UNIVERSITY OF CANTERBURY

A Geometric Approach to Complete Reducibility

by

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Declaration of Authorship

I, DANIEL LOND, declare that this thesis titled, 'A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY' and the work presented in it are my own. I confirm that:

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"A quote."

The author of the quote.

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Abstract

College of Engineering
Department of Mathematics and Statistics

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by Daniel Lond

The Thesis Abstract . . .

Acknowledgements

The acknowledgements and the people to thank . . .

Contents

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	$rads^{-1}$
$\vdots$		

Dedication . . .

Chapter 1

Introduction

What is the thesis about and what are the main results

- *What the thesis is about:*

  This thesis is a contribution to the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz. The main motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer [1]:

  > Let G be a linear algebraic group over an algebraically closed field of characteristic p. Let Γ be a finite group and $\Gamma_p \subset \Gamma$ a Sylow p-subgroup. Fix a conjugacy class of representations $\bar{\rho} : \Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

  A counterexample is known for a non-reductive G, and we investigate the reductive case.

- *Main results*:

  ???

Context, history, literary review

- *K. II - motivation for this:*
- *Work of Liebeck & Seitz, etc, on embedding reductive H inside simple G:*

  Let G be a simple algebraic group of exceptional type over an algebraically closed field of characteristic p.

Methods (can refer forward)

- *Key results e.g. $H^1(SL_2, V) \to H^1(B, V)$:*

- *Use of 1-cohomology to (K. II):*
- *Working in low characteristic:*

Chapter Summary

- *Preliminaries:*
- *1-Cohomology:*
- *K. II:*
- *Calculations:*
- *Summary/Future work:*

Chapter 2

Mathematical Preliminaries

Chapter 3

The 1-Cohomology

3.1 Abelian 1-Cohomology

3.1.1 Definitions

Let H be a group and V an abelian group (vector space) on which H acts homomorphically (linearly). We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \tag{3.1}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call (3.1) the *1-cocycle condition.*

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned}
(\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\
&= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\
&= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\
&= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2),
\end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \rightarrow V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H,V)$ we have

$$\begin{aligned}\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1)\\ &= \sigma(1) + \sigma(1)\\ &= 2\,\sigma(1),\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0\\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1\,(H,V)$ is a $\mathbb{Z}-$module under pointwise addition.

Given a v in V we define a *1-coboundary* $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1\,(H,V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}\chi_v^H(h_1h_2) &= v - (h_1h_2) \cdot v\\ &= v - h_1 \cdot (h_2 \cdot v)\\ &= v - h_1 \cdot (v - v + h_2 \cdot v)\\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v)\\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u+v) - h \cdot (u+v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

We see that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a $\mathbb{Z}-$submodule of $Z^1(H, V)$, so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology.*

Lemma 3.1. *Suppose H is linearly reductive. Then $H^1(H, V)$ is trivial [3].*

3.1.2 Maps between 1-cohomologies

Let ϕ be a homomorphism from $\tilde{H} \to H$, $\tilde{H}$ being another group that acts on V. Suppose that for every h in H, ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V. If σ is a 1-cocycle from $H \to V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \to V$.

Take h_1, h_2 in H. We have

$$\begin{aligned} Z^1(\phi)(\sigma)(h_1h_2) &= (\sigma \circ \phi)(h_1h_2) \\ &= \sigma(\phi(h_1h_2)) \\ &= \sigma(\phi(h_1)\phi(h_2)) \\ &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2) \\ &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\ &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\ &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2). \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \to H^1(\tilde{H}, V),$$

defined by

$$\begin{tikzcd} Z^1(H,V) \arrow[r, "Z^1(\phi)"] \arrow[d, "\pi"'] & Z^1(\tilde{H},V) \arrow[d, "\tilde{\pi}"] \\ H^1(H,V) \arrow[r, "H^1(\phi)"] & H^1(\tilde{H},V) \end{tikzcd}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 3.1. *Let $\tilde{H}$ be a subgroup of H and $i : \tilde{H} \to H$ the inclusion map. Then i gives rise to a well defined map*

$$H^1(i) : H^1(H, V) \to H^1(\tilde{H}, V).$$

Lemma 3.2. *Let V be a vector space over a field of characteristic p. Let H be a finite group and $\tilde{H} = H_p$ a* Sylow p-subgroup *of H. The map*

$$H^1(i) : H^1(H, V) \to H^1(H_p, V)$$

is injective.

Proof. Let x be an element of $H^1(H, V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(H, V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to H_p is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since

$\chi_v^{H_p}$ can be trivially extended to a 1-coboundary χ_v^H from $H \to V$, and

$$\pi(\sigma - \chi_v^H) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^H)$ as a representative for x. Hence there is no harm in assuming that σ is 0 when restricted to H_p. Now choose a set of representatives $h_1, \ldots, h_l$ in H for the coset space H/H_p and set

$$v^* = \sum_{i=1}^{l} \sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by v^*

$$\begin{aligned}
\chi_{v^*}^H(h) &= v^* - h \cdot v^* \\
&= \sum_{i=1}^{l} \sigma(h_i) - h \cdot \sum_{i=1}^{l} \sigma(h_i) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} h \cdot \sigma(h_i).
\end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned}
\sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} h \cdot \sigma(h_i) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} (\sigma(hh_i) - \sigma(h)) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h).
\end{aligned}$$

Now as the value of σ at a fixed h depends only on the value of σ at the representative h_j of the coset containing h we can collapse the middle term to yield

$$\begin{aligned}
\chi_{v^*}^H(h) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(h_i) + \sum_{i=1}^{l} \sigma(h) \\
&= l\,\sigma(h).
\end{aligned}$$

Since $\gcd([H : H_p], p) = \gcd(l, p) = 1$, l is invertible and so

$$l^{-1}\chi_{v^*}^H(h) = \sigma(h).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. $\square$

Example 3.2. *Let*

$$k = \bar{\mathbb{F}_p} = \bigcup_r \mathbb{F}_{p^r},$$

V a vector space on which $SL_2(k)$ acts, and $U(k)$ the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Then $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ for each r, and the map

$$H^1(SL_2(k), V) \to H^1(U(k), V)$$

is injective.

