

UNIVERSITY OF CANTERBURY

A Geometric Approach to Complete Reducibility

by

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Declaration of Authorship

I, DANIEL LOND, declare that this thesis titled, ‘A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY’ and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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“A quote.”

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Abstract

College of Engineering
Department of Mathematics and Statistics

Doctor of Philosophy

by Daniel Lond

The Thesis Abstract ...

Acknowledgements

The acknowledgements and the people to thank ...

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Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	rads^{-1}
\vdots		

Dedication . . .

Chapter 1

Introduction

- What the thesis is about
- Motivation - link with other problems
- Highlight results - lead up to highlights
- Similar to abstract but less formal
- Outline of the contents, chapter by chapter

Chapter 2

Mathematical Preliminaries

Chapter 3

Külshammer's Second Problem

- Külshammer's First Problem
- Külshammer's Second Problem
- Counter example for non-reductive G
- Overture to Ch. 3-5

3.1 Külshammer's First Problem

3.2 Külshammer's Second Problem

3.3 A non-reductive counterexample

3) Look at the nonreductive counterexample in Slodowy's paper on Külshammer's problem. What is special about the 3-dimensional U that makes this counterexample work? Can you find similar structure in the unipotent radical of a reductive group?

Chapter 4

The 1-Cohomology

TODO:

- Abelian 1-Cohomology
- Non-abelian 1-Cohomology [Richardson]
- Brief results of calculations

4.1 Abelian 1-Cohomology

TODO:

- The general setting: Group H , Vector space/Abelian group V , linear action/action of homomorphisms.
- Definition of Z^1 , B^1 , H^1 .
- Definition of $Z^1(f)$, $B^1(f)$, $H^1(f)$ where $f : H \rightarrow H'$. f must respect the actions of H, H' . e.g. i inclusion map.
- $H^1(H, V) \rightarrow H^1(H_P, V)$ is (1-1? onto?)
- What about when H is finite? algebraic? both? Define H^1 for H finite, then H algebraic then H finite algebraic - should be consistent.
- What about $H^1(x)$ where $x : V \rightarrow V'$? Relevant: $B_4 \leq F_4$.
- Explain: if $H = SL_2$ then H^1 is determined by its value on the upper triangular matrices. $B < SL_2$, $H^1(SL_2, V) \rightarrow H^1(B, V)$ is (1-1? onto?) \leftarrow prove this. [Hum G/B].

- $H^1(H, V)$ is trivial if H is linearly reductive [done for H finite].

4.1.1 Definitions

Let H be a group and V an abelian group on which H acts homomorphically, that is the map

$$v \mapsto h \cdot v,$$

is a homomorphism from $V \rightarrow V$ for any h in H . We call a map σ from $H \rightarrow V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \quad (4.1)$$

for all h_1, h_2 in H . Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \rightarrow V$.

We call the (4.1) the *1-cocycle condition*.

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned} (\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\ &= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\ &= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\ &= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2), \end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \rightarrow V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H, V)$ we have

$$\begin{aligned} \sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\sigma(1), \end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) &= \sigma(1) = 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1(H, V)$ is a \mathbb{Z} -module under pointwise addition.

Given a v in V we define a 1-coboundary $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}\chi_v^H(h_1h_2) &= v - (h_1h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u + v) - h \cdot (u + v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

Setting $v = -u$ in the above calculation provides the definition of an inverse of a 1-coboundary and hence shows that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a \mathbb{Z} -submodule of $Z^1(H, V)$,

so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology*.

Lemma 4.1. *Suppose H is linearly reductive. Then $H^1(H, V) = 0$. [Hochschild]*

4.1.2 Maps between 1-cohomologies

Let ϕ be a homomorphism from $\tilde{H} \rightarrow H$, \tilde{H} being another group that acts on V by homomorphisms. Suppose that for every h in H , ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V . If σ is a 1-cocycle from $H \rightarrow V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \rightarrow V$.

