Real Analysis Royden - Fourth Edition

Notes + Solved Exercises :)

Latex Symbols

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May 2024

Contents

I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

Preliminaries on Sets, Mappings, and Relations

Definition. *A relation R on a set X is called an **equivalence relation** provided:*

(i) xRx *for all* $x \in X$ *(reflexive),*

(ii) xRy *implies* yRx *for all* $x, y \in X$ *(symmetric),*

(iii) xRy *and* yRz *imply* xRz *for all* $x, y, z \in X$ *(transitive).*

Partial Ordering on a set X. *A relation R on a nonempty set X is called a **partial ordering** provided:*

(i) xRx *for all* $x \in X$ *(reflexive),*

(ii) xRy *and* yRx *imply* $x = y$ *for all* $x, y \in X$ *(antisymmetric),*

(iii) xRy *and* yRz *imply* xRz *for all* $x, y, z \in X$ *(transitive).*

*A subset E of X is **totally ordered** provided either xRy or yRx for all $x, y \in E$. A member x of X is said to be an **upper bound** for a subset E of X provided that*

$$yRx \text{ for all } y \in E.$$

*A member x of X is said to be **maximal** provided that*

$$xRy \text{ implies that } y = x \text{ for } y \in X.$$

Strict Partial Ordering on a set X. *A relation R on a nonempty set X is called a **strict partial ordering** provided:*

(i) not xRx *for all* $x \in X$ *(irreflexive),*

(ii) xRy *implies not* yRx *for all* $x, y \in X$ *(asymmetric),*

(iii) xRy *and* yRz *imply* xRz *for all* $x, y, z \in X$ *(transitive).*

*A subset E of X is **strictly totally ordered** provided either xRy or yRx if $x \neq y$ for all $x, y \in E$.*

Zorn's Lemma. *Let X be a partially ordered set for which every totally ordered subset has an upper bound. Then X has a maximal member.*

Every vector space has a basis.

Proof. Let V be any vector space, and let L be the collection of all linearly independent subsets of V. L is nonempty as the singleton sets are linearly independent. Define a partial order on L in the form $C \subseteq C'$ for $C, C' \in L$. For any chain (a totally ordered subset of a partially ordered set) $\mathcal{C}$ of L, where $\mathcal{C}$ consists of the sets $C_1 \subseteq C_2 \subseteq \cdots$, we can construct a linearly independent set $C' = \bigcup_{C \in \mathcal{C}} C$ that is an upper bound of $\mathcal{C}$. By Zorn's Lemma, L has a maximal element, say M. This collection M is a basis for V. To show this, suppose by contradiction that there exists a vector $v \in V$ s.t. $v \notin$ Span$\{M\}$. Then $v \cup M$ is linearly independent and $M \subseteq v \cup M$, a contradiction to the fact that M is maximal. $\square$

Chapter 1

The Real Numbers: Sets, Sequences, and Functions

Contents

1.1 The Field, Positivity, and Completeness Axioms

The field axioms
Consider $a, b, c \in \mathbb{R}$:

1. Closure of Addition: $a + b \in \mathbb{R}$.
2. Associativity of Addition: $(a + b) + c = a + (b + c)$.
3. Additive Identity: $0 + a = a + 0 = a$.
4. Additive Inverse: $(-a) + a = a + (-a) = 0$.
5. Commutativity of Addition: $a + b = b + a$.
6. Closure of Multiplication: $ab \in \mathbb{R}$.
7. Associativity of Multiplication: $(ab)c = a(bc)$.
8. Distributive Property: $a(b + c) = ab + ac$.
9. Commutativity of Multiplication: $ab = ba$.
10. Multiplicative Identity: $1a = a1 = a$.
11. No Zero Divisors: $ab = 0 \implies a = 0$ or $b = 0$.

12. Multiplicative Inverse: $a^{-1}a = aa^{-1} = 1$.

13. Nontriviality: $1 \neq 0$.

The positivity axioms
The set of **positive numbers**, $\mathcal{P}$, has the following two properties:

P1 If a and b are positive, then ab and $a + b$ are both positive.

P2 For a real number a, exactly one of the three is true: a is positive, $-a$ is positive, $a = 0$.

We call a nonempty set I of real numbers an **interval** provided for any two points in I, all the points that lie between these two points also lie in I. That is, $\forall x, y \in I, \lambda x + (1 - \lambda)y \in I$ for $\lambda \in [0, 1]$.

The completeness axiom
A nonempty set E of real numbers is said to be **bounded above** provided there is a real number b such that $x \leq b$ for all $x \in E$: the number b is called an **upper bound** for E. We can similarly define a set being **bounded below** and having a **lower bound**. A set that is bounded above need not have a largest member.

The Completeness Axiom. *Let E be a nonempty set of real numbers that is bounded above. The among the set of upper bounds for E there is a smallest, or least, upper bound. (This least upper bound is called the **supremum** of E. Also, it can be shown that any nonempty set E that is bounded below has a greatest lower bound, called the **infimum** of E).*

The extended real numbers
The extended real numbers: $\mathbb{R} \cup \{-\infty, \infty\}$
Every set of real numbers has a supremum and infimum that belongs to the extended real numbers.

PROBLEMS

1. For $a \neq 0$ and $a \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

$$\begin{aligned}
(ab)(ab)^{-1} &= 1 && \text{by multiplicative inverse} \\
a(b(ab)^{-1}) &= 1 && \text{by associativity of multiplication} \\
a^{-1}a(b(ab)^{-1}) &= a^{-1}1 && \\
1(b(ab)^{-1}) &= a^{-1}1 && \text{by multiplicative inverse} \\
b(ab)^{-1} &= a^{-1} && \text{by multiplicative identity} \\
b^{-1}b(ab)^{-1} &= b^{-1}a^{-1} && \\
1(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative inverse} \\
(ab)^{-1} &= b^{-1}a^{-1} && \text{by multiplicative identity} \\
(ab)^{-1} &= a^{-1}b^{-1} && \text{by commutativity of multiplication}
\end{aligned}$$

2. Verify the following:

(i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
By positivity axiom P2, since $a \neq 0$, either a is positive or $-a$ is positive.
In the case a is positive, a^2 is positive by positivity axiom P1.
In the case $-a$ is positive, $(-a)(-a)$ is positive by P1.

$$\begin{aligned}
(-a)(-a) &= (-a)(-a) + a(0) && \text{by additive identity}\\
&= (-a)(-a) + a(-a+a) && \text{by additive inverse}\\
&= (-a)(-a) + a(-a) + a(a) && \text{by distributive property}\\
&= (-a+a)(-a) + a^2 && \text{by distributive property}\\
&= 0(-a) + a^2 && \text{by additive inverse}\\
&= a^2 && \text{by additive identity}
\end{aligned}$$

Therefore a^2 is positive by equality.

(ii) For each positive number a, its multiplicative inverse a^{-1} also is positive.
The multiplication of two positive numbers is positive by positivity axiom P1.
The multiplication of two non-positive numbers is positive: by reformulating the previous result from (i), we can see $0 < (-a)(-b) = ab$ for $a, b \neq 0$.
The multiplication of a positive number and a non-positive number is not positive. To see this, suppose a is positive and b is not positive, but ab is positive. By P1 and P2, $a(-b)$ is also positive. By P1, $ab + a(-b)$ is positive. However,

$$ab + a(-b) = a(b - b) = a(0) = 0.$$

This is a contradiction to P2. Therefore ab is not positive.
The result from (i) shows that 1 is positive. By multiplicative inverse, $aa^{-1} = 1 > 0$. Therefore a^{-1} must be positive because a is positive.

(iii) If $a > b$, then

$$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$

Proof that $a(-1) = -a$ for a real number a:

$$a + (-1)a = 1a + (-1)a = (1 + -1)a = 0a = 0.$$

$a > b$ implies that $a - b$ is positive.
If c is positive, then $(a - b)c = ac - bc$ is positive, and $ac > bc$.
If c is not positive, then $(a - b)c = ac - bc$ is not positive, and $-(ac - bc) = bc - ac$ is positive, so $bc > ac$.

3. For a nonempty set of real numbers E, show that $\inf E = \sup E$ iff E consists of a single point.

($\Longrightarrow$) Suppose $\inf E = \sup E$.

Then $\inf E \leq x \leq \sup E$ for all $x \in E$. But this implies $x = \inf E = \sup E$ for all $x \in E$, so E consists of the single point x.

($\Longleftarrow$) Suppose $E = x$ is a singleton set.

Clearly x is an upper bound and a lower bound for E, as $x \leq x$. By completeness of the reals, there exists $\sup E$ and $\inf E$ s.t. $x \leq \inf E \leq x \leq \sup E \leq x$, as $\inf E$ is the greatest lower bound, and $\sup E$ is the least upper bound. Therefore $\inf E = \sup E$.

4. Let a and b be real numbers.

   (i) Show that if $ab = 0$, then $a = 0$ or $b = 0$.

   Contrapositive: Let $a \neq 0$ and $b \neq 0$. In 2.(ii), it was shown that the multiplication of two nonzero numbers is either positive or not positive. Therefore $ab \neq 0$.

   (ii) Verify that $a^2 - b^2 = (a - b)(a + b)$ and conclude from part (i) that if $a^2 = b^2$, then $a = b$ or $a = -b$.

$$\begin{aligned}
(a-b)(a+b) &= (a-b)(a) + (a-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(a) + (a)(b) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b+b)(a) + (-b)(b) && \text{by distributive property}\\
&= (a)(a) + (-b)(b) && \text{by additive inverse}\\
&= a^2 - b^2
\end{aligned}$$

   Suppose $a^2 = b^2$. Then $(a-b)(a+b) = a^2 - b^2 = 0$, and by (i), $(a-b) = 0 \implies a = b$ or $(a+b) = 0 \implies a = -b$.

   (iii) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (ii) to show that there is a unique $x > 0$ for which $x^2 = c$. It is denoted by $\sqrt{c}$.

   We can consider $0 \in \mathbb{R}$. $0^2 = 0 < c$, so $0 \in E$ and E is nonempty. Also, $c+1$ is a real number and an upper bound for E; thus by the completeness axiom, E has a supremum, say x_0. We can see that for any upper bound b of E, $x \leq x_0 \leq b$ for all $x \in E$. Then $x^2 \leq x_0^2 \leq b^2$ implies $x_0^2 = c$, else x_0 is not the supremum.

   Suppose there exists $x_1, x_2 > 0$ such that $x_1^2 = c$ and $x_2^2 = c$. This implies $x_1^2 = x_2^2$, and by part (ii), $x_1 = x_2$ or $x_1 = -x_2$. Because x_1, x_2 are positive, $x_1 = x_2$.

5. Let a, b, c be real numbers s.t. $a \neq 0$ and consider the quadratic equation

$$ax^2 + bx + c = 0, x \in \mathbb{R}.$$

   (i) Suppose $b^2 - 4ac > 0$. Use the Field Axioms and the preceding problem to complete the square and thereby show that this equation has exactly two solutions given by

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\begin{aligned}
ax^2 + bx + c &= 0\\
4a(ax^2 + bx + c) &= 4a(0)\\
4a^2x^2 + 4abx + 4ac &= 0 && \text{by distributive property}\\
4a^2x^2 + 4abx + 4ac + b^2 - b^2 &= 0 && \text{by additive inverse}\\
4a^2x^2 + 4abx + b^2 &= b^2 - 4ac\\
(2ax + b)^2 &= b^2 - 4ac
\end{aligned}$$

   By 4(iii), because $b^2 - 4ac > 0$, there is a unique $y > 0$ for which $y^2 = b^2 - 4ac$. It is denoted by $y = \sqrt{b^2 - 4ac}$.

