UNIVERSITY OF CANTERBURY

A Geometric Approach to Complete Reducibility

by

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Declaration of Authorship

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"A quote."

The author of the quote.

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Abstract

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The Thesis Abstract . . .

Acknowledgements

The acknowledgements and the people to thank ...

Contents

Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	$rads^{-1}$
$\vdots$		

Dedication . . .

Chapter 1

Introduction

[This thesis is about algebraic groups. Talk a little bit about them.]

A major motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer to do with homomorphisms of finite groups into algebraic groups [1]. We will call these homomorphisms *representations* because of the obvious similarity with the usual kind of representations into GL_n. Külshammer's second question is as follows:

> Let G be a linear algebraic group over an algebraically closed field of characteristic p. Let Γ be a finite group and $\Gamma_p < \Gamma$ a Sylow p-subgroup of Γ. Fix a conjugacy class of representations $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

So far only a non-reductive counterexample is known [1]. We examine a slight variation on the question which we call the *algebraic group* version of Külshammer's question. Instead of a finite group Γ we use a reductive group H, and instead of a Sylow p-subgroup $\Gamma_p < \Gamma$ we use a maximal unipotent subgroup $U < H$:

> Let G, H be connected reductive linear algebraic groups over an algebraically closed field of characteristic p and $U < H$ a maximal unipotent subgroup of H. Fix a conjugacy class of representations $U \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : H \to G$ whose restrictions to U belong to the given class?

Not only is the algebraic group version of Külshammer's question a non-trivial pursuit in its own right but finding an algebraic counterexample might help to find a finite counterexample to Külshammer's original question for a finite subgroup of H and a reductive G. For instance, in our example calculations we pay special attention to $H = SL_2$ and an algebraic counterexample might produce a finite counterexample $\Gamma = SL_2(q)$ for some q.

Our approach to Külshammer's question also means that the work in this thesis contributes to the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz ([2], [3]). Let G be a simple algebraic group over an algebraically closed field of characteristic p. For large enough characteristic ($p = 0$ or $p > 7$ covers all restrictions) Liebeck and Seitz determine explicitly the embeddings of arbitrary connected semisimple groups in G, where G is of exceptional type. We examine the subgroup structure of simple algebraic groups in low characteristic (usually $p = 2$ or $p = 3$) where less is known. We use similar methods to Liebeck and Seitz, calculating a certain 1-cohomology of H with coefficients in V, the unipotent radical of a parabolic subgroup $P < G$.

One of our main results is Theorem 4.8. With this we are able to relate Külshammer's question to a certain 1-cohomology calculation in which Γ acts on the unipotent radical V of a parabolic subgroup $P < G$ via a certain representation $\Gamma \to L$ into a Levi subgroup $L < P$. We show that we can reduce Külshammer's question to another question: is the restriction map of 1-cohomologies

$$H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

injective?

This approach might help settle Külshammer's original question.

It is known that if V is abelian then the restriction map of 1-cohomologies

$$H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

is injective for finite Γ and $\Gamma_p < \Gamma$ a Sylow p-subgroup (see Lemma 3.2). We use this result to show that the restriction map

$$H^1(SL_2(k), V) \to H^1(U_2(k), V)$$

is injective, where $U_2(k) < SL_2(k)$ is the maximal unipotent subgroup of upper unitriangular matrices (see Example 3.2). Hence we will need to investigate non-abelian V which requires us to work with the non-abelian 1-cohomology.

Next we show that if $H = SL_2$ and G is a linear algebraic group then 1-cocycles $H \to V$ that are trivial on a fixed maximal torus $T < H$ have images in an abelian subgroup $W < V$.

The notion of semisimplicity is important in mathematics: that an object can be studied by breaking it up into simple pieces. In representation theory for instance, a semisimple or completely reducible representation is a representation that can be written as a direct sum of irreducible representations.

Serre defined complete reducibility for algebraic groups. Let $G' < G$. We say G' is a G-completely reducible subgroup, or simply G-completely reducible, if whenever G' is contained in a parabolic subgroup $P < G$ then G' is also contained in a Levi subgroup $L < P$. Now let H be an arbitrary group and $\rho : H \to G$ be a representation. If the image of ρ is a G-completely reducible subgroup then we say ρ is G-completely reducible. This definition agrees with the definition from representation theory when we set $G = GL_n$.

Külshammer's question has it's roots Maschke's Theorem of representation theory which shows that any representation from a finte group $\Gamma \to GL_n$ over a field of characteristic not dividing the order of Γ is completely reducible, and that there are only finitely many conjugacy classes of (completely reducible) representations $\Gamma \to GL_n$ [ref Lang].

Let Γ be a finite group and let G be a linear algebraic group over an algebraically closed field of characteristic p. Külshammer's first question reads:

> Suppose p does not divide the order of Γ. Are there only finitely many conjugacy classes of representations $\Gamma \to G$?

The answer is positive [refs] and is essentially contained in a paper of A. Weil [ref]. Külshammer's second question is a refinement of the first:

> Let $\Gamma_p < \Gamma$ be a Sylow p-subgroup and fix a conjugacy class of representations $\Gamma_p \to G$. Are there only finitely many conjugacy classes of representations $\Gamma \to G$ whose restrictions to Γ_p belong to the fixed class?

Note that the condition that p does not divide $|\Gamma|$ is dropped from the first statement. If p does not divide the order of Γ then the answer is "yes", since Γ_p is trivial and so all representations are equal when restricted to Γ_p. If Γ is a p-group then the answer is "yes", as $\Gamma_p = \Gamma$ so restricting to Γ_p does nothing and therefore only representations that come from the fixed class will hit the class. If $G = GL_n$ the answer is also "yes", since by Maschke's theorem there can only be finitely many conjugacy classes of representations $\Gamma \to GL_n$ anyway, regardless of whether or not their restrictions to Γ_p hit the fixed class.

Chapter 2

Mathematical Preliminaries

2.1 Linear Algebraic Groups

Let $k = \overline{k}$ be an algebraically closed field. An affine variety over k is a subset of k^n defined by the vanishing of some polynomial equations. We have such notions as a subvariety of an affine variety, a natural product of affine varieties and maps between affine varieties.

A morphism $\phi : V \to W$ of affine varieties is a map such that the coordinates of $\phi(v) \in W$ are given by polynomial functions in $v \in V$.

An affine algebraic group G is a set G which is an affine algebraic variety and a group such that

$$\begin{aligned} \mu \quad : \quad & G \times G \to G \\ & (x, y) \mapsto x.y, \end{aligned}$$

and

$$\begin{aligned} \iota \quad : \quad & G \to G \\ & x \mapsto x^{-1} \end{aligned}$$

are morphisms of affine varieties.

Example 2.1. *The special linear group of $n \times n$ matrices with entries in k*

$$SL_n(k) = \{(a_{ij}) \in k^{n^2} \,|\, det(a_{ij}) - 1 = 0\}$$

is an affine variety. Furthermore, the general linear group of $n \times n$ matrices with entries in k

$$GL_n(k) = \{(a_{ij}) \in k^{n^2} | \, det(a_{ij}) \neq 0\}$$

is an affine variety, seen more clearly by the inclusion in the affine variety

$$GL_n(k) \subset \{(b, (a_{ij})) \in k^{n^2+1} | \, b.det(a_{ij}) - 1 = 0\}.$$

Of course both examples can be shown to be affine algebraic groups by checking the multiplication and inverse laws.

A homomorphism $\phi : G \to H$ of affine algebraic groups is a morphism of affine varieties and a homomorphism of abstract groups. An isomorphism $\phi : G \to H$ of affine algebraic groups is a bijective homomorphism of affine algebraic groups such that $\phi^{-1} : H \to G$ is also a homomorphism of affine algebraic groups.

Example 2.2. *Let $chark = p$. The map $k \to k$ which sends $x \mapsto x^p$ is bijective, a morphism, but not an isomorphism since the inverse map $x \mapsto x^{1/p}$ is not a morphism of affine varieties (it is not a polynomial).*

Now let $G = GL_n(k)$. The map $F : G \to G$ which sends $(a_{ij}) \mapsto (a_{ij}^q)$, $q = p^z$, $z \in \mathbb{Z}^+$ is a homomorphism of affine algebraic groups, called the Frobenius morphism. It is not an isomorphism.

The subvarieties of an affine variety V form the closed sets of a topology, known as the Zariski topology.

A closed subgroup of an affine algebraic group is itself an affine algebraic group. A closed subgroup of $GL_n(k)$ is called a linear algebraic group. In fact every affine algebraic group is a linear algebraic group.

Example 2.3. *Three important subgroups of the linear algebraic group $G = GL_n(k)$*

$$\begin{aligned} T = T_n(k) &= \{(a_{ij}) \in GL_n(k) | \, a_{ij} = 0 \text{ if } i \neq j\} \\ &\quad \textit{diagonal matrices in } GL_n(k) \end{aligned}$$

$$\begin{aligned} U = U_n(k) &= \{(a_{ij}) \in GL_n(k) | \, a_{ii} = 1, a_{ij} = 0 \text{ if } i < j\} \\ &\quad \textit{upper unitriangular matrices in } GL_n(k) \end{aligned}$$

$$\begin{aligned} B = B_n(k) &= \{(a_{ij}) \in GL_n(k) | \, a_{ij} = 0 \text{ if } i < j\} \\ &\quad \textit{upper triangular matrices in } GL_n(k) \end{aligned}$$

T is an example of a torus of G, U is an example of a unipotent subgroup of G, and B is an example of a Borel subgroup of G.

