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A Geometric Approach to Complete Reducibility

by

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Declaration of Authorship

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“A quote.”

The author of the quote.

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Abstract

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The Thesis Abstract . . .

Acknowledgements

The acknowledgements and the people to thank ...

Contents

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	$rads^{-1}$
$\vdots$		

Dedication . . .

Chapter 1

Introduction

What is the thesis about and what are the main results

– *What the thesis is about:*

  This thesis is a contribution to the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz ([1], [2]). Let G be a simple algebraic group over an algebraically closed field of characteristic p. For large enough characteristic ($p = 0$ or $p > 7$ covers all restrictions) Liebeck and Seitz determine explicitly the embeddings of arbitrary closed connected semisimple subgroups in G where G is of exceptional type. Using similar methods, we examine the subgroup structure of simple algebraic groups in low characteristic (usually $p = 2$ or $p = 3$) where less is known.

  The main motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer [3]:

  > Let G be a linear algebraic group over an algebraically closed field of characteristic p. Let Γ be a finite group and $\Gamma_p \subset \Gamma$ a Sylow p-subgroup. Fix a conjugacy class of representations $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

  We use Lemma 4.2 to reduce Külshammer's question to a 1-cohomology calculation in which Γ acts on the unipotent radical V of a parabolic subgroup $P \subset G$ via an irreducible map $\Gamma \to L$, $L \subset P$ a Levi subgroup of P. We show that we can answer a particular instance of Külshammer's question by examining the restriction map of 1-cohomologies:

$$H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

This is useful for finding a counterexample to Külshammer's question. A counterexample is known for a non-reductive G [3] and we investigate the reductive case.

We broaden our search for a counterexample by defining an *algebraic group* version of Külshammer's question where instead of a finite group Γ we use a reductive group H, and instead of a Sylow p-subgroup $\Gamma_p \subset \Gamma$ we use a unipotent subgroup $U \subset H$. Finding an algebraic counterexample may yield a finite counterexample for a finite subgroup of H. We pay special attention to $H = SL_2$.

– *Main results*:

???

Context, history, literary review

– *K. II - motivation for this:*

Külshammer's question has it's roots Maschke's Theorem which asserts that any representation of a finite group over a field of characteristic not dividing the order of the group is completely reducible. Külshammer's first question ... Link

– *Work of Liebeck & Seitz, etc, on embedding reductive H inside simple G:*

Methods (can refer forward)

– *Use of 1-cohomology to (K. II):*

– *Key results e.g. $H^1(SL_2, V) \to H^1(B, V)$:*

Chapter Summary

– *Preliminaries:*

– *1-Cohomology:*

– *K. II:*

– *Calculations:*

– *Summary/Future work:*

Chapter 2

Mathematical Preliminaries

Chapter 3

The 1-Cohomology

3.1 Abelian 1-Cohomology

3.1.1 Definitions

Let H be a group and V an abelian group (vector space) on which H acts homomorphically (linearly). We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \tag{3.1}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call (3.1) the *1-cocycle condition.*

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned}
(\sigma_1 + \sigma_2)(h_1h_2) &= \sigma_1(h_1h_2) + \sigma_2(h_1h_2) \\
&= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\
&= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\
&= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2),
\end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \to V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H,V)$ we have

$$\begin{aligned}\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\,\sigma(1),\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1\,(H,V)$ is a $\mathbb{Z}-$module under pointwise addition.

Given a v in V we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1\,(H,V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}\chi_v^H(h_1h_2) &= v - (h_1h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u+v) - h \cdot (u+v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

We see that $B^1(H,V)$ is a subgroup of $Z^1(H,V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H,V)$ is a $\mathbb{Z}-$submodule of $Z^1(H,V)$, so we may form the quotient module

$$H^1(H,V) = Z^1(H,V)/B^1(H,V),$$

called the *1-cohomology*.

Lemma 3.1. *Suppose H is linearly reductive. Then $H^1(H,V)$ is trivial [4].*

3.1.2 Maps between 1-cohomologies

Let ϕ be a homomorphism from $\tilde{H} \to H$, $\tilde{H}$ being another group that acts on V. Suppose that for every h in H, ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V. If σ is a 1-cocycle from $H \to V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \to V$.

Take h_1, h_2 in H. We have

$$\begin{aligned} Z^1(\phi)(\sigma)(h_1h_2) &= (\sigma \circ \phi)(h_1h_2) \\ &= \sigma(\phi(h_1h_2)) \\ &= \sigma(\phi(h_1)\phi(h_2)) \\ &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2) \\ &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\ &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\ &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2). \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H,V)$ into $B^1(\tilde{H},V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H,V) \to H^1(\tilde{H},V),$$

defined by

$$\begin{array}{ccc} Z^1(H,V) & \xrightarrow{Z^1(\phi)} & Z^1(\tilde{H},V) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} \\ H^1(H,V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H},V) \end{array}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H,V)$ onto $H^1(H,V)$ and $Z^1(\tilde{H},V)$ onto $H^1(\tilde{H},V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 3.1. *Let $\tilde{H}$ be a subgroup of H and $i : \tilde{H} \to H$ the inclusion map. Then i gives rise to a well defined map*

$$H^1(i) : H^1(H,V) \to H^1(\tilde{H},V).$$

Lemma 3.2. *Let V be a vector space over a field of characteristic p. Let H be a finite group and $\tilde{H} = H_p$ a* Sylow p-subgroup *of H. The map*

$$H^1(i) : H^1(H,V) \to H^1(H_p,V)$$

is injective.

