UNIVERSITY OF CANTERBURY

# A Geometric Approach to Complete Reducibility

by

Daniel Lond

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in the

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Department of Mathematics and Statistics

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*"A quote."*

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# *Abstract*

College of Engineering
Department of Mathematics and Statistics

Doctor of Philosophy

by Daniel Lond

The Thesis Abstract . . .

# *Acknowledgements*

The acknowledgements and the people to thank ...

# Contents

# Symbols

| | | |
|---|---|---|
| $a$ | distance | m |
| $P$ | power | W ($Js^{-1}$) |
| | | |
| $\omega$ | angular frequency | $rads^{-1}$ |
| $\vdots$ | | |

*Dedication ...*

# Chapter 1

# Introduction

- What the thesis is about
- Motivation - link with other problems
- Highlight results - lead up to highlights
- Similar to abstract but less formal
- Outline of the contents, chapter by chapter

# Chapter 2

# Mathematical Preliminaries

# Chapter 3

# Külshammer's Second Problem

- Külshammer's First Problem
- Külshammer's Second Problem
- Counter example for non-reductive $G$
- Overture to Ch. 3-5

## 3.1 Külshammer's First Problem

## 3.2 Külshammer's Second Problem

## 3.3 A non-reductive counterexample

3) Look at the nonreductive counterexample in Slodowy's paper on Kulshammer's problem. What is special about the 3-dimensional U that makes this counterexample work? Can you find similar structure in the unipotent radical of a reductive group?

# Chapter 4

# The 1-Cohomology

TODO:

- Abelian 1-Cohomology
- Non-ablelian 1-Cohomology [Richardson]
- Brief results of calculations

## 4.1 Abelian 1-Cohomology

TODO:

- The general setting: Group $H$, Vector space/Abelian group $V$, linear action/action of homomorphisms.
- Definition of $Z^1$, $B^1$, $H^1$.
- Definition of $Z^1(f)$, $B^1(f)$, $H^1(f)$ where $f : H \to H'$. $f$ must respect the actions of $H, H'$. e.g. $i$ inclusion map.
- $H^1(H, V) \to H^1(H_P, V)$ is (1-1? onto?)
- What about when $H$ is finite? algebraic? both? Define $H^1$ for $H$ finite, then $H$ algebraic then $H$ finite algebraic - should be consistent.
- What about $H^1(x)$ where $x : V \to V'$? Relevant: $B_4 \leq F_4$.
- Explain: if $H = SL_2$ then $H^1$ is determined by its value on the upper triangular matrices. $B < SL_2$, $H^1(SL_2, V) \to H^1(B, V)$ is (1-1? onto?) $\leftarrow$ prove this. [Hum $G/B$].

- $H^1(H,V)$ is trivial if $H$ is linearly reductive [done for $H$ finite].

### 4.1.1 Definitions

Let $H$ be a group and $V$ an abelian group on which $H$ acts homomorphically, that is the map

$$v \mapsto h \cdot v,$$

is a homomorphism from $V \to V$ for any $h$ in $H$. We call a map $\sigma$ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \tag{4.1}$$

for all $h_1, h_2$ in $H$. Denote by $Z^1(H,V)$ the collection of all 1-cocycles from $H \to V$.

We call the (4.1) the *1-cocycle condition*.

For any $\sigma_1, \sigma_2$ in $Z^1(H,V)$

$$\begin{aligned}
(\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\
&= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\
&= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\
&= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2),
\end{aligned}$$

so $Z^1(H,V)$ is closed under pointwise addition.

The trivial map from $H \to V$ that sends every $h$ in $H$ to the identity $0$ in $V$ is a 1-cocycle. Furthermore for any $\sigma$ in $Z^1(H,V)$ we have

$$\begin{aligned}
\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\
&= \sigma(1) + \sigma(1) \\
&= 2\,\sigma(1),
\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each $\sigma$ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1(H, V)$ is a $\mathbb{Z}-$module under pointwise addition.

Given a $v$ in $V$ we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any $v$ in $V$ and any $h_1, h_2$ in $H$

$$\begin{aligned}\chi_v^H(h_1 h_2) &= v - (h_1 h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so that every 1-coboundary is also a 1-cocycle. For any $u, v$ in $V$ and all $h$ in $H$

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u + v) - h \cdot (u + v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

Setting $v = -u$ in the above calculation provides the definition of an inverse of a 1-coboundary and hence shows that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a $\mathbb{Z}-$submodule of $Z^1(H, V)$, so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology.*

**Lemma 4.1.** *Suppose $H$ is linearly reductive. Then $H^1(H,V) = 0$.*

### 4.1.2 Maps between 1-cohomologies

Let $\phi$ be a homomorphism from $\tilde{H} \to H$, $\tilde{H}$ being another group that acts on $V$ by homomorphisms. Suppose that for every $h$ in $H$ $\phi$ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all $v$ in $V$.

If $\sigma$ is a 1-cocycle from $H \to V$ then we claim that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \to V$. For take $h_1, h_2$ in $H$. We have

$$\begin{aligned}
Z^1(\phi)(\sigma)(h_1h_2) &= \sigma(\phi(h_1h_2)) \\
&= \sigma(\phi(h_1)\phi(h_2)) \\
&= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2) \\
&= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\
&= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2).
\end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H,V)$ into $B^1(\tilde{H},V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H,V) \to H^1(\tilde{H},V),$$

defined by

$$\begin{tikzcd}
Z^1(H,V) \arrow[r, "Z^1(\phi)"] \arrow[d, "\pi"'] & Z^1(\tilde{H},V) \arrow[d, "\tilde{\pi}"] \\
H^1(H,V) \arrow[r, "H^1(\phi)"'] & H^1(\tilde{H},V)
\end{tikzcd}$$

where $\pi$ and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H,V)$ onto $H^1(H,V)$ and $Z^1(\tilde{H},V)$ onto $H^1(\tilde{H},V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to show that $Z^1(\phi)$ is a homomorphism:

$$\begin{aligned}
Z^1(\phi)(\sigma_1+\sigma_2)(h) &= (\sigma_1+\sigma_2)(\phi(h)) \\
&= \sigma_1(\phi(h))+\sigma_2(\phi(h)) \\
&= Z^1(\phi)(\sigma_1)(h)+Z^1(\phi)(\sigma_2)(h).
\end{aligned}$$