Let G be a linear algebraic group. The irreducible components of G are disjoint. If G° is the irreducible component containing the identity element of G then G° is a (closed) normal subgroup of G of finite index. The irreducible components of G are the cosets of G° in G. G° is the smallest closed subgroup of G of finite index (every closed subgroup of finite index is open).

G° is called the identity component of G. If $G = G^\circ$ we say G is connected.

Every element $g \in G$ can be uniquely written

$$g = g_s.g_u = g_u.g_s,$$

where g_s is semisimple (diagonalisable) and g_u is unipotent. This is known as the Jordan decomposition.

G has a unique maximal closed normal solvable subgroup $R(G)$, called the radical of G. The set of unipotent elements of $R(G)$ is a maximal closed connected unipotent normal subgroup $R_u(G)$, called the unipotent radical of G.

If $R_u(G) = 1$ we say G is reductive. If $R(G) = 1$ we say G is semisimple. If G is connected and has no proper closed connected normal subgroups then G is simple.

Example 2.4. *$GL_n(k)$ is reductive. $SL_n(k)$ is semisimple (hence reductive). $SL_n(k)$ is simple as an algebraic group but not as an abstract group, since it has a non-trivial center.*

If G is nonabelian and simple then its centre $Z(G)$ is finite.

If G is a reductive linear algebraic group then

$$G = Z(G)^\circ.(G, G),$$

where

$$(G, G) = \langle [g, h] = ghg^{-1}h^{-1} \,|\, g, h \in G \rangle,$$

the commutator subgroup. $Z(G)$ is a torus of G and (G, G) is again reductive.

Every abelian simple algebraic group has dimension 1 and is isomorphic to either

$$G_m(k) = k^* = \text{multiplicative group of } k$$

or

$$G_a(k) = k = \text{additive group of } k.$$

A torus is isomorphic to $k^* \times k^* \cdots k^*$. Any two maximal tori in G are conjugate in G.

If G is connected with maximal torus $T < G$ then $C_G(T) = N_G(T)^\circ$ and hence $N_G(T)/C_G(T)$ is finite. We call $W = N_G(T)/C_G(T)$ the Weyl group of G. Furthermore, if G is also reductive then $T = C_G(T)$ and $W = N_G(T)/T$ is a finite Coxeter group, that is, of the form:

$$W = \langle s_1, \ldots, s_l |\, s_i^2 = 1, (s_i s_j)_{ij}^m = 1\rangle$$

A Borel subgroup of G is a maximal closed connected solvable subgroup of G, any two Borel subgroups of G are conjugate in G. if $T < G$ is a torus of G then there exists a Borel subgroup B of G containing T. Furthermore, we can write $B = T \cdot R_u(B)$.

Let G be a reductive connected linear algebraic group with torus $T < G$. Let $N = N_G(T)$. Then we can write G as

$$G = BNB = \cup_{n \in N} BnB.$$

$BnB = Bn'B$ if and only iff $\pi(n) = \pi(n')$ where $\pi : N \to N/T = W$ so we have the correspondence $B\backslash G/B \leftrightarrow W$ where $BnB \mapsto \pi(n)$.

Suppose $W = \langle s_1, \ldots, s_l\rangle$ and let $J \subset \{1, \ldots, l\}$. We define $W_J = \langle s_j |\, j \in J\rangle < W$ and $N_J = \pi^{-1}(W_J)$. The subgroup of G defined by

$$P_J = BN_JB$$

contains B, and in fact every subgroup of G containing B is of this form. We call $P < G$ a parabolic subgroup of G if $B < P$ for some Borel subgroup $B < G$. Equivalently, P is a parabolic subgroup of G if given a maximal torus $T < G$, P is conjugate to some P_J.

A parabolic subgroup $P < G$ is connected, self-normalizing, and can be decomposed into a semi-direct product of its unipotent radical and a Levi subgroup $L < P$:

$$P = L \cdot R_u(P),$$

with $L \cap R_u(P) = 1$. Any two Levi subgroups of P are conjugate by an element of $R_u(P)$ and will be reductive if G is reductive.

Let T be a maximal torus of a connected reductive linear algebraic group G. We define the character group of T is to be

$$X = \mathrm{Hom}(T, k^*),$$

with the addition law

$$(x_1 + x_2)(t) = x_1(t)x_2(t), \qquad x_1, x_2 \in X, t \in T.$$

The cocharacter group is defined

$$Y = \mathrm{Hom}(k^*, T),$$

with the addition law

$$(y_1 + y_2)(\lambda) = y_1(\lambda)y_2(\lambda), \qquad y_1, y_2 \in Y, \lambda \in k^*.$$

If we compose $x \in X$ with $y \in Y$ we get a morphism

$$k^* \to T \to k^*,$$

that is, of the form $\lambda \mapsto \lambda^n$ for some $n \in \mathbb{Z}$. Hence there exists a pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$ defined

$$(x, y) \mapsto \langle x, y \rangle = n,$$

where $x(y(\lambda)) = \lambda^n$.

Chapter 3

The 1-Cohomology

3.1 Abelian 1-Cohomology

3.1.1 Definitions

Let H be a group and V an abelian group (vector space) on which H acts homomorphically (linearly). We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \tag{3.1}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call (3.1) the *1-cocycle condition*.

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned}
(\sigma_1 + \sigma_2)(h_1h_2) &= \sigma_1(h_1h_2) + \sigma_2(h_1h_2) \\
&= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\
&= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\
&= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2),
\end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \to V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H,V)$ we have

$$\begin{aligned} \sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\,\sigma(1), \end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned} \sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}), \end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1\,(H,V)$ is a $\mathbb{Z}-$module under pointwise addition.

Given a v in V we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1\,(H,V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned} \chi_v^H(h_1h_2) &= v - (h_1h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2), \end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u+v) - h \cdot (u+v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary, and hence $B^1(H,V)$ is also closed under pointwise addition.

We see that $B^1(H,V)$ is a subgroup of $Z^1(H,V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H,V)$ is a $\mathbb{Z}-$submodule of $Z^1(H,V)$, so we may form the quotient module

$$H^1(H,V) = Z^1(H,V)/B^1(H,V),$$

called the *1-cohomology*.

Lemma 3.1. *Suppose H is linearly reductive. Then $H^1(H,V)$ is trivial [4].*

3.1.2 Maps between 1-cohomologies

Let $\phi : \tilde{H} \to H$ be a homomorphism, $\tilde{H}$ being another group that acts on V. Suppose that for every h in $\tilde{H}$, ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V. If σ is a 1-cocycle from $H \to V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \to V$. Thus certain homomorphisms

$$\phi : \tilde{H} \to H$$

give rise to maps of the form

$$Z^1(\phi) : Z^1(H,V) \to Z^1(\tilde{H},V)$$

Take h_1, h_2 in H. We have

$$\begin{aligned} Z^1(\phi)(\sigma)(h_1h_2) &= (\sigma \circ \phi)(h_1h_2) \\ &= \sigma(\phi(h_1h_2)) \\ &= \sigma(\phi(h_1)\phi(h_2)) \\ &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2) \\ &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\ &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\ &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2). \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \to H^1(\tilde{H}, V),$$

defined by

$$\begin{tikzcd} Z^1(H,V) \arrow[r, "Z^1(\phi)"] \arrow[d, "\pi"'] & Z^1(\tilde{H},V) \arrow[d, "\tilde{\pi}"] \\ H^1(H,V) \arrow[r, "H^1(\phi)"] & H^1(\tilde{H},V) \end{tikzcd}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 3.1. *Let $\tilde{H}$ be a subgroup of H and $i : \tilde{H} \to H$ the inclusion map. Then i gives rise to a well defined map*

$$H^1(i) : H^1(H, V) \to H^1(\tilde{H}, V).$$

Lemma 3.2. *Let V be a vector space over a field of characteristic p. Let Γ be a finite group and $\tilde{\Gamma} = \Gamma_p \subset \Gamma$ a* Sylow p-subgroup *of Γ. The map*

$$H^1(i) : H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

is injective.

