# KÜLSHAMMER'S SECOND QUESTION AND THE 1-COHOMOLOGY

Let $\Gamma$ be an algebraic group, $G$ a connected reductive algebraic group over an algebraically closed field $k$ with $\mathrm{char}(k) = p > 0$. Let $P$ be a parabolic subgroup of $G$, with Levi subgroup $L$ and unipotent radical $V$. We have $P = V \rtimes L$, and we denote by $\pi^L$ the canonical projection

$$\pi^L : P \to L. \tag{0.1}$$

Since $L$ normalizes $V$ we have an action by group automorphisms of $L$ on $V$ given by

$$l \cdot v = lvl^{-1}, \tag{0.2}$$

for $l \in L, v \in V$.

Let $\sigma \in \mathrm{Hom}(\Gamma, L)$. Now we have an action of $\Gamma$ on $V$ given by

$$\gamma \cdot v = \sigma(\gamma) \cdot v \tag{0.3}$$

for $\gamma \in \Gamma, v \in V$, using Equation 0.2.

Let $\rho \in \mathrm{Hom}(\Gamma, P)$. We associate with $\rho$ the homomorphism $\rho^L : \Gamma \to L$ defined by

$$\rho^L = \pi^L \circ \rho, \tag{0.4}$$

and $\alpha_\rho : \Gamma \to V$ defined by

$$\alpha_\rho(\gamma) = \rho(\gamma)(\rho^L(\gamma))^{-1}. \tag{0.5}$$

With the action defined in Equation 0.3, $\alpha_\rho$ is a 1-cocycle.

In Equation 0.5 we have a formula associating elements of $\mathrm{Hom}(\Gamma, P)$ with 1-cocycles $\Gamma \to V$. However, to be able to compare 1-cocycles it is important that $\Gamma$ acts on $V$ in the same way. In other words, given $\rho, \varsigma \in \mathrm{Hom}(\Gamma, P)$ it only makes sense to compare $\alpha_\rho, \alpha_\varsigma$ if $\rho^L = \varsigma^L$.

**Definition 0.6** (Notation)**.** Fix $\sigma \in \mathrm{Hom}(\Gamma, L)$ and define

$$\mathrm{Hom}(\Gamma, P)_\sigma = \{\rho \in \mathrm{Hom}(\Gamma, P) \mid \rho^L = \sigma\}. \tag{0.7}$$

More generally, if $R \subset \mathrm{Hom}(\Gamma, P)$ define

$$R_\sigma = \{\rho \in R \mid \rho^L = \sigma\}. \tag{0.8}$$

For a given $\sigma \in \mathrm{Hom}(\Gamma, L)$, we denote by $Z^1(\Gamma, V)_\sigma$ the set of 1-cocycles $\Gamma \to V$ where $\Gamma$ acts on $V$ as in Equation 0.3. Evidently these 1-cocycles are in one-to-one correspondence with elements of $\mathrm{Hom}(\Gamma, P)_\sigma$. Formally, we have the following Lemma.

**Lemma 0.9.** *Define the map* $z : \mathrm{Hom}(\Gamma, P)_\sigma \to Z^1(\Gamma, V)_\sigma$ *by*

$$z(\rho) = \alpha_\rho. \tag{0.10}$$

*Then* $z$ *is a bijection.*

We return to the notion of comparing 1-cocycles. Following the established theory of abelian 1-cohomology, the non-abelian analogue of *equivalent 1-cocycles* is defined in Chapter 3.

**Lemma 0.11.** *The relation on* $Z^1(\Gamma, V)_\sigma$ *given by* $\alpha \sim \beta$ *if there exists* $v \in V$ *such that*

$$\alpha(\gamma) = v\beta(\gamma)(\gamma \cdot v^{-1}), \tag{0.12}$$

*for all* $\gamma \in \Gamma$*, is an equivalence relation.*

We denote the *1-cohomology* by $H^1(\Gamma, V)_\sigma$, defined to be the collection of equivalence classes of $Z^1(\Gamma, V)_\sigma$ under the equivalence relation in Equation 0.12. We denote by $\overline{\alpha}$ the image of $\alpha \in Z^1(\Gamma, V)_\sigma$ under the canonical projection $Z^1(\Gamma, V)_\sigma \to H^1(\Gamma, V)_\sigma$.

Following the theme of Lemma 0.9, we relate elements of $H^1(\Gamma, V)_\sigma$ to certain conjugacy classes of $\mathrm{Hom}(\Gamma, P)_\sigma$.

