}((-\infty, \epsilon))$ is an open set.
Because $\{f_n\}$ converges pointwise to $f$ on $[a, b]$, for any $\epsilon > 0$, for any $x \in [a, b]$, there exists an index $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, we have $|f(x) - f_n(x)| < \epsilon$.
This means that for any $x \in [a, b]$, there exists an index $N_x \in \mathbb{N}$ such that $x \in \{x \in [a, b] \mid f(x) - f_{N_x}(x) < \epsilon\} = E_{N_x}$, and so $x \in \bigcup_{n \in \mathbb{N}} E_n$, which implies
$$\forall x \in [a, b], \exists N_x \in \mathbb{N} \text{ s.t. } x \in \{x \in [a, b] \mid |f(x) - f_{N_x}(x)| < \epsilon\} = E_{N_x} \implies x \in \bigcup_{n \in \mathbb{N}} E_n,$$

and by definition of subset, $[a, b] \subseteq \bigcup_{n\in\mathbb{N}} E_n$. Now, $\{E_n\}$ is an open cover of $[a, b]$ because it is a union of open sets $E_n$ and it covers $[a, b]$. Because $[a, b]$ is compact, there exists a finite subcover $\{E_{n_k}\}_{k=1}^{m} \subseteq \{E_n\}$.
This means that for any $x \in [a, b]$, there exists the index $k \in \{1, \cdots, m\}$ such that $x \in E_{n_k} = \{x \in [a, b] \mid |f(x) - f_{n_k}(x)| < \epsilon\}$.
Then we can let $N_0 = \max\{n_1, \cdots, n_m\}$.
Therefore for any $\epsilon > 0$, there exists the index $N_0$ such that for all $n \geq N_0 \geq n_i$, $i \in \{1, \cdots, m\}$,

$$|f(x) - f_n(x)| < \epsilon \text{ for all } x \in [a, b].$$

Thus we have uniform convergence.

23. Express a measurable function as the difference of nonnegative measurable functions and thereby prove the general Simple Approximation Theorem based on the special case of a nonnegative measurable function.

Let $f$ be a measurable function on $E$, and we have $f = f^+ - f^-$ on $E$, a linear combination of the two nonnegative measurable functions $f^+ = \max\{f, 0\} \geq 0$ and $f^- = \max\{-f, 0\} \geq 0$. In our proof for the Simple Approximation Theorem, we proved the case for $f^+ \geq 0$ and $f^- \geq 0$ that there exist sequences $\{\varphi_n^+\}$ and $\{\varphi_n^-\}$ of simple functions on $E$ that converge pointwise on $E$ to $f^+$ and $f^-$ respectively, and

$$0 \leq \varphi_n^+ \leq f^+ \text{ on } E \text{ for all } n,$$
$$0 \leq \varphi_n^- \leq f^- \text{ on } E \text{ for all } n.$$

then

$$0 \geq -\varphi_n^- \geq -f^- \text{ on } E \text{ for all } n,$$

so that $-f^- \leq -\varphi_n^- \leq f^- \implies -\varphi_n^- \leq |f^-|$.
We have the sets $E^+ = \{x \in E \mid f(x) \geq 0\}$ and $E^- = \{x \in E \mid f(x) \leq 0\}$, so that

$$f(x) = |f(x)| = \begin{cases} 0 & \text{if } x \in E^+ \cap E^- \\ f^+(x) & \text{if } x \in E^+ \text{ only} \\ f^-(x) & \text{if } x \in E^- \text{ only} \end{cases}$$

The function $\varphi_n^+ - \varphi_n^-$ is simple and

$$0 \leq \varphi_n^+ \leq f^+ = 0, 0 \leq \varphi_n^- \leq f^- = 0 \text{ on } E^+ \cap E^- \implies \varphi_n^+, \varphi_n^- = 0,$$

and so we have

$$(\varphi_n^+ - \varphi_n^-)(x) = \begin{cases} 0 \leq f(x) & \text{if } x \in E^+ \cap E^- \\ \varphi_n^+(x) \leq f^+(x) & \text{if } x \in E^+ \text{ only} \\ -\varphi_n^-(x) \leq f^-(x) & \text{if } x \in E^- \text{ only} \end{cases}$$

Then clearly $\varphi_n^+ - \varphi_n^-$ converges pointwise to $f$ on $E$, and therefore we have

$$|\varphi_n^+ - \varphi_n^-| \leq |f| \text{ on } E \text{ for all } n.$$

24. Let $I$ be an interval and $f : I \to \mathbb{R}$ be increasing. Show that $f$ is measurable by first showing that, for each natural number $n$, the strictly increasing function $x \mapsto f(x) + x/n$ is measurable, and then taking pointwise limits.

 Let $f_n(x) = f(x) + x/n$.
 Then each $f_n$ is strictly increasing; that is, for $x, y \in I$, we have $f_n(x) < f_n(y) \iff x < y$. This also tells us that each $f_n$ is injective because

$$x \neq y \implies x < y \text{ or } y < x \implies f_n(x) < f_n(y) \text{ or } f_n(y) < f_n(x) \implies f_n(x) \neq f_n(y).$$

 So for any element $x^* \in dom(f_n)$, we know that $f_n^{-1}(x^*)$ consists of a single element at most.
 Now, for any two $x^*, y^* \in (a, \infty)$, by definition of interval, any point $z^*$ between them is also in $(a, \infty)$.
 We then have $x^* < z^* < y^* \iff f_n^{-1}(x^*) < f_n^{-1}(z^*) < f_n^{-1}(y^*)$, and so the set $f_n^{-1}((a, \infty)) = \{x \in I \mid f_n(x) \in (a, \infty)\}$ is an interval for all $a$, and every interval is measurable, so $f_n$ is measurable.
 To see that $\{f_n\}$ converges pointwise to $f$, let $\epsilon > 0$ and consider $x \in I$. Then there exists an index $N$ such that for all $n \geq N$,

$$|f_n(x) - f(x)| = |x/n| < \epsilon.$$

 Now, because $\{f_n\}$ is a sequence of measurable functions in $I$ that converges pointwise a.e. on $I$ to $f$, then by Proposition 9, $f$ is measurable.

## 3.3 Littlewood's Three Principles, Ergoff's Theorem, and Lusin's Theorem

**Egoroff's Theorem.** *Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise on $E$ to the real-valued function $f$. Then for each $\epsilon > 0$, there is a closed set $F$ contained in $E$ for which*

$$\{f_n\} \to f \textit{ uniformly on } F \textit{ and } m(E \setminus F) < \epsilon.$$

**Lemma 10.** *Under the assumptions of Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$, there is a measurable subset $A$ of $E$ and an index $N$ for which*

$$|f_n - f| < \eta \textit{ on } A \textit{ for all } n \geq N \textit{ and } m(E \setminus A) < \delta.$$

*Proof.* First, we can see that because $\{f_n\}$ is a sequence of measurable functions on $E$ that converges pointwise on $E$ to $f$, by Proposition 9, $f$ is measurable. Then by Theorem 6, the function $f_n - f$ is measurable. Finally by Proposition 7, considering the continuous function $|\cdot|$ and the measurable function $f_n - f$, the composition $|f_n - f|$ is measurable.
This means that the set $\{x \in E \mid |f_n - f| < \eta\}$ is measurable.
Then we see that

$$E_n = \{x \in E \mid |f_k - f| < \eta \text{ for all } k \geq n\} = \bigcap_{k=n}^{\infty} \{x \in E \mid |f_k - f| < \eta\},$$

is also measurable.

Then $\{E_n\}$ is an ascending sequence of measurable sets because

$$E_n = \{x \in E \mid |f_n - f| < \eta\} \cap \left[\bigcap_{k=n+1}^{\infty} \{x \in E \mid |f_k - f| < \eta\}\right] \subseteq \bigcap_{k=n+1}^{\infty} \{x \in E \mid |f_k - f| < \eta\} = E_{n+1}.$$

Also, $E = \bigcup_{n=1}^{\infty} E_n$, because $\{f_n\}$ converges pointwise to $f$ on $E$.
That is, for $\eta > 0$, for $x \in E$, there exists an index $N \in \mathbb{N}$ such that

$$|f_k(x) - f(x)| < \eta \text{ for all } k \geq N,$$

and thus $x \in E_N$.
Now, by continuity of measure, we have

$$m(E) = m(\bigcup_{n=1}^{\infty} E_n) = \lim_{n \to \infty} m(E_n).$$

Then because $m(E) < \infty$, there exists an index $N_0$ for which $m(E) - m(E_{N_0}) < \delta$.
Define $A = E_{N_0}$ so we can use excision to see that

$$m(E \setminus A) = m(E) - m(E_{N_0}) < \delta,$$

and

$$A = \{x \in E \mid |f_k - f| < \eta \text{ for all } k \geq N_0\}.$$

□

*Proof.* To prove Egoroff's Theorem:
Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges pointwise on $E$ to the real-valued function $f$.
For each natural number $n$, we can let $\eta = 1/n$ and $\delta = \epsilon/2^{n+1}$.
By Lemma 10, there exists a subset $A_n$ of $E$ and an index $N_n$ for which

$$|f_k - f| < 1/n \text{ on } A_n \text{ for all } k \geq N_n \text{ and } m(E \setminus A_n) < \epsilon/2^{n+1}.$$

We define

$$A = \bigcap_{n=1}^{\infty} A_n.$$

Then we see that

$$\begin{aligned}
m(E \setminus A) &= m(E \setminus \left[\bigcap_{n=1}^{\infty} A_n\right]) \\
&= m(E \cap \left[\bigcup_{n=1}^{\infty} A_n^c\right]) \\
&= m(\bigcup_{n=1}^{\infty} [E \cap A_n^c]) \\
&\leq \sum_{n=1}^{\infty} m(E \cap A_n^c) \\
&< \sum_{n=1}^{\infty} \epsilon/2^{n+1} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \epsilon/2^n \\
&= \epsilon/2.
\end{aligned}$$

To see that $\{f_n\}$ converges uniformly to $f$ on $A$:
Let $\epsilon > 0$. Then there exists an index $n_0$ such that $1/n_0 < \epsilon$ and

$$|f_k - f| < 1/n_0 < \epsilon \text{ on } A \subseteq A_{n_0} \text{ for all } k \geq N_{n_0}.$$

Finally we can use Chapter 2 Theorem 11 to choose a closed set $F$ contained in $A$ for which $m(A \setminus F) < \epsilon/2$. Then we have

$$\begin{aligned}
m(E \setminus F) &= m(E) - m(F) && \text{by excision} \\
&= m(E) - m(A) + m(A) - m(F) \\
&= m(E \setminus A) + m(A \setminus F) && \text{by excision} \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Therefore $\{f_n\} \to f$ uniformly on $F$ and $m(E \setminus F) < \epsilon$. □

**Proposition 11.** *Let $f$ be a simple function defined on $E$. Then for each $\epsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which*

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \epsilon.$$

**Lusin's Theorem.** *Let $f$ be a real-valued measurable function on $E$. Then for each $\epsilon > 0$, there is a continuous function $g$ on $\mathbb{R}$ and a closed set $F$ contained in $E$ for which*

$$f = g \text{ on } F \text{ and } m(E \setminus F) < \epsilon.$$

*Proof.* Consider the case that $m(E) < \infty$.
The Simple Approximation Theorem tells us that there exists a sequence of simple functions $\{f_n\}$ defined on $E$ that converges to $f$ pointwise on $E$.

By the previous Proposition 11, for each $n$, there exists a continuous function $g_n$ on $\mathbb{R}$ and a closed set $F_n$ contained in $E$ for which

$$f_n = g_n \text{ on } F_n \text{ and } m(E \setminus F_n) < \epsilon/2^{n+1}.$$

Also, Egoroff's Theorem states that because $E$ has finite measure and that $\{f_n\}$ is a sequence of simple (measurable) function on $E$ that converges pointwise on $E$ to the real-valued function $f$, we have that there exists a closed set $F_0$ contained in $E$ such that

$$\{f_n\} \to f \text{ uniformly on } F_0 \text{ and } m(E \setminus F_0) < \epsilon/2.$$

We define $F = \bigcap_{n=0}^{\infty} F_n$, which is closed because it is an intersection of closed sets. Then we see that

$$\begin{aligned}
m(E \setminus F) &= m(E \cap \left[\bigcup_{n=0}^{\infty} F_n^c\right]) \\
&= m(\bigcup_{n=0}^{\infty} [E \cap F_n^c]) \\
&= m([E \cap F_0^c] \cup \bigcup_{n=1}^{\infty} [E \cap F_n^c]) \\
&= m([E \setminus F_0] \cup \bigcup_{n=1}^{\infty} [E \setminus F_n]) \\
&\leq m([E \setminus F_0]) + \sum_{n=1}^{\infty} m(E \setminus F_n) \\
&< \epsilon/2 + \sum_{n=1}^{\infty} \epsilon/2^{n+1} \\
&= \epsilon.
\end{aligned}$$

Each $f_n$ is continuous on $F$ since $F \subseteq F_n$ and $f_n = g_n$ on $F_n$.
Also, $\{f_n\}$ converges to $f$ uniformly on $F$ since $F \subseteq F_0$.
Then we can use the fact that because $\{f_n\}$ is a sequence of continuous functions on $F$ that converges uniformly on $F$ to $f$, then $f$ is continuous on $F$ as well.
Finally see problem 25 to see that there exists a continuous function $g$ that extends $f$ to all of $\mathbb{R}$. Then $f = g$ on $F$ and $m(E \setminus F) < \epsilon$. □

**PROBLEMS**

25. Suppose $f$ is a function that is continuous on a closed set $F$ of real numbers. Show that $f$ has a continuous extension to all of $\mathbb{R}$. This is a special case of the forthcoming Tietze Extension Theorem. (Hint: express $\mathbb{R} \setminus F$ as the union of a countable disjoint collection of open intervals and define $f$ to be linear on the closure of each of these intervals.)

    (See Chapter 1 Problem 47.)
    Let $f$ be a function that is continuous on the closed set $F$. Consider the open set $F^c$. By Chapter 1 Proposition 9, this open $F^c$ is the union of a countable, disjoint collection of intervals.
    In the case that $(-\infty, a)$ [or $(a, \infty)$] is in $F^c$, then $a \in F$ and $f(a)$ is defined. Simply let

$f(x) = f(a)$ be the constant function on $(-\infty, a)$ [or $(a, \infty)$].
In the case that $(a, b) \in F^c$, then $a, b \in F$ and $f(a)$,$f(b)$ are defined. Let

$$f(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ on } (a, b).$$

Then we see that the extension of $f$ is continuous on $\mathbb{R}$.

26. For the function $f$ and the set $F$ in the statement of Lusin's Theorem, show that the restriction of $f$ to $F$ is a continuous function. Must there be any points at which $f$, considered as a function of $E$, is continuous?

    See the proof for Lusin's Theorem; because $\{f_n\}$ is a sequence of continuous functions on $F$ that converges uniformly on $F$ to $f$, then $f$ is continuous on $F$ as well.

27. Show that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.

    Going back to the proof for Egoroff's Theorem, we see that we used the excision property:

$$\begin{aligned} m(E \setminus F) &= m(E) - m(F) && \text{by excision} \\ &= m(E) - m(A) + m(A) - m(F) \\ &= m(E \setminus A) + m(A \setminus F) && \text{by excision} \\ &< \epsilon/2 + \epsilon/2 \\ &= \epsilon. \end{aligned}$$

    The excision property requires that

$$\begin{aligned} m(E \setminus F) &= m(E) - m(F) && \text{if } m(F) < \infty \text{ and } F \subseteq E \\ m(E \setminus A) &= m(E) - m(A) && \text{if } m(A) < \infty \text{ and } A \subseteq E \\ m(A \setminus F) &= m(A) - m(F) && \text{if } m(F) < \infty \text{ and } F \subseteq A \end{aligned}$$

    Specifically, we needed that $m(A), m(F) < \infty$. This was only possible because we assumed $m(E) < \infty$.

28. Show that Egoroff's Theorem continues to hold if the convergence is pointwise a.e. and $f$ is finite a.e.

    In Lemma 10 on the way to proving Egoroff's Theorem, we used Proposition 9, which only requires the convergence to be pointwise a.e.
    In Lemma 10 we also used Theorem 6, which only requires $f_n$ and $f$ to be finite a.e.

29. Prove the extension of Lusin's Theorem to the case that $E$ has infinite measure.

    We needed to assume that $E$ had finite measure because we used Egoroff's Theorem in the proof for Lusin's Theorem, which requires finite measure (Problem 27).

30. Prove the extension of Lusin's Theorem to the case that $f$ is not necessarily real-valued, but is finite a.e.

    We needed to assume that $f$ was real valued because we used Egoroff's Theorem in the proof for Lusin's Theorem, which requires $f$ to be real-valued. However, we showed that Egoroff's Theorem continues to hold if $f$ is finite a.e. (Problem 28).

31. Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converges to the real-valued $f$ pointwise on $E$. Show that $E = \bigcup_{k=1}^{\infty} E_k$, where for each index $k$, $E_k$ is measurable, and $\{f_n\}$ converges uniformly to $f$ on each $E_k$ if $k > 1$, and $m(E_1) = 0$.

    Use Egoroff's Theorem.

# Chapter 4

# Lebesgue Integration

## Contents



## 4.1 The Riemann Integral

In this chapter the Lebesgue integral is defined in four stages:

- define the integral for simple functions over a set of finite measure.
- define the integral for bounded measurable functions $f$ over a set of finite measure, in terms of integrals of upper and lower approximations of $f$ by simple functions.
- define the integral of a general nonnegative measurable function $f$ over $E$ to be the supremum of the integrals of lower approximations of $f$ by bounded measurable functions that vanish outside a set of finite measure; the integral of such a function is nonnegative, but may be infinite.
- define a general measurable function to be integrable over $E$ provided $\int_E |f| < \infty$.

Let $f$ be a bounded real-valued function defined on the closed, bounded interval $[a, b]$. Let $P = \{x_0, x_1, \cdots, x_n\}$ be a partition of $[a, b]$, that is,

$$a = x_0 < x_1 < \cdots < x_n = b.$$

Define the **lower and upper Darboux sums** for $f$ with respect to $P$, respectively, by

$$L(f, P) = \sum_{i=1}^{n} m_i \cdot (x_i - x_{i-1})$$

and

$$U(f, P) = \sum_{i=1}^{n} M_i \cdot (x_i - x_{i-1}),$$

where, for $1 \le i \le n$,

$$m_i = \inf\{f(x) \mid x_{i-1} < x < x_i\} \text{ and } M_i = \sup\{f(x) \mid x_{i-1} < x < x_i\}.$$

We then define the **lower and upper Riemann integrals** of $f$ over $[a, b]$, respectively, by

$$(R)\underline{\int_a^b} f = \sup\Big\{L(f, P) \mid P \text{ a partition of } [a, b]\Big\}$$

and

$$(R)\overline{\int_a^b} f = \inf\Big\{U(f, P) \mid P \text{ a partition of } [a, b]\Big\}.$$

If the upper and lower integrals are equal we say that $f$ is **Riemann integrable** over $[a, b]$ and call this common value the Riemann integral of $f$ over $[a, b]$:

$$(R)\int_a^b f$$

A real-valued function $\psi$ defined on $[a, b]$ is called a **step function** provided there is a partition $P = \{x_0, x_1, \cdots, x_n\}$ of $[a, b]$ and numbers $c_1, \cdots, c_n$ such that for $1 \le i \le n$,

$$\psi(x) = c_i \text{ if } x_{i-1} < x < x_i.$$

Clearly a step function is Riemann integrable:

$$\sum_{i=1}^{n} c_i \cdot (x_i - x_{i-1}) = L(\psi, P) = U(\psi, P) = (R)\int_a^b \psi$$

Then we can reformulate the definition of the lower and upper Riemann integrals:

$$(R)\underline{\int_a^b} f = \sup\Big\{(R)\int_a^b \varphi \mid \varphi \text{ a step function and } \varphi \le f \text{ on } [a, b]\Big\}$$

and

$$(R)\overline{\int_a^b} f = \inf\Big\{(R)\int_a^b \psi \mid \psi \text{ a step function and } \varphi \ge f \text{ on } [a, b]\Big\}.$$

**Example (Dirichlet's Function)** Define $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

...

$$f_n(x) = \begin{cases} 1 & x \in \{q_1, \cdots, q_n\} \\ 0 & \text{else} \end{cases}$$

**PROBLEMS**

1. Show that, in the above Dirichlet function example, $\{f_n\}$ fails to converge to $f$ uniformly on $[0, 1]$.

   Let $\epsilon = 1/2$. Then for any natural number $n$ we choose, there exists $n+1 \ge n$ and $q_{n+1} \in [0, 1] \cap \mathbb{Q}$ such that $|f_n(q_{n+1}) - f(q_{n+1})| = |0 - 1| = 1 > 1/2$. Therefore uniform convergence fails.

2. A partition $P'$ of $[a,b]$ is called a refinement of a partition $P$ provided each partition point of $P$ is also a partition point of $P'$. For a bounded function $f$ on $[a,b]$, show that under refinement lower Darboux sums increase and upper Darboux sums decrease.

3. Use the preceding problem to show that for a bounded function on a closed, bounded interval, each lower Darboux sum is no greater than each upper Darboux sum. From this conclude that the lower Riemann integral is no greater than the upper Riemann integral.

4. Suppose the bounded function $f$ on $[a,b]$ is Riemann integrable over $[a,b]$. Show that there is a sequence $\{P_n\}$ of partitions of $[a,b]$ for which $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$.

5. Let $f$ be a bounded function on $[a,b]$. Suppose there is a sequence $\{P_n\}$ of partitions of $[a,b]$ for which $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$. Show that $f$ is Riemann integrable over $[a,b]$.

6. Use the preceding problem to show that since a continuous function $f$ on a closed, bounded interval $[a,b]$ is uniformly continuous on $[a,b]$, it is Riemann integrable over $[a,b]$.