Proof. Let x be an element of $H^1(\Gamma, V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(\Gamma, V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to Γ_p is equal to a 1-coboundary, say $\chi_v^{\Gamma_p}$. But since

$\chi_v^{\Gamma_p}$ can be trivially extended to a 1-coboundary χ_v^{Γ} from $\Gamma \to V$, and

$$\pi(\sigma - \chi_v^{\Gamma}) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^{\Gamma})$ as a representative for x. Hence there is no harm in assuming that σ is 0 when restricted to Γ_p. Now choose a set of representatives $\gamma_1, \ldots, \gamma_l$ in Γ for the coset space Γ/Γ_p and set

$$v^* = \sum_{i=1}^{l} \sigma(\gamma_i).$$

Consider the 1-coboundary $\chi_{v^*}^{\Gamma}$ given by v^*

$$\begin{aligned} \chi_{v^*}^{\Gamma}(\gamma) &= v^* - \gamma \cdot v^* \\ &= \sum_{i=1}^{l} \sigma(\gamma_i) - \gamma \cdot \sum_{i=1}^{l} \sigma(\gamma_i) \\ &= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \gamma \cdot \sigma(\gamma_i). \end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(\gamma\gamma_i) = \sigma(\gamma) + \gamma \cdot \sigma(\gamma_i),$$

from which we obtain

$$\begin{aligned} \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \gamma \cdot \sigma(\gamma_i) &= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} (\sigma(\gamma\gamma_i) - \sigma(\gamma)) \\ &= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \sigma(\gamma\gamma_i) + \sum_{i=1}^{l} \sigma(\gamma). \end{aligned}$$

Now as the value of σ at a fixed γ depends only on the value of σ at the representative γ_j of the coset containing γ we can collapse the middle term to yield

$$\begin{aligned} \chi_{v^*}^{\Gamma}(\gamma) &= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \sigma(\gamma\gamma_i) + \sum_{i=1}^{l} \sigma(\gamma) \\ &= \sum_{i=1}^{l} \sigma(\gamma_i) - \sum_{i=1}^{l} \sigma(\gamma_i) + \sum_{i=1}^{l} \sigma(\gamma) \\ &= l\,\sigma(\gamma). \end{aligned}$$

Since $\gcd([\Gamma : \Gamma_p], p) = \gcd(l, p) = 1$, l is invertible and so

$$l^{-1}\chi_{v^*}^{\Gamma}(\gamma) = \sigma(\gamma).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. $\square$

Example 3.2. *Let*

$$k = \bar{\mathbb{F}_p} = \bigcup_r \mathbb{F}_{p^r},$$

V a vector space on which $SL_2(k)$ acts, and $U(k)$ the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Then $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ for each r, and the map

$$H^1(SL_2(k), V) \to H^1(U(k), V)$$

is injective.

Proof. The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r}-1)(p^{2r}-p^r)$ since there are $p^{2r}-1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r}-p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \to \mathbb{F}_{p^r}^*,$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First homomorphism theorem for groups

$$GL_2(\mathbb{F}_{p^r}) \,/\, SL_2(\mathbb{F}_{p^r}) \sim \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| \,/\, |\mathbb{F}_{p^r}^*| \\ &= (p^{2r}-1)(p^{2r}-p^r) \,/\, (p^r-1) \\ &= p^r(p^{2r}-1). \end{aligned}$$

Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$.

Fix a non-trivial $y \in H^1(SL_2(k), V)$ and choose a representative $\tau \in Z^1(SL_2(k), V)$ for y. For each $g \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_g^{(r)} : V \to V$ by

$$f_g^{(r)}(v) = \tau(g) - \chi_v(g) = \tau(g) - v + g \cdot v.$$

Consider subsets of V defined by

$$C_r = \{v \in V | f_g^{(r)}(v) = 0\}.$$

Each subset C_r is closed and the inclusion $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$ induces the reverse inclusion $C_{r!} \supset C_{(r+1)!}$. The Noetherian property for V requires that the sequence of subsets of

V defined by

$$\{C_{i!}\}_{i=1}^{\infty}$$

becomes constant. However, $y \neq 0$ so τ is not a 1-coboundary on $SL_2(k)$, which means the C_r's are eventually empty. That is, there exists an integer s such that for any v in V

$$(\tau - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if $y|_{SL_2(\mathbb{F}_{p^r})} = 0$ for all r then $y = 0$.

Take x in the kernel of the map $H^1(SL_2(k), V) \to H^1(U(k), V)$. Then for each r, $x|_{U(\mathbb{F}_{p^r})} = 0$ so by Lemma 3.2 $x|_{SL_2(\mathbb{F}_{p^r})} = 0$. Therefore $x = 0$ and so $H^1(SL_2(k), V) \to H^1(U(k), V)$ is injective. □

We could also consider appropriate maps $f : V \to \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \to H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \to H^1(\tilde{H}, \tilde{V}).$$

3.2 Non-abelian 1-Cohomology

3.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocycles be morphisms of varieties.

3.2.2 Definitions

Let H, V be algebraic groups, H acting on V. We call a map σ from $H \to V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{3.2}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call the (3.2) the *1-cocycle condition.*

Given a v in V we define a *1-coboundary* $\chi_v^H : H \to V$ to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}
\chi_v^H(h_1 h_2) &= v * (h_1 h_2) \cdot v^{-1} \\
&= v * h_1 \cdot \left(h_2 \cdot v^{-1}\right) \\
&= v * h_1 \cdot \left(v v^{-1} h_2 \cdot v\right) \\
&= v * h_1 \cdot v * h_1 \cdot \left(v * h_2 \cdot v^{-1}\right) \\
&= \chi_v^H(h_1) * h_1 \cdot \chi_v^H(h_2),
\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say σ_1, σ_2 in $Z^1(H, V)$ are *equivalent* if there exists a v in V such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \tag{3.3}$$

for all h in H. We call the set of equivalence classes of $Z^1(H, V)$ under the equivalence relation defined by (3.3) the *1-cohomology*, denoted $H^1(H, V)$.

3.2.3 Maps between 1-cohomologies

Lemma 3.3. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V. Then $H^1(i) : H^1(SL_2, V) \to H^1(B, V)$ is injective.*

Proof. Let x be in the kernel of $H^1(i)$ and σ and element of $Z^1(SL_2, V)$ that projects onto the class x. Since $Z^1(i)(\sigma)$ projects to the trivial 1-cohomology class we may as well assume that $\sigma|_B = 1$. For there exists some v in V such that for all b in B

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle $\hat{\sigma} : SL_2 \to V$ defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction $\hat{\sigma}$ also projects to the class x, and for all b in B

$$\begin{aligned}\hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\ &= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\ &= v^{-1} * v * b \cdot (v^{-1} * v) \\ &= 1,\end{aligned}$$

so we may as well have chosen $\hat{\sigma}$ instead as a representative for x.

Now consider the *homogeneous space* SL_2/B [5] and take the map

$$\tilde{\sigma} : SL_2/B \to V,$$

defined in the usual way under the canonical projection $\pi : SL_2 \to SL_2/B$:

$$\begin{tikzcd} SL_2 \arrow[r, "\sigma"] \arrow[d, "\pi"'] & V \\ SL_2/B \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

This map is well defined and is a morphism [6]. Now since SL_2/B is an irreducible projective variety [5], $\tilde{\sigma}$ must be constant [6]. Hence, as σ takes the value 1 for any b in B, $\tilde{\sigma}(hB) = 1$ for all cosets hB. Therefore, for all h in SL_2

$$\sigma(h) = \tilde{\sigma}(hB) = 1.$$

We have shown that σ is the 1-coboundary χ_1 which means that the kernel of $H^1(i)$ is trivial. □

Lemma 3.4. *Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B. Then $H^1(B,V) \to H^1(U,V)$ is injective. Moreover*

$$H^1(SL_2, V) \to H^1(U,V)$$

is injective.

Proof. Let x be an element of the kernel of $H^1(i) : H^1(B,V) \to H^1(U,V)$ and let σ in $Z^1(B,V)$ be a representative for x. We may as well assume that $\sigma|_T = 1$. For any b in

B we can find a u in U and a t in T such that $b = ut$. Hence

$$\begin{aligned} \sigma(b) &= \sigma(ut) \\ &= \sigma(u) * u \cdot \sigma(t) \\ &= \sigma(u). \end{aligned}$$

Since σ represents x, σ must be a 1-coboundary on U. Hence σ is in $B^1(B, V)$ and the kernel of $H^1(i) : H^1(B, V) \to H^1(U, V)$ is trivial. $\square$

Chapter 4

Külshammer's Second Problem

4.1 Külshammer's Second Problem

Two questions were raised by B. Külshammer concerning representations of a finite group Γ into a linear algebraic group G over an algebraically closed field k. The first has a positive answer and is essentially contained a paper by A. Weil [1]:

(K. I) Let $\mathrm{char}(k)$ be prime to the order of Γ. Are there only finitely many representations $\rho : \Gamma \to G$ up to conjugation by G?

(K. II) Let $p = \mathrm{char}(k)$ and $\Gamma_p \subset \Gamma$ be a Sylow p-subgroup. Fix a conjugacy class of representations from $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

(K. II) has positive answer so long as G is reductive and the characteristic of k is good for G [7]. The same paper shows that the answer is "no" in general by way of a counterexample involving a non-reductive G.

We wish to determine whether there exists a reductive counterexample to (K. II).

4.2 The Approach

We are interested in knowing whether there can be infinitely many G-conjugacy classes of representations $\Gamma \to G$ that when restricted to Γ_p hit some fixed G-conjugacy class of representations $\Gamma_p \to G$. A consequence of the following Theorem [reference] is that we will need to study representations into parabolic subgroups $P < G$.

Theorem 4.1. *There are only finitely many G-conjugacy classes of G-completely reducible representations $\Gamma \to G$.*

So by Theorem 4.1, if we have infinitely many G-conjugacy classes of representations $\Gamma \to G$ then infinitely many of those classes must be of non-G-completely reducible representations. The following Lemma states that the finiteness of G-conjugacy classes of a collection of representations $\Gamma \to G$ carries over to P-conjugacy classes for any parabolic subgroup $P < G$ containing the image of the representations.

