

UNIVERSITY OF CANTERBURY

# A Geometric Approach to Complete Reducibility

by

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# Declaration of Authorship

I, DANIEL LOND, declare that this thesis titled, ‘A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY’ and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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*“A quote.”*

The author of the quote.

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*Abstract*

College of Engineering  
Department of Mathematics and Statistics

Doctor of Philosophy

by Daniel Lond

The Thesis Abstract ...

# *Acknowledgements*

The acknowledgements and the people to thank ...

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# Symbols

$a$	distance	m
$P$	power	W ( $\text{Js}^{-1}$ )
$\omega$	angular frequency	$\text{rads}^{-1}$
$\vdots$		



*Dedication . . .*

# Chapter 1

## Introduction

### What is the thesis about and what are the main results

- *What the thesis is about:*

It's about algebraic groups. Talk a little bit about them.

A major motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer to do with homomorphisms of finite groups into algebraic groups [3]. We will call these homomorphisms *representations* because of the obvious similarity with the usual kind of representations into  $GL_n$ . Külshammer's second question is as follows:

Let  $G$  be a linear algebraic group over an algebraically closed field of characteristic  $p$ . Let  $\Gamma$  be a finite group and  $\Gamma_p < \Gamma$  a Sylow  $p$ -subgroup of  $\Gamma$ . Fix a conjugacy class of representations  $\Gamma_p \rightarrow G$ . Are there, up to conjugation by  $G$ , only finitely many representations  $\rho : \Gamma \rightarrow G$  whose restrictions to  $\Gamma_p$  belong to the given class?

So far only a non-reductive counterexample is known [3].

We examine a slight variation on the question which we call the *algebraic group* version of Külshammer's question. Instead of a finite group  $\Gamma$  we use a reductive group  $H$ , and instead of a Sylow  $p$ -subgroup  $\Gamma_p < \Gamma$  we use a maximal unipotent subgroup  $U < H$ :

Let  $G, H$  be connected reductive linear algebraic groups over an algebraically closed field of characteristic  $p$  and  $U < H$  a maximal unipotent subgroup of  $H$ . Fix a conjugacy class of representations  $U \rightarrow G$ . Are there, up to conjugation by  $G$ , only finitely many representations  $\rho : H \rightarrow G$  whose restrictions to  $U$  belong to the given class?

Not only is the algebraic group version of Külshammer's question a non-trivial pursuit in its own right but finding an algebraic counterexample might help to find a finite counterexample to Külshammer's original question for a finite subgroup of  $H$  and a reductive  $G$ . For instance, in our example calculations we pay special attention to  $H = SL_2$  and an algebraic counterexample might produce a finite counterexample  $\Gamma = SL_2(q)$  for some  $q$ .

Our approach to Külshammer's question also means that the work in this thesis contributes to the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz ([1], [2]). Let  $G$  be a simple algebraic group over an algebraically closed field of characteristic  $p$ . For large enough characteristic ( $p = 0$  or  $p > 7$  covers all restrictions) Liebeck and Seitz determine explicitly the embeddings of arbitrary connected semisimple groups in  $G$  where  $G$  is of exceptional type. We examine the subgroup structure of simple algebraic groups in low characteristic (usually  $p = 2$  or  $p = 3$ ) where less is known. We use similar methods to Liebeck and Seitz, calculating a certain 1-cohomology of  $H$  with coefficients in  $V$ , the unipotent radical of a parabolic subgroup  $P < G$ .

– *Main results:*

One of our main results is Lemma 4.2. With this we are able to relate Külshammer's question to a certain 1-cohomology calculation in which  $\Gamma$  acts on the unipotent radical  $V$  of a parabolic subgroup  $P < G$  via a certain representation  $\Gamma \rightarrow L$  into a Levi subgroup  $L < P$ . We show that we can reduce Külshammer's question to another question: is the restriction map of 1-cohomologies

$$H^1(\Gamma, V) \rightarrow H^1(\Gamma_p, V)$$

injective?

This approach might help settle Külshammer's original question.

It is known that if  $V$  is abelian then the restriction map of 1-cohomologies

$$H^1(\Gamma, V) \rightarrow H^1(\Gamma_p, V)$$

is injective for finite  $\Gamma$  and  $\Gamma_p < \Gamma$  a Sylow  $p$ -subgroup (see Lemma 3.2). We use this result to show that the restriction map

$$H^1(SL_2(k), V) \rightarrow H^1(U_2(k), V)$$

is injective, where  $U_2(k) < SL_2(k)$  is the maximal unipotent subgroup of

upper unitriangular matrices (see Example 3.2). Hence we will need to investigate non-abelian  $V$  which requires us to work with the non-abelian 1-cohomology.

Next we show that if  $H = SL_2$  and  $G$  is a linear algebraic group then 1-cocycles  $H \rightarrow V$  that are trivial on a fixed maximal torus  $T < H$  have images in an abelian subgroup  $W < V$ .

### Context, history, literary review

– *K. II - motivation for this:*

The notion of semisimplicity is important in mathematics: that an object can be studied by breaking it up into simple pieces. In representation theory for instance, a semisimple or completely reducible representation is a representation that can be written as a direct sum of irreducible representations.

Serre defined complete reducibility for algebraic groups. Let  $G' < G$ . We say  $G'$  is a  $G$ -completely reducible subgroup, or simply  $G$ -completely reducible, if whenever  $G'$  is contained in a parabolic subgroup  $P < G$  then  $G'$  is also contained in a Levi subgroup  $L < P$ . Now let  $H$  be an arbitrary group and  $\rho : H \rightarrow G$  be a representation. If the image of  $\rho$  is a  $G$ -completely reducible subgroup then we say  $\rho$  is  $G$ -completely reducible. This definition agrees with the definition from representation theory when we set  $G = GL_n$ .

Külshammer's question has its roots Maschke's Theorem of representation theory which shows that any representation from a finite group  $\Gamma \rightarrow GL_n$  over a field of characteristic not dividing the order of  $\Gamma$  is completely reducible, and that there are only finitely many conjugacy classes of (completely reducible) representations  $\Gamma \rightarrow GL_n$  [ref Lang].

Let  $\Gamma$  be a finite group and let  $G$  be a linear algebraic group over an algebraically closed field of characteristic  $p$ . Külshammer's first question reads:

Suppose  $p$  does not divide the order of  $\Gamma$ . Are there only finitely many conjugacy classes of representations  $\Gamma \rightarrow G$ ?

The answer is positive [refs] and is essentially contained in a paper of A. Weil [ref]. Külshammer's second question is a refinement of the first:

Let  $\Gamma_p < \Gamma$  be a Sylow  $p$ -subgroup and fix a conjugacy class of representations  $\Gamma_p \rightarrow G$ . Are there only finitely many conjugacy classes of representations  $\Gamma \rightarrow G$  whose restrictions to  $\Gamma_p$  belong to the fixed class?

Note that the condition that  $p$  does not divide  $|\Gamma|$  is dropped from the first statement. If  $p$  does not divide the order of  $\Gamma$  then the answer is “yes”, since  $\Gamma_p$  is trivial and so all representations are equal when restricted to  $\Gamma_p$ . If  $\Gamma$

is a  $p$ -group then the answer is “yes”, as  $\Gamma_p = \Gamma$  so restricting to  $\Gamma_p$  does nothing and therefore only representations that come from the fixed class will hit the class. If  $G = GL_n$  the answer is also “yes”, since by Maschke’s theorem there can only be finitely many conjugacy classes of representations  $\Gamma \rightarrow GL_n$  anyway, regardless of whether or not their restrictions to  $\Gamma_p$  hit the fixed class.

- *Work of Liebeck & Seitz, etc, on embedding reductive  $H$  inside simple  $G$ :*

### **Methods (can refer forward)**

- *Use of 1-cohomology to (K. II):*
- *Key results e.g.  $H^1(SL_2, V) \rightarrow H^1(B, V)$ :*

### **Chapter Summary**

- *Preliminaries:*
- *1-Cohomology:*
- *K. II:*
- *Calculations:*
- *Summary/Future work:*

## Chapter 2

# Mathematical Preliminaries

## Chapter 3

# The 1-Cohomology

### 3.1 Abelian 1-Cohomology

#### 3.1.1 Definitions

Let  $H$  be a group and  $V$  an abelian group (vector space) on which  $H$  acts homomorphically (linearly). We call a map  $\sigma$  from  $H \rightarrow V$  a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \quad (3.1)$$

for all  $h_1, h_2$  in  $H$ . Denote by  $Z^1(H, V)$  the collection of all 1-cocycles from  $H \rightarrow V$ .

We call (3.1) the *1-cocycle condition*.

For any  $\sigma_1, \sigma_2$  in  $Z^1(H, V)$

$$\begin{aligned} (\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\ &= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\ &= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\ &= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2), \end{aligned}$$

so  $Z^1(H, V)$  is closed under pointwise addition.

The trivial map from  $H \rightarrow V$  that sends every  $h$  in  $H$  to the identity 0 in  $V$  is a 1-cocycle. Furthermore for any  $\sigma$  in  $Z^1(H, V)$  we have

$$\begin{aligned}\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\sigma(1),\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each  $\sigma$  has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore  $Z^1(H, V)$  is a  $\mathbb{Z}$ -module under pointwise addition.

Given a  $v$  in  $V$  we define a 1-coboundary  $\chi_v^H : H \rightarrow V$  to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by  $B^1(H, V)$  the collection of all 1-coboundaries.

For any  $v$  in  $V$  and any  $h_1, h_2$  in  $H$

$$\begin{aligned}\chi_v^H(h_1 h_2) &= v - (h_1 h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.



For any  $u, v$  in  $V$  and all  $h$  in  $H$

$$\begin{aligned}
 (\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\
 &= u - h \cdot u + v - h \cdot v \\
 &= (u + v) - h \cdot (u + v) \\
 &= \chi_{u+v}^H(h)
 \end{aligned}$$

is a 1-coboundary, and hence  $B^1(H, V)$  is also closed under pointwise addition.

We see that  $B^1(H, V)$  is a subgroup of  $Z^1(H, V)$  via the two-step subgroup test. In fact it is easy to show that  $B^1(H, V)$  is a  $\mathbb{Z}$ -submodule of  $Z^1(H, V)$ , so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology*.

