MAPS OF 1-COHOMOLOGIES

Let K, V be algebraic groups such that K acts on V by group automorphisms, so we can define $Z^1(K,V), H^1(K,V)$. Denote the action of K on V by $x \cdot v$ for $x \in K, v \in V$.

Proposition 1. *Let K' be an algebraic group that acts on V by group automorphisms, so we can define $Z^1(K',V), H^1(K',V)$. Denote the action of K' on V by $x * v$ for $x \in K', v \in V$.*

Let $\zeta : K' \to K$ be a homomorphism and suppose that $x * v = \zeta(x) \cdot v$ for all $x \in K', v \in V$. Then for all $\alpha \in Z^1(K,V), \alpha \circ \zeta \in Z^1(K',V)$.

Proof. Let $\alpha \in Z^1(K,V)$ and let $x, y \in K'$. Then

$$\begin{aligned}(\alpha \circ \zeta)(xy) &= \alpha(\zeta(x)\zeta(y) \\ &= \alpha(\zeta(x))\left(\zeta(x) \cdot \alpha(\zeta(y))\right) \\ &= \alpha(\zeta(x))\left(x * \alpha(\zeta(y))\right) \\ &= (\alpha \circ \zeta)(x)\left(x * (\alpha \circ \zeta)(y)\right).\end{aligned}$$

Therefore, since $\alpha \circ \zeta$ is a morphism and satisfies the 1-cocycle condition, $\alpha \circ \zeta \in Z^1(K',V)$. ☐

Proposition 2. *Let V' be an algebraic group on which K acts by group automorphisms. Denote the action of K on V' by $x \wedge v$ for all $x \in K, v \in V'$.*

Let $\xi : V \to V'$ be a K-equivariant homomorphism, that is, $x \wedge \xi(v) = \xi(x \cdot v)$ for all $x \in K, v \in V$. Then for all $\alpha \in Z^1(K,V), \xi \circ \alpha \in Z^1(K,V')$.

Proof. Let $\alpha \in Z^1(K,V)$ and let $x, y \in K$. Then

$$\begin{aligned}(\xi \circ \alpha)(xy) &= \xi\left(\alpha(x)(x \cdot \alpha(y))\right) \\ &= \xi(\alpha(x))\xi(x \cdot \alpha(y)) \\ &= \xi(\alpha(x))\left(x \wedge \xi(\alpha(y))\right) \\ &= (\xi \circ \alpha)(x)\left(x \wedge (\xi \circ \alpha)(y)\right).\end{aligned}$$

Therefore, since $\xi \circ \alpha$ is a morphism and satisfies the 1-cocycle condition, $\xi \circ \alpha \in Z^1(K,V')$. ☐

Proposition 3. *Let K', V' be algebraic groups, such that*

(a) *K' acts on V by group automorphisms, denoted by $x * v$ for all $x \in K', v \in V$,*

(b) *K' acts on V' by group automorphisms, denoted by $x \wedge v$ for all $x \in K', v \in V'$.*

Let $\zeta : K' \to K$ and $\xi : V \to V'$ be homomorphisms such that

(c) *$x * v = \zeta(x) \cdot v$ for all $x \in K', v \in V$,*

(d) *$\xi(x * v) = x \wedge \xi(v)$ for all $x \in K', v \in V$.*

Then the function defined by

$$Z^1(\zeta, \xi)(\alpha) = \xi \circ \alpha \circ \zeta,$$

maps $Z^1(K, V) \to Z^1(K', V')$.

Furthermore, $Z^1(\zeta, \xi)$ descends to give a well-defined map $H^1(\zeta, \xi) : H^1(K, V) \to H^1(K', V')$, defined by

$$H^1(\zeta, \xi)(\psi(\alpha)) = \left(\psi' \circ Z^1(\zeta, \xi)\right)(\alpha),$$

for all $\alpha \in Z^1(K, V)$, where $\psi : Z^1(K, V) \to H^1(K, V)$ and $\psi' : Z^1(K', V') \to H^1(K', V')$ are the canonical projections.

Moreover, the following diagram commutes:

$$\begin{tikzcd}
Z^1(K,V) \arrow[r, "Z^1(\zeta,\xi)"] \arrow[d, "\psi"] & Z^1(K',V') \arrow[d, "\psi'"] \\
H^1(K,V) \arrow[r, "H^1(\zeta,\xi)"] & H^1(K',V').
\end{tikzcd}$$

Proof. Let $\alpha \in Z^1(K, V)$ and let $x, y \in K'$. Then

$$\begin{aligned}
(\xi \circ \alpha \circ \zeta)(xy) &= \xi\left((\alpha \circ \zeta)(x)\right) \xi\left(x * (\alpha \circ \zeta)(y)\right) \quad \text{(by Prop. 1)} \\
&= \xi\left((\alpha \circ \zeta)(x)\right) \left(x \wedge \xi(\alpha \circ \zeta)(y)\right) \quad \text{(by (d))} \\
&= (\xi \circ \alpha \circ \zeta)(x) \left(x \wedge (\xi \circ \alpha \circ \zeta)(y)\right).
\end{aligned}$$

Therefore $(\xi \circ \alpha \circ \zeta) \in Z^1(K', V')$.

It remains to show $H^1(\zeta, \xi)$ is well-defined. Let $\beta \in Z^1(K, V)$ such that $\psi(\alpha) = \psi(\beta)$. Then there exists $v \in V$ such that

$$\beta(x) = v\alpha(x)(x \cdot v^{-1}),$$

for all $x \in K$.

Then, for all $x \in K'$

$$\begin{aligned}
\left(Z^1(\zeta, \xi)(\beta)\right)(x) &= \xi\left(\beta(\zeta(x))\right) \\
&= \xi\left(v\alpha(\zeta(x))\left(\zeta(x) * v^{-1}\right)\right) \\
&= \xi(v)\xi(\alpha(\zeta(x)))\xi\left(\zeta(x) * v^{-1}\right) \\
&= \xi(v)\xi(\alpha(\zeta(x)))\left(x \wedge \xi(v^{-1})\right) \\
&= \xi(v)\left(Z^1(\zeta, \xi)(\alpha)\right)(x)\left(x \wedge \xi(v^{-1})\right).
\end{aligned}$$

This shows that $\psi'\left(Z^1(\zeta,\xi)(\alpha)\right) = \psi'\left(Z^1(\zeta,\xi)(\alpha)\right)$, hence $H^1(\zeta,\xi)$ is well-defined. This completes the proof. $\square$

Remark 1. *The slightly unfortunate choice of notation "$Z^1(K,V)$" doesn't explicitly mention the action. The consequence is that given suitable homomorphisms*

$$\zeta : K \to K,$$
$$\xi : V \to V,$$

the statement

$$H^1(\zeta,\xi) : H^1(K,V) \to H^1(K,V)$$

is misleading on its own: Is the 1-cohomology on the left of the arrow the same as the one on the right? Later we adopt a modified notation for the 1-cocycles and the 1-cohomology which carries the action defining the 1-cocycles.