Lemma 4.2. *Let $R = \{\rho_\lambda : \Gamma \to P \mid \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ, P a fixed parabolic subgroup of G. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Take two elements ρ_μ, ρ_ν of R in a particular G-conjugacy class. Then there exists an element $g \in G$ such that

$$g \cdot \rho_\mu = \rho_\nu.$$

By definition ρ_ν maps into P, but on the other hand $\rho_\nu = g \cdot \rho_\mu$ maps into $Q = gPg^{-1}$. Therefore

$$\rho_\nu : \Gamma \to P \cap Q.$$

Let T be a maximal torus of G contained in $P \cap Q$ [ref], and let $\{n_1, \ldots, n_l\}$ be coset representatives for the Weyl group $W = N_G(T)/T$.

Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists an element $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

By the definition of Q there exists an element $p \in P$ such that $q = gpg^{-1}$, so in fact

$$\begin{aligned} gpg^{-1}Tgp^{-1}g^{-1} &= gTg^{-1} \\ \Rightarrow pg^{-1}Tgp^{-1} &= T. \end{aligned}$$

We see that gp^{-1} lies in $N_G(T)$. Let n_i be the coset representative for the element of W containing gp^{-1} and let $t \in T$ be the element that satisfies

$$gp^{-1} = n_i t.$$

T is a subgroup of P so let $p^{-1}t^{-1} = p' \in P$ and we have

$$\begin{aligned} \rho_\mu &= g^{-1} \cdot \rho_\nu \\ &= (p^{-1}t^{-1}n_i^{-1}) \cdot \rho_\nu \\ &= p' \cdot (n_i^{-1} \cdot \rho_\nu). \end{aligned}$$

Furthermore, as ρ_μ is an arbitrary element of $R \cap (G \cdot \rho_\nu)$ we have

$$R \cap (G \cdot \rho_\nu) \subset \bigcup_{i=1}^{l} P \cdot (n_i^{-1} \cdot \rho_\nu),$$

where $l = |W|$.

Therefore, a G-conjugacy class of R is contained in a union of at most l P-conjugacy classes. Thus it is clear that if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes.

The converse is trivial. $\square$

Although G has infinitely many parabolic subgroups there are only finitely many G-conjugacy classes of parabolic subgroups, so we can choose a finite set $\{Q_i\}$ of representatives. We choose this particular set by fixing a maximal torus $T < G$ and a Borel subgroup $B < G$ containing T. Then a set of representative parabolic subgroups of G can be sought via the root system of G with respect to T. For each Q_i there is a corresponding Levi subgroup $M_i < Q_i$ containing B. By Theorem 4.1 there are only finitely many M_i-conjugacy classes of M_i-irreducible representations $\Gamma \to M_i$ so we can fix a finite set of representatives $\{\sigma^j_{M_i}\}$.

Let $V_i = R_u(Q_i)$ be the unipotent radical of Q_i, so that $Q_i = V_i \rtimes M_i$. We define the projection $\pi_i : Q_i \to M_i$ by $\pi_i(q) = m$, where $q = vm \in Q_i$, $v \in V_i$, $m \in M_i$.

We will show that for each representation $\rho : \Gamma \to G$ there exists an element $g \in G$ such that the representation $\sigma = g \cdot \rho$ fits one of only finitely many commutative diagrams of the following form, determined by the indices i, j:

$$\begin{tikzcd} \Gamma \arrow[r, "\sigma"] \arrow[dr, "\sigma^j_{M_i}"'] & Q_i \arrow[d, "\pi_i"] \\ & M_i \end{tikzcd}$$

We call this construction a *standard commutative diagram* for ρ.

Let $\rho : \Gamma \to G$ be a representation and let P be a minimal parabolic subgroup of G containing $\rho(\Gamma)$. Then there is an element $h \in G$ such that $hPh^{-1} = Q_i$ for some i. Let $\rho' = h \cdot \rho$.

Since Q_i is a minimal parabolic subgroup containing $(\pi_i \circ \rho)(\Gamma)$, ρ'_0 is M_i-irreducible [reference]. Hence there exists an $m \in M_i$ such that

$$m \cdot \rho'_0 = \sigma^j_{M_i},$$

for some j. Let $g = mh \in G$ and define $\sigma = g \cdot \rho$. This verifies what we set out to show.

It is worth pointing out that the element $g \in G$ and the minimal parabolic $P < G$ used in the construction are not necessarily unique, hence the qualifier "*a* standard commutative diagram". As an extreme example, if ρ is the trivial representation then ρ has minimal parabolic $P = B$, and any $g \in G$ could be used to conjugate $\rho(\Gamma)$ into $Q_i = B$. [Example of more than one minimal parabolic?]

For a given parabolic subgroup P of G with Levi subgroup L and unipotent radical V, and a given representation $\rho : \Gamma \to P$ we have a map $\rho_L : \Gamma \to L$ defined by $\rho_L = \pi \circ \rho$. Now define $\alpha_\rho : \Gamma \to V$ by $\alpha_\rho(\gamma) = \rho(\gamma)\rho_L(\gamma)^{-1}$ for all $\gamma \in \Gamma$, so that $\rho = \alpha_\rho \rho_L$.

If ρ is a homomorphism then

$$\begin{aligned} \alpha_\rho(\gamma_1\gamma_2)\rho_L(\gamma_1\gamma_2) = \rho(\gamma_1\gamma_2) &= \rho(\gamma_1)\rho(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1}\rho_L(\gamma_1)\rho_L(\gamma_2) \\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1}\rho_L(\gamma_1\gamma_2), \end{aligned}$$

so that

$$\begin{aligned} \alpha_\rho(\gamma_1\gamma_2) &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1} \\ &= \alpha_\rho(\gamma_1)\left(\gamma_1 \cdot \alpha_\rho(\gamma_2)\right), \end{aligned}$$

where Γ acts on V by conjugation via ρ_L. Therefore α_ρ satisfies the (multiplicative) 1-cocycle condition in (3.2) and so $\alpha_\rho \in Z^1(\Gamma, \rho_L, V)$.

Conversely given a 1-cocycle $\alpha \in Z^1(\Gamma, \rho_L, V)$ we can construct a representation $\rho : \Gamma \to P$ by $\rho(\gamma) = \alpha(\gamma)\rho_L(\gamma)$ for all $\gamma \in \Gamma$.

Given a representation $\rho : \Gamma \to P$, define $Hom(\Gamma, P)_{\rho_L}$ to be the set of representations $\sigma : \Gamma \to P$ such that $\sigma_L = \rho_L$. We formalise the above findings in the following Lemma:

Lemma 4.3. *The map* $h : Hom(\Gamma, P)_{\rho_L} \to Z^1(\Gamma, \rho_L, V)$ *defined by*

$$(h(\sigma))(\gamma) = \sigma(\gamma)\rho_L(\gamma)^{-1},$$

is bijective.

For ease of notation we will often write $h(\sigma)$ as α_σ. Also, since h is bijective we do no harm to use the otherwise suggestive notation α_σ when picking elements from $Z^1(\Gamma, \rho_L, V)$.

Let $v \in V$ and $\sigma \in Hom(\Gamma, P)_{\rho_L}$. Since L normalizes V, $\pi \circ (v \cdot \sigma) = \sigma_L = \rho_L$ and so $v \cdot \sigma \in Hom(\Gamma, P)_{\rho_L}$. Thus V acts on $Hom(\Gamma, P)_{\rho_L}$.

Denote by σV an element of $Hom(\Gamma, P)_{\rho_L}/V$ containing $\sigma \in Hom(\Gamma, P)_{\rho_L}$ and $[\alpha_\sigma]$ an element of $H^1(\Gamma, \rho_L, V)$ containing $\alpha_\sigma \in Z^1(\Gamma, \rho_L, V)$. We show that h gives rise to a bijection $\bar{h} : Hom(\Gamma, P)_{\rho_L}/V \to H^1(\Gamma, \rho_L, V)$.

Lemma 4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} Hom(\Gamma, P)_{\rho_0} & \xrightarrow{h} & Z^1(\Gamma, \rho_L, V) \\ \downarrow & & \downarrow \\ Hom(\Gamma, P)_{\rho_0}/V & \xrightarrow{\bar{h}} & H^1(\Gamma, \rho_L, V). \end{array}$$

Furthermore, $\bar{h}$ *is bijective.*

Proof. Suppose $\sigma V = \tau V$ for some $\sigma, \tau \in Hom(\Gamma, P)_{\rho_L}$, that is to say $\sigma = v \cdot \tau$ for some $v \in V$. Then for all $\gamma \in \Gamma$

$$\begin{aligned} \alpha_\sigma(\gamma) &= \sigma(\gamma)\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)v^{-1}\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)\rho_L(\gamma)^{-1}\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)\rho_L(\gamma)^{-1}(\gamma \cdot v^{-1}) \\ &= v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}). \end{aligned}$$

Hence $[h(\sigma)] = [h(\tau)]$ and so $\overline{h}$ is well-defined.

Conversely, suppose $[\alpha_\sigma] = [\alpha_\tau]$ for some $\alpha_\sigma, \alpha_\tau \in Z^1(\Gamma, \rho_L, V)$, so there exists a $v \in V$ such that

$$\alpha_\sigma(\gamma) = v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}).$$

Then the corresponding representations $\sigma, \tau \in Hom(\Gamma, P)_{\rho_L}$ are V-conjugate:

$$\begin{aligned}
\sigma(\gamma) &= \alpha_\sigma(\gamma)\rho_L(\gamma) \\
&= v\alpha_\tau(\gamma)(\gamma \cdot v^{-1})\rho_L(\gamma) \\
&= v\alpha_\tau(\gamma)\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1}\rho_L(\gamma) \\
&= v \cdot \tau(\gamma).
\end{aligned}$$

That is to say $\sigma V = \tau V$ and so $\bar{h}$ is invertible, hence bijective. □

More generally, we can conjugate $\sigma \in Hom(\Gamma, P)_{\rho_L}$ by $g \in G$ to get an element

$$g \cdot \sigma \in Hom(\Gamma, gPg^{-1})_{g \cdot \rho_L},$$

and

$$\alpha_{g \cdot \sigma} \in Z^1(\Gamma, g \cdot \rho_L, gVg^{-1}),$$

under h.