7. Let $f$ be an increasing real-valued function on $[0,1]$. For a natural number $n$, define $P_n$ to be the partition of $[0,1]$ into $n$ subintervals of length $1/n$. Show that $U(f,P_n)-L(f,P_n)\le 1/n[f(1)-f(0)]$. Use Problem 5 to show that $f$ is Riemann integrable over $[0,1]$.

8. Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to $f$ on the closed, bounded interval $[a,b]$. If each $f_n$ is Riemann integrable over $[a,b]$, show that $f$ also is Riemann integrable over $[a,b]$. Is it true that
$$\lim_{n\to\infty}\int_a^b f_n=\int_a^b f?$$

## 4.2 The Lebesgue Integral of a Bounded Measurable Function over a Set of Finite Measure

**Remark.** *Prior to the proof of the Bounded Convergence Theorem, no use was made of the countable additivity of the Lebesgue measure on the real line. Only finite additivity was used, and it was used just once, in the proof of Lemma 1. Bur for the proof of the Bounded Convergence Theorem we used Egoroff's Theorem. Egoroff's Theorem needed the continuity of Lebesgue measure, a consequence of countable additivity of Lebesgue measure.*

**PROBLEMS**

9. Let $E$ have measure zero. Show that if $f$ is a bounded function on $E$, then $f$ is measurable and $\int_E f=0$.

10. Let $f$ be a bounded measurable function on a set of finite measure $E$. For a measurable subset $A$ of $E$, show that $\int_A f=\int_E f\cdot\chi_A$.

11. Does the Bounded Convergence Theorem hold for the Riemann integral?

12. Let $f$ be a bounded measurable function on a set of finite measure $E$. Assume $g$ is bounded and $f=g$ a.e. on $E$. Show that $\int_E f=\int_E g$.

13. Does the Bounded Convergence Theorem hold if $m(E)<\infty$ but we drop the assumption that the sequence $\{|f_n|\}$ is uniformly bounded on $E$?

14. Show that Proposition 8 is a special case of the Bounded Convergence Theorem.

15. Verify the assertions in the last Remark of this section.

16. Let $f$ be a nonnegative bounded measurable function on a set of finite measure $E$. Assume $\int_E f = 0$. Show that $f = 0$ a.e. on $E$.

## 4.3 The Lebesgue Integral of a Measurable Nonnegative Function

**PROBLEMS**

17. Let $E$ be a set of measure zero and define $f \equiv \infty$ on $E$. Show that $\int_E f = 0$.

18. Show that the integral of a bounded measurable function of finite support is properly defined.

19. For a number $\alpha$, define $f(x) = x^\alpha$ for $0 < x \leq 1$, and $f(0) = 0$. Compute $\int_0^1 f$.

20. Let $\{f_n\}$ be a sequence of nonnegative measurable functions that converges to $f$ pointwise on $E$. Let $M \geq 0$ be such that $\int_E f_n \leq M$ for all $n$. Show that $\int_E f \leq M$. Verify that this property is equivalent to the statement of Fatou's Lemma.

21. Let the function $f$ be nonnegative and integrable over $E$ and $\epsilon > 0$. Show there is a simple function $\eta$ on $E$ that has finite support, $0 \leq \eta \leq f$ on $E$ and $\int_E |f - \eta| < \epsilon$. If $E$ is a closed, bounded interval, show there is a step function $h$ on $E$ that has finite support and $\int_E |f - h| < \epsilon$.

22. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}$ that converges pointwise on $\mathbb{R}$ to $f$ and $f$ be integrable over $\mathbb{R}$. Show that

$$\text{if } \int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} f_n, \text{ then } \int_E f = \lim_{n\to\infty} \int_E f_n \text{ for any measurable set } E.$$

23. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Define the function $f$ on $E = [1, \infty)$ by setting $f(x) = a_n$ if $n \leq x < n + 1$. Show that $\int_E f = \sum_{n=1}^\infty a_n$.

24. Let $f$ be a nonnegative measurable function on $E$.
    (i) Show there is an increasing sequence $\{\varphi_n\}$ of nonnegative simple functions on $E$, each of finite support, which converges pointwise on $E$ to $f$.
    (ii) Show that $\int_E f = \sup\{\int_E \varphi \mid \varphi \text{ simple, of finite support and } 0 \leq \varphi \leq f \text{ on } E\}$.

25. Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $E$ that converges pointwise on $E$ to $f$. Suppose $f_n \leq f$ on $E$ for each $n$. Show that

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

26. Show that the Monotone Convergence Theorem may not hold for decreasing sequences of functions.

27. Prove the following generalization of Fatou's Lemma: If $\{f_n\}$ is a sequence of nonnegative measurable functions on $E$, then

$$\int_E \liminf f_n \leq \liminf \int_E f_n.$$

## 4.4 The General Lebesgue Integral

**PROBLEMS**

28. Let $f$ be integrable over $E$ and let $C$ be a measurable subset of $E$. Show that $\int_C f = \int_E f \cdot \chi_C$.

29. For a measurable function $f$ on $[1, \infty)$ which is bounded on bounded sets, define $a_n = \int_n^{n+1} f$ for each natural number $n$. Is it true that $f$ is integrable over $[1, \infty)$ iff the series $\sum_{n=1}^{\infty} a_n$ converges? Is it true that $f$ is integrable over $[1, \infty)$ iff the series $\sum_{n=1}^{\infty} a_n$ converges absolutely?

30. Let $g$ be a nonnegative integrable function over $E$ and suppose $\{f_n\}$ is a sequence of measurable functions on $E$ such that for each $n$, $|f_n| \leq g$ a.e. on $E$. Show that

$$\int_E \liminf f_n \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E \limsup f_n.$$

31. Let $f$ be a measurable function on $E$ which can be expressed as $f = g + h$ on $E$, where $g$ is finite and integrable over $E$ and $h$ is nonnegative on $E$. Define $\int_E f = \int_E g + \int_E h$. Show that this is properly defined in the sense that it is independent of the particular choice of finite integrable function $g$ and nonnegative function $h$ whose sum is $f$.

32. Prove the General Lebesgue Dominated Convergence Theorem by following the proof of the Lebesgue Dominated Convergence Theorem, but replacing the sequences $\{g - f_n\}$ and $\{g + f_n\}$, respectively, by $\{g_n - f_n\}$ and $\{g_n + f_n\}$.

33. Let $\{f_n\}$ be a sequence of integrable functions on $E$ for which $f_n \to f$ a.e. on $E$ and $f$ is integrable over $E$. Show that $\int_E |f - f_n| \to 0$ iff $\lim_{n\to\infty} \int_E |f_n| = \int_E |f|$. (Hint: use the General Lebesgue Dominated Convergence Theorem.)

34. Let $f$ be a nonnegative measurable function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f.$$

35. Let $f$ be a real-valued function of two variables $(x, y)$ that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of $x$ for each fixed value of $y$. Suppose for each fixed value of $x$, $\lim_{y\to 0} f(x, y) = f(x)$ and that for all $y$, we have $|f(x, y)| \leq g(x)$, where $g$ is integrable over $[0, 1]$. Show that

$$\lim_{y\to 0} \int_0^1 f(x, y)dx = \int_0^1 f(x)dx.$$

Also show that if the function $f(x, y)$ is continuous in $y$ for each $x$, then

$$h(y) = \int_0^1 f(x, y)dx$$

is a continuous function of $y$.

36. Let $f$ be a real-valued function of two variables $(x, y)$ that is defined on the square $Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and is a measurable function of $x$ for each fixed value of $y$. For each $(x, y) \in Q$

let the partial derivative $\partial f/\partial y$ exist. Suppose there is a function $g$ that is integrable over $[0,1]$ and such that

$$\left|\frac{\partial f}{\partial y}(x,y)\right| \leq g(x) \text{ for all } (x,y) \in Q.$$

Prove that

$$\frac{d}{dy}\left[\int_0^1 f(x,y)dx\right] = \int_0^1 \frac{\partial f}{\partial y}(x,y)dx \text{ for all } y \in [0,1].$$

## 4.5 Countable Additivity and Continuity of Integration

**PROBLEMS**

37. Let $f$ be an integrable function on $E$. Show that for each $\epsilon > 0$, there is a natural number $N$ for which if $n \geq N$, then $|int_{E_n} f| < \epsilon$ where $E_n = \{x \in E \mid |x| \geq n\}$.

38. For each of the two functions $f$ on $[1,\infty)$ defined below, show that $\lim_{n\to\infty} \int_1^n f$ exists while $f$ is not integrable over $[1,\infty)$. Does this contradict the continuity of integration?

    (i) Define $f(x) = \frac{(-1)^n}{n}$, for $n \leq x < n+1$.

    (ii) Define $f(x) = \frac{(\sin x)}{x}$ for $1 \leq x < \infty$.

39. Prove the theorem regarding the continuity of integration.

## 4.6 Uniform Integrability: The Vitali Convergence Theorem

**PROBLEMS**

40. Let $f$ be integrable over $\mathbb{R}$. Show that the function $F$ defined by

$$F(x) = \int_{-\infty}^{x} f \text{ for all } x \in \mathbb{R}$$

    is properly defined and continuous. Is it necessarily Lipschitz?

41. Show that Proposition 25 is false if $E = \mathbb{R}$.

42. Show that Theorem 26 is false without the assumption that the $h_n$'s are nonnegative.

43. Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable over $E$. Show that for any $\alpha$ and $\beta$, the sequence of linear combinations $\{\alpha f_n + \beta g_n\}$ also is uniformly integrable over $E$.

44. Let $f$ be integrable over $\mathbb{R}$ and let $\epsilon > 0$. Establish the following three approximation properties.

    (i) There is a simple function $\eta$ on $\mathbb{R}$ which has finite support and $\int_{\mathbb{R}} |f - \eta| < \epsilon$. (Hint: first verify this if $f$ is nonnegative.)

    (ii) There is a step function $s$ on $\mathbb{R}$ which vanishes outside a closed, bounded interval and $\int_{\mathbb{R}} |f - s| < \epsilon$. (Hint: apply part (i) and Problem 18 of Chapter 3.)

    (iii) There is a continuous function $g$ on $\mathbb{R}$ which vanishes outside a bounded set and $\int_{\mathbb{R}} |f - g| < \epsilon$.

45. Let $f$ be integrable over $E$. Define $\hat{f}$ to be the extension of $f$ to all of $\mathbb{R}$ obtained by setting $\hat{f} \equiv 0$ outside of $E$. Show that $\hat{f}$ is integrable over $\mathbb{R}$ and $\int_E f = \int_{\mathbb{R}} \hat{f}$. Use this and part (i) and (iii) of the preceding problem to show that for $\epsilon > 0$, there is a simple function $\eta$ on $E$ and a continuous function $g$ on $E$ for which $\in_E |f - \eta| < \epsilon$ and $\in_E |f - g| < \epsilon$.

46. (Riemann-Lebesgue) Let $f$ be integrable over $(-\infty, \infty)$. Show That

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x) \cos nx dx = 0.$$

(Hint: first show this for $f$ is a step function that vanishes outside a closed, bounded interval and then use the approximation property (ii) of Problem 44.)

47. Let $f$ be integrable over $(-\infty, \infty)$.

   (i) Show that for each $t$,
   $$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f(x+t)dx.$$

   (ii) Let $g$ be a bounded measurable function on $\mathbb{R}$. Show that
   $$\lim_{t\to 0} \int_{-\infty}^{\infty} g(x) \cdot [f(x) - f(x+t)] = 0.$$

   (Hint: first show this, using uniform continuity of $f$ on $\mathbb{R}$, if $f$ is continuous and vanishes outside a bounded set. Then use the approximation property (iii) of Problem 44.)

48. Let $f$ be integrable over $E$ and let $g$ be a bounded measurable function on $E$. Show that $f \cdot g$ is integrable over $E$.

49. Let $f$ be integrable over $\mathbb{R}$. Show that the following four assertions are equivalent:

   (i) $f = 0$ a.e. on $\mathbb{R}$.

   (ii) $\int_{\mathbb{R}} fg = 0$ for every bounded measurable function $g$ on $\mathbb{R}$.

   (iii) $\int_A f = 0$ for every measurable set $A$.

   (iv) $\int_{\mathcal{O}} f = 0$ for every open set $\mathcal{O}$.

50. Let $\mathcal{F}$ be a family of functions, each of which is integrable over $E$. Show that $\mathcal{F}$ is uniformly integrable over $E$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } A \subseteq E \text{ is measurable and } m(A) < \delta, \text{ then } \left| \int_A f \right| < \epsilon.$$

51. Let $\mathcal{F}$ be a family of functions, each of which is integrable over $E$. Show that $\mathcal{F}$ is uniformly integrable over $E$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for each $f \in \mathcal{F}$,

$$\text{if } \mathcal{U} \text{ is open and } m(E \cap \mathcal{U}) < \delta, \text{ then } \int_{E\cap\mathcal{U}} |f| < \epsilon.$$

# Chapter 5

# Lebesgue Integration: Further Topics

**Contents**



## 5.1 Uniform Integrability and Tightness: A General Vitali Convergence Theorem

**PROBLEMS**

1. Let $\{f_n\}_{k=1}^{n}$ be a finite family of functions, each of which is integrable over $E$. Show that $\{f_n\}_{k=1}^{n}$ is uniformly integrable and tight over $E$.

2. Prove Corollary 2.

3. Let the sequences of functions $\{h_n\}$ and $\{g_n\}$ be uniformly integrable and tight over $E$. Show that for any $\alpha$ and $\beta$, $\{\alpha f_n + \beta g_n\}$ is also uniformly integrable and tight over $E$.

4. Let $\{f_n\}$ be a sequence of measurable functions on $E$. Show that f$\{f_n\}$ is uniformly integrable and tight over $E$ iff for each $\epsilon > 0$, there is a measurable subset $E_0$ of $E$ that has finite measure and a $\delta > 0$ such that for each measurable subset $A$ of $E$ and index $n$,
$$\text{if } m(A \cap E_0) < \delta, \text{ then } \int_A |f_n| < \epsilon.$$

5. Let $\{f_n\}$ be a sequence of measurable functions on $\mathbb{R}$. Show that f$\{f_n\}$ is uniformly integrable and tight over $\mathbb{R}$ iff for each $\epsilon > 0$, there are positive numbers $r$ and $\delta$ such that for each open subset $\mathcal{O}$ of $\mathbb{R}$ and index $n$
$$\text{if } m(\mathcal{O} \cap (-r, r)) < \delta, \text{ then } \int_{\mathcal{O}} |f_n| < \epsilon.$$

## 5.2 Convergence in Measure

**PROBLEMS**

6. Let $\{f_n\} \to f$ in measure on $E$ and let $g$ be a measurable function on $E$ that is finite a.e. on $E$. Show that $\{f_n\} \to g$ in measure on $E$ iff $f = g$ a.e. on $E$.

7. Let $E$ have finite measure, let $\{f_n\} \to f$ in measure on $E$ and let $g$ be a measurable function on $E$ that is finite a.e. on $E$. Prove that $\{f_n \cdot g\} \to f \cdot g$ in measure, and use this to show that $\{f_n^2\} \to f^2$ in measure. Infer from this that if $\{g_n\} \to g$ in measure, then $\{f_n \cdot g_n\} \to f \cdot g$ in measure.

8. Show that Fatou's Lemma, the Monotone Convergence Theorem, the Lebesgue Dominated Convergence Theorem, and the Vitali Convergence Theorem remain valid if "pointwise convergence a.e." is replaced by "convergence in measure".

9. Show that Proposition 3 does not necessarily hold for sets $E$ of infinite measure.

10. Show that linear combinations of sequences that converge in measure on a set of finite measure also converge in measure.

11. Assume $E$ has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on $E$ and let $f$ be a measurable function on $E$ for which $f$ and each $f_n$ is finite a.e. on $E$. Prove that $\{f_n\} \to f$ in measure on $E$ iff every subsequence of $\{f_n\}$ has in turn a further subsequence that converges to $f$ pointwise a.e. on $E$.

12. Show that a sequence $\{a_j\}$ of real numbers converges to a real number if $|a_{j+1} - a_j| \leq \frac{1}{2^j}$ for all $j$ by showing that the sequence $\{a_j\}$ must be Cauchy.

13. A sequence $\{f_n\}$ of measurable functions on $E$ is said to be **Cauchy in measure** provided that given $\eta > 0$ and $\epsilon > 0$, there is an index $N$ such that for all $m, n \geq N$,

$$m\{x \in E \mid |f_n(x) - f_m(x)| \geq \eta\} < \epsilon.$$

   Show that if $\{f_n\}$ is Cauchy in measure, then there is a measurable function $f$ on $E$ to which the sequence $\{f_n\}$ converges in measure. (Hint: choose a strictly increasing sequence of natural numbers $\{n_j\}$ such that for each index $j$, if $E_j = \{x \in E \mid |f_{n_{j+1}}(x) - f_{n_j}(x)| > \frac{1}{2^j}\}$, then $m(E_j) < \frac{1}{2^j}$. Now use the Borel-Cantelli Lemma and the preceding problem.)

14. Assume $m(E) < \infty$. For two measurable functions $g$ and $h$ on $E$, Define

$$\rho(g, h) = \int_E \frac{|g - h|}{1 + |g - h|}.$$

   Show that $\{f_n\} \to f$ in measure on $E$ iff $\lim_{n \to \infty} \rho(f_n, f) = 0$.

## 5.3 Characterizations of Riemann and Lebesgue Integrability

**PROBLEMS**

15. Let $f$ and $g$ be bounded functions that are Riemann integrable over $[a, b]$. Show that the product $fg$ is also Riemann integrable over $[a, b]$.

16. Let $f$ be a bounded function on $[a, b]$ whose set of discontinuities has measure zero. Show that $f$ is measurable. Then show that the same holds without the assumption of boundedness.

17. Let $f$ be a function on $[0, 1]$ that is continuous on $(0, 1]$. Show that it is possible for the sequence $\{\int_{[1/n,1]} f\}$ to converge and yet $f$ is not Lebesgue integrable over $[0, 1]$. Can this happen if $f$ is nonnegative?

# Chapter 6

# Differentiation and Integration

## Contents



## 6.1 Continuity of Monotone Functions

**PROBLEMS**

1. Let $C$ be a countable subset of the nondegenerate closed, bounded interval $[a, b]$. Show that there is an increasing function on $[a, b]$ that is continuous only at points in $[a, b] \setminus C$.

2. Show that there is a strictly increasing function on $[0, 1]$ that is continuous only at the irrational numbers in $[0, 1]$.

3. Let $f$ be a monotone function on a subset $E$ of $\mathbb{R}$. Show that $f$ is continuous except possibly at a countable number of points in $E$.

4. Let $E$ be a subset of $\mathbb{R}$ and let $C$ be a countable subset of $E$. Is there a monotone function on $E$ that is continuous only at points in $E \setminus C$?

## 6.2 Differentiability of Monotone Functions: Lebesgue's Theorem

**PROBLEMS**

5. Show that the Vitali Covering Lemma does not extend to the case in which the covering collection has degenerate closed intervals.

6. Show that the Vitali Covering Lemma does extend to the case in which the covering collection consists of nondegenerate general intervals.

7. let $f$ be continuous on $\mathbb{R}$. Is there an open interval on which $f$ is monotone?

8. Let $I$ and $J$ be closed, bounded intervals and $\gamma > 0$ be such that $\ell(I) > \gamma \cdot \ell(J)$. Assume $I \cap J \neq \emptyset$. Show that if $\gamma \geq 1/2$, then $J \subseteq 5 * I$, where $5 * I$ denotes the interval with the same center as $I$ and five times its length. Is the same true if $0 < \gamma < 1/2$?

9. Show that a set $E$ of real numbers has measure zero iff there is a countable collection of open intervals $\{I_k\}_{k=1}^{\infty}$ for which each point in $E$ belongs to infinitely many of the $I_k's$ and $\sum_{k=1}^{\infty} \ell(I_k) < \infty$.

10. (Riesz-Nagy) Let $E$ be a set of measure zero contained in the open interval $(a, b)$. According to the preceding problem, there is a countable collection of open intervals contained in $(a, b)$, $\{(c_k, d_k)\}_{k=1}^{\infty}$, for which each point in $E$ belongs to infinitely many intervals in the collection and $\sum_{k=1}^{\infty}(d_k - c_k) < \infty$. Define

$$f(x) = \sum_{k=1}^{\infty} \ell((c_k, d_k) \cap (-\infty, x)) \text{ for all } x \text{ in } (a, b).$$

Show that $f$ is increasing and fails to be differentiable at each point in $E$.

11. For real numbers $\alpha < \beta$ and $\gamma > 0$, show that if $g$ is integrable over $[\alpha + \gamma, \beta + \gamma]$, Then

$$\int_{\alpha}^{\beta} g(t + \gamma) dt = \int_{\alpha+\gamma}^{\beta+\gamma} g(t) dt.$$

Prove this change of variables formula by successively considering simple functions, bounded measurable functions, nonnegative integrable functions, and general integrable functions. Use it to prove (14).

12. Compute the upper and lower derivatives of the characteristic function of the rationals.

13. Let $E$ be a set of finite outer measure and $\mathcal{F}$ a collection of closed, bounded intervals that cover $E$ in the sense of Vitali. Show that there is a countable disjoint collection $\{I_k\}_{k=1}^{\infty}$ of intervals in $\mathcal{F}$ for which

$$m^* \left[ E \setminus \bigcup_{k=1}^{\infty} I_k \right] = 0.$$

14. Use the Vitali Covering Lemma to show that the union of any collection (countable or uncountable) of closed, bounded, nondegenerate intervals is measurable.

15. Define $f$ on $\mathbb{R}$ by

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find the upper and lower derivatives of $f$ at $x = 0$.

16. Let $g$ be integrable over $[a, b]$. Define the antiderivative of $g$ of $g$ to be the function $f$ defined on $[a, b]$ by

$$f(x) = \int_a^x g \text{ for all } x \in [a, b].$$

Show that $f$ is differentiable almost everywhere on $(a, b)$.