Proof. Let x be an element of $H^1(H,V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(H,V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to H_p is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since

$\chi_v^{H_p}$ can be trivially extended to a 1-coboundary χ_v^H from $H \to V$, and

$$\pi(\sigma - \chi_v^H) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^H)$ as a representative for x. Hence there is no harm in assuming that σ is 0 when restricted to H_p. Now choose a set of representatives $h_1, \ldots, h_l$ in H for the coset space H/H_p and set

$$v^* = \sum_{i=1}^{l} \sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by v^*

$$\begin{aligned}
\chi_{v^*}^H(h) &= v^* - h \cdot v^* \\
&= \sum_{i=1}^{l} \sigma(h_i) - h \cdot \sum_{i=1}^{l} \sigma(h_i) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} h \cdot \sigma(h_i).
\end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned}
\sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} h \cdot \sigma(h_i) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} (\sigma(hh_i) - \sigma(h)) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h).
\end{aligned}$$

Now as the value of σ at a fixed h depends only on the value of σ at the representative h_j of the coset containing h we can collapse the middle term to yield

$$\begin{aligned}
\chi_{v^*}^H(h) &= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(hh_i) + \sum_{i=1}^{l} \sigma(h) \\
&= \sum_{i=1}^{l} \sigma(h_i) - \sum_{i=1}^{l} \sigma(h_i) + \sum_{i=1}^{l} \sigma(h) \\
&= l\, \sigma(h).
\end{aligned}$$

Since $\gcd([H : H_p], p) = \gcd(l, p) = 1$, l is invertible and so

$$l^{-1} \chi_{v^*}^H(h) = \sigma(h).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. $\square$

Example 3.2. *Let*

$$k = \bar{\mathbb{F}_p} = \bigcup_r \mathbb{F}_{p^r},$$

V a vector space on which $SL_2(k)$ acts, and $U(k)$ the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Then $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ for each r, and the map

$$H^1(SL_2(k), V) \to H^1(U(k), V)$$

is injective.

Proof. The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r}-1)(p^{2r}-p^r)$ since there are $p^{2r}-1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r}-p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \to \mathbb{F}_{p^r}^*,$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First homomorphism theorem for groups

$$GL_2(\mathbb{F}_{p^r}) \,/\, SL_2(\mathbb{F}_{p^r}) \sim \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| \,/\, |\mathbb{F}_{p^r}^*| \\ &= (p^{2r}-1)(p^{2r}-p^r) \,/\, (p^r-1) \\ &= p^r(p^{2r}-1). \end{aligned}$$

Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$.

Fix a non-trivial $y \in H^1(SL_2(k), V)$ and choose a representative $\tau \in Z^1(SL_2(k), V)$ for y. For each $g \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_g^{(r)} : V \to V$ by

$$f_g^{(r)}(v) = \tau(g) - \chi_v(g) = \tau(g) - v + g \cdot v.$$

Consider the sequence of subsets of V defined by

$$C_r = \{v \in V | f_g^{(r)}(v) = 0\}.$$

Each subset C_r is closed and the inclusion $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^{r+1}}$ induces the reverse inclusion $C_r \supset C_{r+1}$. The Noetherian property for V requires that the sequence becomes constant.

However, $y \neq 0$ so τ is not a 1-coboundary on $SL_2(k)$, which means the C_r's are eventually empty. That is, there exists an integer s such that for any v in V

$$(\tau - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if $y|_{SL_2(\mathbb{F}_{p^r})} = 0$ for all r then $y = 0$.

Take x in the kernel of the map $H^1(SL_2(k), V) \to H^1(U(k), V)$. Then for each r, $x|_{U(\mathbb{F}_{p^r})} = 0$ so by Lemma 3.2 $x|_{SL_2(\mathbb{F}_{p^r})} = 0$. Therefore $x = 0$ and so $H^1(SL_2(k), V) \to H^1(U(k), V)$ is injective. □

We could also consider appropriate maps $f : V \to \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \to H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \to H^1(\tilde{H}, \tilde{V}).$$

3.2 Non-abelian 1-Cohomology

3.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocyles be morphisms of varieties.

3.2.2 Definitions

Let H, V be algebraic groups, H acting on V. We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{3.2}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call the (3.2) the *1-cocycle condition.*

Given a v in V we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by $B^1(H,V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}
\chi_v^H(h_1h_2) &= v * (h_1h_2) \cdot v^{-1} \\
&= v * h_1 \cdot \left(h_2 \cdot v^{-1}\right) \\
&= v * h_1 \cdot \left(vv^{-1}h_2 \cdot v\right) \\
&= v * h_1 \cdot v * h_1 \cdot \left(v * h_2 \cdot v^{-1}\right) \\
&= \chi_v^H(h_1) * h_1 \cdot \chi_v^H(h_2),
\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say σ_1, σ_2 in $Z^1(H,V)$ are *equivalent* if there exists a v in V such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \tag{3.3}$$

for all h in H. We call the set of equivalence classes of $Z^1(H,V)$ under the equivalence relation defined by (3.3) the *1-cohomology*, denoted $H^1(H,V)$.

3.2.3 Maps between 1-cohomologies

Lemma 3.3. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V. Then $H^1(i) : H^1(SL_2,V) \to H^1(B,V)$ is injective.*

Proof. Let x be in the kernel of $H^1(i)$ and σ and element of $Z^1(SL_2,V)$ that projects onto the class x. Since $Z^1(i)(\sigma)$ projects to the trivial 1-cohomology class we may as well assume that $\sigma|_B = 1$. For there exists some v in V such that for all b in B

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle $\hat{\sigma} : SL_2 \to V$ defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction $\hat{\sigma}$ also projects to the class x, and for all b in B

$$\begin{aligned}
\hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\
&= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\
&= v^{-1} * v * b \cdot (v^{-1} * v) \\
&= 1,
\end{aligned}$$

so we may as well have chosen $\hat{\sigma}$ instead as a representative for x.

Now consider the *homogeneous space* SL_2/B [5] and take the map

$$\tilde{\sigma} : SL_2/B \to V,$$

defined in the usual way under the canonical projection $\pi : SL_2 \to SL_2/B$:

$$\begin{tikzcd} SL_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & V \\ SL_2/B \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

This map is well defined and is a morphism [6]. Now since SL_2/B is an irreducible projective variety [5], $\tilde{\sigma}$ must be constant [6]. Hence, as σ takes the value 1 for any b in B, $\tilde{\sigma}(hB) = 1$ for all cosets hB. Therefore, for all h in SL_2

$$\sigma(h) = \tilde{\sigma}(hB) = 1.$$

We have shown that σ is the 1-coboundary χ_1 which means that the kernel of $H^1(i)$ is trivial. □

Lemma 3.4. *Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B. Then $H^1(B, V) \to H^1(U, V)$ is injective. Moreover*

$$H^1(SL_2, V) \to H^1(U, V)$$

is injective.