If $g \in P$ then $g = vl$ for some $v \in V$ and some $l \in L$, and since $gPg^{-1} = P$ and $gVg^{-1} = V$, conjugating gives rise to the maps

$$Hom(\Gamma, P)_{\rho_L} \to Hom(\Gamma, P)_{l \cdot \rho_L},$$

and

$$Z^1(\Gamma, \rho_L, V) \to Z^1(\Gamma, l \cdot \rho_L, V),$$

again, via h.

Furthermore, if $l \in Z(L)$ then $l \cdot \rho_L = \rho_L$. Indeed $Z(L)$ acts on $Hom(\Gamma, P)_{\rho_L}$ and on $Z^1(\Gamma, \rho_L, V)$ in the following way

$$\begin{aligned}
(z \cdot \sigma)(\gamma) &= z\sigma(\gamma)z^{-1} \\
(z \cdot \alpha_\sigma)(\gamma) &= z\sigma_\alpha(\gamma)z^{-1}.
\end{aligned}$$

Indeed, h is $Z(L)$-equivariant:

$$\begin{aligned}
h(z \cdot \sigma)(\gamma) &= z\sigma(\gamma)z^{-1}\rho_L(\gamma)^{-1} \\
&= z\sigma(\gamma)\rho_L(\gamma)^{-1}z^{-1} \\
&= z \cdot h(\sigma)(\gamma).
\end{aligned}$$

We show that the $Z(L)$-action on $Hom(\Gamma, P)_{\rho_L}$ and $Z^1(\Gamma, \rho_L, V)$ descends to give a $Z(L)$-action on $Hom(\Gamma, P)_{\rho_L}/V$ and $H^1(\Gamma, \rho_L, V)$, respectively. The actions will be well-defined a consequence of the fact that L normalizes V.

Let $z \in Z(L)$ and $\sigma V \in Hom(\Gamma, P)_{\rho_L}$. We define the $Z(L)$-action on $Hom(\Gamma, P)_{\rho_L}$ so that the projection to $Hom(\Gamma, P)_{\rho_L}/V$ is $Z(L)$-equivariant:

$$z \cdot \sigma V = (z \cdot \sigma)V.$$

Suppose $\sigma V = \tau V$. Then there is a $v \in V$ such that $\sigma = v \cdot \tau \in Hom(\Gamma, P)_{\rho_L}$, and a $v' \in V$ such that $zv = v'z$. Therefore

$$\begin{aligned} z \cdot \sigma V &= (z \cdot \sigma)V \\ &= (zv \cdot \tau)V \\ &= (v'z \cdot \tau)V \\ &= (z \cdot \tau)V \\ &= z \cdot \tau V. \end{aligned}$$

Hence the action is well-defined.

Similarly, the $Z(L)$-action on $H^1(\Gamma, \rho_L, V)$ is defined so that the projection to the 1-cohomology is $Z(L)$-equivariant:

$$z \cdot [\alpha_\sigma] = [z \cdot \alpha_\sigma].$$

Suppose $[\alpha_\sigma] = [\alpha_\tau]$. Then there is a $v \in V$ such that for all $\gamma \in \Gamma$,

$$\alpha_\sigma(\gamma) = v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}),$$

and there is a $v' \in V$ such that $zv = v'z$. Therefore, for all $\gamma \in \Gamma$

$$\begin{aligned} z \cdot [\alpha_\sigma(\gamma)] &= [z \cdot \alpha_\sigma(\gamma)] \\ &= [z \cdot (v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}))] \\ &= [zv\alpha_\tau(\gamma)(\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1})z^{-1}] \\ &= [v'z\alpha_\tau(\gamma)z^{-1}(\rho_L(\gamma)v'^{-1}\rho_L(\gamma)^{-1})] \\ &= [v'(z\alpha_\tau(\gamma)z^{-1})(\gamma \cdot v'^{-1})] \\ &= [z \cdot \alpha_\tau(\gamma)] \\ &= z \cdot [\alpha_\tau(\gamma)]. \end{aligned}$$

Hence the action is well-defined.

Since h is $Z(L)$-equivariant, it follows that $\bar{h}$ is also:

$$\begin{aligned} \bar{h}(z \cdot \sigma V) &= [\alpha_{z\cdot\sigma}] \\ &= [z \cdot \alpha_\sigma] \\ &= z \cdot [\alpha_\sigma] \\ &= z \cdot \bar{h}(\sigma V). \end{aligned}$$

Hence the following Lemma:

Lemma 4.5. *The bijection $h : Hom(\Gamma, P)_{\rho_L} \to Z^1(\Gamma, \rho_L, V)$ gives rise to a bijection*

$$\tilde{h} : Hom(\Gamma, P)_{\rho_L}/VZ(L) \to H^1(\Gamma, \rho_L, V)/Z(L).$$

Proof. It remains to show that $\tilde{h}$ is bijective. □

Lemma 4.6. *Let $R = \{\rho_\lambda : \Gamma \to P \,|\, \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ. Given an irreducible representation $\sigma_L : \Gamma \to L$ define*

$$R_{\sigma_L} = \{\rho \in R \,|\, \rho_L = \sigma_L\}.$$

The following statements are equivalent:

(i) R is contained in a finite union of P-conjugacy classes.

(ii) For each irreducible representation $\sigma_L : \Gamma \to L$, R_{σ_L} is contained in a finite union of $VZ(L)^\circ$-conjugacy classes.

(iii) For each irreducible representation $\sigma_L : \Gamma \to L$,

$$\tilde{h}(R_{\sigma_L}/VZ(L)^\circ) \subset H^1(\Gamma, \sigma_L, V)/Z(L)^\circ$$

is finite.

Proof.

$(i) \Rightarrow (ii)$ Assume R is contained in a finite union of P-conjugacy classes and fix an irreducible representation $\sigma_L : \Gamma \to L$. Then R_{σ_L} is contained a finite union of P-conjugacy classes. Take $\rho \in R_{\sigma_L}$ and suppose that $p \cdot \rho \in R_{\sigma_L}$ for some $p \in P$. Writing

$p = vl$ for some $v \in V$ and some $l \in L$, $(vl) \cdot \rho \in R_{\sigma_L}$ implies that in fact $l \in C_L(\sigma_L(\Gamma))$. Furthermore, since σ_L is irreducible it follows that $C_L(\sigma_L(\Gamma))/Z(L)^\circ$ is finite [reference], so we can choose a finite set $\{c_1, \ldots, c_m\}$ of coset representatives for $C_L(\sigma_L(\Gamma))/Z(L)^\circ$. Therefore

$$R_{\sigma_L} \cap (P \cdot \rho) \subset \bigcup_{i=1}^{m} VZ(L)^\circ \cdot (c_i \cdot \rho)\,.$$

$(ii) \Rightarrow (i)$ Assume that for each irreducible representation $\sigma_L : \Gamma \to L$, R_σ is contained in a finite union of $VZ(L)^\circ$-conjugacy classes. Denote by $Hom(\Gamma, L)_{irr}$ the collection of all irreducible representations from $\Gamma \to L$. By Theorem 4.1 we can choose a finite set $S \subset Hom(\Gamma, L)_{irr}$ such that

$$Hom(\Gamma, L)_{irr} = \bigcup_{\sigma \in S} L \cdot \sigma.$$

Hence

$$R = \bigcup_{\sigma \in S} \{R_{l \cdot \sigma} \,|\, l \in L\}.$$

Fix a $\sigma \in S$ and take $\rho_1, \rho_2 \in \{R_{l \cdot \sigma} \,|\, l \in L\}$. Then there exists an $l \in L$ such that $\rho_1, l \cdot \rho_2 \in R_{\sigma_L}$ for some $\sigma_L \in Hom(\Gamma, L)_{irr}$. Since we assumed there are only finitely many $VZ(L)^\circ$-conjugacy classes of R_{σ_L} we may assume ρ_1 and $l \cdot \rho_2$ are $VZ(L)^\circ$-conjugate. Hence ρ_1 and ρ_2 are P-conjugate.

$(ii) \Leftrightarrow (iii)$ This follows directly from the fact that $\tilde{h}$ is a bijection (Lemma 4.5).

□

Theorem 4.7. *Let $R = \{\rho_\lambda : \Gamma \to G \,|\, \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ and define $M_i < Q_i < G$ and $\sigma^j_{0,i}$ as in [reference]. Then R^G is a finite union of G-conjugacy classes if and only if for each i, j the subset of $H^1(\Gamma, \sigma^j_{0,i}, V_i)/Z(L)^\circ$ arising from R is finite.*

Theorem 4.8. *Let Γ be a finite (or algebraic) group and G be an algebraic group over an algebraically closed field k of characteristic p. Define $M_i < Q_i < G$ and $\sigma^j_{0,i} : \Gamma \to M_i$ as in [reference]. The answer to (the algebraic version of) Külshammer's second question is positive if and only if each map*

$$H^1(\Gamma, \sigma^j_{0,i}, V_i) \to H^1(\Gamma_p, \sigma^j_{0,i}, V_i)$$

is injective.

