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On Reductive Subgroups of Algebraic Groups and a Question of Külshammer

by

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Declaration of Authorship

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Abstract

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This thesis concerns embeddings of reductive groups H in a reductive group G complementing the work of M. Liebeck and G. Seitz who have determined explicitly the embeddings of arbitrary connected simple groups in G where G is of exceptional type with certain restrictions on the characteristic of the underlying field k. We work with low-characteristic (usually $p = 2$ or $p = 3$) where examples are exotic and less is know.

We also investigate a question of B. Külshammer which asks whether there can be only finitely many conjugacy classes of representations $\Gamma \to G$ from a finite group Γ which when restricted to a Sylow p-subgroup Γ_p hit some fixed class of representations $\Gamma_p \to G$. A counterexample exists for a particular non-reductive G and we concentrate on the reductive case. We provide a necessary condition for a positive answer to Külshammer's question.

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Contents

To the people of Christchurch affected by the earthquakes of 2010-2011, during which the a lot of this work was written.

Chapter 1

Introduction

A major motivation for the work carried out in this thesis is to investigate a question posed by B. Külshammer to do with homomorphisms of finite groups into algebraic groups [1]. We will call these homomorphisms *representations* because of the obvious similarity with the usual kind of representations into GL_n. Külshammer's second question is as follows:

> Let G be a linear algebraic group over an algebraically closed field of characteristic p. Let Γ be a finite group and $\Gamma_p < \Gamma$ a Sylow p-subgroup of Γ. Fix a conjugacy class of representations $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

So far only a non-reductive counterexample is known [1, Appendix]. We examine Külshammer's second question for reductive G, and we also examine a slight variation on the question which we call the *algebraic group* version of Külshammer's question in which case we substitute a connected reductive group H for the finite group Γ, and instead of a Sylow p-subgroup $\Gamma_p < \Gamma$ we use a maximal unipotent subgroup $U < H$:

> Let G, H be connected reductive linear algebraic groups over an algebraically closed field of characteristic p and $U < H$ a maximal unipotent subgroup of H. Fix a conjugacy class of algebraic group homomorphisms $U \to G$. Are there, up to conjugation by G, only finitely many algebraic group homomorphisms$\rho :$ $H \to G$ whose restrictions to U belong to the given class?

We often overload the term *representation* to also mean algebraic group homomorphism in this setting.

1.1 Külshammer's Questions and Maschke's Theorem

The algebraic group version of Külshammer's question a non-trivial pursuit in its own right as Külshammer's question has it's roots Maschke's Theorem of representation theory. Maschke's Theorem asserts that any representation from a finte group $\Gamma \to GL_n$ over a field of characteristic not dividing the order of Γ is completely reducible, and that there are only finitely many conjugacy classes of (completely reducible) representations $\Gamma \to GL_n$ [ref Lang].

Let Γ be a finite group and let G be a linear algebraic group over an algebraically closed field of characteristic p. Külshammer's first question reads:

> Suppose p does not divide the order of Γ. Are there only finitely many conjugacy classes of representations $\Gamma \to G$?

The answer is positive and is essentially contained in a paper of A. Weil [2]. Külshammer's second question is a refinement of the first:

> Let $\Gamma_p < \Gamma$ be a Sylow p-subgroup and fix a conjugacy class of representations $\Gamma_p \to G$. Are there only finitely many conjugacy classes of representations $\Gamma \to G$ whose restrictions to Γ_p belong to the fixed class?

Note that the condition that p does not divide $|\Gamma|$ is dropped from the hypothesis. If p does not divide the order of Γ then the answer is "yes", since Γ_p is trivial and so all representations are equal when restricted to Γ_p.

If Γ is a p-group then the answer is "yes", as $\Gamma_p = \Gamma$ so restricting to Γ_p does nothing and therefore only representations that come from the fixed class will hit the class.

If $G = GL_n$ and p does not divide $|\Gamma|$ the answer is also "yes", since by Maschke's Theorem there can only be finitely many conjugacy classes of representations $\Gamma \to GL_n$ anyway, regardless of whether or not their restrictions to Γ_p hit the fixed class. If p does divide $|\Gamma|$ the answer has again shown to be positive [1, Theorem].

The following example shows infinitely many conjugacy classes of representations of a finite group into $SL_2(k)$.

Example 1.1. *Let* $\Gamma = C_p \times C_p = \langle a, b \,|\, ab = ba, a^p = b^p = 1\rangle$ *and consider representations* $\rho : \Gamma \to SL_2(k)$*. In particular, for each* $\lambda \in k$ *define* $\rho_\lambda : \Gamma \to SL_2(k)$

by

$$\rho_\lambda(a) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$\rho_\lambda(b) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.$$

It is straightforward to check that if $\lambda_1 \neq \lambda_2$ then ρ_{λ_1} is not $SL_2(k)$-conjugate to ρ_{λ_2}. Hence there are infinitely many $SL_2(k)$-conjugacy classes of representations from $\Gamma \to SL_2(k)$.

1.2 Connection with the Subgroup Structure of Algebraic Groups

Our approach to Külshammer's question also means that the work in this thesis contributes to the study of the subgroup structure of simple algebraic groups, complementing some of the work done by M. Liebeck and G. Seitz ([3], [4]), and D. Stewart ([5]). Let G be a simple algebraic group over an algebraically closed field of characteristic p. For large enough characteristic ($p = 0$ or $p > 7$ covers all restrictions) Liebeck and Seitz determine explicitly the embeddings of arbitrary connected semisimple groups in G, where G is of exceptional type. We examine the subgroup structure of simple algebraic groups in low characteristic (usually $p = 2$ or $p = 3$) where examples are exotic less is known. We use similar methods to Liebeck and Seitz, calculating a certain 1-cohomology of H with coefficients in V, the unipotent radical of a parabolic subgroup $P < G$.

The main difference in our calculations is that we deal with the so-called *non-abelian* 1-cohomology directly where as Liebeck, Seitz and Stewart use results from Representation Theory to study *abelian layers* of the 1-cohomology and then piece the layers back together. Our calculations in Chapter 5 agree with Stewart's G_2 calcualtion, and we acknowledge Stewart's F_4 calculation which provided us with a good example to work with in Chapter 6.

1.3 Chapter Overview

One of our main results is Theorem 4.8. With this we are able to relate Külshammer's question to a certain 1-cohomology calculation in which Γ acts on the unipotent radical V of a parabolic subgroup $P < G$ via a certain representation $\Gamma \to L$ into a Levi subgroup

$L < P$. We show that we can reduce Külshammer's question to another question: is the restriction map of 1-cohomologies

$$H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

injective for all parabolics $P < G$?

This approach might help settle Külshammer's original question.

In Chapter 2 we produce some basic facts to do with Linear Algebraic Groups which could be found in texts such as Humphreys [**?**] and will be well-known to readers with a background in this area. This is an attempt to standardize notation and provide some background for the results to come.

In Chapter 3 we introduce the 1-cohomology, first the well-known abelian case and second the lesser-known non-abelian case. In Example 3.3 we show that the restriction map of 1-cohomologies $H^1(SL_2(k), V) \to H^1(U(k), V)$ is injective for V a vector space, which is the kind of result we can apply Theorem 4.8 to, in answering Külshammer's second question. We also show that $H^1(SL_2(k), V) \to H^1(B, V)$ is injective for B a Borel subgroup of SL_2 and V an algebraic group, not necessarliy abelian, on which SL_2 acts. The final step in applying an argument similar to Theorem 4.8 in an algebraic setting is that $H^1(B, V) \to H^1(U, V)$ is injective, where U is the unipotent radical of B. We were unable to prove this but have evidence for it in our calculations in Chapter 6.

Chapter 4 introduces the approach of finding reductive subgroups in reductive G via the 1-cohomology and finshes with Theorem 4.8 which relates Külshammer's second question to a question of restriction maps of 1-cohomologies as above.

In Chapter 5 we collect together the results of various 1-cohomology calculations for SL_2 in G, G of rank 2. They allude to a perhaps startling conjecture: although the types of 1-cohomology calculations $H^1(SL_2, V)$ involve a nonabelian V, there is evidence that a 1-cocycle in $Z^1(SL_2(k), V)$ that is zero on a maximal torus T has image lying in an abelian subgroup of V. For instance, in Corollary 5.5 we show this is the case for rank 1 parabolics of G not containing G_2 or C_3. Furthermore Examples 5.1, 5.2 verify the conjecture for $G = G_2, C_3$ respectively.

In Chapter 6 we provide two examples which demonstrate the theoretical results captured in Chapter 5.

The future directions are summarised in the final Chapter.

Chapter 2

Mathematical Preliminaries

Let k be an algebraically closed field. An affine variety over k is a subset of k^n defined by the vanishing of some polynomial equations. We have such notions as a subvariety of an affine variety, a natural product of affine varieties and maps between affine varieties.

A morphism $\phi : V \to W$ of affine varieties is a map such that the coordinates of $\phi(v) \in W$ are given by polynomial functions in $v \in V$.

An affine algebraic group G is a set G which is an affine algebraic variety and a group such that

$$\begin{aligned} \mu \ &: \ G \times G \to G \\ &\quad (x, y) \mapsto x.y, \end{aligned}$$

and

$$\begin{aligned} \iota \ &: \ G \to G \\ &\quad x \mapsto x^{-1} \end{aligned}$$

are morphisms of affine varieties.

Example 2.1. *The special linear group of $n \times n$ matrices with entries in k*

$$SL_n(k) = \{(a_{ij}) \in k^{n^2} \,|\, det(a_{ij}) - 1 = 0\}$$

is an affine variety. Furthermore, the general linear group of $n \times n$ matrices with entries in k

$$GL_n(k) = \{(a_{ij}) \in k^{n^2} \,|\, det(a_{ij}) \neq 0\}$$

is an affine variety, seen more clearly by the inclusion in the affine variety

$$GL_n(k) \subset \{(b,(a_{ij})) \in k^{n^2+1} |\, b.det(a_{ij}) - 1 = 0\}.$$

Of course both examples can be shown to be affine algebraic groups by checking the multiplication and inverse laws.

A homomorphism $\phi : G \to H$ of affine algebraic groups is a morphism of affine varieties and a homomorphism of abstract groups. An isomorphism $\phi : G \to H$ of affine algebraic groups is a bijective homomorphism of affine algebraic groups such that $\phi^{-1} : H \to G$ is also a homomorphism of affine algebraic groups.

Example 2.2. *Let p be the characteristic of k. The map $k \to k$ which sends $x \mapsto x^p$ is bijective, a morphism, but not an isomorphism since the inverse map $x \mapsto x^{1/p}$ is not a morphism of affine varieties (it is not a polynomial).*

Now let $G = GL_n(k)$. The map $F : G \to G$ which sends $(a_{ij}) \mapsto (a_{ij}^q)$, $q = p^z$, $z \in \mathbb{Z}^+$ is a homomorphism of affine algebraic groups, called the Frobenius morphism. It is not an isomorphism.

The subvarieties of an affine variety V form the closed sets of a topology, known as the Zariski topology.

A closed subgroup of an affine algebraic group is itself an affine algebraic group. A closed subgroup of $GL_n(k)$ is called a linear algebraic group. In fact every affine algebraic group is a linear algebraic group.

Example 2.3. *Three important subgroups of the linear algebraic group $G = GL_n(k)$*

$$\begin{aligned} T = T_n(k) &= \{(a_{ij}) \in GL_n(k) |\, a_{ij} = 0 \textit{ if } i \neq j\} \\ &\ \textit{diagonal matrices in } GL_n(k) \end{aligned}$$

$$\begin{aligned} U = U_n(k) &= \{(a_{ij}) \in GL_n(k) |\, a_{ii} = 1, a_{ij} = 0 \textit{ if } i < j\} \\ &\ \textit{upper unitriangular matrices in } GL_n(k) \end{aligned}$$

$$\begin{aligned} B = B_n(k) &= \{(a_{ij}) \in GL_n(k) |\, a_{ij} = 0 \textit{ if } i < j\} \\ &\ \textit{upper triangular matrices in } GL_n(k) \end{aligned}$$

T is an example of a torus of G, U is an example of a unipotent subgroup of G, and B is an example of a Borel subgroup of G.