**Lemma 3.1.** *Suppose  $H$  is linearly reductive. Then  $H^1(H, V)$  is trivial [4].*

### 3.1.2 Maps between 1-cohomologies

Let  $\phi : \tilde{H} \rightarrow H$  be a homomorphism,  $\tilde{H}$  being another group that acts on  $V$ . Suppose that for every  $h$  in  $\tilde{H}$ ,  $\phi$  satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all  $v$  in  $V$ . If  $\sigma$  is a 1-cocycle from  $H \rightarrow V$  then we will show that the map denoted  $Z^1(\phi)(\sigma)$  defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from  $\tilde{H} \rightarrow V$ . Thus certain homomorphisms

$$\phi : \tilde{H} \rightarrow H$$

give rise to maps of the form

$$Z^1(\phi) : Z^1(H, V) \rightarrow Z^1(\tilde{H}, V)$$

Take  $h_1, h_2$  in  $H$ . We have

$$\begin{aligned}
 Z^1(\phi)(\sigma)(h_1 h_2) &= (\sigma \circ \phi)(h_1 h_2) \\
 &= \sigma(\phi(h_1 h_2)) \\
 &= \sigma(\phi(h_1)\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\
 &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\
 &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2).
 \end{aligned}$$

Moreover, it can be shown that  $Z^1(\phi)$  maps  $B^1(H, V)$  into  $B^1(\tilde{H}, V)$ . This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \rightarrow H^1(\tilde{H}, V),$$

defined by

$$\begin{array}{ccc}
 Z^1(H, V) & \xrightarrow{Z^1(\phi)} & Z^1(\tilde{H}, V) \\
 \pi \downarrow & & \downarrow \tilde{\pi} \\
 H^1(H, V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H}, V)
 \end{array}$$

where  $\pi$  and  $\tilde{\pi}$  are the respective canonical projections of  $Z^1(H, V)$  onto  $H^1(H, V)$  and  $Z^1(\tilde{H}, V)$  onto  $H^1(\tilde{H}, V)$ . To show that the map  $H^1(\phi)$  is well-defined it is sufficient to notice that  $Z^1(\phi)$  is a homomorphism.

**Example 3.1.** Let  $\tilde{H}$  be a subgroup of  $H$  and  $i : \tilde{H} \rightarrow H$  the inclusion map. Then  $i$  gives rise to a well defined map

$$H^1(i) : H^1(H, V) \rightarrow H^1(\tilde{H}, V).$$

**Lemma 3.2.** Let  $V$  be a vector space over a field of characteristic  $p$ . Let  $\Gamma$  be a finite group and  $\tilde{\Gamma} = \Gamma_p \subset \Gamma$  a Sylow  $p$ -subgroup of  $\Gamma$ . The map

$$H^1(i) : H^1(\Gamma, V) \rightarrow H^1(\Gamma_p, V)$$

is injective.

*Proof.* Let  $x$  be an element of  $H^1(\Gamma, V)$  such that  $H^1(i)(x) = 0$ . Now choose a 1-cocycle  $\sigma$  in  $Z^1(\Gamma, V)$  such that  $\pi(\sigma) = x$ . Hence  $Z^1(i)(\sigma)$  is a 1-coboundary as its image under  $\tilde{\pi}$  is 0. That is to say  $\sigma$  restricted to  $\Gamma_p$  is equal to a 1-coboundary, say  $\chi_v^{\Gamma_p}$ . But since

$\chi_v^{\Gamma_p}$  can be trivially extended to a 1-coboundary  $\chi_v^\Gamma$  from  $\Gamma \rightarrow V$ , and

$$\pi(\sigma - \chi_v^\Gamma) = x,$$

we could well have chosen the 1-cocycle  $(\sigma - \chi_v^\Gamma)$  as a representative for  $x$ . Hence there is no harm in assuming that  $\sigma$  is 0 when restricted to  $\Gamma_p$ . Now choose a set of representatives  $\gamma_1, \dots, \gamma_l$  in  $\Gamma$  for the coset space  $\Gamma/\Gamma_p$  and set

$$v^* = \sum_{i=1}^l \sigma(\gamma_i).$$

Consider the 1-coboundary  $\chi_{v^*}^\Gamma$  given by  $v^*$

$$\begin{aligned} \chi_{v^*}^\Gamma(\gamma) &= v^* - \gamma \cdot v^* \\ &= \sum_{i=1}^l \sigma(\gamma_i) - \gamma \cdot \sum_{i=1}^l \sigma(\gamma_i) \\ &= \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l \gamma \cdot \sigma(\gamma_i). \end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(\gamma\gamma_i) = \sigma(\gamma) + \gamma \cdot \sigma(\gamma_i),$$

from which we obtain

$$\begin{aligned} \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l \gamma \cdot \sigma(\gamma_i) &= \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l (\sigma(\gamma\gamma_i) - \sigma(\gamma)) \\ &= \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l \sigma(\gamma\gamma_i) + \sum_{i=1}^l \sigma(\gamma). \end{aligned}$$

Now as the value of  $\sigma$  at a fixed  $\gamma$  depends only on the value of  $\sigma$  at the representative  $\gamma_j$  of the coset containing  $\gamma$  we can collapse the middle term to yield

$$\begin{aligned} \chi_{v^*}^\Gamma(\gamma) &= \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l \sigma(\gamma\gamma_i) + \sum_{i=1}^l \sigma(\gamma) \\ &= \sum_{i=1}^l \sigma(\gamma_i) - \sum_{i=1}^l \sigma(\gamma_i) + \sum_{i=1}^l \sigma(\gamma) \\ &= l\sigma(\gamma). \end{aligned}$$

Since  $\gcd([\Gamma : \Gamma_p], p) = \gcd(l, p) = 1$ ,  $l$  is invertible and so

$$l^{-1}\chi_{v^*}^\Gamma(\gamma) = \sigma(\gamma).$$

Therefore  $\sigma$  is a 1-coboundary and so the kernel of  $H(i)$  is trivial.  $\square$

**Example 3.2.** Let

$$k = \bar{\mathbb{F}}_p = \bigcup_r \mathbb{F}_{p^r},$$

$V$  a vector space on which  $SL_2(k)$  acts, and  $U(k)$  the subgroup of  $SL_2(k)$  consisting of upper unitriangular matrices. Then  $U(\mathbb{F}_{p^r})$  is a Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_{p^r})$  for each  $r$ , and the map

$$H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$$

is injective.

*Proof.* The group  $GL_2(\mathbb{F}_{p^r})$  has order  $(p^{2r} - 1)(p^{2r} - p^r)$  since there are  $p^{2r} - 1$  choices of vectors for the first column (all choices excluding the zero vector), and  $p^{2r} - p^r$  choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \rightarrow \mathbb{F}_{p^r}^*,$$

with kernel  $SL_2(\mathbb{F}_{p^r})$ . Therefore, by the First homomorphism theorem for groups

$$GL_2(\mathbb{F}_{p^r}) / SL_2(\mathbb{F}_{p^r}) \sim \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| / |\mathbb{F}_{p^r}^*| \\ &= (p^{2r} - 1)(p^{2r} - p^r) / (p^r - 1) \\ &= p^r(p^{2r} - 1). \end{aligned}$$

Since  $|U(\mathbb{F}_{p^r})| = p^r$ ,  $U(\mathbb{F}_{p^r})$  is a Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_{p^r})$ .

Fix a non-trivial  $y \in H^1(SL_2(k), V)$  and choose a representative  $\tau \in Z^1(SL_2(k), V)$  for  $y$ . For each  $g \in SL_2(\mathbb{F}_{p^r})$  define the morphism  $f_g^{(r)} : V \rightarrow V$  by

$$f_g^{(r)}(v) = \tau(g) - \chi_v(g) = \tau(g) - v + g \cdot v.$$

Consider subsets of  $V$  defined by

$$C_r = \{v \in V \mid f_g^{(r)}(v) = 0\}.$$

Each subset  $C_r$  is closed and the inclusion  $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^{(r+1)}}$  induces the reverse inclusion  $C_r \supset C_{(r+1)}$ . The Noetherian property for  $V$  requires that the sequence of subsets of

$V$  defined by

$$\{C_i\}_{i=1}^{\infty}$$

becomes constant. However,  $y \neq 0$  so  $\tau$  is not a 1-coboundary on  $SL_2(k)$ , which means the  $C_r$ 's are eventually empty. That is, there exists an integer  $s$  such that for any  $v$  in  $V$

$$(\tau - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if  $y|_{SL_2(\mathbb{F}_{p^r})} = 0$  for all  $r$  then  $y = 0$ .

Take  $x$  in the kernel of the map  $H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$ . Then for each  $r$ ,  $x|_{U(\mathbb{F}_{p^r})} = 0$  so by Lemma 3.2  $x|_{SL_2(\mathbb{F}_{p^r})} = 0$ . Therefore  $x = 0$  and so  $H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$  is injective.  $\square$

We could also consider appropriate maps  $f : V \rightarrow \tilde{V}$  and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \rightarrow H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \rightarrow H^1(\tilde{H}, \tilde{V}).$$

## 3.2 Non-abelian 1-Cohomology

### 3.2.1 The non-abelian setting

We will be interested in  $H, V$  algebraic groups, where we require that 1-cocycles be morphisms of varieties.

### 3.2.2 Definitions

Let  $H, V$  be algebraic groups,  $H$  acting on  $V$ . We call a map  $\sigma$  from  $H \rightarrow V$  a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{3.2}$$

for all  $h_1, h_2$  in  $H$ . Denote by  $Z^1(H, V)$  the collection of all 1-cocycles from  $H \rightarrow V$ .

We call the (3.2) the *1-cocycle condition*.

Given a  $v$  in  $V$  we define a 1-coboundary  $\chi_v^H : H \rightarrow V$  to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by  $B^1(H, V)$  the collection of all 1-coboundaries.

For any  $v$  in  $V$  and any  $h_1, h_2$  in  $H$

$$\begin{aligned} \chi_v^H(h_1 h_2) &= v * (h_1 h_2) \cdot v^{-1} \\ &= v * h_1 \cdot (h_2 \cdot v^{-1}) \\ &= v * h_1 \cdot (v v^{-1} h_2 \cdot v) \\ &= v * h_1 \cdot v * h_2 \cdot (v * h_2 \cdot v^{-1}) \\ &= \chi_v^H(h_1) * h_2 \cdot \chi_v^H(h_2), \end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say  $\sigma_1, \sigma_2$  in  $Z^1(H, V)$  are *equivalent* if there exists a  $v$  in  $V$  such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \quad (3.3)$$

for all  $h$  in  $H$ . We call the set of equivalence classes of  $Z^1(H, V)$  under the equivalence relation defined by (3.3) the *1-cohomology*, denoted  $H^1(H, V)$ .