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G. The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p. Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G.

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α:

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r \quad : \quad & SL_2(k) \to SL_2(k) \\ & (A_{ij}) \mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r \quad = \quad \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) &= \prod_\delta \epsilon_\delta\left((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_\delta\right) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) &= \prod_\delta n_\alpha \epsilon_\delta\left(\lambda_\delta\right) n_\alpha^{-1}, \end{aligned}$$

where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ and λ_δ are elements of the underlying field k.

Lemma 5.1.

$$\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta\left(u\right)\right),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_\delta \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ. Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \to k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

Therefore

$$x_\delta\left(t^2u\right) = (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta(u).$$

Since x_δ is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle \delta, \alpha \rangle \geq 0$. However, if $\langle \delta, \alpha \rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle \delta, \alpha \rangle > 0$. □

Next we prove a couple of useful facts about root systems not containing G_2 or C_3.

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle \alpha, \beta \rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β. Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. □

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3. Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ $(i = 1, 2)$. If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ, $(i = 1, 2)$ and let θ_3 be the absolute value of the angle between δ_1 and δ_2. Then

$$\begin{aligned} & \langle \delta_i, \gamma \rangle > 0 \qquad (i = 1, 2) \\ \implies & (\delta_i, \gamma) > 0 \\ \implies & \cos(\theta_i) > 0 \\ \implies & \theta_i < \pi/2, \end{aligned}$$

and similarly, using Lemma 5.2

$$\begin{aligned} & \langle \delta_1, \delta_2 \rangle \leq 0 \\ \implies & \theta_3 \geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

$$\begin{array}{llll} \mathbf{1:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = 2\pi/3; \\ \mathbf{2:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \mathbf{3:} & \theta_1 = \pi/4, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \mathbf{4:} & \theta_1 = \pi/4, & \theta_2 = \pi/4, & \theta_3 = \pi/2. \end{array}$$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ, and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3. [Picture?][Wow, is that right? Maybe just say 'we will show that they lie in a C_3 subsystem'.] $\square$

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3.

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \epsilon_\alpha(u_1^{p^r}) \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By Lemma 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α. $\square$

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α.*

Proof. First consider

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta(b)\right).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By Lemma 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$ $(i = 1, 2)$, so Lemma 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix}\right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \\
&= \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^{r}} x_{\delta}(b)\right).
\end{aligned}$$

□

Since the x_δ are homomorphisms from $k \to k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k. Furthermore, combining the calculation in the proof of Lemma 5.1 with the result in Corollary 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta}\left(x_{\delta}\left(a^2 b\right)\right) = \prod_{\delta} \epsilon_{\delta}\left(a^{\langle \delta, \alpha \rangle p^{r}} x_{\delta}(b)\right),$$

severely restricting the possible polynomials x_δ. In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer r in the definition of ρ_r. For suppose x_δ and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned}
\mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\
\implies 2p^j &= \langle \delta, \alpha \rangle p^r.
\end{aligned}$$

In [8] it is shown that the possible pairings of any two roots are bounded by ± 3. Hence by Lemma 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3. It is now clear that if $\langle \delta, \alpha \rangle = 3$ then $x_\delta = 0$.

If $\langle \delta, \alpha \rangle = 1$ the characteristic of k must be 2 and $j = r - 1$. Otherwise $\langle \delta, \alpha \rangle = 2$ and $j = r$, but the characteristic of k is so far unrestricted.

Example 5.1. *Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

We will write v in V_α in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle \;\; := \;\; \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{3\alpha+\beta}(v_4)\epsilon_{3\alpha+2\beta}(v_5) \in V_\alpha$$

The group law for V_α is

$$u * v \quad = \quad \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v \quad = \quad \langle a^{-3p^r} v_1, a^{-p^r} v_2, a^{p^r} v_3, a^{3p^r} v_4, v_5 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), x_4(b), 0 \rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2 b) &= a^{p^r} x_3(b) \\ x_4(a^2 b) &= a^{3p^r} x_4(b). \end{aligned}$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} \quad = \quad \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1 + b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1 + b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r} x_3(b_2). \end{aligned}$$

We see that x_3 is a homomorphism, so it is of the form

$$x_3(b) \quad = \quad \sum_i \mu_i b^{p^i}.$$

Suppose $x_3 \neq 0$. Then some $\mu_j \neq 0$ and

$$\begin{aligned} & \mu_j(a^2b)^{p^j} = a^{p^r}\mu_j b^{p^j} \\ \implies \; & a^{2p^j} = a^{p^r} \\ \implies \; & p = 2. \end{aligned}$$

But then

$$\begin{aligned} x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r}x_3(b) \\ &= b^{2^r}x_3(b), \end{aligned}$$

implies that x_3 is constant, hence zero.

Therefore $x_3 = 0$, so x_4 is a homomorphism:

$$x_4(b) = \sum_i \nu_i b^{p^r}.$$

If $x_4 \neq 0$ then there is a $\nu_j \neq 0$ and we get

$$\begin{aligned} & \nu_j(a^2b)^{p^j} = a^{3p^r}\nu_j b^{p^j} \\ \implies \; & a^{2p^j} = a^{3p^r} \\ \implies \; & 2p^j = 3p^r, \end{aligned}$$

which implies that 2 divides p and 3 divides p, a contradiction. Hence $x_4 = 0$ and

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Example 5.2. *Let $G = C_3$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta, \gamma\}$ with γ being the long root and connected to β. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle.$$

Again we will write v in V_α in angled brackets for ease of notation:

$$\begin{aligned} &\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\rangle := \\ &\epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8) \in V_\alpha \end{aligned}$$

The group law for V_α *is*

$$u * v =$$
$$\langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3v_2, u_6 + v_6 + u_2v_1^2 + 2u_4v_1,$$
$$u_7 + v_7 + u_2u_3v_1 + u_2v_1v_3 + u_5v_1 + u_4v_3, u_8 + v_8 - u_3^2v_2 - 2u_3v_2v_3 + 2u_5v_3\rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-p^r}v_1, v_2, a^{p^r}v_3, a^{-p^r}v_4, a^{p^r}v_5, a^{-2p^r}v_6, v_7, a^{2p^r}v_8\rangle.$$

Let σ *be in* $Z^1(SL_2, V_\alpha)$ *such that*

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b)\rangle.$$

Applying σ *to both sides of the identity*

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r}x_3(b) \\ x_5(a^2b) &= a^{p^r}x_5(b) \\ x_8(a^2b) &= a^{2p^r}x_8(b). \end{aligned}$$

Since the polynomials x_3, x_5, x_8 *are homomorphisms (Lemma 5.2) we get*

$$\begin{aligned} \sum_i \lambda_i(a^2b)^{p^i} &= a^{p^r}\sum_i \lambda_i b^{p^i} \\ \sum_i \mu_i(a^2b)^{p^i} &= a^{p^r}\sum_i \mu_i b^{p^i} \\ \sum_i \nu_i(a^2b)^{p^i} &= a^{2p^r}\sum_i \nu_i b^{p^i}, \end{aligned}$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of SL_2 under σ is $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma}\rangle$ then the characteristic of k must be 2, and so the image is a product of commuting root groups.

We may now state and prove the main result.

[would like]

Theorem 5.6. *Let G be a reductive linear algebraic group over a closed field of positive characteristic p and let $\Gamma = SL_2(k)$. Then the answer to the algebraic interpretation of Külshammer's Second Problem [ref] is "yes".*

Proof. Need to:

- handle arguments above with G possibly containing G_2 and C_3.
- drop the restriction of rank-1 parabolics
- now we have abelian 1-cohomology and can apply result from previous chapter

□

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p. We label the positive roots for B_2 as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [5, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α:

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\alpha$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha, \end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r: write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$
\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(-u^{p^r})\epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_\alpha(-u^{p^r})\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(-2u^{p^r}v_2)\epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r})\epsilon_\alpha(-u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{2p^r}v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r}v_2) \\
&= \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2 - u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r}v_2 + u^{2p^r}v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r}v_1 \\ v_3 - 2u^{p^r}v_2 + u^{2p^r}v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r}))v_3\right) \\
&= \epsilon_\beta\left((t^{p^r})^{\langle\beta,\alpha\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\alpha\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\alpha\rangle}v_3\right) \\
&= \begin{pmatrix} t^{-2p^r}v_1 \\ v_2 \\ t^{2p^r}v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1} \\
&= n_\alpha\epsilon_\beta(v_1)n_\alpha^{-1}n_\alpha\epsilon_{\alpha+\beta}(v_2)n_\alpha^{-1}n_\alpha\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1)\epsilon_{\alpha+\beta}(-v_2)\epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3)\epsilon_{\alpha+\beta}(-v_2)\epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}
$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V:

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r}v_1 - 2(cd)^{p^r}v_2 + c^{2p^r}v_3 \\ (ad+bc)^{p^r}v_2 - (bd)^{p^r}v_1 - (ac)^{p^r}v_3 \\ b^{2p^r}v_1 - 2(ab)^{p^r}v_2 + a^{2p^r}v_3 \end{pmatrix}
$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*. Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.1}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.2}$$

to get further information on the polynomials p_i $(i = 1, 2, 3)$.