17. Let $f$ be an increasing bounded function on the open, bounded interval $(a, b)$. Verify (18).

18. Show that if $f$ is defined on $(a, b)$ and $c \in (a, b)$ is a local minimizer for $f$, then $\underline{D}f(c) \leq 0 \leq \overline{D}f(c)$.

19. Let $f$ be continuous on $[a, b]$ with $\underline{D}f \geq 0$ on $(a, b)$. Show that $f$ is increasing on $[a, b]$. (Hint: first show this for a function $g$ for which $\underline{D}g \geq \epsilon > 0$ on $(a, b)$. Apply this to the function $g(x) = f(x) + \epsilon x$.)

20. Let $f$ and $g$ be real-valued functions on $(a, b)$. Show That
$$\underline{D}f + \underline{D}g \leq \underline{D}(f+g) \leq \overline{D}(f+g) \leq \overline{D}(f) + \overline{D}(g) \text{ on } (a, b).$$

21. Let $f$ be defined on $[a, b]$ and $g$ a continuous function on $[\alpha, \beta]$ that is differentiable at $\gamma \in (\alpha, \beta)$ with $g(\gamma) = c \in (a, b)$. Verify the following.
    (i) If $g'(\gamma) > 0$, then $\overline{D}(f \circ g)(\gamma) = \overline{D}f(c) \cdot g'(\gamma)$.
    (ii) If $g'(\gamma) = 0$ and the upper and lower derivatives of $f$ at $c$ are finite, then $\overline{D}(f \circ g)(\gamma) = 0$.

22. Show that a strictly increasing function that is defined on an interval is measurable and then use this to show that a monotone function that is defined on an interval is measurable.

23. Show that a continuous function $f$ on $[a, b]$ is Lipschitz if its upper and lower derivatives are bounded on $(a, b)$.

24. Show that for $f$ defined in the last remark of this section, $f'$ is not integrable over $[0, 1]$.

# 6.3 Functions of Bounded Variation: Jordan's Theorem

**PROBLEMS**

25. Suppose $f$ is continuous on $[0, 1]$. Must there be a nondegenerate closed subinterval $[a, b]$ of $[0, 1]$ for which the restriction of $f$ to $[a, b]$ is of bounded variation?

26. Let $f$ be the Dirichlet function, the characteristic function of the rationals in $[0, 1]$. Is $f$ of bounded variation on $[0, 1]$?

27. Define $f(x) = \sin x$ on $[0, 2\pi]$. Find two increasing functions $h$ and $g$ for which $f = h - g$ on $[0, 2\pi]$.

28. Let $f$ be a step function on $[a, b]$. Find a formula for its total variation.

29. (i) Define
$$f(x) = \begin{cases} x^2 \cos(1/x^2) & x \neq 0, x \in [-1, 1] \\ 0 & x = 0 \end{cases}$$
    Is $f$ of bounded variation on $[-1, 1]$?
    (ii) Define
$$g(x) = \begin{cases} x^2 \cos(1/x) & x \neq 0, x \in [-1, 1] \\ 0 & x = 0 \end{cases}$$
    Is $g$ of bounded variation on $[-1, 1]$?

30. Show that the linear combination of two functions of bounded variation is also of bounded variation. Is the product of two such functions also of bounded variation?

31. Let $P$ be a partition of $[a,b]$ that is a refinement of the partition $P'$. For a real-valued function $f$ on $[a,b]$, show that $V(f,P') \leq V(f,P)$.

32. Assume $f$ is of bounded variation on $[a,b]$. Show that there is a sequence of partitions $\{P_n\}$ of $[a,b]$ for which the sequence $\{V(f,P_n)\}$ is increasing and converges to $TV(f)$.

33. Let $\{f_n\}$ be a sequence of real-valued functions on $[a,b]$ that converges pointwise on $[a,b]$ to the real-valued function $f$. Show that

$$TV(f) \leq \liminf TV(f_n).$$

34. Let $f$ and $g$ be of bounded variation on $[a,b]$. Show that

$$TV(f+g) \leq TV(f) + TV(g) \text{ and } TV(\alpha f) = |\alpha| TV(f).$$

35. For $\alpha$ and $\beta$ positive numbers, define the function $f$ on $[0,1]$ by

$$f(x) = \begin{cases} x^\alpha \sin(1/x^\beta) & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \end{cases}$$

Show that if $\alpha > \beta$, then $f$ is of bounded variation on $[0,1]$, by showing that $f'$ is integrable over $[0,1]$. Then show that if $\alpha \leq \beta$, then $f$ is not of bounded variation on $[0,1]$.

36. Let $f$ fail to be of bounded variation on $[0,1]$. Show that there is a point $x_0$ in $[0,1]$ such that there are subintervals of $[0,1]$ that contain $x_0$ and have arbitrarily small length on which $f$ fails to be of bounded variation.

## 6.4 Absolutely Continuous Functions

**PROBLEMS**

37. Let $f$ be a continuous function on $[0,1]$ that is absolutely continuous on $[\epsilon, 1]$ for each $0 < \epsilon < 1$.

 (i) Show that $f$ may not be absolutely continuous on $[0,1]$.

 (ii) Show that $f$ is absolutely continuous on $[0,1]$ if it is increasing.

 (iii) Show that the function $f$ on $[0,1]$, defined by $f(x) = \sqrt(x)$ for $0 \leq x \leq 1$, is absolutely continuous, but not Lipschitz, on $[0,1]$.

38. Show that $f$ is absolutely continuous on $[a,b]$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for every countable disjoint collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals in $(a,b)$,

$$\sum_{k=1}^{\infty} |f(b_k) - f(a_k)| < \epsilon \text{ if } \sum_{k=1}^{\infty} [b_k - a_k] < \delta.$$

39. Use the preceding problem to show that if $f$ is continuous and increasing on $[a,b]$, then $f$ is absolutely continuous on $[a,b]$ iff for each $\epsilon$, there is a $\delta > 0$ such that for a measurable subset $E$ of $[a,b]$,

$$m^*(f(E)) < \epsilon \text{ if } m(E) < \delta.$$

40. Use the preceding problem to show that an increasing absolutely continuous function $f$ on $[a, b]$ maps sets of measure zero onto sets of measure zero. Conclude that the Cantor-Lebesgue function $\varphi$ is not absolutely continuous on $[0, 1]$ since the function $\psi$, defined by $\psi(x) = x + \varphi(x)$ for $0 \le x \le 1$, maps the Cantor set to a set of measure 1 (page 52).

41. Let $f$ be an increasing absolutely continuous function on $[a, b]$. Use (i) and (ii) below to conclude that $f$ maps measurable sets to measurable sets.

    (i) Infer from the continuity of $f$ and the compactness of $[a, b]$ that $f$ maps closed sets to closed sets and therefore maps $F_\sigma$ sets to $F_\sigma$ sets.

    (ii) The preceding problem tells us that $f$ maps sets of measure zero to sets of measure zero.

42. Show that both the sum and product of absolutely continuous functions are absolutely continuous.

43. Define the functions $f$ and $g$ on $[-1, 1]$ by $f(x) = x^{\frac{1}{3}}$ for $-1 \le x \le 1$ and

$$g(x) = \begin{cases} x^2 \cos(\pi/2x) & \text{if } x \neq 0, x \in [-1, 1] \\ 0 & \text{if } x = 0 \end{cases}$$

    (i) Show that both $f$ and $g$ are absolutely continuous on $[-1, 1]$.

    (ii) For the partition $P_n = \{-1, 0, 1/2n, 1/[2n-1], \cdots, 1/3, 1/2, 1\}$ of $[-1, 1]$, examine $V(f \circ g, P_n)$.

    (iii) Show that $f \circ g$ fails to be of bounded variation, and hence also fails to be absolutely continuous, on $[-1, 1]$.

44. Let $f$ be Lipschitz on $\mathbb{R}$ and $g$ be absolutely continuous on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

45. Let $f$ be absolutely continuous on $\mathbb{R}$ and $g$ be absolutely continuous and strictly monotone on $[a, b]$. Show that the composition $f \circ g$ is absolutely continuous on $[a, b]$.

46. Verify the assertions made in the final remark of this section.

47. Show that a function $f$ is absolutely continuous on $[a, b]$ iff for each $\epsilon > 0$, there is a $\delta > 0$ such that for every finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in $(a, b)$,

$$\left|\sum_{k=1}^{n}[f(b_k) - f(a_k)]\right| < \epsilon \text{ if } \sum_{k=1}^{n}[b_k - a_k] < \delta.$$

## 6.5 Integrating Derivatives: Differentiating Indefinite Integrals

**PROBLEMS**

48. The Cantor-Lebesgue function $\varphi$ is continuous and increasing on $[0, 1]$. Conclude from Theorem 10 that $\varphi$ is not absolutely continuous on $[0, 1]$. Compare this reasoning with that proposed in Problem 40.

49. Let $f$ be continuous on $[a, b]$ and differentiable almost everywhere on $(a, b)$. Show that

$$\int_a^b f' = f(b) - f(a)$$

if and only if

$$\int_a^b [\lim_{n\to\infty} \text{Diff}_{1/n} f] = \lim_{n\to\infty} [\int_a^b \text{Diff}_{1/n} f].$$

50. Let $f$ be continuous on $[a, b]$ and differentiable almost everywhere on $(a, b)$. Show that if $\{\text{Diff}_{1/n} f\}$ is uniformly integrable over $[a, b]$, then

$$\int_a^b f' = f(b) - f(a).$$

51. Let $f$ be continuous on $[a, b]$ and differentiable almost everywhere on $(a, b)$. Suppose there is a nonnegative function $g$ that is integrable over $[a, b]$ and

$$|\text{Diff}_{1/n} f| \leq g \text{ a.e. on } [a, b] \text{ for all } n.$$

Show that

$$\int_a^b f' = f(b) - f(a).$$

52. Let $f$ and $g$ be absolutely continuous on $[a, b]$. Show that

$$\int_a^b f \cdot g' = f(b)g(b) - f(a)g(a) - \int_a^b f' \cdot g.$$

53. Let the function $f$ be absolutely continuous on $[a, b]$. Show that $f$ is Lipschitz on $[a, b]$ iff there is a $c > 0$ for which $|f'| \leq c$ a.e. on $[a, b]$.

54. (i) Let $f$ be a singular increasing function on $[a, b]$. Use the Vitali Covering Lemma to show that $f$ has the following property: Given $\epsilon > 0, \delta > 0$, there is a finite disjoint collection $\{(a_k, b_k)\}_{k=1}^n$ of open intervals in $(a, b)$ for which

$$\sum_{k=1}^n [b_k - a_k] < \delta \text{ and } \sum_{k=1}^n [f(b_k) - f(a_k)] > f(b) - f(a) - \epsilon.$$

(ii) Let $f$ be an increasing function on $[a, b]$ with the property described in part (i). Show that $f$ is singular.

(iii) Let $\{f_n\}$ be a sequence of singular increasing functions on $[a, b]$ for which the series $\sum_{n=1}^\infty f_n(x)$ converges to a finite value for each $x \in [a, b]$. Define

$$f(c) = \sum_{n=1}^\infty f_n(x) \text{ for } x \in [a, b].$$

Show that $f$ is also singular.

55. Let $f$ be of bounded variation on $[a, b]$, and define $v(x) = TV(f_{[a,x]})$ for all $x \in [a, b]$.

(i) Show that $|f'| \leq v'$ a.e. on $[a, b]$, and infer from this that

$$\int_a^b |f'| \leq TV(f).$$

(ii) Show that the above is an equality iff $f$ is absolutely continuous on $[a, b]$.

(iii) Compare parts (i) and (ii) with Corollaries 4 and 12, respectively.

56. Let $g$ be strictly increasing and absolutely continuous on $[a, b]$.

(i) Show that for any open subset $\mathcal{O}$ of $(a, b)$,

$$m(g(\mathcal{O})) = \int_{\mathcal{O}} g'(x)dx.$$

(ii) Show that for any $G_\delta$ subset $E$ of $(a, b)$,

$$m(g(E)) = \int_E g'(x)dx.$$

(iii) Show that for any subset $E$ of $[a, b]$ that has measure 0, its image $g(E)$ also has measure 0, so that

$$m(g(E)) = 0 = \int_E g'(x)dx.$$

(iv) Show that for any measurable subset $A$ of $[a, b]$,

$$m(g(A)) = \int_A g'(x)dx.$$

(v) Let $c = g(a)$ and $d = g(b)$. Show that for any simple function $\varphi$ on $[c, d]$,

$$\int_c^d \varphi(y)dy = \int_a^b \varphi(g(x))g'(x)dx.$$

(vi) Show that for any nonnegative integrable function $f$ over $[c, d]$,

$$\int_c^d f(y)dy = \int_a^b f(g(x))g'(x)dx.$$

(vii) Show that part (i) follows from (vi) in the case that $f$ is the characteristic function of $g(\mathcal{O})$ and the composition is defined.

57. Is the change of variables formula in part (vi) of the preceding problem true if we just assume $g$ is increasing, not necessarily strictly?

58. Construct an absolutely continuous strictly increasing function $f$ on $[0, 1]$ for which $f' = 0$ on a set of positive measure. (Hint: Let $E$ be the relative complement in $[0, 1]$ of a generalized Cantor set of positive measure and $f$ the indefinite integral of $\chi_E$. See Problem 39 of Chapter 2 for the construction of such a Cantor set.)

59. For a nonnegative integrable function $f$ over $[c, d]$, and a strictly increasing absolutely continuous function $g$ on $[a, b]$ such that $g([a, b]) \subseteq [c, d]$, is it possible to justify the change of variables formula

$$\int_{g(a)}^{g(b)} f(y)dy = \int_a^b f(g(x))g'(x)dx$$

by showing that

$$\frac{d}{dx}\left[\int_{g(a)}^{g(x)} f(s)ds - \int_a^x f(g(t))g'(t)dt\right] = 0 \text{ for almost all } x \in (a, b)?$$

60. Let $f$ be absolutely continuous and singular on $[a, b]$. Show that $f$ is constant. Also show that the Lebesgue decomposition of a function of bounded variation is unique if the singular function is required to vanish a t $x = a$.

## 6.6 Convex Functions

**PROBLEMS**

61. Show that a real-valued function $\varphi$ on $(a, b)$ is convex iff for points $x_1, \cdots, x_n$ in $(a, b)$ and nonnegative numbers $\lambda_1, \cdots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$,
$$\varphi\left(\sum_{k=1}^{n} \lambda_k x_k\right) \leq \sum_{k=1}^{n} \lambda_k \varphi(x_k).$$
Use this to directly prove Jensen's Inequality for $f$ a simple function.

62. Show that a continuous function on $(a, b)$ is convex iff
$$\varphi(\frac{x_1 + x_2}{2}) \leq \frac{\varphi(x_1) + \varphi(x_2)}{2} \text{ for all } x_1, x_2 \in (a, b).$$

63. A function on a general interval $I$ is said to be convex provided it is continuous on $I$ and (38) holds for all $x_1, x_2 \in I$. Is a convex function on a closed, bounded interval $[a, b]$ necessarily Lipschitz on $[a, b]$?

64. Let $\varphi$ have a second derivative at each point in $(a, b)$. Show that $\varphi$ is convex iff $\varphi''$ is nonnegative.

65. Suppose $a \geq 0$ and $b \geq 0$. Show that the function $\varphi(t) = (a + bt)^p$ is convex on $[0, \infty)$ for $1 \leq p < \infty$.

66. For what functions $\varphi$ is Jensen's Inequality always an equality?

67. State and prove a version of Jensen's Inequality on a general closed, bounded interval $[a, b]$.

68. Let $f$ be integrable over $[0, 1]$. Show that
$$\exp\left[\int_0^1 f(x)dx\right] \leq \int_0^1 \exp(f(x))dx.$$

69. Let $\{\alpha_n\}$ be a sequence of nonnegative numbers whose sum is 1 and $\{\zeta_n\}$ is a sequence of positive numbers. Show that
$$\prod_{n=1}^{\infty} \zeta_n^{\alpha_n} \leq \sum_{n=1}^{\infty} \alpha_n \zeta_n.$$

70. Let $g$ be a positive measurable function on $[0, 1]$. Show that $\log(\int_0^1 g(x)dx) \geq \int_0^1 \log(g(x))dx$ whenever each side is defined.

71. (Nemytskii) Let $\varphi$ be a continuous function on $\mathbb{R}$. Show that if there are constants for which (43) holds, then $\varphi \circ f$ is integrable over $[0, 1]$ whenever $f$ is. Then show that if $\varphi \circ f$ is integrable over $[0, 1]$ whenever $f$ is, then there are constants $c_1$ and $c_2$ for which (43) holds.

# Chapter 7

# The $L^p$ Spaces: Completeness and Approximation

**Contents**



## 7.1 Normed Linear Spaces

Throughout this chapter $E$ denotes a measurable set of real numbers. Define $\mathcal{F}$ to be the collection of all measurable extended real-valued functions on $E$ that are finite a.e. on $E$. We can say that two functions $f, g \in \mathcal{F}$ are equivalent, denoted by $f \cong g$, provided

$$f(x) = g(x) \text{ for almost all } x \in E.$$

This is an equivalence relation and induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes, denoted by $\mathcal{F}/\cong$, which is a linear space. There is a natural family $\{L^p(E)\}_{1\leq p\leq\infty}$ of subspaces of $\mathcal{F}/\cong$.

For $1 \leq p < \infty$, define $L^p(E)$ to be the collection of equivalence class $[f]$ for which

$$\int_E |f|^p < \infty.$$

Then if $f \cong g$, then $\int_E |f|^p = \int_E |g|^p$. Showing that $L^p(E)$ is closed under linear combinations will prove that $L^p(E)$ is a linear subspace. To do this, let $c = \max\{|a|, |b|\}$ so that

$$|a + b| \leq |a| + |b| \leq 2c,$$

which implies

$$|a + b|^p \leq 2^p c^p \leq 2^p(|a|^p + |b|^p).$$

This inequality, together with the linearity and monotonicity of integration tells us that

$$\int_E |\alpha f + \beta g|^p \leq 2^p(|\alpha|^p \int_E |f|^p + |\beta|^p \int_E |g|^p) < \infty.$$

That is, for $[f], [g] \in L^p(E)$, then $\alpha[f] + \beta[g] \in L^p(E)$.
We call a function $f \in \mathcal{F}$ **essentially bounded** provided there is some $M \geq 0$, called an **essential upper bound** for $f$, for which

$$|f(x)| \leq M \text{ for almost all } x \in E.$$

Then we can define $L^\infty(E)$ to be the collection of equivalence classes $[f]$ for which $f$ is essentially bounded. Clearly $L^\infty(E)$ is a linear subspace because

$$|\alpha f(x) + \beta g(x)| \leq |\alpha||f(x)| + |\beta||g(x)| \leq |\alpha|M + |\beta|M' = M'' \text{ a.e. on } E$$

To state that a function $f$ in $L^p[a, b]$ is continuous means that there is a continuous function that agrees with $f$ a.e. on $[a, b]$. There is only one such continuous function and it is often convenient to consider this unique continuous function as the representative of $[f]$.
It is useful to consider real-valued functions that have as their domain linear spaces of functions: such functions are called **functionals**.

**Definition.** *Let $X$ be a linear space. A real-valued functional $\|\cdot\|$ on $X$ is called a **norm** provided for each $f$ and $g$ in $X$ and each real number $\alpha$,*
*(The Triangle Inequality)*

$$\|f + g\| \leq \|f\| + \|g\|,$$

*(Positive Homogeneity)*

$$\|\alpha f\| = |\alpha|\|f\|,$$

*(Nonnegativity)*

$$\|f\| \geq 0 \text{ and } \|f\| = 0 \iff f = 0.$$

A **normed linear space** is a linear space together with a norm. If $X$ is a linear space normed by $\|\cdot\|$ we say that a function $f$ in $X$ is a **unit function** provided $\|f\| = 1$. For any $f \in X, f \neq 0$, the function $\frac{f}{\|f\|}$ is a unit function: it is a scalar multiple of $f$ which we call the **normalization** of $f$.