**Lemma 4.2.** *Let $\tilde{H}$ be a subgroup of $H$ and $i:\tilde{H}\to H$ the inclusion map. Then $i$ gives rise to a well defined map*

$$H^1(i): H^1(H,V)\to H^1(\tilde{H},V).$$

**Lemma 4.3.** *Let $H$ be a finite group and $\tilde{H}=H_p$ a* Sylow $p$-subgroup *of $H$. The map*

$$H^1(i): H^1(H,V)\to H^1(H_p,V)$$

*is injective.*

*Proof.* Let $x$ be an element of $H^1(H,V)$ such that $H^1(i)(x)=0$. Now choose a 1-cocycle $\sigma$ in $Z^1(H,V)$ such that $\pi(\sigma)=x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say $\sigma$ restricted to $H_p$ is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since $\chi_v^{H_p}$ can be trivially extended to a 1-coboundary $\chi_v^H$ from $H\to V$, and

$$\pi(\sigma-\chi_v^H)=x,$$

we could well have chosen the 1-cocycle $(\sigma-\chi_v^H)$ as a representative for $x$. Hence there is no harm in assuming that $\sigma$ is 0 when restricted to $H_p$. Now choose a set of representatives $h_1,\ldots,h_l$ in $H$ for the coset space $H/H_p$ and set

$$v^*=\sum_{i=1}^{l}\sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by $v^*$

$$\begin{aligned}
\chi_{v^*}^H(h) &= v^*-h\cdot v^* \\
&= \sum_{i=1}^{l}\sigma(h_i)-h\cdot\sum_{i=1}^{l}\sigma(h_i) \\
&= \sum_{i=1}^{l}\sigma(h_i)-\sum_{i=1}^{l}h\cdot\sigma(h_i).
\end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned}\sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} h \cdot \sigma(h_i) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} (\sigma(hh_i) - \sigma(h)) \\ &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h).\end{aligned}$$

Now as the value of $\sigma$ at a fixed $h$ depends only on the value of $\sigma$ at the representative $h_j$ of the coset containing $h$ we can collapse the middle term to yield

$$\begin{aligned}\chi_{v^*}^{H}(h) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h) \\ &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(h_i) + \sum_{i=1}^{l} \sigma(h) \\ &= l\,\sigma(h).\end{aligned}$$

Since $l < p$, *[okay, I should have been talking about $V$ vector space here]* $l$ is invertible and so

$$l^{-1}\chi_{v^*}^{H}(h) = \sigma(h).$$

That is, $\sigma$ is a 1-coboundary, so the kernel of $H(i)$ is trivial. □

We could also let $\tilde{V}$ be another abelian group and $f : V \to \tilde{V}$ a homomorphism of groups satisfying

$$f(h \cdot v) = h \cdot f(v).$$

Following a similar chain of arguments as before we can define a map

$$H^1(f) : H^1(H, V) \to H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \to H^1(\tilde{H}, \tilde{V}).$$

It is worth reminding the reader here of the directions of the underlying homomorphisms for the above construction to work:

$$\phi : \tilde{H} \to H$$
$$f : V \to \tilde{V}.$$

## 4.2 Non-abelian 1-Cohomology

TODO:

- set up: Group $H$, non-abelian group $V$.
- define: $Z^1$ as before, $B^1$ - not really the same as before, $H^1$ - harder.
- $\phi : H \to H'$, $Z^1(H', V) \to Z^1(H, V)$? $H^1(H', V) \to H^1(H, V)$? - could put in some details.
- $H$ finite, $H$ algebraic - as before.
- $H^1(H, V) \to H^1(H_p, V)$ different to abelian case? Link to counter example to KII?
- $H^1(SL_2, V) \to H^1(B, V)$ (1-1? onto?)
- $H^1(H, V)$, $H$ linearly reductive - done.
- $V \to V'$, $Z^1(H, V) \to Z^1(H, V')$ ? exists? $H^1(H, V) \to H^1(H, V')$?? could put in details. $V$ abelian group, $R = \mathbb{Z}$. $V$ is a $\mathbb{Z}$- module.

### 4.2.1 The non-abelian setting

### 4.2.2 Maps of non-abelian 1-cohomologies

OLD STUFF: Let $G$ be an algebraic group, $P$ a parabolic subgroup of $G$, and $L$ a Levi subgroup of $P$. Let $\rho : H \to L$ be a homomorphism, $H$ an abstract group.

We are interested in functions $\rho_\alpha : H \to P$ of the form $\rho_\alpha(h) = \alpha(h)\rho(h)$, where $\alpha : H \to R_u(P)$.

What properties must $\alpha$ satisfy for $\rho_\alpha$ to be a homomorphism?

$$\begin{aligned}\alpha(gh)\rho(gh) = \rho_\alpha(gh) &= \rho_\alpha(g)\rho_\alpha(h)\\ &= \alpha(g)\rho(g)\alpha(h)\rho(h)\\ &= \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}\rho(g)\rho(h)\\ &= \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}\rho(gh),\end{aligned}$$

that is

$$\alpha(gh) = \alpha(g)\rho(g)\alpha(h)\rho(g)^{-1}.$$

Since $L$ normalises $R_u(P)$, we choose write this as

$$\alpha(gh) = \alpha(g) * g \cdot \alpha(h), \tag{4.2}$$

where the action of $H$ on $R_u(P)$ is defined by $\rho$, and $* : R_u(P) \times R_u(P) \to R_u(P)$.