Let G be a linear algebraic group. The irreducible components of G are disjoint. If G° is the irreducible component containing the identity element of G then G° is a (closed) normal subgroup of G of finite index. The irreducible components of G are the cosets of G° in G. G° is the smallest closed subgroup of G of finite index (every closed subgroup of finite index is open).

G° is called the identity component of G. If $G = G^\circ$ we say G is connected.

Every element $g \in G$ can be uniquely written

$$g = g_s.g_u = g_u.g_s,$$

where g_s is semisimple (diagonalisable) and g_u is unipotent. This is known as the Jordan decomposition.

G has a unique maximal closed normal solvable subgroup $R(G)$, called the radical of G. The set of unipotent elements of $R(G)$ is a maximal closed connected unipotent normal subgroup $R_u(G)$, called the unipotent radical of G.

If $R_u(G) = 1$ we say G is reductive. If $R(G) = 1$ we say G is semisimple. If G is connected and has no proper closed connected normal subgroups then G is simple.

Example 2.4. *$GL_n(k)$ is reductive. $SL_n(k)$ is semisimple (hence reductive). $SL_n(k)$ is simple as an algebraic group but not as an abstract group, since it has a non-trivial center.*

If G is nonabelian and simple then its centre $Z(G)$ is finite.

If G is a reductive linear algebraic group then

$$G = Z(G)^\circ.(G, G),$$

where

$$(G, G) = \langle [g, h] = ghg^{-1}h^{-1} | \, g, h \in G \rangle,$$

the commutator subgroup. $Z(G)$ is a torus of G and (G, G) is again reductive.

Every abelian simple algebraic group has dimension 1 and is isomorphic to either

$$G_m(k) = k^* = \text{multiplicative group of } k$$

or

$$G_a(k) = k = \text{additive group of } k.$$

A torus is isomorphic to $k^* \times k^* \cdots k^*$. Any two maximal tori in G are conjugate in G.

If G is connected with maximal torus $T < G$ then the centralizer of T in G, $C_G(T)$, is equal to the identity component of the normalizer of T in G, $N_G(T)^\circ$, and hence $N_G(T)/C_G(T)$ is finite. We call $W = N_G(T)/C_G(T)$ the Weyl group of G. Furthermore, if G is also reductive then $T = C_G(T)$ and $W = N_G(T)/T$ is a finite Coxeter group, that is, of the form:

$$W = \langle s_1, \ldots, s_l |\, s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1\rangle,$$

for some m_{ij}.

A Borel subgroup of G is a maximal closed connected solvable subgroup of G, any two Borel subgroups of G are conjugate in G. If $T < G$ is a torus of G then there exists a Borel subgroup B of G containing T. Furthermore, we can write $B = T \cdot R_u(B)$.

Let G be a reductive connected linear algebraic group with torus $T < G$. Let $N = N_G(T)$. Then we can write G as

$$G = BNB = \cup_{n \in N} BnB.$$

$BnB = Bn'B$ if and only iff $\pi(n) = \pi(n')$ where $\pi : N \to N/T = W$ so we have the correspondence $B \backslash G/B \leftrightarrow W$ where $BnB \mapsto \pi(n)$.

Suppose $W = \langle s_1, \ldots, s_l\rangle$ and let $J \subset \{1, \ldots, l\}$. We define $W_J = \langle s_j |\, j \in J\rangle < W$ and $N_J = \pi^{-1}(W_J)$. The subgroup of G defined by

$$P_J = BN_JB$$

contains B, and in fact every subgroup of G containing B is of this form. We call $P < G$ a parabolic subgroup of G if $B < P$ for some Borel subgroup $B < G$. Equivalently, P is a parabolic subgroup of G if given a maximal torus $T < G$, P is conjugate to some P_J.

A parabolic subgroup $P < G$ is connected, self-normalizing, and can be decomposed into a semi-direct product of its unipotent radical and a Levi subgroup $L < P$:

$$P = L \cdot R_u(P),$$

with $L \cap R_u(P) = 1$. Any two Levi subgroups of P are conjugate by an element of $R_u(P)$ and will be reductive if G is reductive.

Let T be a maximal torus of a connected reductive linear algebraic group G. We define the character group of T is to be

$$X = \mathrm{Hom}(T, k^*),$$

with the addition law

$$(x_1 + x_2)(t) = x_1(t)x_2(t), \qquad x_1, x_2 \in X, t \in T.$$

The cocharacter group is defined

$$Y = \mathrm{Hom}(k^*, T),$$

with the addition law

$$(y_1 + y_2)(\lambda) = y_1(\lambda)y_2(\lambda), \qquad y_1, y_2 \in Y, \lambda \in k^*.$$

If we compose $x \in X$ with $y \in Y$ we get a morphism

$$k^* \to T \to k^*,$$

that is, of the form $\lambda \mapsto \lambda^n$ for some $n \in \mathbb{Z}$. Hence there exists a pairing $\langle , \rangle : X \times Y \to \mathbb{Z}$ defined

$$(x, y) \mapsto \langle x, y \rangle = n,$$

where $x(y(\lambda)) = \lambda^n$.

Chapter 3

The 1-Cohomology

3.1 Abelian 1-Cohomology

The abelian 1-cohomology is standard. We present this section as a foundation for the less-known non-abelian theory to appear later in this chapter. We will employ additive notation for the abelian section.

3.1.1 Definitions

Let H be an algebraic group and V an abelian group on which H acts homomorphically. We call a morphism α from $H \to V$ a *1-cocycle* if it satisfies

$$\alpha(h_1 h_2) = \alpha(h_1) + h_1 \cdot \alpha(h_2), \tag{3.1}$$

for all h_1, h_2 in H. Denote by $Z^1(H, V)$ the collection of all 1-cocycles that are morphisms from $H \to V$.

We call Equation 3.1 the *1-cocycle condition.*

For any α_1, α_2 in $Z^1(H, V)$

$$\begin{aligned}
(\alpha_1 + \alpha_2)(h_1 h_2) &= \alpha_1(h_1 h_2) + \alpha_2(h_1 h_2) \\
&= \alpha_1(h_1) + h_1 \cdot \alpha_1(h_2) + \alpha_2(h_1) + h_1 \cdot \alpha_2(h_2) \\
&= (\alpha_1(h_1) + \alpha_2(h_1)) + h_1 \cdot (\alpha_1(h_2) + \alpha_2(h_2)) \\
&= (\alpha_1 + \alpha_2)(h_1) + h_1 \cdot (\alpha_1 + \alpha_2)(h_2).
\end{aligned}$$

It is easy to check that $\alpha_1 + \alpha_2$ is a morphism, so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \to V$ that sends every $h \in H$ to the identity $0 \in V$ is a 1-cocycle. Furthermore for any $\alpha \in Z^1(H,V)$ we have

$$\begin{aligned} \alpha(1) = \alpha(1 \cdot 1) &= \alpha(1) + 1 \cdot \alpha(1) \\ &= \alpha(1) + \alpha(1) \\ &= 2\,\alpha(1), \end{aligned}$$

which implies that

$$\alpha(1) = 0.$$

From this we deduce that

$$\begin{aligned} \alpha(hh^{-1}) = \alpha(1) &= 0 \\ &= \alpha(h) + h \cdot \alpha(h^{-1}), \end{aligned}$$

and so each α has a negative defined by

$$-\alpha(h) = h \cdot \alpha(h^{-1}).$$

Therefore $Z^1(H,V)$ is an abelian $\mathbb{Z}-$module under pointwise addition.

Given a $v \in V$ we define a *1-coboundary* $\chi_v^H : H \to V$ to be the morphism

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1(H,V)$ the collection of all 1-coboundaries.

For any $v \in V$ and any $h_1, h_2 \in H$

$$\begin{aligned} \chi_v^H(h_1 h_2) &= v - (h_1 h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2), \end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any $u, v \in V$ and all $h \in H$

$$\begin{aligned}(\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\ &= u - h \cdot u + v - h \cdot v \\ &= (u+v) - h \cdot (u+v) \\ &= \chi_{u+v}^H(h)\end{aligned}$$

is a 1-coboundary and a morphism, and hence $B^1(H, V)$ is also closed under pointwise addition.

We see that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a $\mathbb{Z}-$submodule of $Z^1(H, V)$, so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology*.

Lemma 3.1. *Suppose H is linearly reductive. Then $H^1(H, V)$ is trivial [6, Proposition 1].*

Example 3.1. *If V is a vector space and H acts linearly on V then $Z^1(H, V)$ is a vector space and $B^1(H, V)$ is a vector subspace.*

3.1.2 Maps between 1-cohomologies

Let $\phi : \tilde{H} \to H$ be a homomorphism, $\tilde{H}$ being another group that acts on V. Suppose that for every $h \in \tilde{H}$, ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all $v \in V$. We call such a map ϕ $\tilde{H}$-equivariant.

If α is a 1-cocycle from $H \to V$ then we will show that the map denoted $Z^1(\phi)(\alpha)$ defined by

$$Z^1(\phi)(\alpha) = \alpha \circ \phi,$$

is a 1-cocycle from $\tilde{H} \to V$. Thus $\tilde{H}$-equivariant homomorphisms of the form

$$\phi : \tilde{H} \to H$$

give rise to maps of the form

$$Z^1(\phi) : Z^1(H, V) \to Z^1(\tilde{H}, V).$$

Take $h_1, h_2 \in H$. We have

$$\begin{aligned}
\left(Z^1(\phi)(\alpha)\right)(h_1 h_2) &= (\alpha \circ \phi)(h_1 h_2) \\
&= \alpha(\phi(h_1 h_2)) \\
&= \alpha(\phi(h_1)\phi(h_2)) \\
&= \alpha(\phi(h_1)) + \phi(h_1) \cdot \alpha(\phi(h_2) \\
&= \alpha(\phi(h_1)) + h_1 \cdot \alpha(\phi(h_2)) \\
&= (\alpha \circ \phi)(h_1) + h_1 \cdot (\alpha \circ \phi)(h_2) \\
&= \left(Z^1(\phi)(\alpha)\right)(h_1) + h_1 \cdot \left(Z^1(\phi)(\alpha)\right)(h_2).
\end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \to H^1(\tilde{H}, V),$$

defined by the commutative diagram

$$\begin{array}{ccc}
Z^1(H,V) & \xrightarrow{Z^1(\phi)} & Z^1(\tilde{H},V) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\
H^1(H,V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H},V)
\end{array}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$.

Example 3.2. *Let $\tilde{H}$ be a subgroup of H and $\iota : \tilde{H} \to H$ the inclusion map. Then ι gives rise to the well-defined maps*

$$Z^1(\iota) : Z^1(H, V) \to Z^1(\tilde{H}, V),$$
$$H^1(\iota) : H^1(H, V) \to H^1(\tilde{H}, V).$$

The next Lemma is standard [7, Theorem 10.3] but we give our own proof here.

Lemma 3.2. *Let V be a vector space over a field of characteristic p. Let Γ be a finite group and $\Gamma_p < \Gamma$ a* Sylow p-subgroup *of Γ. The map*

$$H^1(\iota) : H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

is injective.

