

UNIVERSITY OF CANTERBURY

A Geometric Approach to Complete Reducibility

by

Daniel Lond

A thesis submitted in partial fulfillment for the
degree of Doctor of Philosophy

in the
College of Engineering
Department of Mathematics and Statistics

May 2010

Declaration of Authorship

I, DANIEL LOND, declare that this thesis titled, ‘A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY’ and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

“A quote.”

The author of the quote.

UNIVERSITY OF CANTERBURY

Abstract

College of Engineering
Department of Mathematics and Statistics

Doctor of Philosophy

by Daniel Lond

The Thesis Abstract ...

Acknowledgements

The acknowledgements and the people to thank ...

Contents

Declaration of Authorship	i
Abstract	iii
Acknowledgements	iv
Symbols	vii
1 Introduction	1
2 Mathematical Preliminaries	2
3 Külshammer's Second Problem	3
3.1 Külshammer's First Problem	3
3.2 Külshammer's Second Problem	3
3.3 A non-reductive counterexample	3
4 The 1-Cohomology	4
4.1 Abelian 1-Cohomology	4
4.1.1 Definitions	4
4.1.2 Maps between 1-cohomologies	6
4.2 Non-abelian 1-Cohomology	9
4.2.1 The non-abelian setting	9
4.2.2 Definitions	9
4.2.3 Maps between 1-cohomologies	10
5 1-Cohomology Calculation	12
5.1 The method	12
5.2 A rank 1 calculation	18
5.2.1 Example	19
5.2.2 Example	30
5.3 A rank 2 calculation	37
6 Conclusion	49

A Further Calculations	50
B Source Code	51
Bibliography	52

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	rads^{-1}
\vdots		

Dedication . . .

Chapter 1

Introduction

- What the thesis is about
- Motivation - link with other problems
- Highlight results - lead up to highlights
- Similar to abstract but less formal
- Outline of the contents, chapter by chapter

Chapter 2

Mathematical Preliminaries

Chapter 3

Külshammer's Second Problem

- Külshammer's First Problem
- Külshammer's Second Problem
- Counter example for non-reductive G
- Overture to Ch. 3-5

3.1 Külshammer's First Problem

3.2 Külshammer's Second Problem

3.3 A non-reductive counterexample

3) Look at the nonreductive counterexample in Slodowy's paper on Külshammer's problem. What is special about the 3-dimensional U that makes this counterexample work? Can you find similar structure in the unipotent radical of a reductive group?

Chapter 4

The 1-Cohomology

4.1 Abelian 1-Cohomology

4.1.1 Definitions

Let H be a group and V an abelian group (vector space) on which H acts homomorphically (linearly). We call a map σ from $H \rightarrow V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \quad (4.1)$$

for all h_1, h_2 in H . Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \rightarrow V$.

We call the (4.1) the *1-cocycle condition*.

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned} (\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\ &= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\ &= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\ &= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2), \end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \rightarrow V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H, V)$ we have

$$\begin{aligned}\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\sigma(1),\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1(H, V)$ is a \mathbb{Z} -module under pointwise addition.

Given a v in V we define a 1-coboundary $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}\chi_v^H(h_1 h_2) &= v - (h_1 h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}
 (\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\
 &= u - h \cdot u + v - h \cdot v \\
 &= (u + v) - h \cdot (u + v) \\
 &= \chi_{u+v}^H(h)
 \end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

Setting $v = -u$ in the above calculation provides the definition of an inverse of a 1-coboundary and hence shows that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a \mathbb{Z} -submodule of $Z^1(H, V)$, so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology*.

Lemma 4.1. *Suppose H is linearly reductive. Then $H^1(H, V) = 0$. [Hochschild]*

4.1.2 Maps between 1-cohomologies

Let ϕ be a homomorphism from $\tilde{H} \rightarrow H$, \tilde{H} being another group that acts on V . Suppose that for every h in H , ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V . If σ is a 1-cocycle from $H \rightarrow V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \rightarrow V$.