We call (1) the 1-cocycle condition. A morphism $\alpha : H \to R_u(P)$ that satisfies the 1-cocycle condition for all $g, h \in H$ is said to be a 1-cocycle, and we denote by $Z^1(H, R_u(P))$ the set of all 1-cocycles from $H$ into $R_u(P)$.

Note that in the case that $R_u(P)$ is abelian, $Z^1(H, R_u(P))$ is a vector space under pointwise addition and scalar multiplication, and we write the 1-cocycle condition in additive notation: $\alpha(gh) = \alpha(g) + g \cdot \alpha(h)$.

When is $\rho$ $R_u(P)$-conjugate to $\rho_\alpha$?

Suppose there exists a $v \in R_u(P)$ such that $\rho_\alpha(h) = v\rho(h)v^{-1}$ for all $h \in H$. Then

$$\begin{aligned}\alpha(h)\rho(h) = \rho_\alpha(h) &= v\rho(h)v^{-1}\\ &= v\rho(h)v^{-1}\rho(h)^{-1}\rho(h).\end{aligned}$$

Therefore, $\alpha$ is of the form

$$\alpha(h) = v * h \cdot v^{-1}.$$

This leads us to the next definition.

For a fixed $v \in R_u(P)$ we define a 1-coboundary to be a morphism $\chi_v : H \to R_u(P)$ of the form

$$\chi_v(h) = v * h \cdot v^{-1}, \tag{4.3}$$

and denote the collection of all 1-coboundaries from $H$ into $R_u(P)$ by $B^1(H, R_u(P))$. Indeed,

$$\begin{aligned}
\chi_v(gh) &= v\rho(gh)v^{-1}\rho(gh)^{-1} \\
&= v\rho(g)\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1} \\
&= v\rho(g)\left[v^{-1}\rho(g)^{-1}\rho(g)v\right]\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1} \\
&= \left[v\rho(g)v^{-1}\rho(g)^{-1}\right]\left[\rho(g)v\rho(h)v^{-1}\rho(h)^{-1}\rho(g)^{-1}\right] \\
&= \left[v * g \cdot v^{-1}\right] * g \cdot \left[v * h \cdot v^{-1}\right] \\
&= \chi_v(g) * g \cdot \chi_v(h),
\end{aligned}$$

so that $B^1(H, R_u(P)) \subset Z^1(H, R_u(P))$.

As before, if $R_u(P)$ is abelian then we write (2) in additive notation, $\chi_v(h) = v - h \cdot v$, and note that $B^1(H, R_u(P))$ is a vector subspace of $Z^1(H, R_u(P))$.

When is $\rho_\alpha$ $R_u(P)$-conjugate to $\rho_\beta$?

Let $\alpha, \beta \in Z^1(H, R_u(P))$ and suppose there exists a $v \in R_u(P)$ such that $\rho_\beta(h) = v\rho_\alpha(h)v^{-1}$. Then

$$\begin{aligned}
\beta(h)\rho(h) &= v\alpha(h)\rho(h)v^{-1} \\
&= v\alpha(h)\rho(h)v^{-1}\rho(h)^{-1}\rho(h),
\end{aligned}$$

that is

$$\beta(h) = v\alpha(h) * h \cdot v^{-1}. \tag{4.4}$$

We show that (3) gives rise to an equivalence relation on $Z^1(H, R_u(P))$.

Reflexivity:

$\alpha(h) = v\alpha(h) * h \cdot v^{-1}$ with the choice $v = 1$.

Symmetry:

If $\alpha(h) = v\beta(h) * h \cdot v^{-1}$, then $\beta(h) = v^{-1}\beta(h) * h \cdot v$.

Transitivity:

If $\alpha(h) = v\beta(h) * h \cdot v^{-1}$ and $\beta(h) = w\gamma(h) * h \cdot w^{-1}$ then $\alpha(h) = (vw)\gamma(h) * h \cdot (vw)^{-1}$.

Now we define the 1-cohomology, denoted by $H^1(H, R_u(P))$, to be the set of equivalence classes of $Z^1(H, R_u(P))$, where $\alpha \sim \beta$ if and only if (3) holds.

In the abelian case, (3) becomes $\beta(h) = \alpha(h) + \chi_v(h)$, so that two 1-cocycles are equivalent when they differ by a 1-coboundary, and $H^1(H, R_u(P)) = Z^1(H, R_u(P))/B^1(H, R_u(P))$.

BEN'S SUGGESTIONS:

1) Understand the basic properties of nonabelian 1-cohomology. (Definitions of 1-cocycles, 1-coboundaries and 1-cohomology. Maps $H^1(F, U) \to H^1(F_p, U)$, where $F_p$ is a Sylow $p$-subgroup of $F$. Etc. Some of this will be in Richardson's paper.)

2) Try to show that nonabelian $U$ is needed if we are to find a counterexample to Kulshammer's problem. There are various issues here. For instance, is the following true? Suppose we have $\rho_0 : H \to L$, and suppose $U = R_u(P)$ is abelian. Let $\{\rho_\alpha\}$ be a family of representations of H constructed from $\rho_0$ using 1-cocycles $\alpha$. Show that if the restrictions $\rho_\alpha|H_p$ are all $U$-conjugate then the $\rho_\alpha$ are all $U$-conjugate. Here $H$ is a finite group with Sylow $p$-subgroup $H_p$, but one could ask the same question with $H = A_1$ and replacing $H_p$ with the group of upper unitriangular matrices.

# Chapter 5

# 1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup $P$ of a reductive group $G$. The motivation for this, as outlined in previous chapters, is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into $G$ and then examine various finite subgroups $H$ of $SL_2(k)$ in the hope of finding a counterexample for Külshammer's Second Problem.

## 5.1 The method

Let $G$ be a reductive group over an algebraically closed field $k$ of characteristic $p$. Let $\Phi$ be the roots for $G$ with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus $T$ of $G$.