**Example** (The Normed Linear Space $L^1(E)$)**.** *For a function $f$ in $L^1(E)$, define*

$$\|f\|_1 = \int_E |f|.$$

*Then $\|\cdot\|$ is a norm on $L^1(E)$.*
*For $f, g \in L^1(E) \subseteq \mathcal{F}$, since $f$ and $g$ are finite a.e. on $E$, the triangle inequality for real numbers tells us that*

$$|f + g| \leq |f| + |g| \text{ a.e. on } E.$$

*Then by the monotonicity and linearity of integration, we have subadditivity:*

$$\|f + g\|_1 = \int_E |f + g| \leq \int_E [|f| + |g|] = \int_E |f| + \int_E |g| = \|f\|_1 + \|g\|_1.$$

*By the linearity of integration, clearly we have absolute homogeneity:*

$$\|\alpha f\|_1 = \int_E |\alpha f| = \int_E |\alpha||f| = |\alpha| \int_E |f| = |\alpha|\|f\|_1.$$

*Clearly $\|f\|$ is nonnegative. Finally, if $f \in L^1(E)$ and $\|f\|_1 = 0$, then $f = 0$ a.e. on $E$. Therefore $[f]$ is the zero element of the linear space $L^1(E) \subseteq \mathcal{F}/\cong$, that is $f = 0$.*

**Example** (The Normed Linear Space $L^\infty(E)$)**.** *For a function $f$ in $L^\infty(E)$, define $\|f\|_\infty$ to be the infimum of the essential upper bounds for $f$.*

$$\|f\|_\infty = \inf\{M \ : \ |f(x)| \leq M \text{ a.e. on } E\}.$$

*We call $\|f\|_\infty$ the **essential supremum** of $f$ and claim that $\|\cdot\|_\infty$ is a norm on $L^\infty(E)$.*
*Nonnegativity and positive homogeneity are clear.*
*To show that the triangle inequality holds, we see that for each natural number $n$, there is a subset $E_n$ of $E$ for which*

$$|f| \leq \|f\|_\infty + \frac{1}{n} \text{ on } E \setminus E_n \text{ and } m(E_n) = 0.$$

*This is true because $\|f\|_\infty$ is the infimum, the greatest lower bound, so $\|f\|_\infty + \frac{1}{n}$ is not a lower bound and thus there exists a real number $M$ in the set of upper bounds a.e. of $f$ for which*
*$\|f\|_\infty \leq M < \|f\|_\infty + \frac{1}{n}$ a.e. on $E$, and so $|f| \leq M < \|f\|_\infty + \frac{1}{n}$ a.e. on $E$.*
*Accepting that the union of sets of measure zero is also measure zero, we can let $E_\infty = \bigcup_{n=1}^\infty E_n$, and so*

$$|f| \leq \|f\|_\infty \text{ on } E \setminus E_\infty \text{ and } m(E_n\infty) = 0.$$

*Thus we have that $|f| \leq \|f\|_\infty$ a.e. on $E$; i.e., ess. sup$f$ is the smallest essential upper bound for $f$.*
*Now, for $f, g \in L^\infty(E)$,*

$$|f + g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty \text{ a.e. on } E.$$

*Therefore $\|f\|_\infty + \|g\|_\infty$ is an essential bound for $f + g$ and thus the smallest essential upper bound, $\|f + g\|_\infty$, is such that*

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

**Example** (The Normed Linear Spaces $\ell^1$ and $\ell^\infty$)**.** *For $1 \leq p < \infty$, define $\ell^p$ to be the collection of real sequences $a = (a_1, a_2, \cdots)$ for which*

$$\sum_{k=1}^\infty |a_k|^p < \infty.$$

*Let $a, b \in \ell^p$, and let $\alpha, \beta$ be real numbers. Then we have that $\sum_{k=1}^\infty |a_k|^p < \infty$ and $\sum_{k=1}^\infty |b_k|^p < \infty$.*
*Using the inequality $|a + b|^p \leq 2^p(|a|^p + |b|^p)$, we have*

$$\begin{aligned}
\sum_{k=1}^\infty |\alpha a_k + \beta b_k|^p &\leq \sum_{k=1}^\infty [2^p(|\alpha a_k|^p + |\beta b_k|^p)] \\
&= \sum_{k=1}^\infty 2^p |\alpha|^p |a_k|^p + \sum_{k=1}^\infty 2^p |\beta|^p |b_k|^p \\
&= 2^p|\alpha|^p \sum_{k=1}^\infty |a_k|^p + 2^p|\beta|^p \sum_{k=1}^\infty |b_k|^p \\
&< 2^p|\alpha|^p \infty + 2^p|\beta|^p \infty \\
&= \infty.
\end{aligned}$$

*Thus $\ell^p$ is a linear space.*
*We define $\ell^\infty$ to be the linear space of real bounded sequences: that is, for any $\{a_k\}$ in $\ell^\infty$, there exists a real number $M$ for which $|a_k| \leq M$ for all $k$. We can define the following norms:*

*For* $\{a_k\} \in \ell^1$*:*

$$\|\{a_k\}\|_1 = \sum_{k=1}^{\infty} |a_k|$$

*For* $\{a_k\} \in \ell^\infty$*:*

$$\|\{a_k\}\|_\infty = \sup_{1 \le k < \infty} |a_k|$$

**Example** (The Normed Linear Space $C[a,b]$)**.** *Let* $[a,b]$ *be a closed, bounded interval. The the linear space of continuous real-valued functions on* $[a,b]$ *is denoted by* $C[a,b]$*. Since a continuous function on a compact set takes on a maximum value (ch1 problem 52), we can Define*

$$\|f\|_{\max} = \max_{x \in [a,b]} |f(x)|.$$

**PROBLEMS**

1. For $f$ in $C[a,b]$, Define
$$\|f\|_1 = \int_a^b |f|.$$
Show that this is a norm on $C[a,b]$.

Let $f, g \in C[a,b]$. For each $x \in [a.b]$, we have the inequality $|f(x)+g(x)| \le |f(x)| + |g(x)|$, so by monotonicity and linearity of integration,

$$\|f+g\|_1 = \int_a^b |f(x)+g(x)| \le \int_a^b [|f(x)| + |g(x)|] = \int_a^b |f(x)| + \int_a^b |g(x)| = \|f\|_1 + \|g\|_1.$$

Therefore subadditivity holds.
Also, by linearity of integration, we have

$$\|\alpha f\|_1 = \int_a^b |\alpha f| = \int_a^b |\alpha||f| = |\alpha| \int_a^b |f| = |\alpha| \|f\|_1.$$

Therefore absolute homogeneity holds.
Finally, by definition of absolute value, $0 \le |f(x)|$ for all $x \in [a,b]$, and by monotonicity of integration,

$$0 = \int_a^b 0 \le \int_a^b |f| = \|f\|_1.$$

Clearly $\int_a^b |f| = 0$ iff $f \equiv 0$ on $[a,b]$. Therefore positive definiteness holds.
Thus $\|\cdot\|_1$ is a norm on $C[a,b]$.
Also show that there is no number $c \ge 0$ for which

$$\|f\|_{\max} \le c\|f\|_1 \text{ for all } f \text{ in } C[a,b],$$

Consider the interval $[a,b] = [0,1]$. For any $c > 0$ we choose, there exists an $n \in \mathbb{N}$ such that $n > c$, with the continuous function $f_n : [0,1] \to \mathbb{R}$ defined as

$$f_n(x) = \begin{cases} \frac{n-0}{1/n-0}x & \text{if } x \in [0, \frac{1}{n}] \\ \frac{0-n}{2/n-1/n}(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases} = \begin{cases} n^2 x & \text{if } x \in [0, \frac{1}{n}] \\ -n^2(x - \frac{1}{n}) + n & \text{if } x \in (\frac{1}{n}, \frac{2}{n}] \\ 0 & \text{if } x \in (\frac{2}{n}, 1] \end{cases}$$

(This is a triangle-shaped function that reaches its peak $n$ at $x = \frac{1}{n}$.)
Now, for any $n$, we have $\|f_n\|_1 = \int_0^1 |f_n| = 1$, and $\|f_n\|_{\max} = n$.
Then $\|f\|_{\max} = n > c = c\|f\|_1$.

but there is a $c \geq 0$ for which

$$\|f\|_1 \leq c\|f\|_{\max} \text{ for all } f \text{ in } C[a,b].$$

We can see that for any $f$ in $C[a,b]$, by monotonicity of the integral,

$$\begin{aligned}
\|f\|_1 &= \int_a^b |f(x)| \\
&\leq \int_a^b \max_{x\in[a,b]} |f(x)| \\
&= \max_{x\in[a,b]} |f(x)| \int_a^b 1 \\
&= \max_{x\in[a,b]} |f(x)| \cdot m([a,b]) \\
&= \|f\|_{\max} \cdot m([a,b]).
\end{aligned}$$

Therefore $\|f\|_1 \leq m([a,b])\|f\|_{\max}$ for all $f \in C[a,b]$.

2. Let $X$ be the family of all polynomials with real coefficients defined on $\mathbb{R}$. Show that this is a linear space. For a polynomial $p$, define $\|p\|$ to be the sum of the absolute values of the coefficients of $p$. Is this a norm?
For any two polynomials $p, q \in X$, there exists natural numbers $n, m$ (suppose without loss of generality that $n \leq m$) such that

$$\begin{aligned}
p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n + \cdots + 0x^m \\
q(x) &= b_0 + b_1x + b_2x^2 + \cdots + b_{n-1}x^{n-1} + b_nx^n + \cdots + b_mx^m
\end{aligned}$$

Now, considering any scalars $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned}
\alpha p(x) + \beta q(x) &= \alpha(a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + a_nx^n) \\
&\quad + \beta(b_0 + b_1x + b_2x^2 + \cdots + b_{m-1}x^{m-1} + b_mx^m) \\
&= (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_{n-1})x^{n-1} + (\alpha a_n)x^n \\
&\quad + (\beta b_0) + (\beta b_1)x + (\beta b_2)x^2 + \cdots + (\beta b_{n-1})x^{n-1} + (\beta b_n)x^n + \cdots + (\beta b_m)x^m \\
&= (\alpha a_0 + \beta b_0) + (\alpha a_1 + \beta b_1)x + \cdots + (\alpha a_n + \beta b_n)x^n + \cdots + (\beta b_m)x^m
\end{aligned}$$

This is also a polynomial, as for each $i$, we have $(\alpha a_i + \beta b_i) \in \mathbb{R}$, so $X$ is a linear space.
Now, for any polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n,$$

we can define $\|p\| = |a_0| + |a_1| + |a_2| + \cdots + |a_n| = \sum_{i=0}^n |a_i|$.
The triangle inequality is clear because

$$\|p + q\| = \sum_{i=0}^m |a_i + b_i| \leq \sum_{i=0}^m [|a_i| + |b_i|] = \sum_{i=0}^m |a_i| + \sum_{i=0}^m |b_i| = \|p\| + \|q\|.$$

Absolute homogeneity is clear because

$$\|\alpha p\| = \sum_{i=0}^{n} |\alpha a_i| = \sum_{i=0}^{n} |\alpha||a_i| = |\alpha| \sum_{i=0}^{n} |a_i| = |\alpha|\|p\|.$$

Finally, positive definiteness is clear because

$$0 \leq |a_i| \implies 0 \leq \sum_{i=0}^{n} |a_i| = \|p\|,$$

And $\|p\| = 0$ if and only if $p(x) = 0 + 0x + 0x^2 + \cdots 0x^n = 0$.

3. For $f$ in $L^1[a,b]$, define $\|f\| = \int_a^b x^2|f(x)|dx$. Show that this is a norm on $L^1[a,b]$.
For $f \in L^1[a,b]$, then $f$ is measurable and finite a.e. on $[a,b]$, and $\int_a^b |f(x)|dx < \infty$.
Let $f, g \in L^1[a,b]$, and let $\alpha$ be a real number.
Because the triangle inequality holds a.e. on $[a,b]$, by monotonicity and linearity of the integral, we have

$$\begin{aligned}\|f+g\| &= \int_a^b x^2|f(x)+g(x)|dx \\ &\leq \int_a^b x^2[|f(x)|+|g(x)|]dx \\ &= \int_a^b [x^2|f(x)|+x^2|g(x)|]dx \\ &= \int_a^b x^2|f(x)|dx + \int_a^b x^2|g(x)|dx \\ &= \|f\|+\|g\|.\end{aligned}$$

Therefore $\|\cdot\|$ is subadditive.
By linearity of the integral, we have

$$\|\alpha f\| = \int_a^b x^2|\alpha f(x)|dx = \int_a^b x^2|\alpha||f(x)|dx = |\alpha| \int_a^b x^2|f(x)|dx = |\alpha|\|f\|.$$

Therefore $\|\cdot\|$ satisfies absolute homogeneity.
We can use the fact that $0 \leq x^2$ and $0 \leq |f(x)|$ implies $0 \leq x^2|f(x)|$. By monotonicity of the integral, we have

$$0 = \int_a^b 0dx \leq \int_a^b x^2|f(x)|dx = \|f\|.$$

Clearly $\|f\| = 0$ if and only if $f = 0$ a.e. on $[a,b]$ because $x^2 \cdot 0 = 0$.
Therefore $\|\cdot\|$ satisfies positive definiteness.

4. For $f$ in $L^\infty[a,b]$, show that

$$\|f\|_\infty = \min\left\{M \;\middle|\; m\{x \in [a,b] \mid |f(x)| > M\} = 0\right\},$$

That is, the sup norm is the smallest real number $M$ such that $|f(x)| > M$ only on a set of measure

zero. In an above example, we showed that $\|f\|_\infty$ is the smallest essential upper bound for $f$. That is, $|f| \leq \|f\|_\infty$ a.e. on $E$ (That is, the inequality is true for $E \setminus E_0$, where $m(E_0) = 0$.)
and if, furthermore, $f$ is continuous on $[a, b]$, that

$$\|f\|_\infty = \|f\|_{\max}.$$

If $f$ is continuous, then there are no jump discontinuities ($f$ is continuous at $x_0$ iff $f(x_0^-) = f(x_0) = f(x_0^+)$). Then $|f| \leq \|f\|_\infty$ everywhere on $E$.

5. Show that $\ell^\infty$ and $\ell^1$ are normed linear spaces.
$\ell^\infty$:
Let $a, b \in \ell^\infty$, and let $\alpha, \beta$ be real numbers.
Then for some real numbers $M, N$, we have that $|a_k| \leq M$ and $|b_k| \leq N$ for all $k$.

$$\begin{aligned}\alpha a + \beta b &= \alpha(a_1, a_2, \cdots) + \beta(b_1, b_2, \cdots)\\ &= (\alpha a_1, \alpha a_2, \cdots) + (\beta b_1, \beta b_2, \cdots)\\ &= (\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2, \cdots)\end{aligned}$$

Then $|\alpha a_k + \beta b_k| \leq \alpha M + \beta N$ for all $k$, and $\ell^\infty$ is a linear space.
To show that $\|a\|_\infty = \sup_{1 \leq k < \infty} |a_k|$ is a norm:

$$\|a+b\|_\infty = \sup_{1\leq k<\infty} |a_k + b_k| \leq \sup_{1\leq i<\infty} |a_i| + \sup_{1\leq j<\infty} |b_j| = \|a\|_\infty + \|b\|_\infty,$$

$$\|\alpha a\|_\infty = \sup_{1\leq k<\infty} |\alpha a_k| = \sup_{1\leq k<\infty} |\alpha||a_k| = |\alpha| \sup_{1\leq k<\infty} |a_k| = |\alpha|\|a\|_\infty,$$

$$0 \leq \sup_{1\leq k<\infty} |a_k| = \|a\|_\infty, \text{ and } \sup_{1\leq k<\infty} |a_k| = 0 \text{ iff } a_k = 0 \text{ for all } k.$$

$\ell^1$:
Let $a, b \in \ell^1$, and let $\alpha, \beta$ be real numbers.
Then we have that $\sum_{k=1}^\infty |a_k| < \infty$ and $\sum_{k=1}^\infty |b_k| < \infty$.
By the triangle inequality for real numbers, we have

$$\sum_{k=1}^\infty |\alpha a_k + \beta b_k| \leq \sum_{k=1}^\infty [|\alpha||a_k| + |\beta||b_k|] = |\alpha| \sum_{k=1}^\infty |a_k| + |\beta| \sum_{k=1}^\infty |b_k| < |\alpha|\infty + |\beta|\infty = \infty.$$

Therefore $\ell^1$ is a linear space.
To show that $\|a\|_1 = \sum_{k=1}^\infty |a_k|$ is a norm:

$$\|a+b\|_1 = \sum_{k=1}^\infty |a_k + b_k| \leq \sum_{k=1}^\infty [|a_k| + |b_k|] = \sum_{k=1}^\infty |a_k| + \sum_{k=1}^\infty |b_k| < \infty + \infty = \infty,$$

$$\|\alpha a\|_1 = \sum_{k=1}^\infty |\alpha a_k| = \sum_{k=1}^\infty |\alpha||a_k| = |\alpha| \sum_{k=1}^\infty |a_k| = |\alpha|\|a\|_1,$$

$$0 \leq |a_k| \implies 0 \leq \sum_{k=1}^\infty |a_k| = \|a\|_1, \text{ and } \sum_{k=1}^\infty |a_k| = 0 \text{ iff } a_k = 0 \text{ for all } k.$$

## 7.2 The Inequalities of Young, Hölder, and Minkowski

**PROBLEMS**

6. Show that if Hölder's Inequality is true for normalized functions it is true in general.

7. Verify the assertions in the above two examples regarding the membership of the function $f$ in $L^p(E)$.

8. Let $f$ and $g$ belong to $L^2(E)$. From the linearity of integration show that for any number $\lambda$,
$$\lambda^2 \int_E f^2 + 2\lambda \int_E f \cdot g + \int_E g^2 = \int_E (\lambda f + g)^2 \geq 0.$$
From this and the quadratic formula directly derive the Cauchy-Schwarz Inequality.

9. Show that in Young's Inequality there is equality iff $a^p = b^q$.

10. Show that in Hölder's Inequality there is equality iff there are constants $\alpha, \beta$ not both zero, for which
$$\alpha|f|^p = \beta|g|^q \text{ a.e. on } E.$$
For a point $x = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$, define $T_x$ to be the step function on the interval ...

## 7.3 $L^p$ is Complete: The Riesz-Fischer Theorem

## 7.4 Approximation and Separability

# Chapter 8

# The $L^p$ Spaces: Duality and Weak Convergence

**Contents**



## 8.1 The Riesz Representation for the Dual of $L^p, a \leq p \leq \infty$

**PROBLEMS**

1. Verify (8).

2. Prove Proposition 1.

3. Let $T$ be a linear functional on a normed linear space $X$. Show that $T$ is bounded iff the continuity property (7) holds.

4. A functional $T$ on a normed linear space $X$ is said to be Lipschitz provided there is a $c \geq 0$ such that
$$|T(g) - T(h)| \leq c\|g - h\| \text{ for all } g, h \in X.$$
The infimum of such $c$'s is called the Lipschitz constant for $T$. Show that a linear functional is bounded iff it is Lipschitz, in which case its Lipschitz constant is $\|T\|_*$.

5. Let $E$ be a measurable set and $1 \leq 0 < \infty$. Show that the functions in $L^p(E)$ that vanish outside a bounded set are dense in $L^p(E)$. Show that this is false for $L^\infty(\mathbb{R})$.

6. Establish the Riesz Representation Theorem in the case $p = 1$ by first showing, in the notation of the proof of the theorem, that the function $\Phi$ is Lipschitz and therefore it is absolutely continuous. Then follow the $p > 1$ proof.

7. State and prove a Riesz Representation Theorem for the bounded linear functionals on $\ell^p$, $1 \leq p < \infty$.

8. Let $c$ be the linear space of real sequences that converge to a real number and $c_0$ the subspace of $c$ comprising sequences that converge to 0. Norm each of these linear spaces with the $\ell^\infty$ norm. Determine the dual space of $c$ and $c_0$.

9. Let $[a, b]$ be a closed, bounded interval and $C[a, b]$ be normed by the maximum norm. Let $x_0$ belong to $[a, b]$. Define the linear functional $T$ on $C[a, b]$ by $T(f) = f(x_0)$. Show that $T$ is bounded and is given by Riemann-Stieltjes integration against a function of bounded variation.

10. Let $f$ belong to $C[a, b]$. Show that there is a function $g$ that is of bounded variation on $[a, b]$ for which
$$\int_a^b f dg = \|f\|_{\max} \text{ and } TV(g) = 1.$$

11. Let $[a, b]$ be a closed, bounded interval and $C[a, b]$ be normed by the maximum norm. Let $T$ be a bounded linear functional on $C[a, b]$. For $x \in [a, b]$, let $g_x$ be the member of $C[a, b]$ that is linear on $[a, x]$ and on $[x, b]$ with $g_x(a) = 0, g_x(x) = x - a$ and $g_x(b) = x - a$. Define $\Phi(x) = T(g_x)$ for $x \in [a, b]$. Show that $\Phi$ is Lipschitz on $[a, b]$.

## 8.2 Weak Sequential Convergence in $L^p$

**PROBLEMS**

12. f

## 8.3 Weak Sequential Compactness

**PROBLEMS**

37. f

## 8.4 The Minimization of Convex Functionals

**PROBLEMS**

37. f

# II ABSTRACT SPACES: METRIC, TOPOLOGICAL, BANACH, AND HILBERT SPACES

# Chapter 9

# Metric Spaces: General Properties

## Contents



## 9.1 Examples of Metric Spaces

"The object of the present chapter is to study general spaces called metric spaces for which the notion of distance between two points is fundamental."

**Definition.** *Let $X$ be a nonempty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** provided for all $x, y, z \in X$,*

*(i)* $\rho(x, y) \geq 0$,

*(ii)* $\rho(x, y) = 0$ *iff* $x = y$,

*(iii)* $\rho(x, y) = \rho(y, x)$,

*(iv)* $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

*A nonempty set together with a metric on the set is called a **metric space**, often denoted by $(X, \rho)$.*

An example of a metric space is the set $\mathbb{R}$ of all real numbers with the metric $\rho(x, y) = |x - y|$.

A linear space with a norm is called a normed linear space. A norm $\|\cdot\|$ on a linear space $X$ induces a metric $\rho$ on $X$ by defining

$$\rho(x, y) = \|x - y\| \text{ for all } x, y \in X.$$

To show this, let $x, y \in X$. Because $X$ is a linear space, $x - y \in X$, and $\|x - y\|$ is defined.

(i) $\|x - y\| \geq 0$ by positive definiteness of norm

(ii) $\|x-y\| = 0$ iff $x - y = 0 \implies x = y$ by positive definiteness of norm

(iii) $\|x-y\| = \|-1(y-x)\| = |-1|\|y-x\| = \|y-x\|$ by absolute homogeneity of norm

(iv) $\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ by subadditivity of norm

Three prominent examples of normed linear spaces: the Euclidean spaces $\mathbb{R}^n$, the $L^p(E)$ spaces, $C[a,b]$. For a natural number $n$, consider the linear space $\mathbb{R}^n$, whose points are $n$-tuples of real numbers. For $x = (x_1, \cdots, c_n) \in \mathbb{R}^n$, the Euclidean norm $\|x\|$ is defined by

$$\|x\| = [x_1^2 + \cdots + x_n^2]^{1/2}.$$

*The Discrete Metric* For any nonempty set $X$, the discrete metric $\rho$ is defined by setting $\rho(x,y) = 0$ if $x = y$ and $\rho(x,y) = 1$ if $x \neq y$.

(i) $\rho(x,y) \in \{0,1\} \implies \rho(x,y) \geq 0.$

(ii) $\rho(x,y) = 0 \iff x = y$ by definition.