Proof. Let x be an element of $H^1(\Gamma, V)$ such that $\left(H^1(\iota)\right)(x) = 0$. Now choose a 1-cocycle $\alpha \in Z^1(\Gamma, V)$ such that $[\alpha] = x$, where $[\alpha]$ denotes the projection to the 1-cohomology. Hence $\left(Z^1(\iota)\right)(\alpha)$ is a 1-coboundary as $\left[\left(Z^1(\iota)\right)(\alpha)\right] = 0$. That is to say α restricted to Γ_p is equal to a 1-coboundary, say $\chi_v^{\Gamma_p}$. But since $\chi_v^{\Gamma_p}$ can be trivially extended to a 1-coboundary χ_v^{Γ} from $\Gamma \to V$, and

$$\left[\alpha - \chi_v^{\Gamma}\right] = x,$$

we could well have chosen the 1-cocycle $(\alpha - \chi_v^{\Gamma})$ as a representative for x. Hence there is no harm in assuming that α is 0 when restricted to Γ_p. Now choose a set of representatives $\gamma_1, \ldots, \gamma_l \in \Gamma$ for the coset space Γ/Γ_p and set

$$v^* = \sum_{i=1}^{l} \alpha(\gamma_i).$$

Consider the 1-coboundary $\chi_{v^*}^{\Gamma}$:

$$\begin{aligned} \chi_{v^*}^{\Gamma}(\gamma) &= v^* - \gamma \cdot v^* \\ &= \sum_{i=1}^{l} \alpha(\gamma_i) - \gamma \cdot \sum_{i=1}^{l} \alpha(\gamma_i) \\ &= \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \gamma \cdot \alpha(\gamma_i). \end{aligned}$$

By the 1-cocycle condition we have for all $\gamma \in \Gamma$

$$\alpha(\gamma\gamma_i) = \alpha(\gamma) + \gamma \cdot \alpha(\gamma_i),$$

for each $1 \leq i \leq l$. Therefore

$$\begin{aligned} \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \gamma \cdot \alpha(\gamma_i) &= \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \left(\alpha(\gamma\gamma_i) - \alpha(\gamma)\right) \\ &= \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \alpha(\gamma\gamma_i) + \sum_{i=1}^{l} \alpha(\gamma). \end{aligned}$$

The value of α at a fixed γ depends only on the value of α at the representative γ_j of the coset containing γ, and for two cosets $\gamma_i\Gamma_p, \gamma_j\Gamma_p$ and a fixed $\gamma \in \Gamma$

$$\gamma\gamma_i\Gamma_p = \gamma\gamma_j\Gamma_p \Leftrightarrow \gamma_i\Gamma_p = \gamma_j\Gamma_p,$$

so that $\{\gamma\gamma_i \,|\, 1 \leq i \leq l\}$ meets each coset in Γ/Γ_p exactly once. Hence we can collapse the middle term to yield

$$\begin{aligned} \chi^{\Gamma}_{v^*}(\gamma) &= \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \alpha(\gamma\gamma_i) + \sum_{i=1}^{l} \alpha(\gamma) \\ &= \sum_{i=1}^{l} \alpha(\gamma_i) - \sum_{i=1}^{l} \alpha(\gamma_i) + \sum_{i=1}^{l} \alpha(\gamma) \\ &= l\,\alpha(\gamma). \end{aligned}$$

Since $\gcd([\Gamma : \Gamma_p], p) = \gcd(l, p) = 1$, l has an inverse $l^{-1} = m$ and so

$$m\chi^{\Gamma}_{v^*}(\gamma) = \alpha(\gamma).$$

Therefore α is the 1-coboundary

$$m\chi^{\Gamma}_{v^*} = \chi^{\Gamma}_{mv^*}$$

and so the kernel of $H(\iota)$ is trivial. □

Example 3.3. *Let*

$$k = \overline{\mathbb{F}_p} = \bigcup_r \mathbb{F}_{p^r}.$$

Note that in general

$$\mathbb{F}_{p^r} \not\subset \mathbb{F}_{p^{r+1}},$$

but we do have

$$\mathbb{F}_{p^r} \subset \mathbb{F}_{p^{(r+1)!}}.$$

Let V a be vector space on which $SL_2(k)$ acts, and $U(k)$ the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Then $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ for each r, and the map

$$H^1(\iota) : H^1(SL_2(k), V) \to H^1(U(k), V)$$

is injective.

Proof. The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r}-1)(p^{2r}-p^r)$ since there are $p^{2r}-1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r}-p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \to \mathbb{F}_{p^r}^*,$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First Homomorphism Theorem for groups

$$GL_2(\mathbb{F}_{p^r}) \,/\, SL_2(\mathbb{F}_{p^r}) \simeq \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| \,/\, |\mathbb{F}_{p^r}^*| \\ &= (p^{2r}-1)(p^{2r}-p^r) \,/\, (p^r-1) \\ &= p^r(p^{2r}-1). \end{aligned}$$

Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$.

Fix a non-trivial $y \in H^1(SL_2(k), V)$ and choose a representative $\beta \in Z^1(SL_2(k), V)$ for y. For each $g \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_g : V \to V$ by

$$f_g(v) = \beta(g) - \chi_v(g) = \beta(g) - v + g \cdot v.$$

Consider sequence of subsets of V defined by

$$C_r = \{v \in V \mid f_g(v) = 0\}.$$

Each subset C_r is closed and the inclusion $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$ induces the reverse inclusion $C_{r!} \supset C_{(r+1)!}$. The Noetherian property for V requires that the sequence of subsets of V defined by

$$\{C_{i!}\}_{i=1}^{\infty}$$

becomes constant. However, $y \neq 0$ so β is not a 1-coboundary on $SL_2(k)$, which means the C_r's are eventually empty. That is, there exists an integer s such that for any v in V

$$(\beta - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if $y|_{SL_2(\mathbb{F}_{p^r})} = 0$ for all r then $y = 0$.

Take x in the kernel of the map

$$H^1(\iota) : H^1(SL_2(k), V) \to H^1(U(k), V).$$

Then for each r, $x|_{U(\mathbb{F}_{p^r})} = 0$ so by Lemma 3.2, $x|_{SL_2(\mathbb{F}_{p^r})} = 0$. Therefore $x = 0$ and so $H^1(\iota)$ is injective. $\square$

We could also consider H-equivariant maps $f : V \to \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \to H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \to H^1(\tilde{H}, \tilde{V}).$$

3.2 Non-abelian 1-Cohomology

3.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocycles be morphisms of varieties. We will employ multiplicative notation for the non-abelian setting.

3.2.2 Definitions

Let H, V be algebraic groups, H acting on V by group automorphisms. We call a morphism α from $H \to V$ a *1-cocycle* if it satisfies

$$\alpha(h_1 h_2) = \alpha(h_1)(h_1 \cdot \alpha(h_2)), \tag{3.2}$$

for all $h_1, h_2 \in H$. Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \to V$.

We call Equation 3.2 the *1-cocycle condition.* Note that unlike the previous section there is no natural addition operation on $Z^1(H, V)$ for non-abelian V.

Given a $v \in V$ we define the *1-coboundary* $\chi_v^H : H \to V$

$$\chi_v^H(h) = v(h \cdot v^{-1}),$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries. It is easy to show that 1-coboundaries are morphisms.

For any $v \in V$ and any $h_1, h_2 \in H$,

$$\begin{aligned}
\chi_v^H(h_1h_2) &= v((h_1h_2)\cdot v^{-1}) \\
&= v(h_1\cdot v)^{-1}(h_1\cdot v)((h_1h_2)\cdot v^{-1}) \\
&= v(h_1\cdot v^{-1})(h_1\cdot v)((h_1h_2)\cdot v^{-1}) \\
&= v(h_1\cdot v^{-1})(h_1\cdot v)(h_1\cdot(h_2\cdot v^{-1})) \\
&= v(h_1\cdot v^{-1})(h_1\cdot(v(h_2\cdot v^{-1}))) \\
&= \chi_v^H(h_1)(h_1\cdot\chi_v^H(h_2)),
\end{aligned}$$

so, noting that 1-coboundaries are morphisms, we see that every 1-coboundary is also a 1-cocycle.

We say $\alpha_1, \alpha_2 \in Z^1(H,V)$ are *equivalent* if there exists a $v \in V$ such that

$$\alpha_1(h) = v\alpha_2(h)(h\cdot v^{-1}), \tag{3.3}$$

for all $h \in H$.

It is straighforward to check that Equation 3.3 is an equivalence relation.

We call the set of equivalence classes of $Z^1(H,V)$ under the equivalence relation defined by Equation 3.3 the *1-cohomology*, denoted $H^1(H,V)$.

R. Richardson provides a result analogous to Lemma 3.1:

Lemma 3.3. *Suppose H is linearly reductive and V is unipotent. Then $H^1(H,V)$ is trivial [8, Lemma 6.2.6].*

3.2.3 Maps between 1-cohomologies

We obtain maps of 1-cohomologies in just the same way as in the abelian setting. Let $\phi : \tilde{H} \to H$ be a morphism, $\tilde{H}$ being another group that acts on V by group automorphisms. If ϕ is $\tilde{H}$-equivariant then the map $Z^1(\phi)$ defined by

$$Z^1(\phi)(\alpha) = \alpha\circ\phi,$$

for all $\alpha \in Z^1(H,V)$ is a 1-cocycle from $\tilde{H} \to V$. Thus we obtain maps $Z^1(H,V) \to Z^1(\tilde{H},V)$ from $\tilde{H}$-equivariant morphisms $\phi : \tilde{H} \to H$.

It follows that $Z^1(\phi)$ gives rise to a well-defined map of 1-cohomologies $H^1(\phi)$ as in the abelian setting.

Lemma 3.4. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V and let $\iota : B \to SL_2$ be the inclusion map. Then $H^1(\iota) : H^1(SL_2, V) \to H^1(B, V)$ is injective.*

Proof. Let x be in the kernel of $H^1(\iota)$ and α and element of $Z^1(SL_2, V)$ that projects onto the class x. Since $Z^1(\iota)(\alpha)$ projects to the trivial 1-cohomology class we may as well assume that $\alpha|_B = 1$. For there exists some $v \in V$ such that for all $b \in B$

$$\left(Z^1(\iota)(\alpha)\right)(b) = v(b \cdot v^{-1}).$$

Consider the 1-cocycle $\hat{\alpha} : SL_2 \to V$ defined by

$$\hat{\alpha}(h) = v^{-1}\alpha(h)(h \cdot v).$$

Then by construction $\hat{\alpha}$ also projects to the class x, and for all $b \in B$

$$\begin{aligned} \hat{\alpha}(b) &= v^{-1}\alpha(b)(b \cdot v) \\ &= v^{-1}(v(b \cdot v^{-1}))(b \cdot v) \\ &= v^{-1}v(b \cdot v)^{-1}(b \cdot v) \\ &= 1, \end{aligned}$$

so we may as well have chosen $\hat{\alpha}$ instead as a representative for x.

Now consider the *homogeneous space* SL_2/B and take the map

$$\tilde{\alpha} : SL_2/B \to V,$$

to be the unique morphism such that the following diagram commutes:

$$\begin{tikzcd} SL_2 \arrow[r, "\alpha"] \arrow[d, "\pi"'] & V \\ SL_2/B \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd},$$

π the canonical projection $\pi : SL_2 \to SL_2/B$. That is, $\tilde{\alpha}(hB) = \alpha(h)$ for all $h \in SL_2$.

Now since SL_2/B is an irreducible projective variety [9, Theorem 21.3], $\tilde{\alpha}$ must be constant [10]. Hence, as α takes the value 1 for any $b \in B$, $\tilde{\alpha}(hB) = 1$ for all cosets hB. Therefore, for all $h \in SL_2$

$$\alpha(h) = \tilde{\alpha}(hB) = 1.$$

We have shown that α is the trivial 1-coboundary χ_1 which means that the kernel of $H^1(\iota)$ is trivial. ∎

We end this section with the following conjecture:

> Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B. Let V be a unipotent group on which SL_2 acts. Then $H^1(\iota) : H^1(B,V) \to H^1(U,V)$ is injective.