By 4(ii), $(2ax+b)^2 = b^2 - 4ac = y^2$ implies $(2ax+b) = \sqrt{b^2-4ac} = y$ or $(2ax+b) = -\sqrt{b^2-4ac} = -y$.

$$\begin{aligned} 2ax + b &= \pm\sqrt{b^2 - 4ac} \\ 2ax &= -b \pm \sqrt{b^2 - 4ac} \\ x &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \end{aligned}$$

6. Use the Completeness Axiom to show that every nonempty set of real numbers that is bounded below has an infimum and that

$$\inf E = -\sup\{-x \mid x \in E\}.$$

Let E be a set that is bounded below; that is, there exists $l \in \mathbb{R}$ such that $l \leq x$ for all $x \in E$. Then $-l \geq -x$ for all $x \in E$, and $-l$ is an upper bound for $-E = \{-x \mid x \in E\}$. Therefore the set $-E$ is bounded above, and by the completeness axiom, there exists a least upper bound $c = \sup(-E)$. Then for any upper bound u of $-E$, $u \geq c \geq -x$ for all $x \in E$. Then $-u$ is a lower bound of E, and $-u \leq c \leq x$ for all $x \in E$, and c is the greatest lower bound and thus infimum of E.

7. For real numbers a and b, verify the following:

(i) $|ab| = |a||b|$.
We have
$$|ab| = \begin{cases} ab & \text{if } ab \geq 0, \\ -(ab) & \text{if } ab < 0. \end{cases}$$
The case where either a or b are zero is trivial. In problem 2(ii), it was shown that $ab > 0$ if a, b are the same sign, and $ab < 0$ if a, b are opposite signs.
Case $a, b > 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = a$ and $|b| = b$ so $|a||b| = ab$.
Case $a, b < 0$: Then $ab > 0$ so $|ab| = ab$, and $|a| = -a$ and $|b| = -b$ so $|a||b| = (-a)(-b) = ab$.
Case $a < 0, b > 0$: Then $ab < 0$ so $|ab| = -(ab) = (-1)ab$, and $|a| = -a = (-1)a$ and $|b| = b$ so $|a||b| = (-1)ab$.

(ii) $|a + b| \leq |a| + |b|$.
The case where both $a, b = 0$ is trivial.
Case $a, b > 0$: Then $a + b > 0$, so $|a + b| = a + b$ and $|a| + |b| = a + b$.
Case $a > 0, b = 0$: Then $a + b = a + 0 = a > 0$, so $|a + b| = a$ and $|a| + |b| = a + 0 = a$.
Case $a < 0, b = 0$: Then $a+b = a+0 = a < 0$, so $|a+b| = -a$ and $|a|+|b| = -a+0 = -a$.
Case $a, b < 0$: Then $a + b < 0$, so $|a + b| = -(a + b) = -a - b$ and $|a| + |b| = -a - b$.
That is, equality holds except for the case where a, b are nonzero opposite signs:
Case $a > 0, b < 0$: $|a + b| \in \{a + b, -(a + b)\}$.
$b < 0 < -b \implies a+b < a < a-b$, and $-a < 0 < a \implies -(a+b) = -a-b < -b < a-b$.
$|a| + |b| = a - b$, so $|a + b| < |a| + |b|$.

(iii) For $\epsilon > 0$,
$$|x - a| < \epsilon \text{ iff } a - \epsilon < x < a + \epsilon.$$

We have

$$|x-a| = \begin{cases} x-a & \text{if } x-a \geq 0, \\ -(x-a) & \text{if } x-a < 0. \end{cases}$$

($\Longrightarrow$) Suppose $|x-a| < \epsilon$.
Then $x-a < \epsilon$ and $a-x < \epsilon$.
Then $x < a+\epsilon$ and $a-\epsilon < x$.
($\Longleftarrow$) Suppose $a-\epsilon < x < a+\epsilon$.
Then

$$\begin{aligned} a-\epsilon-a < x-a < a+\epsilon-a \\ -\epsilon < x-a < \epsilon \end{aligned}$$

So $x-a < \epsilon$ and $-\epsilon < x-a \implies -(x-a) < \epsilon$, so $|x-a| < \epsilon$.

1.2 The Natural and Rational Numbers

Definition. *A set E of real numbers is said to be **inductive** provided it contains 1 and if the number x belongs to E, the number $x+1$ also belongs to E.*
The set of **natural numbers**, denoted by $\mathbb{N}$, is defined to be the intersection of all inductive subsets of $\mathbb{R}$.

Theorem 1. *Every nonempty set of natural numbers has a smallest member.*

Proof. Let E be a nonempty set of natural numbers. Since the set $\{x \in \mathbb{R} \mid x \geq 1\}$ is an inductive set, by definition of intersection, $\mathbb{N} \subseteq \{x \in \mathbb{R} \mid x \geq 1\}$, and the natural numbers are bounded below by 1. Therefore E is bounded below by 1. By the Completeness Axiom, E has an infimum; let $c = \inf E$. Since $c+1$ is not a lower bound for E, there is an $m \in E$ for which $m < c+1$. We claim that m is the smallest member of E. Otherwise, there is an $n \in E$ for which $n < m$. Since $n \in E$, $c \leq n$. Thus $c \leq n < m < c+1$ and therefore $m-n < 1$. Therefore the natural number m belongs to the interval $(n, n+1)$. However, an induction argument shows that $(n, n+1) \cap \mathbb{N} = \emptyset$ (Problem 8). This is a contradiction to $m \in E$. Therefore m is the smallest member of E. □

Archimedean Property. *For each pair of positive real numbers a and b, there is a natural number n for which $na > b$. This can be reformulated: for each positive real number ϵ, there is a natural number n for which $\dfrac{1}{n} < \epsilon$.*

The set of **integers**, denoted $\mathbb{Z}$, is defined to be the set of numbers consisting of the natural numbers, their negatives, and zero.

Consider the number 2. From problem 4(iii), there is a unique $x > 0$ for which $x^2 = 2$. It is denoted by $\sqrt{2}$. This number is not rational. Suppose that x is rational: then there exist $p, q \in \mathbb{Z}$ such that $(\dfrac{p}{q})^2 = 2$. Then $p^2 = 2q^2$. By the unique prime factorizations of p and q, p^2 is divisible by 2^{2k} for some $k \in \mathbb{Z}_{\geq 0}$, while $2q^2$ is divisible by $2 \cdot 2^{2j} = 2^{2j+1}$ for some $j \in \mathbb{Z}_{\geq 0}$. $2^{2k} \neq 2^{2j+1}$ for any combinations of k, j so $p^2 = 2q^2$ is not possible, and $\sqrt{2}$ is not rational.

Definition. *A set E of real numbers is said to be **dense** in $\mathbb{R}$ provided that between any two real numbers there lies a member of E.*

Theorem 2. *The rational numbers are dense in* $\mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ with $a < b$.
Case $a > 0$:
By the Archimedean Property, for $(b-a) > 0$, there exists $q \in \mathbb{N}$ for which $\frac{1}{q} < b - a$.
Again by the Archimedean Property, for $b, \frac{1}{q} > 0$, there exists $n \in \mathbb{N}$ for which $n(\frac{1}{q}) > b$.
Therefore the set $S = \{n \in \mathbb{N} \mid \frac{n}{q} \geq b\}$ is nonempty. Because S is a set of natural numbers, by Theorem 1, S has a smallest member p. Noticing $\frac{1}{q} < b - a < b$, we see that $1 \notin S$ and $p > 1$. Therefore $p - 1$ is a natural number (Problem 9). Because p is the smallest member of S, $p - 1 \notin S$ and $\frac{(p-1)}{q} < b$. Also,

$$a = b - (b - a) < \frac{p}{q} - (\frac{1}{q}) = \frac{(p-1)}{q}.$$

Therefore the rational number $\frac{(p-1)}{q}$ lies between a and b.
Case $a < 0$:
By the Archimedean Property, for $1, -a > 0$, there exists $n \in \mathbb{N}$ for which $n(1) > -a$, which implies $n + a > 0$, and $b > a$ implies $n + b > n + a > 0$. Then we can use the first case to show that there exists a rational number r such that $n + a < r < n + b$. Therefore the rational number $r - n$ lies between a and b. ☐

PROBLEMS

8. Use an induction argument to show that for each natural number n, the interval $(n, n+1)$ fails to contain any natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(n, n+1) \cap \mathbb{N} = \emptyset$.

   $P(1)$: $(1, 2) = \{x \mid 1 < x < 2\}$. Suppose there exists a natural number $q \in (1, 2)$. Then $q > 1$ and by problem 9, $q - 1$ is a natural number. However, $1 < q < 2 \implies 0 < q - 1 < 1$, which is a contradiction to the fact that the natural numbers are bounded below by 1 (Theorem 1). Therefore there are no natural numbers in $(1, 2)$.

   Suppose $P(k)$ is true for some natural number k.

   $P(k+1)$: Suppose there exists a natural number $p \in (k+1, (k+1)+1)$; that is, $k+1 < p < k+2$.

   Case $p = 1$: then $k + 1 < 1 < k + 2$. but $k \in \mathbb{N}$ so $k + 1 > 1$. Thus $p = 1$ is not possible.

   Case $p > 1$: then by problem 9, $p - 1 \in \mathbb{N}$, so $k + 1 < p < k + 2 \implies k < p - 1 < k + 1$. This is a contradiction to $P(k)$, the assumption that there are no natural numbers between $(k, k + 1)$. Therefore $P(k + 1)$ is true.

9. Use an induction argument to show that if $n > 1$ is a natural number, then $n - 1$ also is a natural number. The use another induction argument to show that if m and n are natural numbers with $n > m$, then $n - m$ is a natural number.

   For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $n = 1$ or $n - 1 \in \mathbb{N}$.

   $P(1)$: $1 = 1$, true.

   Suppose $P(k)$ is true for some $k \in \mathbb{N}$.

   $P(k + 1)$: $(k + 1) - 1 = k \in \mathbb{N}$, true.

For $n \in \mathbb{N}$, let $Q(n)$ be the assertion that for all $m \in \mathbb{N}$ such that $n > m$, then $n - m \in \mathbb{N}$.

$Q(1)$: true trivially, because there are no natural numbers less than 1.

Suppose $Q(k)$ is true for some $k \in \mathbb{N}$; that is, for all $m \in \mathbb{N}$ such that $k > m$, then $k - m \in \mathbb{N}$.

$Q(k+1)$: For all the m from $Q(k)$, we have $(k+1) > k > m$.

We want to show that $(k+1) - m \in \mathbb{N}$.

This is clearly true for $m = 1$ because $(k+1) - 1 = k \in \mathbb{N}$. Otherwise, $m > 1$, so by $P(m)$, $m - 1 \in \mathbb{N}$ and $(k+1) - m = k - (m-1)$. $Q(k)$ is true tells us that because $(m-1) \in \mathbb{N}$ and $k > m > m - 1$, then $k - (m-1) \in \mathbb{N}$. Therefore $Q(k+1)$ is true.

10. Show that for any real number r, there is exactly one integer in the interval $[r, r+1)$.

    This is trivial if $r \in \mathbb{Z}$.

    Consider the smallest integer p less than $[r, r+1)$. Then $p < r < r+1$ (and $r < p+1$, because $r = p+1 \implies r \in \mathbb{Z}$ and $r > p+1 \implies p$ is not the smallest integer less than $[r, r+1)$), therefore $r < p+1 < r+1$. Because the integers are inductive, $p+1 \in \mathbb{Z}$.

    To show that there is not more than one integer between $[r, r+1)$: let q be a natural number such that $r \le q < r+1$. Then $q - 1 < r \le q$ and $q < r+1 \le q+1$. From problem 8, we see that there are no integers between $(q-1, q)$ and $(q, q+1)$, so there is only one integer in $(q-1, q) \cup q \cup (q, q+1) \supseteq [r, r+1)$.

11. Show that any nonempty set of integers that is bounded above has a largest member.

    Let E be a nonempty set of integers that is bounded above. By the completeness axiom, there exists $c = \sup E$. That is, $x \le c$ for all $x \in E$. Then $c - 1 < z \le c$ for some $z \in E$ because $c - 1$ is not an upper bound of E. Suppose c is not in E. Then $c - 1 < z < c$. This implies that $c - 1 < z < w \le c$ for some $w \in E$ because z is not an upper bound of E. But then there exists two integers in the interval $(c-1, c]$, which is a contradiction to problem 10. Therefore c is an element of E, and it is the largest member.

12. Show that the irrational numbers are dense in $\mathbb{R}$.

    Choose any two real numbers a, b and any irrational number z. Then $\frac{a}{z}, \frac{b}{z}$ are real numbers. By density of the rationals in $\mathbb{R}$, there exists a rational r such that $\frac{a}{z} < r < \frac{b}{z}$. This implies $a < rz < b$, where rz is an irrational number.

