Real Analysis Royden - Fourth Edition

Notes + Solved Exercises :)
Latex Symbols

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Contents

I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

Preliminaries on Sets, Mappings, and Relations

Definition. *A relation R on a set X is called an **equivalence relation** provided:*

(i) xRx for all $x \in X$ (reflexive),

(ii) xRy implies yRx for all $x, y \in X$ (symmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

Partial Ordering on a set X. *A relation R on a nonempty set X is called a **partial ordering** provided:*

(i) xRx for all $x \in X$ (reflexive),

(ii) xRy and yRx imply $x = y$ for all $x, y \in X$ (antisymmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

*A subset E of X is **totally ordered** provided either xRy or yRx for all $x, y \in E$. A member x of X is said to be an **upper bound** for a subset E of X provided that*

$$yRx \text{ for all } y \in E.$$

*A member x of X is said to be **maximal** provided that*

$$xRy \text{ implies that } y = x \text{ for } y \in X.$$

Strict Partial Ordering on a set X. *A relation R on a nonempty set X is called a **strict partial ordering** provided:*

(i) not xRx for all $x \in X$ (irreflexive),

(ii) xRy implies not yRx for all $x, y \in X$ (asymmetric),

(iii) xRy and yRz imply xRz for all $x, y, z \in X$ (transitive).

*A subset E of X is **strictly totally ordered** provided either xRy or yRx if $x \neq y$ for all $x, y \in E$.*

Zorn's Lemma. *Let X be a partially ordered set for which every totally ordered subset has an upper bound. Then X has a maximal member.*

Every vector space has a basis.

Proof. Let V be any vector space, and let L be the collection of all linearly independent subsets of V. L is nonempty as the singleton sets are linearly independent. Define a partial order on L in the form $C \subseteq C'$ for $C, C' \in L$. For any chain (a totally ordered subset of a partially ordered set) $\mathcal{C}$ of L, where $\mathcal{C}$ consists of the sets $C_1 \subseteq C_2 \subseteq \cdots$, we can construct a linearly independent set $C' = \bigcup_{C \in \mathcal{C}} C$ that is an upper bound of $\mathcal{C}$. By Zorn's Lemma, L has a maximal element, say M. This collection M is a basis for V. To show this, suppose by contradiction that there exists a vector $v \in V$ s.t. $v \notin$ Span$\{M\}$. Then $v \cup M$ is linearly independent and $M \subseteq v \cup M$, a contradiction to the fact that M is maximal. $\square$

Chapter 1

The Real Numbers: Sets, Sequences, and Functions

Contents

1.1 The Field, Positivity, and Completeness Axioms

The field axioms
Consider $a, b, c \in \mathbb{R}$:

1. Closure of Addition: $a + b \in \mathbb{R}$.
2. Associativity of Addition: $(a + b) + c = a + (b + c)$.
3. Additive Identity: $0 + a = a + 0 = a$.
4. Additive Inverse: $(-a) + a = a + (-a) = 0$.
5. Commutativity of Addition: $a + b = b + a$.
6. Closure of Multiplication: $ab \in \mathbb{R}$.
7. Associativity of Multiplication: $(ab)c = a(bc)$.
8. Distributive Property: $a(b + c) = ab + ac$.
9. Commutativity of Multiplication: $ab = ba$.
10. Multiplicative Identity: $1a = a1 = a$.
11. No Zero Divisors: $ab = 0 \implies a = 0$ or $b = 0$.

12. Multiplicative Inverse: $a^{-1}a = aa^{-1} = 1$.

13. Nontriviality: $1 \neq 0$.

The positivity axioms
The set of **positive numbers**, $\mathcal{P}$, has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are both positive.

P2 For a real number a, exactly one of the three is true: a is positive, $-a$ is positive, $a = 0$.

We call a nonempty set I of real numbers an **interval** provided for any two points in I, all the points that lie between these two points also lie in I. That is, $\forall x, y \in I, \lambda x + (1 - \lambda)y \in I$ for $\lambda \in [0, 1]$.

The completeness axiom
A nonempty set E of real numbers is said to be **bounded above** provided there is a real number b such that $x \leq b$ for all $x \in E$: the number b is called an **upper bound** for E. We can similarly define a set being **bounded below** and having a **lower bound**. A set that is bounded above need not have a largest member.