We have calculations to support the conjecture (see Example) but no such proof. If the conjecture holds then by Lemma 3.4 we have that

$$H^1(\iota) : H^1(SL_2, V) \to H^1(U,V)$$

is injective. Then in the next Chapter we will see by Theorem 4.8 the answer to the algebraic version of Külshammer's second question would be positive for SL_2 and any reductive G, regardless of the characteristic of the underlying field k.

Chapter 4

Külshammer's Second Problem

4.1 Külshammer's Second Problem

Two questions were raised by B. Külshammer concerning representations of a finite group Γ into a linear algebraic group G over an algebraically closed field k.

1. Let $\mathrm{char}(k)$ be prime to the order of Γ. Are there only finitely many representations $\rho : \Gamma \to G$ up to conjugation by G?

2. Let $p = \mathrm{char}(k)$ and $\Gamma_p < \Gamma$ be a Sylow p-subgroup. Fix a conjugacy class of representations from $\Gamma_p \to G$. Are there, up to conjugation by G, only finitely many representations $\rho : \Gamma \to G$ whose restrictions to Γ_p belong to the given class?

As pointed out in the Introduction, the first has a positive answer and the second has positive answer so long as G is reductive and the characteristic of k is good for G. We wish to determine whether there exists a reductive counterexample to Külshammer's second question. We also investigate the algebraic version of Külshammer's second question and conclude this chapter with a necessary condition for a positive answer to both versions of Külshammer's second question.

Throughout this Chapter we let H, G be reductive linear algebraic groups, and Γ a finite group.

4.2 The Approach

We are interested in knowing whether there can be infinitely many G-conjugacy classes of representations $\Gamma \to G$ that when restricted to Γ_p hit some fixed G-conjugacy class of representations $\Gamma_p \to G$. A consequence of the following Theorem [11, Theorem 1.2] is that we will need to study representations into proper parabolic subgroups $P < G$.

Theorem 4.1. *Let Γ be a finite group There are only finitely many G-conjugacy classes of G-completely reducible representations $\Gamma \to G$.*

So by Theorem 4.1, if we have infinitely many G-conjugacy classes of representations $\Gamma \to G$ then infinitely many of those classes must be of non-G-completely reducible representations. The following Lemma states that the finiteness of G-conjugacy classes of a collection of representations $\Gamma \to G$ carries over to P-conjugacy classes for any parabolic subgroup $P < G$ containing the image of the representations.

Lemma 4.2. *Let $R = \{\rho_\lambda : \Gamma \to P \mid \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ, P a fixed parabolic subgroup of G. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Take two elements ρ_μ, ρ_ν of R in a particular G-conjugacy class. Then there exists an element $g \in G$ such that

$$g \cdot \rho_\mu = \rho_\nu.$$

By definition $\rho_\nu(\Gamma) \subset P$, but on the other hand $\rho_\nu = g \cdot \rho_\mu$, therefore $\rho_\nu(\Gamma) \subset P \cap Q$.

Let T be a maximal torus of G contained in $P \cap Q$ and let $\{n_1, \dots, n_l\}$ be coset representatives for the Weyl group $W = N_G(T)/T$.

Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists an element $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

By the definition of Q there exists an element $p \in P$ such that $q = gpg^{-1}$, so in fact

$$\begin{aligned} gpg^{-1}Tgp^{-1}g^{-1} &= gTg^{-1} \\ \Rightarrow pg^{-1}Tgp^{-1} &= T. \end{aligned}$$

We see that gp^{-1} lies in $N_G(T)$. Let n_i be the coset representative for the element of W containing gp^{-1} and let $t \in T$ be the element that satisfies

$$gp^{-1} = n_i t.$$

T is a subgroup of P so let $p^{-1}t^{-1} = p' \in P$ and we have

$$\begin{aligned} \rho_\mu &= g^{-1} \cdot \rho_\nu \\ &= (p^{-1}t^{-1}n_i^{-1}) \cdot \rho_\nu \\ &= p' \cdot (n_i^{-1} \cdot \rho_\nu). \end{aligned}$$

Furthermore, as ρ_μ is an arbitrary element of $R \cap (G \cdot \rho_\nu)$ we have

$$R \cap (G \cdot \rho_\nu) \subset \bigcup_{i=1}^{l} P \cdot (n_i^{-1} \cdot \rho_\nu),$$

where $l = |W|$.

Therefore, a G-conjugacy class of R is contained in a union of at most l P-conjugacy classes. Thus it is clear that if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes.

The converse is trivial. $\square$

We direct the reader's attention to the following notation: For a given parabolic subgroup P of G with Levi subgroup L and unipotent radical V, and a given representation $\rho : \Gamma \to P$ we have a map $\rho_L : \Gamma \to L$ defined by

$$\rho_L = \pi \circ \rho, \tag{4.1}$$

where π is the projection $Q \to L$.

Now define $\alpha_\rho : \Gamma \to V$ by

$$\alpha_\rho(\gamma) = \rho(\gamma)\rho_L(\gamma)^{-1},$$

for all $\gamma \in \Gamma$. Hence $\rho = \alpha_\rho \rho_L$.

Γ acts on V by conjugation via ρ_L.

$$\begin{aligned}\alpha_\rho(\gamma_1\gamma_2)\rho_L(\gamma_1\gamma_2) = \rho(\gamma_1\gamma_2) &= \rho(\gamma_1)\rho(\gamma_2)\\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_2)\\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1}\rho_L(\gamma_1)\rho_L(\gamma_2)\\ &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1}\rho_L(\gamma_1\gamma_2),\end{aligned}$$

so that

$$\begin{aligned}\alpha_\rho(\gamma_1\gamma_2) &= \alpha_\rho(\gamma_1)\rho_L(\gamma_1)\alpha_\rho(\gamma_2)\rho_L(\gamma_1)^{-1}\\ &= \alpha_\rho(\gamma_1)\left(\gamma_1\cdot\alpha_\rho(\gamma_2)\right),\end{aligned}$$

where Γ acts on V by conjugation via ρ_L. Therefore α_ρ satisfies the 1-cocycle condition (Equation 3.2) and so α_ρ is a 1-cocycle from $\Gamma \to V$. At this point we make a change to our notation in previous Chapters to make it explicit that the action of Γ on V depends on ρ_L and write $\alpha_\rho \in Z^1(\Gamma, \rho_L, V)$.

Conversely given a 1-cocycle $\alpha \in Z^1(\Gamma, \rho_L, V)$ we can construct a map $\rho : \Gamma \to P$ by $\rho(\gamma) = \alpha(\gamma)\rho_L(\gamma)$ for all $\gamma \in \Gamma$. The construction is a homomorphism from $\Gamma \to P$, for take $\gamma_1, \gamma_2 \in \Gamma$:

$$\begin{aligned}\rho(\gamma_1\gamma_2) &= \alpha(\gamma_1\gamma_2)\rho_L(\gamma_1\gamma_2)\\ &= \alpha(\gamma_1)(\gamma_1\cdot\alpha(\gamma_2))\rho_L(\gamma_1)\rho_L(\gamma_2)\\ &= \alpha(\gamma_1)\rho_L(\gamma_1)\alpha(\gamma_2)\rho_L(\gamma_1)^{-1}\rho_L(\gamma_1)\rho_L(\gamma_2)\\ &= \alpha(\gamma_1)\rho_L(\gamma_1)\alpha(\gamma_2)\rho_L(\gamma_2)\\ &= \rho(\gamma_1)\rho(\gamma_2).\end{aligned}$$

Given a representation $\rho : \Gamma \to P$, define $Hom(\Gamma, P)_{\rho_L}$ to be the set of representations $\sigma : \Gamma \to P$ such that $\sigma_L = \rho_L$. We formalise the above findings in the following Lemma:

Lemma 4.3. *The map $h : Hom(\Gamma, P)_{\rho_L} \to Z^1(\Gamma, \rho_L, V)$ defined by*

$$(h(\sigma))(\gamma) = \sigma(\gamma)\rho_L(\gamma)^{-1},$$

is bijective.

For ease of notation we will often write $h(\sigma)$ as α_σ. Also, since h is bijective we do no harm to use the otherwise suggestive notation α_σ when picking elements from $Z^1(\Gamma, \rho_L, V)$.

Let $v \in V$ and $\sigma \in Hom(\Gamma, P)_{\rho_L}$. Since π kills V, $\pi \circ (v \cdot \sigma) = \sigma_L = \rho_L$ and so $v \cdot \sigma \in Hom(\Gamma, P)_{\rho_L}$. Thus V acts on $Hom(\Gamma, P)_{\rho_L}$.

Denote by $\overline{\sigma}$ an element of $Hom(\Gamma, P)_{\rho_L}/V$ with a representative element $\sigma \in Hom(\Gamma, P)_{\rho_L}$ and $[\alpha_\sigma]$ an element of $H^1(\Gamma, \rho_L, V)$ with a representative element $\alpha_\sigma \in Z^1(\Gamma, \rho_L, V)$. We show that h gives rise to a bijection $\overline{h} : Hom(\Gamma, P)_{\rho_L}/V \to H^1(\Gamma, \rho_L, V)$ defined by

$$\overline{h}(\overline{\sigma}) = [h(\sigma)] = [\alpha_\sigma],$$

for all $\overline{\sigma} \in Hom(\Gamma, P)_{\rho_L}/V$.

Lemma 4.4. *The following diagram is commutative:*

$$\begin{array}{ccc} Hom(\Gamma, P)_{\rho_L} & \xrightarrow{h} & Z^1(\Gamma, \rho_L, V) \\ \downarrow & & \downarrow \\ Hom(\Gamma, P)_{\rho_L}/V & \xrightarrow{\overline{h}} & H^1(\Gamma, \rho_L, V). \end{array}$$

Furthermore, $\overline{h}$ is bijective.

Proof. Suppose $\overline{\sigma} = \overline{\tau}$ for some $\sigma, \tau \in Hom(\Gamma, P)_{\rho_L}$, that is to say $\sigma = v \cdot \tau$ for some $v \in V$. Then for all $\gamma \in \Gamma$

$$\begin{aligned} \alpha_\sigma(\gamma) &= \sigma(\gamma)\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)v^{-1}\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)\rho_L(\gamma)^{-1}\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1} \\ &= v\tau(\gamma)\rho_L(\gamma)^{-1}(\gamma \cdot v^{-1}) \\ &= v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}). \end{aligned}$$

Hence $[h(\sigma)] = [h(\tau)]$ and so $\overline{h}$ is well-defined.

Since h is onto and $Z^1(\Gamma, \rho_L, V) \to H^1(\Gamma, \rho_L, V)$ is onto, so is $\overline{h}$.

Now suppose $[\alpha_\sigma] = [\alpha_\tau]$ for some $\alpha_\sigma, \alpha_\tau \in Z^1(\Gamma, \rho_L, V)$. Then there exists $v \in V$ such that

$$\alpha_\sigma(\gamma) = v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}),$$

for all $\gamma \in \Gamma$.

Therefore corresponding representations $\sigma, \tau \in Hom(\Gamma, P)_{\rho_L}$ are V-conjugate:

$$\begin{aligned}\sigma(\gamma) &= \alpha_\sigma(\gamma)\rho_L(\gamma) \\ &= v\alpha_\tau(\gamma)(\gamma \cdot v^{-1})\rho_L(\gamma) \\ &= v\alpha_\tau(\gamma)\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1}\rho_L(\gamma) \\ &= v\tau(\gamma)v^{-1} \\ &= (v \cdot \tau)(\gamma).\end{aligned}$$

That is to say $\overline{\sigma} = \overline{\tau}$ and so $\overline{h}$ is bijective. □

More generally, we can conjugate $\sigma \in Hom(\Gamma, P)_{\rho_L}$ by $g \in G$ to get an element

$$g \cdot \sigma \in Hom(\Gamma, gPg^{-1})_{g \cdot \rho_L},$$

and

$$\alpha_{g\cdot\sigma} \in Z^1(\Gamma, g \cdot \rho_L, gVg^{-1}),$$

under h.