Let $P_\alpha < G$ be the parabolic subgroup of $G$ corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup $L_\alpha$ and unipotent radical $V_\alpha$:

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism $\rho_0$ from $SL_2(k)$ into $L_\alpha$ under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define $\rho_r$ to be the homomorphism from $SL_2(k)$ into $L_\alpha$ composed of $\rho_0$ and the Frobenius map,

$$\begin{aligned} F_r \quad : \quad & SL_2(k) \to SL_2(k) \\ & (A_{ij}) \mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r \quad = \quad \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \quad &= \quad \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \quad &= \quad \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on $V_\alpha$ via $\rho_r$ and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma \left( \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) \quad &= \quad \prod_\delta \epsilon_\delta \left( (t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_\delta \right) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) \quad &= \quad \prod_\delta n_\alpha \epsilon_\delta \left( \lambda_\delta \right) n_\alpha^{-1}, \end{aligned}$$

where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ and $\lambda_\delta$ are elements of the underlying field $k$.

**Lemma 5.1.**

$$\sigma \left( \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \prod_\delta \epsilon_\delta \left( x_\delta \left( u \right) \right),$$

*where $\delta$ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_\delta \in k[T]$ are polynomials in one variable.*

*Proof.* We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where $i$ is the inclusion map and $\pi_\delta$ the projection onto the root subgroup $V_\delta$. Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k$ to $k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

Therefore

$$x_\delta\left(t^2u\right) = (t^{p^r})^{\langle\delta,\alpha\rangle} x_\delta(u).$$

Since $x$ is a polynomial function there can only be non-negative powers of $t$ on the left-hand side of the equality which forces $\langle\delta,\alpha\rangle \geq 0$. However, if $\langle\delta,\alpha\rangle = 0$ then $x_\delta$ is constant and hence zero, as $\sigma$ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero $x_\delta$ occur precisely when $\langle\delta,\alpha\rangle > 0$. □

**Lemma 5.2.** *Suppose $\Phi$ is not of type $G_2$ and let $\alpha, \beta \in \Phi$. If $\alpha+\beta \in \Phi$ then $\langle\alpha,\beta\rangle \leq 0$.*

*Proof.*

$$\langle \alpha, \beta \rangle > 0 \Longleftrightarrow (\alpha, \beta) > 0 \Longleftrightarrow \cos(\theta) > 0,$$

where $\theta$ is the angle between $\alpha$ and $\beta$. Hence acute angles correspond to positive pairs. Referring to the $A_2$ and $B_2$ root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. □

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots ($\alpha$ short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$. Since the following results depend upon 5.2 we will henceforth assume that $\Phi \neq G_2$ and deal with the case $\Phi = G_2$ by way of an example.

**Lemma 5.3.** *Let $\delta_1, \delta_2 \in \Phi \neq G_2$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ $(i = 1, 2)$. If $\delta_1 + \delta_2$ is a root, then $\delta_1$ and $\delta_2$ are of opposite sign.*

*Proof.* Suppose $\delta_1 + \delta_2 \in \Phi$. Let $\theta_i$ be the absolute value of the angle between $\delta_i$ and $\gamma$, $(i = 1, 2)$ and let $\theta_3$ be the absolute value of the angle between $\delta_1$ and $\delta_2$. Then

$$\begin{aligned} & \langle \delta_i, \gamma \rangle > 0 \qquad (i = 1, 2) \\ \Longrightarrow \;& (\delta_i, \gamma) > 0 \\ \Longrightarrow \;& \cos(\theta_i) > 0 \\ \Longrightarrow \;& \theta_i < \pi/2, \end{aligned}$$

and similarly, using 5.2

$$\begin{aligned} & \langle \delta_1, \delta_2 \rangle \leq 0 \\ \Longrightarrow \;& \theta_3 \geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

$$\begin{array}{llll} \textbf{1:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = 2\pi/3; \\ \textbf{2:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{3:} & \theta_1 = \pi/4, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{4:} & \theta_1 = \pi/4, & \theta_2 = \pi/4, & \theta_3 = \pi/2. \end{array}$$

For the cases in which $\theta_3 = \pi/2$ we can see that this means $\delta_1$ and $\delta_2$ lie in a $B_2$ subsystem of $\Phi$, and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that $\delta_1$ and $\delta_2$ are short roots and their sum is a long root.

We are able rule out the third case. For if $\theta_1 = \pi/4$ then $\delta_1$ and $\gamma$ are roots of different length in a $B_2$ subsystem, but $\theta_2 = \pi/3$ implies that $\delta_2$ and $\gamma$ are roots of the same length in an $A_2$ subsystem, which is a contradiction.

In the second case we see that $\delta_1, \delta_2$ and $\gamma$ do not lie together in a rank 2 subsystem but a rank 3 subsystem, and that these roots are the same length which implies that $\gamma$ is a short root. In fact, since a pair short roots lie in subsystems of type $A_2$ it must be that the rank 3 subsystem in which the four roots lie is of type $C_3$. [Picture?]