    Proof that rz is irrational:
    Let $r = \frac{p}{q}$ and suppose that rz is rational; then $rz = \frac{m}{n}$.

    $$\begin{aligned} \frac{p}{q}z &= \frac{m}{n} \\ z &= \frac{m}{n}\frac{q}{p} \\ z &= \frac{mq}{np} \end{aligned}$$

    Then z is rational, a contradiction.

13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

Choose any real number a. Let $S = \{r \in \mathbb{Q} \mid r \leq a\}$. Then a is an upper bound for this set. To show that a is the supremum, suppose by contradiction that it is not. Then there exists $c \in \mathbb{R}$ such that $r \leq c < a$. However, the rational numbers are dense in $\mathbb{R}$, so there exists a rational between c and a, a contradiction to the assumption that c is an upper bound to S.

The same argument can easily be shown for the irrational numbers.

14. Show that if $r > 0$, then, for each natural number n, $(1 + r)^n \geq 1 + n \cdot r$.

    Let $r > 0$.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that $(1 + r)^n \geq 1 + n \cdot r$.

    $P(1)$: $(1 + r)^1 = 1 + 1 \cdot r$, true.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $(1 + r)^k \geq 1 + k \cdot r$.

    $P(k + 1)$:

    $(1+r)^{k+1} = (1+r)^k(1+r) \geq (1+kr)(1+r) = 1+kr+r+kr^2 > 1+kr+r = 1+(k+1)\cdot r.$

15. Use induction arguments to prove that for every natural number n,

    (i)
    $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

    $P(1)$: $\sum_{j=1}^{1} j^2 = 1 = \dfrac{1(1+1)(2+1)}{6}$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.
    $P(k + 1)$:

    $$\begin{aligned}\sum_{j=1}^{k+1} j^2 &= \sum_{j=1}^{k} j^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(2k^2+k+2k+1)}{6} + \frac{6(k^2+2k+1)}{6} \\ &= \frac{(2k^3+k^2+2k^2+k)+(6k^2+12k+6)}{6} \\ &= \frac{2k^3+9k^2+13k+6}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(k+2)(2(k+1)+1)}{6}.\end{aligned}$$

    (ii)
    $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

    $P(1)$: $a^3 = 1 = (1)^3$.
    Suppose $P(k)$ is true for $k \in \mathbb{N}$.

$P(k+1)$:

$$\begin{aligned}
1^3+2^3+\cdots+(k+1)^3 &= 1^3+2^3+\cdots+k^3+(k+1)^3\\
&= (1+2+\cdots+k)^2+(k+1)^3\\
&= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3\\
&= \frac{k^2(k+1)^2}{4}+\frac{4(k+1)^3}{4}\\
&= \frac{k^2(k+1)^2+(4k+4)(k+1)^2}{4}\\
&= \frac{(k^2+4k+4)(k+1)^2}{4}\\
&= \frac{(k+2)^2(k+1)^2}{2^2}\\
&= \left(\frac{(k+2)(k+1)}{2}\right)^2\\
&= \left(\frac{((k+1)+1)(k+1)}{2}\right)^2\\
&= \Big(1+2+\cdots+(k+1)\Big)^2.
\end{aligned}$$

(iii)

$$1+r+\cdots+r^n=\frac{1-r^{n+1}}{1-r} \text{ if } r\neq 1.$$

$P(1)$: $1+r^1=\dfrac{(1+r)(1-r)}{1-r}=\dfrac{1-r^2}{1-r}$.
Suppose $P(k)$ is true for $k\in\mathbb{N}$.
$P(k+1)$:

$$\begin{aligned}
1+r+\cdots+r^{k+1} &= 1+r+\cdots+r^k+r^{k+1}\\
&= \frac{1-r^{k+1}}{1-r}+r^{k+1}\\
&= \frac{1-r^{k+1}}{1-r}+\frac{(1-r)r^{k+1}}{1-r}\\
&= \frac{1-r^{k+1}+r^{k+1}-r^{(k+1)+1}}{1-r}\\
&= \frac{1-r^{(k+1)+1}}{1-r}.
\end{aligned}$$

1.3 Countable and Uncountable Sets

Two sets A and B are **equipotent** provided there exists a bijection between them.
A set E is **countable** if it is equipotent to a set of natural numbers.
For a countably infinite set X, we say that $\{x_n \mid n\in\mathbb{N}\}$ is an **enumeration** of X provided

$$X=\{x_n \mid n\in\mathbb{N}\} \text{ and } x_n\neq x_m \text{ if } n\neq m.$$

Theorem 3. *A subset of a countable set is countable. In particular, every set of natural numbers is countable.*

Corollary 4. *The following sets are countably infinite:*

(i) For each natural numbers n, the Cartesian product $\mathbb{N}^n = \mathbb{N} \times \cdots \times \mathbb{N}$.

(ii) The set of natural numbers $\mathbb{Q}$.

The rationals are countable: $\mathbb{Q} = \{0, \frac{1}{1}, -\frac{1}{1}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}, \frac{3}{1}, -\frac{3}{1}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \cdots\}$.

Corollary 6. *The union of a countable collection of countable sets is countable.*

An interval of real numbers is called degenerate if it is empty or contains a single member.

Theorem 7. *A nondegenerate interval of real numbers is uncountable.*

Proof. Let I be a nondegenerate interval of real numbers. Clearly I is not finite. Suppose I is countably infinite. Let $\{x_n \mid n \in \mathbb{N}\}$ be an enumeration of I. For each $n \in \mathbb{N}$, choose a nondegenerate compact subinterval $[a_n, b_n] \subseteq I$ such that $x_n \notin [a_n, b_n]$. Let the set of such intervals $\{[a_n, b_n]\}_{n=1}^{\infty}$ be descending: $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$ (That is, $a_n \leq a_{n+1} < b_{n+1} \leq b_n$.) Now, the nonempty set $E = \{a_n \mid n \in \mathbb{N}\}$ is bounded above by b_1. Then the Completeness Axiom implies that E has a supremum, say $x^* = \sup E$. Then for each n, $a_n \leq x^* \leq b_n$ because x^* is the supremum of E and each b_n is an upper bound for E. Therefore x^* belongs to $[a_n, b_n]$ for each n. But then x^* is an element of I and thus has an index $n_0 \in \mathbb{N}$ such that $x^* = x_{n_0}$. But $x^* \in [a_{n_0}, b_{n_0}]$, a contradiction. Therefore I is not countable. $\square$

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16. Show that the set $\mathbb{Z}$ of integers is countable.

    There exists a bijection $\phi : \mathbb{Z} \to \mathbb{N}$ with

$$\phi(x) = \begin{cases} 2x & \text{if } x > 0, \\ -2x + 1 & \text{if } x \leq 0. \end{cases}$$

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, 4, -4, \cdots\}$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \cdots\}$$

17. Show that a set A is countable iff there is an injective mapping of A to $\mathbb{N}$.

    ($\Longrightarrow$) Suppose A is countable.

    Then either A is equipotent to $\mathbb{N}$, or there is an $n \in \mathbb{N}$ such that A is equipotent to $\{1, 2, \cdots, n\}$. In the case A is countably infinite, we have a bijection with $\mathbb{N}$ and thus an injection. In the case A is finite, we have an injection with a subset of $\mathbb{N}$, and thus an injection with $\mathbb{N}$ (injection: $f(a) = f(b) \implies a = b$ for $a, b \in A$).

    ($\Longleftarrow$) Suppose there is an injective mapping of A to $\mathbb{N}$.

    Then there is a bijection from A to some subset B of $\mathbb{N}$. By Theorem 3, every subset of natural numbers is countable, and because A is equipotent to this countable set B, then A is countable.

18. Use an induction argument to complete the proof of part (i) of Corollary 4.

    (Not an induction argument)

    Consider the function $f : \mathbb{N}^2 \to \mathbb{N}$, where $f(m, n) = 2^m 3^n$. By the Fundamental Theorem of Arithmetic, $2^m 3^n = 2^{m'} 3^{n'} \implies m = m', n = n'$. Then clearly f is an injection. By problem 17, we see that $\mathbb{N}^2$ is countable.

    For any $k \in \mathbb{N}$ we can construct a function $f : \mathbb{N}^k \to \mathbb{N}$, where we have n primes such that $f(m_1, m_2, \cdots, m_k) = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$. By the fundamental theorem of arithmetic, this is an injection and thus $\mathbb{N}^k$ is countable.

19. Prove Corollary 6 in the case of a finite family of countable sets.

    Let $\{S_n\}_{n=1}^k$ be a finite family of countable sets. Then each set S_n is countable, and we can enumerate as follows: $S_n = \{s_{nm} \mid m \in \mathbb{N}\}$. Then because there is only a finite number of countable sets, we can construct a function $f : \bigcup_{n=1}^k S_n \to \mathbb{N}$ seeing that

$$\bigcup_{n=1}^{k} S_n = \{s_{11}, s_{21}, s_{31}, \cdots, s_{k1}, s_{12}, s_{22}, s_{32}, \cdots, s_{k2}, s_{13}, \cdots\}.$$

20. Let both $f : A \to B$ and $g : B \to C$ be injective and surjective. Show that the composition $g \circ f : A \to B$ and the inverse $f^{-1} : B \to A$ are also injective and surjective.

    $g \circ f$:

    By surjectivity of g, for all $c \in C$, there exists a $b \in B$ such that $g(b) = c$. Then by surjectivity of f, there exists an $a \in A$ such that $f(a) = b$.

    Therefore for any $c \in C$:

$$\begin{aligned} c &= g(b) && \text{for some } b \in B \\ &= g(f(a)) && \text{for some } a \in A \\ &= g \circ f(a) \end{aligned}$$

    Therefore $g \circ f$ is surjective.

    By injectivity of g, $g(b) = g(b') \implies b = b'$.

    By injectivity of f, $f(a) = f(a') \implies a = a'$.

$$\begin{aligned} g \circ f(a) &= g \circ f(a') \\ g(f(a)) &= g(f(a')) \\ f(a) &= f(a') && \text{by injectivity of } g \\ a &= a' && \text{by injectivity of } f \end{aligned}$$

    Therefore $g \circ f$ is injective.

    f^{-1}:

    Because f is a function from A to B, $f(a) \subseteq B$ is defined for all $a \in A$. That is, for all $a \in A$, there exists a $b \in B$ such that $f^{-1}(b) = a$. Thus f^{-1} is surjective.

    Because f is a function, for each $a \in A$, $f(a) = b$ and $f(a) = b'$ imply $b = b'$. That is, $f^{-1}(b) = f^{-1}(b') \implies b = b'$. Thus f^{-1} is injective.

21. Use an induction argument to establish the pigeonhole principle.

    For $n \in \mathbb{N}$, let $P(n)$ be the assertion that for any $m \in \mathbb{N}$, the set $\{1, 2, \cdots, n\}$ is not equipotent to the set $\{1, 2, \cdots, n+m\}$.

    $P(1)$: We have the sets $A = \{1\}$ and $B = \{1, 2, \cdots, 1+m\}$, for $m \in \mathbb{N}$. In the case $m = 1$, $B = \{1, 1+1\} = \{1, 2\}$, and clearly A is not equipotent to B. Clearly A is also not equipotent to B for any other natural number $m > 1$.

    Suppose $P(k)$ is true for some $k \in \mathbb{N}$. Then $\{1, 2, \cdots, k\}$ is not equipotent to the set $\{1, 2, \cdots, k+m\}$, for any $m \in \mathbb{N}$.

    $P(k+1)$: Then clearly $\{1, 2, \cdots, k+1\}$ is not equipotent to the set $\{1, 2, \cdots, (k+1), \cdots, (k+1)+m\}$, for any $m \in \mathbb{N}$.

22. Show that $2^{\mathbb{N}}$, the collection of all sets of natural numbers, is uncountable.

    (Cantor's Theorem: for a set A, any function $f : A \to \mathcal{P}(A)$ is not surjective.)