If $g \in P$ then $g = vl$ for some $v \in V$ and some $l \in L$, and since $gPg^{-1} = P$ and $gVg^{-1} = V$, conjugating gives rise to the maps

$$Hom(\Gamma, P)_{\rho_L} \to Hom(\Gamma, P)_{l\cdot\rho_L},$$

and

$$Z^1(\Gamma, \rho_L, V) \to Z^1(\Gamma, l \cdot \rho_L, V),$$

again, via h.

Furthermore, if $l \in Z(L)^\circ$ then $l \cdot \rho_L = \rho_L$. Indeed $Z(L)^\circ$ acts on $Hom(\Gamma, P)_{\rho_L}$ and on $Z^1(\Gamma, \rho_L, V)$ in the following way

$$\begin{aligned}(z \cdot \sigma)(\gamma) &= z\sigma(\gamma)z^{-1} \\ (z \cdot \alpha_\sigma)(\gamma) &= z\alpha_\sigma(\gamma)z^{-1},\end{aligned}$$

and h is $Z(L)^\circ$-equivariant:

$$\begin{aligned}h(z \cdot \sigma)(\gamma) &= z\sigma(\gamma)z^{-1}\rho_L(\gamma)^{-1} \\ &= z\sigma(\gamma)\rho_L(\gamma)^{-1}z^{-1} \\ &= (z \cdot h(\sigma))(\gamma),\end{aligned}$$

that is

$$\alpha_{z\cdot\sigma} \quad = \quad z \cdot \alpha_\sigma, \tag{4.2}$$

for all $\alpha_\sigma \in Z^1(\Gamma, \rho_L, V)$ and all $z \in Z(L)^\circ$.

We show that the $Z(L)^\circ$-action on $Hom(\Gamma, P)_{\rho_L}$ and $Z^1(\Gamma, \rho_L, V)$ descends to give a $Z(L)^\circ$-action on $Hom(\Gamma, P)_{\rho_L}/V$ and $H^1(\Gamma, \rho_L, V)$, respectively. The actions will be well-defined as a consequence of the fact that L normalizes V.

Let $z \in Z(L)^\circ$ and $\overline{\sigma} \in Hom(\Gamma, P)_{\rho_L}/V$. We define the $Z(L)^\circ$-action on $Hom(\Gamma, P)_{\rho_L}/V$ so that the projection $Hom(\Gamma, P)_{\rho_L} \to Hom(\Gamma, P)_{\rho_L}/V$ is $Z(L)^\circ$-equivariant:

$$z \cdot \overline{\sigma} = \overline{z \cdot \sigma}. \tag{4.3}$$

Suppose $\overline{\sigma} = \overline{\tau}$. Then there is $v \in V$ such that $\sigma = v \cdot \tau \in Hom(\Gamma, P)_{\rho_L}$, and $v' \in V$ such that $zv = v'z$. Therefore

$$\begin{aligned} \overline{z \cdot \sigma} &= \overline{zv \cdot \tau} \\ &= \overline{v'z \cdot \tau} \\ &= \overline{z \cdot \tau}. \end{aligned}$$

Hence the $Z(L)^\circ$-action on $Hom(\Gamma, P)_{\rho_L}/V$ in Equation 4.3 is well-defined.

Similarly, the $Z(L)^\circ$-action on $H^1(\Gamma, \rho_L, V)$ is defined so that the projection to the 1-cohomology $Z^1(\Gamma, \rho_L, V) \to H^1(\Gamma, \rho_L, V)$ is $Z(L)^\circ$-equivariant:

$$z \cdot [\alpha_\sigma] = [z \cdot \alpha_\sigma]. \tag{4.4}$$

Suppose $[\alpha_\sigma] = [\alpha_\tau]$. Then there is $v \in V$ such that for all $\gamma \in \Gamma$,

$$\alpha_\sigma(\gamma) = v\alpha_\tau(\gamma)(\gamma \cdot v^{-1}),$$

and there is $v' \in V$ such that $zv = v'z$. Therefore, for all $\gamma \in \Gamma$

$$\begin{aligned} (z \cdot \alpha_\sigma)(\gamma) &= zv\alpha_\tau(\gamma)(\gamma \cdot v^{-1})z^{-1} \\ &= zv\alpha_\tau(\gamma)\rho_L(\gamma)v^{-1}\rho_L(\gamma)^{-1}z^{-1} \\ &= v'z\alpha_\tau(\gamma)z^{-1}\rho_L(\gamma)v'^{-1}\rho_L(\gamma)^{-1} \\ &= v'z\alpha_\tau(\gamma)z^{-1}(\gamma \cdot v'^{-1}) \\ &= v'((z \cdot \alpha_\tau)(\gamma))(\gamma \cdot v'^{-1}). \end{aligned}$$

Therefore $[z \cdot \alpha_\sigma] = [z \cdot \alpha_\tau]$ and the $Z(L)^\circ$-action on $H^1(\Gamma, \rho_L, V)$ in Equation 4.4 is well-defined.

Since h is $Z(L)^\circ$-equivariant, it follows that $\overline{h}$ is also:

$$\begin{aligned} \overline{h}(z \cdot \overline{\sigma}) &= \overline{h}(\overline{z \cdot \sigma}) \\ &= [\alpha_{z \cdot \sigma}] \\ &= [z \cdot \alpha_\sigma] \qquad \text{(Equation 4.2)} \\ &= z \cdot [\alpha_\sigma] \\ &= z \cdot \overline{h}(\overline{\sigma}). \end{aligned}$$

In summary we have a well-defined $Z(L)^\circ$-action on $Hom(\Gamma, P)_{\rho_L}/V$ and $H^1(\Gamma, \rho_L, V)$, and a $Z(L)^\circ$-equivariant bijection

$$\overline{h} : Hom(\Gamma, P)_{\rho_L}/V \to H^1(\Gamma, \rho_L, V).$$

Hence the following Lemma:

Lemma 4.5. *The bijection $h : Hom(\Gamma, P)_{\rho_L} \to Z^1(\Gamma, \rho_L, V)$ gives rise to a bijection*

$$\tilde{h} : Hom(\Gamma, P)_{\rho_L}/VZ(L)^\circ \to H^1(\Gamma, \rho_L, V)/Z(L)^\circ.$$

Proof. It remains to show that $\left(Hom(\Gamma, P)_{\rho_L}/V\right)/Z(L)^\circ \simeq Hom(\Gamma, P)_{\rho_L}/VZ(L)^\circ$.

Denote by π_Z the canonical projection from $Hom(\Gamma, P)_{\rho_L}/V \to \left(Hom(\Gamma, P)_{\rho_L}/V\right)/Z(L)^\circ$ and $\tilde{\sigma}$ the element of $Hom(\Gamma, P)_{\rho_L}/VZ(L)^\circ$ with representative $\sigma \in Hom(\Gamma, P)_{\rho_L}$. We show that the map

$$\varphi : (Hom(\Gamma, P)_{\rho_L}/V)/Z(L)^\circ \to Hom(\Gamma, P)_{\rho_L}/VZ(L)^\circ,$$

defined by

$$\varphi(\pi_Z(\overline{\sigma})) = \tilde{\sigma},$$

is well defined and a bijection. To this end consider the following statements:

$$\begin{aligned}
&\pi_Z(\overline{\sigma}) = \pi_Z(\overline{\tau}) \\
\iff\ &\text{there exists } z \in Z(L)^\circ \text{ such that } \overline{\sigma} = z \cdot \overline{\tau} \\
\iff\ &\text{there exists } z \in Z(L)^\circ \text{ such that } \overline{\sigma} = \overline{z \cdot \tau} \qquad \text{(Equation 4.4)} \\
\iff\ &\text{there exists } v \in V, z \in Z(L)^\circ \text{ such that } \sigma = v \cdot (z \cdot \tau) \\
\iff\ &\text{there exists } v \in V, z \in Z(L)^\circ \text{ such that } \sigma = (vz) \cdot \tau \\
\iff\ &\tilde{\sigma} = \tilde{\tau}.
\end{aligned}$$

Hence the forward direction implies that φ is well-defined and the reverse direction implies that φ is injective. Surjectivity follows from the fact that the canonical projections:

$$Hom(\Gamma, P)_{\rho_L} \to Hom(\Gamma, P)_{\rho_L}/V \to (Hom(\Gamma, P)_{\rho_L}/V)/Z(L)^\circ,$$

and

$$Hom(\Gamma, P)_{\rho_L} \to Hom(\Gamma, P)_{\rho_L}/VZ(L)^\circ,$$

are surjective. □

Let $R = \{\rho_\lambda : \Gamma \to G \,|\, \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ. For $\rho \in R$, we say that a parabolic subgroup P of G is ρ-minimal if P is minimal among the parabolic subgroups of G that contain $\rho(\Gamma)$. For a parabolic subgroup $P < G$ define

$$R_P = \{\rho \in R \,|\, P \text{ is } \rho\text{-minimal}\}.$$

Since there are only finitely many G-conjugacy classes of parabolic subgroups of G (a standard result, e.g. [9, Theorem 30.1(a)]), we can choose a finite set of representative parabolic subgroups $\{Q_i\}_{i=0}^n$ of G such that every parabolic subgroup $P < G$ is G-conjugate to precisely one Q_i. Every $\rho \in R$ has a minimal parabolic subgroup and so there exists an element of each G-conjugacy class in R with minimal parabolic Q_i for some i, hence

$$G \cdot R = \bigcup_i G \cdot R_{Q_i}. \tag{4.5}$$

Furthermore, fix a particular Q_i with Levi subgroup M_i. Since Q_i is minimal for each $\rho \in R_{Q_i}$, ρ_{M_i} (Equation 4.1) is M_i-irreducible [12, Lemma 6.2(ii)].

For an M_i-irreducible representation $\sigma : \Gamma \to M_i$ define

$$R_\sigma = \{\rho \in R_{Q_i} \,|\, \rho_{M_i} = \sigma\}.$$

Since there are only finitely many M_i-conjugacy classes of M_i-irreducible representations $\Gamma \to M_i$ (Theorem 4.1), we can choose a finite set of representative M_i-irreducible representations $\{\sigma_i^j : \Gamma \to M_i\}_{j=0}^{n_i}$, such that every M_i-irreducible representation $\Gamma \to M_i$ is M_i-conjugate to precisely one σ_i^j. Every M_i-conjugacy class in R_{Q_i} has an element ρ such that $\rho_{M_i} = \sigma_i^j$ for some j, hence

$$M_i \cdot R_{Q_i} = \bigcup_j M_i \cdot R_{\sigma_i^j},$$

and therefore

$$G \cdot R = \bigcup_i G \cdot R_{Q_i} = \bigcup_i \bigcup_j G \cdot R_{\sigma_i^j}. \tag{4.6}$$

Fix an M_i-irreducible representation $\sigma : \Gamma \to M_i$. We have a map from $Hom(\Gamma, P_i)_\sigma \to Hom(\Gamma, P_i)_\sigma / V_i Z(M_i)^\circ$ given by the canonical projection and a map $\tilde{h} : Hom(\Gamma, P_i)_\sigma / V_i Z(M_i)^\circ \to H^1(\Gamma, \sigma, V_i)/Z(M_i)^\circ$. We define the map

$$\mathcal{H} : Hom(\Gamma, P_i)_\sigma \to H^1(\Gamma, \sigma, V_i)/Z(M_i)^\circ$$

to be the composition of the above canonical projection with $\tilde{h}$. That is, $\mathcal{H}(\rho) = \tilde{h}(\tilde{\rho})$ for all $\rho \in Hom(\Gamma, P_i)_\sigma$, where $\tilde{\rho}$ is the projection of ρ to $Hom(\Gamma, P_i)_\sigma / V_i Z(M_i)^\circ$.