The Completeness Axiom. *Let E be a nonempty set of real numbers that is bounded above. The among the set of upper bounds for E there is a smallest, or least, upper bound. (This least upper bound is called the **supremum** of E. Also, it can be shown that any nonempty set E that is bounded below has a greatest lower bound, called the **infimum** of E).*

The extended real numbers
The extended real numbers: $\mathbb{R} \cup \{-\infty, \infty\}$
Every set of real numbers has a supremum and infimum that belongs to the extended real numbers.

PROBLEMS

1. For $a \neq 0$ and $a \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

$$\begin{aligned}
(ab)(ab)^{-1} &= 1 && \text{by multiplicative inverse} \\
a(b(ab)^{-1}) &= 1 && \text{by associativity of multiplication} \\
a^{-1}a(b(ab)^{-1}) &= a^{-1}1 && \\
1(b(ab)^{-1}) &= a^{-1}1 && \text{by multiplicative inverse} \\
b(ab)^{-1} &= a^{-1} && \text{by multiplicative identity} \\
b^{-1}b(ab)^{-1} &= b^{-1}a^{-1} && \\
1(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative inverse} \\
(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative identity} \\
(ab)^{-1} &= a^{-1}b^{-1} && \text{by commutativity of multiplication}
\end{aligned}$$

2. Verify the following:

(i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
By positivity axiom P2, since $a \neq 0$, either a is positive or $-a$ is positive.
In the case a is positive, a^2 is positive by positivity axiom P1.
In the case $-a$ is positive, $(-a)(-a)$ is positive by P1.

$$\begin{aligned}
(-a)(-a) &= (-a)(-a) + a(0) && \text{by additive identity}\\
&= (-a)(-a) + a(-a + a) && \text{by additive inverse}\\
&= (-a)(-a) + a(-a) + a(a) && \text{by distributive property}\\
&= (-a + a)(-a) + a^2 && \text{by distributive property}\\
&= 0(-a) + a^2 && \text{by additive inverse}\\
&= a^2 && \text{by additive identity}
\end{aligned}$$

Therefore a^2 is positive by equality.

(ii) For each positive number a, its multiplicative inverse a^{-1} also is positive.
The multiplication of two positive numbers is positive by positivity axiom P1.
The multiplication of two non-positive numbers is positive: by reformulating the previous result from (i), we can see $0 < (-a)(-b) = ab$ for $a, b \neq 0$.
The multiplication of a positive number and a non-positive number is not positive. To see this, suppose a is positive and b is not positive, but ab is positive. By P1 and P2, $a(-b)$ is also positive. By P1, $ab + a(-b)$ is positive. However,

$$ab + a(-b) = a(b - b) = a(0) = 0.$$

This is a contradiction to P2. Therefore ab is not positive.
The result from (i) shows that 1 is positive. By multiplicative inverse, $aa^{-1} = 1 > 0$. Therefore a^{-1} must be positive because a is positive.

(iii) If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$

Proof that $a(-1) = -a$ for a real number a:

$$a + (-1)a = 1a + (-1)a = (1 + -1)a = 0a = 0.$$

$a > b$ implies that $a - b$ is positive.
If c is positive, then $(a - b)c = ac - bc$ is positive, and $ac > bc$.
If c is not positive, then $(a - b)c = ac - bc$ is not positive, and $-(ac - bc) = bc - ac$ is positive, so $bc > ac$.

3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ iff E consists of a single point.

($\Longrightarrow$) Suppose $\inf E = \sup E$.

Then $\inf E \leq x \leq \sup E$ for all $x \in E$. But this implies $x = \inf E = \sup E$ for all $x \in E$, so E consists of the single point x.

($\Longleftarrow$) Suppose $E = x$ is a singleton set.

Clearly x is an upper bound and a lower bound for E, as $x \leq x$. By completeness of the reals, there exists $\sup E$ and $\inf E$ s.t. $x \leq \inf E \leq x \leq \sup E \leq x$, as $\inf E$ is the greatest lower bound, and $\sup E$ is the least upper bound. Therefore $\inf E = \sup E$.