Proof. The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r}-1)(p^{2r}-p^r)$ since there are $p^{2r}-1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r}-p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \to \mathbb{F}_{p^r}^*,$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First homomorphism theorem for groups

$$GL_2(\mathbb{F}_{p^r}) \,/\, SL_2(\mathbb{F}_{p^r}) \sim \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| \,/\, |\mathbb{F}_{p^r}^*| \\ &= (p^{2r}-1)(p^{2r}-p^r) \,/\, (p^r-1) \\ &= p^r(p^{2r}-1). \end{aligned}$$

Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$.

Fix a non-trivial $y \in H^1(SL_2(k), V)$ and choose a representative $\tau \in Z^1(SL_2(k), V)$ for y. For each $g \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_g^{(r)} : V \to V$ by

$$f_g^{(r)}(v) = \tau(g) - \chi_v(g) = \tau(g) - v + g \cdot v.$$

Consider the sequence of subsets of V defined by

$$C_r = \{v \in V | f_g^{(r)}(v) = 0\}.$$

Each subset C_r is closed and the inclusion $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^{r+1}}$ induces the reverse inclusion $C_r \supset C_{r+1}$. The Noetherian property for V requires that the sequence becomes constant.

However, $y \neq 0$ so τ is not a 1-coboundary on $SL_2(k)$, which means the C_r's are eventually empty. That is, there exists an integer s such that for any v in V

$$(\tau - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if $y|_{SL_2(\mathbb{F}_{p^r})} = 0$ for all r then $y = 0$.

Take x in the kernel of the map $H^1(SL_2(k), V) \to H^1(U(k), V)$. Then for each r, $x|_{U(\mathbb{F}_{p^r})} = 0$ so by Lemma 3.2 $x|_{SL_2(\mathbb{F}_{p^r})} = 0$. Therefore $x = 0$ and so $H^1(SL_2(k), V) \to H^1(U(k), V)$ is injective. □

We could also consider appropriate maps $f : V \to \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \to H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \to H^1(\tilde{H}, \tilde{V}).$$

3.2 Non-abelian 1-Cohomology

3.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocyles be morphisms of varieties.

3.2.2 Definitions

Let H, V be algebraic groups, H acting on V. We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{3.2}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call the (3.2) the *1-cocycle condition.*

Given a v in V we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by $B^1(H,V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}
\chi_v^H(h_1h_2) &= v * (h_1h_2) \cdot v^{-1} \\
&= v * h_1 \cdot \left(h_2 \cdot v^{-1}\right) \\
&= v * h_1 \cdot \left(vv^{-1}h_2 \cdot v\right) \\
&= v * h_1 \cdot v * h_1 \cdot \left(v * h_2 \cdot v^{-1}\right) \\
&= \chi_v^H(h_1) * h_1 \cdot \chi_v^H(h_2),
\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say σ_1, σ_2 in $Z^1(H,V)$ are *equivalent* if there exists a v in V such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \tag{3.3}$$

for all h in H. We call the set of equivalence classes of $Z^1(H,V)$ under the equivalence relation defined by (3.3) the *1-cohomology*, denoted $H^1(H,V)$.

3.2.3 Maps between 1-cohomologies

Lemma 3.3. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V. Then $H^1(i) : H^1(SL_2,V) \to H^1(B,V)$ is injective.*

Proof. Let x be in the kernel of $H^1(i)$ and σ and element of $Z^1(SL_2,V)$ that projects onto the class x. Since $Z^1(i)(\sigma)$ projects to the trivial 1-cohomology class we may as well assume that $\sigma|_B = 1$. For there exists some v in V such that for all b in B

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle $\hat{\sigma} : SL_2 \to V$ defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction $\hat{\sigma}$ also projects to the class x, and for all b in B

$$\begin{aligned}
\hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\
&= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\
&= v^{-1} * v * b \cdot (v^{-1} * v) \\
&= 1,
\end{aligned}$$

so we may as well have chosen $\hat{\sigma}$ instead as a representative for x.

Now consider the *homogeneous space* SL_2/B [4] and take the map

$$\tilde{\sigma} : SL_2/B \to V,$$

defined in the usual way under the canonical projection $\pi : SL_2 \to SL_2/B$:

$$\begin{tikzcd} SL_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & V \\ SL_2/B \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

This map is well defined and is a morphism [5]. Now since SL_2/B is an irreducible projective variety [4], $\tilde{\sigma}$ must be constant [5]. Hence, as σ takes the value 1 for any b in B, $\tilde{\sigma}(hB) = 1$ for all cosets hB. Therefore, for all h in SL_2

$$\sigma(h) = \tilde{\sigma}(hB) = 1.$$

We have shown that σ is the 1-coboundary χ_1 which means that the kernel of $H^1(i)$ is trivial. $\square$

Lemma 3.4. *Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B. Then $H^1(B,V) \to H^1(U,V)$ is injective. Moreover*

$$H^1(SL_2, V) \to H^1(U, V)$$

is injective.

Proof. Let x be an element of the kernel of $H^1(i) : H^1(B,V) \to H^1(U,V)$ and let σ in $Z^1(B,V)$ be a representative for x. We may as well assume that $\sigma|_T = 1$. For any b in B we can find a u in U and a t in T such that $b = ut$. Hence

$$\begin{aligned} \sigma(b) &= \sigma(ut) \\ &= \sigma(u) * u \cdot \sigma(t) \\ &= \sigma(u). \end{aligned}$$

Since σ represents x, σ must be a 1-coboundary on U. Hence σ is in $B^1(B,V)$ and the kernel of $H^1(i) : H^1(B,V) \to H^1(U,V)$ is trivial. $\square$

Chapter 4

Külshammer's Second Problem

4.1 Külshammer's Second Problem

Two questions raised by B. Külshammer concerning representations of a finite group Γ into a linear algebraic group G over an algebraically closed field k. The first has a positive answer and is essentially contained a paper by A. Weil [1]:

(K. I) Let $\mathrm{char}(k)$ be prime to the order of Γ. Are there only finitely many representations $\rho : \Gamma \to G$ up to conjugation by G?