(iii) By symmetry of the equality relation,

$$\rho(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases} = \begin{cases} 1 & y = x \\ 0 & y \neq x \end{cases} = \rho(y,x)$$

(iv) In the case $x \neq y$, the triangle inequality is trivial.
In the case $x = y$,
if $x = z$, then by transitivity, $z = y$ and
$\rho(x,y) = 0, \rho(x,z) = 0, \rho(z,y) = 0$, and the triangle inequality is obvious: $0 \leq 0$.
if $x \neq z$, then by transitivity, $z \neq y$ and
$\rho(x,y) = 0, \rho(x,z) = 1, \rho(z,y) = 1 \implies \rho(x,y) = 0 \leq 2 = \rho(x,z) + \rho(z,y).$

*Metric Subspaces* For a metric space $(X, \rho)$, let $Y$ be a nonempty subset of $X$. Then the restriction of $\rho$ to $Y \times Y$ defines a metric on $Y$ and we call such a metric space a metric **subspace**. Therefore every nonempty subset of Euclidean space, of and $L^p(E)$ space, $1 \leq p \leq \infty$, and of $C[a,b]$ is a metric space.

*Metric Products* For metric spaces $(X_1, \rho_1)$ and $(X_2, \rho_2)$, we define the **product metric** $\tau$ on the Cartesian product $X_1 \times X_2$ by setting, for $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$,

$$\tau((x_1,x_2),(y_1,y_2)) = \{[\rho(x_1,y_1)]^2 + [\rho(x_2,y_2)]^2\}^{1/2}.$$

To see that this is a metric:

(i) By metric property (i) of $\rho_1, \rho_2$, we have

$$\begin{aligned} [\rho_1(x_1,y_1)]^2 \geq 0 \text{ and } [\rho_2(x_2,y_2)]^2 \geq 0 &\iff \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2} \geq 0 \\ &\iff \tau((x_1,x_2),(y_1,y_2)) \geq 0 \end{aligned}$$

(ii) By metric property (ii) of $\rho_1, \rho_2$, we have

$$\begin{aligned} \tau((x_1,x_2),(y_1,y_2)) &= \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2} = 0 \\ &\iff [\rho_1(x_1,y_1)]^2 = 0 \text{ and } [\rho_2(x_2,y_2)]^2 = 0 \\ &\iff x_1 = y_1 \text{ and } x_2 = y_2 \\ &\iff (x_1,x_2) = (y_1,y_2). \end{aligned}$$

(iii) By metric property (iii) (symmetry) of $\rho_1, \rho_2$, we have

$$\begin{aligned}\tau((x_1,x_2),(y_1,y_2)) &= \{[\rho_1(x_1,y_1)]^2 + [\rho_2(x_2,y_2)]^2\}^{1/2} \\ &= \{[\rho_1(y_1,x_1)]^2 + [\rho_2(y_2,x_2)]^2\}^{1/2} \\ &= \tau((y_1,y_2),(x_1,x_2)).\end{aligned}$$

(iv) We must first prove an inequality $\star$. Because $x^2 \geq 0$ for any real number $x$, we have

$$\begin{aligned} 0 &\leq [\rho_1(x_1,z_1)\rho_2(z_2,y_2) - \rho_2(x_2,z_2)\rho_1(z_1,y_1)]^2 \\ 0 &\leq \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2 \\ &\quad - 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2) \\ 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2 \end{aligned}$$

Adding $\rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2$ to both sides, we have

$$\begin{aligned} &\rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2 + 2\rho_1(x_1,z_1)\rho_1(z_1,y_1)\rho_2(x_2,z_2)\rho_2(z_2,y_2) \\ &\leq \rho_1(x_1,z_1)^2\rho_1(z_1,y_1)^2 + \rho_2(x_2,z_2)^2\rho_2(z_2,y_2)^2 + \rho_1(x_1,z_1)^2\rho_2(z_2,y_2)^2 + \rho_2(x_2,z_2)^2\rho_1(z_1,y_1)^2 \end{aligned}$$

Therefore we end up with the inequality: $\star$

$$\begin{aligned} [\rho_1(x_1,z_1)\rho_1(z_1,y_1) + \rho_2(x_2,z_2)\rho_2(z_2,y_2)]^2 &\leq [\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2][\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2] \\ \rho_1(x_1,z_1)\rho_1(z_1,y_1) + \rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq \sqrt{[\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2][\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2]} \\ 2\rho_1(x_1,z_1)\rho_1(z_1,y_1) + 2\rho_2(x_2,z_2)\rho_2(z_2,y_2) &\leq 2\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2}\sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2} \quad \star \end{aligned}$$

Now, to prove the triangle inequality of the product metric, we use the metric property (iv) (triangle inequality) of $\rho_1, \rho_2$:

$$\begin{aligned} &\rho(x_1,y_1)^2 + \rho(x_2,y_2)^2 \\ &\leq [\rho(x_1,z_1) + \rho(z_1,y_1)]^2 + [\rho(x_2,z_2) + \rho(z_2,y_2)]^2 \\ &= \rho(x_1,z_1)^2 + \rho(z_1,y_1)^2 + 2\rho(x_1,z_1)\rho(z_1,y_1) + \rho(x_2,z_2)^2 + \rho(z_2,y_2)^2 + 2\rho(x_2,z_2)\rho(z_2,y_2) \\ &\leq \rho(x_1,z_1)^2 + \rho(z_1,y_1)^2 + \rho(x_2,z_2)^2 + \rho(z_2,y_2)^2 + 2\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2}\sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2} \quad \star \\ &= \left[\sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2} + \sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2}\right]^2 \end{aligned}$$

Therefore we have

$$\sqrt{\rho(x_1,y_1)^2 + \rho(x_2,y_2)^2} \leq \sqrt{\rho_1(x_1,z_1)^2 + \rho_2(x_2,z_2)^2} + \sqrt{\rho_1(z_1,y_1)^2 + \rho_2(z_2,y_2)^2}$$

and thus

$$\tau((x_1,x_2),(y_1,y_2)) \leq \tau((x_1,x_2),(z_1,z_2)) + \tau((z_1,z_2),(y_1,y_2)).$$

This construction extends to countable products (problem 10).

**Definition.** *Two metrics $\rho$ and $\sigma$ on a set $X$ are said to be **equivalent** provided there are positive numbers $c_1, c_2$ such that for all $x_1, x_2 \in X$,*

$$c_1 \cdot \sigma(x_1, x_2) \leq \rho(x_1, x_2) \leq c_2 \cdot \sigma(x_1, x_2).$$

**Definition.** *A mapping $f$ from a metric space $(X, \rho)$ to a metric space $(Y, \sigma)$ is said to be an **isometry** provided it maps $X$ onto $Y$ and for all $x_1, x_2 \in X$,*

$$\sigma(f(x_1), f(x_2)) = \rho(x_1, x_2).$$

Two metric spaces are called **isometric** provided there is an isometry from one onto the other. To be isometric is an equivalence relation among metric spaces. From the viewpoint of metric spaces, two isometric metric spaces are exactly the same, an isometry amounting merely to a relabeling of the points.

In the definition of a metric $\rho$ on a set $X$ it is often convenient to relax the condition that $\rho(x, y) = 0$ if and only if $x = y$. When we allow the possibility that $\rho(x, y) = 0$ for some $x \neq y$, we call $\rho$ a **pseudometric** and $(X, \rho)$ a pseudometric space. On such a space, define the relation $x \cong y$ provided $\rho(x, y) = 0$. This is an equivalence relation that separates $X$ into a disjoint collection of equivalence classes $X/\cong$.

## PROBLEMS

1. Show that two metrics $\rho$ and $\tau$ on the same set $X$ are equivalent iff there is a $c > 0$ such that for all $u, v \in X$,

$$\frac{1}{c}\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

Let $\rho$ and $\tau$ be two metrics on the same set $X$.
( $\implies$ ) Suppose $\rho$ and $\tau$ are equivalent.
Then there exist $c_1, c_2 > 0$ such that for all $u, v \in X$,

$$c_1 \cdot \tau(u, v) \leq \rho(u, v) \leq c_2 \cdot \tau(u, v).$$

By the Archimedean Property of $\mathbb{R}$, for the positive real number $c_1$, there exists a natural number $n$ for which $\frac{1}{n} < c_1$.
Let $c = \max\{n, c_2\}$ so that $n \leq c \implies \frac{1}{c} \leq \frac{1}{n} < c_1$ and also $c_2 \leq c$, so we have

$$\frac{1}{c} \cdot \tau(u, v) < c_1 \cdot \tau(u, v) \leq \rho(u, v) \leq c_2 \cdot \tau(u, v) \leq c\tau(u, v).$$

( $\impliedby$ ) Suppose that there is a $c > 0$ such that for all $u, v \in X$,

$$\frac{1}{c}\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

We showed that for a positive number $c$, its multiplicative inverse $\frac{1}{c}$ is also positive [ch1, 2(ii)], and so we have $\frac{1}{c}, c > 0$ such that for all $u, v \in X$,

$$(\frac{1}{c})\tau(u, v) \leq \rho(u, v) \leq c\tau(u, v).$$

Therefore $\rho$ and $\tau$ are equivalent.

2. Show that the following define equivalent metrics on $\mathbb{R}^n$:

$$\rho^*(x,y) = |x_1 - y_1| + \cdots + |x_n - y_n|;$$
$$\rho^+(x,y) = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}.$$

For some $j \in \{1, \cdots, n\}$, we have $|x_j - y_j| = \max\{|x_1 - y_1|, \cdots, |x_n - y_n|\}$.
Then

$$\begin{aligned}\rho^+(x,y) &= |x_j - y_j| \\ &= 0 + \cdots + |x_j - y_j| + \cdots + 0 \\ &\le |x_1 - y_1| + \cdots |x_j - y_j| + \cdots + |x_n - y_n| \\ &= \rho^*(x,y).\end{aligned}$$

Also,

$$\begin{aligned}\rho^*(x,y) &= |x_1 - y_1| + \cdots |x_j - y_j| + \cdots + |x_n - y_n| \\ &\le |x_j - y_j| + \cdots |x_j - y_j| + \cdots + |x_j - y_j| \\ &= n|x_j - y_j| \\ &= n\rho^+(x,y).\end{aligned}$$

Therefore we have $1, n > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$\rho^+(x,y) \le \rho^*(x,y) \le n\rho^+(x,y).$$

3. Find a metric on $\mathbb{R}^n$ that fails to be equivalent to either of those defined in the preceding problem.
Let

$$\rho(x,y) = \begin{cases} \min\{|x-y|, 1\} & \text{if } x, y \text{ are both rational or } x, y \text{ are both irrational} \\ 1 & \text{else} \end{cases}$$

4. For a closed, bounded interval $[a,b]$, consider the set $X = C[a,b]$ of continuous real-valued functions on $[a,b]$. Show that the metric induced by the maximum norm and that induced by the $L^1[a,b]$ norm are not equivalent.
From Chapter 7 Problem 1, we proved that there is no number $c \ge 0$ for which

$$\|f\|_{\max} \le c\|f\|_1 \text{ for all } f \text{ in } C[a,b].$$

Therefore there exists no $c_1, c_2 > 0$ such that for all $f, g$ in $C[a,b]$,

$$c_1\|f-g\|_1 \le \|f-g\|_{\max} \le c_2\|f-g\|_1,$$

and the metrics induced by the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_1$ are not equivalent.

5. *The Nikodym Metric*. Let $E$ be a Lebesgue measurable set of real numbers of finite measure, $X$ the set of measurable subsets of $E$, and $m$ Lebesgue measure. For $A, B \in X$, define $\rho(A,B) = m(A\Delta B)$, where $A\Delta B = [A \setminus B] \cup [B \setminus A]$, the symmetric difference of $A$ and $B$. Show that this is a pseudometric on $X$. Define two measurable sets to be equivalent provided their symmetric difference has measure zero. Show that $\rho$ induces a metric on the collection of equivalence classes. Finally, show that for $A, B \in X$,

$$\rho(A,B) = \int_E |\chi_A - \chi_B|,$$

where $\chi_A$ and $\chi_B$ are the characteristic functions of $A$ and $B$, respectively.

6. Show that for $a, b, c \geq 0$,

$$\text{if } a \leq b+c, \text{ then } \frac{a}{1+a} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

Because $a \leq b+c$, we see that $\frac{1}{1+b+c} \leq \frac{1}{1+a}$, and $-\frac{1}{1+a} \leq -\frac{1}{1+b+c}$ so

$$\frac{a}{1+a} = 1 - \frac{1}{1+a} \leq 1 - \frac{1}{1+b+c} = \frac{b+c}{1+b+c} = \frac{b}{1+b+c} + \frac{c}{1+b+c} \leq \frac{b}{1+b} + \frac{c}{1+c}.$$

7. Let $E$ be a Lebesgue measurable set of real numbers that has finite measure and $X$ the set of Lebesgue measurable real-valued functions on $E$. For $f, g \in X$, define

$$\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|}.$$

Use the preceding problem to show that this is a pseudometric on $X$. Define two measurable functions to be equivalent provided they are equal a.e. on $E$. Show that $\rho$ induces a metric on the collection of equivalence classes.

Let $f, g, h \in X$.
Then for all $x \in E$, we have the triangle inequality $|f(x)-g(x)| \leq |f(x)-h(x)| + |h(x)-g(x)|$.
By Problem 6 and monotonicity and linearity of integration, we have

$$\int_E \frac{|f-g|}{1+|f-g|} \leq \int_E \frac{|f-h|}{1+|f-h|} + \int_E \frac{|h-g|}{1+|h-g|},$$

Therefore $\rho(f,g) \leq \rho(f,h) + \rho(h,g)$ and (iv) holds.
Clearly $\rho(f,g) \geq 0$ is true so (i) holds.
We also have $\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|} = \int_E \frac{|g-h|}{1+|g-h|} = \rho(g,f)$ so (iii) holds.
However, (ii) does not hold. We can consider two functions $f, g \in X$ such that $f = g$ on $E \setminus E_0$, where $m(E_0) = 0$. Then $\rho(f,g) = \int_E \frac{|f-g|}{1+|f-g|} = \int_{E \setminus E_0} \frac{|f-f|}{1+|f-f|} = 0$ but $f \neq g$.
If we consider defining the equivalence

$$f \cong g \text{ when } f = g \text{ on } E \setminus E_0 \text{ where } m(E_0) = 0,$$

Then $\rho$ now induces a metric on the collection of equivalence classes $X/\cong$, because we now have $\rho(f,g) = 0 \iff f \cong g$.

8. For $0 < p < 1$, show that

$$(a+b)^p \leq a^p + b^p \text{ for all } a, b \geq 0.$$

9. For $E$ a Lebesgue measurable set of real numbers, $0 < p < 1$, and $g, h$ Lebesgue measurable functions on $E$ that have integrable $p^{th}$ powers, define

$$\rho_p(h,g) = \int_E |g-h|^p.$$

Use the preceding problem to show that this is a pseudometric on the collection of Lebesgue measurable functions on $E$ that have integrable $p^{th}$ powers. Define two such functions to be equivalent provided they are equal a.e. on $E$. Show that $\rho_p(\cdot,\cdot)$ induces a metric on the collection of equivalence classes.

Let $f, g, h : E \to \mathbb{R}$ such that $f, g, h$ are Lebesgue measurable and have integrable $p^{th}$ powers. For all $x \in E$, Problem 8 tells us that $|g(x) - h(x)|^p \leq |g(x) - f(x)|^p + |f(x) - h(x)|^p$. By monotonicity and linearity of integration, we have

$$\int_E |g - h|^p \leq \int_E |g - f|^p + \int_E |f - h|^p.$$

Therefore $\rho_p(h, g) \leq \rho_p(f, g) + \rho_p(h, f)$ and (iv) holds.
Clearly $\rho_p(h, g) \geq 0$ is true so (i) holds.
We also have $\rho_p(h, g) = \int_E |g - h|^p = \int_E |h - g|^p = \rho_p(g, h)$ so (iii) holds.
However, (ii) does not hold. We can consider two functions $h, g$ that are equivalent a.e. on $E$ but not equal. Then $\rho_p(h, g) = \int_E |g - h|^p = \int_{E \setminus E_0} |g - h|^p = 0$ with $m(E_0) = 0$ but $h \neq g$.
If we define an equivalence when two functions are equal a.e. on $E$, then $\rho_p$ is a metric on the collection of such equivalence classes.

10. Let $\{(X_n, \rho_n)\}_{n=1}^{\infty}$ be a countable collection of metric spaces. Use problem 6 to show that $\rho_*$ defines a metric on the Cartesian product $\prod_{n=1}^{\infty} X_n$, where for points $x = \{x_n\}$ and $y = \{y_n\}$ in $\prod_{n=1}^{\infty} X_n$,
$$\rho_*(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x_n, y_n)}{1 + \rho_n(x_n, y_n)}.$$

11. Let $(X, \rho)$ be a metric space and $A$ any set for which there is a one-to-one (injective) mapping $f$ of $A$ onto (surjective?) the set $X$ (bijection?). Show that there is a unique metric on $A$ for which $f$ is an isometry of metric spaces. (This is the sense in which an isometry amounts merely to a relabeling of the points in a space.)

12. Show that the triangle inequality for Euclidean space $\mathbb{R}^n$ follows from the triangle inequality for $L^2[0, 1]$.

## 9.2 Open Sets, Closed Sets, and Convergent Sequences

## 9.3 Continuous Mappings Between Metric Spaces

## 9.4 Complete Metric Spaces

## 9.5 Compact Metric Spaces

## 9.6 Separable Metric Spaces

# Chapter 10

# Metric Spaces: Three Fundamental Theorems

**Contents**



## 10.1 The Arzelá-Ascoli Theorem

## 10.2 The Baire Category Theorem

## 10.3 The Banach Contraction Principle

# Chapter 11

# Topological Spaces: General Properties

**Contents**



## 11.1 Open Sets, Closed Sets, Bases, and Subbases

## 11.2 The Separation Properties

## 11.3 Countability and Separability

## 11.4 Continuous Mappings Between Topological Spaces

## 11.5 Compact Topological Spaces

## 11.6 Connected Topological Spaces

# Chapter 12

# Topological Spaces: Three Fundamental Theorems

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## 12.1 Urysohn's Lemma and the Tietze Extension Theorem

## 12.2 The Tychonoff Product Theorem

## 12.3 Thye Stone-Weierstrass Theorem

# Chapter 13

# Continuous Linear Operators Between Banach Spaces

**Contents**



## 13.1 Normed Linear Spaces

**PROBLEMS**

1. Show that a nonempty subset $S$ of a linear space $X$ is a subspace iff $S + S = S$ and $\lambda \cdot S = S$ for each $\lambda \in \mathbb{R}, \lambda \neq 0$.

2. If $Y$ and $Z$ are subspaces of the linear space $X$, show that $T + Z$ is also a subspace and $Y + Z = \text{span}[Y \cup Z]$.

3. Let $S$ be a subset of a normed linear space $X$.

   (i) Show that the intersection of a collection of linear subspaces of $X$ is also a linear subspace of $X$.

   (ii) Show that span$[S]$ is the intersection of all the linear subspaces of $X$ that contain $S$ and therefore is a linear subspace of $X$.

   (iii) Show that $\overline{\text{span}}[S]$ is the intersection of all the closed linear subspaces of $X$ that contain $S$ and is therefore a closed linear subspace of $X$.

4. For a normed linear space $X$, show that the function $\|\cdot\| : X \to \mathbb{R}$ is continuous.

5. For two normed linear spaces $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$, define a linear structure on the Cartesian product $X \times Y$ by ...

## 13.2 Linear Operators

## 13.3 Compactness Lost: Infinite Dimensional Normed Linear Spaces

## 13.4 The Open Mapping and Closed Graph Theorems

## 13.5 The Uniform Boundedness Principle

# Chapter 14

# Duality for Normed Linear Spaces

**Contents**



## 14.1 Linear Functionals, Bounded Linear Functionals, and Weak Topologies

## 14.2 The Hahn-Banach Theorem

## 14.3 Reflexive Banach Spaces and Weak Sequential Convergence

## 14.4 Locally Convex Topological Vector Spaces

## 14.5 The Separation of Convex Sets and Mazur's Theorem

## 14.6 The Krein-Milman Theorem

# Chapter 15

# Compactness Regained: The Weak Topology

## Contents



## 15.1 Alaoglu's Extension of Helley's Theorem

**PROBLEMS**

1. For $X$ a normed linear space with closed unit ball $B$, suppose the function $f : B \to [-1, 1]$ has the property that whenever $u, v, u + v, \lambda u$ belong to $B$, $f(u + v) = f(u) + f(v)$ and $f(\lambda u) = \lambda f(u)$. Show that $f$ is the restriction to $B$ of a linear functional on all of $X$ which belongs to the closed unit ball of $X^*$.

2. Let $X$ be a normed linear space and $K$ be a bounded convex weak-$*$ closed subset of $X^*$. Show that $K$ possesses an extreme point.

3. Show that any nonempty weakly open set in an infinite dimensional normed linear spae is unbounded with respect to the norm.

4. Use the Baire Category Theorem and the preceding problem to show that the weak topology on an infinite dimensional Banach space is not metrizable by a complete metric.

5. Is every Banach space isomorphic to the dual of a Banach space?

## 15.2 Reflexivity and Weak Compactness: Kakutani's Theorem

**PROBLEMS**

6. Show that every weakly compact subset of a normed linear space is bounded with respect to the norm.

7. Show that the closed unit ball $B^*$ of the dual $X^*$ of a Banach space $X$ has an extreme point.

8. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two compact, Hausdorff topologies on a set $\mathcal{S}$ for which $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Show that $\mathcal{T}_1 = \mathcal{T}_2$.

9. Let $X$ be a normed linear space containing the subspace $Y$. For $A \subseteq Y$, show that the weak topology on $A$ induced by $Y^*$ is the same as the topology $A$ inherits as a subspace of $X$ with its weak topology.

10. Argue as follows to show that q Banach space $X$ is reflexive iff its dual space $X^*$ is reflexive.

    (i) If $X$ is reflexive, show that the weak and weak-$*$ topologies on $B^*$ are the same, and infer from this that $X^*$ is reflexive.