### 3.2.3 Maps between 1-cohomologies

**Lemma 3.3.** *Let  $B$  be a Borel subgroup of  $SL_2$  acting on an algebraic group  $V$ . Then  $H^1(i) : H^1(SL_2, V) \rightarrow H^1(B, V)$  is injective.*

*Proof.* Let  $x$  be in the kernel of  $H^1(i)$  and  $\sigma$  an element of  $Z^1(SL_2, V)$  that projects onto the class  $x$ . Since  $Z^1(i)(\sigma)$  projects to the trivial 1-cohomology class we may as well assume that  $\sigma|_B = 1$ . For there exists some  $v$  in  $V$  such that for all  $b$  in  $B$

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle  $\hat{\sigma} : SL_2 \rightarrow V$  defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction  $\hat{\sigma}$  also projects to the class  $x$ , and for all  $b$  in  $B$

$$\begin{aligned}
 \hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\
 &= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\
 &= v^{-1} * v * b \cdot (v^{-1} * v) \\
 &= 1,
 \end{aligned}$$

so we may as well have chosen  $\hat{\sigma}$  instead as a representative for  $x$ .

Now consider the *homogeneous space*  $SL_2/B$  [5] and take the map

$$\tilde{\sigma} : SL_2/B \rightarrow V,$$

defined in the usual way under the canonical projection  $\pi : SL_2 \rightarrow SL_2/B$ :

$$\begin{array}{ccc}
 SL_2 & \xrightarrow{\sigma} & V \\
 \pi \downarrow & \nearrow \tilde{\sigma} & \\
 SL_2/B & & 
 \end{array}$$

This map is well defined and is a morphism [6]. Now since  $SL_2/B$  is an irreducible projective variety [5],  $\tilde{\sigma}$  must be constant [6]. Hence, as  $\sigma$  takes the value 1 for any  $b$  in  $B$ ,  $\tilde{\sigma}(hB) = 1$  for all cosets  $hB$ . Therefore, for all  $h$  in  $SL_2$

$$\sigma(h) = \tilde{\sigma}(hB) = 1.$$

We have shown that  $\sigma$  is the 1-coboundary  $\chi_1$  which means that the kernel of  $H^1(i)$  is trivial. □

**Lemma 3.4.** *Let  $B$  be a Borel subgroup of  $SL_2$  and  $U$  be the unipotent radical of  $B$ . Then  $H^1(B, V) \rightarrow H^1(U, V)$  is injective. Moreover*

$$H^1(SL_2, V) \rightarrow H^1(U, V)$$

*is injective.*

*Proof.* Let  $x$  be an element of the kernel of  $H^1(i) : H^1(B, V) \rightarrow H^1(U, V)$  and let  $\sigma$  in  $Z^1(B, V)$  be a representative for  $x$ . We may as well assume that  $\sigma|_U = 1$ . For any  $b$  in

$B$  we can find a  $u$  in  $U$  and a  $t$  in  $T$  such that  $b = ut$ . Hence

$$\begin{aligned}\sigma(b) &= \sigma(ut) \\ &= \sigma(u) * u \cdot \sigma(t) \\ &= \sigma(u).\end{aligned}$$

Since  $\sigma$  represents  $x$ ,  $\sigma$  must be a 1-coboundary on  $U$ . Hence  $\sigma$  is in  $B^1(B, V)$  and the kernel of  $H^1(i) : H^1(B, V) \rightarrow H^1(U, V)$  is trivial.  $\square$



## Chapter 4

# Külshammer's Second Problem

### 4.1 Külshammer's Second Problem

Two questions raised by B. Külshammer concerning representations of a finite group  $\Gamma$  into a linear algebraic group  $G$  over an algebraically closed field  $k$ . The first has a positive answer and is essentially contained a paper by A. Weil [3]:

- (K. I) Let  $\text{char}(k)$  be prime to the order of  $\Gamma$ . Are there only finitely many representations  $\rho : \Gamma \rightarrow G$  up to conjugation by  $G$ ?
- (K. II) Let  $p = \text{char}(k)$  and  $\Gamma_p \subset \Gamma$  be a Sylow  $p$ -subgroup. Fix a conjugacy class of representations from  $\Gamma_p \rightarrow G$ . Are there, up to conjugation by  $G$ , only finitely many representations  $\rho : \Gamma \rightarrow G$  whose restrictions to  $\Gamma_p$  belong to the given class?

(K. II) has positive answer so long as  $G$  is reductive and the characteristic of  $k$  is good for  $G$  [7]. The same paper shows that the answer is “no” in general by way of a counterexample involving a non-reductive  $G$ .

We will explore the possibility of a reductive counterexample to (K. II).

### 4.2 The Approach

We are interested in knowing whether there can be infinitely many  $G$ -conjugacy classes of representations  $\Gamma \rightarrow G$  that when restricted to  $\Gamma_p$  hit some fixed  $G$ -conjugacy class of representations  $\Gamma_p \rightarrow G$ .

**Theorem 4.1.** *There are only finitely many  $G$ -conjugacy classes of  $G$ -completely reducible representations  $\Gamma \rightarrow G$ .*

Reference something.  

Although  $G$  has infinitely many parabolic subgroups there are only finitely many  $G$ -conjugacy classes of parabolic subgroups, so we can choose a finite set  $\{Q_i\}$  of representatives. We choose a set of Levi subgroups  $\{M_i\}$ ,  $M_i$  being a Levi subgroup of  $Q_i$ . By Theorem 4.1 there are only finitely many  $M_i$ -conjugacy classes of  $M_i$ -completely reducible representations  $\sigma_{0,i} : \Gamma \rightarrow M_i$ , so we fix a set of representatives  $\{\sigma_{0,i}^{j_i}\}$ .

We will show that for each representation  $\rho : \Gamma \rightarrow G$  there exists a representation  $\sigma$  that is  $G$ -conjugate to  $\rho$  and that fits one of only finitely many commutative diagrams

$$\begin{array}{ccc} \Gamma & \xrightarrow{\sigma} & Q_i \\ & \searrow \sigma_{0,i}^{j_i} & \downarrow \\ & & M_i \end{array}$$

Let  $\rho$  be a representation from  $\Gamma \rightarrow G$  and let  $P$  be a minimal parabolic subgroup of  $G$  containing  $\rho(\Gamma)$ . Then there is a  $g$  in  $G$  such that  $P = g \cdot Q$ , where  $Q \in \{Q_i\}$ . Let  $\rho' = g \cdot \rho$ .

Define  $\rho_0 : \Gamma \rightarrow M$  by composing  $\rho'$  with the projection  $Q \rightarrow M$ ,  $M \in \{M_i\}$  the chosen Levi subgroup for  $Q$ :

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho'} & Q \\ & \searrow \rho_0 & \downarrow \\ & & M \end{array}$$

Since  $Q$  is a minimal parabolic containing  $\rho'(\Gamma)$ ,  $\rho_0$  is  $M$ -irreducible [reference] and therefore  $M$ -completely reducible. Hence we can choose an  $m \in M$  such that  $\sigma_0 = m \cdot \rho_0$  where  $\sigma_0 \in \{\sigma_{0,i}^{j_i}\}$ . Let  $\sigma = m \cdot \rho' = mg \cdot \rho$ . This verifies what we set out to show.

[Say something about  $\rho \rightsquigarrow (i, j) \rightsquigarrow H^1(\Gamma, V_i)$  so that I can state the next lemma]

For a given parabolic  $P < G$ , Levi  $L < P$  and representation  $\rho : \Gamma \rightarrow P$  we have defined the map  $\rho_0 : \Gamma \rightarrow L$  by the projection  $P \rightarrow L$ . Now define  $\alpha_\rho : \Gamma \rightarrow R_u(P)$  by projecting on to  $R_u(P)$ , so that  $\rho = \alpha_\rho \rho_0$ . If  $\rho_0$  is a homomorphism then

$$\begin{aligned} \alpha_\rho(\gamma_1 \gamma_2) \rho_0(\gamma_1 \gamma_2) &= \rho(\gamma_1 \gamma_2) = \rho(\gamma_1) \rho(\gamma_2) \\ &= \alpha_\rho(\gamma_1) \rho_0(\gamma_1) \alpha_\rho(\gamma_2) \rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1) \rho_0(\gamma_1) \alpha_\rho(\gamma_2) \rho_0(\gamma_1)^{-1} \rho_0(\gamma_1) \rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1) \rho_0(\gamma_1) \alpha_\rho(\gamma_2) \rho_0(\gamma_1)^{-1} \rho_0(\gamma_1 \gamma_2), \end{aligned}$$

so that

$$\alpha_\rho(\gamma_1\gamma_2) = \alpha_\rho(\gamma_1)\rho_0(\gamma_2) \cdot \alpha_\rho(\gamma_2),$$

which is the 1-cocycle condition in (3.1).

Hence for the given  $\rho$  we have a corresponding 1-cocycle  $\alpha_\rho : \Gamma \rightarrow R_u(P)$  where  $\Gamma$  acts on  $R_u(P)$  via  $\rho_0$ .

Suppose we conjugate  $\rho$  by an element  $g \in G$ . Then  $\rho' = g \cdot \rho$  has corresponding 1-cocycle  $\alpha_{\rho'} : \Gamma \rightarrow gR_u(P)g^{-1}$ .

So the  $G$ -action on  $\rho : \Gamma \rightarrow P$  almost corresponds to maps of 1-cocycles of the form:

$$Z^1(\Gamma, R_u(P)) \rightarrow Z^1(\Gamma, gR_u(P)g^{-1}),$$

the catch being that conjugating  $\rho$  by an arbitrary element of  $g$  changes the action of  $\Gamma$  on  $R_u(P)$ . If we choose  $g \in Z(L)^\circ$  and consider the  $Z(L)^\circ$ -action:

$$\begin{array}{ccc} \Gamma & \xrightarrow{\rho} & P \\ & \searrow \rho_0 & \downarrow \\ & & L \end{array} \quad \mapsto \quad \begin{array}{ccc} \Gamma & \xrightarrow{g \cdot \rho} & gPg^{-1} \\ & \searrow \rho_0 & \downarrow \\ & & L \end{array}$$

then we really do get a map of 1-cocycles.