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

That is,

$$
\begin{aligned}
p_1(t^2u) &= t^{-2p^r}p_1(u) && (5.3)\\
p_2(t^2u) &= p_2(u) && (5.4)\\
p_3(t^2u) &= t^{2p^r}p_3(u). && (5.5)
\end{aligned}
$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k. Now notice that on the left hand side of (5.3) there are only non-negative powers of t, and on the right hand side there are only non-positive powers of t. This equality is only satisfied if $p_1(x) = 0$ for all x in k, so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$
\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right).
\end{aligned}
$$

That is,

$$
\begin{aligned}
p_2(u_1+u_2) &= p_2(u_1)+p_2(u_2) && (5.6)\\
p_3(u_1+u_2) &= p_3(u_1)+p_3(u_2) - 2u_1^{p^r}p_2(u_2). && (5.7)
\end{aligned}
$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$
p_3(u_1+u_2) = p_3(u_1)+p_3(u_2).
$$

Hence p_3 is a homomorphism, that is, of the form

$$
p_3(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{5.8}
$$

for some u_i in k.

Now combining (5.5) and (5.8) yields

$$
\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{2p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{5.9}
$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating the coefficients of u in (5.9) we get

$$\begin{aligned} \mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies \quad p^l &= p^r. \end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r.

Letting $\mu_l = \mu$ in k, we have

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}. \end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k, p, is not equal to 2 then by setting $\mathbf{v}$ in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
\chi_v\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
&= \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right).
\end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2. By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \to V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \tag{5.10}$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\ 0\\ \mu\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ \mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ 0\\ \mu\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\mu\\ \mu\\ \mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ 0\\ \mu\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\mu+\mu'\\ \mu\\ \mu\end{pmatrix}\\
&= \begin{pmatrix}0\\ 0\\ \mu\end{pmatrix} + \begin{pmatrix}\mu+\mu'\\ \mu'\\ \mu'\end{pmatrix} = \begin{pmatrix}\mu+\mu'\\ \mu'\\ \mu+\mu'\end{pmatrix}.
\end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}t^{-1} & 0\\ 0 & t\end{pmatrix},$$

we get

$$\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right).$$

Therefore $\sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ is an element of V that is fixed by the action of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix}0\\ *\\ 0\end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

If $c = 0$ then we already have

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r}\mu(cd)^{2^r} \\ (ac^{-1})^{2^{r+1}}\mu(cd)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1+ad)^{2^r} \\ \mu(ac^{-1})^{2^r}(1+ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

In fact, we see that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering $\mathbf{v}$ that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$ where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a $\mathbf{v}$ in V such that

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \to P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α. Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \,|\, x \in k\rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma\rangle = 0. \tag{5.11}$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n, we have

$$\langle \alpha, \gamma\rangle = \langle \alpha, m\alpha + n\beta\rangle \tag{5.12}$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta\rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha\rangle &= 0 \\ \iff m\langle \alpha, \alpha\rangle + n\langle \beta, \alpha\rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \,|\, x \in k\rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s}\cdot\sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\ &= \epsilon_\beta\left(s^{\langle\beta,\alpha+\beta\rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle\alpha+\beta,\alpha+\beta\rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta,\alpha+\beta\rangle}\mu(ab)^{2^r}\right) \\ &= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \to V$ collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V-conjugacy to P_α-conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}\,. \end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β:

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\beta$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows. Let $\mathbf{v}, \mathbf{w}$ in V. We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\ &= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r. Let $\mathbf{v}$ be an element of V:

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*, and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k.

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{5.13}$$

$$\begin{pmatrix} 1 & u_1+u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{5.14}$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \tag{5.15}$$

$$p_2(t^2u) = t^{p^r} p_2(u) \tag{5.16}$$

$$p_3(t^2u) = p_3(u). \tag{5.17}$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k. From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)+p_2(u_2)\\ 2\lambda\end{pmatrix}
\end{aligned}$$

That is,

$$p_2(u_1+u_2) = p_2(u_1)+p_2(u_2) \tag{5.18}$$

$$\lambda = 2\lambda. \tag{5.19}$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{5.20}$$

for some μ_i in k.

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{5.21}$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
\mu_l t^{2p^l} &= \mu_l t^{p^r}\\
\Longrightarrow\quad 2p^l &= p^r.
\end{aligned}$$

Thus 2 divides p^r, and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r, and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right) + \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right) + \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\right) + \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix} + \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix} + \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\begin{pmatrix}0\\\mu\\0\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix} + \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix} + \begin{pmatrix}\mu\\\mu\\\mu^2\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix} + \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}\mu'+\mu\\\mu\\\mu^2\end{pmatrix}\\
&= \begin{pmatrix}0\\\mu\\0\end{pmatrix} + \begin{pmatrix}\mu'+\mu\\\mu'\\\mu'^2\end{pmatrix}\\
&= \begin{pmatrix}\mu'+\mu\\\mu'+\mu\\\mu'^2\end{pmatrix}.
\end{aligned}$$

Since $\sigma\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\right)$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$ for all t in k^* we must have $\mu' = \mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r}\mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{p^r}\left(\mu(cd)^{2^{r-1}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ac^{-1} + a^2c^{-1}d\right)^{2^{r-1}} \\ \mu^2\left(1 + ad\right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a $\mathbf{v}$ in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix},
\end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$, where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
(\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)\,(ab)^{2^{r-1}} \\ (s\mu)^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

our infinitely many V-conjugacy classes collapse to just two P_β-conjugacy classes:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\}, \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}.
\end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\to Im(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ.

1) Know about non G-cr in B_2, can I put them in an A_1A_1?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $Im(\rho_{r,s})$, can we conjugate it into P_γ or P_δ?

Let char$(k) = 2$ and set $V := \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma+\delta, \phi \neq \gamma+2\delta\}\rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)$ $\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} \quad = \quad \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\begin{aligned}
\rho_{r,s}\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\
\rho_{r,s}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma+\delta)^\vee(t^{2^s}) \\
\rho_{r,s}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta}
\end{aligned}$$

from which we obtain an action of SL_2 on V:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so we let

$$\sigma\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each p_i $(1 \leq i \leq 12)$ is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1,6,7,12 \\ t^{-2^{r+1}} p_i(u), & i = 4,5 \\ t^{-2^{s+1}} p_i(u), & i = 2,3 \\ t^{2^{r+1}} p_i(u), & i = 8,9 \\ t^{2^{s+1}} p_i(u), & i = 10,11 \end{cases} \tag{5.22}$$

It is clear that for $i = 1,6,7,12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t, p_i must be the zero polynomial for $i = 2,3,4,5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned}
p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\
p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\
p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\
p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j},
\end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned}
\sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\
\Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\
\Rightarrow \quad l &= r.
\end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned}
p_8(u) = w u^{2^r}, \quad p_9(u) = x u^{2^r}, \\
p_{10}(u) = y u^{2^s}, \quad p_{11}(u) = z u^{2^s},
\end{aligned}$$

for some w, x, y, z in k.

So, we have

$$\sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p_i'(u)$ for all u in k, to get

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k.

From this we deduce that

$$\begin{aligned}
\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1&1\\0&1\end{pmatrix} * \begin{pmatrix}1&1\\0&1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1&0\\1&1\end{pmatrix} * \begin{pmatrix}1&0\\1&1\end{pmatrix}\cdot\sigma\begin{pmatrix}1&1\\0&1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}$$

Furthermore, since $\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}$, we have

$$\sigma\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}n_1\\0\\0\\0\\0\\n_6\\n_7\\0\\0\\0\\0\\n_{12}\end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k. So in fact

$$\begin{aligned} w' &= w \\ x' &= x \\ y' &= y \\ z' &= z \\ n_1 &= 0 \\ n_6 &= w+y \\ n_7 &= x+z \\ n_{12} &= wx+yz. \end{aligned}$$

Consider $\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c=0$ then we already have

$$\begin{aligned} \sigma\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \to V$ of the form

$$\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a $\mathbf{v}$ in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g.\mathbf{v}^{-1}$$

for all g in SL_2. Furthermore, τ' is conjugate to some 1-cocycle τ, where τ has the added property that

$$\tau\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some $\mathbf{v}$ in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned}
\tau\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\
&= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix}
\end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \quad \sim \quad (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P-conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ s^{-1}t^2 y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2 w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2 w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2 y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

5.4 A Non-Reductive Counterexample

In [7] a counterexample to [ref KII] is presented for a closed field k of characteristic $p = 2$ and a non-reductive algebraic group G.

Example 5.3. *Let Q be the algebraic group isomorphic to the affine space $\mathbf{A}^3$ with the group multiplication law:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1 v_1 + u_2 v_2 + u_1 v_2 \end{pmatrix}.$$

Let $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$ and $\Gamma_2 = \langle \tau \rangle$ the Sylow 2-subgroup of Γ. Γ acts on Q via

$$\tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix}$$

$$\sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}.$$

Let $G = Q \rtimes \Gamma$. Then there are infinitely many pairwise G-conjugate classes of extensions to the representation $\rho : \Gamma_2 \to G$ defined by the natural inclusion $\Gamma_2 \to \Gamma \to G$ [7, Appendix].