    Let $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be any map. Let $E = \{n \in \mathbb{N} \mid n \notin f(n)\}$. Then E is a subset of the naturals that is not in the image of f, so f is not surjective. Therefore there is no bijection between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

23. Show that the Cartesian product of a finite collection of countable sets is countable. Use the preceding theorem to show that $\mathbb{N}^{\mathbb{N}}$, the collection of all mappings of $\mathbb{N}$ into $\mathbb{N}$, is not countable.

    In problem 18, we showed that for any $k \in \mathbb{N}$, the set $\mathbb{N}^k = \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is countable. It is then trivial to see that the Cartesian product of any finite collection of countable sets $S_1 \times S_2 \times \cdots \times S_k$ is countable.

    Notation:
    $$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, 3 = \{0, 1, 2\}, \cdots$$
    We can let $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ be the set of functions $f : \mathbb{N} \to \{0, 1\}$.

    Then, for any subset $A \subseteq \mathbb{N}$, there exists a function $f \in \{0, 1\}^{\mathbb{N}}$ such that
    $$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{cases}$$
    and we have a bijection between the elements of $\{0, 1\}^{\mathbb{N}}$ and the subsets of $\mathbb{N}$ ("Two sets that are equipotent are, from a set-theoretic point of view, indistinguishable"). Therefore $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ can be used to represent the collection of subsets of $\mathbb{N}$.

    Now, because the set of functions $f : \mathbb{N} \to \{0, 1\}$ is uncountable, then clearly the set of functions $f : \mathbb{N} \to \mathbb{N} \supseteq \{0, 1\}$ is uncountable (including zero in the naturals for notation convenience).

24. Show that a nondegenerate interval of real numbers fails to be finite.

    Theorem 7 tells us that a nondegenerate interval of real numbers is uncountable, and thus, finite.

25. Show that any two nondegenerate intervals of real numbers are equipotent.

    We can prove this by showing that any interval is equipotent to the interval $(0, 1)$.

    For any bounded interval $(a, b), (a, b], [a, b), [a, b]$, there exists a bijection to $(0, 1), (0, 1], [0, 1), [0, 1]$ respectively, of the form $f(x) = \dfrac{1}{b-a}(x - a)$.

26. Is the set $\mathbb{R} \times \mathbb{R}$ equipotent to $\mathbb{R}$?

    yes (Schröder-Bernstein theorem)

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

Proposition 9. *Every nonempty open set is the union of a countable, disjoint collection of open intervals.*

The Heine-Borel Theorem. *Let F be a closed and bounded set of real numbers. Then every open cover of F has a finite subcover.*

Proof. Let F be the closed, bounded interval $[a, b]$. Let $\mathcal{F}$ be an open cover of $[a, b]$. Define E to be the set of numbers $x \in [a.b]$ with the property that the interval $[a, x]$ can be covered by a finite number of the sets of $\mathcal{F}$. Since $a \in [a, b] \subseteq \mathcal{F}$ implies that a is in one of the sets $\mathcal{O}' \subseteq \mathcal{F}$ by definition of union, $\mathcal{O}'$ is a finite subcover of $[a, a] = \{a\}$, and thus $a \in E$ and E is nonempty. Since $E \subseteq [a, b] = \{x \mid a \leq x \leq b\}$, E is bounded above by b, so by the completeness of $\mathbb{R}$, E has a supremum $c = \sup E$. Because $c \leq b$, clearly c belongs to $[a, b]$, and this implies that there is an $\mathcal{O} \subseteq \mathcal{F}$ that contains c. Since $\mathcal{O}$ is open, there is an $\epsilon > 0$ such that that the interval $(c-\epsilon, c+\epsilon) \subseteq \mathcal{O}$. Now $c-\epsilon$ is not an upper bound for E, and so there must be an $x \in E$ with $c - \epsilon < x$. Because $x \in E$, there exists a finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k\}$ of sets in $\mathcal{F}$ that covers $[a, x]$. Then clearly the finite collection $\{\mathcal{O}_1, \cdots, \mathcal{O}_k, \mathcal{O}\}$ covers the interval $[a, c + \epsilon)$. Therefore $c = b$, otherwise there exists a number $c + \frac{1}{2}\epsilon$ that has a finite subcover and $c < c + \frac{1}{2}\epsilon$ implies that c is not an upper bound for E. Thus $[a, b] \in E$ and $[a, b]$ can be covered by a finite number of sets of $\mathcal{F}$. □

The Heine-Borel Theorem ($\Longleftarrow$). *Let F be a real set such that every open cover of F has a finite subcover. Then F is closed and bounded.*

Proof. Let K be a compact subset of a metric space X. Proving that $X \setminus K$ is open will show that K is closed. Consider any $p \in X \setminus K$. For a $k \in K$, let O_k and I_k be neighborhoods of p and k respectively, with radius less than $\frac{1}{2}d(p, q)$. Because K is compact, there are finitely many points $k_1, \cdots, k_n$ in K such that $K \subseteq I_{k_1} \cup \cdots \cup I_{k_n}$. Let $O = O_{k_1} \cap \cdots \cap O_{k_n}$ so that O is an open neighborhood of p that does not intersect K. Then $O \subseteq X \setminus K$ and $X \setminus K$ is open. Therefore K is closed. □

The Nested Set Theorem. *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof. By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$, and we have an open cover of $\mathbb{R}$ and thus an open cover of $F_1 \subseteq \mathbb{R}$. By the Heine-Borel Theorem, there exists an $N \in \mathbb{N}$ such that $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c$. Because $\{F_n\}$ is descending, $F_n \supseteq F_{n+1}$ for any $n \geq 1$. This implies $F_n^c \subseteq F_{n+1}^c$, and thus $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c = F_N^c = \mathbb{R} \setminus F_N$. This is a contradiction to the assumption that F_N is a nonempty subset of F_1. □

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27. Is the set of rational numbers open or closed?

    The set of rationals is neither open nor closed. The rationals is not open because the irrationals are dense in the rationals; that is, between any two rationals there is an irrational. The rationals is not closed because it does not contain all its limit points; a sequence of rationals can be constructed that converges to an irrational. (Thus we see that the irrationals is neither open nor closed as well.)

28. What are the sets of real numbers that are both open and closed?

    It is clear that $\mathbb{R}$ is open, and $\emptyset$ is open (vacuously). Then because the complement of an open set is closed, $\mathbb{R}$ and $\emptyset$ are both closed as well.

29. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

    Let $A = (4, 5)$ and $B = (5, 20)$. Then $(4, 5) \cap (5, 20) = \emptyset$ and $\overline{A} = [4, 5]$ and $\overline{B} = [5, 20]$ so $[4, 5] \cap [5, 20] = \{5\} \neq \emptyset$.

    Let $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$. Then $\mathbb{Q} \cap \mathbb{Q}^c = \emptyset$ and $\overline{A} = \mathbb{R}$ and $\overline{B} = \mathbb{R}$ so $\mathbb{R} \cap \mathbb{R} = \mathbb{R} \neq \emptyset$.

30. A point x is called an **accumulation point** of a set E provided it is a point of closure of $E \setminus \{x\}$.

    (i) Show that the set E' of accumulation points of E is a closed set.
    Then for $x \in E'$, every open interval that contains x also contains a point in $E \setminus \{x\}$.
    Suppose E' is not closed. Then there exists an element $y \notin E'$ such that every open interval that contains y also contains a point $x \in E'$. Then every open interval that contains x contains a point $z \in E \setminus \{x\}$... It can be shown that $y \in E'$ and so E' contains all its points of closure and is thus closed.

    (ii) Show that $\overline{E} = E \cup E'$.
    E includes all the isolated points not included in E'.

31. A point x is called an **isolated point** of a set E provided there is an $r > 0$ for which $(x - r, x + r) \cap E = \{x\}$. Show that if a set E consists of isolated points, then it is countable.

    Each singleton set $\{x\}$ can be enumerated.

32. A point x is called an **interior point** of a set E if there is an $r > 0$ such that the open interval $(x - r, x + r)$ is contained in E. The set of interior points of E is called the **interior** of E denoted by int E. Show that

    (i) E is open iff $E =$ int E.
    ($\Longrightarrow$) Suppose E is open.
    Then clearly every point of E is an interior point.
    ($\Longleftarrow$) Suppose $E =$ int E.
    Then every point has an open neighborhood contained in E, so E is open.

    (ii) E is dense iff int $(\mathbb{R} \setminus E) = \emptyset$.

33. Show that the nested set theorem is false if F_1 is unbounded.

    The nested set theorem works because the compactness of F_1 allows us to reach a contradiction to the fact that the intersection is empty (see the proof above).

    Consider
    $$\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset.$$
    This intersection is empty because for any x, there exists an $n \in \mathbb{N}$ such that $x < n$ and thus $x \notin [n, \infty)$.

34. Show that the assertion of the Heine-Borel Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Nested Set Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Heine-Borel Theorem States that Closed and bounded sets are compact; that is, every open cover of a closed and bounded set has a finite subcover. If a set E is bounded, then for any open cover $E \subseteq \mathcal{F}$ there exists a finite open subcover $\mathcal{O} \subseteq \mathcal{F}$. We can consider the intersection of all such $\mathcal{O}$ so that $E \subseteq \bigcap_{\mathcal{O} \subseteq \mathcal{F}} \mathcal{O} \subseteq \mathcal{O}$, and this intersection is the supremum.

    Clearly the descending sets from the nested set theorem are closed and bounded, so the Heine-Borel Theorem discussed above can be used to imply the Completeness Axiom.

35. Show that the collection of Borel sets is the smallest σ-algebra that contains the closed sets.

    The Borel sets is defined to be the smallest σ-algebra that contains all the open sets of real numbers. Any sigma-algebra that contains the closed sets contains the open sets by the complement property of a sigma-algebra, so the Borel sets is the smallest sigma-algebra that contains the closed sets as well.

36. Show that the collection of Borel sets is the smallest σ-algebra that contains the intervals of the form $[a, b)$, where $a < b$.

    Any interval $[a, b)$ can be written in the form

$$[a, b) = \bigcup_{n=1}^{\infty} [a, b - \tfrac{1}{n}]$$

37. Show that each open set is an F_σ set.

    Any open set (a, b) can be written in the form

$$(a, b) = \bigcup_{n=1}^{\infty} [a + \tfrac{1}{n}, b - \tfrac{1}{n}].$$

1.5 Sequences of Real Numbers

Proposition 14. *Let the sequence of real numbers $\{a_n\}$ converge to the real number a. Then the limit is unique, the sequence is bounded, and, for a real number c,*

$$\textit{if } a_n \leq c \textit{ for all } n, \textit{ then } a \leq c.$$

Proof. Suppose there exist a and b such that $\{a_n\} \to a$ and $\{a_n\} \to b$. Then For any $\epsilon > 0$, there exists the index $N = \max\{N_a, N_b\}$ such that for all $n \geq N \geq N_a, N_b$, then $|a - a_n| < \epsilon$ and $|b - a_n| < \epsilon$. By the triangle inequality, $|a - b| \leq |a - a_n| + |a_n - b| < \epsilon + \epsilon = 2\epsilon = \epsilon'$. Therefore $a = b$, and the limit is unique.

Consider $\epsilon = 1$. Then there exists an index $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < 1$. Also, $|a_n| - |a| \leq |a_n - a| < 1 \implies |a_n| < |a| + 1$. Let $M = \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\}$. The maximum exists because this is a finite set of real numbers (totally ordered). Considering any $n \in \mathbb{N}$, if $n \geq N$, then $|a_n - a| < 1 \implies |a_n| < |a| + 1 \leq M$, and if $n < N$, then $|a_n| \leq \max\{|a_1|, |a_2|, \cdots, |a_N|, |a| + 1\} = M$, so M is a bound for this sequence.