Proof. Let x be an element of the kernel of $H^1(i) : H^1(B, V) \to H^1(U, V)$ and let σ in $Z^1(B, V)$ be a representative for x. We may as well assume that $\sigma|_T = 1$. For any b in B we can find a u in U and a t in T such that $b = ut$. Hence

$$\begin{aligned} \sigma(b) &= \sigma(ut) \\ &= \sigma(u) * u \cdot \sigma(t) \\ &= \sigma(u). \end{aligned}$$

Since σ represents x, σ must be a 1-coboundary on U. Hence σ is in $B^1(B, V)$ and the kernel of $H^1(i) : H^1(B, V) \to H^1(U, V)$ is trivial. □

Chapter 4

Külshammer's Second Problem

4.1 Külshammer's Second Problem

Two questions raised by B. Külshammer concerning representations of a finite group Γ into a linear algebraic group G over an algebraically closed field k. The first has a positive answer and is essentially contained a paper by A. Weil [3]:

(K. I) Let $\mathrm{char}(k)$ be prime to the order of Γ. Are there only finitely many representations $\rho : \Gamma \to G$ up to conjugation by G?

(K. II) Let $p = \mathrm{char}(k)$ and $\Gamma_p \subset \Gamma$ be a Sylow p-subgroup. Fix a conjugacy class of representations from $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

(K. II) has positive answer so long as G is reductive and the characteristic of k is good for G [7]. The same paper shows that the answer is "no" in general by way of a counterexample involving a non-reductive G.

We will explore the possibility of a reductive counterexample to (K. II).

4.2 The Approach

We are interested in knowing whether there can be infinitely many G-conjugacy classes of representations $\Gamma \to G$ that when restricted to Γ_p hit some fixed G-conjugacy class of representations $\Gamma_p \to G$.

Theorem 4.1. *There are only finitely many G-conjugacy classes of G-completely reducible representations $\Gamma \to G$.*

Reference something. □

Although G has infinitely many parabolic subgroups there are only finitely many G-conjugacy classes of parabolic subgroups, so we can choose a finite set $\{Q_i\}$ of representatives. We choose a set of Levi subgroups $\{M_i\}$, M_i being a Levi subgroup of Q_i. By Theorem 4.1 there are only finitely many M_i-conjugacy classes of M_i-completely reducible representations $\sigma_0^{(i)} : \Gamma \to M_i$, so we fix a set of representatives $\{\sigma_{0,j}^{(i)}\}$.

We will show that for each representation $\rho : \Gamma \to G$ there exists a representation σ that is G-conjugate to ρ and that fits one of only finitely many commutative diagrams

$$\begin{tikzcd} \Gamma \arrow[r, "\sigma"] \arrow[dr, "\sigma_{0,j}^{(i)}"'] & Q_i \arrow[d] \\ & M_i \end{tikzcd}$$

Let ρ be a representation from $\Gamma \to G$ and let P be a minimal parabolic subgroup of G containing $\rho(\Gamma)$. Then there is a g in G such that $P = g \cdot Q$, where $Q \in \{Q_i\}$. Let $\rho' = g \cdot \rho$.

Define $\rho_0 : \Gamma \to M$ by composing ρ' with the projection $Q \to M$, $M \in \{M_i\}$ the chosen Levi subgroup for Q:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho'"] \arrow[dr, "\rho_0"'] & Q \arrow[d] \\ & M \end{tikzcd}$$

Since Q is a minimal parabolic containing $\rho'(\Gamma)$, ρ_0 is M-irreducible [reference] and therefore M-completely reducible. Hence we can choose an $m \in M$ such that $\sigma_0 = m \cdot \rho_0$ where $\sigma_0 \in \{\sigma_{0,j}^{(i)}\}$. Let $\sigma = m \cdot \rho' = mg \cdot \rho$. This verifies what we set out to show.

[Say something about $\rho \rightsquigarrow (i,j) \rightsquigarrow H^1(\Gamma, V_i)$ so that I can state the next lemma]

For a given parabolic $P < G$, Levi $L < P$ and representation $\rho : \Gamma \to P$ we have defined the map $\rho_0 : \Gamma \to L$ by the projection $P \to L$. Now define $\alpha_\rho : \Gamma \to R_u(P)$ by projecting on to $R_u(P)$, so that $\rho = \alpha_\rho \rho_0$. If ρ_0 is a homomorphism then

$$\begin{aligned} \alpha_\rho(\gamma_1\gamma_2)\rho_0(\gamma_1\gamma_2) = \rho(\gamma_1\gamma_2) &= \rho(\gamma_1)\rho(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_1)^{-1}\rho_0(\gamma_1)\rho_0(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_0(\gamma_1)\alpha_\rho(\gamma_2)\rho_0(\gamma_1)^{-1}\rho_0(\gamma_1\gamma_2), \end{aligned}$$

so that

$$\alpha_\rho(\gamma_1\gamma_2) = \alpha_\rho(\gamma_1)\rho_0(\gamma_1) \cdot \alpha_\rho(\gamma_2),$$

which is the 1-cocycle condition in (3.1).

Hence for the given ρ we have a corresponding 1-cocycle $\alpha_\rho : \Gamma \to R_u(P)$ where Γ acts on $R_u(P)$ via ρ_0.

Suppose we conjugate ρ by an element $g \in G$. Then $\rho' = g \cdot \rho$ has corresponding 1-cocycle $\alpha_{\rho'} : \Gamma \to gR_u(P)g^{-1}$.

So the G-action on $\rho : \Gamma \to P$ almost corresponds to maps of 1-cocycles of the form:

$$Z^1(\Gamma, R_u(P)) \to Z^1(\Gamma, gR_u(P)g^{-1}),$$

the catch being that conjugating ρ by an arbitrary element of g changes the action of Γ on $R_u(P)$. If we choose $g \in Z(L)^\circ$ and consider the $Z(L)^\circ$-action:

$$\begin{tikzcd} \Gamma \arrow[r, "\rho"] \arrow[dr, "\rho_0"'] & P \arrow[d] \\ & L \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} \Gamma \arrow[r, "g\cdot\rho"] \arrow[dr, "\rho_0"'] & gPg^{-1} \arrow[d] \\ & L \end{tikzcd}$$

then we really do get a map of 1-cocycles.

Lemma 4.2. *Let $\{\rho_\mu \,|\, \mu \in I\}$ be a collection of representations $\Gamma \to P$ for a fixed parabolic subgroup $P < G$. The following statements are equivalent:*

(i) There are only finitely many P-conjugacy classes of $\{\rho_\mu\}$.

(ii) There are only finitely many G-conjugacy classes of $\{\rho_\mu\}$.

(iii) For each i, the number of elements of $H^1(\Gamma, V_i)$ modulo $Z(M_i^\circ)$ that come from the ρ_μ's is finite.