**Lemma 4.2.** *Let  $R = \{\rho_\lambda \mid \lambda \in \Lambda\}$  be a collection of representations  $\Gamma \rightarrow P$  for a fixed parabolic subgroup  $P < G$ . The following statements are equivalent:*

- (i)  *$R$  is a finite union of  $G$ -conjugacy classes.*
- (ii)  *$R$  is a finite union of  $P$ -conjugacy classes.*
- (iii) *For each  $i$ , the number of elements of  $H^1(\Gamma, V_i)$  modulo  $Z(M_i^\circ)$  that come from the elements of  $R$  is finite.*

*Proof.* (i)  $\Rightarrow$  (ii): Suppose there are only finitely many  $G$ -conjugacy classes of  $R$  and take  $\rho_\mu, \rho_\nu \in R$  such that

$$g \cdot \rho_\mu = \rho_\nu$$

for some  $g \in G$ . By definition  $\rho_\mu$  maps into  $P$  but on the other hand  $\rho_\nu = g \cdot \rho_\mu$  maps into  $Q = gPg^{-1}$ , so

$$\rho_\nu : \Gamma \rightarrow P \cap Q$$

Let  $T < P \cap Q$  be a maximal torus of  $G$  [ref]. Since  $T, gTg^{-1}$  are maximal tori of  $Q$  they must be  $Q$ -conjugate, say

$$qTq^{-1} = gTg^{-1},$$

for some  $q \in Q$ . Then

$$q^{-1}g \in N_G(T),$$

and so

$$q^{-1}g = (gp^{-1}g^{-1})g = gp^{-1} \in N_G(T),$$

for some  $p \in P$ . Therefore

$$\begin{aligned} g &= (gp^{-1})p \\ &= n(tp), \end{aligned}$$

where  $n \in G$  is one of the finitely many representatives of the Weyl group  $W = N_G(T)/T$  such that  $\pi(gp^{-1}) = nT \in W$ , and  $t \in T$  such that  $nt = gp^{-1}$ .

Therefore, a  $G$ -conjugacy class of  $R$  splits up into at most  $|N_G(T)/T|$   $P$ -conjugacy classes.

(ii)  $\Rightarrow$  (iii): □

Let  $K$  be a collection of representations  $\Gamma \rightarrow G$  whose restrictions to  $\Gamma_p$  belong to some fixed class. Let  $\sigma \in K$ , then  $\sigma|_{\Gamma_p}$  and  $\sigma_{0,i}^{j_i}|_{\Gamma_p}$  are conjugate. We have a 1-cocycle  $\alpha$  corresponding to  $\sigma$ , while  $\sigma_{0,j}^{(i)}$  corresponds to the trivial 1-cocycle. Hence  $\alpha|_{\Gamma_p}$  is conjugate to the trivial 1-cocycle, that is,  $\alpha|_{\Gamma_p}$  is a 1-coboundary.

[Therefore the problem is controlled by the 1-cohomology]

### 4.3 An algebraic group version

In an attempt to gain further insight into (K. II) we adjust the original question by letting  $\Gamma$  be an infinite group  $H$ . The advantage being that a negative answer in the algebraic group version may provide a negative answer to (K. II) by choosing an appropriate finite subgroup  $\Gamma$  of  $H$ . In many of the examples to follow we set  $H = SL_2(K)$  with Sylow  $p$ -subgroup  $H_p = U_2(K)$  consisting of upper unitriangular matrices.

Let  $P \subset G$  be a parabolic subgroup and  $L \subset P$  the corresponding Levi subgroup. Fix a representation  $\rho_0 : H \rightarrow L$ . We can assume  $\rho_0(H)$  is  $L$ -irreducible, that is, not contained in a proper parabolic of  $L$ .

Now define  $\rho_\alpha : H \rightarrow P$  by  $\rho_\alpha(h) = \alpha(h)\rho_0(h)$  where  $\alpha : H \rightarrow R_u(P)$ ,  $R_u(P)$  the unipotent radical of  $P$ .

For  $\rho_\alpha$  to be a homomorphism

$$\begin{aligned} \alpha(h_1 h_2) \rho_0(h_1 h_2) &= \alpha(h_1) \rho_0(h_1) \alpha(h_2) \rho_0(h_2) \\ &= \alpha(h_1) \rho_0(h_1) \alpha(h_2) \rho_0(h_1)^{-1} \rho_0(h_1) \rho_0(h_2) \\ &= \alpha(h_1) \rho_0(h_1) \alpha(h_2) \rho_0(h_1)^{-1} \rho_0(h_1 h_2). \end{aligned}$$

That is  $\alpha(h_1 h_2) = \alpha(h_1) h_1 \cdot \alpha(h_2)$ , where the action  $H \times R_u(P) \rightarrow R_u(P)$  is conjugation via  $\rho_0$ . This is a 1-cocycle condition;  $\alpha \in Z^1(H, R_u(P))$ .  $R_u(P)$  will not be abelian in general.

Now suppose  $\rho_\alpha$  is  $R_u(P)$ -conjugate to some  $\rho_\beta$ ,  $\alpha, \beta \in Z^1(H, R_u(P))$ . That is, there exists a  $v \in R_u(P)$  such that for all  $h \in H$

$$\begin{aligned} \alpha(h) \rho_0(h) &= v \beta(h) \rho_0(h) v^{-1} \\ &= v \beta(h) \rho_0(h) v^{-1} \rho_0(h)^{-1} \rho_0(h). \end{aligned}$$

That is  $\alpha(h) = v \beta(h) h \cdot v^{-1}$ . In particular if  $\rho_\alpha$  is  $R_u(P)$ -conjugate to  $\rho_0$ , that is  $\beta$  is trivial, then  $\alpha$  takes the form of a 1-coboundary. Generally speaking  $\alpha$  and  $\beta$  project to the same 1-cohomology class. In the abelian case this reads “ $\alpha$  and  $\beta$  differ by a 1-coboundary”:

$$\begin{aligned} \alpha(h) = v \beta(h) h \cdot v^{-1} &\rightsquigarrow \alpha(h) = v + \beta(h) - h \cdot v \\ &= \beta(h) + v - h \cdot v \\ &= \beta(h) + \chi_v(h). \end{aligned}$$

# Chapter 5

## 1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology  $H^1(SL_2(k), V)$  where  $V = R_u(P)$  is the unipotent radical of a parabolic subgroup  $P$  of a reductive group  $G$ . The motivation for this is to look for infinitely many conjugacy classes of representations of  $SL_2(k)$  into  $G$  in the hope of finding a finite subgroup  $H$  of  $SL_2(k)$  as a counterexample for Külshammer's Second Problem.

### 5.1 The method

Let  $G$  be a reductive group over an algebraically closed field  $k$  of characteristic  $p$ . Let  $\Phi$  be the roots for  $G$  with  $\Delta \subset \Phi^+ \subset \Phi$  the simple and positive roots, respectively, associated to a fixed maximal torus  $T$  of  $G$ .

[I want to see if this works for arbitrary rank] Let  $P_\alpha < G$  be the parabolic subgroup of  $G$  corresponding to the simple root  $\alpha \in \Delta$ , with Levi subgroup  $L_\alpha$  and unipotent radical  $V_\alpha$ :

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism  $\rho_0$  from  $SL_2(k)$  into  $L_\alpha$  under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer  $r > 0$  and define  $\rho_r$  to be the homomorphism from  $SL_2(k)$  into  $L_\alpha$  composed of  $\rho_0$  and the Frobenius map,

$$\begin{aligned} F_r &: SL_2(k) \rightarrow SL_2(k) \\ (A_{ij}) &\mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r = \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let  $SL_2(k)$  act on  $V_\alpha$  via  $\rho_r$  and we consider 1-cocycles  $\sigma \in Z^1(SL_2(k), V_\alpha)$ . As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of  $SL_2(k)$  [reference], so let  $\sigma \in Z^1(SL_2(k), V_\alpha)$  such that

$$\sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = 0,$$

for all  $t \in k^*$ . We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} \epsilon_{\delta}((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_{\delta}) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} n_{\alpha} \epsilon_{\delta}(\lambda_{\delta}) n_{\alpha}^{-1}, \end{aligned}$$

where  $n_{\alpha} = \epsilon_{\alpha}(1)\epsilon_{-\alpha}(-1)\epsilon_{\alpha}(1)$  and  $\lambda_{\delta}$  are elements of the underlying field  $k$ .

**Lemma 5.1.**

$$\sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_{\delta}(x_{\delta}(u)),$$

where  $\delta$  ranges  $\Phi^+ - \{\alpha\}$  such that  $\langle \delta, \alpha \rangle > 0$ , and  $x_{\delta} \in k[T]$  are polynomials in one variable.

*Proof.* We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where  $i$  is the inclusion map and  $\pi_\delta$  the projection onto the root subgroup  $V_\delta$ . Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from  $k \rightarrow k$ .

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left( \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore

$$x_\delta(t^2 u) = (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta(u).$$

Since  $x_\delta$  is a polynomial function there can only be non-negative powers of  $t$  on the left-hand side of the equality which forces  $\langle \delta, \alpha \rangle \geq 0$ . However, if  $\langle \delta, \alpha \rangle = 0$  then  $x_\delta$  is constant and hence zero, as  $\sigma$  is zero on  $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ . Therefore the non-zero  $x_\delta$  occur precisely when  $\langle \delta, \alpha \rangle > 0$ .  $\square$

Next we prove a couple of useful facts about root systems not containing  $G_2$  or  $C_3$ .

**Lemma 5.2.** *Suppose  $\Phi$  is not of type  $G_2$  and let  $\alpha, \beta \in \Phi$ . If  $\alpha + \beta \in \Phi$  then  $\langle \alpha, \beta \rangle \leq 0$ .*



*Proof.*

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where  $\theta$  is the angle between  $\alpha$  and  $\beta$ . Hence acute angles correspond to positive pairs. Referring to the  $A_2$  and  $B_2$  root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if  $\langle \alpha, \beta \rangle > 0$  then  $\alpha + \beta \notin \Phi$ .  $\square$

We must exclude the case  $\Phi = G_2$  here since  $\alpha, 2\alpha + \beta$  and  $3\alpha + \beta$  are all roots ( $\alpha$  short) but  $\langle \alpha, 2\alpha + \beta \rangle = 1$ .