Suppose that for all n, $a_n \leq c$ but $a > c$. Then $a_n \leq c < a$ for all n, and $0 \leq c - a_n < a - a_n$. Choosing $\epsilon = c - a_n$, there exists an index such that $|a - a_n| < c - a_n$. But this is a contradiction. Therefore $a \leq c$. □

Theorem 15 (the Monotone Convergence Criterion for Real Sequences)**.** *A monotone sequence of real numbers converges iff it is bounded.*

Proof. ($\implies$) Suppose a monotone sequence converges.
By the above proposition, it is bounded.
($\impliedby$) Suppose a monotone sequence $\{a_n\}$ is bounded.
By the Completeness Axiom, there exists a supremum say a such that $a_n \leq a$ for all n. Consider any $\epsilon > 0$. Now, $a - \epsilon$ is not an upper bound, and because the sequence is increasing, there exists an index N for which $a_n \geq a_N > a - \epsilon$ for all $n \geq N$. Then $\epsilon > a - a_n$ and the sequence converges to a. The proof is the same for a decreasing sequence. □

Theorem 16 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let a_n be a bounded sequence of real numbers. Choose $M > 0$ s.t. $|a_n| \leq M$ for all n. Define $E_n = \overline{\{a_j \mid j \geq n\}}$. Then we also have $E_n \subseteq [-M, M]$ and E_n is closed since it is the closure of a set. Therefore $\{E_n\}$ is a descending sequence of nonempty closed bounded subsets of real numbers. The Nested Set Theorem tells us that $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$, so there exists $a \in \bigcap_{n=1}^{\infty} E_n$. For each natural number k, a is a point of closure of $\{a_j \mid j \geq k\}$. Thus for infinitely many indices $j \geq n$, a_j belongs to $(a - \frac{1}{k}, a + \frac{1}{k})$. By induction, choose a strictly increasing subsequence of natural numbers n_k such that $|a - a_{n_k}| < \frac{1}{k}$ for all k. From the Archimedean Property of the reals, the subsequence $\{a_{n_k}\}$ converges to a. □

Proposition 19. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.*

(i) $\limsup\{a_n\} = \ell \in \mathbb{R}$ *iff for each $\epsilon > 0$, there are infinitely many indices n for which $a_n > l - \epsilon$ and only finitely many indices n for which $a_n < l - \epsilon$.*

($\Longrightarrow$) *Suppose* $\limsup\{a_n\} = \ell \in \mathbb{R}$.

Then by problem 38, ℓ is a cluster point of the sequence. This means that for all $\epsilon > 0$, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all n_k greater than some index, and thus $\ell - \epsilon < a_{n_k}$ for infinitely many indices n_k.

Suppose by contradiction that for $\epsilon > 0$, there are infinitely many indices n for which $a_n < l - \epsilon$. That is, no matter how large the epsilon we choose, there exists a subsequence $\{a_{n_k}\}$ such that $\epsilon < l - a_{n_k}$ for all n_k after a certain index. This implies that $\{a_n\}$ is not bounded, so by Proposition 14, the sequence does not converge to a real number. This is a contradiction to $\ell \in \mathbb{R}$.

($\Longleftarrow$) *Suppose for $\epsilon > 0$, there are infinitely many indices n for which $a_n > l - \epsilon$ and only finitely many indices n for which $a_n < l - \epsilon$.*

Then choosing specific indices n_k, there exists a subsequence $\{a_{n_k}\}$ such that $\ell - a_{n_k} < \epsilon$ for all n_k, and this implies the subsequence converges to ℓ. If we suppose that $\ell \neq \limsup\{a_n\}$, then there exists some $\delta > 0$ such that $\ell > \ell - \delta = \limsup\{a_n\}$.

Now, $\ell - \delta = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. That means for any n, $a_k \leq \ell - \delta$ for $k \geq n$. However, this is a contradiction to the fact that there are only finitely many such indices k for which this is true. Therefore $\ell = \limsup\{a_n\}$.

(ii) $\limsup\{a_n\} = \infty$ *iff $\{a_n\}$ is not bounded above.*

($\Longrightarrow$) *Suppose* $\limsup\{a_n\} = \infty$.

This implies that $\infty = \limsup\{a_n\}$ is a cluster point and there exists a subsequence that converges to infinity. Therefore $\{a_n\}$ is not bounded above.

($\Longleftarrow$) *Suppose $\{a_n\}$ is not bounded above.*

By Proposition 4, $\{a_n\}$ does not converge to a real number. Also,$\{a_n\}$ is not bounded above implies that for any real number c, there exists an index such that $a_n > c$. Then the only upper bound of this sequence is ∞ and thus $\limsup\{a_n\} = \infty$.

(iii)

$$\limsup\{a_n\} = -\liminf\{-a_n\}.$$

Definitions of limsup and liminf:

$\limsup\{a_n\} = \lim_{n\to\infty}[\sup\{a_k \mid k \geq n\}] \implies$ *for any $n \in \mathbb{N}$, $\sup\{a_k \mid k \geq n\} \geq a_k$ for $k \geq n$.*

$\liminf\{a_n\} = \lim_{n\to\infty}[\inf\{a_k \mid k \geq n\}]$. $\implies$ *for any* $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k$ *for* $k \geq n$.
Now we have
$\liminf\{-a_n\} = \lim_{n\to\infty}[\inf\{-a_k \mid k \geq n\}]$.
$\implies$ *for any* $n \in \mathbb{N}$, $\inf\{-a_k \mid k \geq n\} \leq -a_k$ *for* $k \geq n$.
$\implies$ *for any* $n \in \mathbb{N}$, $-\inf\{-a_k \mid k \geq n\} \geq a_k$ *for* $k \geq n$*, the definition of limsup.*

(iv) A sequence of real numbers $\{a_n\}$ *converges to an extended real number* a *iff*

$$\liminf\{a_n\} = \limsup\{a_n\} = a.$$

($\implies$) *Suppose a sequence of real numbers* $\{a_n\}$ *converges to an extended real number* a.
Clearly $\liminf\{a_n\} \leq a \leq \limsup\{a_n\}$.
If $\liminf\{a_n\} < a < \sup\{a_n\}$*, then we reach a contradiction to the infimum and supremum respectively.*
Therefore $\liminf\{a_n\} = a = \limsup\{a_n\}$.
($\impliedby$) *Suppose* $\liminf\{a_n\} = \limsup\{a_n\} = a$.
Then for any $n \in \mathbb{N}$, $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ *for* $k \geq n$*, which implies*

$$a = \liminf\{a_n\} = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\} \leq \lim_{n\to\infty} a_k \leq \lim_{n\to\infty} \sup\{a_k \mid k \geq n\} = \limsup\{a_n\} = a$$

Clearly $\{a_n\}$ *converges to* a.

(v) If $a_n \leq b_n$ *for all* n*, then*

$$\limsup\{a_n\} \leq \limsup\{b_n\}.$$

For any $n \in \mathbb{N}$, $a_k \leq \sup\{a_k \mid k \geq n\}$ *and* $b_k \leq \sup\{b_k \mid k \geq n\}$ *for all* $k \geq n$.
If we suppose $\limsup\{a_n\} > \limsup\{b_n\}$*, then there exists a natural number* n *such that* $\sup\{a_k \mid k \geq n\} > \sup\{b_k \mid k \geq n\} \geq b_k \geq a_k$ *for all* $k \geq n$*. However, by problem 38, we see that* $\limsup\{a_n\}$ *is a cluster point of* $\{a_n\}$*, and we reach a contradiction. (or contradiction to def of supremum?)*

Proposition 20. *Let* $\{a_n\}$ *be a sequence of real numbers.*

(i) The series $\sum_{k=1}^{\infty} a_k$ *is summable iff for each* $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \text{ for } n \geq N \text{ and any natural number } m.$$

($\implies$) *Suppose the series* $\sum_{k=1}^{\infty} a_k$ *is summable.*
That is, there exists an s *such that* $\{\sum_{k=1}^{n} a_k\}$ *converges to* s*. Convergent sequences are Cauchy, so for any* $\epsilon > 0$*, there exists and index* N *such that for all* $n + m \geq n - 1 \geq N$*,*

$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$
$$\left|\sum_{k=1}^{n-1} a_k + \sum_{k=n}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$
$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon.$$

($\Longleftarrow$) *Suppose that for each* $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| < \epsilon \textit{ for } n \geq N \textit{ and any natural number } m.$$

Then

$$\left|\sum_{k=n}^{n+m} a_k + \sum_{k=1}^{n-1} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$
$$\left|\sum_{k=1}^{n+m} a_k - \sum_{k=1}^{n-1} a_k\right| < \epsilon$$

Without loss of generality, we can suppose that $n - 1 \geq N$*, and because* m *is a natural number,* $n + m > n - 1 \geq N$*. Clearly this describes a Cauchy Sequence, and because the real numbers is complete, this sequence converges and thus the series is summable.*

(ii) If the series $\sum_{k=1}^{\infty} |a_k|$ *is summable, then* $\sum_{k=1}^{\infty} a_k$ *is also summable.*

By subadditivity of absolute value, we can show that for each $\epsilon > 0$*, there is an index* N *for which*

$$\left|\sum_{k=n}^{n+m} a_k\right| \leq \left|\sum_{k=n}^{n+m} |a_k|\right| < \epsilon \textit{ for } n \geq N \textit{ and any natural number } m.$$

(iii) If each term a_k *is nonnegative, then the series* $\sum_{k=1}^{\infty} a_k$ *is summable iff the sequence of partial sums is bounded.*

Let $\{a_k\}$ *be a sequence of nonnegative numbers.*

($\Longrightarrow$) *Suppose the series* $\sum_{k=1}^{\infty} a_k$ *is summable.*

Then the sequence of partial sums converges to a real number. By Proposition 14, the sequence of partial sums is bounded.

($\Longleftarrow$) *Suppose the sequence of partial sums is bounded.*

Because each a_k *is positive, the sequence of partial sums is positive monotonic:*

$$\sum_{k=1}^{n} a_k < \sum_{k=1}^{n} a_k + a_{n+1} = \sum_{k=1}^{n+1} a_k.$$

Therefore by Theorem 15, the sequence of partial sums converges; that is, the series is summable.

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38. We call an extended real number a **cluster point** of a sequence $\{a_n\}$ if a subsequence converges to this extended real number. Show that $\liminf\{a_n\}$ is the smallest cluster point of $\{a_n\}$ and $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$.

    Let $s = \limsup\{a_n\} = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$. Suppose there exists a subsequence $\{a_{n_k}\}$ that converges to an extended real number a. Fix $\epsilon > 0$. Then there exists an index M such that $|a - a_{n_m}| < \epsilon$ when $n_m \geq M$, and $a_{n_m} \leq \sup\{a_k \mid k \geq M\}$.

    Then $\lim_{M\to\infty} a_{n_m} \leq \lim_{M\to\infty} \sup\{a_k \mid k \geq M\} \implies a \leq s$.

    Therefore $\limsup\{a_n\}$ is the largest cluster point of $\{a_n\}$. ($\limsup\{a_n\}$ is itself a cluster point else we reach a contradiction to the supremum.) The same method can be used to prove $\liminf\{a_n\}$.

39. Prove proposition 19.

    See above for proof.

40. Show that a sequence $\{a_n\}$ is convergent to an extended real number iff there is exactly one extended real number that is a cluster point of the sequence.

    ($\Longrightarrow$) Suppose $\{a_n\}$ is convergent to an extended real number a.

    By Proposition 19(iv), we have $\liminf\{a_n\} = \limsup\{a_n\} = a$, so clearly any cluster point is equal to a.

    ($\Longleftarrow$) Suppose there is exactly one extended real number a that is a cluster point of $\{a_n\}$.

    Then there exists a subsequence that converges to a. Suppose that $\{a_n\}$ does not converge to a. Then there exists an $\epsilon > 0$ such that there are infinitely many indices n for which $a - a_n > \epsilon$. Collect these indices to construct a subsequence $\{a_{n_k}\}$. In the case that $\{a_{n_k}\}$ is bounded, there exists another subsequence of $\{a_{n_k}\}$ that converges to a real number $b \neq a$. But this is also a subsequence of the original sequence $\{a_n\}$, which implies $\{a_n\}$ has two cluster points a and b, a contradiction. In the case that $\{a_{n_k}\}$ is unbounded, then for any real number c, there exists an index n such that $|a_n| > c$. Then we can construct a subsequence that converges to $+\infty \neq a$ or $-\infty \neq a$, which is again a contradiction to the fact that $\{a_n\}$ has only one cluster point.