Proof. (i) $\Rightarrow$ (ii) is obvious. □

Let K be a collection of representations $\Gamma \to G$ whose restrictions to Γ_p belong to some fixed class. Let $\sigma \in K$, then $\sigma|_{\Gamma_p}$ and $\sigma_{0,j}^{(i)}|_{\Gamma_p}$ are conjugate. We have a 1-cocycle α corresponding to σ, while $\sigma_{0,j}^{(i)}$ corresponds to the trivial 1-cocycle. Hence $\alpha|_{\Gamma_p}$ is conjugate to the trivial 1-cocycle, that is, $\alpha|_{\Gamma_p}$ is a 1-coboundary.

[Therefore the problem is controlled by the 1-cohomology]

4.3 An algebraic group version

In an attempt to gain further insight into (K. II) we adjust the original question by letting Γ be an infinite group H. The advantage being that a negative answer in the algebraic group version may provide a negative answer to (K. II) by choosing an appropriate finite subgroup Γ of H. In many of the examples to follow we set $H = SL_2(K)$ with Sylow p-subgroup $H_p = U_2(K)$ consisting of upper unitriangular matrices.

Let $P \subset G$ be a parabolic subgroup and $L \subset P$ the corresponding Levi subgroup. Fix a representation $\rho_0 : H \to L$. We can assume $\rho_0(H)$ is L-irreducible, that is, not contained in a proper parabolic of L.

Now define $\rho_\alpha : H \to P$ by $\rho_\alpha(h) = \alpha(h)\rho_0(h)$ where $\alpha : H \to R_u(P)$, $R_u(P)$ the unipotent radical of P.

For ρ_α to be a homomorphism

$$\begin{aligned}
\alpha(h_1h_2)\rho_0(h_1h_2) &= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_2) \\
&= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_1)^{-1}\rho_0(h_1)\rho_0(h_2) \\
&= \alpha(h_1)\rho_0(h_1)\alpha(h_2)\rho_0(h_1)^{-1}\rho_0(h_1h_2).
\end{aligned}$$

That is $\alpha(h_1h_2) = \alpha(h_1)h_1 \cdot \alpha(h_2)$, where the action $H \times R_u(P) \to R_u(P)$ is conjugation via ρ_0. This is a 1-cocycle condtion; $\alpha \in Z^1(H, R_u(P))$. $R_u(P)$ will not be abelian in general.

Now suppose ρ_α is $R_u(P)$-conjugate to some ρ_β, $\alpha, \beta \in Z^1(H, R_u(P))$. That is, there exists a $v \in R_u(P)$ such that for all $h \in H$

$$\begin{aligned}
\alpha(h)\rho_0(h) &= v\beta(h)\rho_0(h)v^{-1} \\
&= v\beta(h)\rho_0(h)v^{-1}\rho_0(h)^{-1}\rho_0(h).
\end{aligned}$$

That is $\alpha(h) = v\beta(h)h \cdot v^{-1}$. In particular if ρ_α is $R_u(P)$-conjugate to ρ_0, that is β is trivial, then α takes the form of a 1-coboundary. Generally speaking α and β project to the same 1-cohomology class. In the abelian case this reads "α and β differ by a 1-coboundary":

$$\alpha(h) = v\beta(h)h \cdot v^{-1} \quad \rightsquigarrow \quad \begin{aligned}
\alpha(h) &= v + \beta(h) - h \cdot v \\
&= \beta(h) + v - h \cdot v \\
&= \beta(h) + \chi_v(h).
\end{aligned}$$

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G. The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p. Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G.

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α:

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r \quad : \quad & SL_2(k) \to SL_2(k) \\ & (A_{ij}) \mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r \quad = \quad \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) &= \prod_\delta \epsilon_\delta\left((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_\delta\right) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) &= \prod_\delta n_\alpha \epsilon_\delta\left(\lambda_\delta\right) n_\alpha^{-1}, \end{aligned}$$

where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ and λ_δ are elements of the underlying field k.

Lemma 5.1.

$$\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta\left(u\right)\right),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_\delta \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ. Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \to k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned} \sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

Therefore

$$x_\delta\left(t^2u\right) = (t^{p^r})^{\langle\delta,\alpha\rangle} x_\delta(u).$$

Since x_δ is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle\delta,\alpha\rangle \geq 0$. However, if $\langle\delta,\alpha\rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle\delta,\alpha\rangle > 0$. □

Next we prove a couple of useful facts about root systems not containing G_2 or C_3.

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha,\beta \in \Phi$. If $\alpha+\beta \in \Phi$ then $\langle\alpha,\beta\rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β. Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. □

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3. Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ $(i = 1, 2)$. If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ, $(i = 1, 2)$ and let θ_3 be the absolute value of the angle between δ_1 and δ_2. Then

$$\begin{aligned} & \langle \delta_i, \gamma \rangle > 0 \qquad (i = 1, 2) \\ \implies & (\delta_i, \gamma) > 0 \\ \implies & \cos(\theta_i) > 0 \\ \implies & \theta_i < \pi/2, \end{aligned}$$

and similarly, using Lemma 5.2

$$\begin{aligned} & \langle \delta_1, \delta_2 \rangle \leq 0 \\ \implies & \theta_3 \geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

$$\begin{array}{llll} \mathbf{1:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = 2\pi/3; \\ \mathbf{2:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \mathbf{3:} & \theta_1 = \pi/4, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \mathbf{4:} & \theta_1 = \pi/4, & \theta_2 = \pi/4, & \theta_3 = \pi/2. \end{array}$$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ, and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3. [Picture?][Wow, is that right? Maybe just say 'we will show that they lie in a C_3 subsystem'.] □

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3.

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \epsilon_\alpha(u_1^{p^r}) \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By Lemma 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α. □

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α.*

Proof. First consider

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta(b)\right).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By Lemma 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$ $(i = 1, 2)$, so Lemma 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix}\right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \\ &= \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b)\right). \end{aligned}$$

□

Since the x_δ are homomorphisms from $k \to k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k. Furthermore, combining the calculation in the proof of Lemma 5.1 with the result in Corollary 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta}\left(x_{\delta}\left(a^2 b\right)\right) = \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b)\right),$$

severely restricting the possible polynomials x_δ. In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer r in the definition of ρ_r. For suppose x_δ and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned} \mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\ \Longrightarrow 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

In [8] it is shown that the possible pairings of any two roots are bounded by ± 3. Hence by Lemma 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3. It is now clear that if $\langle \delta, \alpha \rangle = 3$ then $x_\delta = 0$.