Take h_1, h_2 in H . We have

$$\begin{aligned} Z^1(\phi)(\sigma)(h_1 h_2) &= (\sigma \circ \phi)(h_1 h_2) \\ &= \sigma(\phi(h_1 h_2)) \\ &= \sigma(\phi(h_1) \phi(h_2)) \\ &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2)) \\ &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\ &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\ &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2). \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \rightarrow H^1(\tilde{H}, V),$$

defined by

$$\begin{array}{ccc} Z^1(H, V) & \xrightarrow{Z^1(\phi)} & Z^1(\tilde{H}, V) \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ H^1(H, V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H}, V) \end{array}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 4.1. Let \tilde{H} be a subgroup of H and $i : \tilde{H} \rightarrow H$ the inclusion map. Then i gives rise to a well defined map

$$H^1(i) : H^1(H, V) \rightarrow H^1(\tilde{H}, V).$$

Lemma 4.2. Let H be a finite group and $\tilde{H} = H_p$ a Sylow p -subgroup of H . The map

$$H^1(i) : H^1(H, V) \rightarrow H^1(H_p, V)$$

is injective.

Proof. Let x be an element of $H^1(H, V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(H, V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to H_p is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since $\chi_v^{H_p}$ can be trivially extended to a 1-coboundary χ_v^H from $H \rightarrow V$, and

$$\pi(\sigma - \chi_v^H) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^H)$ as a representative for x . Hence there is no harm in assuming that σ is 0 when restricted to H_p . Now choose a set of representatives h_1, \dots, h_l in H for the coset space H/H_p and set

$$v^* = \sum_{i=1}^l \sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by v^*

$$\begin{aligned} \chi_{v^*}^H(h) &= v^* - h \cdot v^* \\ &= \sum_{i=1}^l \sigma(h_i) - h \cdot \sum_{i=1}^l \sigma(h_i) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i). \end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned} \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l (\sigma(hh_i) - \sigma(h)) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h). \end{aligned}$$

Now as the value of σ at a fixed h depends only on the value of σ at the representative h_j of the coset containing h we can collapse the middle term to yield

$$\begin{aligned} \chi_{v^*}^H(h) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(h_i) + \sum_{i=1}^l \sigma(h) \\ &= l \sigma(h). \end{aligned}$$

Since $\gcd([H : H_p], p) = \gcd(l, p) = 1$, [okay, I should have been talking about V vector space here] l is invertible and so

$$l^{-1} \chi_{v^*}^H(h) = \sigma(h).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. \square

We could also let \tilde{V} be another abelian group and $f : V \rightarrow \tilde{V}$ a homomorphism of groups satisfying

$$f(h \cdot v) = h \cdot f(v).$$

Following a similar chain of arguments as before we can define a map

$$H^1(f) : H^1(H, V) \rightarrow H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \rightarrow H^1(\tilde{H}, \tilde{V}).$$

It is worth reminding the reader here of the directions of the underlying homomorphisms for the above construction to work:

$$\begin{aligned}\phi : \tilde{H} &\rightarrow H \\ f : V &\rightarrow \tilde{V}.\end{aligned}$$

4.2 Non-abelian 1-Cohomology

TODO:

- set up: Group H , non-abelian group V .
- define: Z^1 as before, B^1 - not really the same as before, H^1 - harder.
- $\phi : H \rightarrow H'$, $Z^1(H', V) \rightarrow Z^1(H, V)$? $H^1(H', V) \rightarrow H^1(H, V)$? - could put in some details.
- H finite, H algebraic - as before.
- $H^1(H, V) \rightarrow H^1(H_p, V)$ different to abelian case? Link to counter example to KII?
- $H^1(SL_2, V) \rightarrow H^1(B, V)$ (1-1? onto?)
- $H^1(H, V)$, H linearly reductive - done.
- $V \rightarrow V'$, $Z^1(H, V) \rightarrow Z^1(H, V')$? exists? $H^1(H, V) \rightarrow H^1(H, V')$?? could put in details. V abelian group, $R = \mathbb{Z}$. V is a \mathbb{Z} - module.

4.2.1 The non-abelian setting

4.2.2 Maps of non-abelian 1-cohomologies

OLD STUFF: Let G be an algebraic group, P a parabolic subgroup of G , and L a Levi subgroup of P . Let $\rho : H \rightarrow L$ be a homomorphism, H an abstract group.

We are interested in functions $\rho_\alpha : H \rightarrow P$ of the form $\rho_\alpha(h) = \alpha(h)\rho(h)$, where $\alpha : H \rightarrow R_u(P)$.