4. Let a and b be real numbers.

   (i) Show that if $ab = 0$, then $a = 0$ or $b = 0$.

   Contrapositive: Let $a \neq 0$ and $b \neq 0$. In 2.(ii), it was shown that the multiplication of two nonzero numbers is either positive or not positive. Therefore $ab \neq 0$.

   (ii) Verify that $a^2 - b^2 = (a-b)(a+b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

$$\begin{aligned}
(a-b)(a+b) &= (a-b)(a) + (a-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(a) + (a)(b) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b+b)(a) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(b) && \text{by additive inverse}\\
&= a^2 - b^2
\end{aligned}$$

   Suppose $a^2 = b^2$. Then $(a-b)(a+b) = a^2 - b^2 = 0$, and by (i), $(a-b) = 0 \implies a = b$ or $(a+b) = 0 \implies a = -b$.

   (iii) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$.

   We can consider $0 \in \mathbb{R}$. $0^2 = 0 < c$, so $0 \in E$ and E is nonempty. Also, $c+1$ is a real number and an upper bound for E; thus by the completeness axiom, E has a supremum, say x_0. We can see that for any upper bound b of E, $x \leq x_0 \leq b$ for all $x \in E$. Then $x^2 \leq x_0^2 \leq b^2$ implies $x_0^2 = c$, else x_0 is not the supremum.

   Suppose there exists $x_1, x_2 > 0$ such that $x_1^2 = c$ and $x_2^2 = c$. This implies $x_1^2 = x_2^2$, and by part (ii), $x_1 = x_2$ or $x_1 = -x_2$. Because x_1, x_2 are positive, $x_1 = x_2$.

5. Let a, b, c be real numbers s.t. $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}.$$

   (i) Suppose $b^2 - 4ac > 0$. Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\begin{aligned}
ax^2 + bx + c &= 0\\
4a(ax^2 + bx + c) &= 4a(0)\\
4a^2x^2 + 4abx + 4ac &= 0 && \text{by distributive property}\\
4a^2x^2 + 4abx + 4ac + b^2 - b^2 &= 0 && \text{by additive inverse}\\
4a^2x^2 + 4abx + b^2 &= b^2 - 4ac\\
(2ax + b)^2 &= b^2 - 4ac
\end{aligned}$$

   By 4(iii), because $b^2 - 4ac > 0$, there is a unique $y > 0$ for which $y^2 = b^2 - 4ac$. It is denoted by $y = \sqrt{b^2 - 4ac}$.

By 4(ii), $(2ax+b)^2 = b^2 - 4ac = y^2$ implies $(2ax+b) = \sqrt{b^2-4ac} = y$ or $(2ax+b) = -\sqrt{b^2-4ac} = -y$.

$$\begin{aligned} 2ax + b &= \pm\sqrt{b^2-4ac} \\ 2ax &= -b \pm \sqrt{b^2-4ac} \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \end{aligned}$$

6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x \mid x \in E\}.$$

Let E be a set that is bounded below; that is, there exists $l \in \mathbb{R}$ such that $l \leq x$ for all $x \in E$. Then $-l \geq -x$ for all $x \in E$, and $-l$ is an upper bound for $-E = \{-x \mid x \in E\}$. Therefore the set $-E$ is bounded above, and by the completeness axiom, there exists a least upper bound $c = \sup(-E)$. Then for any upper bound u of $-E$, $u \geq c \geq -x$ for all $x \in E$. Then $-u$ is a lower bound of E, and $-u \leq c \leq x$ for all $x \in E$, and c is the greatest lower bound and thus infimum of E.

7. For real numbers a and b, verify the following:

(i) $|ab| = |a||b|$.
We have

$$|ab| = \begin{cases} ab & \text{if } ab \geq 0, \\ -(ab) & \text{if } ab < 0. \end{cases}$$

The case where either a or b are zero is trivial. In problem 2(ii), it was shown that $ab > 0$ if a, b are the same sign, and $ab < 0$ if a, b are opposite signs.

Case $a, b > 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = a$ and $|b| = b$ so $|a||b| = ab$.

Case $a, b < 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = -a$ and $|b| = -b$ so $|a||b| = (-a)(-b) = ab$.