If $\langle \delta, \alpha \rangle = 1$ the characteristic of k must be 2 and $j = r - 1$. Otherwise $\langle \delta, \alpha \rangle = 2$ and $j = r$, but the characteristic of k is so far unrestricted.

Example 5.1. *Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

We will write v in V_α in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle \quad := \quad \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{3\alpha+\beta}(v_4)\epsilon_{3\alpha+2\beta}(v_5) \in V_\alpha$$

The group law for V_α is

$$u * v \quad = \quad \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v \quad = \quad \langle a^{-3p^r} v_1, a^{-p^r} v_2, a^{p^r} v_3, a^{3p^r} v_4, v_5 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), x_4(b), 0 \rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2 b) &= a^{p^r} x_3(b) \\ x_4(a^2 b) &= a^{3p^r} x_4(b). \end{aligned}$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1 + b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1 + b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r} x_3(b_2). \end{aligned}$$

We see that x_3 is a homomorphism, so it is of the form

$$x_3(b) \quad = \quad \sum_i \mu_i b^{p^i}.$$

Suppose $x_3 \neq 0$*. Then some* $\mu_j \neq 0$ *and*

$$\begin{aligned} & \mu_j(a^2b)^{p^j} = a^{p^r}\mu_j b^{p^j} \\ \Longrightarrow\ & a^{2p^j} = a^{p^r} \\ \Longrightarrow\ & p = 2. \end{aligned}$$

But then

$$\begin{aligned} x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r}x_3(b) \\ &= b^{2^r}x_3(b), \end{aligned}$$

implies that x_3 *is constant, hence zero.*

Therefore $x_3 = 0$*, so* x_4 *is a homomorphism:*

$$x_4(b) = \sum_i \nu_i b^{p^r}.$$

If $x_4 \neq 0$ *then there is a* $\nu_j \neq 0$ *and we get*

$$\begin{aligned} & \nu_j(a^2b)^{p^j} = a^{3p^r}\nu_j b^{p^j} \\ \Longrightarrow\ & a^{2p^j} = a^{3p^r} \\ \Longrightarrow\ & 2p^j = 3p^r, \end{aligned}$$

which implies that 2 divides p and 3 divides p, a contradiction. Hence $x_4 = 0$ *and*

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Example 5.2. *Let* $G = C_3$*. Fix a maximal torus, labeling the positive simple roots* $\Delta = \{\alpha, \beta, \gamma\}$ *with* γ *being the long root and connected to* β*. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle.$$

Again we will write v in V_α *in angled brackets for ease of notation:*

$$\begin{aligned} &\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\rangle := \\ &\epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8) \in V_\alpha \end{aligned}$$

The group law for V_α *is*

$$
\begin{aligned}
&u * v = \\
&\langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3v_2, u_6 + v_6 + u_2v_1^2 + 2u_4v_1, \\
&\quad u_7 + v_7 + u_2u_3v_1 + u_2v_1v_3 + u_5v_1 + u_4v_3, u_8 + v_8 - u_3^2v_2 - 2u_3v_2v_3 + 2u_5v_3\rangle.
\end{aligned}
$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-p^r}v_1, v_2, a^{p^r}v_3, a^{-p^r}v_4, a^{p^r}v_5, a^{-2p^r}v_6, v_7, a^{2p^r}v_8\rangle.$$

Let σ *be in* $Z^1(SL_2, V_\alpha)$ *such that*

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b)\rangle.$$

Applying σ *to both sides of the identity*

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$
\begin{aligned}
x_3(a^2b) &= a^{p^r}x_3(b) \\
x_5(a^2b) &= a^{p^r}x_5(b) \\
x_8(a^2b) &= a^{2p^r}x_8(b).
\end{aligned}
$$

Since the polynomials x_3, x_5, x_8 *are homomorphisms (Lemma 5.2) we get*

$$
\begin{aligned}
\sum_i \lambda_i(a^2b)^{p^i} &= a^{p^r}\sum_i \lambda_i b^{p^i} \\
\sum_i \mu_i(a^2b)^{p^i} &= a^{p^r}\sum_i \mu_i b^{p^i} \\
\sum_i \nu_i(a^2b)^{p^i} &= a^{2p^r}\sum_i \nu_i b^{p^i},
\end{aligned}
$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of SL_2 under σ is $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle$ then the characteristic of k must be 2, and so the image is a product of commuting root groups.

We may now state and prove the main result.

[would like]

Theorem 5.6. *Let G be a reductive linear algebraic group over a closed field of positive characteristic p and let $\Gamma = SL_2(k)$. Then the answer to the algebraic interpretation of Külshammer's Second Problem [ref] is "yes".*

Proof. Need to:

- handle arguments above with G possibly containing G_2 and C_3.
- drop the restriction of rank-1 parabolics
- now we have abelian 1-cohomology and can apply result from previous chapter

□

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p. We label the positive roots for B_2 as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [5, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α:

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\alpha$ defined by

$$\begin{aligned}\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,\end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r: write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(-u^{p^r})\epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_\alpha(-u^{p^r})\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(-2u^{p^r}v_2)\epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\alpha(-u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{2p^r}v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r}v_2) \\
&= \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2 - u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r}v_2 + u^{2p^r}v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r}v_1 \\ v_3 - 2u^{p^r}v_2 + u^{2p^r}v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r}))v_3\right) \\
&= \epsilon_\beta\left((t^{p^r})^{\langle\beta,\alpha\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\alpha\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\alpha\rangle}v_3\right) \\
&= \begin{pmatrix} t^{-2p^r}v_1 \\ v_2 \\ t^{2p^r}v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1} \\
&= n_\alpha\epsilon_\beta(v_1)n_\alpha^{-1}n_\alpha\epsilon_{\alpha+\beta}(v_2)n_\alpha^{-1}n_\alpha\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1)\epsilon_{\alpha+\beta}(-v_2)\epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3)\epsilon_{\alpha+\beta}(-v_2)\epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r}v_1 - 2(cd)^{p^r}v_2 + c^{2p^r}v_3 \\ (ad+bc)^{p^r}v_2 - (bd)^{p^r}v_1 - (ac)^{p^r}v_3 \\ b^{2p^r}v_1 - 2(ab)^{p^r}v_2 + a^{2p^r}v_3 \end{pmatrix}$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*. Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

to get further information on the polynomials p_i $(i = 1, 2, 3)$.