What properties must α satisfy for ρ_α to be a homomorphism?

$$\begin{aligned}
 \alpha(gh)\rho(gh) = \rho_\alpha(gh) &= \rho_\alpha(g)\rho_\alpha(h) \\
 &= \alpha(g)\rho(g)\alpha(h)\rho(h) \\
 &= \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}\rho(g)\rho(h) \\
 &= \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}\rho(gh),
 \end{aligned}$$

that is

$$\alpha(gh) = \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}.$$

Since L normalises $R_u(P)$, we choose write this as

$$\alpha(gh) = \alpha(g) * g \cdot \alpha(h), \quad (4.2)$$

where the action of H on $R_u(P)$ is defined by ρ , and $*$: $R_u(P) \times R_u(P) \rightarrow R_u(P)$.

We call (1) the 1-cocycle condition. A morphism $\alpha : H \rightarrow R_u(P)$ that satisfies the 1-cocycle condition for all $g, h \in H$ is said to be a 1-cocycle, and we denote by $Z^1(H, R_u(P))$ the set of all 1-cocycles from H into $R_u(P)$.

Note that in the case that $R_u(P)$ is abelian, $Z^1(H, R_u(P))$ is a vector space under pointwise addition and scalar multiplication, and we write the 1-cocycle condition in additive notation: $\alpha(gh) = \alpha(g) + g \cdot \alpha(h)$.

When is ρ $R_u(P)$ -conjugate to ρ_α ?

Suppose there exists a $v \in R_u(P)$ such that $\rho_\alpha(h) = v\rho(h)v^{-1}$ for all $h \in H$. Then

$$\begin{aligned}
 \alpha(h)\rho(h) = \rho_\alpha(h) &= v\rho(h)v^{-1} \\
 &= v\rho(h)v^{-1}\rho(h)^{-1}\rho(h).
 \end{aligned}$$

Therefore, α is of the form

$$\alpha(h) = v * h \cdot v^{-1}.$$

This leads us to the next definition.

For a fixed $v \in R_u(P)$ we define a 1-coboundary to be a morphism $\chi_v : H \rightarrow R_u(P)$ of the form

$$\chi_v(h) = v * h \cdot v^{-1}, \quad (4.3)$$

and denote the collection of all 1-coboundaries from H into $R_u(P)$ by $B^1(H, R_u(P))$. Indeed,

$$\begin{aligned}
 \chi_v(gh) &= v\rho(gh)v^{-1}\rho(gh)^{-1} \\
 &= v\rho(g)\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1} \\
 &= v\rho(g)[v^{-1}\rho(g)^{-1}\rho(g)v]\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1} \\
 &= [v\rho(g)v^{-1}\rho(g)^{-1}][\rho(g)v\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1}] \\
 &= [v * g \cdot v^{-1}] * g \cdot [v * h \cdot v^{-1}] \\
 &= \chi_v(g) * g \cdot \chi_v(h),
 \end{aligned}$$

so that $B^1(H, R_u(P)) \subset Z^1(H, R_u(P))$.

As before, if $R_u(P)$ is abelian then we write (2) in additive notation, $\chi_v(h) = v - h \cdot v$, and note that $B^1(H, R_u(P))$ is a vector subspace of $Z^1(H, R_u(P))$.

When is ρ_α $R_u(P)$ -conjugate to ρ_β ?

Let $\alpha, \beta \in Z^1(H, R_u(P))$ and suppose there exists a $v \in R_u(P)$ such that $\rho_\beta(h) = v\rho_\alpha(h)v^{-1}$. Then

$$\begin{aligned}
 \beta(h)\rho(h) &= v\alpha(h)\rho(h)v^{-1} \\
 &= v\alpha(h)\rho(h)v^{-1}\rho(h)^{-1}\rho(h),
 \end{aligned}$$

that is

$$\beta(h) = v\alpha(h) * h \cdot v^{-1}. \quad (4.4)$$

We show that (3) gives rise to an equivalence relation on $Z^1(H, R_u(P))$.

Reflexivity:

$$\alpha(h) = v\alpha(h) * h \cdot v^{-1} \text{ with the choice } v = 1.$$

Symmetry:

$$\text{If } \alpha(h) = v\beta(h) * h \cdot v^{-1}, \text{ then } \beta(h) = v^{-1}\alpha(h) * h \cdot v.$$

Transitivity:

$$\text{If } \alpha(h) = v\beta(h) * h \cdot v^{-1} \text{ and } \beta(h) = w\gamma(h) * h \cdot w^{-1} \text{ then } \alpha(h) = (vw)\gamma(h) * h \cdot (vw)^{-1}.$$

Now we define the 1-cohomology, denoted by $H^1(H, R_u(P))$, to be the set of equivalence classes of $Z^1(H, R_u(P))$, where $\alpha \sim \beta$ if and only if (3) holds.