(K. II) Let $p = \mathrm{char}(k)$ and $\Gamma_p \subset \Gamma$ be a Sylow p-subgroup. Fix a conjugacy class of representations from $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

(K. II) has positive answer so long as G is reductive and the characteristic of k is good for G [2]. The same paper shows that the answer is "no" in general by way of a counterexample involving a non-reductive G.

We will explore the possibility of a reductive counterexample to (K. II).

4.2 The Approach

We are interested in knowing whether there can be infinitely many G-conjugacy classes of representations $\Gamma \to G$ that when restricted to Γ_p hit some fixed G-conjugacy class of representations $\Gamma_p \to G$.

Theorem 4.1. *There are only finitely many G-conjugacy classes of G-completely reducible representations $\Gamma \to G$.*

Reference something. □

Although G has infinitely many parabolic subgroups there are only finitely many G-conjugacy classes of parabolic subgroups, so we can choose a finite set $\{Q_i\}$ of representatives. We choose a set of Levi subgroups $\{M_i\}$, M_i being a Levi subgroup of Q_i. By Theorem 4.1 there are only finitely many M_i-conjugacy classes of M_i-completely reducible representations $\sigma_0^{(i)} : \Gamma \to M_i$, so we fix a set of representatives $\{\sigma_{0,j}^{(i)}\}$.

We will show that for each representation $\rho : \Gamma \to G$ there exists a representation σ that is G-conjugate to ρ and that fits one of only finitely many commutative diagrams

$$\begin{tikzcd} \Gamma \arrow[r, "\sigma"] \arrow[dr, "\sigma_{0,j}^{(i)}"'] & Q_i \arrow[d] \\ & M_i \end{tikzcd}$$

Let ρ be a representation from $\Gamma \to G$ and let P be a minimal parabolic subgroup of G containing $\rho(\Gamma)$. Then there is a g in G such that $P = g \cdot Q$, where $Q \in \{Q_i\}$. Let $\rho' = g \cdot \rho$.

Define $\rho_0 : \Gamma \to M$ by composing ρ' with the projection $Q \to M$, $M \in \{M_i\}$ the chosen Levi subgroup for Q:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho'"] \arrow[dr, "\rho_0"'] & Q \arrow[d] \\ & M \end{tikzcd}$$

Since Q is a minimal parabolic containing $\rho'(\Gamma)$, ρ_0 is M-irreducible [reference] and therefore M-completely reducible. Hence we can choose an $m \in M$ such that $\sigma_0 = m \cdot \rho_0$ where $\sigma_0 \in \{\sigma_{0,j}^{(i)}\}$. Let $\sigma = m \cdot \rho' = mg \cdot \rho$. This verifies what we set out to show.

[Say something about $\rho \rightsquigarrow (i,j) \rightsquigarrow H^1(\Gamma, V_i)$ so that I can state the next lemma]

For a given parabolic $P < G$, Levi $L < P$ and representation $\rho : \Gamma \to P$ we have defined the map $\rho_0 : \Gamma \to L$ by the projection $P \to L$. Now define $\alpha_\rho : \Gamma \to R_u(P)$ by projecting on to $R_u(P)$, so that $\rho = \alpha_\rho \rho_0$. If ρ_0 is a homomorphism then

$$\begin{aligned} \alpha_\rho(\gamma_1\gamma_2)\rho_0(\gamma_1\gamma_2) = \rho(\gamma_1\gamma_2) &= \rho(\gamma_1)\rho(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_1)^{-1}\rho_0(\gamma_1)\rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_1)^{-1}\rho_0(\gamma_1\gamma_2), \end{aligned}$$

so that

$$\alpha_\rho(\gamma_1\gamma_2) = \alpha_\rho(\gamma_1)\rho_0(\gamma_1) \cdot \alpha_\rho(\gamma_2),$$

which is the 1-cocycle condition in (3.1).

Hence for the given ρ we have a corresponding 1-cocycle $\alpha_\rho : \Gamma \to R_u(P)$ where Γ acts on $R_u(P)$ via ρ_0.

Suppose we conjugate ρ by an element $g \in G$. Then $\rho' = g \cdot \rho$ has corresponding 1-cocycle $\alpha_{\rho'} : \Gamma \to gR_u(P)g^{-1}$.

So the G-action on $\rho : \Gamma \to P$ almost corresponds to maps of 1-cocycles of the form:

$$Z^1(\Gamma, R_u(P)) \to Z^1(\Gamma, gR_u(P)g^{-1}),$$

the catch being that conjugating ρ by an arbitrary element of g changes the action of Γ on $R_u(P)$. If we choose $g \in Z(L)^\circ$ and consider the $Z(L)^\circ$-action:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[dr, "\rho_0"'] & P \arrow[d] \\ & L \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \Gamma \arrow[r, "g\cdot\rho"] \arrow[dr, "\rho_0"'] & gPg^{-1} \arrow[d] \\ & L \end{tikzcd}$$

then we really do get a map of 1-cocycles.

Lemma 4.2. *Let $\{\rho_\mu \,|\, \mu \in I\}$ be a collection of representations $\Gamma \to P$ for a fixed parabolic subgroup $P < G$. The following statements are equivalent:*

(i) There are only finitely many P-conjugacy classes of $\{\rho_\mu\}$.

(ii) There are only finitely many G-conjugacy classes of $\{\rho_\mu\}$.

(iii) For each i, the number of elements of $H^1(\Gamma, V_i)$ modulo $Z(M_i^\circ)$ that come from the ρ_μ's is finite.

Proof. (i) $\Rightarrow$ (ii) is obvious. ☐

[Therefore the problem is controlled by the 1-cohomology]

4.3 An algebraic group version

In an attempt to gain further insight into (K. II) we adjust the original question by letting Γ be an infinite group H. The advantage being that a negative answer in the algebraic group version may provide a negative answer to (K. II) by choosing an appropriate finite subgroup Γ of H. In many of the examples to follow we set $H = SL_2(K)$ with Sylow p-subgroup $H_p = U_2(K)$ consisting of upper unitriangular matrices.

Let $P \subset G$ be a parabolic subgroup and $L \subset P$ the corresponding Levi subgroup. Fix a representation $\rho_0 : H \to L$. We can assume $\rho_0(H)$ is L-irreducible, that is, not contained in a proper parabolic of L.