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned}
\sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).
\end{aligned}$$

That is,

$$
\begin{aligned}
p_1(t^2u) &= t^{-2p^r}p_1(u) && (5.3)\\
p_2(t^2u) &= p_2(u) && (5.4)\\
p_3(t^2u) &= t^{2p^r}p_3(u). && (5.5)
\end{aligned}
$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k. Now notice that on the left hand side of (5.3) there are only non-negative powers of t, and on the right hand side there are only non-positive powers of t. This equality is only satisfied if $p_1(x) = 0$ for all x in k, so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
p_2(u_1+u_2) &= p_2(u_1)+p_2(u_2) && (5.6)\\
p_3(u_1+u_2) &= p_3(u_1)+p_3(u_2) - 2u_1^{p^r}p_2(u_2). && (5.7)
\end{aligned}
$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$p_3(u_1+u_2) = p_3(u_1)+p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \qquad (5.8)$$

for some u_i in k.

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{2p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \qquad (5.9)$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating the coefficients of u in (5.9) we get

$$\begin{aligned} \mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies \quad p^l &= p^r. \end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r.

Letting $\mu_l = \mu$ in k, we have

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}. \end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k, p, is not equal to 2 then by setting $\mathbf{v}$ in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
\chi_v\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
&= \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right).
\end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2. By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \to V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \tag{5.10}$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu+\mu'\\\mu\\\mu\end{pmatrix}\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix}+\begin{pmatrix}\mu+\mu'\\\mu'\\\mu'\end{pmatrix}=\begin{pmatrix}\mu+\mu'\\\mu'\\\mu+\mu'\end{pmatrix}.
\end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}t^{-1}&0\\0&t\end{pmatrix},$$

we get

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right).$$

Therefore $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is an element of V that is fixed by the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix}0\\ *\\0\end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

If $c = 0$ then we already have

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r}\mu(cd)^{2^r} \\ (ac^{-1})^{2^{r+1}}\mu(cd)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1+ad)^{2^r} \\ \mu(ac^{-1})^{2^r}(1+ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

In fact, we see that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering $\mathbf{v}$ that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$ where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a $\mathbf{v}$ in V such that

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \to P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α. Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \,|\, x \in k \rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle = 0. \tag{5.11}$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n, we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \tag{5.12}$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \,|\, x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\ &= \epsilon_\beta\left(s^{\langle \beta, \alpha+\beta \rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle \alpha+\beta, \alpha+\beta \rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle}\mu(ab)^{2^r}\right) \\ &= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \to V$ collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V-conjugacy to P_α-conjugacy:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\} \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}\,.
\end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β:

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\beta$ defined by

$$\begin{aligned}
\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\
\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\
\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta,
\end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows. Let $\mathbf{v}, \mathbf{w}$ in V. We have, using notation similar to the previous example

$$\begin{aligned}
\mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\
&= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\
&= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1 \end{pmatrix}.
\end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r. Let $\mathbf{v}$ be an element of V:

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*, and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k.

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.13}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \tag{5.15}$$

$$p_2(t^2u) = t^{p^r} p_2(u) \tag{5.16}$$

$$p_3(t^2u) = p_3(u). \tag{5.17}$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k. From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)+p_2(u_2)\\ 2\lambda\end{pmatrix}
\end{aligned}$$

That is,

$$p_2(u_1+u_2) = p_2(u_1)+p_2(u_2) \tag{5.18}$$

$$\lambda = 2\lambda. \tag{5.19}$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{5.20}$$

for some μ_i in k.

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{5.21}$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
\mu_l t^{2p^l} &= \mu_l t^{p^r}\\
\Longrightarrow\quad 2p^l &= p^r.
\end{aligned}$$

Thus 2 divides p^r, and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r, and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

We have

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right)+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\\mu\\0\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix}+\begin{pmatrix}\mu\\\mu\\\mu^2\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu'+\mu\\\mu\\\mu^2\end{pmatrix}\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix}+\begin{pmatrix}\mu'+\mu\\\mu'\\\mu'^2\end{pmatrix}\\
&= \begin{pmatrix}\mu'+\mu\\\mu'+\mu\\\mu'^2\end{pmatrix}.
\end{aligned}
$$

Since $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$ for all t in k^* we must have $\mu'=\mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r}\mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{p^r}\left(\mu(cd)^{2^{r-1}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ac^{-1} + a^2c^{-1}d\right)^{2^{r-1}} \\ \mu^2\left(1 + ad\right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a $\mathbf{v}$ in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix},
\end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$, where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
(\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)\,(ab)^{2^{r-1}} \\ (s\mu)^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

our infinitely many V-conjugacy classes collapse to just two P_β-conjugacy classes:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\}, \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}.
\end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow Im(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ.

1) Know about non G-cr in B_2, can I put them in an A_1A_1?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $Im(\rho_{r,s})$, can we conjugate it into P_γ or P_δ?

Let char$(k) = 2$ and set $V := \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma+\delta, \phi \neq \gamma+2\delta\}\rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)$ $\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\begin{aligned}
\rho_{r,s}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} &= \epsilon_\delta(u^{2^r})\cdot\epsilon_{\gamma+\delta}(u^{2^s})\\
\rho_{r,s}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} &= \delta^\vee(t^{2^r})\cdot(\gamma+\delta)^\vee(t^{2^s})\\
\rho_{r,s}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= n_\delta\cdot n_{\gamma+\delta}
\end{aligned}$$

from which we obtain an action of SL_2 on V:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}
v_1\\
c^{2^{s+1}}v_{10}+d^{2^{s+1}}v_2\\
c^{2^{s+1}}v_{11}+d^{2^{s+1}}v_3\\
c^{2^{r+1}}v_8+d^{2^{r+1}}v_4\\
c^{2^{r+1}}v_9+d^{2^{r+1}}v_5\\
v_6+(bd)^{2^r}v_4+(bd)^{2^s}v_2+(ac)^{2^r}v_8+(ac)^{2^s}v_{10}\\
v_7+(bd)^{2^r}v_5+(bd)^{2^s}v_3+(ac)^{2^r}v_9+(ac)^{2^s}v_{11}\\
a^{2^{r+1}}v_8+b^{2^{r+1}}v_4\\
a^{2^{r+1}}v_9+b^{2^{r+1}}v_5\\
a^{2^{s+1}}v_{10}+b^{2^{s+1}}v_2\\
a^{2^{s+1}}v_{11}+b^{2^{s+1}}v_3\\
v_{12}+(bd)^{2^{r+1}}v_4v_5+(bd)^{2^{s+1}}v_2v_3+(bc)^{2^{r+1}}(v_4v_9+v_5v_8)\\
+(bc)^{2^{s+1}}(v_2v_{11}+v_3v_{10})+(ac)^{2^{r+1}}(v_8v_9)+(ac)^{2^{s+1}}(v_{10}v_{11})
\end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ should be a polynomial function of u, so we let

$$\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}p_1(u)\\ \vdots\\ p_{12}(u)\end{pmatrix},$$

where each p_i $(1 \leq i \leq 12)$ is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1,6,7,12 \\ t^{-2^{r+1}} p_i(u), & i = 4,5 \\ t^{-2^{s+1}} p_i(u), & i = 2,3 \\ t^{2^{r+1}} p_i(u), & i = 8,9 \\ t^{2^{s+1}} p_i(u), & i = 10,11 \end{cases} \tag{5.22}$$

