Real Analysis Royden - Fourth Edition

Notes + Solved Exercises :)

Latex Symbols

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I LEBESGUE INTEGRATION FOR FUNCTIONS OF A SINGLE REAL VARIABLE

Preliminaries on Sets, Mappings, and Relations

Chapter 1

The Real Numbers: Sets, Sequences, and Functions

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1.1 The Field, Positivity, and Completeness Axioms

PROBLEMS

1. For $a \neq 0$ and $a \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$.

2. Verify the following:

   (i) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.

   (ii) For each positive number a, its multiplicative inverse a^{-1} also is positive.

   (iii) If $a > b$, then

   $$ac > bc \text{ if } c > 0 \text{ and } ac < bc \text{ if } c < 0.$$

1.2 The Natural and Rational Numbers

PROBLEMS

8. Use an induction argument to show that for each natural number n, the interval $(n, n+1)$ fails to contain any natural number.

9. Use an induction argument to show that if $n > 1$ is a natural number, then $n - 1$ also is a natural number. The use another induction argument to show that if m and n are natural numbers with $n > m$, then $n - m$ is a natural number.

10. Show that for any real number r, there is exactly one integer in the interval $[r, r + 1)$.

11. Show that ay nonempty set of integers that is bounded above has a largest member.

12. Show that the irrational numbers are dense in $\mathbb{R}$.

13. Show that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

14. Show that if $r > 0$, then, for each natural number n, $(1 + r)^n \geq 1 + n \cdot r$.

15. Use induction arguments to prove that for every natural number n,

    (i)
    $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6},$$

    (ii)
    $$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2,$$

    (iii)
    $$1 + r + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \text{ if } r \neq 1.$$

1.3 Countable and Uncountable Sets

1.4 Open Sets, Closed Sets, and Borel Sets of Real Numbers

The Nested Set Theorem. *Let $\{F_n\}_{n=1}^{\infty}$ be a descending countable collection of nonempty closed sets of real numbers for which F_1 is bounded. Then*

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Proof. By contradiction, suppose that $\bigcap_{n=1}^{\infty} F_n = \emptyset$. Then $\bigcup_{n=1}^{\infty} F_n^c = (\bigcap_{n=1}^{\infty} F_n)^c = \emptyset^c = \mathbb{R}$, and we have an open cover of $\mathbb{R}$ and thus an open cover of $F_1 \subseteq \mathbb{R}$. By the Heine-Borel Theorem, there exists an $N \in \mathbb{N}$ such that $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c$. Because $\{F_n\}$ is descending, $F_n \supseteq F_{n+1}$ for any $n \geq 1$. This implies $F_n^c \subseteq F_{n+1}^c$, and thus $F_1 \subseteq \bigcup_{n=1}^{N} F_n^c = F_N^c = \mathbb{R} \setminus F_N$. This is a contradiction to the assumption that F_N is a nonempty subset of F_1. □

PROBLEMS

27. Is the set of rational numbers open or closed?

28. What are the sets of real numbers that are both open and closed?

29. Find two sets A and B such that

1.5 Sequences of Real Numbers

1.6 Continuous Real-Valued Functions of a Real Variable

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