Now define $\rho_\alpha : H \to P$ by $\rho_\alpha(h) = \alpha(h)\rho_0(h)$ where $\alpha : H \to R_u(P)$, $R_u(P)$ the unipotent radical of P.

For ρ_α to be a homomorphism

$$\begin{aligned}
\alpha(h_1h_2)\rho_0(h_1h_2) &= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_2) \\
&= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_1)^{-1}\rho_0(h_1)\rho_0(h_2) \\
&= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_1)^{-1}\rho_0(h_1h_2).
\end{aligned}$$

That is $\alpha(h_1h_2) = \alpha(h_1)h_1 \cdot \alpha(h_2)$, where the action $H \times R_u(P) \to R_u(P)$ is conjugation via ρ_0. This is a 1-cocycle condtion; $\alpha \in Z^1(H, R_u(P))$. $R_u(P)$ will not be abelian in general.

Now suppose ρ_α is $R_u(P)$-conjugate to some ρ_β, $\alpha, \beta \in Z^1(H, R_u(P))$. That is, there exists a $v \in R_u(P)$ such that for all $h \in H$

$$\begin{aligned}
\alpha(h)\rho_0(h) &= v\beta(h)\rho_0(h)v^{-1} \\
&= v\beta(h)\rho_0(h)v^{-1}\rho_0(h)^{-1}\rho_0(h).
\end{aligned}$$

That is $\alpha(h) = v\beta(h)h \cdot v^{-1}$. In particular if ρ_α is $R_u(P)$-conjugate to ρ_0, that is β is trivial, then α takes the form of a 1-coboundary. Generally speaking α and β project to the same 1-cohomology class. In the abelian case this reads "α and β differ by a 1-coboundary":

$$\alpha(h) = v\beta(h)h \cdot v^{-1} \quad \rightsquigarrow \quad \begin{aligned}
\alpha(h) &= v + \beta(h) - h \cdot v \\
&= \beta(h) + v - h \cdot v \\
&= \beta(h) + \chi_v(h).
\end{aligned}$$

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G. The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p. Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G.

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α:

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r \quad : \quad & SL_2(k) \to SL_2(k) \\ & (A_{ij}) \mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r \quad = \quad \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad &= \quad \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad &= \quad \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) \quad &= \quad \prod_\delta \epsilon_\delta\left((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_\delta\right) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) \quad &= \quad \prod_\delta n_\alpha \epsilon_\delta\left(\lambda_\delta\right) n_\alpha^{-1}, \end{aligned}$$

where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ and λ_δ are elements of the underlying field k.

Lemma 5.1.

$$\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta\left(u\right)\right),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_\delta \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ. Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \to k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

Therefore

$$x_\delta\left(t^2u\right) = (t^{p^r})^{\langle\delta,\alpha\rangle} x_\delta(u).$$

Since x_δ is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle\delta,\alpha\rangle \geq 0$. However, if $\langle\delta,\alpha\rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle\delta,\alpha\rangle > 0$. $\square$

Next we prove a couple of useful facts about root systems not containing G_2 or C_3.

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha, \beta \in \Phi$. If $\alpha+\beta \in \Phi$ then $\langle\alpha,\beta\rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β. Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. □

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3. Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ $(i = 1, 2)$. If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ, $(i = 1, 2)$ and let θ_3 be the absolute value of the angle between δ_1 and δ_2. Then

$$\begin{aligned} & \langle \delta_i, \gamma \rangle > 0 \qquad (i = 1, 2) \\ \Longrightarrow \quad & (\delta_i, \gamma) > 0 \\ \Longrightarrow \quad & \cos(\theta_i) > 0 \\ \Longrightarrow \quad & \theta_i < \pi/2, \end{aligned}$$

and similarly, using Lemma 5.2

$$\begin{aligned} & \langle \delta_1, \delta_2 \rangle \leq 0 \\ \Longrightarrow \quad & \theta_3 \geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

$$\begin{array}{llll} \textbf{1:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = 2\pi/3; \\ \textbf{2:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{3:} & \theta_1 = \pi/4, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{4:} & \theta_1 = \pi/4, & \theta_2 = \pi/4, & \theta_3 = \pi/2. \end{array}$$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ, and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3. [Picture?][Wow, is that right? Maybe just say 'we will show that they lie in a C_3 subsystem'.] □

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3.

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \epsilon_\alpha(u_1^{p^r}) \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By Lemma 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α. □

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α.*

Proof. First consider

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta(b)\right).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By Lemma 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$ $(i = 1, 2)$, so Lemma 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix}\right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \\
&= \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b)\right).
\end{aligned}$$

□

Since the x_δ are homomorphisms from $k \to k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k. Furthermore, combining the calculation in the proof of Lemma 5.1 with the result in Corollary 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta}\left(x_{\delta}\left(a^2 b\right)\right) = \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b)\right),$$

severely restricting the possible polynomials x_δ. In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer r in the definition of ρ_r. For suppose x_δ and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned}
\mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\
\Longrightarrow 2p^j &= \langle \delta, \alpha \rangle p^r.
\end{aligned}$$

In [6] it is shown that the possible pairings of any two roots are bounded by ± 3. Hence by Lemma 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3. It is now clear that if $\langle \delta, \alpha \rangle = 3$ then $x_\delta = 0$.

If $\langle \delta, \alpha \rangle = 1$ the characteristic of k must be 2 and $j = r - 1$. Otherwise $\langle \delta, \alpha \rangle = 2$ and $j = r$, but the characteristic of k is so far unrestricted.