Proof. Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle $\alpha \in Z^1(\Gamma, Q)$ such that $\alpha|_{\langle\sigma\rangle} = 1$. Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned}
1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1 u_2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}.
\end{aligned}$$

This shows $u_1 = u_2$, so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\sigma\tau = \sigma^2$ we obtain

$$\begin{aligned}
1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
&= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
\end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle\sigma\rangle$ satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \qquad (u \in k).$$

Now we calculate the class $[\alpha_u] \in H^1(\Gamma, Q)$. Suppose $\alpha_v \sim \alpha_u$. Then there is a $q \in Q$ fixed under the action of σ, that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
\end{aligned}$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(\Gamma, Q)$ is infinite. [to finish]. □

[note how to make my example look like the Slodowy one]

It is natural to ask whether this leads to a reductive counterexample, although we can quickly verify that the answer is "not immediately". For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$\begin{aligned}
&\ldots \epsilon_\alpha(u_\alpha) \ldots \epsilon_\beta(u_\beta) \ldots \epsilon_\gamma(u_\gamma) \times \ldots \epsilon_\alpha(v_\alpha) \ldots \epsilon_\beta(v_\beta) \ldots \epsilon_\gamma(v_\gamma) \\
&= \ldots \epsilon_\alpha(u_\alpha + v_\alpha) \ldots \epsilon_\beta(u_\beta + v_\beta) \ldots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
\end{aligned}$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \alpha$ gives

$$\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) \quad = \quad \epsilon_\alpha(u_\alpha + v_\alpha)\epsilon_\gamma(u_\alpha v_\alpha),$$

which is absurd. [try find more examples]

Chapter 6

Conclusion

Appendix A

Further Calculations

G	P	Z^1	H^1	V-conj	P-conj
B_2 (α short)	P_α	✓	✓	✓	✓
	P_β	✓	✓	✓	✓
G_2 (α short)	P_α	✓			
C_3 (γ long)	P_α	✓			
[7]	$Q \rtimes SL(2,2)$	✓	✓	✓	

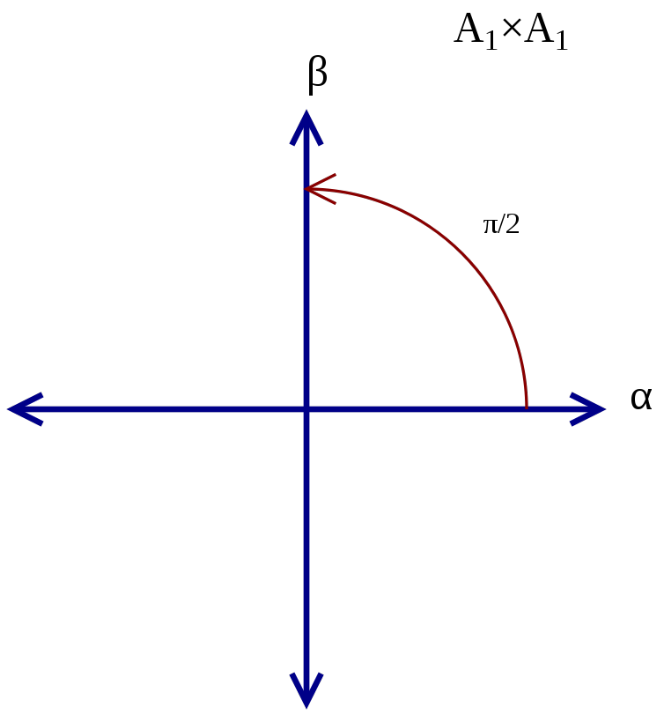
Appendix B

Source Code

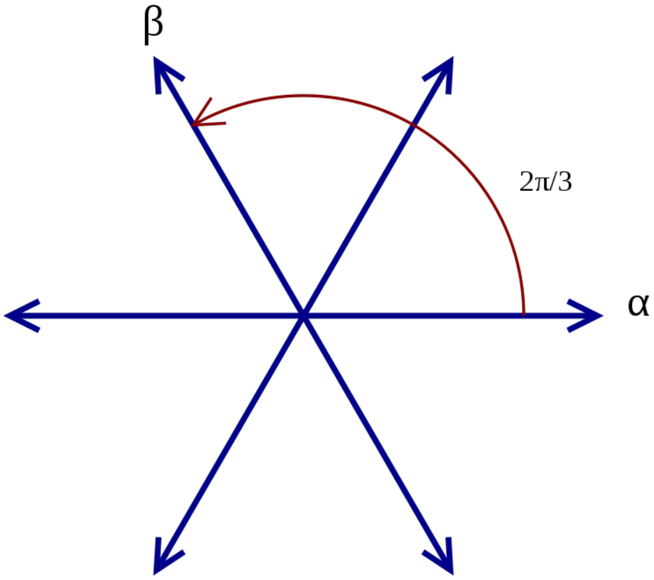
Put source code here . . .

Appendix C

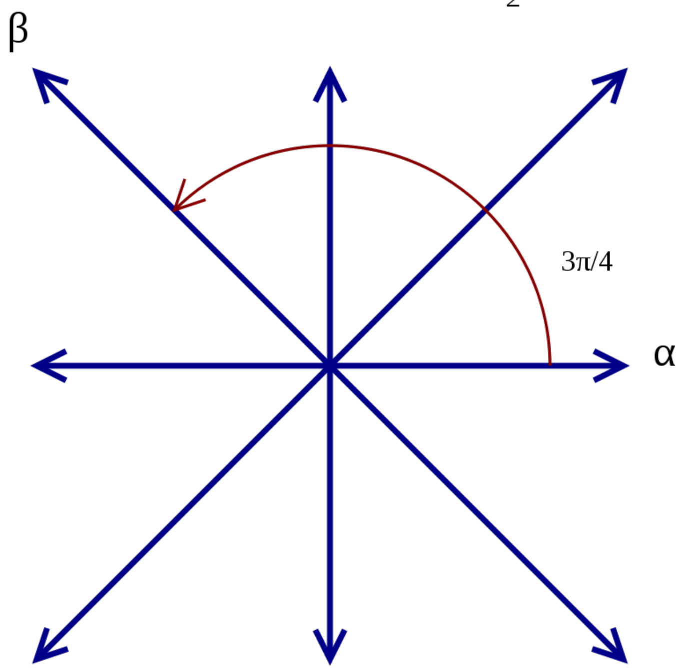
Rank 2 Root System Diagrams

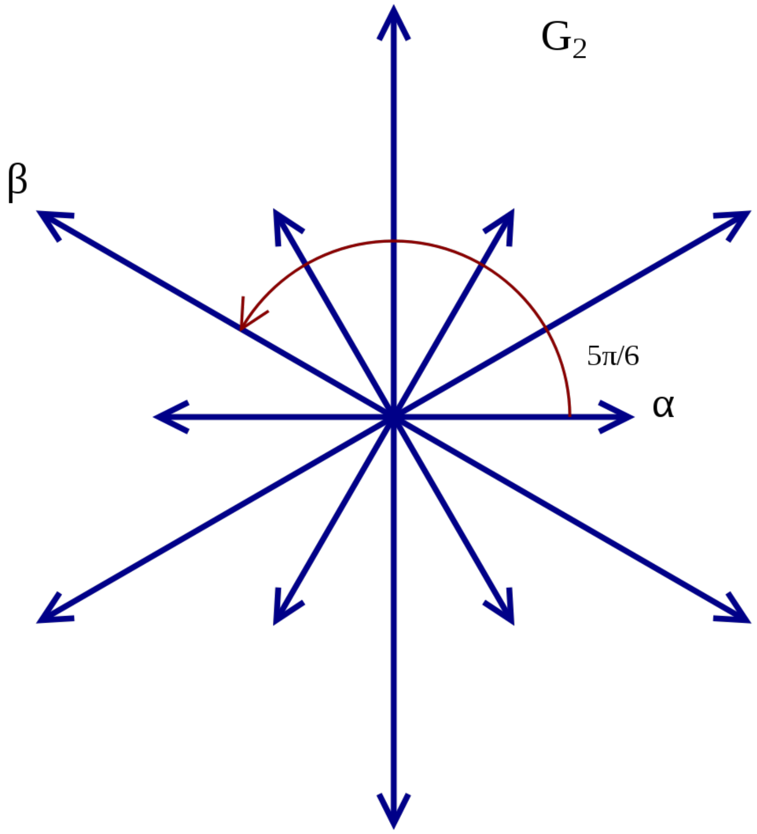


A_2



B_2




G₂
β
5π/6
α

Bibliography

[1] A. Weil. Remarks on the cohomology of groups. *Annals of Mathematics*, 80(1): 149–157, 1964.

[2] M.W. Liebeck and G.M. Seitz. *Reductive subgroups of exceptional algebraic groups.* American Mathematical Society, 1996. ISBN 0821804618.

[3] M.W. Liebeck and G.M. Seitz. *The maximal subgroups of positive dimension in exceptional algebraic groups.* American Mathematical Society, 2004. ISBN 0821834827.

[4] G.P. Hochschild. *The structure of Lie groups.* Holden-Day, 1965.

[5] J.E. Humphreys. *Linear algebraic groups.* Springer, 1975.

[6] A. Borel. *Linear algebraic groups.* Springer, 1991.

[7] P. Slodowy. Two notes on a finiteness problem in the representation theory of finite groups. *Algebraic groups and Lie groups: a volume of papers in honour of the late RW Richardson*, page 331, 1997.

[8] R.W. Carter. *Simple groups of Lie type.* John Wiley & Sons Inc, 1989.