**Lemma 0.13.** *The bijective map* $z : \mathrm{Hom}(\Gamma, P)_\sigma \to Z^1(\Gamma, V)_\sigma$ *defined in Equation 0.14 descends to give a bijective map* $h : \mathrm{Hom}(\Gamma, P)_\sigma/V \to H^1(\Gamma, V)_\sigma$*, defined by*

$$h(V \cdot \rho) = \overline{\alpha_\rho}, \tag{0.14}$$

*where* $\rho \in \mathrm{Hom}(\Gamma, P)_\sigma$*.*

[some explaination for the following]

**Lemma 0.15.** *The bijective map* $h$ *defined in Equation* **??** *descends to give a bijective map* $\widetilde{h} : [\mathrm{Hom}(\Gamma, P)_\sigma/V]\,/C_L(\sigma) \to H^1(\Gamma, V)_\sigma/C_L(\sigma)$*, defined by*

$$\widetilde{h}((C_L(\sigma)V) \cdot \rho) = \widetilde{\alpha_\rho}, \tag{0.16}$$

*where* $(C_L(\sigma)V) \cdot \rho \in [\mathrm{Hom}(\Gamma, P)_\sigma/V]\,/C_L(\sigma) = \mathrm{Hom}(\Gamma, P)_\sigma/VC_L(\sigma)$*.*

**Definition 0.17** (Notation)**.** We define

$$\mathrm{Hom}(\Gamma, P)^L = \{\rho^L \mid \rho \in \mathrm{Hom}(\Gamma, P)\}. \tag{0.18}$$

More generally, when $R \subset \mathrm{Hom}(\Gamma, P)$ we define

$$R^L = \{\rho^L \mid \rho \in R\}. \tag{0.19}$$

**Lemma 0.20.** *Let* $R \subset \mathrm{Hom}(\Gamma, P)$*. Suppose* $R = P \cdot \rho$ *for some* $\rho \in R$*. Then* $R^L = L \cdot \rho^L$*.*

*More generally, if* $R = P \cdot R$ *then* $R^L = L \cdot R^L$*.*

**Definition 0.21.** Define the map

$$\mathcal{H} : \mathrm{Hom}(\Gamma, P)_\sigma \to H^1(\Gamma, V)_\sigma / C_L(\sigma) \tag{0.22}$$

by the composition of $h : \mathrm{Hom}(\Gamma, P)_\sigma \to Z^1(\Gamma, V)_\sigma$, followed by the canonical projection $Z^1(\Gamma, V)_\sigma \to H^1(\Gamma, V)_\sigma$, followed by the canonical projection $H^1(\Gamma, V)_\sigma \to H^1(\Gamma, V)_\sigma / C_L(\sigma)$.

**Theorem 0.23.** *Let $R \subset \mathrm{Hom}(\Gamma, P)$ and suppose $P \cdot R = R$. Then $R$ is a finite union of $P$-conjugacy classes if and only if*

(i) *$R^L$ is a finite union of $L$-conjugacy classes, and*
(ii) *for each $\sigma \in \mathrm{Hom}(\Gamma, L)$, $\mathcal{H}(R_\sigma) \subset H^1(\Gamma, V)_\sigma / C_L(\sigma)$ is finite.*

*Remark* 0.24. Conditions (i) and (ii) are equivalent to

(i′) $R_\sigma = \emptyset$ for all but finitely many $L$-conjugacy classes of $\sigma \in \mathrm{Hom}(\Gamma, L)$, and
(ii′) for each $\sigma \in \mathrm{Hom}(\Gamma, L)$, $R_\sigma$ is a finite union of $VC_L(\sigma)$-conjugacy classes,

respectively. We obtain (ii) $\Leftrightarrow$ (ii′) by appealing to the bijection $\widetilde{h}$, while (i) $\Leftrightarrow$ (i′) is self-evident.

*Proof.* First an observation. Suppose $R = P \cdot R$. Fix $\sigma \in \mathrm{Hom}(\Gamma, L)$ and let $\rho \in R_\sigma$, so that $\rho^L = \sigma$. Suppose $p \cdot \rho \in R_\sigma$ for some $p \in P$, and let $v \in V, l \in L$ such that $p = vl$. Then

$$\begin{aligned}
p \cdot \rho \in R_\sigma &\Leftrightarrow (vl) \cdot \rho \in R_\sigma && (0.25)\\
&\Leftrightarrow [(vl) \cdot \rho]^L = \sigma && (0.26)\\
&\Leftrightarrow l \cdot \rho^L = \sigma && (0.27)\\
&\Leftrightarrow l \in C_L(\sigma). && (0.28)
\end{aligned}$$

This shows that

$$R_\sigma \cap P \cdot \rho \subset (VC_L(\sigma)) \cdot \rho. \tag{0.29}$$

The reverse inclusion follows since $R = P \cdot R$ and $R_\sigma$ is stable under conjugation by $V$ and $C_L(\sigma)$. Hence

$$R_\sigma \cap P \cdot \rho = (VC_L(\sigma)) \cdot \rho. \tag{0.30}$$

Now suppose $R$ is a finite union of $P$-conjugacy classes, so there exists a finite set $\mathscr{P} \subset \mathrm{Hom}(\Gamma, P)$ such that

$$R = \bigcup_{\rho \in \mathscr{P}} P \cdot \rho \tag{0.31}$$

Lemma 0.20 shows that (i) holds. Furthermore

$$\begin{aligned}
R_\sigma &= R_\sigma \cap R && (0.32)\\
&= R_\sigma \cap \Big( \bigcup_{\rho \in \mathscr{P}} P \cdot \rho \Big) && (0.33)\\
&= \bigcup_{\rho \in \mathscr{P}} (R_\sigma \cap P \cdot \rho), && (0.34)
\end{aligned}$$

and by Equation 0.30

$$R_\sigma = \bigcup_{\rho \in \mathscr{P}} (VC_L(\sigma)) \cdot \rho. \tag{0.35}$$

Hence (ii′), and therefore (ii), holds. This proves the forward direction.