Example 5.1. *Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

We will write v in V_α in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle \quad := \quad \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{3\alpha+\beta}(v_4)\epsilon_{3\alpha+2\beta}(v_5) \in V_\alpha$$

The group law for V_α is

$$u * v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-3p^r} v_1, a^{-p^r} v_2, a^{p^r} v_3, a^{3p^r} v_4, v_5 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), x_4(b), 0 \rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2 b) &= a^{p^r} x_3(b) \\ x_4(a^2 b) &= a^{3p^r} x_4(b). \end{aligned}$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1 + b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1 + b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r} x_3(b_2). \end{aligned}$$

We see that x_3 is a homomorphism, so it is of the form

$$x_3(b) = \sum_i \mu_i b^{p^i}.$$

Suppose $x_3 \neq 0$*. Then some* $\mu_j \neq 0$ *and*

$$\begin{aligned} & \mu_j(a^2b)^{p^j} = a^{p^r}\mu_j b^{p^j} \\ \implies & a^{2p^j} = a^{p^r} \\ \implies & p = 2. \end{aligned}$$

But then

$$\begin{aligned} x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r}x_3(b) \\ &= b^{2^r}x_3(b), \end{aligned}$$

implies that x_3 *is constant, hence zero.*

Therefore $x_3 = 0$*, so* x_4 *is a homomorphism:*

$$x_4(b) = \sum_i \nu_i b^{p^r}.$$

If $x_4 \neq 0$ *then there is a* $\nu_j \neq 0$ *and we get*

$$\begin{aligned} & \nu_j(a^2b)^{p^j} = a^{3p^r}\nu_j b^{p^j} \\ \implies & a^{2p^j} = a^{3p^r} \\ \implies & 2p^j = 3p^r, \end{aligned}$$

which implies that 2 divides p and 3 divides p, a contradiction. Hence $x_4 = 0$ *and*

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Example 5.2. *Let* $G = C_3$*. Fix a maximal torus, labeling the positive simple roots* $\Delta = \{\alpha, \beta, \gamma\}$ *with* γ *being the long root and connected to* β*. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle.$$

Again we will write v *in* V_α *in angled brackets for ease of notation:*

$$\begin{aligned} & \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\rangle := \\ & \epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8) \in V_\alpha \end{aligned}$$

The group law for V_α *is*

$$
\begin{aligned}
&u * v = \\
&\langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3 v_2, u_6 + v_6 + u_2 v_1^2 + 2u_4 v_1, \\
&\quad u_7 + v_7 + u_2 u_3 v_1 + u_2 v_1 v_3 + u_5 v_1 + u_4 v_3, u_8 + v_8 - u_3^2 v_2 - 2u_3 v_2 v_3 + 2u_5 v_3\rangle.
\end{aligned}
$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v \quad = \quad \langle a^{-p^r} v_1, v_2, a^{p^r} v_3, a^{-p^r} v_4, a^{p^r} v_5, a^{-2p^r} v_6, v_7, a^{2p^r} v_8\rangle.$$

Let σ *be in* $Z^1(SL_2, V_\alpha)$ *such that*

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b)\rangle.$$

Applying σ *to both sides of the identity*

$$\begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$
\begin{aligned}
x_3(a^2 b) &= a^{p^r} x_3(b) \\
x_5(a^2 b) &= a^{p^r} x_5(b) \\
x_8(a^2 b) &= a^{2p^r} x_8(b).
\end{aligned}
$$

Since the polynomials x_3, x_5, x_8 *are homomorphisms (Lemma 5.2) we get*

$$
\begin{aligned}
\sum_i \lambda_i (a^2 b)^{p^i} &= a^{p^r} \sum_i \lambda_i b^{p^i} \\
\sum_i \mu_i (a^2 b)^{p^i} &= a^{p^r} \sum_i \mu_i b^{p^i} \\
\sum_i \nu_i (a^2 b)^{p^i} &= a^{2p^r} \sum_i \nu_i b^{p^i},
\end{aligned}
$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of SL_2 under σ is $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle$ then the characteristic of k must be 2, and so the image is a product of commuting root groups.

We may now state and prove the main result.

[would like]

Theorem 5.6. *Let G be a reductive linear algebraic group over a closed field of positive characteristic p and let $\Gamma = SL_2(k)$. Then the answer to the algebraic interpretation of Külshammer's Second Problem [ref] is "yes".*

Proof. Need to:

- handle arguments above with G possibly containing G_2 and C_3.
- drop the restriction of rank-1 parabolics
- now we have abelian 1-cohomology and can apply result from previous chapter

□

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p. We label the positive roots for B_2 as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [4, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α:

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\alpha$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha, \end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r: write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_\alpha(-u^{p^r}) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_\alpha(-u^{p^r}) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(-2u^{p^r} v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\alpha(-u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(-u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{2p^r} v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2) \\
&= \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2 - u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r} v_1 \\ v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r})) v_1\right) \epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r})) v_2\right) \epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r})) v_3\right) \\
&= \epsilon_\beta\left((t^{p^r})^{\langle \beta, \alpha \rangle} v_1\right) \epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle \alpha+\beta, \alpha \rangle} v_2\right) \epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta, \alpha \rangle} v_3\right) \\
&= \begin{pmatrix} t^{-2p^r} v_1 \\ v_2 \\ t^{2p^r} v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) n_\alpha^{-1} n_\alpha \epsilon_{\alpha+\beta}(v_2) n_\alpha^{-1} n_\alpha \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1) \epsilon_{\alpha+\beta}(-v_2) \epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3) \epsilon_{\alpha+\beta}(-v_2) \epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad+bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*. Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

to get further information on the polynomials p_i $(i = 1, 2, 3)$.

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

That is,

$$
\begin{aligned}
p_1(t^2u) &= t^{-2p^r}p_1(u) && (5.3)\\
p_2(t^2u) &= p_2(u) && (5.4)\\
p_3(t^2u) &= t^{2p^r}p_3(u). && (5.5)
\end{aligned}
$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k. Now notice that on the left hand side of (5.3) there are only non-negative powers of t, and on the right hand side there are only non-positive powers of t. This equality is only satisfied if $p_1(x) = 0$ for all x in k, so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
p_2(u_1+u_2) &= p_2(u_1)+p_2(u_2) && (5.6)\\
p_3(u_1+u_2) &= p_3(u_1)+p_3(u_2)-2u_1^{p^r}p_2(u_2). && (5.7)
\end{aligned}
$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$p_3(u_1+u_2) = p_3(u_1)+p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{5.8}$$

for some u_i in k.

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{2p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{5.9}$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating the coefficients of u in (5.9) we get

$$\begin{aligned} \mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies p^l &= p^r. \end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r.