Case $a < 0, b > 0$: Then $ab < 0$ so $|ab| = -(ab) = (-1)ab$, and $|a| = -a = (-1)a$ and $|b| = b$ so $|a||b| = (-1)ab$.

(ii) $|a+b| \leq |a| + |b|$.
The case where both $a, b = 0$ is trivial.

Case $a, b > 0$: Then $a + b > 0$, so $|a+b| = a+b$ and $|a| + |b| = a + b$.

Case $a > 0, b = 0$: Then $a + b = a + 0 = a > 0$, so $|a+b| = a$ and $|a| + |b| = a + 0 = a$.

Case $a < 0, b = 0$: Then $a+b = a+0 = a < 0$, so $|a+b| = -a$ and $|a|+|b| = -a+0 = -a$.

Case $a, b < 0$: Then $a + b < 0$, so $|a+b| = -(a+b) = -a-b$ and $|a| + |b| = -a - b$.

That is, equality holds except for the case where a, b are nonzero opposite signs:

Case $a > 0, b < 0$: $|a+b| \in \{a+b, -(a+b)\}$.

$b < 0 < -b \implies a+b < a < a-b$, and $-a < 0 < a \implies -(a+b) = -a-b < -b < a-b$.

$|a| + |b| = a - b$, so $|a+b| < |a| + |b|$.

(iii) For $\epsilon > 0$,

$$|x - a| < \epsilon \text{ iff } a - \epsilon < x < a + \epsilon.$$

We have

$$|x-a| = \begin{cases} x-a & \text{if } x-a \geq 0, \\ -(x-a) & \text{if } x-a < 0. \end{cases}$$

($\Longrightarrow$) Suppose $|x-a| < \epsilon$.
Then $x-a < \epsilon$ and $a-x < \epsilon$.
Then $x < a+\epsilon$ and $a-\epsilon < x$.
($\Longleftarrow$) Suppose $a-\epsilon < x < a+\epsilon$.
Then

$$\begin{aligned} a-\epsilon-a < x-a < a+\epsilon-a \\ -\epsilon < x-a < \epsilon \end{aligned}$$

So $x-a < \epsilon$ and $-\epsilon < x-a \implies -(x-a) < \epsilon$, so $|x-a| < \epsilon$.

1.2 The Natural and Rational Numbers

Definition. *A set E of real numbers is said to be* ***inductive*** *provided it contains 1 and if the number x belongs to E, the number $x+1$ also belongs to E.*
The set of **natural numbers**, denoted by $\mathbb{N}$, is defined to be the intersection of all inductive subsets of $\mathbb{R}$.

Theorem 1. *Every nonempty set of natural numbers has a smallest member.*

Proof. Let E be a nonempty set of natural numbers. Since the set $\{x \in \mathbb{R} \mid x \geq 1\}$ is an inductive set, by definition of intersection, $\mathbb{N} \subseteq \{x \in \mathbb{R} \mid x \geq 1\}$, and the natural numbers are bounded below by 1. Therefore E is bounded below by 1. By the Completeness Axiom, E has an infimum; let $c = \inf E$. Since $c+1$ is not a lower bound for E, there is an $m \in E$ for which $m < c+1$. We claim that m is the smallest member of E. Otherwise, there is an $n \in E$ for which $n < m$. Since $n \in E$, $c \leq n$. Thus $c \leq n < m < c+1$ and therefore $m-n < 1$. Therefore the natural number m belongs to the interval $(n, n+1)$. However, an induction argument shows that $(n, n+1) \cap \mathbb{N} = \emptyset$ (Problem 8). This is a contradiction to $m \in E$. Therefore m is the smallest member of E. $\square$

Archimedean Property. *For each pair of positive real numbers a and b, there is a natural number n for which $na > b$. This can be reformulated: for each positive real number ϵ, there is a natural number n for which $\frac{1}{n} < \epsilon$.*

The set of **integers**, denoted $\mathbb{Z}$, is defined to be the set of numbers consisting of the natural numbers, their negatives, and zero.