41. Show that $\liminf a_n \leq \limsup a_n$.

    For any natural number n, we have $\inf\{a_k \mid k \geq n\} \leq a_k \leq \sup\{a_k \mid k \geq n\}$ for all $k \geq n$. Taking the limit with respect to n clearly proves the statement.

42. Prove that if, for all n, $a_n \geq 0$ and $b_n \geq 0$, then

    $$\limsup[a_n \cdot b_n] \leq (\limsup a_n) \cdot (\limsup b_n),$$

    provided the product on the right is not of the form $0 \cdot \infty$.

    For any natural number n, we can see that

    $$\{a_k \cdot b_k \mid k \geq n\} \subseteq \{a_i \cdot b_j \mid i, j \geq n\}.$$

    Then this clearly implies

    $$\begin{aligned}\sup\{a_k \cdot b_k \mid k \geq n\} &\leq \sup\{a_i \cdot b_j \mid i, j \geq n\} \\ &= \sup\{a_i \mid i \geq n\} \cdot \sup\{b_j \mid j \geq n\}.\end{aligned}$$

    Taking the limit on both sides proves the inequality.

43. Show that every real sequence has a monotone subsequence. Use this to provide another proof of the Bolzano-Weierstrass Theorem.

    Let $\{a_n\}$ be any sequence of real numbers. Supposing that there exist no monotone subsequences of $\{a_n\}$, then there are only finitely many indices n for which $a_n \leq a_{n+1}$, and only finitely many indices n for which $a_n \geq a_{n+1}$. Clearly we see a contradiction so there must exist a monotone subsequence.

    Now, in the case that $\{a_n\}$ is bounded, then the monotone subsequence $\{a_{n_k}\}$ is also bounded. By Theorem 15, $\{a_{n_k}\}$ converges. Thus $\{a_n\}$ has a convergent subsequence.

44. Let p be a natural number greater than 1, and x a real number $0 \leq x \leq 1$. Show that there is a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ for each n such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

and that this sequence is unique except when x is of the form q/p^n, $0 < q < p^n$, in which case there are exactly two such sequences. Show that, conversely, if $\{a_n\}$ is any sequence of integers with $0 \leq a_n < p$, the series

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to a real number x with $0 \leq x \leq 1$. If $p = 10$, this sequence is called the *decimal* expansion of x. For $p = 2$ it is called the *binary* expansion; and for $p = 3$, the *ternary* expansion.

For each $m \in \mathbb{N}$, we can construct a partial sum:

$$\sum_{n=1}^{m} \frac{a_n}{p^n} = \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m}$$

We choose each a_m in the following way:

(The $\sum_{n=1}^{m-1} \frac{a_n}{p^n}$ is a fixed value found from the previous iteration, so for each step, we are simply choosing the best a_m).

Case $\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} = x$ for some $a_m \in \{0, 1, \cdots, p\}$: Then set $a_k = 0$ for all $k \geq m$, and the equality is clear.

Else: Choose $a_m \in \{0, 1, \cdots, p\}$ such that:

$$\sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m}{p^m} < x < \sum_{n=1}^{m-1} \frac{a_n}{p^n} + \frac{a_m + 1}{p^m}.$$

In this way we can construct a monotone sequence (of partial sums) that is bounded above by x:

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq \sum_{n=1}^{k+1} \frac{a_n}{p^n} \leq x \text{ for all } k \in \mathbb{N}.$$

By showing that x is the supremum, we can apply Theorem 15 to show that this sequence of partial sums converges to its supremum:

$$\lim_{k \to \infty} \sum_{n=1}^{k} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = x.$$

Suppose that x is not the supremum. Then there exists an $\epsilon > 0$ such that $\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x$ for all k. Now, by the Archimedean Property, there exists a natural number m such that $\frac{1}{m} < \epsilon$; therefore $0 < \epsilon - \frac{1}{m}$. Now, because $p > 1$, there exists a natural number l such that $m < p^l$, so $0 < \frac{1}{p^l} < \frac{1}{m}$ and thus $x - (\epsilon - \frac{1}{p^l}) < x - (\epsilon - \frac{1}{m}) < x$.

Then for all natural numbers k,

$$\sum_{n=1}^{k} \frac{a_n}{p^n} \leq x - \epsilon < x - \epsilon + \frac{1}{p^l} < x - \epsilon + \frac{1}{m} < x.$$

However, there exists the natural number l such that

$$\sum_{n=1}^{l} \frac{a_n}{p^n} = \sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l}{p^l} \leq x - \epsilon < x - \epsilon + \frac{1}{p^l} < x$$
$$\sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l + 1}{p^l} \leq x - \epsilon + \frac{1}{p^l} < x.$$

This is a contradiction to our choice of a_l so that

$$\sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l}{p^l} < x < \sum_{n=1}^{l-1} \frac{a_n}{p^n} + \frac{a_l + 1}{p^l}.$$

Therefore x is the supremum, and the series $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is summable to x.

In the case that x is of the form q/p^n, the obvious solution would be to set $a_n = q$ (assuming q is an integer), and all other $a_k = 0$. The second solution would be to use the method described above.

For the converse, $0 \leq a_n \leq p - 1$ implies that

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n} \leq \sum_{n=1}^{\infty} \frac{p-1}{p^n} = (p-1) \sum_{n=1}^{\infty} \frac{1}{p^n}$$

Showing that $(p-1) \sum_{n=1}^{\infty} \frac{1}{p^n} < 1$ implies that $\sum_{n=1}^{k} \frac{a_n}{p^n}$ is a bounded, monotone sequence of partial sums, and therefore it converges to a number in $[0, 1]$.

Ex: $x = .547$; decimal expansion:

$$x = \frac{5}{10^1} + \frac{4}{10^2} + \frac{7}{10^3} + \frac{0}{10^4} + \frac{0}{10^5} + \cdots = .5 + .04 + .007 + 0 + 0 + \cdots$$

45. Prove Proposition 20.

    See above.

46. Show that the assertion of the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom for the real numbers. Show that the assertion of the Monotone Convergence Theorem is equivalent to the Completeness Axiom for the real numbers.

    The Bolzano-Weierstrass Theorem asserts that every bounded sequence of real numbers has a convergent subsequence.

    The Completeness Axiom asserts that every nonempty set of real numbers that is bounded above has a supremum.

    The Monotone Convergence Theorem asserts that a monotone sequence of real numbers converges iff it is bounded.

1.6 Continuous Real-Valued Functions of a Real Variable

Proposition 21. *A real-valued function f defined on a set E of real numbers is continuous at the point $x_* \in E$ iff whenever a sequence $\{x_n\}$ in E converges to x_*, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.*

Proof. Let f be a real-valued function defined on a set E.
($\Longrightarrow$) Suppose that f is continuous at the point $x_* \in E$.
Then for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\text{if } x' \in E \text{ and } |x - x'| < \delta, \text{ then } |f(x) - f(x')| < \epsilon.$$

Suppose that a sequence $\{x_n\}$ in E converges to x_*. Then for any $\delta > 0$, there exists an index N such that when $n \geq N$, $|x_* - x_n| < \delta$. Then by continuity of f, $|f(x_*) - f(x_n)| < \epsilon$, and thus the image sequence converges.
($\Longleftarrow$) Suppose that whenever a sequence $\{x_n\}$ in E converges to x_*, its image sequence $\{f(x_n)\}$ converges to $f(x_*)$.
That is, for any $\delta > 0$, there exists an index N such that $|x_* - x_n| < \delta$ whenever $n \geq N$, and this implies that for any $\epsilon > 0$, there exists an index M such that $|f(x_*) - f(x_n)| < \epsilon$ whenever $n \geq M$.
Thus continuity is clear. □

Proposition 22. *Let f be a real-valued function defined on a set E of real numbers. Then f is continuous on E iff for each open set $\mathcal{O}$,*

$$f^{-1}(\mathcal{O}) = E \cap \mathcal{U} \textit{ where } \mathcal{U} \textit{ is an open set.}$$

The Extreme Value Theorem. *A continuous real-valued function on a nonempty, closed, bounded set of real numbers takes a minimum and a maximum value.*

Proof. Let f be a continuous real-valued function on a nonempty, closed, bounded set E of real numbers. Suppose by contradiction that f is not bounded. Then for any $n \in \mathbb{N}$, there exists $x_n \in E$ such that $f(x_n) > n$. With this we can construct a sequence $\{x_n\}$ in E. Because E is bounded, $\{x_n\}$ is bounded, and so by the Bolzano Weierstrass Theorem, there exists a convergent subsequence $\{x_{n_k}\}$. Because f is continuous, $\{x_{n_k}\}$ is convergent implies $\{f(x_{n_k})\}$ is convergent. However, for each element in the image sequence, $f(x_{n_k}) > n_k$, and $\{f(x_{n_k})\}$ is unbounded, thus it cannot converge, and we reach a contradiction.
Because f is bounded, then it has a supremum s such that $f(x) \leq s$ for all $x \in E$. Suppose that f does not have a maximum. Then there is no $x \in E$ such that $f(x) = s$. Then $f(x) < s \implies f(x) \in (-\infty, s)$ for all $x \in E$. (We can use the fact that $(-\infty, s)$ is open $\implies f^{-1}(-\infty, s)$ is open): Then we reach a contradiction because E is closed. The same proof can be used for the minimum. □

The Intermediate Value Theorem. *Let f be a continuous real-valued function on the closed, bounded interval $[a, b]$ for which $f(a) < c < f(b)$. The there is a point x_0 in (a, b) at which $f(x_0) = c$.*

Theorem 23. *A continuous real-valued function on a closed, bounded set of real numbers is uniformly continuous.*

Proof. Let E be a closed, bounded set of real numbers, and let f be a continuous real-valued function on E.
Fix some $\epsilon > 0$.
By the continuity of f, for all $x \in E$, there exists $\delta_x > 0$ such that for $y \in E$ satisfying $|x - y| < 2\delta_x$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then we can construct an open cover of E consisting of the open balls $\mathbb{B}(x, \delta_x)$ for all $x \in E$.
Because E is compact, there exists a finite subcover $\{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$.

Let $\delta_* = \min\{\delta_1, \cdots, \delta_n\}$.
Consider $x, y \in E$ such that $|x - y| < \delta_*$.
Because $y \in E \subseteq \{\mathbb{B}(x_1, \delta_1), \cdots, \mathbb{B}(x_n, \delta_1)\}$, there exists an index $j \in \{1, \cdots, n\}$ such that $y \in \mathbb{B}(x_j, \delta_j)$; therefore

$$|x_j - y| < \delta_j < 2\delta_j.$$

By continuity of f, $|f(x_j) - f(y)| < \frac{\epsilon}{2}$. (A)
By the triangle inequality,

$$|x - x_j| \leq |x - y| + |y - x_j| < \delta_* + \delta_j \leq 2\delta_j.$$

By continuity of f, $|f(x) - f(x_j)| < \frac{\epsilon}{2}$. (B)
By the triangle inequality using (A) and (B):

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

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47. Let E be a closed set of real numbers and f a real-valued function that is defined and continuous on E. Show that there is a function g defined and continuous on all of $\mathbb{R}$ such that $f(x) = g(x)$ for each $x \in E$. (Hint: Take g to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)

    Because E is closed, then $\mathbb{R} \setminus E$ is open. In the case that $E = \mathbb{R}$, then $\mathbb{R} \setminus E = \emptyset$ and the conclusion is trivial. Else $\mathbb{R} \setminus E$ is nonempty. By proposition 9, $\mathbb{R} \setminus E$ is the union of a countable, disjoint collection of open intervals.
    In the case that $(-\infty, a)$ [or (a, ∞)] is in $\mathbb{R} \setminus E$, then $a \in E$ and $f(a)$ is defined. Simply let $g(x) = f(a)$ be the constant function on $(-\infty, a)$ [or (a, ∞)].
    In the case that $(a, b) \in \mathbb{R} \setminus E$, then $a, b \in E$ and $f(a)$,$f(b)$ are defined. Let

    $$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a) \text{ on } (a, b).$$

    Also let $g(x) = f(x)$ whenever $x \in E$. Then we see that g is continuous.

48. Define the real-valued function f on $\mathbb{R}$ by setting

    $$f(x) = \begin{cases} x & \text{if x irrational} \\ p \sin \dfrac{1}{q} & \text{if } x = \dfrac{p}{q} \text{ in lowest terms.} \end{cases}$$

    At what points is f continuous?