Take h_1, h_2 in H . We have

$$\begin{aligned}
 Z^1(\phi)(\sigma)(h_1 h_2) &= (\sigma \circ \phi)(h_1 h_2) \\
 &= \sigma(\phi(h_1 h_2)) \\
 &= \sigma(\phi(h_1)\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\
 &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\
 &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2).
 \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \rightarrow H^1(\tilde{H}, V),$$

defined by

$$\begin{array}{ccc}
 Z^1(H, V) & \xrightarrow{Z^1(\phi)} & Z^1(\tilde{H}, V) \\
 \pi \downarrow & & \downarrow \tilde{\pi} \\
 H^1(H, V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H}, V)
 \end{array}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 4.1. Let \tilde{H} be a subgroup of H and $i : \tilde{H} \rightarrow H$ the inclusion map. Then i gives rise to a well defined map

$$H^1(i) : H^1(H, V) \rightarrow H^1(\tilde{H}, V).$$

Lemma 4.2. Let H be a finite group and $\tilde{H} = H_p$ a Sylow p -subgroup of H . If V is a vector space then the map

$$H^1(i) : H^1(H, V) \rightarrow H^1(H_p, V)$$

is injective.

Proof. Let x be an element of $H^1(H, V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(H, V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to H_p is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since

$\chi_v^{H_p}$ can be trivially extended to a 1-coboundary χ_v^H from $H \rightarrow V$, and

$$\pi(\sigma - \chi_v^H) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^H)$ as a representative for x . Hence there is no harm in assuming that σ is 0 when restricted to H_p . Now choose a set of representatives h_1, \dots, h_l in H for the coset space H/H_p and set

$$v^* = \sum_{i=1}^l \sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by v^*

$$\begin{aligned} \chi_{v^*}^H(h) &= v^* - h \cdot v^* \\ &= \sum_{i=1}^l \sigma(h_i) - h \cdot \sum_{i=1}^l \sigma(h_i) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i). \end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned} \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l (\sigma(hh_i) - \sigma(h)) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h). \end{aligned}$$

Now as the value of σ at a fixed h depends only on the value of σ at the representative h_j of the coset containing h we can collapse the middle term to yield

$$\begin{aligned} \chi_{v^*}^H(h) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(h_i) + \sum_{i=1}^l \sigma(h) \\ &= l\sigma(h). \end{aligned}$$

Since $\gcd([H : H_p], p) = \gcd(l, p) = 1$, l is invertible and so

$$l^{-1}\chi_{v^*}^H(h) = \sigma(h).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. \square

We could also consider appropriate maps $f : V \rightarrow \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \rightarrow H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \rightarrow H^1(\tilde{H}, \tilde{V}).$$

4.2 Non-abelian 1-Cohomology

4.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocycles be morphisms of varieties.

4.2.2 Definitions

Let H, V be algebraic groups, H acting on V . We call a map σ from $H \rightarrow V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{4.2}$$

for all h_1, h_2 in H . Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \rightarrow V$.

We call the (4.2) the *1-cocycle condition*.

Given a v in V we define a *1-coboundary* $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}
 \chi_v^H(h_1 h_2) &= v * (h_1 h_2) \cdot v^{-1} \\
 &= v * h_1 \cdot (h_2 \cdot v^{-1}) \\
 &= v * h_1 \cdot (v v^{-1} h_2 \cdot v) \\
 &= v * h_1 \cdot v * h_1 \cdot (v * h_2 \cdot v^{-1}) \\
 &= \chi_v^H(h_1) * h_1 \cdot \chi_v^H(h_2),
 \end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say σ_1, σ_2 in $Z^1(H, V)$ are equivalent if there exists a v in V such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \quad (4.3)$$

for all h in H . We call the set of equivalence classes of $Z^1(H, V)$ under (4.3) the *1-cohomology*, denoted $H^1(H, V)$.