We note that each subset $R_{\sigma_i^j} \subset R_{Q_i}$ is a $VZ(M_i)^\circ$-stable subset of $Hom(\Gamma, Q_i)_{\sigma_i^j}$, so it makes sense to calculate

$$\mathcal{H}(R_{\sigma_i^j}) \subset H^1(\Gamma, \sigma_i^j, V_i)/Z(M_i)^\circ.$$

Lemma 4.6. *Let $R_P = \{\rho_\lambda : \Gamma \to P \,|\, \lambda \in \Lambda\}$ be a collection of representations such that P ρ_λ-minimal for each ρ_λ.*

The following statements are equivalent:

(i) R_P is contained in a finite union of P-conjugacy classes.

(ii) For each irreducible representation $\sigma : \Gamma \to L$, R_σ is contained in a finite union of $VZ(L)^\circ$-conjugacy classes.

(iii) For each irreducible representation $\sigma : \Gamma \to L$*,*

$$\mathcal{H}(R_\sigma) \subset H^1(\Gamma, \sigma, V)/Z(L)^\circ$$

is finite.

Proof.

$(i) \Rightarrow (ii)$ Assume R_P is contained in a finite union of P-conjugacy classes and fix an irreducible representation $\sigma : \Gamma \to L$. Then R_σ is contained a finite union of P-conjugacy classes. Take $\rho \in R_\sigma$ and suppose that $p \cdot \rho \in R_\sigma$ for some $p \in P$. Writing $p = vl$ for some $v \in V$ and some $l \in L$, $(vl) \cdot \rho \in R_\sigma$ implies that in fact $l \in C_L(\sigma(\Gamma))$. Furthermore, since σ is irreducible it follows that $C_L(\sigma(\Gamma))/Z(L)^\circ$ is finite [11, Lemma 6.2], so we can choose a finite set $\{c_1, \ldots, c_m\}$ of coset representatives for $Z(L)^\circ \backslash C_L(\sigma(\Gamma))$. Therefore

$$R_\sigma \cap (P \cdot \rho) \subset \bigcup_{i=1}^{m} VZ(L)^\circ \cdot (c_i \cdot \rho)\,.$$

Since R_σ is contained in a finite number of P-conjugacy classes, we are done.

$(ii) \Rightarrow (i)$ Assume that for each irreducible representation $\sigma : \Gamma \to L$, R_σ is contained in a finite union of $VZ(L)^\circ$-conjugacy classes, so for each σ there is a finite set $\Phi^\sigma \subset R_P$ such that

$$R_\sigma \subset \bigcup_{\phi \in \Phi^\sigma} VZ(L)^\circ \cdot \phi.$$

We do no harm to assume that $R_P = P \cdot R_P$. Denote by $Hom(\Gamma, L)_{irr}$ the collection of all irreducible representations from $\Gamma \to L$. By Theorem 4.1 we can choose a finite set $\Sigma \subset Hom(\Gamma, L)_{irr}$ such that

$$Hom(\Gamma, L)_{irr} = \bigcup_{\sigma \in \Sigma} L \cdot \sigma.$$

For each $\sigma \in \Sigma$ define

$$R_{L \cdot \sigma} = \bigcup_{l \in L} R_{l \cdot \sigma}.$$

Since P is minimal for each $\rho \in R_P$, ρ_L is L-irreducible. Hence

$$R_P = \bigcup_{\sigma \in \Sigma} R_{L \cdot \sigma}.$$

Suppose $\rho \in R_{L \cdot \sigma}$. Then there exists $l \in L$ such that $\rho_L = l \cdot \sigma$, so that $l^{-1} \cdot \rho_L = \sigma$. Since we assumed $R_P = P \cdot R_P$, $l^{-1} \cdot \rho \in R_P$ and therefore $l^{-1} \cdot \rho \in R_\sigma$. Conversely if

$\rho \in L \cdot R_\sigma$ then $l \cdot \rho \in R_{L\cdot\sigma}$ for some $l \in L$. Hence

$$R_{L\cdot\sigma} = L \cdot R_\sigma.$$

Therefore

$$R_P = \bigcup_{\sigma\in\Sigma} L \cdot R_\sigma \subset \bigcup_{\sigma\in\Sigma} \bigcup_{\phi\in\Phi^\sigma} LVZ(L)^\circ \cdot \phi = \bigcup_{\sigma\in\Sigma} \bigcup_{\phi\in\Phi^\sigma} P \cdot \phi.$$

$(ii) \Leftrightarrow (iii)$ This follows directly from the fact that $\tilde{h}$ is a bijection (Lemma 4.5).

□

We are now ready to state precisely the connection with G-conjugacy classes of representations and the 1-cohomology.

Theorem 4.7. *Let $R = \{\rho_\lambda : \Gamma \to G \,|\, \lambda \in \Lambda\}$ be a collection of representations indexed by the set Λ.*

Suppose $R = G \cdot R$. Then R is a finite union of G-conjugacy classes if and only if for each i, j the subset $\mathcal{H}(R_{\sigma_i^j}) \subset H^1(\Gamma, \sigma_i^j, V_i)/Z(M_i)^\circ$ is finite.

Proof. Assume R is a finite union of G-conjugacy classes. Then for each Q_i, R_{Q_i} is contained in a finite union of G-conjugacy classes. By Lemma 4.2 R_{Q_i} is contained in a finite union of Q_i-conjugacy classes, and by Lemma 4.6 $\tilde{h}(R_{\sigma_i^j}/VZ(M_i)^\circ)$ is finite for each j.

Conversely, suppose that for each i, for each j, $\tilde{h}(R_{\sigma_i^j}/VZ(M_i)^\circ)$ is finite. Then by Lemma 4.6, each R_{Q_i} is contained in a finite union of Q_i-conjugacy classes. Since

$$G \cdot R = \bigcup_i G \cdot R_{Q_i} \qquad \text{(Equation 4.5)}$$

we are done. □

Theorem 4.8. *Let G be an algebraic group over an algebraically closed field k of characteristic p, Γ a finite group and $\Gamma_p < \Gamma$ a Sylow p-subgroup. Define $M_i < Q_i < G$ and $\sigma_i^j : \Gamma \to_{irr} M_i$ as above. Let ι be the inclusion map $\iota : \Gamma_p \to \Gamma$ and for each i, j let $Z^1(\iota), H^1(\iota)$ be the corresponding 1-cocycle and 1-cohomology restriction maps*

$$Z^1(\iota) : Z^1(\Gamma, \sigma_i^j, V_i) \to Z^1(\Gamma_p, \sigma_i^j, V_i),$$

and

$$H^1(\iota) : H^1(\Gamma, \sigma_i^j, V_i) \to H^1(\Gamma_p, \sigma_i^j, V_i).$$

The answer to Külshammer's second question for Γ, G *is positive only if* $H^1(\iota)$ *is injective for each* i, j.

Proof. Fix a representation $\rho_0 : \Gamma_p \to G$ and let $X = G \cdot \rho_0$. For a map φ from Γ denote by φ^p its restriction to Γ_p. Let

$$\begin{aligned} R &= \{\rho \in \mathrm{Hom}(\Gamma, G) \,|\, \rho^p \in X\}, \\ R_{Q_i} &= \{\rho \in R \,|\, Q_i \text{ is } \rho\text{-minimal}\}, \\ R_{\sigma_i^j} &= \{\rho \in R_{Q_i} \,|\, \rho_{M_i} = \sigma_i^j\}, \end{aligned}$$

as done previously.

Fix $g \in G$ and consider the set R_j^p defined

$$R_j^p = \{\rho \in R_{\sigma_i^j} \,|\, \rho^p = g \cdot \rho_0\}.$$

Then

$$R_{\sigma_i^j} \subset G \cdot R_j^p. \tag{4.7}$$

Assume $H^1(\iota)$ is injective. Since $Z^1(\iota)(h(\rho))$ is equal to a fixed 1-cocycle for all $\rho \in R_j^p$ the image of R_j^p in $H^1(\Gamma_p, \sigma_i^j, V_i)$ is finite, hence the image of R_j^p in $H^1(\Gamma, \sigma_i^j, V_i)$ is finite.

Therefore, the image of R_j^p in $H^1(\Gamma, \sigma_i^j, V_i)/Z(M_i)^\circ$ is finite. By Lemma 4.6, R_j^p is contained in a finite union of Q_i-conjugacy classes, hence by Equation 4.7, $R_{\sigma_i^j}$ is contained in a finite union of G-conjugacy classes.

So by Equation 4.6, $G \cdot R = R$ is contained in a finite union of G-conjugacy classes. Therefore the answer to Külshammer's second question for Γ, G is positive.

$\square$

Chapter 5

1-Cohomology Calculations: Theoretical Results

In this Chapter we present some theoretical results of 1-cohomology calculations of the form $H^1(SL_2(k), V)$.

Following the approach to the algebraic version of Külshammer's second question in the previous Chapter we let G be a reductive group and fix representative parabolic subgroups. In particular we fix a Borel subgroup $B < G$ containing a maximal torus T, hence fix a base Δ of Φ the root system for G. Then we can choose representative parabolic subgroups P_I, $I \subset \Delta$ as in [9, §30].

5.1 Calculations for Rank-1 Parabolic Subgroups

Let G be a reductive group over an algebraically closed field k of characteristic p. Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus $T < G$ contained by a fixed Borel subgroup $B < G$.

Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α:

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \,|\, \delta \in \Phi^+, \delta \neq \alpha \rangle, \\ P_\alpha &= V_\alpha L_\alpha \end{aligned}$$

There exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u), \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u), \end{aligned}$$

where $\epsilon_\alpha : k \to U_\alpha$ is an isomorphism [9, Theorem 26.3(c)].

We fix an integer $r > 0$ and define $\rho_r : SL_2(k) \to L_\alpha$ composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r \quad : \quad & SL_2(k) \to SL_2(k) \\ & (A_{ij}) \mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r = \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), \rho_r, V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ (Lemma 3.3), so let $\sigma \in Z^1(SL_2(k), \rho_r, V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = 0,$$

for all $t \in k^*$. We fix an ordering of the roots so that expressions such as $\prod_\delta U_\delta$ are unambiguious.

We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. By [9, Theorem 26.3(c)]

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_\delta \epsilon_\delta(\lambda_\delta) = \prod_\delta \epsilon_\delta\left((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_\delta\right), \tag{5.1}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_{\delta} \epsilon_\delta(\lambda_\delta) \quad = \quad \prod_{\delta} n_\alpha \epsilon_\delta \left(\lambda_\delta\right) n_\alpha^{-1}, \tag{5.2}$$

where $n_\alpha = \epsilon_\alpha(1)\epsilon_{-\alpha}(-1)\epsilon_\alpha(1)$ and $\lambda_\delta \in k$.

Lemma 5.1.

$$\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) = \prod_{\delta} \epsilon_\delta\left(x_\delta\left(u\right)\right),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_\delta \in k[X]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \simeq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{\iota} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where ι is the inclusion map and π_δ the projection onto the root subgroup V_δ. Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ \iota$$

is a morphism from $k \to k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix}, \tag{5.3}$$

we use the 1-cocycle condition (Equation 3.2) to obtain

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

So by Equation 5.1,

$$x_\delta \left(t^2 u\right) \quad = \quad (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta \left(u\right).$$

Since x_δ is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle \delta, \alpha \rangle \geq 0$. However, if $\langle \delta, \alpha \rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle \delta, \alpha \rangle > 0$. □

Next we prove a couple of useful facts about root systems not containing G_2 or C_3.