    See Thomae's Function for something similar.

    f should be discontinuous at each rational number and continuous at each irrational number.

49. Let f and g be continuous real-valued functions with a common domain E.

    (i) Show that the sum, $f + g$, and product, fg, are also continuous functions.
    Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{f(x_n)\}$ converges to $f(x)$ and $\{g(x_n)\}$ converges to $g(x)$ by continuity of f, g.

That is, for any $\epsilon > 0$, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(x_n) - f(x)| < \frac{\epsilon}{2}$ and $|g(x_n) - g(x)| < \frac{\epsilon}{2}$ whenever $|x_n - x| < \delta$. By the triangle inequality,

$$\begin{aligned}
|(f+g)(x_n) - (f+g)(x))| &= |(f(x_n) + g(x_n)) - (f(x) + g(x))| \\
&\leq |f(x_n) - f(x)| + |g(x_n) + g(x)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Fix any $\epsilon > 0$. By continuity of f, g, there exists a $0 < \delta \leq \delta_f, \delta_g$ such that $|f(y) - f(x)| < \frac{\epsilon}{2|g(x)|}$ and $|g(y) - g(x)| < \frac{\epsilon}{2|f(y)|}$ whenever $|y - x| < \delta$.

$$\begin{aligned}
|fg(y) - fg(x)| &= |f(y)g(y) - f(x)g(x)| \\
&= |f(y)g(y) - f(y)g(x) + f(y)g(x) - f(x)g(x)| \\
&= |f(y)(g(y) - g(x)) + g(x)(f(y) - f(x))| \\
&\leq |f(y)(g(y) - g(x))| + |g(x)(f(y) - f(x))| \\
&= |f(y)||g(y) - g(x)| + |g(x)||f(y) - f(x)| \\
&< |f(y)|\frac{\epsilon}{2|f(y)|} + |g(x)|\frac{\epsilon}{2|g(x)|} \\
&= \epsilon
\end{aligned}$$

(This one is a bit janky).

(ii) If h is a continuous function with image contained in E, show that the composition $f \circ h$ is continuous.

Suppose $\{x_n\} \in E$ converges to $x \in E$. Then $\{h(x_n)\} \in E$ converges to $h(x) \in E$ by continuity of h. Then $\{f \circ h(x_n)\} = \{f(h(x_n))\}$ converges to $f \circ h(x) = f(h(x))$ by continuity of f. Therefore the composition is continuous.

(iii) Let $\max\{f, g\}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, for $x \in E$. Show that $\max\{f, g\}$ is continuous.

Fix $\epsilon > 0$. By continuity of f, g, there exists a $0 < \delta \leq \delta_f, \delta_g$ s.t. whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$ and $|g(x) - g(y)| < \frac{\epsilon}{2}$.

We can write

$$\max\{f(x), g(x)\} = \frac{f(x) + g(x)}{2} + \frac{|f(x) - g(x)|}{2}.$$

This is by the identity:

$$\begin{aligned}
\max(x, y) + \min(x, y) &= x + y \\
\max(x, y) - \min(x, y) &= |x - y| \\
\max(x, y) &= \frac{1}{2}(x + y + |x - y|) \\
\min(x, y) &= \frac{1}{2}(x + y - |x - y|)
\end{aligned}$$

Now, $|\max\{f(x), g(x)\} - \max\{f(y), g(y)\}|$ is equal to

$$\begin{aligned}
&\left|\frac{f(x)+g(x)}{2}+\frac{|f(x)-g(x)|}{2}-(\frac{f(y)+g(y)}{2}+\frac{|f(y)-g(y)|}{2})\right| \\
&= \left|\frac{f(x)-f(y)+g(x)-g(y)+|f(x)-g(x)|-|f(y)-g(y)|}{2}\right| \\
&\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|\,|f(x)-g(x)|-|f(y)-g(y)|\,|}{2} \\
&\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-g(x)-f(y)+g(y)|}{2} \\
&\leq \frac{|f(x)-f(y)|+|g(x)-g(y)|+|f(x)-f(y)|+|g(y)+g(x)|}{2} \\
&< \frac{\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}+\frac{\epsilon}{2}}{2} \\
&= \epsilon.
\end{aligned}$$

(iv) Show that $|f|$ is continuous.
For any $\epsilon > 0$, there exists a delta such that whenever $|x-y| < \delta$, by the reverse triangle inequality:

$$|\,|f(x)| - |f(y)|\,| \leq |f(x) - f(y)| < \epsilon.$$

50. Show that a Lipschitz function is uniformly continuous but there are uniformly continuous functions that are not Lipschitz.

Lipschitz: there exists $L \geq 0$ s.t. for all x, x':

$$|f(x) - f(x')| \leq L|x - x'|$$

Fixing any $\epsilon > 0$, whenever $|x - x'| \leq \delta$, we have

$$|f(x) - f(x')| \leq L|x - x'| < L\delta,$$

so we can set $\delta = \dfrac{\epsilon}{L}$. The δ is the same for any values of x, so f is uniformly continuous.

The function $\sqrt{x}$ is uniformly continuous but not Lipschitz.

51. A continuous function ϕ on $[a, b]$ is called **piecewise linear** provided there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ for which ϕ is linear on each interval $[x_i, x_{i+1}]$. Let f be a continuous function on $[a, b]$ and ϵ a positive number. Show that there is a piecewise linear function ϕ on $[a, b]$ with $|f(x) - \phi(x)| < \epsilon$ for all $x \in [a, b]$.
Start with $f(x_0)$, and choose x_1 so that $f(x_1) = f(x_0) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0) + f(x_0)$ on $[x_0, x_1]$.
Repeat this process to choose each interval:
Start with $f(x_i)$, and choose x_{i+1} so that $f(x_{i+1}) = f(x_i) \pm \epsilon$.
Define $\phi(x) = \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i}(x - x_i) + f(x_i)$ on $[x_i, x_{i+1}]$.
Then we see that f and ϕ are always within ϵ of each other, and ϕ is continuous and piecewise linear.

52. Show that a nonempty set E of real numbers is closed and bounded if and only if every continuous real-valued function on E takes a maximum value.

Let E be a nonempty set of real numbers.
($\Longrightarrow$) Suppose E is closed and bounded.
By the Extreme Value Theorem, every continuous real-valued function on E takes a maximum (and minimum) value.
($\Longleftarrow$) Suppose every continuous real-valued function on E takes a maximum value.
Suppose that E is not closed. The continuous real-valued function $f(x) = \frac{1}{x}$ on the open set $E = (0, 1)$ does not take on a maximum value. Contradiction.
Suppose E is not bounded. The continuous real-valued function $f(x) = x^2$ on the unbounded set $E = [0, \infty)$ does not take on a maximum value. Contradiction.
Therefore E must be closed and bounded. (Not the right way to do this...)

53. Show that a set E of real numbers is closed and bounded iff every open cover of E has a finite subcover.

    Let E be a set of real numbers.

    ($\Longrightarrow$) Suppose E is closed and bounded.

    By the Heine-Borel Theorem, every open cover of E has a finite subcover.

    ($\Longleftarrow$) Suppose every open cover of E has a finite subcover.

    See the proof in 1.4 after the Heine-Borel Theorem.

54. Show that a nonempty set E of real numbers is an interval iff every continuous real-valued function on E has an interval as its image.
    Let E be a nonempty set of real numbers.
    ($\Longrightarrow$) Suppose E is an interval.
    Then for any two points $x, y \in E$, the set $[x, y]$ is in E. Let f be a continuous real-valued function on E. Now, we have $f(x), f(y)$ in the image of f. Suppose, without loss of generality, that $f(x) < f(y)$. By the Intermediate Value Theorem, for any c such that $f(x) < c < f(y)$, there exists $x_0 \in (x, y) \subseteq E$ such that $f(x_0) = c$. That is, for any two points in the image of f, every point between them is also in the image of f. Therefore the image of f is an interval.
    ($\Longleftarrow$) Suppose every continuous real-valued function on E has an interval as its image.
    Suppose E is not an interval. Then there exist two points $x, y \in E$ such that $x < a < y$ but $a \notin E$. Let f be a continuous real-valued function on E, and without loss of generality, let f be monotonically increasing. Because $x, y \in E$, then $f(x)$,$f(y)$ are defined, so $[f(x), f(y)]$ is in the image of f.
    Define two disjoint collections of subsets of E: $I_{<a} = \{I \subseteq E \mid x < a\ \forall x \in I\}$ and $I_{>a} = \{I \subseteq E \mid x > a\ \forall x \in I\}$, so that $I_{<a} \cap I_{>a} = \emptyset$. These collections are nonempty because $\{x\} \in I_{<a}$ and $\{y\} \in I_{>a}$. Consider $\bigcup I_{<a} \subseteq E$, the union of all elements of $I_{<a}$, and $\bigcup I_{>a} \subseteq E$, the union of all elements of $I_{>a}$. By monotonicity of f, $f(\bigcup I_{<a}) < f(\bigcup I_{>a})$, so $[f(x), f(y)] \not\subseteq f(\bigcup I_{<a}) \bigcup f(\bigcup I_{>a}) = f(E)$, a contradiction.

55. Show that a monotone function on an open interval is continuous iff its image is an interval.
    Let f be a monotone function on an open interval $E = (a, b)$.
    ($\Longrightarrow$) Suppose f is continuous.
    Then by Problem 54, E is an interval implies that the continuous real-valued function f has an interval as its image.
    ($\Longleftarrow$) Suppose the image of f is an interval.
    Let x_0 be a point in the open interval E, so that $f(x_0)$ is defined. For any sequence $\{x_n\}$ in $E \cap (x_0, \infty)$ that converges to x_0, then $\{f(x_n)\}$ converges to $f(x_0^+)$.
    Similarly, for any sequence $\{x_n\}$ in $E \cap (-\infty, x_0)$ that converges to x_0, then $\{f(x_n)\}$ converges to$f(x_0^-)$.

Then $f(x_0^-) = f(x_0) = f(x_0^+)$ by monotonicity.
(messy)

56. Let f be a real-valued function defined on $\mathbb{R}$. Show that the set of points at which f is continuous is a G_δ set.
A G_δ set is a set that is a countable intersection of open sets.
f is continuous at a point x if for any open set in the image containing $f(x)$, the inverse image is an open set containing x.

57. Let $\{f_n\}$ be a sequence of continuous functions defined on $\mathbb{R}$. Show that the set of points x at which the sequence $\{f_n(x)\}$ converges to a real number is the intersection of a countable collection of F_σ sets.
An F_σ set is a set that is a countable union of closed sets.

58. Let f be a continuous real-valued function on $\mathbb{R}$. Show that the inverse image with respect to f of an open set is open, of a closed set is closed, and of a Borel set is Borel.
The inverse image of an open set is open (See prop 22).
Suppose that the inverse image of a closed set is not closed. That is, let B be a closed set of real numbers and let $f^{-1}(B) = \{x \in \mathbb{R} \mid f(x) \in B\}$ not be closed. Then there exists a sequence $x_n \in f^{-1}(B)$ that converges to $x \notin f^{-1}(B)$. However, by continuity of f, $f(x_n) \in B$ converges to $f(x) \notin B$. This implies that B does not contain all its limit points, and thus B is not closed, a contradiction. Therefore $f^{-1}(B)$ must be closed.