In the abelian case, (3) becomes $\beta(h) = \alpha(h) + \chi_v(h)$, so that two 1-cocycles are equivalent when they differ by a 1-coboundary, and $H^1(H, R_u(P)) = Z^1(H, R_u(P))/B^1(H, R_u(P))$.

BEN'S SUGGESTIONS:

1) Understand the basic properties of nonabelian 1-cohomology. (Definitions of 1-cocycles, 1-coboundaries and 1-cohomology. Maps $H^1(F, U) \rightarrow H^1(F_p, U)$, where F_p is a Sylow p -subgroup of F . Etc. Some of this will be in Richardson's paper.)

2) Try to show that nonabelian U is needed if we are to find a counterexample to Kulshammer's problem. There are various issues here. For instance, is the following true? Suppose we have $\rho_0 : H \rightarrow L$, and suppose $U = R_u(P)$ is abelian. Let $\{\rho_\alpha\}$ be a family of representations of H constructed from ρ_0 using 1-cocycles α . Show that if the restrictions $\rho_\alpha|_{H_p}$ are all U -conjugate then the ρ_α are all U -conjugate. Here H is a finite group with Sylow p -subgroup H_p , but one could ask the same question with $H = A_1$ and replacing H_p with the group of upper unitriangular matrices.

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G . The motivation for this, as outlined in previous chapters, is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p . Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G .

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α :

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r &: SL_2(k) \rightarrow SL_2(k) \\ (A_{ij}) &\mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r = \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} \epsilon_{\delta}((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_{\delta}) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} n_{\alpha} \epsilon_{\delta}(\lambda_{\delta}) n_{\alpha}^{-1}, \end{aligned}$$

where $n_{\alpha} = \epsilon_{\alpha}(1)\epsilon_{-\alpha}(-1)\epsilon_{\alpha}(1)$ and λ_{δ} are elements of the underlying field k .

Lemma 5.1.

$$\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_{\delta}(x_{\delta}(u)),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_{\delta} \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ . Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \rightarrow k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore

$$x_\delta(t^2 u) = (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta(u).$$

Since x is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle \delta, \alpha \rangle \geq 0$. However, if $\langle \delta, \alpha \rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle \delta, \alpha \rangle > 0$. \square

Next we prove a couple of useful facts about root systems not containing G_2 or C_3 .

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle \alpha, \beta \rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β . Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. \square

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3 . Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ ($i = 1, 2$). If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ , ($i = 1, 2$) and let θ_3 be the absolute value of the angle between δ_1 and δ_2 . Then

$$\begin{aligned} \langle \delta_i, \gamma \rangle &> 0 & (i = 1, 2) \\ \implies (\delta_i, \gamma) &> 0 \\ \implies \cos(\theta_i) &> 0 \\ \implies \theta_i &< \pi/2, \end{aligned}$$

and similarly, using 5.2

$$\begin{aligned} \langle \delta_1, \delta_2 \rangle &\leq 0 \\ \implies \theta_3 &\geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

- 1:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = 2\pi/3;$
- 2:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 3:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 4:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/4, \quad \theta_3 = \pi/2.$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ , and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length

in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3 . [Picture?][Wow, is that right? Maybe just say ‘we will show that they lie in a C_3 subsystem’.] \square

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3 .

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \epsilon_\alpha(u_1^{p^r}) \prod_{\delta} \epsilon_\delta(x_\delta(u_2)) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α . \square

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α .*

Proof. First consider

$$\sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_\delta(x_\delta(b)).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$, so 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \\ &= \prod_{\delta} \epsilon_{\delta} \left((a^{p^r})^{\langle \delta, \alpha \rangle} x_{\delta}(b) \right). \end{aligned}$$

□

Since the x_{δ} are homomorphisms from $k \rightarrow k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k . Furthermore, combining the calculation in the proof of 5.1 with the result 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta} (x_{\delta}(a^2 b)) = \prod_{\delta} \epsilon_{\delta} (a^{\langle \delta, \alpha \rangle} x_{\delta}(b)),$$

severely restricting the possible polynomials x_{δ} . In fact, they are confined to be polynomials involving just one term, and the exponent has already been decided upon fixing the integer r in the definition of ρ_r . For suppose x_{δ} and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned} \mu_j (a^2)^{p^j} &= \mu_j (a^{p^r})^{\langle \delta, \alpha \rangle} \\ \implies 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

Since 2 divides that on the right hand side, if $\langle \delta, \alpha \rangle$ is not a multiple of 2 then p , the characteristic of k , must be 2 or else $x_{\delta} = 0$.