    (ii) If $X^*$ is reflexive, use part (i) and Proposition 15 of Chapter 14 to show that $X$ is reflexive.

11. For $X$ a Banach space, by the preceding problem, if $X$ is reflexive, then so is $X^*$. Conclude that $X$ is not reflexive if there is a closed subspace of $X^*$ that is not reflexive. Let $K$ be an infinite compact Hausdorff space and $\{x_n\}$ an enumeration of a countably infinite subset of $K$. Define the operator $T : l^1 \to [C(K)]^*$ by

$$[T(\{n_k\})](f) = \sum_{k=1}^{\infty} \eta_k \cdot f(x_k) \text{ for all } \{\eta_k\} \in l^1 \text{ and } f \in C(k).$$

    Show that $T$ is an isometry and therefore, since $l^1$ is not reflexive, neither is $T(l^1)$ and therefore neither is $C(K)$. Use a dimension counting argument to show that $C(K)$ is reflexive if $K$ is a finite set.

12. If $Y$ is a linear subspace of a Banach space $X$, we define the *annihilator* $Y^\perp$ to be the subspace of $X^*$ consisting of those $\psi \in X^*$ for which $\psi = 0$ on $Y$. If $Y$ is a subspace of $X^*$, we define $Y^0$ to be the subspace of vectors in $X$ for which $\psi(x) = 0$ for all $\psi \in Y$.

    (i) Show that $Y^\perp$ is a closed linear subspace of $X^*$.

    (ii) Show that $(Y^\perp)^0 = \overline{Y}$.

    (iii) If $X$ is reflexive and $Y$ is a subspace of $X^*$, show that $Y^\perp = J(Y^0)$.

## 15.3 Compactness and Weak Sequential Compactness: The Eberlein-Šmulian Theorem

**PROBLEMS**

13. In a general topological space that is not metrizable a sequence may converge to more than one point. Show that this cannot occur for the $W$-weak topology on a normed linear space $X$, where $W$ is a subspace of $X^*$ that separates points in $X$.

14. Show that there is a bounded sequence in $L^\infty[0, 1]$ that fails to have a weakly convergent subsequence. Show that the closed unit ball of $C[a, b]$ is not weakly compact.

15. Let $K$ be a compact metric space with infinitely many points. Show that there is a bounded sequence in $C(K)$ that fails to have a weakly convergent subsequence (see Problem 11), but every bounded sequence of continuous linear functionals on $C(K)$ has a subsequence that converges pointwise to a continuous linear functional on $C(K)$.

## 15.4 Metrizability of Weak Topologies

**PROBLEMS**

16. Show that the dual of an infinite dimensional normed linear space also is infinite dimensional.

17. Complete the last step of the proof of Theorem 10 by showing that the inequalities (12) imply that the metric $\rho$ induces the $W$-weak topology.

18. Let $X$ be a Banach space, $W$ a closed subspace of its dual $X^*$, and $\psi_0$ belong to $X^* \setminus W$. Show that if either $W$ is finite dimensional or $X$ is reflexive, then there is a vector $x_0$ in $X$ for which $\psi_0(x_0) \neq 0$ but $\psi(x_0) = 0$ for all $\psi \in W$. Exhibit an example of an infinite dimensional closed subspace $W$ of $X^*$ for which this separation property fails.

# Chapter 16

# Continuous Linear Operators on Hilbert Spaces

## Contents



## 16.1 The Inner Product and Orthogonality

**PROBLEMS**

In the following problems, $H$ is a Hilbert space.

1. Let $[a,b]$ be a closed, bounded interval of real numbers. Show that the $L^2[a,b]$ inner product is also an inner product on $C[a,b]$. Is $C[a,b]$ considered as an inner product space with the $L^2[a,b]$ inner product, a Hilbert space?

2. Show that the maximum norm on $C[a,b]$ is not induced by an inner product and neither is the usual norm on $\ell^1$.

3. Let $H_1$ and $H_2$ be Hilbert spaces. Show that the Cartesian product $H_1 \times H_2$ is also a Hilbert space with an inner product with respect to which $H_1 \times \{0\} = [\{0\} \times H_2]^{\perp}$.

4. Show that if $S$ is a subset of an inner product space $H$, then $S^{\perp}$ is a closed subspace of $H$.

5. Let $S$ be a subset of $H$. Show that $S = (S^{\perp})^{\perp}$ iff $S$ is a closed subspace of $H$.

6. (Polarization Identity) Show that for any two vectors $u, v \in X$,

$$\langle u, v\rangle = \frac{1}{4}[\|u+v\|^2 - \|u-v\|^2].$$

7. (Jordan-von Neumann) Let $X$ be a linear space normed bu $\|\cdot\|$. Use the polarization identity to show that a norm $\|\cdot\|$ is induced by an inner product iff the parallelogram identity holds.

8. Let $V$ be a closed subspace of $H$ and $P$ a projection of $H$ onto $V$. Show that $P$ is the orthogonal projection of $H$ onto $V$ iff (4) holds.

9. Let $T$ belong to $\mathcal{L}(H)$. Show that $T$ is an isometry iff
$$\langle T(u), T(v)\rangle = \langle u, v\rangle \text{ for all } u, v \in H.$$
Let $V$ be a finite dimensional subspace of $H$ and $\varphi_1, \cdots, \varphi_n$ a basis for $V$ consisting of unit vectors, each pair of which is orthogonal. Show that the orthogonal projection $P$ of $H$ onto $V$ is given by
$$P(h) = \sum_{k=1}^{n} \langle h, \varphi_k\rangle \varphi_k \text{ for all } h \in V.$$

10. For $h$ a vector in $H$, show that the function $u \mapsto \langle h, u\rangle$ belongs to $H^*$.

11. For any vector $h \in H$, show that there is a bounded linear functional $\psi \in H^*$ for which
$$\|\psi\| = 1 \text{ and } \psi(h) = \|h\|.$$

12. Let $V$ be a closed subspace of $H$ and $P$ the orthogonal projection of $H$ onto $V$. For any normed linear space $X$ and $T \in \mathcal{L}(V, X)$, show that $T \circ P$ belongs to $\mathcal{L}(H, X)$, and is an extension of $T : V \to X$ for which $\|T \circ P\| = \|T\|$.

13. Prove the Hyperplane Separation Theorem for $H$, considered as a locally convex topological vector space with respect to the strong topology, by directly using Proposition 2.

14. Use Proposition 2 to prove the Krein-Milman Lemma in a Hilbert space.

## 16.2 The Dual Space and Weak Sequential Convergence

**PROBLEMS**

In the following problems, $H$ is a Hilbert space.

16. Show that neither $\ell^1, \ell^\infty, L^1[a, b]$ nor $L^\infty[a, b]$ is Hilbertable.

17. Prove Proposition 7.

18. Let $H$ be an inner product space. Show that since $H$ is a dense subset of a Banach space $X$ whose norm restricts to the norm induced by the inner product on $H$, the inner product on $H$ extends to $X$ and induces the norm on $X$. Thus inner product spaces have Hilbert space completions.

## 16.3 Bessel's Inequality and Orthonormal Bases

**PROBLEMS**

In the following problems, $H$ is a Hilbert space.

19. Show that an orthonormal subset of a separable Hilbert space $H$ must be countable.

20. Let $\{\varphi_k\}$ be an orthonormal sequence in a Hilbert space $H$. Show that $\{\varphi_k\}$ converges weakly to 0 in $H$.

21. Let $\{\varphi_k\}$ be an orthonormal basis for the separable Hilbert space $H$. Show that $\{u_n\} \to u$ in $H$ iff $\{u_n\}$ is bounded and, for each $k$, $\lim_{n\to\infty}\langle u_n, \varphi_k\rangle = \langle u, \varphi_k\rangle$.

22. Show that any two infinite dimensional separable Hilbert spaces are isometrically isomorphic and that any such isomorphism preserves the inner product.

23. Let $H$ be a Hilbert space and $V$ a closed separable subspace of $H$ for which $\{\varphi_k\}$ is an orthonormal basis. Show that the orthogonal projection of $H$ onto $V$, $P$, is given By

$$P(h) = \sum_{k=1}^{\infty}\langle\varphi_k, h\rangle\varphi_k \text{ for all } h \in H.$$

24. (Parseval's Indentities) Let $\{\varphi_k\}$ be an orthonormal basis for a Hilbert space $H$. Verify that

$$\|h\|^2 = \sum_{k=1}^{\infty}\langle\varphi_k, h\rangle^2 \text{ for all } h \in H.$$

Also verify that

$$\langle u, v\rangle = \sum_{k=1}^{\infty} a_k \cdot b_k \text{ for all } u, v \in H,$$

where, for each natural number $k$, $a_k = \langle u, \varphi_k\rangle$ and $b_k = \langle v, \varphi_k\rangle$.

25. Verify the assertions in the example of the orthonormal basis for $L^2[0, 2\pi]$.

26. Use Proposition 10 and the Stone-Weierstrass Theorem to show that for each $f \in L^2[-\pi, \pi]$,

$$f(x) = a_0/2 + \sum_{k=1}^{\infty}[a_k \cdot \cos kx + b_k \cdot \sin kx],$$

where the convergence is in $L^2[-\pi, \pi]$ and each

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx dx \text{ and } b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx dx.$$

## 16.4 Adjoints and Symmetry for Linear Operators

**PROBLEMS**

In the following problems, $H$ is a Hilbert space.

27. Verify (12).

28. Let $T$ and $S$ belong to $\mathcal{L}(H)$ and be symmetric. ...

## 16.5 Compact Operators

## 16.6 The Hilbert-Schmidt Theorem

## 16.7 The Riesz-Schauder Theorem: Characterization of Fredholm Operators

# III MEASURE AND INTEGRATION: GENERAL THEORY

# Chapter 17

# General Measure Spaces: Their Properties and Construction

## Contents



## 17.1 Measures and Measurable Sets

**Definition.** *By a **measurable space** we mean a couple $(X, \mathcal{M})$ consisting of a set $X$ and a $\sigma$-algebra $\mathcal{M}$ of subsets of $X$. A subset $E$ of $X$ is called **measurable** (or measurable with respect to $\mathcal{M}$) provided $E$ belongs to $\mathcal{M}$.*

**Definition.** *By a **measure** $\mu$ on a measurable space $(X, \mathcal{M})$ we mean an extended real-valued nonnegative set function $\mu : \mathcal{M} \to [0, \infty]$ for which $\mu(\emptyset) = 0$ and which is **countably additive** in the sense that for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,*

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

**Definition.** *By a **measure space** $(X, \mathcal{M}, \mu)$ we mean a measurable space $(X, \mathcal{M})$ together with a measure $\mu$ defined on $\mathcal{M}$.*

**Proposition 1.** *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

*(Finite Additivity) For any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets,*

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k).$$

*(Monotonicity) If $A$ and $B$ are measurable sets and $A \subseteq B$, then*

$$\mu(A) \leq \mu(B).$$

*(Excision) If, moreover, $A \subseteq B$ and $\mu(A) < \infty$, then*

$$\mu(B \setminus A) = \mu(B) - \mu(A),$$

*so that if $\mu(A) = 0$, then*

$$\mu(B \setminus A) = \mu(B).$$

*(Countable Monotonicity) For any countable collection $\{E_k\}_{k=1}^n$ of measurable sets that covers a measurable set $E$,*

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

**Definition.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure $\mu$ is called* ***finite*** *provided $\mu(X) < \infty$. It is called* ***$\sigma$-finite*** *provided $X$ is the union of a countable collection of measurable sets, each of which has finite measure. A measurable set $E$ is said to be of* ***finite measure*** *provided $\mu(E) < \infty$, and is said to be* ***$\sigma$-finite*** *provided $E$ is the union of a countable collection of measurable sets, each of which has finite measure.*

**Definition.** *A measure space $(X, \mathcal{M}, \mu)$ is said to be* ***complete*** *provided $\mathcal{M}$ contains all subsets of sets of measure zero, that is, if $E$ belongs to $\mathcal{M}$ and $\mu(E) = 0$, then every subset of $E$ also belongs to $\mathcal{M}$.*

For example, the Lebesgue measure $m$ on the real line is complete. Moreover, in Chapter 2 Proposition 22, we showed that the Cantor set $C$, a Borel set that has Lebesgue measure zero, contains a Lebesgue measurable set that is not a Borel set. Therefore the Lebesgue measure restricted to the Borel $\sigma$-algebra $\mathcal{B}$ is not complete because $C$ belongs to $\mathcal{B}$ and $m(C) = 0$ but there exists a subset $A \subseteq C$ such that $A \notin \mathcal{B}$.

The following proposition tells us that each measure space can be completed.

**Proposition 3.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Define $\mathcal{M}_0$ to be the collection of subsets $E$ of $X$ of the form $E = A \cup B$ where $B \in \mathcal{M}$ and $A \subseteq C$ for some $C \in \mathcal{M}$ for which $\mu(C) = 0$. For such a set $E$ define $\mu_0(E) = \mu(B)$. Then $\mathcal{M}_0$ is a $\sigma$-algebra that contains $\mathcal{M}$, $\mu_0$ is a measure that extends $\mu$, and $(X, \mathcal{M}_0, \mu_0)$ (the* ***completion*** *of $(X, \mathcal{M}, \mu)$) is a complete measure space.*

## PROBLEMS

1. Let $f$ be a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each Lebesgue measurable subset $E$ of $\mathbb{R}$, define $\mu(E) = \int_E f$, the Lebesgue integral of $f$ over $E$. Show that $\mu$ is a measure on the $\sigma$-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

   Because $f$ is nonnegative, by monotonicity of integration, for any Lebesgue measurable set $E$,

   $$0 \leq f \implies 0 = \int_E 0 \leq \int_E f = \mu(E).$$

   Check Chapter 4 Problem 28 to see that for $f$ Lebesgue integrable over $\mathbb{R}$ and $\emptyset$ a Lebesgue measurable subset of $\mathbb{R}$, we have that

   $$\mu(\emptyset) = \int_{\emptyset} f = \int_{\mathbb{R}} f \cdot \chi_{\emptyset} = \int_{\mathbb{R}} f \cdot \chi_{\emptyset} = \int_{\mathbb{R}} 0 = 0.$$

Let $\{E_n\}_{n=1}^{\infty}$ be a disjoint countable collection of Lebesgue measurable sets so that each $\mu(E_n) = \int_{E_n} f$ is defined. Then $E = \bigcup_{n=1}^{\infty} E_n$ is Lebesgue measurable, and $\mu(E) = \int_E f$ is defined.
Then by Chapter 4 Theorem 20,

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(E) = \int_E f = \sum_{n=1}^{\infty} \int_{E_n} f = \sum_{n=1}^{\infty} \mu(E_n).$$

Therefore $\mu$ is a measure on the $\sigma$-algebra of Lebesgue measurable sets.

2. Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of a set $X$ and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive. Prove that $\mu$ is a measure iff whenever $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, then
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

( $\Longrightarrow$ ) Suppose that $\mu$ is a measure.
Then by Continuity of Measure, the conclusion follows.

( $\Longleftarrow$ ) Suppose that whenever $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, then $\mu(\bigcup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$. (See Chapter 2 Problem 28.)
Finite additivity of $\mu$ means that for any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets, we have $\mu(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu(E_k)$.
We define $F_n = \bigcup_{k=1}^{n} E_k$ so that $\{F_n\}_{n=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, and thus $\mu(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n)$.
Thus we see
$$\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \mu(F_n) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^{n} E_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) = \sum_{k=1}^{\infty} \mu(E_k),$$
that is, $\mu$ satisfies countable additivity, and thus $\mu$ is a measure.

3. Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of a set $X$. Formulate and establish a correspondent of the preceding problem for descending sequences of sets in $\mathcal{M}$.

Let $\mathcal{M}$ be a $\sigma$-algebra of subsets of a set $X$ and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive. Prove that $\mu$ is a measure iff whenever $\{A_k\}_{k=1}^{\infty}$ is a descending sequence of sets in $\mathcal{M}$ with $m(A_1) < \infty$, then
$$\mu\left(\bigcap_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

( $\Longrightarrow$ ) Suppose that $\mu$ is a measure.
Then by Continuity of Measure, the conclusion follows.

( $\Longleftarrow$ ) Suppose that whenever $\{A_k\}_{k=1}^{\infty}$ is a descending sequence of sets in $\mathcal{M}$ with $\mu(A_1) < \infty$, then $\mu(\bigcap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \mu(A_k)$.
Finite additivity of $\mu$ means that for any finite disjoint collection $\{E_k\}_{k=1}^{n}$ of measurable sets, we have $\mu(\bigcup_{k=1}^{n} E_k) = \sum_{k=1}^{n} \mu(E_k)$.
We can consider $\{\bigcup_{k=n+1}^{\infty} E_k\}_{n=1}^{\infty}$, a descending sequence of sets in $\mathcal{M}$ with $\mu(\bigcup_{k=2}^{\infty} E_k) < \infty$, and then because $\{E_k\}_{k=1}^{n}$ is disjoint,
$$\mu(\bigcap_{n=1}^{\infty} [\bigcup_{k=n+1}^{\infty} E_k]) = \lim_{n\to\infty} \mu(\bigcup_{k=n+1}^{\infty} E_k) = 0.$$

Thus we see

$$\begin{aligned}
\mu(\bigcup_{k=1}^{\infty} E_k) &= \mu([\bigcup_{k=1}^{n} E_k] \cup [\bigcup_{k=n+1}^{\infty} E_k]) \\
&= \mu(\bigcup_{k=1}^{n} E_k) + \mu(\bigcup_{k=n+1}^{\infty} E_k) && \text{by disjoint additivity} \\
&= \sum_{k=1}^{n} \mu(E_k) + \mu(\bigcup_{k=n+1}^{\infty} E_k) && \text{by disjoint additivity}
\end{aligned}$$

The left hand side is independent of $n$, so taking the limit, we have

$$\begin{aligned}
\mu(\bigcup_{k=1}^{\infty} E_k) &= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(E_k) + \lim_{n\to\infty} \mu(\bigcup_{k=n+1}^{\infty} E_k) \\
&= \sum_{k=1}^{\infty} \mu(E_k) + 0 \\
&= \sum_{k=1}^{\infty} \mu(E_k),
\end{aligned}$$

that is, $\mu$ satisfies countable additivity, and thus $\mu$ is a measure.

4. Let $\{(X_\lambda, \mathcal{M}_\lambda, \mu_\lambda)\}_{\lambda\in\Lambda}$ be a collection of measure spaces parametrized by the set $\Lambda$. Assume the collection of sets $\{X_\lambda\}_{\lambda\in\Lambda}$ is disjoint. Then we can form a new measure space (called their union) $(X, \mathcal{B}, \mu)$ by letting $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, letting $\mathcal{B}$ be the collection of subsets $B$ of $X$ such that $B \cap X_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda$, and defining $\mu(B) = \sum_{\lambda\in\Lambda} \mu_\lambda[B \cap X_\lambda]$ for $B \in \mathcal{B}$.

   (i) Show that $\mathcal{B}$ is a $\sigma$-algebra.

   We have:

   (i) $X \in \mathcal{B}$ because $X \subseteq X$ such that for any $\lambda' \in \Lambda$,

   $$X \cap X_{\lambda'} = \bigcup_{\lambda\in\Lambda} X_\lambda \cap X_{\lambda'} = X_{\lambda'},$$

   where $X_{\lambda'} \in \mathcal{M}_{\lambda'}$ because $(X_{\lambda'}, \mathcal{M}_{\lambda'}, \mu_{\lambda'})$ is a measure space.

   (ii) if $B \in \mathcal{B}$, then $B \subseteq X$ such that for any $\lambda' \in \Lambda$, $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$.
   Then $B^c \subseteq X$ and because $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ and $X_{\lambda'} \in \mathcal{M}_{\lambda'}$,

   $$\mathcal{M}_{\lambda'} \ni [B \cap X_{\lambda'}]^c \cap X_{\lambda'} = [B^c \cup X_{\lambda'}^c] \cap X_{\lambda'} = [B^c \cap X_{\lambda'}] \cup [X_{\lambda'}^c \cap X_{\lambda'}] = B^c \cap X_{\lambda'}.$$

   Therefore $B^c \in \mathcal{B}$.

   (iii) if $B_i \in \mathcal{B}$, then $B_i \in X$ such that for any $\lambda' \in \Lambda$, $B_i \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all $i$.
   Then $\bigcup_{i=1}^{\infty} B_i \in X$ and $[\bigcup_{i=1}^{\infty} B_i] \cap X_{\lambda'} = \bigcup_{i=1}^{\infty} [B_i \cap X_{\lambda'}] \in \mathcal{M}_{\lambda'}$.
   Therefore $\bigcup_{i=1}^{\infty} B_i \in \mathcal{B}$.

   (ii) Show that $\mu$ is a measure.

   For $B \in \mathcal{B}$, we have $B \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all $\lambda' \in \Lambda$, and so $\mu_{\lambda'}[B \cap X_{\lambda'}]$ is defined.

Then $\mu_{\lambda'}[B \cap X_{\lambda'}] \geq 0$ for all $\lambda' \in \Lambda$, which implies $\mu(B) = \sum_{\lambda \in \Lambda} \mu_\lambda[B \cap X_\lambda] \geq 0$.
Then because $\emptyset = X^c$ is in $\mathcal{B}$, then $\emptyset = \emptyset \cap X_{\lambda'} \in \mathcal{M}_{\lambda'}$ for all $\lambda' \in \Lambda$, and then $\mu(\emptyset) = \sum_{\lambda \in \Lambda} \mu_\lambda[\emptyset \cap X_\lambda] = \sum_{\lambda \in \Lambda} 0 = 0$.
Finally, consider any countable disjoint collection $\{B_k\}_{k=1}^\infty$ in $\mathcal{B}$. Then for any $\lambda' \in \Lambda$, the collection $\{B_k \cap X_{\lambda'}\}_{k=1}^\infty$ is disjoint so that

$$\begin{aligned}
\mu(\bigcup_{k=1}^\infty B_k) &= \sum_{\lambda \in \Lambda} \mu_\lambda[(\bigcup_{k=1}^\infty B_k) \cap X_\lambda] \\
&= \sum_{\lambda \in \Lambda} \mu_\lambda[\bigcup_{k=1}^\infty (B_k \cap X_\lambda)] \\
&= \sum_{\lambda \in \Lambda} \sum_{k=1}^\infty \mu_\lambda(B_k \cap X_\lambda) \\
&= \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(B_k \cap X_\lambda) \\
&= \sum_{k=1}^\infty \mu(B_k).
\end{aligned}$$

Therefore $\mu$ is a measure.