**Lemma 5.3.** *Suppose  $\Phi$  does not contain  $G_2$  or  $C_3$ . Let  $\delta_1, \delta_2 \in \Phi$  and  $\gamma \in \Delta$  be roots such that  $\langle \delta_i, \gamma \rangle > 0$  ( $i = 1, 2$ ). If  $\delta_1 + \delta_2$  is a root, then  $\delta_1$  and  $\delta_2$  are of opposite sign.*

*Proof.* Suppose  $\delta_1 + \delta_2 \in \Phi$ . Let  $\theta_i$  be the absolute value of the angle between  $\delta_i$  and  $\gamma$ , ( $i = 1, 2$ ) and let  $\theta_3$  be the absolute value of the angle between  $\delta_1$  and  $\delta_2$ . Then

$$\begin{aligned} \langle \delta_i, \gamma \rangle &> 0 & (i = 1, 2) \\ \implies (\delta_i, \gamma) &> 0 \\ \implies \cos(\theta_i) &> 0 \\ \implies \theta_i &< \pi/2, \end{aligned}$$

and similarly, using Lemma 5.2

$$\begin{aligned} \langle \delta_1, \delta_2 \rangle &\leq 0 \\ \implies \theta_3 &\geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

- 1:**  $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = 2\pi/3;$
- 2:**  $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 3:**  $\theta_1 = \pi/4, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 4:**  $\theta_1 = \pi/4, \quad \theta_2 = \pi/4, \quad \theta_3 = \pi/2.$

[Wow, probably need more explanation there]

For the cases in which  $\theta_3 = \pi/2$  we can reason from the root system diagrams that  $\delta_1$  and  $\delta_2$  lie in a  $B_2$  subsystem of  $\Phi$ , and they have the same length. Since  $\delta_1 + \delta_2$  is a root it must be that  $\delta_1$  and  $\delta_2$  are short roots and their sum is a long root. However we must rule out the third case. For if  $\theta_1 = \pi/4$  then  $\delta_1$  and  $\gamma$  are roots of different length

in a  $B_2$  subsystem, but  $\theta_2 = \pi/3$  implies that  $\delta_2$  and  $\gamma$  are roots of the same length in an  $A_2$  subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem;  $A_2$  and  $B_2$  respectively. Consulting the root system diagrams yields  $\gamma = \delta_1 + \delta_2$  and the result holds.

In the second case we see that  $\delta_1, \delta_2$  and  $\gamma$  do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that  $\gamma$  is a short root. In fact, since a pair short roots lie in subsystems of type  $A_2$  it must be that the rank 3 subsystem in which the four roots lie is of type  $C_3$ . [Picture?][Wow, is that right? Maybe just say ‘we will show that they lie in a  $C_3$  subsystem’.]  $\square$

We return to the 1-cohomology calculation but assume that  $G$  does not contain  $G_2$  or  $C_3$ .

**Corollary 5.4.** *For any  $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \sigma \left( \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right).$$

Furthermore, the  $x_\delta$  are homomorphisms.

*Proof.* We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \epsilon_\alpha(u_1^{p^r}) \prod_{\delta} \epsilon_\delta(x_\delta(u_2)) \epsilon_\alpha(-u_1^{p^r}),$$

with  $\langle \delta, \alpha \rangle > 0$ . By Lemma 5.2  $\alpha + \delta \notin \Phi$  so each  $\epsilon_\delta$  commutes with the  $\epsilon_\alpha$ .  $\square$

**Corollary 5.5.** *The image of the group of upper triangular matrices of  $SL_2(k)$  under  $\sigma$  lies in a product of commuting root groups of  $V_\alpha$ .*

*Proof.* First consider

$$\sigma \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_\delta(x_\delta(b)).$$

Suppose the roots  $\delta_1$  and  $\delta_2$  appear on the right hand side. By Lemma 5.1  $\delta_i \in \Phi^+ - \{\alpha\}$  and  $\langle \delta_i, \alpha \rangle > 0$  ( $i = 1, 2$ ), so Lemma 5.3 asserts that  $\delta_1 + \delta_2$  is no root, hence,  $\epsilon_{\delta_1}$  and  $\epsilon_{\delta_2}$  commute.

Therefore, for any  $a, b \in k$  with  $a \neq 0$

$$\begin{aligned} \sigma \left( \begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix} \right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \\ &= \prod_{\delta} \epsilon_{\delta} \left( a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b) \right). \end{aligned}$$

□

Since the  $x_{\delta}$  are homomorphisms from  $k \rightarrow k$  they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some  $\mu_i$  in  $k$ . Furthermore, combining the calculation in the proof of Lemma 5.1 with the result in Corollary 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta} (x_{\delta}(a^2 b)) = \prod_{\delta} \epsilon_{\delta} \left( a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b) \right),$$

severely restricting the possible polynomials  $x_{\delta}$ . In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer  $r$  in the definition of  $\rho_r$ . For suppose  $x_{\delta}$  and hence some  $\mu_j$  is non-zero. Then equating the coefficients of  $b$  in the equality directly above yields

$$\begin{aligned} \mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\ \implies 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

In [8] it is shown that the possible pairings of any two roots are bounded by  $\pm 3$ . Hence by Lemma 5.1  $\langle \delta, \alpha \rangle = 1, 2$  or  $3$ . It is now clear that if  $\langle \delta, \alpha \rangle = 3$  then  $x_{\delta} = 0$ .

If  $\langle \delta, \alpha \rangle = 1$  the characteristic of  $k$  must be 2 and  $j = r - 1$ . Otherwise  $\langle \delta, \alpha \rangle = 2$  and  $j = r$ , but the characteristic of  $k$  is so far unrestricted.

**Example 5.1.** Let  $G = G_2$ . Fix a maximal torus, labeling the positive simple roots  $\Delta = \{\alpha, \beta\}$  with  $\beta$  being the long root. Let

$$V_{\alpha} = R_u(P_{\alpha}) = \langle U_{\beta}, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

We will write  $v$  in  $V_{\alpha}$  in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle := \epsilon_{\beta}(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_{3\alpha+\beta}(v_4) \epsilon_{3\alpha+2\beta}(v_5) \in V_{\alpha}$$

The group law for  $V_\alpha$  is

$$u * v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-3p^r}v_1, a^{-p^r}v_2, a^{p^r}v_3, a^{3p^r}v_4, v_5 \rangle.$$

Let  $\sigma$  be in  $Z^1(SL_2, V_\alpha)$  such that

$$\sigma \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = 0.$$

By Lemma 5.1

$$\sigma \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \langle 0, 0, x_3(b), x_4(b), 0 \rangle.$$

Applying  $\sigma$  to both sides of the identity

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r}x_3(b) \\ x_4(a^2b) &= a^{3p^r}x_4(b). \end{aligned}$$

Applying  $\sigma$  to both sides of the identity

$$\begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1 + b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1 + b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r}x_3(b_2). \end{aligned}$$

We see that  $x_3$  is a homomorphism, so it is of the form

$$x_3(b) = \sum_i \mu_i b^{p^i}.$$

Suppose  $x_3 \neq 0$ . Then some  $\mu_j \neq 0$  and

$$\begin{aligned}\mu_j(a^2b)^{p^j} &= a^{p^r} \mu_j b^{p^j} \\ \implies a^{2p^j} &= a^{p^r} \\ \implies p &= 2.\end{aligned}$$

But then

$$\begin{aligned}x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r} x_3(b) \\ &= b^{2^r} x_3(b),\end{aligned}$$

implies that  $x_3$  is constant, hence zero.

Therefore  $x_3 = 0$ , so  $x_4$  is a homomorphism:

$$x_4(b) = \sum_i \nu_i b^{p^i}.$$

If  $x_4 \neq 0$  then there is a  $\nu_j \neq 0$  and we get

$$\begin{aligned}\nu_j(a^2b)^{p^j} &= a^{3p^r} \nu_j b^{p^j} \\ \implies a^{2p^j} &= a^{3p^r} \\ \implies 2p^j &= 3p^r,\end{aligned}$$

which implies that 2 divides  $p$  and 3 divides  $p$ , a contradiction. Hence  $x_4 = 0$  and

$$\sigma \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = 0.$$

**Example 5.2.** Let  $G = C_3$ . Fix a maximal torus, labeling the positive simple roots  $\Delta = \{\alpha, \beta, \gamma\}$  with  $\gamma$  being the long root and connected to  $\beta$ . Let

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle.$$

Again we will write  $v$  in  $V_\alpha$  in angled brackets for ease of notation:

$$\begin{aligned}\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \rangle &:= \\ \epsilon_\beta(v_1) \epsilon_\gamma(v_2) \epsilon_{\alpha+\beta}(v_3) \epsilon_{\beta+\gamma}(v_4) \epsilon_{\alpha+\beta+\gamma}(v_5) \epsilon_{2\beta+\gamma}(v_6) \epsilon_{\alpha+2\beta+\gamma}(v_7) \epsilon_{2\alpha+2\beta+\gamma}(v_8) &\in V_\alpha\end{aligned}$$

The group law for  $V_\alpha$  is

$$\begin{aligned} u * v = & \\ \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3v_2, u_6 + v_6 + u_2v_1^2 + 2u_4v_1, \\ & u_7 + v_7 + u_2u_3v_1 + u_2v_1v_3 + u_5v_1 + u_4v_3, u_8 + v_8 - u_3^2v_2 - 2u_3v_2v_3 + 2u_5v_3 \rangle. \end{aligned}$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-p^r}v_1, v_2, a^{p^r}v_3, a^{-p^r}v_4, a^{p^r}v_5, a^{-2p^r}v_6, v_7, a^{2p^r}v_8 \rangle.$$

Let  $\sigma$  be in  $Z^1(SL_2, V_\alpha)$  such that

$$\sigma \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = 0.$$

By Lemma 5.1

$$\sigma \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b) \rangle.$$

Applying  $\sigma$  to both sides of the identity

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r} x_3(b) \\ x_5(a^2b) &= a^{p^r} x_5(b) \\ x_8(a^2b) &= a^{2p^r} x_8(b). \end{aligned}$$

Since the polynomials  $x_3, x_5, x_8$  are homomorphisms (Lemma 5.2) we get

$$\begin{aligned} \sum_i \lambda_i (a^2b)^{p^i} &= a^{p^r} \sum_i \lambda_i b^{p^i} \\ \sum_i \mu_i (a^2b)^{p^i} &= a^{p^r} \sum_i \mu_i b^{p^i} \\ \sum_i \nu_i (a^2b)^{p^i} &= a^{2p^r} \sum_i \nu_i b^{p^i}, \end{aligned}$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of  $SL_2$  under  $\sigma$  is  $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle$  then the characteristic of  $k$  must be 2, and so the image is a product of commuting root groups.

We may now state and prove the main result.