In [Carter] it is shown that the possible pairings of any two roots are bounded by ± 3 . Hence by 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3 .

We had to discard G_2 and C_3 from the little result here. Next we will dispense with them by way of example to prove the following fact. [What is it?]

Things to do here:

- Refer to Structure/Classification Theorem to get the homomorphisms ρ_r
- Choosing σ s.t. $\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = 0$

- Letting $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}_i = p_i(u)$
- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot (p_i(u)) = (p_i(t^2 u))$
- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \Rightarrow p_i(u_1 + u_2) = p_i(u_1) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot p_i(u_2)$. Usually end up with p_i homomorphisms.
- Know $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Calc. $\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}$
- Can get $\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}$ by a similar argument.
- Calc. $\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$
- Compare with fact $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now we know σ exactly on B and n_γ .
- Already know $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $c = 0$. Now calc.

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix} \right)$$
- We now have fact $\sigma' \in Z^1(SL_2, V) \Rightarrow \sigma' \sim \sigma$ and know the form of σ . To check “ \Leftarrow ” direction apply σ to the Steinberg relations.
- Find all $\tau \in Z^1(SL_2, V)$ conj. to σ and also zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ by calculating $\tau(g) = v * \sigma(g) * g \cdot v^{-1}$.
- Can now state conj. classes of 1-cocycles by inspection.
- Extend classes to P -conjugacy by action of $Z(L)$. Explain why ...
- G -conjugacy ...

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p . We label the positive roots for B_2 as $\alpha, \beta, \alpha + \beta, 2\alpha + \beta$. We have from [reference Humphreys 33.4]:

$$\begin{aligned}\epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy),\end{aligned}$$

and

$$\begin{aligned}n_\alpha\epsilon_\beta(x)n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\ n_\alpha\epsilon_{\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\ n_\alpha\epsilon_{2\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_\beta(x) \\ n_\beta\epsilon_\alpha(x)n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\ n_\beta\epsilon_{\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_\alpha(-x) \\ n_\beta\epsilon_{2\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}P_\alpha &= \langle B, U_{-\alpha} \rangle \\ P_\beta &= \langle B, U_{-\beta} \rangle,\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\ &= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\ P_\beta &= L_\beta \ltimes R_u(P_\beta) \\ &= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α :

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\alpha$ defined by

$$\begin{aligned}\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,\end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r : write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_\alpha(-u^{p^r}) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_\alpha(-u^{p^r}) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(-2u^{p^r} v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\alpha(-u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(-u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{2p^r} v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2) \\
&= \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2 - u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r} v_1 \\ v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta(\beta(\alpha^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\alpha^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\alpha^\vee(t^{p^r})) v_3) \\
&= \epsilon_\beta((t^{p^r})^{\langle \beta, \alpha \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha+\beta, \alpha \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha+\beta, \alpha \rangle} v_3) \\
&= \begin{pmatrix} t^{-2p^r} v_1 \\ v_2 \\ t^{2p^r} v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) n_\alpha^{-1} n_\alpha \epsilon_{\alpha+\beta}(v_2) n_\alpha^{-1} n_\alpha \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1) \epsilon_{\alpha+\beta}(-v_2) \epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3) \epsilon_{\alpha+\beta}(-v_2) \epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad + bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* . Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.1)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.2)$$

to get further information on the polynomials p_i ($i = 1, 2, 3$).

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_1(t^2u) = t^{-2p^r} p_1(u) \quad (5.3)$$

$$p_2(t^2u) = p_2(u) \quad (5.4)$$

$$p_3(t^2u) = t^{2p^r} p_3(u). \quad (5.5)$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k . Now notice that on the left hand side of (5.3) there are only non-negative powers of t , and on the right hand side there are only non-positive powers of t . This equality is only satisfied if $p_1(x) = 0$ for all x in k , so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.6)$$

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2) - 2u_1^{p^r} p_2(u_2). \quad (5.7)$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.8)$$

for some μ_i in k .