It is clear that for $i = 1,6,7,12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t, p_i must be the zero polynomial for $i = 2,3,4,5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned}
p_1(u_1+u_2) &= p_1(u_1) + p_1(u_2) \\
p_6(u_1+u_2) &= p_6(u_1) + p_6(u_2) \\
p_7(u_1+u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\
p_8(u_1+u_2) &= p_8(u_1) + p_8(u_2) \\
p_9(u_1+u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\
p_{10}(u_1+u_2) &= p_{10}(u_1) + p_{10}(u_2) \\
p_{11}(u_1+u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\
p_{12}(u_1+u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2).
\end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\ \Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow \quad l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) = w u^{2^r}, \quad p_9(u) = x u^{2^r}, \\ p_{10}(u) = y u^{2^s}, \quad p_{11}(u) = z u^{2^s}, \end{aligned}$$

for some w, x, y, z in k.

So, we have

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p'_i(u)$ for all u in k, to get

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k.

From this we deduce that

$$
\begin{aligned}
\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1&0\\1&1\end{pmatrix} * \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}
$$

Furthermore, since $\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$, we have

$$
\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}n_1\\0\\0\\0\\0\\n_6\\n_7\\0\\0\\0\\0\\n_{12}\end{pmatrix},
$$

for some n_1, n_6, n_7, n_{12} in k. So in fact

$$\begin{aligned} w' &= w \\ x' &= x \\ y' &= y \\ z' &= z \\ n_1 &= 0 \\ n_6 &= w+y \\ n_7 &= x+z \\ n_{12} &= wx+yz. \end{aligned}$$

Consider $\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\begin{aligned} \sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \to V$ of the form

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a $\mathbf{v}$ in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g.\mathbf{v}^{-1}$$

for all g in SL_2. Furthermore, τ' is conjugate to some 1-cocycle τ, where τ has the added property that

$$\tau\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some $\mathbf{v}$ in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned}
\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\
&= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + y v_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + w v_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + w v_1)(bc)^{2^r} + (z + y v_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + w v_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + y v_1)(ab)^{2^r} \\ w(x + w v_1)(bc)^{2^{r+1}} + y(z + y v_1)(bc)^{2^{s+1}} \end{pmatrix}
\end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \quad \sim \quad (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P-conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ s^{-1}t^2 y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2 w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2 w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2 y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

5.4 A Non-Reductive Counterexample

In [7] a counterexample to [ref KII] is presented for a closed field k of characteristic $p = 2$ and a non-reductive algebraic group G.

Example 5.3. *Let Q be the algebraic group isomorphic to the affine space $\mathbf{A}^3$ with the group multiplication law:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1 v_1 + u_2 v_2 + u_1 v_2 \end{pmatrix}.$$

Let $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$ and $\Gamma_2 = \langle \tau \rangle$ the Sylow 2-subgroup of Γ. Γ acts on Q via

$$\tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix}$$
$$\sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}.$$

Let $G = Q \rtimes \Gamma$. Then there are infinitely many pairwise G-conjugate classes of extensions to the representation $\rho : \Gamma_2 \to G$ defined by the natural inclusion $\Gamma_2 \to \Gamma \to G$ [7, Appendix].

Proof. Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle $\alpha \in Z^1(\Gamma, Q)$ such that $\alpha|_{\langle\sigma\rangle} = 1$. Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned}
1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1 u_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}.
\end{aligned}$$

This shows $u_1 = u_2$, so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\sigma\tau = \sigma^2$ we obtain

$$\begin{aligned}
1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
&= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
\end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle\sigma\rangle$ satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \qquad (u \in k).$$

Now we calculate the class $[\alpha_u] \in H^1(\Gamma, Q)$. Suppose $\alpha_v \sim \alpha_u$. Then there is a $q \in Q$ fixed under the action of σ, that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u+\lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
\end{aligned}$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(\Gamma, Q)$ is infinite. [to finish]. □

[note how to make my example look like the Slodowy one]

It is natural to ask whether this leads to a reductive counterexample, although we can quickly verify that the answer is "not immediately". For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$\begin{aligned}
&\dots \epsilon_\alpha(u_\alpha) \dots \epsilon_\beta(u_\beta) \dots \epsilon_\gamma(u_\gamma) \times \dots \epsilon_\alpha(v_\alpha) \dots \epsilon_\beta(v_\beta) \dots \epsilon_\gamma(v_\gamma) \\
&= \dots \epsilon_\alpha(u_\alpha + v_\alpha) \dots \epsilon_\beta(u_\beta + v_\beta) \dots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
\end{aligned}$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \alpha$ gives

$$\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) \quad = \quad \epsilon_\alpha(u_\alpha + v_\alpha)\epsilon_\gamma(u_\alpha v_\alpha),$$

which is absurd. [try find more examples]

Chapter 6

Conclusion

Appendix A

Further Calculations

G	P	Z^1	H^1	V-conj	P-conj
B_2 (α short)	P_α	✓	✓	✓	✓
	P_β	✓	✓	✓	✓
G_2 (α short)	P_α	✓			
C_3 (γ long)	P_α	✓			
[7]	$Q \rtimes SL(2,2)$	✓	✓	✓	