Letting $\mu_l = \mu$ in k, we have

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}. \end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k, p, is not equal to 2 then by setting $\mathbf{v}$ in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
\chi_v\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\cdot\mathbf{v} \\
&= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
&= \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right).
\end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2. By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \to V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \tag{5.10}$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d\end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu+\mu'\\\mu\\\mu\end{pmatrix}\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}\mu+\mu'\\\mu'\\\mu'\end{pmatrix}=\begin{pmatrix}\mu+\mu'\\\mu'\\\mu+\mu'\end{pmatrix}.
\end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}t^{-1}&0\\0&t\end{pmatrix},$$

we get

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right).$$

Therefore $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is an element of V that is fixed by the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix}0\\ *\\0\end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

If $c = 0$ then we already have

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r}\mu(cd)^{2^r} \\ (ac^{-1})^{2^{r+1}}\mu(cd)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1+ad)^{2^r} \\ \mu(ac^{-1})^{2^r}(1+ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

In fact, we see that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering $\mathbf{v}$ that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$ where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a $\mathbf{v}$ in V such that

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \;=\; \mathbf{v} + \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \to P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α. Now,

$$Z(L_\alpha)^\circ \;=\; \langle \gamma^\vee(x) \,|\, x \in k \rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle \;=\; 0. \tag{5.11}$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n, we have

$$\langle \alpha, \gamma \rangle \;=\; \langle \alpha, m\alpha + n\beta \rangle \tag{5.12}$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \,|\, x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\ &= \epsilon_\beta\left(s^{\langle \beta, \alpha+\beta \rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle \alpha+\beta, \alpha+\beta \rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle}\mu(ab)^{2^r}\right) \\ &= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \to V$ collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V-conjugacy to P_α-conjugacy:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\} \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}\,.
\end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β:

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\beta$ defined by

$$\begin{aligned}
\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\
\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\
\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta,
\end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows. Let $\mathbf{v}, \mathbf{w}$ in V. We have, using notation similar to the previous example

$$\begin{aligned}
\mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\
&= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1 \end{pmatrix}.
\end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r. Let $\mathbf{v}$ be an element of V:

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*, and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k.

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.13}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \tag{5.15}$$

$$p_2(t^2u) = t^{p^r} p_2(u) \tag{5.16}$$

$$p_3(t^2u) = p_3(u). \tag{5.17}$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k. From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\right) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ p_2(u_1)+p_2(u_2) \\ 2\lambda \end{pmatrix}
\end{aligned}$$

That is,

$$p_2(u_1+u_2) = p_2(u_1)+p_2(u_2) \tag{5.18}$$

$$\lambda = 2\lambda. \tag{5.19}$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^{N} \mu_i x^{p^i}, \tag{5.20}$$

for some μ_i in k.

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^{N} \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^{N} \mu_i u^{p^i}. \tag{5.21}$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
\mu_l t^{2p^l} &= \mu_l t^{p^r} \\
\Longrightarrow \quad 2p^l &= p^r.
\end{aligned}$$

Thus 2 divides p^r, and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r, and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

We have

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\\mu\\0\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu^2\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu'+\mu\\\mu\\\mu^2\end{pmatrix}\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}\mu'+\mu\\\mu'\\\mu'^2\end{pmatrix}\\
&= \begin{pmatrix}\mu'+\mu\\\mu'+\mu\\\mu'^2\end{pmatrix}.
\end{aligned}
$$

Since $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$ for all t in k^* we must have $\mu'=\mu$.

Suppose $c \neq 0$. We have

$$
\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r}\mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{p^r}\left(\mu(cd)^{2^{r-1}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ac^{-1} + a^2c^{-1}d\right)^{2^{r-1}} \\ \mu^2\left(1 + ad\right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
\end{aligned}
$$

But the above result holds when $c = 0$ too, so we conclude that

$$
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
$$

[Show converse is true]

As in the previous example, we choose a $\mathbf{v}$ in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix},
\end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$, where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
(\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)\,(ab)^{2^{r-1}} \\ (s\mu)^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

our infinitely many V-conjugacy classes collapse to just two P_β-conjugacy classes:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\}, \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}.
\end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow Im(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ.

1) Know about non G-cr in B_2, can I put them in an A_1A_1?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $Im(\rho_{r,s})$, can we conjugate it into P_γ or P_δ?

Let char$(k) = 2$ and set $V := \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma+\delta, \phi \neq \gamma+2\delta\}\rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)$ $\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} \quad = \quad \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\begin{aligned}
\rho_{r,s}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\
\rho_{r,s}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma+\delta)^\vee(t^{2^s}) \\
\rho_{r,s}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta}
\end{aligned}$$

from which we obtain an action of SL_2 on V:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix}
v_1 \\
c^{2^{s+1}} v_{10} + d^{2^{s+1}} v_2 \\
c^{2^{s+1}} v_{11} + d^{2^{s+1}} v_3 \\
c^{2^{r+1}} v_8 + d^{2^{r+1}} v_4 \\
c^{2^{r+1}} v_9 + d^{2^{r+1}} v_5 \\
v_6 + (bd)^{2^r} v_4 + (bd)^{2^s} v_2 + (ac)^{2^r} v_8 + (ac)^{2^s} v_{10} \\
v_7 + (bd)^{2^r} v_5 + (bd)^{2^s} v_3 + (ac)^{2^r} v_9 + (ac)^{2^s} v_{11} \\
a^{2^{r+1}} v_8 + b^{2^{r+1}} v_4 \\
a^{2^{r+1}} v_9 + b^{2^{r+1}} v_5 \\
a^{2^{s+1}} v_{10} + b^{2^{s+1}} v_2 \\
a^{2^{s+1}} v_{11} + b^{2^{s+1}} v_3 \\
v_{12} + (bd)^{2^{r+1}} v_4 v_5 + (bd)^{2^{s+1}} v_2 v_3 + (bc)^{2^{r+1}} (v_4 v_9 + v_5 v_8) \\
+ (bc)^{2^{s+1}} (v_2 v_{11} + v_3 v_{10}) + (ac)^{2^{r+1}} (v_8 v_9) + (ac)^{2^{s+1}} (v_{10} v_{11})
\end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so we let

$$\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each p_i ($1 \leq i \leq 12$) is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1,6,7,12 \\ t^{-2^{r+1}} p_i(u), & i = 4,5 \\ t^{-2^{s+1}} p_i(u), & i = 2,3 \\ t^{2^{r+1}} p_i(u), & i = 8,9 \\ t^{2^{s+1}} p_i(u), & i = 10,11 \end{cases} \tag{5.22}$$

It is clear that for $i = 1,6,7,12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t, p_i must be the zero polynomial for $i = 2,3,4,5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned}
p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\
p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\
p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\
p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\
p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\
p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\
p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\
p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2).
\end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\ \Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow \quad l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$p_8(u) = w u^{2^r}, \quad p_9(u) = x u^{2^r},$$
$$p_{10}(u) = y u^{2^s}, \quad p_{11}(u) = z u^{2^s},$$

for some w, x, y, z in k.