4.2.3 Maps between 1-cohomologies

Lemma 4.3. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V . Then $H^1(i) : H^1(SL_2, V) \rightarrow H^1(B, V)$ is injective.*

Proof. Let x be in the kernel of $H^1(i)$ and σ an element of $Z^1(SL_2, V)$ that projects onto the class x . Since $Z^1(i)(\sigma)$ projects to the trivial 1-cohomology class we may as well assume that $\sigma|_B = 1$. For there exists some v in V such that for all b in B

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle $\hat{\sigma} : SL_2 \rightarrow V$ defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction $\hat{\sigma}$ also projects to the class x , and for all b in B

$$\begin{aligned}
 \hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\
 &= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\
 &= v^{-1} * v * b \cdot (v^{-1} * v) \\
 &= 1,
 \end{aligned}$$

so we may as well have chosen $\hat{\sigma}$ instead as a representative for x .

Now consider the *homogeneous space* SL_2/B [ref Humphreys] and take the map

$$\tilde{\sigma} : SL_2/B \rightarrow V,$$

defined in the usual way under the canonical projection $SL_2 \rightarrow SL_2/B$. This map is well defined and is a morphism [ref Borel]. Now since SL_2/B is an irreducible projective variety [ref Humphreys], $\tilde{\sigma}$ must be constant [ref Borel]. Hence, as σ takes the value 1 for any b in B , $\tilde{\sigma}(gB) = 1$ for all cosets gB . Therefore, for all g in SL_2

$$\sigma(g) = \tilde{\sigma}(gB) = 1.$$

We have shown that σ is the 1-coboundary χ_1 which means that the kernel of $H^1(i)$ is trivial. □

Lemma 4.4. *Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B . Then $H^1(B, V) \rightarrow H^1(U, V)$ is injective. Moreover*

$$H^1(SL_2, V) \rightarrow H^1(U, V)$$

is injective.

Proof. As in the previous example, let x be an element of the kernel of the map $H^1(i) : H^1(B, V) \rightarrow H^1(U, V)$ and let σ in $Z^1(B, V)$ be a representative for x such that $\sigma|_U = 1$. Let T be a maximal torus for B . For any u in U and t in T there is a u' in U such that

$$ut = tu'.$$

Hence U acts trivially on the image of T under σ :

$$\begin{aligned} \sigma(ut) &= \sigma(tu') \\ \sigma(u) * u \cdot \sigma(t) &= \sigma(t) * t \cdot \sigma(u') \\ u \cdot \sigma(t) &= \sigma(t). \end{aligned}$$

□

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G . The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p . Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G .

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α :

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r &: SL_2(k) \rightarrow SL_2(k) \\ (A_{ij}) &\mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r = \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} \epsilon_{\delta}((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_{\delta}) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} n_{\alpha} \epsilon_{\delta}(\lambda_{\delta}) n_{\alpha}^{-1}, \end{aligned}$$

where $n_{\alpha} = \epsilon_{\alpha}(1)\epsilon_{-\alpha}(-1)\epsilon_{\alpha}(1)$ and λ_{δ} are elements of the underlying field k .

Lemma 5.1.

$$\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_{\delta}(x_{\delta}(u)),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_{\delta} \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ . Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \rightarrow k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore

$$x_\delta(t^2 u) = (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta(u).$$

Since x is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle \delta, \alpha \rangle \geq 0$. However, if $\langle \delta, \alpha \rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle \delta, \alpha \rangle > 0$. \square

Next we prove a couple of useful facts about root systems not containing G_2 or C_3 .