Let $\theta_4$ be the absolute value of the angle between $\delta_1 + \delta_2$ and $\gamma$. Then

$$\begin{aligned}
\cos(\theta_4) &= \frac{(\delta_1+\delta_2,\gamma)}{|\delta_1+\delta_2||\gamma|} \\
&> \frac{(\delta_1,\gamma)}{2|\delta_1||\gamma|} + \frac{(\delta_2,\gamma)}{2|\delta_2||\gamma|} \quad \text{since } |\delta_1+\delta_2| < |\delta_1| + |\delta_2| \\
&= \cos(\pi/3).
\end{aligned}$$

Hence $\theta_4 < \pi/3$. The only possibility is for $\theta_4 = \pi/4$ which means that $\delta_1 + \delta_2$ is a long root adjacent to $\gamma$ in a $B_2$ subsystem. $\square$

**Corollary 5.4.** *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right).$$

*Furthermore, the $x_\delta$ are homomorphisms.*

*Proof.* We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \epsilon_\alpha\left(u_1^{p^r}\right) \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \epsilon_\alpha\left(-u_1^{p^r}\right),$$

with $\langle \delta, \alpha \rangle > 0$. By 5.2 $\alpha + \delta \notin \Phi$ so each $\epsilon_\delta$ commutes with the $\epsilon_\alpha$. $\square$

**Corollary 5.5.** *The image of the group of upper triangular matrices of $SL_2(k)$ under $\sigma$ lies in an abelian subgroup of $V_\alpha$.*

*Proof.* First consider

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta(b)\right).$$

Suppose the roots $\delta_1$ and $\delta_2$ appear on the right hand side. By 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$, so 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, $\epsilon_{\delta_1}$ and $\epsilon_{\delta_2}$ commute. Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \\ &= \prod_\delta \epsilon_\delta \left((a^{p^r})^{\langle \delta, \alpha \rangle} x_\delta\left(b\right)\right). \end{aligned}$$

□

Since the $x_\delta$ are homomorphisms from $k \to k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some $\mu_i$ in $k$. Furthermore, combining the calculation in the proof of 5.1 with the result 5.4 we get that

$$\prod_\delta \epsilon_\delta \left(x_\delta\left(a^2 b\right)\right) = \prod_\delta \epsilon_\delta \left(a^{\langle \delta, \alpha \rangle} x_\delta\left(b\right)\right),$$

severely restricting the possible polynomials $x_\delta$. In fact, they are confined to be polynomials involving just one term, and the exponent has already be decided when fixing the integer $r$ in the definition of $\rho_r$. For suppose $x_\delta$ and hence some $\mu_j$ is non-zero. Then equating the coefficients of $b$ in the equality directly above yields

$$\begin{aligned} \mu_j (a^2)^{p^j} &= \mu_j \left(a^{p^r}\right)^{\langle \delta, \alpha \rangle} \\ \implies 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

Since 2 divides that on the right hand side of the above, if $\langle \delta, \alpha \rangle$ is not a multiple of 2 then $p$, the characteristic of $k$, must be 2 or else $x_\delta = 0$, which implies that the 1-cohomology of the restriction of $SL_2(k)$ to the upper triangular matrices is trivial, and this implies that the 1-cohomology of the whole of $SL_2(k)$ is trivial.

In [Carter] it is shown that the possible pairings of any two roots are bounded by $\pm 3$. Hence by 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3. In fact $\langle \delta, \alpha \rangle = 3$ occurs only in the particularly tricky case that excludes the root system $G_2$ from the result in 5.2.

Things to do here:

- Refer to Structure/Classification Theorem to get the homomorphisms $\rho_r$
- Choosing $\sigma$ s.t. $\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = 0$

- Letting $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}_i = p_i(u)$

- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot (p_i(u)) = \big(p_i(t^2u)\big)$

- $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \Rightarrow p_i(u_1+u_2) = p_i(u_1) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot p_i(u_2)$. Usually end up with $p_i$ homomorphisms.

- Know $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Calc. $\sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}c \\ 0 & 1 \end{pmatrix}$

- Can get $\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}$ by a similar argument.

- Calc. $\sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \left( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$

- Compare with fact $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sigma \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now we know $\sigma$ exactly on $B$ and $n_\gamma$.

- Already know $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if $c = 0$. Now calc.

  $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \sigma \left( \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -c & -d \\ 0 & -c^{-1} \end{pmatrix} \right)$

- We now have fact $\sigma' \in Z^1(SL_2, V) \Rightarrow \sigma' \sim \sigma$ and know the form of $\sigma$. To check "$\Leftarrow$" direction apply $\sigma$ to the Steinberg relations.

- Find all $\tau \in Z^1(SL_2, V)$ conj. to $\sigma$ and also zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ by calculating $\tau(g) = v * \sigma(g) * g \cdot v^{-1}$.

- Can now state conj. classes of 1-cocycles by inspection.

- Extend classes to $P$-conjugacy by action of $Z(L)$. Explain why ...

- $G$-conjugacy ...

## 5.2 A rank 1 calculation

[INCLUDE $G_2$ OR $B_2$ CALCULATIONS]

Let $T$ be a maximal torus of $B_2$ over an algebraically closed field $k$ of characteristic $p$. We label the positive roots for $B_2$ as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [reference Humphreys 33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned} n_\alpha\epsilon_\beta(x)n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\ n_\alpha\epsilon_{\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\ n_\alpha\epsilon_{2\alpha+\beta}(x)n_\alpha^{-1} &= \epsilon_\beta(x) \\ n_\beta\epsilon_\alpha(x)n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\ n_\beta\epsilon_{\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_\alpha(-x) \\ n_\beta\epsilon_{2\alpha+\beta}(x)n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x) \end{aligned}$$

A proper parabolic subgroup of $B_2$ is conjugate to one of

$$\begin{aligned} P_\alpha &= \langle B, U_{-\alpha}\rangle \\ P_\beta &= \langle B, U_{-\beta}\rangle, \end{aligned}$$

where $B$ is the Borel subgroup of $B_2$ containing $T$

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned} P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\ &= \langle T, U_\alpha, U_{-\alpha}\rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle \\ P_\beta &= L_\beta \ltimes R_u(P_\beta) \\ &= \langle T, U_\beta, U_{-\beta}\rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle \end{aligned}$$

### 5.2.1 Example

Let $V$ be the unipotent radical of the parabolic subgroup of $B_2$ defined by the (short) root $\alpha$:

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle,$$

and let $\rho_r$ be the homomorphism from $SL_2 \to L_\alpha$ defined by

$$\begin{aligned}
\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\
\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\
\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,
\end{aligned}$$

where $r$ is some non-negative integer.