So, we have

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} = \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}$ is a polynomial function, say $p_i'(u)$ for all u in k, to get

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d\end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k.

From this we deduce that

$$\begin{aligned}
\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} * \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} * \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} * \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}$$

Furthermore, since $\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$, we have

$$\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}n_1\\ 0\\ 0\\ 0\\ 0\\ n_6\\ n_7\\ 0\\ 0\\ 0\\ 0\\ n_{12}\end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k. So in fact

$$\begin{aligned} w' &= w \\ x' &= x \\ y' &= y \\ z' &= z \\ n_1 &= 0 \\ n_6 &= w+y \\ n_7 &= x+z \\ n_{12} &= wx+yz. \end{aligned}$$

Consider $\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c=0$ then we already have

$$\begin{aligned} \sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$
\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}
$$

We see that in any case

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \to V$ of the form

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a $\mathbf{v}$ in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g.\mathbf{v}^{-1}$$

for all g in SL_2. Furthermore, τ' is conjugate to some 1-cocycle τ, where τ has the added property that

$$\tau\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0\end{pmatrix}.$$

Thus σ is conjugate to τ by some $\mathbf{v}$ in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1}\end{pmatrix}$:

$$\begin{aligned}
\tau\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \mathbf{v} * \sigma\begin{pmatrix} a & b \\ c & d\end{pmatrix} * \begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \mathbf{v}^{-1} \\
&= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12}\end{pmatrix} * \sigma\begin{pmatrix} a & b \\ c & d\end{pmatrix} * \begin{pmatrix} a & b \\ c & d\end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2\end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}}\end{pmatrix}
\end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \quad \sim \quad (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P-conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ s^{-1}t^2 y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2 w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2 w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2 y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

5.4 A Non-Reductive Counterexample

In [2] a counterexample to [ref KII] is presented for a closed field k of characteristic $p = 2$ and a non-reductive algebraic group G.

Example 5.3. *Let Q be the algebraic group isomorphic to the affine space $\mathbf{A}^3$ with the group multiplication law:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1 v_1 + u_2 v_2 + u_1 v_2 \end{pmatrix}.$$

Let $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$ and $\Gamma_2 = \langle \tau \rangle$ the Sylow 2-subgroup of Γ. Γ acts on Q via

$$\tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix}$$
$$\sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}.$$

Let $G = Q \rtimes \Gamma$. Then there are infinitely many pairwise G-conjugate classes of extensions to the representation $\rho : \Gamma_2 \to G$ defined by the natural inclusion $\Gamma_2 \to \Gamma \to G$ [2, Appendix].

Proof. Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle $\alpha \in Z^1(\Gamma, Q)$ such that $\alpha|_{\langle\sigma\rangle} = 1$. Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned} 1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1 u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}. \end{aligned}$$

This shows $u_1 = u_2$, so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\sigma\tau = \sigma^2$ we obtain

$$\begin{aligned}
1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
&= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
\end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle\sigma\rangle$ satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \qquad (u \in k).$$

Now we calculate the class $[\alpha_u] \in H^1(\Gamma, Q)$. Suppose $\alpha_v \sim \alpha_u$. Then there is a $q \in Q$ fixed under the action of σ, that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
\end{aligned}$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(\Gamma, Q)$ is infinite. [to finish]. □

[note how to make my example look like the Slodowy one]

It is natural to ask whether this leads to a reductive counterexample, although we can quickly verify that the answer is "not immediately". For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$\begin{aligned}
&\ldots \epsilon_\alpha(u_\alpha) \ldots \epsilon_\beta(u_\beta) \ldots \epsilon_\gamma(u_\gamma) \times \ldots \epsilon_\alpha(v_\alpha) \ldots \epsilon_\beta(v_\beta) \ldots \epsilon_\gamma(v_\gamma) \\
&= \ldots \epsilon_\alpha(u_\alpha + v_\alpha) \ldots \epsilon_\beta(u_\beta + v_\beta) \ldots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
\end{aligned}$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \alpha$ gives

$$\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) = \epsilon_\alpha(u_\alpha + v_\alpha)\epsilon_\gamma(u_\alpha v_\alpha),$$

which is absurd. [try find more examples]

Chapter 6

Conclusion

Appendix A

Further Calculations

G	P	Z^1	H^1	V-conj	P-conj
B_2 (α short)	P_α	✓	✓	✓	✓
	P_β	✓	✓	✓	✓
G_2 (α short)	P_α	✓			
C_3 (γ long)	P_α	✓			
[2]	$Q \rtimes SL(2,2)$	✓	✓	✓	