Lemma 5.2. *Suppose Φ does not contain G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle \alpha, \beta \rangle \leq 0$.*

Proof. α, β lie in a rank-2 subsystem of Φ. We have

$$\langle \alpha, \beta \rangle > 0 \Longleftrightarrow (\alpha, \beta) > 0 \Longleftrightarrow \cos(\theta) > 0,$$

where θ is the angle between α and β. Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams (Appendix B) we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. □

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3. Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ $(i = 1, 2)$. If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ, $(i = 1, 2)$ and let θ_3 be the absolute value of the angle between δ_1 and δ_2. Then

$$\begin{aligned} & \langle \delta_i, \gamma \rangle > 0 \qquad (i = 1, 2) \\ \Longrightarrow\ & (\delta_i, \gamma) > 0 \\ \Longrightarrow\ & \cos(\theta_i) > 0 \\ \Longrightarrow\ & \theta_i < \pi/2, \end{aligned}$$

and similarly, using Lemma 5.2

$$\begin{aligned} & \langle \delta_1, \delta_2 \rangle \leq 0 \\ \Longrightarrow \quad & \theta_3 \geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

$$\begin{array}{llll} \textbf{1:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = 2\pi/3; \\ \textbf{2:} & \theta_1 = \pi/3, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{3:} & \theta_1 = \pi/4, & \theta_2 = \pi/3, & \theta_3 = \pi/2; \\ \textbf{4:} & \theta_1 = \pi/4, & \theta_2 = \pi/4, & \theta_3 = \pi/2. \end{array}$$

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams (Appendix B) that δ_1 and δ_2 lie in a B_2 subsystem of Φ, and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we can rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is, they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams, recall that $\gamma \in \Delta$, yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair of short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots, $\delta_1, \delta_2, \delta_1 + \delta_2, \gamma$, lie is of type C_3, but this is impossible by assumption. □

We excluded Φ containing C_3 for berevity. The particular roots of C_3 which result in case two of Lemma 5.3 arises with γ being the short simple root that is not connected to the long simple root (see Example 5.2).

We return to the 1-cohomology calculation but assume that the root system for G does not contain G_2 or C_3.

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) = \epsilon_\alpha(u_1^{p^r}) \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By Lemma 5.2, $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α. Hence

$$\begin{aligned} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right) &= \prod_\delta \epsilon_\delta\left(x_\delta\left(u_2\right)\right) \\ &= \sigma\left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}\right). \end{aligned}$$

□

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α.*

Proof. First consider

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \prod_\delta \epsilon_\delta\left(x_\delta(b)\right).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By Lemma 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$ $(i = 1, 2)$, so Lemma 5.3 asserts that $\delta_1 + \delta_2$ is not a root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

For any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix}\right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \\ &= \prod_\delta \epsilon_\delta\left(a^{\langle \delta, \alpha \rangle p^r} x_\delta\left(b\right)\right). \end{aligned}$$

□

Since the x_δ are homomorphisms from $k \to k$ they must take the form

$$k \mapsto \sum_i \mu_i k^{p^i},$$

for some μ_i in k. Furthermore, combining Equation 5.3 with the result in Corollary 5.4 we get that

$$\prod_{\delta} \epsilon_\delta \left(x_\delta \left(a^2 b\right)\right) = \prod_{\delta} \epsilon_\delta \left(a^{\langle \delta, \alpha \rangle p^r} x_\delta \left(b\right)\right), \tag{5.4}$$

severely restricting the possible polynomials x_δ. In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer r in the definition of ρ_r. For suppose x_δ and hence some μ_j is non-zero. Then equating the coefficients of b in Equation 5.4 yields

$$\begin{aligned} \mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\ \Longrightarrow 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

In [13, §3.4] it is shown that the possible pairings of any two roots are bounded by ± 3. Hence by Lemma 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3. It is now clear that if $\langle \delta, \alpha \rangle = 3$ then $x_\delta = 0$.

If $\langle \delta, \alpha \rangle = 1$ the characteristic of k must be 2 and $j = r - 1$. Otherwise $\langle \delta, \alpha \rangle = 2$ and $j = r$, but the characteristic of k is so far unrestricted.

In order to capture the excluded cases where the root system for G contains G_2 or C_3 we provide the following examples.

Example 5.1. *Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

Note that this example captures the the root calculation that was excluded from Lemma 5.2. We will write v in V_α in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle \quad := \quad \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{3\alpha+\beta}(v_4)\epsilon_{3\alpha+2\beta}(v_5) \in V_\alpha$$

We use the commutation relations in [9, §33.5] to compute the group law for V_α:

$$u * v \quad = \quad \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v \quad = \quad \langle a^{-3p^r} v_1, a^{-p^r} v_2, a^{p^r} v_3, a^{3p^r} v_4, v_5 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), x_4(b), 0\rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r}x_3(b) \\ x_4(a^2b) &= a^{3p^r}x_4(b). \end{aligned}$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & b_1+b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1+b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1+b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r}x_3(b_2). \end{aligned}$$

We see that x_3 is a homomorphism, so it is of the form

$$x_3(b) = \sum_i \mu_i b^{p^i}.$$

Suppose $x_3 \neq 0$. Then some $\mu_j \neq 0$ and

$$\begin{aligned} & \mu_j(a^2b)^{p^j} = a^{p^r}\mu_j b^{p^j} \\ \Longrightarrow\ & a^{2p^j} = a^{p^r} \\ \Longrightarrow\ & p = 2. \end{aligned}$$

But then

$$\begin{aligned} x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r} x_3(b) \\ &= b^{2^r} x_3(b), \end{aligned}$$

implies that x_3 is constant, hence zero.

Therefore $x_3 = 0$, so x_4 is a homomorphism:

$$x_4(b) = \sum_i \nu_i b^{p^r}.$$

If $x_4 \neq 0$ then there is a $\nu_j \neq 0$ and we get

$$\begin{aligned} & \nu_j (a^2 b)^{p^j} = a^{3p^r} \nu_j b^{p^j} \\ \Longrightarrow \quad & a^{2p^j} = a^{3p^r} \\ \Longrightarrow \quad & 2p^j = 3p^r, \end{aligned}$$

which implies that 2 divides p and 3 divides p, a contradiction. Hence $x_4 = 0$ and

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Example 5.2. *Let $G = C_3$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta, \gamma\}$ with γ being the long root and connected to β. Let*

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle.$$

Note that this example captures the root calculation that was excluded in Lemma 5.3. Again we will write v in V_α in angled brackets for ease of notation:

$$\begin{aligned} & \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \rangle := \\ & \epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8) \in V_\alpha \end{aligned}$$

We use the commutation relations in [9, §33.3, §33.4] to calculate the group law for V_α:

$$\begin{aligned} & u * v = \\ & \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3 v_2, u_6 + v_6 + u_2 v_1^2 + 2u_4 v_1, \\ & \quad u_7 + v_7 + u_2 u_3 v_1 + u_2 v_1 v_3 + u_5 v_1 + u_4 v_3, u_8 + v_8 - u_3^2 v_2 - 2u_3 v_2 v_3 + 2u_5 v_3 \rangle. \end{aligned}$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-p^r} v_1, v_2, a^{p^r} v_3, a^{-p^r} v_4, a^{p^r} v_5, a^{-2p^r} v_6, v_7, a^{2p^r} v_8 \rangle.$$

Let σ *be in* $Z^1(SL_2, V_\alpha)$ *such that*

$$\sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) = 0.$$

By Lemma 5.1

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b) \rangle.$$

Applying σ *to both sides of the identity*

$$\begin{pmatrix} 1 & a^2 b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2 b) &= a^{p^r} x_3(b) \\ x_5(a^2 b) &= a^{p^r} x_5(b) \\ x_8(a^2 b) &= a^{2p^r} x_8(b). \end{aligned}$$

Since the polynomials x_3, x_5, x_8 *are homomorphisms (Lemma 5.2) we get*

$$\begin{aligned} \sum_i \lambda_i (a^2 b)^{p^i} &= a^{p^r} \sum_i \lambda_i b^{p^i} \\ \sum_i \mu_i (a^2 b)^{p^i} &= a^{p^r} \sum_i \mu_i b^{p^i} \\ \sum_i \nu_i (a^2 b)^{p^i} &= a^{2p^r} \sum_i \nu_i b^{p^i}, \end{aligned}$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of SL_2 *under* σ *is* $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle$ *then the characteristic of* k *must be 2, and so the image is contained in a product of commuting root groups.*

The results leading up to Corollary 5.5 provide a general set of tools for calculating the 1-cohomology of a rank 2 algebraic group G which we employ in Chapter 6, but they allude to something much grander which remains unproven: we conjecture

> Let G be a reductive group over a closed field of positive characteristic p and let $\Gamma = SL_2(k)$. Let $I \subset \Delta$ and $\sigma \in Z^1(SL_2(k), V_I)$ such that $\sigma(T) = 0$. Then the image of σ lies in a product of commuting root groups.

There is some work involved in extending the results in this Chapter to parabolics of rank > 1. Furthermore we need a way to extend the results to G with root system containing G_2 or C_3, as Examples 5.1, 5.2 only verify the conjecture for G with root system equal to G_2, C_3, respectively.

If the conjecture was true then we could attempt to apply Lemma 3.2 together with an algebraic version of Theorem 4.8 to show that the answer to the algebraic version of Külshammer's second question is positive for $SL_2(k)$ and any reductive G, with no restrictions on the characteristic of k.

5.2 A Non-Reductive Counterexample

In [1] a counterexample to Külshammer's second question is presented for a closed field k of characteristic $p = 2$ and a non-reductive algebraic group G.

Example 5.3. *Let Q be the algebraic group isomorphic to the affine space $\mathbf{A}^3$ with the group multiplication law:*

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1 v_1 + u_2 v_2 + u_1 v_2 \end{pmatrix}.$$

Let $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$ and $\Gamma_2 = \langle \tau \rangle$, a Sylow 2-subgroup of Γ. Γ acts on Q via

$$\tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix}$$

$$\sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}.$$

Let $G = Q \rtimes \Gamma$ and fix the representation $\rho : \Gamma_2 \to G$ defined by the natural inclusion $\Gamma_2 \to \Gamma \to G$. Then there are infinitely many pairwise G-conjugate classes of extensions of ρ to representations of Γ into G [1, Appendix].

Proof. Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle $\alpha \in Z^1(\Gamma, Q)$ such that $\alpha|_{\langle\sigma\rangle} = 1$. Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned} 1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\ &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1u_2 \end{pmatrix} \\ &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1u_2 \end{pmatrix}. \end{aligned}$$

This shows $u_1 = u_2$, so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\sigma\tau = \sigma^2$ we obtain

$$\begin{aligned}
1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
&= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
&= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
&= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
\end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle\sigma\rangle$ satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \qquad (u \in k).$$

This is a necessary condition. To show it is sufficient one can apply [14, Proposition 2] which involves looking at the presentation of Γ. Now we calculate the class $[\alpha_u] \in H^1(\Gamma, Q)$. Suppose $[\alpha_v] = [\alpha_u]$. Then there is a $q \in Q$ fixed under the action of σ, that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$
\begin{aligned}
\begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
\end{aligned}
$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(\Gamma, Q)$ is infinite. □

In view of Theorem 4.8 one could show that the map $H^1(\Gamma, Q) \to H^1(\Gamma_p, Q)$ is not injective. Furthermore, it is natural to ask whether Example 5.3 leads to a reductive counterexample to Külshammer's second question, although we can quickly verify that the answer is "not immediately". For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$
\begin{aligned}
&\ldots \epsilon_\alpha(u_\alpha) \ldots \epsilon_\beta(u_\beta) \ldots \epsilon_\gamma(u_\gamma) \times \ldots \epsilon_\alpha(v_\alpha) \ldots \epsilon_\beta(v_\beta) \ldots \epsilon_\gamma(v_\gamma) \\
&= \ldots \epsilon_\alpha(u_\alpha + v_\alpha) \ldots \epsilon_\beta(u_\beta + v_\beta) \ldots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
\end{aligned}
$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \alpha$ gives

$$
\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) \quad = \quad \epsilon_\alpha(u_\alpha + v_\alpha)\epsilon_\gamma(u_\alpha v_\alpha),
$$

which is absurd.