59. A sequence $\{f_n\}$ of real-valued functions defined on a set E is said to converge uniformly on E to a function f iff given $\epsilon > 0$, there is an N such that for all $x \in E$ and all $n \geq N$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\{f_n\}$ be a sequence of continuous functions defined on a set E. Prove that if $\{f_n\}$ converges uniformly to f on E, then f is continuous on E.
We want to show that uniform convergence preserves continuity.
Fix $\epsilon > 0$.
By uniform convergence of $\{f_n\}$, there exists an index N such that $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ for all $x \in E$ and all $n \geq N$.
By continuity of each f_n, for all $x \in E$, there exists a $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$ whenever $|x - y| < \delta$.
Therefore we have:
$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Chapter 2

Lebesgue Measure

Contents

2.1 Introduction

In this chapter we construct a collection of sets called **Lebesgue measurable sets**, and a set function of this collection called **Lebesgue measure**, denoted by m. The collection of Lebesgue measurable sets is a σ-algebra which contains all open sets and all closed sets. The set function m possesses the following three properties:

The measure of an interval is its length. *Each nonempty interval I is Lebesgue measurable and*

$$m(I) = \ell(I).$$

Measure is translation invariant. *If E is Lebesgue measurable and y is any number then the translate of E by y, $E + y = \{x + y \mid x \in E\}$, also is Lebesgue measurable and*

$$m(E + y) = m(E).$$

Measure is countably additive over countable disjoint unions of sets. *If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of Lebesgue measurable sets, then*

$$m(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} m(E_k).$$

It is not possible to construct a set function that possesses the above three properties and is defined for all sets of real numbers (See Vitali sets). We first construct a set function called **outer measure**, denoted by m^*, such that:

(i) the outer measure of an interval is its length,

(ii) outer measure is translation invariant,

(iii) outer measure is countably subadditive.

Then the Lebesgue measure m is the restriction of m^* to the Lebesgue measurable sets.

PROBLEMS

In the first three problems, let m be a set function defined for all sets in a σ-algebra $\mathcal{A}$ with values in $[0, \infty]$. Assume m is countably additive over countable disjoint collections of sets in $\mathcal{A}$.

1. Prove that if A and B are two sets in $\mathcal{A}$ with $A \subseteq B$, then $m(A) \leq m(B)$. This property is called *monotonicity*.

   $A \subseteq B \implies B = A \cup (B \cap A^c)$, where $A \cap (B \cap A^c) = \emptyset$. The set $(B \cap A^c)$ is measurable because A^c is measurable and countable intersection is measurable, so $m(B) = m(A \cup (B \cap A^c)) = m(A) + m(B \cap A^c)$ by countable additivity, and thus $m(B) \geq m(A)$.

2. Prove that if there is a set A in the collection $\mathcal{A}$ for which $m(A) < \infty$, then $m(\emptyset) = 0$.

   We have $A \cap \emptyset = \emptyset$ and $A \cup \emptyset = A$.

   $$\begin{aligned} m(A) &= m(A \cup \emptyset) \\ m(A) &= m(A) + m(\emptyset) && \text{by disjoint additivity} \\ 0 &= m(\emptyset). \end{aligned}$$

3. Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets in $\mathcal{A}$. Prove that $m(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m(E_k)$. For any two measurable sets A, B, we have $A \cup B = (A \setminus B) \cup (B)$. By disjoint additivity,

   $$m(A \cup B) = m(A \setminus B) + m(B)$$

   Now, by problem 1, $(A \setminus B) \subseteq A$ implies that $m(A \setminus B) \leq m(A)$. Therefore

   $$m(A \cup B) \leq m(A) + m(B).$$

4. A set function c, defined on all subsets of $\mathbb{R}$, is defined as follows. Define $c(E)$ to be ∞ if E has infinitely many members and $c(E)$ to be equal to the number of elements in E if E is finite; define $c(\emptyset) = 0$. Show that c is a countably additive and translation invariant set function. This set function is called the **counting measure**.
   Suppose $E = \{x_1, \cdots, x_n\}$.
   Then $m(E) = n$. For any real number y, $y + E = \{y + x_1, \cdots, y + x_n\}$, so $m(y + E) = n$.
   Suppose E has infinitely many members.
   Then $y + E$ has infinitely members as well, so $m(E) = m(y + E) = \infty$.
   Let $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of sets of real numbers. In the case that there exists an E_k with infinitely many members, then the countable additivity is clear.
   In the case that all sets E_k are finite, for any two sets E_i, E_j:
   $E_i = \{x_1, \cdots, x_n\}$
   $E_j = \{y_1, \cdots, y_m\}$
   Then $E_i \cup E_j = \{x_1, \cdots, x_n, y_1, \cdots, y_m\}$ and $m(E_i \cup E_j) = n + m = m(E_i) + m(E_j)$.

2.2 Lebesgue Outer Measure

PROBLEMS

5. By using properties of outer measure, prove that the interval $[0, 1]$ is not countable.

6. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

7. A set of real numbers is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Show that for any bounded set E, there is a G_δ set G for which
$$E \subseteq G \text{ and } m^*(G) = m^*(E).$$

8. Let B be the set of rational numbers in the interval $[0, 1]$, and let $\{I_k\}_{k=1}^n$ be a finite collection of open intervals that covers B. Prove that $\sum_{k=1}^n m^*(I_k) \geq 1$.

9. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

10. Let A and B be bounded sets for which there is an $\alpha > 0$ such that $|a-b| \geq \alpha$ for all $a \in A, b \in B$. Prove that $m^*(A \cup B) = m^*(a) + m^*(B)$.

2.3 The σ-Algebra of Lebesgue Measurable Sets

PROBLEMS

11. Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a, ∞), then it contains all intervals.

12. Show that every interval is a Borel set.

13. Show that
    (i) the translate of an F_σ set is also F_σ,
    (ii) the translate of a G_δ set is also G_δ,
    (iii) the translate of a set of measure zero also has measure zero.

14. Show that if a set E has positive outer measure, then there is a bounded subset of E that also has positive outer measure.

15. Show that if E has finite measure and $\epsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ϵ.

2.4 Outer and Inner Approximation of Lebesgue Measurable Sets

Theorem 11. *Let E be any set of real numbers. Then each of the following four assertions is equivalent to the measurability of E.*
(Outer Approximation by Open Sets and G_δ sets)

(i) For each $\epsilon > 0$, there is an open set $\mathcal{O}$ containing E for which $m^*(\mathcal{O} \setminus E) < \epsilon$.

(ii) There is a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

(Inner Approximation by Closed Sets and F_σ sets)

(iii) For each $\epsilon > 0$, there is a closed set F contained in E for which $m^*(E \setminus F) < \epsilon$.

(iv) There is a F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

PROBLEMS

16. Complete the proof of Theorem 11 by showing that measurability is equivalent to (iii) and also equivalent to (iv).

17. Show that a set E is measurable iff for each $\epsilon > 0$, there is a closed set F and open set $\mathcal{O}$ for which $F \subseteq E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \setminus F) < \epsilon$.

18. Let E have finite outer measure. Show that there is a G_δ set $G \supseteq E$ with $m(G) = m^*(E)$. Show that E is measurable iff there is an F_σ set $F \subseteq E$ with $m(F) = m^*(E)$.

19. Let E have finite outer measure.

2.5 Countable Additivity, Continuity, and the Borel-Cantelli Lemma

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The L^p Spaces: Completeness and Approximation

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7.1 Normed Linear Spaces

Throughout this chapter E denotes a measurable set of real numbers. Define $\mathcal{F}$ to be the collection of all measurable extended real-valued functions on E that are finite a.e. on E. We can say that two functions $f, g \in \mathcal{F}$ are equivalent, denoted by $f \cong g$, provided

$$f(x) = g(x) \text{ for almost all } x \in E.$$

This is an equivalence relation and induces a partition of $\mathcal{F}$ into a disjoint collection of equivalence classes, denoted by $\mathcal{F}/\cong$, which is a linear space. There is a natural family $\{L^p(E)\}_{1 \leq p \leq \infty}$ of subspaces of $\mathcal{F}/\cong$.
For $1 \leq p < \infty$, define $L^p(E)$ to be the collection of equivalence class $[f]$ for which

$$\int_E |f|^p < \infty.$$

Then if $f \cong g$, then $\int_E |f|^p = \int_E |g|^p$. Showing that $L^p(E)$ is closed under linear combinations will prove that $L^p(E)$ is a linear subspace. To do this, let $c = \max\{|a|, |b|\}$ so that

$$|a + b| \leq |a| + |b| \leq 2c,$$

which implies

$$|a + b|^p \leq 2^p c^p \leq 2^p(|a|^p + |b|^p).$$

This inequality, together with the linearity and monotonicity of integration tells us that

$$\int_E |\alpha f + \beta g|^p \leq 2^p(|\alpha|^p \int_E |f|^p + |\beta|^p \int_E |g|^p) < \infty.$$

That is, for $[f], [g] \in L^p(E)$, then $\alpha[f] + \beta[g] \in L^p(E)$.
We call a function $f \in \mathcal{F}$ **essentially bounded** provided there is some $M \geq 0$, called an **essential upper bound** for f, for which

$$|f(x)| \leq M \text{ for almost all } x \in E.$$

Then we can define $L^\infty(E)$ to be the collection of equivalence classes $[f]$ for which f is essentially bounded. Clearly $L^\infty(E)$ is a linear subspace because

$$|\alpha f(x) + \beta g(x)| \leq |\alpha||f(x)| + |\beta||g(x)| \leq |\alpha|M + |\beta|M' = M'' \text{ a.e. on } E$$

To state that a function f in $L^p[a, b]$ is continuous means that there is a continuous function that agrees with f a.e. on $[a, b]$. There is only one such continuous function and it is often convenient to consider this unique continuous function as the representative of $[f]$.
It is useful to consider real-valued functions that have as their domain linear spaces of functions: such functions are called **functionals**.

Definition. *Let X be a linear space. A real-valued functional $\|\cdot\|$ on X is called a* ***norm*** *provided for each f and g in X and each real number α,*
(The Triangle Inequality)

$$\|f + g\| \leq \|f\| + \|g\|,$$

(Positive Homogeneity)

$$\|\alpha f\| = |\alpha|\|f\|,$$

(Nonnegativity)

$$\|f\| \geq 0 \textit{ and } \|f\| = 0 \iff f = 0.$$

A **normed linear space** is a linear space together with a norm. If X is a linear space normed by $\|\cdot\|$ we say that a function f in X is a **unit function** provided $\|f\| = 1$. For any $f \in X, f \neq 0$, the function $\frac{f}{\|f\|}$ is a unit function: it is a scalar multiple of f which we call the **normalization** of f.

Example (The Normed Linear Space $L^1(E)$)**.** *For a function f in $L^1(E)$, define*

$$\|f\|_1 = \int_E |f|.$$

Then $\|\cdot\|$ is a norm on $L^1(E)$.

Example (The Normed Linear Space $L^\infty(E)$)**.**

PROBLEMS

1. For f in $C[a, b]$, Define

$$\|f\|_1 = \int_a^b |f|.$$

Show that this is a norm on $C[a, b]$. Also show that there is no number $c \geq 0$ for which

$$\|f\|_{\max} \leq c\|f\|_1 \text{ for all } f \text{ in } C[a, b],$$

but there is a $c \geq 0$ for which

$$\|f\|_1 \leq c\|f\|_{\max} \text{ for all } f \text{ in } C[a, b].$$

2. Let X be the family of all polynomials with real coefficients defined on $\mathbb{R}$. Show that this is a linear space. For a polynomial p, define $\|p\|$ to be the sum of the absolute values of the coefficients of p. Is this a norm?

3. For f in $L^1[a,b]$, define $\|f\| = \int_a^b x^2 |f(x)| dx$. Show that this is a norm on $L^1[a,b]$.

4. For f in $L^\infty[a,b]$, show that

$$\|f\|_\infty = \min\left\{ M \,\middle|\, m\{x \in [a,b] \mid |f(x)| > M\} = 0 \right\},$$

and if, furthermore, f is continuous on $[a,b]$, that

$$\|f\|_\infty = \|f\|_{\max}.$$

5. Show that ℓ^∞ and ℓ^1 are normed linear spaces.

7.2 The Inequalities of Young, Hölder, and Minkowski

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II ABSTRACT SPACES: METRIC, TOPOLOGICAL, BANACH, AND HILBERT SPACES

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PROBLEMS

1. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each Lebesgue measurable subset E of $\mathbb{R}$, define $\mu(E) = \int_E f$, the Lebesgue integral of f over E. Show that μ is a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

2. Let $\mathcal{M}$ be a σ-algebra of subsets of a set X and the set function $\mu : \mathcal{M} \to [0, \infty)$ be finitely additive. Prove that μ is a measure iff whenever $\{A_k\}_{k=1}^{\infty}$ is an ascending sequence of sets in $\mathcal{M}$, then

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} \mu(A_k).$$

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