[would like]

**Theorem 5.6.** *Let  $G$  be a reductive linear algebraic group over a closed field of positive characteristic  $p$  and let  $\Gamma = SL_2(k)$ . Then the answer to the algebraic interpretation of Külshammer's Second Problem [ref] is "yes".*

*Proof.* Need to:

- handle arguments above with  $G$  possibly containing  $G_2$  and  $C_3$ .
- drop the restriction of rank-1 parabolics
- now we have abelian 1-cohomology and can apply result from previous chapter

□

## 5.2 A rank 1 calculation

[INCLUDE  $G_2$  OR  $B_2$  CALCULATIONS]

Let  $T$  be a maximal torus of  $B_2$  over an algebraically closed field  $k$  of characteristic  $p$ . We label the positive roots for  $B_2$  as  $\alpha, \beta, \alpha + \beta, 2\alpha + \beta$ . We have from [5, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of  $B_2$  is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where  $B$  is the Borel subgroup of  $B_2$  containing  $T$

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

### 5.2.1 Example

Let  $V$  be the unipotent radical of the parabolic subgroup of  $B_2$  defined by the (short) root  $\alpha$ :

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$



and let  $\rho_r$  be the homomorphism from  $SL_2 \rightarrow L_\alpha$  defined by

$$\begin{aligned}\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,\end{aligned}$$

where  $r$  is some non-negative integer.

Note that  $V$  is abelian. Now  $SL_2$  acts on  $V$  via  $\rho_r$ : write  $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$  in  $V$  as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_\alpha(-u^{p^r}) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_\alpha(-u^{p^r}) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(-2u^{p^r} v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\alpha(-u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(-u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{2p^r} v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2) \\
&= \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2 - u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r} v_1 \\ v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta(\beta(\alpha^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\alpha^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\alpha^\vee(t^{p^r})) v_3) \\
&= \epsilon_\beta((t^{p^r})^{\langle \beta, \alpha \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha + \beta, \alpha \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha + \beta, \alpha \rangle} v_3) \\
&= \begin{pmatrix} t^{-2p^r} v_1 \\ v_2 \\ t^{2p^r} v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) n_\alpha^{-1} n_\alpha \epsilon_{\alpha+\beta}(v_2) n_\alpha^{-1} n_\alpha \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1) \epsilon_{\alpha+\beta}(-v_2) \epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3) \epsilon_{\alpha+\beta}(-v_2) \epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of  $SL_2$  on  $V$ :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad + bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Now let  $\sigma'$  in  $Z^1(SL_2, V)$  be a 1-cocycle from  $SL_2 \rightarrow V$ . By [some reference]  $\sigma'$  is conjugate to a 1-cocycle  $\sigma$  that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all  $t$  in  $k^*$ . Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with  $\sigma$  instead.

Since  $\sigma$  is a morphism of varieties, each component of  $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$  should be a polynomial function of  $u$ , so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.1)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.2)$$

to get further information on the polynomials  $p_i$  ( $i = 1, 2, 3$ ).

If we apply  $\sigma$  to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned} \sigma \left( \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left( \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_1(t^2u) = t^{-2p^r} p_1(u) \quad (5.3)$$

$$p_2(t^2u) = p_2(u) \quad (5.4)$$

$$p_3(t^2u) = t^{2p^r} p_3(u). \quad (5.5)$$

From (5.4) it is clear that  $p_2$  is constant, so there is a  $\lambda$  in  $k$  such that  $p_2(x) = \lambda$  for all  $x$  in  $k$ . Now notice that on the left hand side of (5.3) there are only non-negative powers of  $t$ , and on the right hand side there are only non-positive powers of  $t$ . This equality is only satisfied if  $p_1(x) = 0$  for all  $x$  in  $k$ , so  $p_1$  is the zero polynomial.

We apply  $\sigma$  to (5.2) and using the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left( \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.6)$$

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2) - 2u_1^{p^r} p_2(u_2). \quad (5.7)$$

Since  $p_2$  is constant, (5.6) implies that  $p_2$  is the zero polynomial, which means (5.7) becomes

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2).$$

Hence  $p_3$  is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.8)$$

for some  $\mu_i$  in  $k$ .

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^N \mu_i (t^2u)^{p^i} = t^{2p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.9)$$

If  $p_3$  is not the zero polynomial then there is a non-zero  $\mu_l$  for some index  $l$ . By equating the coefficients of  $u$  in (5.9) we get

$$\begin{aligned}\mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies p^l &= p^r.\end{aligned}$$

Therefore  $l = r$ . This means that the only non-zero  $\mu_i$  is already specified by the choice of  $r$  in defining  $\rho_r$ .

Letting  $\mu_l = \mu$  in  $k$ , we have

$$\begin{aligned}\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}.\end{aligned}$$

If we are to find a non-trivial 1-cohomology  $H^1(SL_2, V)$  then  $\sigma$  cannot be a 1-coboundary. But if the characteristic of  $k$ ,  $p$ , is not equal to 2 then by setting  $\mathbf{v}$  in  $V$  as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all  $a$  in  $k^*$  and all  $b$  in  $k$

$$\begin{aligned}
 \chi_v \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
 &= \sigma \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right).
 \end{aligned}$$

That is,  $\sigma$  takes the value of a 1-coboundary on the subgroup of upper triangular matrices of  $SL_2$ . By [some reference], this means that  $\sigma$  is a 1-coboundary from the whole of  $SL_2 \rightarrow V$ , and hence the 1-cohomology  $H^1(SL_2, V)$  is trivial. Therefore it is necessary to proceed with  $p = 2$ :

$$\sigma \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \quad (5.10)$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma \left( \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some  $\mu'$  in  $k$ .

We are now interested in the value of

$$\sigma \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

remembering that  $k$  now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu + \mu' \\ \mu \\ \mu \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu' \end{pmatrix} = \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu + \mu' \end{pmatrix}.
\end{aligned}$$

On the other hand, by applying  $\sigma$  to both sides of the equality

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

we get

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Therefore  $\sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$  is an element of  $V$  that is fixed by the action of  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ . Referring to the formula for the action of  $SL_2$  on  $V$  we see that such an element of  $V$  is of the form

$$\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix},$$

which implies that  $\mu = \mu'$ .

Finally, consider

$$\sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

If  $c = 0$  then we already have

$$\sigma \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise  $c^{-1}$  exists and we can compute

$$\begin{aligned} \sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r} \mu(cd)^{2^r} \\ (ac^{-1})^{2^r+1} \mu(cd)^{2^r} \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1 + ad)^{2^r} \\ \mu(ac^{-1})^{2^r} (1 + ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$



In fact, we see that

$$\sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if  $\sigma$  is in the same conjugacy class as  $\tau$  then by [some reference]

$$\tau \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ , so this means considering  $\mathbf{v}$  that is fixed by the action of  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ :

$$\begin{aligned} \tau \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each  $\mu$  in  $k$  corresponds to a conjugacy class of 1-cocycles  $[\sigma_\mu]$  from  $SL_2 \rightarrow V$  where

$$\sigma_\mu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle  $\tau$  is in the class  $[\sigma_\mu]$  if there is a  $\mathbf{v}$  in  $V$  such that

$$\tau \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma_\mu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from  $SL_2 \rightarrow P_\alpha$  by considering the action of  $Z(L_\alpha)^\circ$ , the connected centre of the Levi subgroup  $L_\alpha$ . Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \mid x \in k \rangle$$

where  $\gamma$  is a root in  $\Phi_{\alpha,\beta}$  such that

$$\langle \alpha, \gamma \rangle = 0. \quad (5.11)$$

Since  $\gamma = m\alpha + n\beta$  for some integers  $m, n$ , we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \quad (5.12)$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore  $Z(L_\alpha)^\circ = \langle (\alpha + \beta)^\vee(x) \mid x \in k \rangle$ . Taking an element  $\mathbf{s} = (\alpha + \beta)^\vee(s)$  of  $Z(L_\alpha)^\circ$  we compute the action of  $\mathbf{s}$  on the 1-cocycle  $\sigma_\mu$  as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha + \beta)^\vee(s) \epsilon_\beta (\mu(cd)^{2^r}) \epsilon_{\alpha+\beta} (\mu(bc)^{2^r}) \epsilon_{2\alpha+\beta} (\mu(ab)^{2^r}) (\alpha + \beta)^\vee(s)^{-1} \\ &= \epsilon_\beta \left( s^{\langle \beta, \alpha+\beta \rangle} \mu(cd)^{2^r} \right) \epsilon_{\alpha+\beta} \left( s^{\langle \alpha+\beta, \alpha+\beta \rangle} \mu(bc)^{2^r} \right) \epsilon_{2\alpha+\beta} \left( s^{\langle 2\alpha+\beta, \alpha+\beta \rangle} \mu(ab)^{2^r} \right) \\ &= \begin{pmatrix} (s^2 \mu)(cd)^{2^r} \\ (s^2 \mu)(bc)^{2^r} \\ (s^2 \mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from  $SL_2 \rightarrow V$  collapse

to just two classes when we consider the action of  $Z(L_\alpha)^\circ$ , that is, moving from  $V$ -conjugacy to  $P_\alpha$ -conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}. \end{aligned}$$

### 5.2.2 Example

Let  $V$  be the unipotent radical of the parabolic subgroup of  $B_2$  defined by the (long) root  $\beta$ :

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let  $\rho_r$  be the homomorphism from  $SL_2 \rightarrow L_\beta$  defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where  $r$  is some non-negative integer.

Note that  $V$  is not abelian in general. The Group Law for  $V$  can be computed as follows.

Let  $\mathbf{v}, \mathbf{w}$  in  $V$ . We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1 + w_1)\epsilon_{\alpha+\beta}(v_2 + w_2)\epsilon_{2\alpha+\beta}(v_3 + w_3 + 2v_2w_1) \\ &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 + 2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of  $SL_2$  on  $V$  via  $\rho_r$ . Let  $\mathbf{v}$  be an element of  $V$ :

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\beta(u^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(u^{p^r}) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \\
&= \begin{pmatrix} v_1 \\ v_2 + u^{p^r} v_1 \\ v_3 + u^{p^r} v_1^2 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \beta^\vee(t^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\beta^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\alpha(\alpha(\beta^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\beta^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\beta^\vee(t^{p^r})) v_3) \\
&= \epsilon_\alpha((t^{p^r})^{\langle \alpha, \beta \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha+\beta, \beta \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha+\beta, \beta \rangle} v_3) \\
&= \begin{pmatrix} t^{-p^r} v_1 \\ t^{p^r} v_2 \\ v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left( \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) n_\beta^{-1} n_\beta \epsilon_{\alpha+\beta}(v_2) n_\beta^{-1} n_\beta \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= \epsilon_{\alpha+\beta}(v_1) \epsilon_\alpha(-v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(-v_2) \epsilon_{\alpha+\beta}(v_1) \epsilon_{2\alpha+\beta}(v_3 - 2v_1 v_2) \\
&= \begin{pmatrix} -v_2 \\ v_1 \\ v_3 - 2v_1 v_2 \end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} c^{p^r} v_2 + d^{p^r} v_1 \\ a^{p^r} v_2 + b^{p^r} v_1 \\ v_3 + (ac)^{p^r} v_2^2 + (bd)^{p^r} v_1^2 + 2(bc)^{p^r} v_1 v_2 \end{pmatrix}.$$

As in the previous example we let  $\sigma$  in  $Z^1(SL_2, V)$  be a 1-cocycle from  $SL_2 \rightarrow V$  such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all  $t$  in  $k^*$ , and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all  $u$  in  $k$ .