Note that $V$ is abelian. Now $SL_2$ acts on $V$ via $\rho_r$: write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in $V$ as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$
\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(-u^{p^r})\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(-u^{p^r})\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_\alpha(-u^{p^r})\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(-2u^{p^r}v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\alpha(-u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{2p^r}v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2)\\
&= \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2+u^{2p^r}v_1)\\
&= \begin{pmatrix}v_1\\ v_2-u^{p^r}v_1\\ v_3-2u^{p^r}v_2+u^{2p^r}v_1\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \alpha^\vee(t^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\alpha^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\beta\left((t^{p^r})^{\langle\beta,\alpha\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\alpha\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\alpha\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-2p^r}v_1\\ v_2\\ t^{2p^r}v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)n_\alpha^{-1}n_\alpha\epsilon_{\alpha+\beta}(v_2)n_\alpha^{-1}n_\alpha\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= \epsilon_{2\alpha+\beta}(v_1)\epsilon_{\alpha+\beta}(-v_2)\epsilon_\beta(v_3)\\
&= \epsilon_\beta(v_3)\epsilon_{\alpha+\beta}(-v_2)\epsilon_{2\alpha+\beta}(v_1)\\
&= \begin{pmatrix}v_3\\ -v_2\\ v_1\end{pmatrix}.
\end{aligned}
$$

We can combine the above calculations to get an explicit formula for the action of $SL_2$ on $V$:

$$
\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}d^{2p^r}v_1-2(cd)^{p^r}v_2+c^{2p^r}v_3\\ (ad+bc)^{p^r}v_2-(bd)^{p^r}v_1-(ac)^{p^r}v_3\\ b^{2p^r}v_1-2(ab)^{p^r}v_2+a^{2p^r}v_3\end{pmatrix}
$$

Now let $\sigma'$ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$. By [some reference] $\sigma'$ is conjugate to a 1-cocycle $\sigma$ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all $t$ in $k^*$. Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with $\sigma$ instead.

Since $\sigma$ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of $u$, so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

to get further information on the polynomials $p_i$ ($i = 1, 2, 3$).

If we apply $\sigma$ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

That is,

$$\begin{aligned} p_1(t^2u) &= t^{-2p^r}p_1(u) && (5.3)\\ p_2(t^2u) &= p_2(u) && (5.4)\\ p_3(t^2u) &= t^{2p^r}p_3(u). && (5.5)\end{aligned}$$

From (5.4) it is clear that $p_2$ is constant, so there is a $\lambda$ in $k$ such that $p_2(x) = \lambda$ for all $x$ in $k$. Now notice that on the left hand side of (5.3) there are only non-negative powers of $t$, and on the right hand side there are only non-positive powers of $t$. This equality is only satisfied if $p_1(x) = 0$ for all $x$ in $k$, so $p_1$ is the zero polynomial.

We apply $\sigma$ to (5.2) and using the 1-cocycle condition to obtain

$$\begin{aligned}\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\ &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right).\end{aligned}$$

That is,

$$\begin{aligned} p_2(u_1+u_2) &= p_2(u_1)+p_2(u_2) && (5.6)\\ p_3(u_1+u_2) &= p_3(u_1)+p_3(u_2)-2u_1^{p^r}p_2(u_2). && (5.7)\end{aligned}$$

Since $p_2$ is constant, (5.6) implies that $p_2$ is the zero polynomial, which means (5.7) becomes

$$p_3(u_1+u_2) = p_3(u_1)+p_3(u_2).$$

Hence $p_3$ is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \qquad (5.8)$$

for some $u_i$ in $k$.

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{2p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \qquad (5.9)$$

If $p_3$ is not the zero polynomial then there is a non-zero $\mu_l$ for some index $l$. By equating the coefficients of $u$ in (5.9) we get

$$\begin{aligned} \mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies \quad p^l &= p^r. \end{aligned}$$

Therefore $l = r$. This means that the only non-zero $\mu_i$ is already specified by the choice of $r$ in defining $\rho_r$.

Letting $\mu_l = \mu$ in $k$, we have

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}. \end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then $\sigma$ cannot be a 1-coboundary. But if the characteristic of $k$, $p$, is not equal to 2 then by setting $\mathbf{v}$ in $V$ as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all $a$ in $k^*$ and all $b$ in $k$

$$\begin{aligned}
\chi_v\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
&= \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right).
\end{aligned}$$

That is, $\sigma$ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of $SL_2$. By [some reference], this means that $\sigma$ is a 1-coboundary from the whole of $SL_2 \to V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \tag{5.10}$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some $\mu'$ in $k$.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

remembering that $k$ now has characteristic 2. On the one hand

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu+\mu'\\\mu\\\mu\end{pmatrix}\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}\mu+\mu'\\\mu'\\\mu'\end{pmatrix}=\begin{pmatrix}\mu+\mu'\\\mu'\\\mu+\mu'\end{pmatrix}.
\end{aligned}
$$

On the other hand, by applying $\sigma$ to both sides of the equality

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}t^{-1}&0\\0&t\end{pmatrix},$$

we get

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right).$$

Therefore $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is an element of $V$ that is fixed by the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$. Referring to the formula for the action of $SL_2$ on $V$ we see that such an element of $V$ is of the form

$$\begin{pmatrix}0\\ *\\0\end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

If $c = 0$ then we already have

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise $c^{-1}$ exists and we can compute

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r}\mu(cd)^{2^r} \\ (ac^{-1})^{2^{r+1}}\mu(cd)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1+ad)^{2^r} \\ \mu(ac^{-1})^{2^r}(1+ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

In fact, we see that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

*[Show converse - Steinberg relations]*

Now if $\sigma$ is in the same conjugacy class as $\tau$ then by [some reference]

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering $\mathbf{v}$ that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each $\mu$ in $k$ corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$ where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle $\tau$ is in the class $[\sigma_\mu]$ if there is a $\mathbf{v}$ in $V$ such that