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^N \mu_i (t^2u)^{p^i} = t^{2p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.9)$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating the coefficients of u in (5.9) we get

$$\begin{aligned}\mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies p^l &= p^r.\end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r .

Letting $\mu_l = \mu$ in k , we have

$$\begin{aligned}\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}.\end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k , p , is not equal to 2 then by setting \mathbf{v} in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
 \chi_v \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
 &= \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right).
 \end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2 . By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \rightarrow V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \quad (5.10)$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu + \mu' \\ \mu \\ \mu \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu' \end{pmatrix} = \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu + \mu' \end{pmatrix}.
 \end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

we get

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Therefore $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is an element of V that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

If $c = 0$ then we already have

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r} \mu(cd)^{2^r} \\ (ac^{-1})^{2^r+1} \mu(cd)^{2^r} \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1 + ad)^{2^r} \\ \mu(ac^{-1})^{2^r} (1 + ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

In fact, we see that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering

\mathbf{v} that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$ where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a \mathbf{v} in V such that

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \rightarrow P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α . Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \mid x \in k \rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle = 0. \quad (5.11)$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n , we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \quad (5.12)$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha + \beta)^\vee(x) \mid x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha + \beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of \mathbf{s} on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha + \beta)^\vee(s) \epsilon_\beta (\mu(cd)^{2^r}) \epsilon_{\alpha+\beta} (\mu(bc)^{2^r}) \epsilon_{2\alpha+\beta} (\mu(ab)^{2^r}) (\alpha + \beta)^\vee(s)^{-1} \\ &= \epsilon_\beta \left(s^{\langle \beta, \alpha+\beta \rangle} \mu(cd)^{2^r} \right) \epsilon_{\alpha+\beta} \left(s^{\langle \alpha+\beta, \alpha+\beta \rangle} \mu(bc)^{2^r} \right) \epsilon_{2\alpha+\beta} \left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle} \mu(ab)^{2^r} \right) \\ &= \begin{pmatrix} (s^2 \mu)(cd)^{2^r} \\ (s^2 \mu)(bc)^{2^r} \\ (s^2 \mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \rightarrow V$ collapse

to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V -conjugacy to P_α -conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}. \end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β :

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\beta$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows.

Let \mathbf{v}, \mathbf{w} in V . We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1 + w_1)\epsilon_{\alpha+\beta}(v_2 + w_2)\epsilon_{2\alpha+\beta}(v_3 + w_3 + 2v_2w_1) \\ &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 + 2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r . Let \mathbf{v} be an element of V :

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\beta(u^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(u^{p^r}) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \\
&= \begin{pmatrix} v_1 \\ v_2 + u^{p^r} v_1 \\ v_3 + u^{p^r} v_1^2 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \beta^\vee(t^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\beta^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\alpha(\alpha(\beta^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\beta^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\beta^\vee(t^{p^r})) v_3) \\
&= \epsilon_\alpha((t^{p^r})^{\langle \alpha, \beta \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha+\beta, \beta \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha+\beta, \beta \rangle} v_3) \\
&= \begin{pmatrix} t^{-p^r} v_1 \\ t^{p^r} v_2 \\ v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) n_\beta^{-1} n_\beta \epsilon_{\alpha+\beta}(v_2) n_\beta^{-1} n_\beta \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= \epsilon_{\alpha+\beta}(v_1) \epsilon_\alpha(-v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(-v_2) \epsilon_{\alpha+\beta}(v_1) \epsilon_{2\alpha+\beta}(v_3 - 2v_1 v_2) \\
&= \begin{pmatrix} -v_2 \\ v_1 \\ v_3 - 2v_1 v_2 \end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} c^{p^r} v_2 + d^{p^r} v_1 \\ a^{p^r} v_2 + b^{p^r} v_1 \\ v_3 + (ac)^{p^r} v_2^2 + (bd)^{p^r} v_1^2 + 2(bc)^{p^r} v_1 v_2 \end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* , and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k .

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.13)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.14)$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \quad (5.15)$$

$$p_2(t^2u) = t^{p^r} p_2(u) \quad (5.16)$$

$$p_3(t^2u) = p_3(u). \quad (5.17)$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k . From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) + p_2(u_2) \\ 2\lambda \end{pmatrix}
 \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.18)$$

$$\lambda = 2\lambda. \quad (5.19)$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.20)$$

for some μ_i in k .