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.13)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.14)$$

Applying  $\sigma$  to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma \left( \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \quad (5.15)$$

$$p_2(t^2u) = t^{p^r} p_2(u) \quad (5.16)$$

$$p_3(t^2u) = p_3(u). \quad (5.17)$$

From (5.17) we find that  $p_3$  is constant-valued, say  $p_3(x) = \lambda$  in  $k$  for all  $x$  in  $k$ . From (5.15) we see that there are only non-negative powers of  $t$  on the left hand side and only non-positive powers the right hand side. Therefore  $p_1$  is the zero polynomial.

Now applying  $\sigma$  to both sides of (5.14):

$$\begin{aligned}
 \sigma \left( \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left( \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) + p_2(u_2) \\ 2\lambda \end{pmatrix}
 \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.18)$$

$$\lambda = 2\lambda. \quad (5.19)$$

By (5.19) we see that  $p_3$  is in fact the zero polynomial, and (5.18) implies that  $p_2$  is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.20)$$

for some  $\mu_i$  in  $k$ .

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^N \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.21)$$

If  $p_2$  is not the zero polynomial then there is a non-zero  $\mu_l$  for some index  $l$ . By equating coefficients of  $u^{p^i}$  in (5.21) we get

$$\begin{aligned}
 \mu_l t^{2p^l} &= \mu_l t^{p^r} \\
 \implies 2p^l &= p^r.
 \end{aligned}$$

Thus 2 divides  $p^r$ , and since  $p$  is a prime,  $p = 2$ . Furthermore  $l = r - 1$ . This means that the non-zero  $\mu_l$  is already specified by the choice of  $r$  in defining  $\rho_r$ , and that  $r$  must be non-zero if  $p_2$  is to be non-zero.

Referring to the Group Law we see that  $V$  is abelian in characteristic 2, so we will use the ‘+’ symbol for combining elements of  $V$  from now on.

Proceeding with  $p = 2$ ,  $r > 0$  and letting  $\mu_l = \mu$ , we have

$$\begin{aligned}
 \sigma \left( \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma \left( \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some  $\mu'$  in  $k$ .

We are now interested in the value of

$$\sigma \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

We have

$$\begin{aligned}
\sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu^2 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' + \mu \\ \mu \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu' + \mu \\ \mu' \\ \mu'^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu' + \mu \\ \mu' + \mu \\ \mu'^2 \end{pmatrix}.
\end{aligned}$$

Since  $\sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$  is fixed under the action of  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$  for all  $t$  in  $k^*$  we must have  $\mu' = \mu$ .



Suppose  $c \neq 0$ . We have

$$\begin{aligned}
\sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r} \mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{2^r} (\mu(cd)^{2^{r-1}})^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ac^{-1} + a^2c^{-1}d)^{2^{r-1}} \\ \mu^2(1+ad)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when  $c = 0$  too, so we conclude that

$$\sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a  $\mathbf{v}$  in  $V$  that is fixed by  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$  and compute

$$\begin{aligned}
 \tau \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \mathbf{v} + \sigma \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix},
 \end{aligned}$$

which tells us that for each  $\mu$  in  $k$  we get a distinct conjugacy class of 1-cocycles  $[\sigma_\mu]$  from  $SL_2 \rightarrow V$ , where

$$\sigma_\mu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of  $Z(L_\beta)$  on our 1-cocycles

$$\begin{aligned}
 (\mathbf{s} \cdot \sigma_\mu) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \\
 &= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)(ab)^{2^{r-1}} \\ (s\mu)^2(bc)^{2^r} \end{pmatrix}.
 \end{aligned}$$

our infinitely many  $V$ -conjugacy classes collapse to just two  $P_\beta$ -conjugacy classes:

$$\begin{aligned}
 [\sigma_0] &= \{\sigma_0\}, \\
 [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}.
 \end{aligned}$$

### 5.3 A rank 2 calculation

Is  $Im(\rho_{r,s})$  irred in  $L_{\gamma,\delta}$ ?

No  $\rightarrow \text{Im}(\rho_{r,s})$  inside (a conjugate of)  $P_\gamma(B_2)$  or  $P_\delta(B_2)$ . Then it's inside  $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$  or  $P_\delta = L_\delta \ltimes R_u(P_\delta)$ , so it's inside  $L_\gamma$  or  $L_\delta$ .

1) Know about non G-cr in  $B_2$ , can I put them in an  $A_1 A_1$ ?

1a) Can this sit inside a rank 1 Levi?

2) Use  $B_2 = SO_5$ .

3) Take  $\text{Im}(\rho_{r,s})$ , can we conjugate it into  $P_\gamma$  or  $P_\delta$ ?

Let  $\text{char}(k) = 2$  and set  $V := \langle U_\phi \mid \phi \in \Phi^+, \phi \neq \gamma + \delta, \phi \neq \gamma + 2\delta \rangle$ . We will write  $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$  as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on  $V$  is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2 v_1 \\ 0 \\ u_4 v_1 \\ 0 \\ u_6 v_1 \\ 0 \\ u_8 v_1 \\ 0 \\ u_{10} v_1 \\ u_{10} v_1 v_2 + u_8 v_1 v_4 + u_6^2 v_1 + u_{11} v_2 + u_{10} v_3 + u_9 v_4 + u_8 v_5 \end{pmatrix}.$$

For integers  $r, s \geq 0$  we have a homomorphism  $\rho_{r,s} : SL_2 \rightarrow \tilde{A}_1 \tilde{A}_1 < L_{\{\gamma, \delta\}}$  defined by

$$\begin{aligned}\rho_{r,s} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\ \rho_{r,s} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma + \delta)^\vee(t^{2^s}) \\ \rho_{r,s} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta}\end{aligned}$$

from which we obtain an action of  $SL_2$  on  $V$ :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let  $\sigma$  be a 1-cocycle from  $SL_2$  to  $V$  such that for all  $t$  in  $k^*$

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since  $\sigma$  is a morphism of varieties, each component of  $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$  should be a polynomial function of  $u$ , so we let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each  $p_i$  ( $1 \leq i \leq 12$ ) is as required. Applying  $\sigma$  to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \quad (5.22)$$

It is clear that for  $i = 1, 6, 7, 12$  the polynomials  $p_i$  must be constant-valued, say  $\lambda_i$  for some fixed  $\lambda_i$  in  $k$  (resp). Furthermore, since  $p_i(t^2 u)$  involves only non-negative powers of  $t$ ,  $p_i$  must be the zero polynomial for  $i = 2, 3, 4, 5$ . Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying  $\sigma$  to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials  $p_1, p_6, p_7, p_{12}$  must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from  $k \rightarrow k$ . That is

for some  $w_j, x_j, y_j, z_j$  in  $k$  and all  $u$  in  $k$

$$\begin{aligned} p_8(u) &= \sum_{j=0}^N w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^N x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^N y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^N z_j u^{2^j}, \end{aligned}$$

If  $\sigma$  is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that  $p_8$  is not the zero polynomial, so that  $w_l \neq 0$  for some index  $l \geq 0$ . By (5.22)

$$\begin{aligned} \sum_{j=0}^N w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^N w_j u^{2^j} \\ \Rightarrow w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) &= w u^{2^r}, & p_9(u) &= x u^{2^r}, \\ p_{10}(u) &= y u^{2^s}, & p_{11}(u) &= z u^{2^s}, \end{aligned}$$

for some  $w, x, y, z$  in  $k$ .

So, we have

$$\begin{aligned} \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

We apply the same argument using the fact that each component of  $\sigma \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$  is a polynomial function, say  $p'_i(u)$  for all  $u$  in  $k$ , to get

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some  $w', x', y', z'$  in  $k$ .

From this we deduce that

$$\begin{aligned}
 \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ y + y' \\ z + z' \\ w + w' \\ x + x' \\ w' + y' \\ x' + z' \\ w + w' \\ x + x' \\ y + y' \\ z + z' \\ w'x' + y'z' \end{pmatrix}.
 \end{aligned}$$

Furthermore, since  $\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is fixed under the action of  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ , we have

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} n_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_6 \\ n_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_{12} \end{pmatrix},$$



for some  $n_1, n_6, n_7, n_{12}$  in  $k$ . So in fact

$$\begin{aligned}
 w' &= w \\
 x' &= x \\
 y' &= y \\
 z' &= z \\
 n_1 &= 0 \\
 n_6 &= w + y \\
 n_7 &= x + z \\
 n_{12} &= wx + yz.
 \end{aligned}$$

Consider  $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . If  $c = 0$  then we already have

$$\begin{aligned}
 \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

Otherwise,  $c \neq 0$  and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left( \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map  $\sigma : SL_2 \rightarrow V$  of the form

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some  $w, x, y, z$  in  $k$  and integers  $r, s \geq 0$ .

[Show  $\sigma$  is a 1-cocycle]

Next we shall describe  $H^1(SL_2, V)$ . Recall that a 1-cocycle  $\tau'$  is in the same conjugacy class as  $\sigma$  if there is a  $\mathbf{v}$  in  $V$  such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g \cdot \mathbf{v}^{-1}$$

for all  $g$  in  $SL_2$ . Furthermore,  $\tau'$  is conjugate to some 1-cocycle  $\tau$ , where  $\tau$  has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus  $\sigma$  is conjugate to  $\tau$  by some  $\mathbf{v}$  in  $V$  that is fixed under the action of  $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ :

$$\begin{aligned} \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\ &= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix} \end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \sim (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple  $(w, x, y, z)$  represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each  $x, z$  in  $k$  the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider  $P$ -conjugacy. An element  $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$  acts on the 1-cocycle  $\sigma$  by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ s^{-1}t^2y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

## 5.4 A Non-Reductive Counterexample

In [7] a counterexample to [ref KII] is presented for a closed field  $k$  of characteristic  $p = 2$  and a non-reductive algebraic group  $G$ .