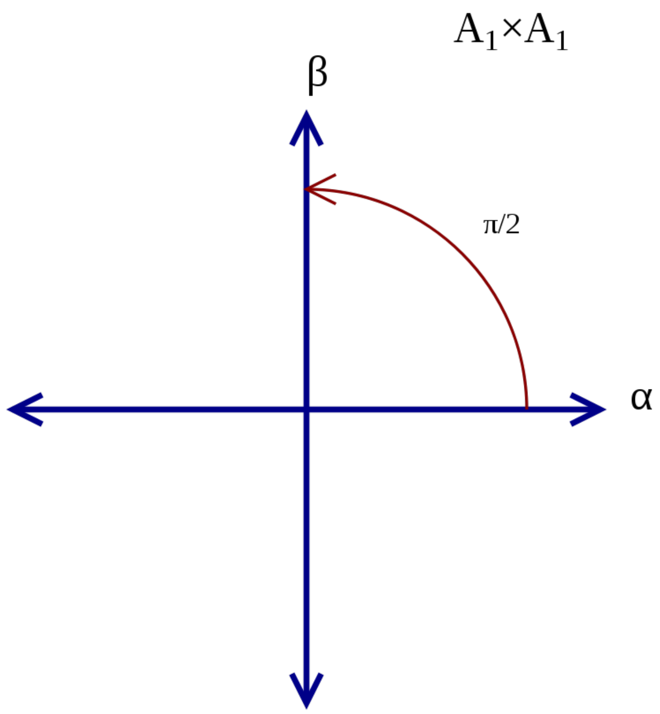
Appendix B

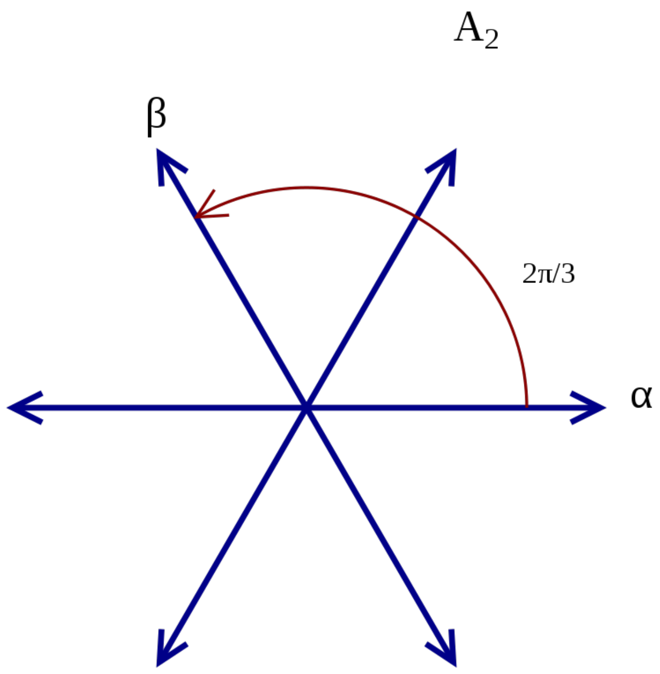
Source Code

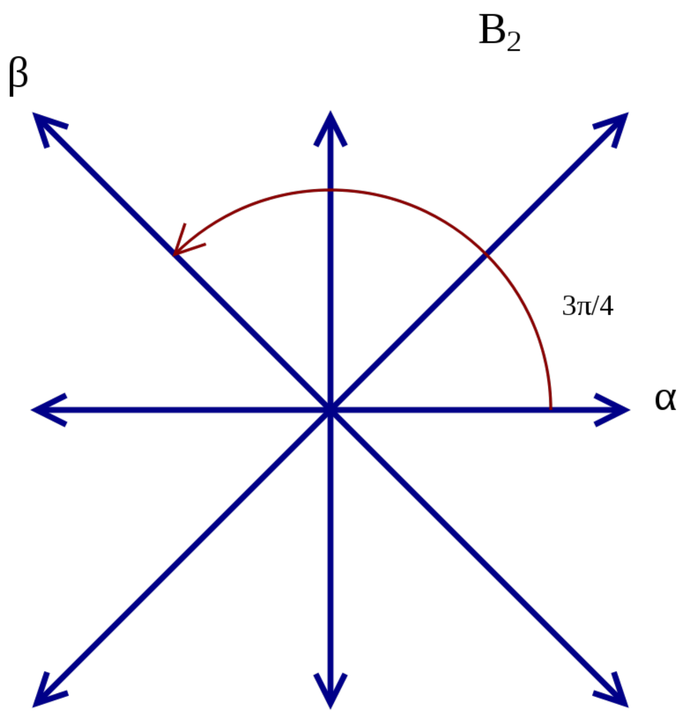
Put source code here ...

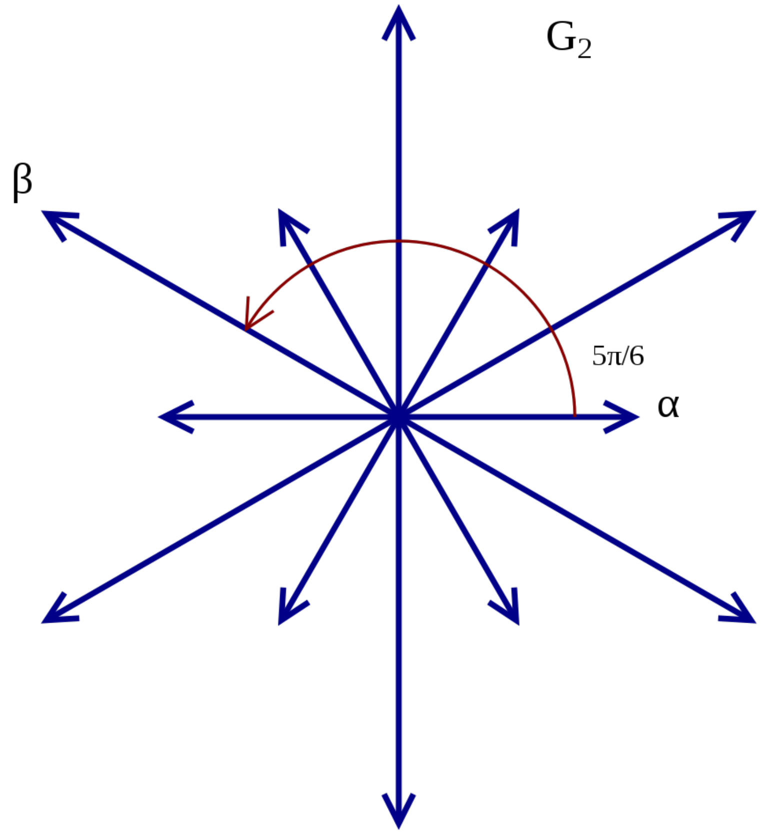
Appendix C

Rank 2 Root System Diagrams




A2
β
2π/3
α


B2
β
3π/4
α



G2
β
5π/6
α

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