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle \alpha, \beta \rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β . Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. \square

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3 . Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ ($i = 1, 2$). If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ , ($i = 1, 2$) and let θ_3 be the absolute value of the angle between δ_1 and δ_2 . Then

$$\begin{aligned} \langle \delta_i, \gamma \rangle &> 0 & (i = 1, 2) \\ \implies (\delta_i, \gamma) &> 0 \\ \implies \cos(\theta_i) &> 0 \\ \implies \theta_i &< \pi/2, \end{aligned}$$

and similarly, using 5.2

$$\begin{aligned} \langle \delta_1, \delta_2 \rangle &\leq 0 \\ \implies \theta_3 &\geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

- 1:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = 2\pi/3;$
- 2:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 3:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 4:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/4, \quad \theta_3 = \pi/2.$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ , and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length

in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3 . [Picture?][Wow, is that right? Maybe just say ‘we will show that they lie in a C_3 subsystem’.] \square

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3 .

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \epsilon_\alpha(u_1^{p^r}) \prod_{\delta} \epsilon_\delta(x_\delta(u_2)) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α . \square

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α .*

Proof. First consider

$$\sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_\delta(x_\delta(b)).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$, so 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \\ &= \prod_{\delta} \epsilon_{\delta} \left((a^{p^r})^{\langle \delta, \alpha \rangle} x_{\delta}(b) \right). \end{aligned}$$

□

Since the x_{δ} are homomorphisms from $k \rightarrow k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k . Furthermore, combining the calculation in the proof of 5.1 with the result 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta} (x_{\delta}(a^2 b)) = \prod_{\delta} \epsilon_{\delta} (a^{\langle \delta, \alpha \rangle} x_{\delta}(b)),$$

severely restricting the possible polynomials x_{δ} . In fact, they are confined to be polynomials involving just one term, and the exponent has already been decided upon fixing the integer r in the definition of ρ_r . For suppose x_{δ} and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned} \mu_j (a^2)^{p^j} &= \mu_j (a^{p^r})^{\langle \delta, \alpha \rangle} \\ \implies 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

Since 2 divides that on the right hand side, if $\langle \delta, \alpha \rangle$ is not a multiple of 2 then p , the characteristic of k , must be 2 or else $x_{\delta} = 0$.

In [Carter] it is shown that the possible pairings of any two roots are bounded by ± 3 . Hence by 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3 .

We had to discard G_2 and C_3 from the little result here. Next we will dispense with them by way of example to prove the following fact. [What is it?]

Things to do here:

- Refer to Structure/Classification Theorem to get the homomorphisms ρ_r
- Choosing σ s.t. $\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = 0$

- Letting $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}_i = p_i(u)$
- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot (p_i(u)) = (p_i(t^2 u))$
- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \Rightarrow p_i(u_1 + u_2) = p_i(u_1) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot p_i(u_2)$. Usually end up with p_i homomorphisms.
- Know $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Calc. $\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}$
- Can get $\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}$ by a similar argument.
- Calc. $\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$
- Compare with fact $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now we know σ exactly on B and n_γ .
- Already know $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $c = 0$. Now calc.

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix} \right)$$
- We now have fact $\sigma' \in Z^1(SL_2, V) \Rightarrow \sigma' \sim \sigma$ and know the form of σ . To check “ \Leftarrow ” direction apply σ to the Steinberg relations.
- Find all $\tau \in Z^1(SL_2, V)$ conj. to σ and also zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ by calculating $\tau(g) = v * \sigma(g) * g \cdot v^{-1}$.
- Can now state conj. classes of 1-cocycles by inspection.
- Extend classes to P -conjugacy by action of $Z(L)$. Explain why ...
- G -conjugacy ...