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \to P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup $L_\alpha$. Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \,|\, x \in k \rangle$$

where $\gamma$ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle = 0. \tag{5.11}$$

Since $\gamma = m\alpha + n\beta$ for some integers $m, n$, we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \tag{5.12}$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \,|\, x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle $\sigma_\mu$ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\ &= \epsilon_\beta\left(s^{\langle \beta, \alpha+\beta \rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle \alpha+\beta, \alpha+\beta \rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle}\mu(ab)^{2^r}\right) \\ &= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \to V$ collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from $V$-conjugacy to $P_\alpha$-conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}\,. \end{aligned}$$

### 5.2.2 Example

Let $V$ be the unipotent radical of the parabolic subgroup of $B_2$ defined by the (long) root $\beta$:

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let $\rho_r$ be the homomorphism from $SL_2 \to L_\beta$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where $r$ is some non-negative integer.

Note that $V$ is not abelian in general. The Group Law for $V$ can be computed as follows. Let $\mathbf{v}, \mathbf{w}$ in $V$. We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\ &= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of $SL_2$ on $V$ via $\rho_r$. Let $\mathbf{v}$ be an element of $V$:

$$
\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}
$$

Or, more explicitly

$$
\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.
$$

As in the previous example we let $\sigma$ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$ such that

$$\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all $t$ in $k^*$, and

$$\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all $u$ in $k$.

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.13}$$

$$\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

Applying $\sigma$ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\cdot\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).$$

That is

$$p_1(t^2u) = t^{-p^r}p_1(u) \tag{5.15}$$

$$p_2(t^2u) = t^{p^r}p_2(u) \tag{5.16}$$

$$p_3(t^2u) = p_3(u). \tag{5.17}$$

From (5.17) we find that $p_3$ is constant-valued, say $p_3(x) = \lambda$ in $k$ for all $x$ in $k$. From (5.15) we see that there are only non-negative powers of $t$ on the left hand side and only non-positive powers the right hand side. Therefore $p_1$ is the zero polynomial.

Now applying $\sigma$ to both sides of (5.14):

$$\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)+p_2(u_2)\\ 2\lambda\end{pmatrix}
\end{aligned}$$

That is,

$$p_2(u_1+u_2) = p_2(u_1)+p_2(u_2) \tag{5.18}$$

$$\lambda = 2\lambda. \tag{5.19}$$

By (5.19) we see that $p_3$ is in fact the zero polynomial, and (5.18) implies that $p_2$ is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{5.20}$$

for some $\mu_i$ in $k$.

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{5.21}$$

If $p_2$ is not the zero polynomial then there is a non-zero $\mu_l$ for some index $l$. By equating coefficients of $u^{p^i}$ in (5.21) we get

$$\begin{aligned}
\mu_l t^{2p^l} &= \mu_l t^{p^r}\\
\Longrightarrow\quad 2p^l &= p^r.
\end{aligned}$$

Thus 2 divides $p^r$, and since $p$ is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero $\mu_l$ is already specified by the choice of $r$ in defining $\rho_r$, and that $r$ must be non-zero if $p_2$ is to be non-zero.

Referring to the Group Law we see that $V$ is abelian in characteristic 2, so we will use the '+' symbol for combining elements of $V$ from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some $\mu'$ in $k$.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\\mu\\0\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu^2\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu'+\mu\\\mu\\\mu^2\end{pmatrix}\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}\mu'+\mu\\\mu'\\\mu'^2\end{pmatrix}\\
&= \begin{pmatrix}\mu'+\mu\\\mu'+\mu\\\mu'^2\end{pmatrix}.
\end{aligned}$$

Since $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$ for all $t$ in $k^*$ we must have $\mu'=\mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r}\mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{p^r}\left(\mu(cd)^{2^{r-1}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ac^{-1} + a^2c^{-1}d\right)^{2^{r-1}} \\ \mu^2\left(1 + ad\right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.$$

*[Show converse is true]*

As in the previous example, we choose a $\mathbf{v}$ in $V$ that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix},
\end{aligned}$$

which tells us that for each $\mu$ in $k$ we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$, where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
(\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)\,(ab)^{2^{r-1}} \\ (s\mu)^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

our infinitely many $V$-conjugacy classes collapse to just two $P_\beta$-conjugacy classes:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\}, \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}.
\end{aligned}$$

## 5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow Im(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside $L_\gamma$ or $L_\delta$.

1) Know about non G-cr in $B_2$, can I put them in an $A_1A_1$?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $Im(\rho_{r,s})$, can we conjugate it into $P_\gamma$ or $P_\delta$?

Let char$(k) = 2$ and set $V := \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma+\delta, \phi \neq \gamma + 2\delta\}\rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)$ $\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on $V$ is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\begin{aligned}
\rho_{r,s}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} &= \epsilon_\delta(u^{2^r})\cdot\epsilon_{\gamma+\delta}(u^{2^s})\\
\rho_{r,s}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} &= \delta^\vee(t^{2^r})\cdot(\gamma+\delta)^\vee(t^{2^s})\\
\rho_{r,s}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= n_\delta\cdot n_{\gamma+\delta}
\end{aligned}$$

from which we obtain an action of $SL_2$ on $V$:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}
v_1\\
c^{2^{s+1}}v_{10}+d^{2^{s+1}}v_2\\
c^{2^{s+1}}v_{11}+d^{2^{s+1}}v_3\\
c^{2^{r+1}}v_8+d^{2^{r+1}}v_4\\
c^{2^{r+1}}v_9+d^{2^{r+1}}v_5\\
v_6+(bd)^{2^r}v_4+(bd)^{2^s}v_2+(ac)^{2^r}v_8+(ac)^{2^s}v_{10}\\
v_7+(bd)^{2^r}v_5+(bd)^{2^s}v_3+(ac)^{2^r}v_9+(ac)^{2^s}v_{11}\\
a^{2^{r+1}}v_8+b^{2^{r+1}}v_4\\
a^{2^{r+1}}v_9+b^{2^{r+1}}v_5\\
a^{2^{s+1}}v_{10}+b^{2^{s+1}}v_2\\
a^{2^{s+1}}v_{11}+b^{2^{s+1}}v_3\\
v_{12}+(bd)^{2^{r+1}}v_4v_5+(bd)^{2^{s+1}}v_2v_3+(bc)^{2^{r+1}}(v_4v_9+v_5v_8)\\
+(bc)^{2^{s+1}}(v_2v_{11}+v_3v_{10})+(ac)^{2^{r+1}}(v_8v_9)+(ac)^{2^{s+1}}(v_{10}v_{11})
\end{pmatrix}$$