Definition. *A set E of real numbers is said to be* ***dense*** *in $\mathbb{R}$ provided that between any two real numbers there lies a member of E.*

Theorem 2. *The rational numbers are dense in $\mathbb{R}$.*

Proof. Let $a, b \in \mathbb{R}$ with $a < b$.
Case $a > 0$:
By the Archimedean Property, for $(b-a) > 0$, there exists $q \in \mathbb{N}$ for which $\frac{1}{q} < b-a$.
Again by the Archimedean Property, for $b, \frac{1}{q} > 0$, there exists $n \in \mathbb{N}$ for which $n(\frac{1}{q}) > b$.

Therefore the set $S = \{n \in \mathbb{N} \mid \frac{n}{q} \geq b\}$ is nonempty. Because S is a set of natural numbers, by Theorem 1, S has a smallest member p. Noticing $\frac{1}{q} < b - a < b$, we see that $1 \notin S$ and $p > 1$. Therefore $p - 1$ is a natural number (Problem 9). Because p is the smallest member of S, $p - 1 \notin S$ and $\frac{(p-1)}{q} < b$. Also,

$$a = b - (b - a) < \frac{p}{q} - (\frac{1}{q}) = \frac{(p-1)}{q}.$$

Therefore the rational number $\frac{(p-1)}{q}$ lies between a and b.
Case $a < 0$:
By the Archimedean Property, for $1, -a > 0$, there exists $n \in \mathbb{N}$ for which $n(1) > -a$, which implies $n + a > 0$, and $b > a$ implies $n + b > n + a > 0$. Then we can use the first case to show that there exists a rational number r such that $n + a < r < n + b$. Therefore the rational number $r - n$ lies between a and b. □

PROBLEMS

8. Use an induction argument to show that for each natural number n, the interval $(n, n+1)$ fails to contain any natural number.

   Let $P(n)$ be the assertion that $(n, n+1) \cap \mathbb{N} = \emptyset$.

   $P(1)$: $(1, 2) = \{x \mid 1 < x < 2\}$,so 1 is a lower bound

   Suppose $P(k)$ is true for some natural number k.

   $P(k+1)$: $(k+1, (k+1)+1)$

9. Use an induction argument to show that if $n > 1$ is a natural number, then $n - 1$ also is a natural number. The use another induction argument to show that if m and n are natural numbers with $n > m$, then $n - m$ is a natural number.

   Let $P(n)$ be the assertion that $(n, n+1) \cap \mathbb{N} = \emptyset$.

   $P(1)$: $(1, 2)$

10. Show that for any real number r, there is exactly one integer in the interval $[r, r+1)$.

11. Show that ay nonempty set of integers that is bounded above has a largest member.

12. Show that the irrational numbers are dense in $\mathbb{R}$.

13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

14. Show that if $r > 0$, then, for each natural number n, $(1+r)^n \geq 1 + n \cdot r$.

15. Use induction arguments to prove that for every natural number n,

    (i)
    $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

    (ii)
    $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

    (iii)
    $$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \text{ if } r \neq 1.$$

1.3 Countable and Uncountable Sets

PROBLEMS

16. Show that the set $\mathbb{Z}$ of integers is countable.
17. Show that a set A is countable iff there is an injective mapping of A to $\mathbb{N}$.
18. Use an induction argument to complete the proof of part (i) of Corollary 4.
19. Prove Corollary 6 in the case of a finite family of countable sets.
20. Let both $f : A \to B$ and $g : B \to C$ be injective and surjective. Show that the composition $g \circ f : A \to B$ and the inverse $f^{-1} : B \to A$ are also injective and surjective.
21. Use an induction argument to establish the pigeonhole principle.
22. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable.
23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding theorem to show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.
24. Show that a degenerate interval of real numbers fails to be finite.
25. Show that any two nondegenerate intervals of real numbers are equipotent.
26. Is the set $\mathbb{R} \times \mathbb{R}$ equipotent to $\mathbb{R}$?