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^N \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.21)$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
 \mu_l t^{2p^l} &= \mu_l t^{p^r} \\
 \implies 2p^l &= p^r.
 \end{aligned}$$

Thus 2 divides p^r , and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r , and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu^2 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' + \mu \\ \mu \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu' + \mu \\ \mu' \\ \mu'^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu' + \mu \\ \mu' + \mu \\ \mu'^2 \end{pmatrix}.
\end{aligned}$$

Since $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for all t in k^* we must have $\mu' = \mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r} \mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{2^r} (\mu(cd)^{2^{r-1}})^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ac^{-1} + a^2c^{-1}d)^{2^{r-1}} \\ \mu^2(1+ad)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a \mathbf{v} in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
 \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix},
 \end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$, where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
 (\mathbf{s} \cdot \sigma_\mu) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \\
 &= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)(ab)^{2^{r-1}} \\ (s\mu)^2(bc)^{2^r} \end{pmatrix}.
 \end{aligned}$$

our infinitely many V -conjugacy classes collapse to just two P_β -conjugacy classes:

$$\begin{aligned}
 [\sigma_0] &= \{\sigma_0\}, \\
 [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}.
 \end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow \text{Im}(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B_2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ .

1) Know about non G-cr in B_2 , can I put them in an $A_1 A_1$?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $\text{Im}(\rho_{r,s})$, can we conjugate it into P_γ or P_δ ?

Let $\text{char}(k) = 2$ and set $V := \langle U_\phi \mid \phi \in \Phi^+, \phi \neq \gamma + \delta, \phi \neq \gamma + 2\delta \rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2 v_1 \\ 0 \\ u_4 v_1 \\ 0 \\ u_6 v_1 \\ 0 \\ u_8 v_1 \\ 0 \\ u_{10} v_1 \\ u_{10} v_1 v_2 + u_8 v_1 v_4 + u_6^2 v_1 + u_{11} v_2 + u_{10} v_3 + u_9 v_4 + u_8 v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \rightarrow \tilde{A}_1 \tilde{A}_1 < L_{\{\gamma, \delta\}}$ defined by

$$\begin{aligned} \rho_{r,s} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\ \rho_{r,s} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma + \delta)^\vee(t^{2^s}) \\ \rho_{r,s} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta} \end{aligned}$$

from which we obtain an action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so we let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each p_i ($1 \leq i \leq 12$) is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \quad (5.22)$$

It is clear that for $i = 1, 6, 7, 12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t , p_i must be the zero polynomial for $i = 2, 3, 4, 5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \rightarrow k$. That is

for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^N w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^N x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^N y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^N z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^N w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^N w_j u^{2^j} \\ \Rightarrow w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) &= w u^{2^r}, & p_9(u) &= x u^{2^r}, \\ p_{10}(u) &= y u^{2^s}, & p_{11}(u) &= z u^{2^s}, \end{aligned}$$

for some w, x, y, z in k .

So, we have

$$\begin{aligned} \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

We apply the same argument using the fact that each component of $\sigma \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p'_i(u)$ for all u in k , to get

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k .

From this we deduce that

$$\begin{aligned}
 \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ y + y' \\ z + z' \\ w + w' \\ x + x' \\ w' + y' \\ x' + z' \\ w + w' \\ x + x' \\ y + y' \\ z + z' \\ w'x' + y'z' \end{pmatrix}.
 \end{aligned}$$

Furthermore, since $\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, we have

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} n_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_6 \\ n_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_{12} \end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k . So in fact

$$\begin{aligned}
 w' &= w \\
 x' &= x \\
 y' &= y \\
 z' &= z \\
 n_1 &= 0 \\
 n_6 &= w + y \\
 n_7 &= x + z \\
 n_{12} &= wx + yz.
 \end{aligned}$$

Consider $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\begin{aligned}
 \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \rightarrow V$ of the form

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a \mathbf{v} in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g \cdot \mathbf{v}^{-1}$$

for all g in SL_2 . Furthermore, τ' is conjugate to some 1-cocycle τ , where τ has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some \mathbf{v} in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\ &= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix} \end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \sim (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P -conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ s^{-1}t^2y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

Chapter 6

Conclusion

Appendix A

Further Calculations

- G_2 calculation?
- The rest of the B_4 calculations

Appendix B

Source Code

Put source code here ...

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