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p . We label the positive roots for B_2 as $\alpha, \beta, \alpha + \beta, 2\alpha + \beta$. We have from [reference Humphreys 33.4]:

$$\begin{aligned}\epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy),\end{aligned}$$

and

$$\begin{aligned}n_\alpha\epsilon_\beta(x)n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\ n_\alpha\epsilon_{\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\ n_\alpha\epsilon_{2\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_\beta(x) \\ n_\beta\epsilon_\alpha(x)n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\ n_\beta\epsilon_{\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_\alpha(-x) \\ n_\beta\epsilon_{2\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}P_\alpha &= \langle B, U_{-\alpha} \rangle \\ P_\beta &= \langle B, U_{-\beta} \rangle,\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\ &= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\ P_\beta &= L_\beta \ltimes R_u(P_\beta) \\ &= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α :

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\alpha$ defined by

$$\begin{aligned}\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,\end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r : write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_\alpha(-u^{p^r}) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_\alpha(-u^{p^r}) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(-2u^{p^r} v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\alpha(-u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(-u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{2p^r} v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2) \\
&= \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2 - u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r} v_1 \\ v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta(\beta(\alpha^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\alpha^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\alpha^\vee(t^{p^r})) v_3) \\
&= \epsilon_\beta((t^{p^r})^{\langle \beta, \alpha \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha + \beta, \alpha \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha + \beta, \alpha \rangle} v_3) \\
&= \begin{pmatrix} t^{-2p^r} v_1 \\ v_2 \\ t^{2p^r} v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) n_\alpha^{-1} n_\alpha \epsilon_{\alpha+\beta}(v_2) n_\alpha^{-1} n_\alpha \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1) \epsilon_{\alpha+\beta}(-v_2) \epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3) \epsilon_{\alpha+\beta}(-v_2) \epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad + bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* . Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.1)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.2)$$

to get further information on the polynomials p_i ($i = 1, 2, 3$).

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_1(t^2u) = t^{-2p^r} p_1(u) \quad (5.3)$$

$$p_2(t^2u) = p_2(u) \quad (5.4)$$

$$p_3(t^2u) = t^{2p^r} p_3(u). \quad (5.5)$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k . Now notice that on the left hand side of (5.3) there are only non-negative powers of t , and on the right hand side there are only non-positive powers of t . This equality is only satisfied if $p_1(x) = 0$ for all x in k , so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.6)$$

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2) - 2u_1^{p^r} p_2(u_2). \quad (5.7)$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.8)$$

for some μ_i in k .

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^N \mu_i (t^2u)^{p^i} = t^{2p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.9)$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating the coefficients of u in (5.9) we get

$$\begin{aligned}\mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies p^l &= p^r.\end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r .

Letting $\mu_l = \mu$ in k , we have

$$\begin{aligned}\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}.\end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k , p , is not equal to 2 then by setting \mathbf{v} in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
 \chi_v \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
 &= \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right).
 \end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2 . By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \rightarrow V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \quad (5.10)$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu + \mu' \\ \mu \\ \mu \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu' \end{pmatrix} = \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu + \mu' \end{pmatrix}.
 \end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

we get

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Therefore $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is an element of V that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

If $c = 0$ then we already have

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r} \mu(cd)^{2^r} \\ (ac^{-1})^{2^r+1} \mu(cd)^{2^r} \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1 + ad)^{2^r} \\ \mu(ac^{-1})^{2^r} (1 + ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

In fact, we see that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering

\mathbf{v} that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$ where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a \mathbf{v} in V such that

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \rightarrow P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α . Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \mid x \in k \rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle = 0. \quad (5.11)$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n , we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \quad (5.12)$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha + \beta)^\vee(x) \mid x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha + \beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of \mathbf{s} on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha + \beta)^\vee(s) \epsilon_\beta (\mu(cd)^{2^r}) \epsilon_{\alpha+\beta} (\mu(bc)^{2^r}) \epsilon_{2\alpha+\beta} (\mu(ab)^{2^r}) (\alpha + \beta)^\vee(s)^{-1} \\ &= \epsilon_\beta \left(s^{\langle \beta, \alpha+\beta \rangle} \mu(cd)^{2^r} \right) \epsilon_{\alpha+\beta} \left(s^{\langle \alpha+\beta, \alpha+\beta \rangle} \mu(bc)^{2^r} \right) \epsilon_{2\alpha+\beta} \left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle} \mu(ab)^{2^r} \right) \\ &= \begin{pmatrix} (s^2 \mu)(cd)^{2^r} \\ (s^2 \mu)(bc)^{2^r} \\ (s^2 \mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \rightarrow V$ collapse

to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V -conjugacy to P_α -conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}. \end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β :

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\beta$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows.