(iii) Show that $\mu$ is $\sigma$-finite iff all but a countable number of the measures $\mu_\lambda$ have $\mu(X_\lambda) = 0$ and the remainder are $\sigma$-finite.

( $\Longrightarrow$ ) Suppose $\mu$ is $\sigma$-finite.
Then $X$ can be written as the countable union of disjoint measurable sets, each of which has finite measure under $\mu$.
That is, we have $X = \bigcup_{k=1}^\infty A_k$, with $A_k \in \mathcal{B}$ s.t. $\mu(A_k) < \infty$ for each $k$.
So $A_k \in \mathcal{B} \implies A_k \cap X_\lambda \in \mathcal{M}_\lambda$ for each $\lambda$, and $\sum_{\lambda \in \Lambda} \mu_\lambda(A_k \cap X_\lambda) = \mu(A_k) < \infty \implies \mu_\lambda(A_k \cap X_\lambda) < \infty$ for each $\lambda$.
Then

$$\mu(X) = \mu(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \mu(A_k) = \sum_{k=1}^\infty \sum_{\lambda \in \Lambda} \mu_\lambda(A_k \cap X_\lambda)$$

Then all but a countable number of the measures $\mu_\lambda$ can be nonzero, and the remainder must be $\sigma$-finite.

( $\Longleftarrow$ ) Suppose all but a countable number of the measures $\mu_\lambda$ have $\mu(X_\lambda) = 0$ and the remainder are $\sigma$-finite.
Let $\Lambda^* \subseteq \Lambda$ be the set of measures $\mu_\lambda$ such that $\mu_\lambda(X_\lambda) = 0$.
Let $\Lambda^{*c} = \{\lambda_k\}_{k=1}^\infty \subseteq \Lambda$ be a countable collection such that each $\mu_{\lambda_k}$ is $\sigma$-finite.
By definition of $\sigma$-finite, for each $k$, we have $X_{\lambda_k} = \bigcup_{i=1}^\infty [A_{\lambda_k}]_i$, with $[A_{\lambda_k}]_i \in \mathcal{M}_{\lambda_k}$ s.t. $\mu_{\lambda_k}([A_{\lambda_k}]_i) < \infty$ for each $i$.
Then because the collection $\{X_\lambda\}_{\lambda \in \Lambda}$ is disjoint, then $[A_{\lambda_k}]_i \cap X_\lambda = ([A_{\lambda_k}]_i \cap X_{\lambda_k}) \cap X_\lambda = \emptyset$ for $\lambda \neq \lambda_k$.
Also $[A_{\lambda_k}]_i = [A_{\lambda_k}]_i \cap X_{\lambda_k} \in \mathcal{M}_{\lambda_k}$ so that $[A_{\lambda_k}]_i \in \mathcal{B}$ and $\mu([A_{\lambda_k}]_i)$ is defined.

Then we have for each $i$,

$$\begin{aligned}
\mu([A_{\lambda_k}]_i) &= \sum_{\lambda\in\Lambda} \mu_\lambda([A_{\lambda_k}]_i \cap X_\lambda) \\
&= \sum_{\lambda\neq\lambda_k} \mu_\lambda([A_{\lambda_k}]_i \cap X_\lambda) + \mu_{\lambda_k}([A_{\lambda_k}]_i \cap X_\lambda) \\
&= \sum_{\lambda\neq\lambda_k} 0 + \mu_{\lambda_k}([A_{\lambda_k}]_i) \\
&= \mu_{\lambda_k}([A_{\lambda_k}]_i).
\end{aligned}$$

Therefore $\mu_{\lambda_k}([A_{\lambda_k}]_i) = \mu([A_{\lambda_k}]_i) < \infty$.
Then we can write

$$\begin{aligned}
\mu(X) &= \sum_{\lambda\in\Lambda} \mu_\lambda(X_\lambda) \\
&= \sum_{\lambda\in\Lambda*} \mu_\lambda(X_\lambda) + \sum_{\lambda\notin\Lambda^*} \mu_\lambda(X_\lambda) \\
&= \sum_{\lambda\in\Lambda*} 0 + \sum_{k=1}^{\infty} \mu_{\lambda_k}(X_{\lambda_k}) \\
&= \sum_{k=1}^{\infty} \mu_{\lambda_k}(\bigcup_{i=1}^{\infty}[A_{\lambda_k}]_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \mu_{\lambda_k}([A_{\lambda_k}]_i) \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \mu([A_{\lambda_k}]_i),
\end{aligned}$$

and thus $X$ can be written as a countable disjoint union of measurable sets $[A_{\lambda_k}]_i$, each of which has finite measure under $\mu$.
Therefore $\mu$ is $\sigma$-finite.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space. The symmetric difference, $E_1 \Delta E_2$, of two subsets $E_1$ and $E_2$ of $X$ is defined by
$$E_1 \Delta E_2 = [E_1 \setminus E_2] \cup [E_2 \setminus E_1].$$

(i) Show that if $E_1$ and $E_2$ are measurable and $\mu(E_1 \Delta E_2) = 0$, then $\mu(E_1) = \mu(E_2)$.

We can see that

$$\mu(E_1 \cup E_2) = \mu([E_1 \Delta E_2] \cup [E_1 \cap E_2]) = \mu(E_1 \Delta E_2) + \mu(E_1 \cap E_2) = \mu(E_1 \cap E_2).$$

Then we also know that by monotonicity we have

$$E_1 \cap E_2 \subseteq E_1, E_2 \subseteq E_1 \cup E_2 \implies \mu(E_1 \cap E_2) \leq \mu(E_1), \mu(E_2) \leq \mu(E_1 \cup E_2),$$

and therefore $\mu(E_1) = \mu(E_2)$.

(ii) Show that if $\mu$ is complete and $E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ if $\mu(E_1 \Delta E_2) = 0$.

Because $\mu(E_1 \Delta E_2) = 0$, then because $\mu$ is complete, the subsets $[E_1 \setminus E_2] \subseteq E_1 \Delta E_2$ and $[E_2 \setminus E_1] \subseteq E_1 \Delta E_2$ are measurable. Therefore the set $[E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c$ is also measurable, and

$$\begin{aligned}
[E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c &= [E_2 \cup E_1] \cap [E_1^c \cup E_1] \cap [E_1^c \cup E_2] \\
&= [E_2 \cup E_1] \cap [E_1^c \cup E_2] \\
&= ([E_2 \cup E_1] \cap E_1^c) \cup ([E_2 \cup E_1] \cap E_2) \\
&= ([E_2 \cap E_1^c] \cup [E_1 \cap E_1^c]) \cup E_2 \\
&= (E_2 \cap E_1^c) \cup E_2 \\
&= E_2,
\end{aligned}$$

therefore $E_2 = [E_2 \setminus E_1] \cup [E_1] \cap [E_1 \setminus E_2]^c$ is measurable.

6. Let $(X, \mathcal{M}, \mu)$ be a measure space and $X_0$ belong to $\mathcal{M}$. Define $\mathcal{M}_0$ to be the collection of sets in $\mathcal{M}$ that are subsets of $X_0$ and $\mu_0$ the restriction of $\mu$ to $\mathcal{M}_0$. Show that $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

   We want to show that $(X_0, \mathcal{M}_0)$ is a measurable space (i.e., that $\mathcal{M}_0$ is a $\sigma$-algebra of subsets of $X_0$) and that $\mu_0$ is a measure on $\mathcal{M}_0$.
   To see that $\mathcal{M}_0$ is a $\sigma$-algebra:

   (i) $X_0 \in \mathcal{M}_0$ because $X_0 \in \mathcal{M}$ and $X_0 \subseteq X_0$.

   (ii) if $A \in \mathcal{M}_0$, then $A \in \mathcal{M}$ and $A \subseteq X_0$.
   Then $X_0 \cap A^c \in \mathcal{M}$ and $X_0 \cap A^c \subseteq X_0$ imply that $X_0 \cap A^c \in \mathcal{M}_0$.

   (iii) if $A_i \in \mathcal{M}_0$, then $A_i \in \mathcal{M}$ and $A_i \subseteq X_0$ for all $i$.
   Then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}$ and $\bigcup_{i=1}^{\infty} A_i \subseteq X_0$ imply that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{M}_0$.

   Therefore $(X_0, \mathcal{M}_0)$ is a measurable space.
   Clearly $\mu_0$ is a measure on $\mathcal{M}_0$, because it inherits the properties of a measure from $\mu$.
   Thus $(X_0, \mathcal{M}_0, \mu_0)$ is a measure space.

7. Let $(X, \mathcal{M})$ be a measurable space. Verify the following:

   (i) If $\mu$ and $\nu$ are measures defined on $\mathcal{M}$, then set set function $\lambda$ defined on $\mathcal{M}$ by $\lambda(E) = \mu(E) + \nu(E)$ also is a measure. We denote $\lambda$ by $\mu + \nu$.

   Because $\mu(E) \geq 0$ and $\nu(E) \geq 0$ for any $E \in \mathcal{M}$, then $\lambda(E) = \mu(E) + \nu(E) \geq 0$.
   Also, $\mu(\emptyset) = 0$ and $\nu(\emptyset) = 0$ imply that $\lambda(\emptyset) = \mu(\emptyset) + \nu(\emptyset) = 0$.
   Finally, for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets,

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \sum_{k=1}^{\infty} \mu(E_k) + \sum_{k=1}^{\infty} \nu(E_k) \\
&= \sum_{k=1}^{\infty} [\mu(E_k) + \nu(E_k)] \\
&= \sum_{k=1}^{\infty} \lambda(E_k).
\end{aligned}$$

Therefore $\lambda$ is a measure.

(ii) If $\mu$ and $\nu$ are measures on $\mathcal{M}$ and $\mu \geq \nu$, then there is a measure $\lambda$ on $\mathcal{M}$ for which $\mu = \nu + \lambda$.

In the case $\mu(E) < \infty$, then we also have $\nu(E) \leq \mu(E) < \infty$, and we can let $\lambda = \mu - \nu$. We clearly see that $\mu \geq \nu \implies \mu - \nu \geq 0$ so that $\lambda(E) = \mu(E) - \nu(E) \geq 0$ for any $E \in \mathcal{M}$ (of finite measure under $\mu$).
Also, $\mu(\emptyset) = 0$ and $\nu(\emptyset) = 0$ imply that $\lambda(\emptyset) = \mu(\emptyset) - \nu(\emptyset) = 0$.
Finally, for any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets each of finite measure,

$$\begin{aligned}
\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) - \nu\left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \sum_{k=1}^{\infty} \mu(E_k) - \sum_{k=1}^{\infty} \nu(E_k) \\
&= \sum_{k=1}^{\infty} [\mu(E_k) - \nu(E_k)] \\
&= \sum_{k=1}^{\infty} \lambda(E_k).
\end{aligned}$$

In the case $\mu(E) = \infty$, we can let $\lambda(E) = \infty$ so that $\nu(E) + \lambda(E) = \mu(E)$.
Then $\lambda(E) = \infty \geq 0$.
For any countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ of measurable sets, supposing there exists an index $j$ such that $\mu(E_j) = \infty$, then we defined $\lambda(E_j) = \infty$ so that by monotonicity, we have

$$\infty = \lambda(E_j) \leq \lambda\left(\bigcup_{k=1}^{\infty} E_k\right),$$

so $\lambda\left(\bigcup_{k=1}^{\infty} E_k\right) = \infty = \sum_{k=1}^{\infty} \lambda(E_k)$.
Then we also have $\sum_{k=1}^{\infty} \mu(E_k) = \infty$ and

$$\begin{aligned}
\mu\left(\bigcup_{k=1}^{\infty} E_k\right) &= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \lambda\left(\bigcup_{k=1}^{\infty} E_k\right) \\
&= \mu\left(\bigcup_{k=1}^{\infty} E_k\right) + \infty \\
&= \infty.
\end{aligned}$$

In conclusion, we have defined

$$\lambda(E) = \begin{cases} \mu(E) - \nu(E) & \text{if } \mu(E) < \infty \\ \infty & \text{if } \mu(E) = \infty, \end{cases}$$

and we have proved that $\lambda$ is a measure.

(iii) If $\nu$ is $\sigma$-finite, the measure $\lambda$ in (ii) is unique.

Because $\nu$ is $\sigma$-finite, then $X$ is the union of a countable collection of measurable sets (may be taken to be disjoint), each of which has finite measure under $\nu$. That is, $X = \bigcup_{k=1}^{\infty} X_k$, where $\nu(X_k) < \infty$. Then for any $E \in \mathcal{M}$, we have

$$E = E \cap X = E \cap \bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} [E \cap X_k],$$

where by monotonicity of measure we have $\nu(E \cap X_k) \leq \nu(X_k) < \infty$, and thus any measurable set $E$ is also $\sigma$-finite when $\nu$ is $\sigma$-finite.
Now, suppose there exist measures $\lambda_1$ and $\lambda_2$ such that $\mu = \nu + \lambda_1$ and $\mu = \nu + \lambda_2$. Then $\nu + \lambda_1 = \nu + \lambda_2$ and thus $\nu - \nu = \lambda_2 - \lambda_1$.
For any $E \in \mathcal{M}$ such that $\nu(E) < \infty$, then clearly $\lambda_1(E) = \lambda_2(E)$.
For any $E \in \mathcal{M}$ such that $\nu(E) = \infty$, $\nu(E) - \nu(E) = \infty - \infty$ is not defined.
However, because $\nu$ is $\sigma$-finite, there exists a countable disjoint collection $\{E_k\}_{k=1}^{\infty}$ such that $E = \bigcup_{k=1}^{\infty} E_k$ and $\nu(E_k) < \infty$ for each $k$. Then we see that $\nu(E_k) - \nu(E_k)$ is defined for all $k$, and

$$\nu(E) = \nu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \nu(E_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k).$$

Then we can write

$$\begin{aligned}
\lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k) - \lim_{n\to\infty} \sum_{k=1}^{n} \nu(E_k) &= \lambda_2(E) - \lambda_1(E) \\
\lim_{n\to\infty} \sum_{k=1}^{n} [\nu(E_k) - \nu(E_k)] &= \lambda_2(E) - \lambda_1(E) \\
\lim_{n\to\infty} \sum_{k=1}^{n} 0 &= \lambda_2(E) - \lambda_1(E) \\
0 &= \lambda_2(E) - \lambda_1(E).
\end{aligned}$$

Therefore $\lambda_1(E) = \lambda_2(E)$, and the measure $\lambda$ is unique.

(iv) Show that in general the measure $\lambda$ need not be unique but that there is always a smallest such $\lambda$.

Suppose there exists a set $E \in \mathcal{M}$ such that $\mu(E) = \infty$ and $\nu(E) = \infty$. Then regardless of the number $\lambda(E) \in [0, \infty]$ we define $\lambda$ to be, we always have $\infty = \mu(E) = \nu(E) + \lambda(E)$. Then $\lambda(E) = 0$ is the smallest value that we can set $\lambda$ to be, and we can define the smallest $\lambda$ in the following way:

$$\lambda(E) = \begin{cases} \mu(E) - \nu(E) & \text{if } \mu(E) < \infty \text{ ( forces } \nu(E) < \infty) \\ \infty & \text{if } \mu(E) = \infty, \nu(E) < \infty \\ 0 & \text{if } \mu(E) = \infty, \nu(E) = \infty \end{cases}$$

8. Let $(X, \mathcal{M}, \mu)$ be a measure space. The measure $\mu$ is said to be **semifinite** provided each measurable set of infinite measure contains measurable sets of arbitrarily large finite measure.

(i) Show that each $\sigma$-finite measure is semifinite.

If we suppose $\mu$ is $\sigma$-finite, then we can write any $E \in \mathcal{M}$ as the countable disjoint union of measurable sets of finite measure under $\mu$: $E = \bigcup_{k=1}^{\infty} E_k$ with $\mu(E_k) < \infty$.
Consider any measurable set $E$ such that $\mu(E) = \infty$. Then

$$\mu(E) = \mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) = \infty.$$

Then the sequence of partial sums $\sum_{k=1}^{n} \mu(E_k)$ converges to infinity. That is, for any real number $x$, there exists an index $j$ such that $\sum_{k=1}^{j} \mu(E_k) > x$. Because each $E_k$ is disjoint and measurable, we have that $E_x := \bigcup_{k=1}^{j} E_k \in \mathcal{M}$, and we can write

$$x < \sum_{k=1}^{j} \mu(E_k) = \mu(\bigcup_{k=1}^{j} E_k) = \mu(E_x) < \infty.$$

That is, for any real number we choose, there exists a measurable set $E_x \subseteq E$ of finite measure that is larger than $x$.
Therefore $\mu$ is semifinite.

(ii) For $E \in \mathcal{M}$, define $\mu_1(E) = \sup\{\mu(F) \mid F \subseteq E, \mu(F) < \infty\}$. Show that $\mu_1$ is a semifinite measure: it is called the semifinite part of $\mu$.

Consider any measurable set $E$ such that $\mu_1(E) = \infty$. Then for any subset $F$ of $E$ such that $\mu(F) < \infty$, we have that $\mu_1(E) \geq \mu(F) = \mu_1(F)$ by definition of supremum. (We have $\mu_1(F) = \mu(F)$ because $F$ is the largest subset of itself). However, because $\mu_1$ is the least upper bound, for any real number $x$, there exists a subset $F_x$ of $E$ such that $x < \mu_1(F_x) \leq \mu_1(E)$, else we reach a contradiction to the supremum. Therefore for any real number $x$ we choose, there exists a measurable set $F_x \subseteq E$ of finite measure that is larger than $x$.

(iii) Find a measure $\mu_2$ on $\mathcal{M}$ that only takes the values 0 and $\infty$ and $\mu = \mu_1 + \mu_2$.

We can define, for any $E \in \mathcal{M}$,

$$\mu_2(E) = \begin{cases} 0 & \text{if } \mu_1(E) < \infty \\ \infty & \text{if } \mu_1(E) = \infty \end{cases}$$

So that we have

$$\mu(E) = \begin{cases} \mu_1(E) + \mu_2(E) = \mu(E) + 0 & \text{if } \mu_1(E) < \infty \\ \mu_1(E) + \mu_2(E) = \mu(E) + \infty & \text{if } \mu_1(E) = \infty \end{cases}$$

9. Prove Proposition 3; that is, show that $\mathcal{M}_0$ is a $\sigma$-algebra, $\mu_0$ is properly defined, and $(X, \mathcal{M}_0, \mu_0)$ is complete. In what sense is $\mathcal{M}_0$ minimal?

We can see

(i) $X \in \mathcal{M}_0$ because $X \subseteq X$, and $X = \emptyset \cup X$ with $X \in \mathcal{M}$ and $\emptyset \subseteq \emptyset$ for $\emptyset \in \mathcal{M}$ where $\mu(\emptyset) = 0$.

(ii) If $E \in \mathcal{M}_0$, then $E \subseteq X$, and $E = A \cup B$ with $B \in \mathcal{M}$ and $A \subseteq C$ for $C \in \mathcal{M}$ where $\mu(C) = 0$.
Then $A \subseteq C \implies A^c \supseteq C^c$, and $A^c = [A^c \cap C] \cup [A^c \cap C^c] = [A^c \cap C] \cup C^c$.

Now, $X \cap E^c \subseteq X$.
We can write

$$\begin{aligned} E^c &= A^c \cap B^c \\ &= ([A^c \cap C] \cup C^c) \cap B^c \\ &= ([A^c \cap C] \cap B^c) \cup (C^c \cap B^c), \end{aligned}$$

Where $C^c \cap B^c \in \mathcal{M}$ and $[A^c \cap C] \cap B^c \subseteq C$ for $C \in \mathcal{M}$ where $\mu(C) = 0$.
Therefore $E^c \in \mathcal{M}_0$.

(iii) If $E_k \in \mathcal{M}_0$, then $E_k \subseteq X$, and $E_k = A_k \cup B_k$ with $B_k \in \mathcal{M}$ and $A_k \subseteq C_k$ for $C_k \in \mathcal{M}$ where $\mu(C_k) = 0$ for all $k$.
Then $\bigcup_{k=1}^{\infty} E_k \subseteq X$, and

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} [A_k \cup B_k] = [\bigcup_{k=1}^{\infty} A_k] \cup [\bigcup_{k=1}^{\infty} B_k],$$

Where $\bigcup_{k=1}^{\infty} B_k \in \mathcal{M}$ and $A_k \subseteq C_k \implies \bigcup_{k=1}^{\infty} A_k \subseteq \bigcup_{k=1}^{\infty} C_k$ for $\bigcup_{k=1}^{\infty} C_k \in \mathcal{M}$ with $\mu(\bigcup_{k=1}^{\infty} C_k) \leq \sum_{k=1}^{\infty} \mu(C_k) = \sum_{k=1}^{\infty} 0 = 0$.
Thus $\bigcup_{k=1}^{\infty} E_k \in \mathcal{M}_0$.

Thus $\mathcal{M}_0$ is a $\sigma$-algebra of subsets of $X$.