Now let $\sigma$ be a 1-cocycle from $SL_2$ to $V$ such that for all $t$ in $k^*$

$$\sigma\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Since $\sigma$ is a morphism of varieties, each component of $\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ should be a polynomial function of $u$, so we let

$$\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}p_1(u)\\ \vdots\\ p_{12}(u)\end{pmatrix},$$

where each $p_i$ $(1 \leq i \leq 12)$ is as required. Applying $\sigma$ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1,6,7,12 \\ t^{-2^{r+1}} p_i(u), & i = 4,5 \\ t^{-2^{s+1}} p_i(u), & i = 2,3 \\ t^{2^{r+1}} p_i(u), & i = 8,9 \\ t^{2^{s+1}} p_i(u), & i = 10,11 \end{cases} \tag{5.22}$$

It is clear that for $i = 1,6,7,12$ the polynomials $p_i$ must be constant-valued, say $\lambda_i$ for some fixed $\lambda_i$ in $k$ (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of $t$, $p_i$ must be the zero polynomial for $i = 2,3,4,5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying $\sigma$ to both sides yields

$$\begin{aligned} p_1(u_1+u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1+u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1+u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1+u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1+u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1+u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1+u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1+u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials $p_1, p_6, p_7, p_{12}$ must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some $w_j, x_j, y_j, z_j$ in $k$ and all $u$ in $k$

$$\begin{aligned} p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j}, \end{aligned}$$

If $\sigma$ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that $p_8$ is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\ \Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow \quad l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$p_8(u) = wu^{2^r}, \quad p_9(u) = xu^{2^r},$$
$$p_{10}(u) = yu^{2^s}, \quad p_{11}(u) = zu^{2^s},$$

for some $w, x, y, z$ in $k$.

So, we have

$$\begin{aligned}\sigma\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix} &= \sigma\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} * \begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\cdot\sigma\begin{pmatrix}1 & a^{-1}b\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\ w(ab)^{2^{r+1}}\\ x(ab)^{2^{r+1}}\\ y(ab)^{2^{s+1}}\\ z(ab)^{2^{s+1}}\\ 0\end{pmatrix}.\end{aligned}$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}$ is a polynomial function, say $p_i'(u)$ for all $u$ in $k$, to get

$$\sigma\begin{pmatrix}d^{-1} & 0\\ c & d\end{pmatrix} = \begin{pmatrix}0\\ y'(cd)^{2^s}\\ z'(cd)^{2^s}\\ w'(cd)^{2^r}\\ x'(cd)^{2^r}\\ 0\\0\\0\\0\\0\\0\\0\end{pmatrix},$$

for some $w', x', y', z'$ in $k$.

From this we deduce that

$$\begin{aligned}
\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1&0\\1&1\end{pmatrix} * \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}$$

Furthermore, since $\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$, we have

$$\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}n_1\\0\\0\\0\\0\\n_6\\n_7\\0\\0\\0\\0\\n_{12}\end{pmatrix},$$

for some $n_1, n_6, n_7, n_{12}$ in $k$. So in fact

$$\begin{aligned} w' &= w \\ x' &= x \\ y' &= y \\ z' &= z \\ n_1 &= 0 \\ n_6 &= w+y \\ n_7 &= x+z \\ n_{12} &= wx+yz. \end{aligned}$$

Consider $\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\begin{aligned} \sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$
\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}
$$

We see that in any case

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \to V$ of the form

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some $w, x, y, z$ in $k$ and integers $r, s \geq 0$.

*[Show $\sigma$ is a 1-cocycle]*

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle $\tau'$ is in the same conjugacy class as $\sigma$ if there is a $\mathbf{v}$ in $V$ such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g.\mathbf{v}^{-1}$$

for all $g$ in $SL_2$. Furthermore, $\tau'$ is conjugate to some 1-cocycle $\tau$, where $\tau$ has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus $\sigma$ is conjugate to $\tau$ by some $\mathbf{v}$ in $V$ that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned}
\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\
&= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix}
\end{aligned}$$

We can denote this relationship by

$$(w,x,y,z) \quad \sim \quad (w, x+\lambda w, y, z+\lambda y),$$

where the 4-tuple $(w,x,y,z)$ represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each $x, z$ in $k$ the family of classes of the form

$$[(0,x,0,z)] = \{(0,x,0,z)\}.$$

Now we consider $P$-conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta+\gamma+\delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle $\sigma$ by

$$(\mathbf{s}\cdot\sigma)\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ s^{-1}t^2 y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2 w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2 w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2 y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

# Chapter 6

# Complete Reducibility

- Discuss $B_4$ results relating to $G$-cr

# Chapter 7

# Conclusion

# Appendix A

# Further Calculations

- $G_2$ calculation?
- The rest of the $B_4$ calculations

# Appendix B

# Source Code

Put source code here ...

# Bibliography

[1] A. S. Arnold, J. S. Wilson, and M. G. Boshier. A simple extended-cavity diode laser. *Review of Scientific Instruments*, 69(3):1236–1239, March 1998. URL http://link.aip.org/link/?RSI/69/1236/1.

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