**Example 5.3.** Let  $Q$  be the algebraic group isomorphic to the affine space  $\mathbf{A}^3$  with the group multiplication law:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1v_1 + u_2v_2 + u_1v_2 \end{pmatrix}.$$

Let  $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$  and  $\Gamma_2 = \langle \tau \rangle$  the Sylow 2-subgroup of  $\Gamma$ .  $\Gamma$  acts on  $Q$  via

$$\begin{aligned} \tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1u_2 \end{pmatrix} \\ \sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}. \end{aligned}$$

Let  $G = Q \rtimes \Gamma$ . Then there are infinitely many pairwise  $G$ -conjugate classes of extensions to the representation  $\rho : \Gamma_2 \rightarrow G$  defined by the natural inclusion  $\Gamma_2 \rightarrow \Gamma \rightarrow G$  [7, Appendix].

*Proof.* Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle  $\alpha \in Z^1(\Gamma, Q)$  such that  $\alpha|_{\langle \sigma \rangle} = 1$ . Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some  $u_1, u_2, u_3 \in k$ . Since  $\tau$  is an involution we have

$$\begin{aligned}
 1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\
 &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1 u_2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}.
 \end{aligned}$$

This shows  $u_1 = u_2$ , so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as  $\tau\sigma\tau = \sigma^2$  we obtain

$$\begin{aligned}
 1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
 &= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
 &= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
 &= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
 \end{aligned}$$

Therefore  $u_1 = 0$ . Hence a typical 1-cocycle that is trivial on  $\langle \sigma \rangle$  satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \quad (u \in k).$$

Now we calculate the class  $[\alpha_u] \in H^1(\Gamma, Q)$ . Suppose  $\alpha_v \sim \alpha_u$ . Then there is a  $q \in Q$  fixed under the action of  $\sigma$ , that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$



such that  $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$ . In particular, for  $\gamma = \tau$

$$\begin{aligned}
 \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
 \end{aligned}$$

Hence only if  $u = v$  are two 1-cocycles of the particular form in the same class, and therefore  $H^1(\Gamma, Q)$  is infinite. [to finish].  $\square$

[note how to make my example look like the Slodowy one]

It is natural to ask whether this leads to a reductive counterexample, although we can quickly verify that the answer is “not immediately”. For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$\begin{aligned}
 &\dots \epsilon_\alpha(u_\alpha) \dots \epsilon_\beta(u_\beta) \dots \epsilon_\gamma(u_\gamma) \times \dots \epsilon_\alpha(v_\alpha) \dots \epsilon_\beta(v_\beta) \dots \epsilon_\gamma(v_\gamma) \\
 &= \dots \epsilon_\alpha(u_\alpha + v_\alpha) \dots \epsilon_\beta(u_\beta + v_\beta) \dots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
 \end{aligned}$$

Then setting  $u_\delta = v_\delta = 0$  whenever  $\delta \neq \alpha$  gives

$$\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) = \epsilon_\alpha(u_\alpha + v_\alpha) \epsilon_\gamma(u_\alpha v_\alpha),$$

which is absurd. [try find more examples]

## Chapter 6

## Conclusion

# Appendix A

## Further Calculations

$G$	$P$	$Z^1$	$H^1$	$V\text{-conj}$	$P\text{-conj}$
$B_2$ ( $\alpha$ short)	$P_\alpha$	✓	✓	✓	✓
	$P_\beta$	✓	✓	✓	✓
$G_2$ ( $\alpha$ short)	$P_\alpha$	✓			
$C_3$ ( $\gamma$ long)	$P_\alpha$	✓			
[7]	$Q \rtimes SL(2, 2)$	✓	✓	✓	

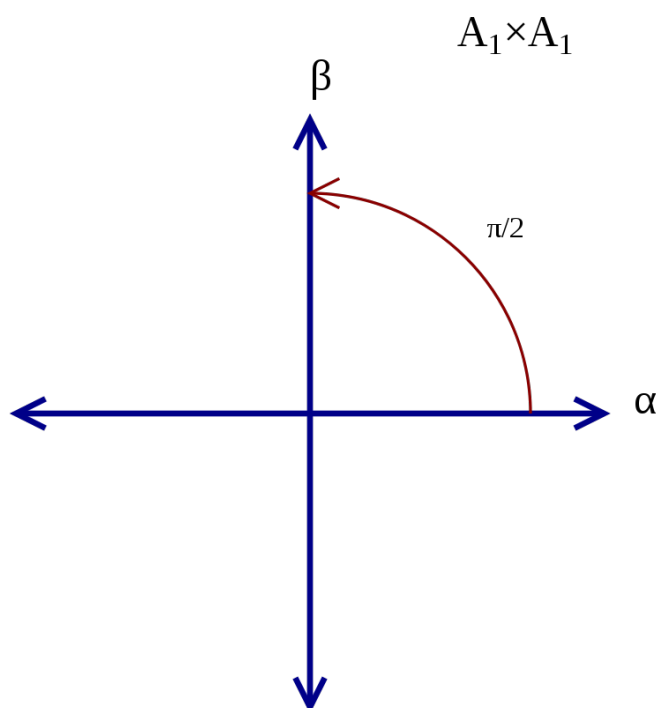
## Appendix B

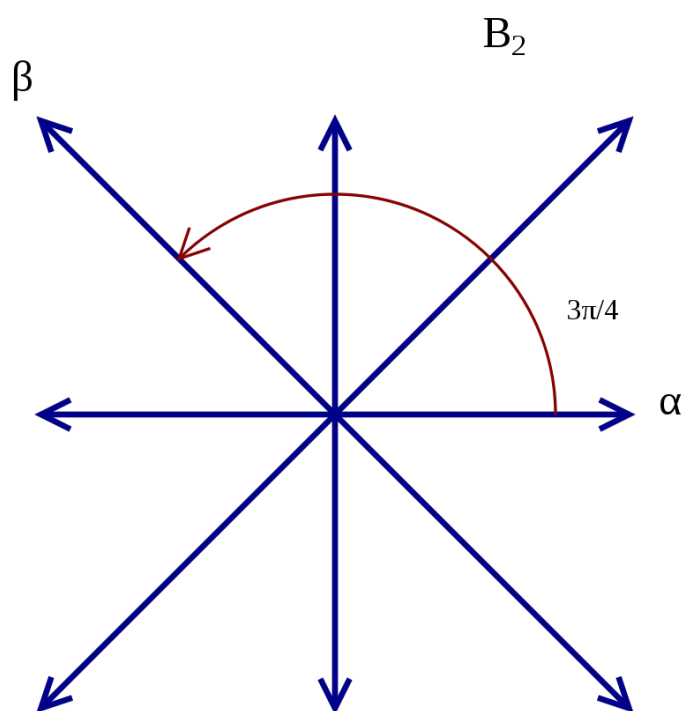
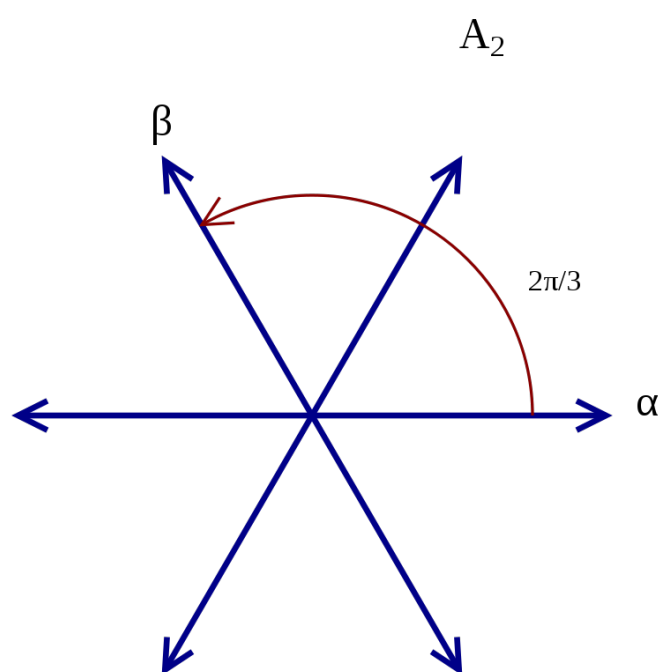
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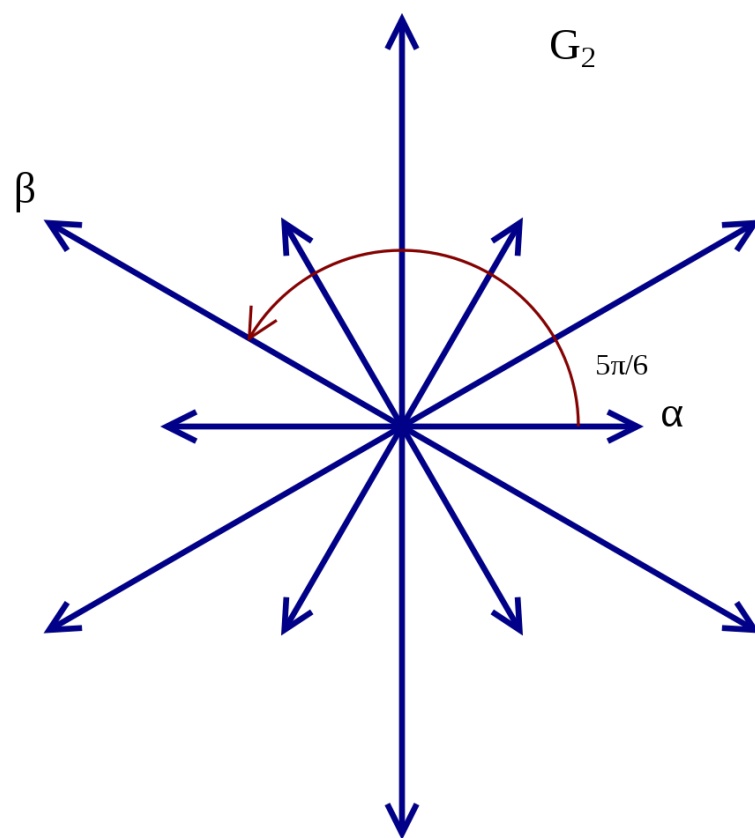
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## Appendix C

### Rank 2 Root System Diagrams







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