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

The Nested Set Theorem. *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof. By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$, and we have an open cover of $\mathbb{R}$ and thus an open cover of $F_1 \subseteq \mathbb{R}$. By the Heine-Borel Theorem, there exists an $N \in \mathbb{N}$ such that $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c$. Because $\{F_n\}$ is descending, $F_n \supseteq F_{n+1}$ for any $n \geq 1$. This implies $F_n^c \subseteq F_{n+1}^c$, and thus $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c = F_N^c = \mathbb{R} \setminus F_N$. This is a contradiction to the assumption that F_N is a nonempty subset of F_1. □

PROBLEMS

27. Is the set of rational numbers open or closed?
28. What are the sets of real numbers that are both open and closed?
29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.
30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \setminus \{x\}$.
    (i) Show that the set E' of accumulation points of E is a closed set.

(ii) Show that $\overline{E} = E \cup E'$.

31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.

32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E. The set of interior points of E is called the **interior** of E denoted by int E. Show that

    (i) E is open iff $E =$ int E.

    (ii) E is dense iff int $(\mathbb{R} \setminus E) = \emptyset$.

33. Show that the nested set theorem is false if F_1 is unbounded.

34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

35. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

36. Show that the collection of Borel sets is the smallest σ-algebra that contains the intervals of the form $[a, b)$, where $a < b$.

37. Show that each open set is an F_σ set.

1.5 Sequences of Real Numbers

PROBLEMS

38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

39. Prove proposition 19.

40. Show that a sequence $\{a_n\}$ is convergent to an extended real number iff there is exactly one extended real number that is a cluster point of the sequence.

41. Show that $\liminf a_n \leq \limsup a_n$.

42. Prove that if, for all n, $a_n \geq 0$ and $b_n \geq 0$, then

$$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

    provided the product on the right is not of the form $0 \cdot \infty$.

43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

44. Let p be a natural number greater than 1, and x a real number $0 \leq x \leq 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n, $0 < q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal* expansion of x. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

45. Prove Proposition 20.

46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

1.6 Continuous Real-Valued Functions of a Real Variable

PROBLEMS

47. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E. Show that there is a function g defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take g to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)

48. Define the real-valued function f on $\mathbb{R}$ by setting

$$f(x) = \begin{cases} x & \text{if x irrational} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

At what points is f continuous?

49. Let f and g be continuous real-valued functions with a common domain E.
    (i) Show that the sum, $f + g$, and product, fg, are also continuous functions.
    (ii) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.
    (iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.
    (iv) Show that $|f|$ is continuous.

50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

51. A continuous function ϕ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which ϕ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function ϕ on $[a, b]$ with $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a, b]$.

52. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

53. Show that a set E of real numbers is closed and bounded iff every open cover of E has a finite subcover.

54. Show that a nonempty set E of real numbers is an interval iff every continuous real-valued function on E has an interval as its image.

55. Show that a monotone function on an open interval is continuous iff its image is an interval.

56. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.

57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.

58. Let f be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.

59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f iff given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.

Chapter 2

Lebesgue Measure

Contents

2.1 Introduction

In this chapter we construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure**, denoted by m. The collection of Lebesgue measurable sets is a σ-algebra which contains all open sets and all closed sets. The set function m possesses the following three properties:

The measure of an interval is its length. *Each nonempty interval I is Lebesgue measurable and*

$$m(I) = \ell(I).$$

Measure is translation invariant. *If E is Lebesgue measurable and y is any number then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and*

$$m(E + y) = m(E).$$

Measure is countably additive over countable disjoint unions of sets. *IF $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then*

$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (See Vitali sets). We first construct a set function called **outer measure**, denoted by m^*, such that:

(i) the outer measure of an interval is its length,

(ii) outer measure is translation invariant,

(iii) outer measure is countably subadditive.

Then the Lebesgue measure m is the restriction of m^* to the Lebesgue measurable sets.

PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

   $A \subseteq B \implies B = A \cup (B \cap A^c)$, where $A \cap (B \cap A^c) = \emptyset$. The set $(B \cap A^c)$ is measurable because A^c is measurable and countable intersection is measurable, so $m(B) = m(A \cup (B \cap A^c)) = m(A) + m(B \cap A^c)$ by countable additivity, and thus $m(B) \geq m(A)$.

2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$.

4. A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.

2.2 Lebesgue Outer Measure

PROBLEMS

5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.

6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which

$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

2.3 The σ-Algebra of Lebesgue Measurable Sets

2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

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