Let \mathbf{v}, \mathbf{w} in V . We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1 + w_1)\epsilon_{\alpha+\beta}(v_2 + w_2)\epsilon_{2\alpha+\beta}(v_3 + w_3 + 2v_2w_1) \\ &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 + 2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r . Let \mathbf{v} be an element of V :

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\beta(u^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(u^{p^r}) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \\
&= \begin{pmatrix} v_1 \\ v_2 + u^{p^r} v_1 \\ v_3 + u^{p^r} v_1^2 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \beta^\vee(t^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\beta^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\alpha(\alpha(\beta^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\beta^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\beta^\vee(t^{p^r})) v_3) \\
&= \epsilon_\alpha((t^{p^r})^{\langle \alpha, \beta \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha+\beta, \beta \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha+\beta, \beta \rangle} v_3) \\
&= \begin{pmatrix} t^{-p^r} v_1 \\ t^{p^r} v_2 \\ v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) n_\beta^{-1} n_\beta \epsilon_{\alpha+\beta}(v_2) n_\beta^{-1} n_\beta \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= \epsilon_{\alpha+\beta}(v_1) \epsilon_\alpha(-v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(-v_2) \epsilon_{\alpha+\beta}(v_1) \epsilon_{2\alpha+\beta}(v_3 - 2v_1 v_2) \\
&= \begin{pmatrix} -v_2 \\ v_1 \\ v_3 - 2v_1 v_2 \end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} c^{p^r} v_2 + d^{p^r} v_1 \\ a^{p^r} v_2 + b^{p^r} v_1 \\ v_3 + (ac)^{p^r} v_2^2 + (bd)^{p^r} v_1^2 + 2(bc)^{p^r} v_1 v_2 \end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* , and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k .

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.13)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.14)$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \quad (5.15)$$

$$p_2(t^2u) = t^{p^r} p_2(u) \quad (5.16)$$

$$p_3(t^2u) = p_3(u). \quad (5.17)$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k . From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) + p_2(u_2) \\ 2\lambda \end{pmatrix}
 \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.18)$$

$$\lambda = 2\lambda. \quad (5.19)$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.20)$$

for some μ_i in k .

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^N \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.21)$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
 \mu_l t^{2p^l} &= \mu_l t^{p^r} \\
 \implies 2p^l &= p^r.
 \end{aligned}$$

Thus 2 divides p^r , and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r , and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu^2 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' + \mu \\ \mu \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu' + \mu \\ \mu' \\ \mu'^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu' + \mu \\ \mu' + \mu \\ \mu'^2 \end{pmatrix}.
\end{aligned}$$

Since $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for all t in k^* we must have $\mu' = \mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r} \mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{2^r} (\mu(cd)^{2^{r-1}})^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ac^{-1} + a^2c^{-1}d)^{2^{r-1}} \\ \mu^2(1+ad)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a \mathbf{v} in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
 \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix},
 \end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$, where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
 (\mathbf{s} \cdot \sigma_\mu) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \\
 &= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)(ab)^{2^{r-1}} \\ (s\mu)^2(bc)^{2^r} \end{pmatrix}.
 \end{aligned}$$

our infinitely many V -conjugacy classes collapse to just two P_β -conjugacy classes:

$$\begin{aligned}
 [\sigma_0] &= \{\sigma_0\}, \\
 [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}.
 \end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow \text{Im}(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B_2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ .

1) Know about non G-cr in B_2 , can I put them in an $A_1 A_1$?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $\text{Im}(\rho_{r,s})$, can we conjugate it into P_γ or P_δ ?

Let $\text{char}(k) = 2$ and set $V := \langle U_\phi \mid \phi \in \Phi^+, \phi \neq \gamma + \delta, \phi \neq \gamma + 2\delta \rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2 v_1 \\ 0 \\ u_4 v_1 \\ 0 \\ u_6 v_1 \\ 0 \\ u_8 v_1 \\ 0 \\ u_{10} v_1 \\ u_{10} v_1 v_2 + u_8 v_1 v_4 + u_6^2 v_1 + u_{11} v_2 + u_{10} v_3 + u_9 v_4 + u_8 v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \rightarrow \tilde{A}_1 \tilde{A}_1 < L_{\{\gamma, \delta\}}$ defined by

$$\begin{aligned} \rho_{r,s} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\ \rho_{r,s} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma + \delta)^\vee(t^{2^s}) \\ \rho_{r,s} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta} \end{aligned}$$

from which we obtain an action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so we let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each p_i ($1 \leq i \leq 12$) is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \quad (5.22)$$

It is clear that for $i = 1, 6, 7, 12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t , p_i must be the zero polynomial for $i = 2, 3, 4, 5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \rightarrow k$. That is

for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^N w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^N x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^N y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^N z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^N w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^N w_j u^{2^j} \\ \Rightarrow w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) &= w u^{2^r}, & p_9(u) &= x u^{2^r}, \\ p_{10}(u) &= y u^{2^s}, & p_{11}(u) &= z u^{2^s}, \end{aligned}$$

for some w, x, y, z in k .

So, we have

$$\begin{aligned} \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

We apply the same argument using the fact that each component of $\sigma \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p'_i(u)$ for all u in k , to get

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k .

From this we deduce that

$$\begin{aligned}
 \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ y + y' \\ z + z' \\ w + w' \\ x + x' \\ w' + y' \\ x' + z' \\ w + w' \\ x + x' \\ y + y' \\ z + z' \\ w'x' + y'z' \end{pmatrix}.
 \end{aligned}$$

Furthermore, since $\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, we have

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} n_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_6 \\ n_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_{12} \end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k . So in fact

$$\begin{aligned}
 w' &= w \\
 x' &= x \\
 y' &= y \\
 z' &= z \\
 n_1 &= 0 \\
 n_6 &= w + y \\
 n_7 &= x + z \\
 n_{12} &= wx + yz.
 \end{aligned}$$

Consider $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\begin{aligned}
 \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \rightarrow V$ of the form

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a \mathbf{v} in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g \cdot \mathbf{v}^{-1}$$

for all g in SL_2 . Furthermore, τ' is conjugate to some 1-cocycle τ , where τ has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some \mathbf{v} in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\ &= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix} \end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \sim (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P -conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ s^{-1}t^2y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

Chapter 6

Conclusion

Appendix A

Further Calculations

- G_2 calculation?
- The rest of the B_4 calculations

Appendix B

Source Code

Put source code here ...

Bibliography

- [1] A. S. Arnold, J. S. Wilson, and M. G. Boshier. A simple extended-cavity diode laser. *Review of Scientific Instruments*, 69(3):1236–1239, March 1998. URL <http://link.aip.org/link/?RSI/69/1236/1>.
- [2] Carl E. Wieman and Leo Hollberg. Using diode lasers for atomic physics. *Review of Scientific Instruments*, 62(1):1–20, January 1991. URL <http://link.aip.org/link/?RSI/62/1/1>.
- [3] C. J. Hawthorn, K. P. Weber, and R. E. Scholten. Littrow configuration tunable external cavity diode laser with fixed direction output beam. *Review of Scientific Instruments*, 72(12):4477–4479, December 2001. URL <http://link.aip.org/link/?RSI/72/4477/1>.