Conversely, suppose (i) and (ii) hold. By (ii) there exists a finite set $\mathscr{Q} \subset \mathrm{Hom}(\Gamma, P)$ and by Equation 0.30

$$R_\sigma = \bigcup_{\rho \in \mathscr{Q}} (VC_L(\sigma)) \cdot \rho \tag{0.36}$$

$$= \bigcup_{\rho \in \mathscr{Q}} (R_\sigma \cap P \cdot \rho) \tag{0.37}$$

$$= R_\sigma \cap \Big( \bigcup_{\rho \in \mathscr{Q}} P \cdot \rho \Big). \tag{0.38}$$

Hence $R_\sigma$ is contained in a finite union of $P$-conjugacy classes. By (i) there exists a finite set $\mathscr{S} \subset \mathrm{Hom}(\Gamma, L)$ such that

$$R^L = \bigcup_{\tau \in \mathscr{S}} L \cdot \tau. \tag{0.39}$$

For $\sigma \in \mathrm{Hom}(\Gamma, L)$, define $L \cdot R_\sigma = \{L \cdot \rho \,|\, \rho \in R_\sigma\}$. Evidently $L \cdot R_\sigma$ is contained in a finite union of $P$-conjugacy classes.

Now let $\rho \in R$. By Equation 0.39 there exists $l \in L$, $\tau \in \mathscr{S}$ such that $\rho^L = l \cdot \tau$. Hence $l^{-1} \cdot \rho^L = \tau$ which implies $\rho \in L \cdot R_\tau$. This shows that $R$ is contained in a finite union of $P$-conjugacy classes. The reverse inclusion is satisfied since $R = P \cdot R$. This completes the proof. □

**Definition 0.40.** Let $\Gamma' < \Gamma$ and denote by $\iota$ the inclusion map $\Gamma' \hookrightarrow \Gamma$. We denote by $\rho^\iota$ the homomorphism $\Gamma' \to P$ defined by

$$\rho^\iota = \rho \circ \iota, \tag{0.41}$$

and define

$$\mathrm{Hom}(\Gamma, P)^\iota = \{\rho^\iota \,|\, \rho \in \mathrm{Hom}(\Gamma, P)\}. \tag{0.42}$$

More generally, if $R \subset \mathrm{Hom}(\Gamma, P)$ then define

$$R^\iota = \{\rho^\iota \,|\, \rho \in R\}. \tag{0.43}$$

**Theorem 0.44.** *Let $R \subset \mathrm{Hom}(\Gamma, P)$ such that $R = P \cdot R$. Suppose*

(i) *$R^L$ is a finite union of $L$-conjugacy classes,*

(ii) *for all $\sigma \in \mathrm{Hom}(\Gamma, L)$ such that $R_\sigma \neq \emptyset$, the map*

$$H^1(\iota) : H^1(\Gamma, V)_\sigma / C_L(\sigma) \to H^1(\Gamma', V)_{\sigma^\iota} / C_L(\sigma)$$

*has finite fibres, and*

(iii) *$R^\iota$ is a finite union of $P$-conjugacy classes.*

*Then $R$ is a finite union of $P$-conjugacy classes.*

*Remark* 0.45. Since $R = P \cdot R$, $R^L$ is already a union of $L$-conjugacy classes by Lemma 0.20, so the point of (i) is that the union is finite.

*Proof.* Since $R^\iota \subset \mathrm{Hom}(\Gamma', V)_\sigma$, $R^\iota = P \cdot R^\iota$ and $R^\iota$ is a finite union of $P$-conjugacy classes, by Theorem 0.23

(iv) $(R^\iota)^L$ is a finite union of $L$-conjugacy classes, and

(v) for each $\tau \in \mathrm{Hom}(\Gamma', V)_{\sigma^\iota}$, $\mathcal{H}(R_\tau)$ is finite.

Let $\sigma \in \mathrm{Hom}(\Gamma, V)$. If $R_\sigma = \emptyset$ then $\mathcal{H}(R_\sigma)$ is certainly finite. On the other hand, if $R_\sigma \neq \emptyset$ then

$$\mathcal{H}(R_\sigma) \subset \widetilde{H^1}(\iota)^{-1}(R_{\sigma^\iota}) \tag{0.46}$$

□