To see that $\mu_0$ is a measure on the measurable space $(X, \mathcal{M}_0)$:
For any $E \in \mathcal{M}_0$, we have $E = A \cup B, B \in \mathcal{M}$, so that $\mu_0(E) = \mu(B) \geq 0$.
Then for $\emptyset \in \mathcal{M}_0$, we have $\emptyset = \emptyset \cup \emptyset, \emptyset \in \mathcal{M}$, so that $\mu_0(\emptyset) = \mu(\emptyset) = 0$.
Finally, consider a disjoint collection $\{E_k\}_{k=1}^{\infty}$ of sets in $\mathcal{M}_0$.
See (iii) to see that $\bigcup_{k=1}^{\infty} E_k = [\bigcup_{k=1}^{\infty} A_k] \cup [\bigcup_{k=1}^{\infty} B_k]$, where $\{E_k\}_{k=1}^{\infty}$ disjoint implies $\{B_k\}_{k=1}^{\infty}$ disjoint and we have $\bigcup_{k=1}^{\infty} B_k \in \mathcal{M}$.
Then

$$\mu_0(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \mu(B_k) = \sum_{k=1}^{\infty} \mu_0(E_k).$$

Therefore $(X, \mathcal{M}_0, \mu_0)$ is a measure space.

To see that $(X, \mathcal{M}_0, \mu_0)$ is complete, consider any set $E \in \mathcal{M}_0$ such that $\mu_0(E) = 0$.

$$E \in \mathcal{M}_0 \implies E \subseteq X, E = A \cup B, B \in \mathcal{M} \text{ and } A \subseteq C \text{ with } C \in \mathcal{M}, \mu(C) = 0.$$

Then $A \subseteq C \implies A \cup B \subseteq C \cup B$, and $C, B \in \mathcal{M} \implies C \cup B \in \mathcal{M}$. Thus $\mu(C \cup B) \leq \mu(C) + \mu(B) = 0$ is well-defined.
Consider any $D \subseteq E$.

$$D \subseteq E \subseteq X, D = D \cup \emptyset, \emptyset \in \mathcal{M} \text{ and } D \subseteq A \cup B \subseteq C \cup B \text{ with } C \cup B \in \mathcal{M}, \mu(C \cup B) = 0.$$

Therefore $D \in \mathcal{M}_0$ and $(X, \mathcal{M}_0, \mu_0)$ is complete.

10. If $(X, \mathcal{M}, \mu)$ is a measure space, we say that a subset $E$ of $X$ is **locally measurable** provided for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, the intersection $E \cap B$ belongs to $\mathcal{M}$. The measure $\mu$ is called **saturated** provided every locally measurable set is measurable.

(i) Show that each $\sigma$-finite measure is saturated.

Suppose $\mu$ is $\sigma$-finite, then $X$ can be taken to be the union of a countable collection of measurable sets, each of which has finite measure under $\mu$.
That is, $X = \bigcup_{k=1}^{\infty} X_k$, where $\mu(X_k) < \infty$.
Then for any $E \in X$, we have

$$E = E \cap X = E \cap \bigcup_{k=1}^{\infty} X_k = \bigcup_{k=1}^{\infty} [E \cap X_k],$$

In the case that $E$ is locally measurable, then each intersection $E \cap X_k$ is measurable. Then the countable intersection of measurable sets $\bigcup_{k=1}^{\infty}[E \cap X_k] = E$ is measurable.
Thus when $\mu$ is $\sigma$-finite, every locally measurable set is measurable, and thus $\mu$ is saturated.

(ii) Show that the collection $\mathcal{C}$ of locally measurable sets is a $\sigma$-algebra.

We have

(i) $X \in \mathcal{C}$ because for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $X \cap B = B \in \mathcal{M}$.

(ii) if $E \in \mathcal{C}$, then for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $E \cap B \in \mathcal{M}$.
Then we have the two measurable sets $E \cap B$ and $B$ so that $[E \cap B]^c \cap B$ is also measurable, and

$$\mathcal{M} \ni [E \cap B]^c \cap B = [E^c \cup B^c] \cap B = [E^c \cap B] \cup [B^c \cap B] = E^c \cap B,$$

and thus $E^c \in \mathcal{C}$.

(iii) if $E_i \in \mathcal{C}$, then for all $B \in \mathcal{M}$ with $\mu(B) < \infty$, then $E_i \cap B \in \mathcal{M}$ for all $i$.
Then $[\bigcup_{i=1}^{\infty} E_i] \cap B = \bigcup_{i=1}^{\infty}[E_i \cap B] \in \mathcal{M}$ and thus $\bigcup_{i=1}^{\infty} E_i \in \mathcal{C}$.

(iii) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\mathcal{C}$ the $\sigma$-algebra of locally measurable sets. For $E \in \mathcal{C}$, define $\overline{\mu}(E) = \mu(E)$ if $E \in \mathcal{M}$ and $\overline{\mu}(E) = \infty$ if $E \notin \mathcal{M}$. Show that $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

In (ii) we showed that $\mathcal{C}$ is a $\sigma$-algebra of subsets of $X$.
Therefore $(X, \mathcal{C})$ is a measurable space.

We have defined

$$\overline{\mu}(E) = \begin{cases} \mu(E) & \text{if } E \in \mathcal{M} \\ \infty & \text{if } E \notin \mathcal{M} \end{cases}$$

We have $\overline{\mu}(E) \in \{\mu(E), \infty\} \geq 0$ for all $E \in \mathcal{C}$. We have $\overline{\mu}(\emptyset) = \mu(\emptyset) = 0$ because $\emptyset \in \mathcal{M}$ and $\emptyset \in \mathcal{C}$. Finally, consider a countable disjoint collection of sets $\{E_k\}_{k=1}^{\infty}$ in $\mathcal{C}$.

(i) If for all $k$ we have $E_k \in \mathcal{M}$, then $\mu(\bigcup_{k=1}^{\infty} E_k)$ is measurable, $\overline{\mu}(E_k) = \mu(E_k)$, and

$$\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{\infty} \overline{\mu}(E_k).$$

(ii) If there exists an index $j$ such that $E_j \notin \mathcal{M}$, then $\overline{\mu}(E_j) = \infty$ and $\sum_{k=1}^{\infty} \overline{\mu}(E_k) = \infty$.
Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is measurable and $\mu(\bigcup_{k=1}^{\infty} E_k) < \infty$.
Then because for any $j$, we have $E_j \in \mathcal{C}$, then $E_j = E_j \cap \bigcup_{k=1}^{\infty} E_k \in \mathcal{M}$.
Then (i) must hold.

Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is measurable and $\mu(\bigcup_{k=1}^{\infty} E_k) = \infty$.
If (i) holds, then $\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \overline{\mu}(E_k)$.
If (ii) holds, then $\overline{\mu}(\bigcup_{k=1}^{\infty} E_k) = \mu(\bigcup_{k=1}^{\infty} E_k) = \infty = \sum_{k=1}^{\infty} \overline{\mu}(E_k)$.

Consider the case that $\bigcup_{k=1}^{\infty} E_k \in \mathcal{C}$ is not measurable. Then $\mu(\bigcup_{k=1}^{\infty} E_k) = \infty$.
Then (ii) must hold else we reach a contradiction to $\bigcup_{k=1}^{\infty} E_k \notin \mathcal{M}$.

Therefore $(X, \mathcal{C}, \overline{\mu})$ is a measure space.

We can use the definition of $\overline{\mu}$ to see that

$$B \in \mathcal{C} \text{ with } \overline{\mu}(B) < \infty \iff B \in \mathcal{M} \text{ with } \mu(B) < \infty.$$

Consider a set $E \subseteq X$ such that $E \cap B \in \mathcal{C}$ for any such $B$.
Then by monotonicity, $\overline{\mu}(E \cap B) \leq \overline{\mu}(B) < \infty$.
Because $\overline{\mu}(E \cap B) < \infty$, see the definition of $\overline{\mu}$ to see that $E \cap B \in \mathcal{M}$.
Then we see that $E \in \mathcal{C}$ because for each $B \in \mathcal{M}$ with $\mu(B) < \infty$, we have $E \cap B \in \mathcal{M}$.

Therefore $(X, \mathcal{C}, \overline{\mu})$ is a saturated measure space.

(iv) If $\mu$ is semifinite and $E \in \mathcal{C}$, the set $\underline{\mu}(E) = \sup\{\mu(B) \mid B \in \mathcal{M}, B \subseteq E\}$. Show that $(X, \mathcal{C}, \underline{\mu})$ is a saturated measure space and that $\underline{\mu}$ is an extension of $\mu$. Give an example to show that $\overline{\mu}$ and $\underline{\mu}$ may be different.

We first want to show that $\underline{\mu}$ is a measure on the measurable space $(X, \mathcal{C})$:
For any $E \in \mathcal{C}$, we have $\underline{\mu}(E) \geq \mu(B) \geq 0$ for $B \in \mathcal{M}, B \subseteq E$.
For $\emptyset \in \mathcal{C}$, we have $\underline{\mu}(\emptyset) = \mu(\emptyset) = 0$ because $\{\emptyset\} = \{B \in \mathcal{M} \mid B \subseteq \emptyset\}$.
Finally, for any disjoint collection $\{E_k\}_{k=1}^{\infty}$ in $\mathcal{C}$,

Therefore $(X, \mathcal{C}, \underline{\mu})$ is a measure space.

Consider any $E \subseteq X$ such that $E \cap B \in \mathcal{C}$ whenever $B \in \mathcal{C}$ with $\underline{\mu}(B) < \infty$.
Then $E \cap B \in \mathcal{C}$ implies that $[E \cap B] \cap B' \in \mathcal{M}$ whenever $B' \in \mathcal{M}$ with $\mu(B') < \infty$.

11. Let $\mu$ and $\eta$ be measures on the measurable space $(X, \mathcal{M})$. For $E \in \mathcal{M}$, define $\nu(E) = \max\{\mu(E), \eta(E)\}$. Is $\nu$ a measure on $(X, \mathcal{M})$?

We have $0 \leq \mu(E), \eta(E) \leq \max\{\mu(E), \eta(E)\}$ for any $E \in \mathcal{M}$.
We have $\max\{\mu(E), \eta(E)\} \in [0, \infty]$ for any $E \in \mathcal{M}$.
Counterexample: Let $E_1, E_2$ be nonempty disjoint measurable (singleton) sets such that

$$\mu(E) = \begin{cases} 1 & E \supseteq E_1 \\ 0 & E \not\supseteq E_1 \end{cases} \quad \text{and} \quad \eta(E) = \begin{cases} 1 & E \supseteq E_2 \\ 0 & E \not\supseteq E_2 \end{cases}$$

Then $\mu(E) \in \{0, 1\} \geq 0$, $\mu(\emptyset) = 0$ because $\emptyset \not\supseteq E_1$, and for any countable disjoint collection $\{A_k\}_{k=1}^{\infty}$ of measurable sets, in the case that for all $k$, $A_k \not\supseteq E_1$, then $\bigcup_{k=1}^{\infty} A_k \not\supseteq E_1$ and

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = 0.$$

In the case that there exists an index $j$ such that $A_j \supseteq E_1$, then $\bigcup_{k=1}^{\infty} A_k \supseteq E_1$, and because the sets are disjoint, $A_i \not\supseteq E_1$ for $i \neq j$.

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) = \sum_{k\in\mathbb{N}\setminus\{j\}} \mu(A_k) + \mu(A_j) = 0 + \mu(A_j) = 1.$$

Then $\eta$ can also be shown to be a measure in the exact same way.
Then we see that

$$\nu(E_1 \cup E_2) = \max\{\mu(E_1 \cup E_2), \eta(E_1 \cup E_2)\} = \max\{1, 1\} = 1,$$
$$\nu(E_1) + \nu(E_2) = \max\{\mu(E_1), \eta(E_1)\} + \max\{\mu(E_2), \eta(E_2)\} = \max\{1, 0\} + \max\{0, 1\} = 2.$$

Thus $\nu$ is not a measure because it does not satisfy countable additivity.

## 17.2 Signed Measures: The Hahn and Jordan Decompositions

**Definition.** *By a **signed measure** $\nu$ on the measurable space $(X, \mathcal{M})$ we mean an extended real-valued set function $\nu : \mathcal{M} \to [-\infty, \infty]$ that possesses the following properties:*

*(i) $\nu$ assumes at most one of the values $+\infty, -\infty$.*

*(ii) $\nu(\emptyset) = 0$.*

*(iii) For any countable collection $\{E_k\}_{k=1}^{\infty}$ of disjoint measurable sets,*

$$\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k),$$

*where the series $\sum_{k=1}^{\infty} \nu(E_k)$ converges absolutely if $\nu(\bigcup_{k=1}^{\infty} E_k)$ is finite (convergence must hold for any rearrangement).*

Monotonicity for signed measures:

$$A \subseteq B \text{ and } |\nu(B)| < \infty \implies |\nu(A)| < \infty.$$

**PROBLEMS**

12. In the above example, let $E$ be a Lebesgue measurable set such that $0 < \nu(E) < \infty$. Find a positive set $A$ contained in $E$ for which $\nu(A) > 0$.

13. Let $\mu$ be a measure and $\mu_1$ an $\mu_2$ be mutually singular measures on a measurable space $(X, \mu)$ for which $\mu = \mu_1 - \mu_2$. Use this to establish the uniqueness assertion of the Jordan Decomposition Theorem.

14. Show that if $E$ is any measurable set, then

$$-\nu^-(E) \leq \nu(E) \leq \nu^+(E) \text{ and } |\nu(E)| \leq |\nu|(E).$$

15. Show that if $\nu_1$ and $\nu_2$ are any two finite signed measures, then so it $\alpha\nu_1 + \beta\nu_2$, where $\alpha$ and $\beta$ are real numbers. Show that

$$|\alpha\nu| = |\alpha||\nu| \text{ and } |\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|,$$

where $\nu \leq \mu$ means $\nu(E) \leq \mu(E)$ for all measurable sets $E$.

16. Prove (4).

17. Let $\mu$ and $\nu$ be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$ and $\mu \vee \nu = \mu + \nu - \mu \wedge \nu$.

    (i) Show that the signed measure $\mu \wedge \nu$ is smaller than $\mu$ and $\nu$ but larger than any other signed measure that is smaller than $\mu$ and $\nu$.

    (ii) Show that the signed measure $\mu \vee \nu$ is larger than $\mu$ and $\nu$ but smaller than any other signed measure that is larger than $\mu$ and $\nu$.

    (iii) If $\mu$ and $\nu$ are positive measures, show that they are mutually singular iff $\mu \wedge \nu = 0$.

## 17.3 The Cathéodory Measure Induced by an Outer Measure

## 17.4 The Construction of Outer Measures

**PROBLEMS**

18. Let $\mu^* : 2^X \to [0, \infty]$ be an outer measure. Let $A \subseteq X$, $\{E_k\}_{k=1}^{\infty}$ be a disjoint countable collection of measurable sets and $E = \bigcup_{k=1}^{\infty} E_k$. Show that

$$\mu^*(A \cap E) = \sum_{k=1}^{\infty} \mu^*(A \cap E_k).$$

19. Show that any measure that is induced by an outer measure is complete.

20. Let $X$ be any set. Define $\eta^* : 2^X \to [0, \infty]$ by defining $\eta(\emptyset) = 0$ and for $E \subseteq X, E \neq \emptyset$, defining $\eta(E) = \infty$. Show that $\eta$ is an outer measure. Also show that the set function that assigns 0 to every subset of $X$ is an outer measure.

21. Let $X$ be a set, $\mathcal{S} = \{\emptyset, X\}$, and define $\mu(\emptyset) = 0, \mu(X) = 1$. Determine the outer measure $\mu^*$ induced by the set function $\mu : \mathcal{S} \to [0, \infty)$ and the $\sigma$-algebra of measurable sets.

22. On the collection $\mathcal{S} = \{\emptyset, [1, 2]\}$ of subsets of $\mathbb{R}$, define the set function $\mu : \mathcal{S} \to [0, \infty)$ as follows: $\mu(\emptyset) = 0, \mu([1, 2]) = 1$. Determine the outer measure $\mu^*$ induced by $\mu$ and the $\sigma$-algebra of measurable sets.

23. On the collection $\mathcal{S}$ of all subsets of $\mathbb{R}$, define the set function $\mu : \mathcal{S} \to \mathbb{R}$ by setting $\mu(A)$ to be the number of integers in $A$. Determine the outer measure $\mu^*$ induced by $\mu$ and the $\sigma$-algebra of measurable sets.

24. Let $\mathcal{S}$ be a collection of subsets of $X$ and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Is every set in $\mathcal{S}$ measurable with respect to the outer measure induced by $\mu$?

## 17.5 The Cathéodory-Hahn Theorem: The Extension of a Premeasure to a Measure

**PROBLEMS**

25. Let $X$ be any set containing more than one point and $A$ a proper nonempty subset of $X$. Define $\mathcal{S} = \{A, X\}$ and the set function $\mu : \mathcal{S} \to [0, \infty]$ by $\mu(A) = 1$ and $\mu(X) = 2$. Show that $\mu : \mathcal{S} \to [0, \infty]$ is a premeasure. Can $\mu$ be extended to a measure? What are the subsets of $X$ that are measurable with respect to the outer measure $\mu^*$ induced by $\mu$?

26. Consider the collection $\mathcal{S} = \{\emptyset, [0,1], [0,3], [2,3]\}$ of subsets of $\mathbb{R}$ and define $\mu(\emptyset) = 0, \mu([0,1]) = 1, \mu([0,3]) = 1, \mu([2,3]) = 1$. Show that $\mu : \mathcal{S} \to [0, \infty]$ is a premeasure. Can $\mu$ be extended to a measure? What are the subsets of $\mathbb{R}$ that are measurable with respect to the outer measure $\mu^*$ induced by $\mu$?

27. Let $\mathbb{S}$ be a collection of subsets of a set $X$ and $\mu : \mathcal{S} \to [0, \infty]$ a set function. Show that $\mu$ is countably monotone iff $\mu^*$ is an extension of $\mu$.

28. Show that a set function is a premeasure if it has an extension that is a measure.

29. Show that a set function on a $\sigma$-algebra is a measure iff it is a premeasure.

30. Let $\mathcal{S}$ be a collection of sets that is closed with respect to the formation of finite unions and finite intersections.
    (i) Show that $\mathcal{S}_\sigma$ is closed with respect to the formation of countable unions and finite intersections.
    (ii) Show that each set in $\mathcal{S}_{\sigma\delta}$ is the intersection of a decreasing sequence of $\mathcal{S}_\sigma$ sets.

31. Let $\mathcal{S}$ be a semialgebra of subsets of a set $X$ and $\mathcal{S}'$ the collection of unions of finite disjoint collections of sets in $\mathcal{S}$.
    (i) Show that $\mathcal{S}'$ is an algebra.
    (ii) Show that $\mathcal{S}_\sigma = \mathcal{S}'_\sigma$ and therefore $\mathcal{S}_{\sigma\delta} = \mathcal{S}'_{\sigma\delta}$.
    (iii) Let $\{E_k\}_{k=1}^\infty$ be a collection of sets in $\mathcal{S}'$. Show that we can express
$$\sum_{k=1}^{\infty} \mu'(E'_k) \geq \sum_{k=1}^{\infty} \mu(E_k).$$
    (iv) Let $A$ belong to $\mathcal{S}'_{\sigma\delta}$. Show that $A$ is the intersection of a descending sequence $\{A_k\}_{k=1}^\infty$ of sets in $\mathcal{S}_\sigma$.

32. Let $\mathbb{Q}$ be the set of rational numbers and and $\mathcal{S}$ the collection of all finite unions of intervals of the form $(a, b] \cap \mathbb{Q}$, where $a, b \in \mathbb{Q}$ and $a \leq b$. Define $\mu((a, b]) = \infty$ if $a < b$ and $\mu(\emptyset) = 0$. Show that $\mathcal{S}$ is closed with respect to the formation of relative complements and $\mu : \mathcal{S} \to [0, \infty]$ is a premeasure. Then show that the extension of $\mu$ to the smallest $\sigma$-algebra containing $\mathcal{S}$ is not unique.

33. By a bounded interval of real numbers we mean a set of the form $[a, b], [a, b), (a, b]$, or $(a, b)$ for real numbers $a \leq b$. Thus we consider the empty-set and a set consisting of a single point to be a bounded interval. Show that each of the following three collections of sets $\mathcal{S}$ is a semiring.

(i) Let $\mathcal{S}$ be the collection of all bounded intervals of real numbers.

(ii) Let $\mathcal{S}$ be the collection of all subsets of $\mathbb{R} \times \mathbb{R}$ that are products of bounded intervals of real numbers.

(iii) Let $n$ be a natural number an $X$ be the $n$-fold Cartesian product of $\mathbb{R}$:

$$X = \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n.$$

Let $\mathcal{S}$ be the collection of all subsets of $X$ that are $n$-fold Cartesian products of bounded intervals of real numbers.

34. If we start with an outer measure $\mu^*$ on $2^X$ and form the induced measure $\overline{\mu}$ on the $\mu^*$-measurable sets, we can view $\overline{\mu}$ as a set function and denote by $\mu^+$ the outer measure induced by $\overline{\mu}$.

   (i) Show that for each set $E \subset X$ we have $\mu^+(E) \geq \mu^*(E)$.

   (ii) For a given set $E$, show that $\mu^+(E) = \mu^*(E)$ iff there is a $\mu^*$-measurable set $A \supseteq E$ with $\mu^*(A) = \mu^*(E)$.

35. Let $\mathcal{S}$ be a $\sigma$-algebra of subsets of $X$ and $\mu : \mathcal{S} \to [0, \infty]$ a measure. Let $\overline{\mu} : \mathcal{M} \to [0, \infty]$ be the measure induced by $\mu$ via the Carathéodory construction. Show that $\mathcal{S}$ is a subcollection of $\mathcal{M}$ and it may be a proper subcollection.

36. Let $\mu$ be a finite premeasure on an algebra $\mathcal{S}$, and $\mu^*$ the induced outer measure. Show that a subset $E$ of $X$ is $\mu^*$-measurable iff for each $\epsilon > 0$ there is a set $A \in \mathcal{S}_\delta$, $A \subseteq E$, such that $\mu^*(E \setminus A) < \epsilon$.

# Chapter 18

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