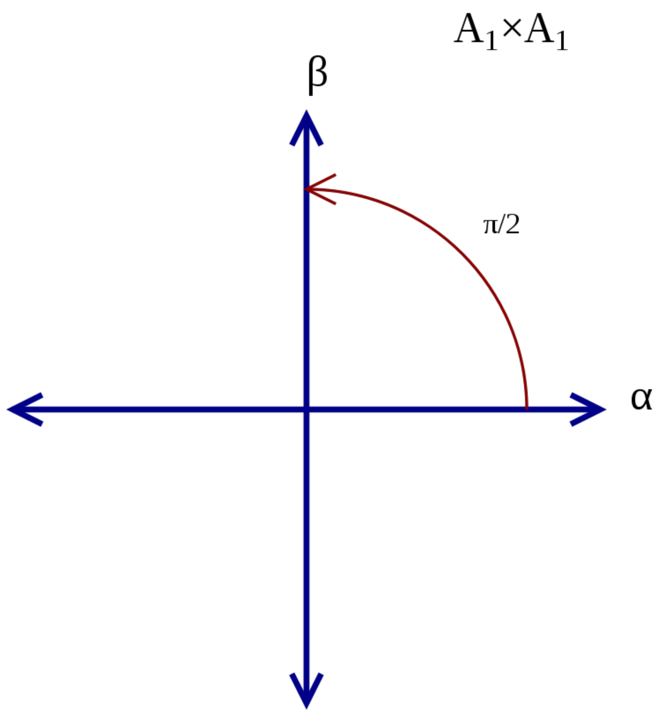
Appendix B

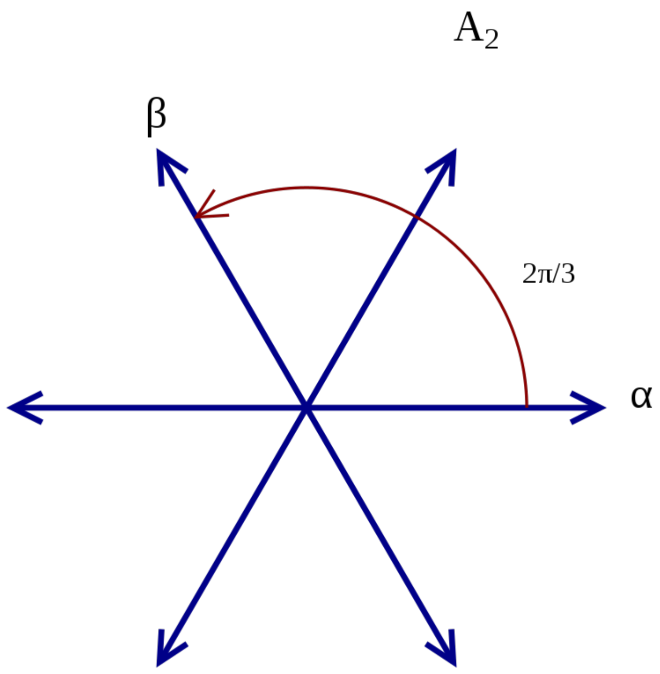
Source Code

Put source code here ...

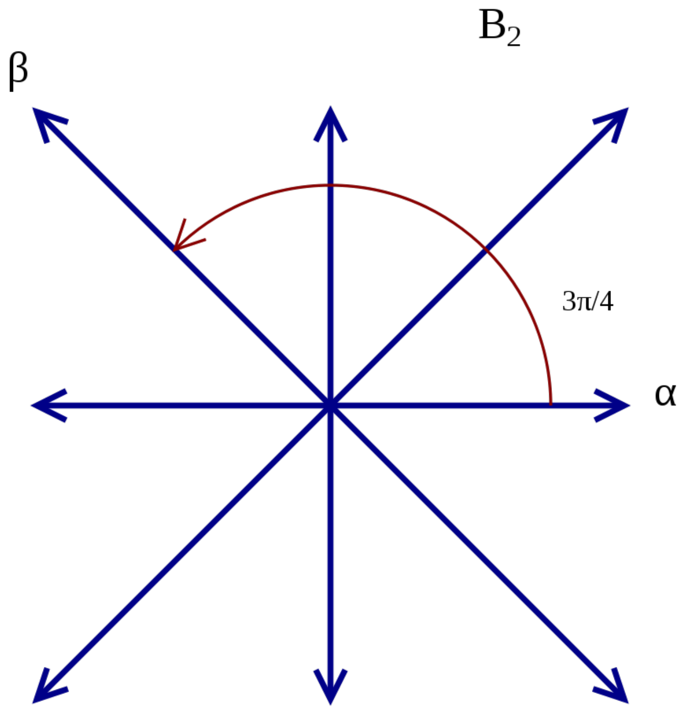
Appendix C

Rank 2 Root System Diagrams

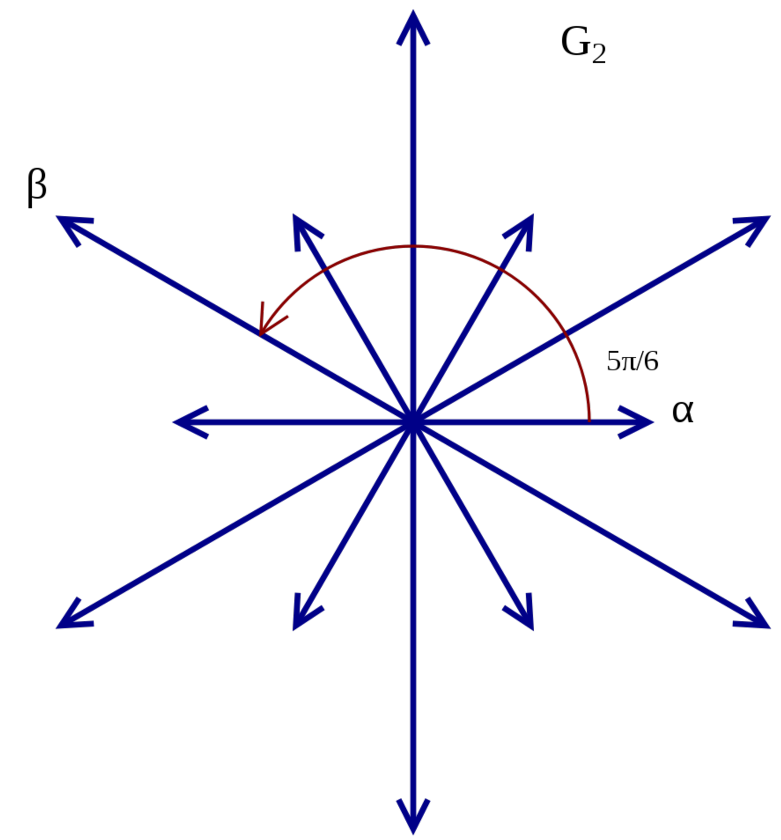




A2
β
2π/3
α



B2
β
3π/4
α


G$_2$
β
5π/6
α

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