Chapter 6

Example 1-Cohomology Calculations

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G. The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

Throughout we use the notation of Humphreys [9, Chapter XI].

6.1 A rank 1 calculation

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p. We label the positive roots for B_2 as $\alpha, \beta, \alpha+\beta, 2\alpha+\beta$. We have from [9, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

6.1.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α:

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\alpha$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha, \end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r: write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$
\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(-u^{p^r})\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(-u^{p^r})\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\beta(v_1)\epsilon_\alpha(-u^{p^r})\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(-2u^{p^r}v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(u^{p^r})\epsilon_\alpha(-u^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{2p^r}v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2)\\
&= \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2-u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3-2u^{p^r}v_2+u^{2p^r}v_1)\\
&= \begin{pmatrix} v_1\\ v_2-u^{p^r}v_1\\ v_3-2u^{p^r}v_2+u^{2p^r}v_1\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \alpha^\vee(t^{p^r})\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\alpha^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\beta\left(\beta(\alpha^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\alpha^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\alpha^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\beta\left((t^{p^r})^{\langle\beta,\alpha\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\alpha\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\alpha\rangle}v_3\right)\\
&= \begin{pmatrix} t^{-2p^r}v_1\\ v_2\\ t^{2p^r}v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= n_\alpha\epsilon_\beta(v_1)n_\alpha^{-1}n_\alpha\epsilon_{\alpha+\beta}(v_2)n_\alpha^{-1}n_\alpha\epsilon_{2\alpha+\beta}(v_3)n_\alpha^{-1}\\
&= \epsilon_{2\alpha+\beta}(v_1)\epsilon_{\alpha+\beta}(-v_2)\epsilon_\beta(v_3)\\
&= \epsilon_\beta(v_3)\epsilon_{\alpha+\beta}(-v_2)\epsilon_{2\alpha+\beta}(v_1)\\
&= \begin{pmatrix} v_3\\ -v_2\\ v_1\end{pmatrix}.
\end{aligned}
$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V:

$$
\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix} d^{2p^r}v_1-2(cd)^{p^r}v_2+c^{2p^r}v_3\\ (ad+bc)^{p^r}v_2-(bd)^{p^r}v_1-(ac)^{p^r}v_3\\ b^{2p^r}v_1-2(ab)^{p^r}v_2+a^{2p^r}v_3\end{pmatrix}
$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$. By 3.3 σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*. Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

We could apply the results in Chapter 5 at this point but provide the full calculation as a demonstration.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u, so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{6.1}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{6.2}$$

to get further information on the polynomials p_i $(i = 1, 2, 3)$.

If we apply σ to both sides of (6.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned}
\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

That is,

$$\begin{aligned} p_1(t^2u) &= t^{-2p^r}p_1(u) && (6.3)\\ p_2(t^2u) &= p_2(u) && (6.4)\\ p_3(t^2u) &= t^{2p^r}p_3(u). && (6.5) \end{aligned}$$

From (6.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k. Now notice that on the left hand side of (6.3) there are only non-negative powers of t, and on the right hand side there are only non-positive powers of t. This equality is only satisfied if $p_1(x) = 0$ for all x in k, so p_1 is the zero polynomial.

We apply σ to (6.2) and using the 1-cocycle condition to obtain

$$\begin{aligned} \sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\ &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right). \end{aligned}$$

That is,

$$\begin{aligned} p_2(u_1+u_2) &= p_2(u_1)+p_2(u_2) && (6.6)\\ p_3(u_1+u_2) &= p_3(u_1)+p_3(u_2) - 2u_1^{p^r}p_2(u_2). && (6.7) \end{aligned}$$

Since p_2 is constant, (6.6) implies that p_2 is the zero polynomial, which means (6.7) becomes

$$p_3(u_1+u_2) = p_3(u_1)+p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^{N}\mu_i x^{p^i}, \tag{6.8}$$

for some u_i in k.

Now combining (6.5) and (6.8) yields

$$\sum_{i=0}^{N}\mu_i(t^2u)^{p^i} = t^{2p^r}\sum_{i=0}^{N}\mu_i u^{p^i}. \tag{6.9}$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating the coefficients of u in (6.9) we get

$$\begin{aligned} \mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \Longrightarrow \quad p^l &= p^r. \end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r.

Letting $\mu_l = \mu$ in k, we have

$$\begin{aligned} \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}. \end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k, p, is not equal to 2 then by setting $\mathbf{v}$ in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
\chi_v\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
&= \sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right).
\end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2. In view of Lemma 3.2 this means that σ is a 1-coboundary from the whole of $SL_2 \to V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \tag{6.10}$$

We can use an entirely similar argument to the one in calculating (6.10) to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix} + \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix} + \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix} + \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\0\\0\end{pmatrix} + \begin{pmatrix}\mu\\\mu\\\mu\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix} + \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\cdot\begin{pmatrix}\mu+\mu'\\\mu\\\mu\end{pmatrix}\\
&= \begin{pmatrix}0\\0\\\mu\end{pmatrix} + \begin{pmatrix}\mu+\mu'\\\mu'\\\mu'\end{pmatrix} = \begin{pmatrix}\mu+\mu'\\\mu'\\\mu+\mu'\end{pmatrix}.
\end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix}t & 0\\0 & t^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}t^{-1} & 0\\0 & t\end{pmatrix},$$

we get

$$\begin{pmatrix}t & 0\\0 & t^{-1}\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right) = \sigma\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right).$$

Therefore $\sigma\left(\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\right)$ is an element of V that is fixed by the action of $\begin{pmatrix}t & 0\\0 & t^{-1}\end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix}0\\ *\\0\end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

If $c = 0$ then we already have

$$\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r}\mu(cd)^{2^r} \\ (ac^{-1})^{2^{r+1}}\mu(cd)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1+ad)^{2^r} \\ \mu(ac^{-1})^{2^r}(1+ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}.
\end{aligned}$$

In fact, we see that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

This is a necessary condition for a 1-cocycle. To show it is sufficient there is a calculation to check involving the Steinberg relations for $SL_2(k)$ [14, Proposition 2].

Now if σ is in the same 1-cohomology class as τ then by Equation 3.3

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering $\mathbf{v}$ that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$ where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a $\mathbf{v}$ in V such that

$$\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \mathbf{v} + \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

In view of Lemma 4.6 we consider the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α. Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \,|\, x \in k\rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma\rangle = 0. \tag{6.11}$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n, we have

$$\langle \alpha, \gamma\rangle = \langle \alpha, m\alpha + n\beta\rangle \tag{6.12}$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta\rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha\rangle &= 0 \\ \iff m\langle \alpha, \alpha\rangle + n\langle \beta, \alpha\rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha+\beta)^\vee(x) \,|\, x \in k\rangle$. Taking an element $\mathbf{s} = (\alpha+\beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of $\mathbf{s}$ on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s}\cdot\sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha+\beta)^\vee(s)\epsilon_\beta\left(\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(\mu(ab)^{2^r}\right)(\alpha+\beta)^\vee(s)^{-1} \\ &= \epsilon_\beta\left(s^{\langle \beta,\alpha+\beta\rangle}\mu(cd)^{2^r}\right)\epsilon_{\alpha+\beta}\left(s^{\langle \alpha+\beta,\alpha+\beta\rangle}\mu(bc)^{2^r}\right)\epsilon_{2\alpha+\beta}\left(s^{\langle 2\alpha+\beta,\alpha+\beta\rangle}\mu(ab)^{2^r}\right) \\ &= \begin{pmatrix} (s^2\mu)(cd)^{2^r} \\ (s^2\mu)(bc)^{2^r} \\ (s^2\mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \to V$ collapse to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V-conjugacy to P_α-conjugacy:

$$\begin{aligned}[\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}\,.\end{aligned}$$

6.1.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β:

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta}\rangle,$$

and let ρ_r be the homomorphism from $SL_2 \to L_\beta$ defined by

$$\begin{aligned}\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} &= n_\beta,\end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows. Let $\mathbf{v}, \mathbf{w}$ in V. We have, using notation similar to the previous example

$$\begin{aligned}\mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1+w_1)\epsilon_{\alpha+\beta}(v_2+w_2)\epsilon_{2\alpha+\beta}(v_3+w_3+2v_2w_1) \\ &= \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3+2v_2w_1\end{pmatrix}.\end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r. Let $\mathbf{v}$ be an element of V:

$$\begin{aligned}
\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)^{-1}\\
&= \epsilon_\beta(u^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_\beta(u^{p^r})\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(u^{p^r}v_1)\epsilon_{2\alpha+\beta}(u^{p^r}v_1^2)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\beta(u^{p^r})\epsilon_\beta(-u^{p^r})\\
&= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2+u^{p^r}v_1)\epsilon_{2\alpha+\beta}(v_3+u^{p^r}v_1^2)\\
&= \begin{pmatrix}v_1\\ v_2+u^{p^r}v_1\\ v_3+u^{p^r}v_1^2\end{pmatrix}\\
\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}\right)^{-1}\\
&= \beta^\vee(t^{p^r})\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)(\beta^\vee(t^{p^r}))^{-1}\\
&= \epsilon_\alpha\left(\alpha(\beta^\vee(t^{p^r}))v_1\right)\epsilon_{\alpha+\beta}\left((\alpha+\beta)(\beta^\vee(t^{p^r}))v_2\right)\epsilon_{2\alpha+\beta}\left((2\alpha+\beta)(\beta^\vee(t^{p^r}))v_3\right)\\
&= \epsilon_\alpha\left((t^{p^r})^{\langle\alpha,\beta\rangle}v_1\right)\epsilon_{\alpha+\beta}\left((t^{p^r})^{\langle\alpha+\beta,\beta\rangle}v_2\right)\epsilon_{2\alpha+\beta}\left((t^{p^r})^{\langle 2\alpha+\beta,\beta\rangle}v_3\right)\\
&= \begin{pmatrix}t^{-p^r}v_1\\ t^{p^r}v_2\\ v_3\end{pmatrix}\\
\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\mathbf{v} &= \rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\mathbf{v}\left(\rho_r\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= n_\beta\epsilon_\alpha(v_1)n_\beta^{-1}n_\beta\epsilon_{\alpha+\beta}(v_2)n_\beta^{-1}n_\beta\epsilon_{2\alpha+\beta}(v_3)n_\beta^{-1}\\
&= \epsilon_{\alpha+\beta}(v_1)\epsilon_\alpha(-v_2)\epsilon_{2\alpha+\beta}(v_3)\\
&= \epsilon_\alpha(-v_2)\epsilon_{\alpha+\beta}(v_1)\epsilon_{2\alpha+\beta}(v_3-2v_1v_2)\\
&= \begin{pmatrix}-v_2\\ v_1\\ v_3-2v_1v_2\end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\cdot\mathbf{v} = \begin{pmatrix}c^{p^r}v_2+d^{p^r}v_1\\ a^{p^r}v_2+b^{p^r}v_1\\ v_3+(ac)^{p^r}v_2^2+(bd)^{p^r}v_1^2+2(bc)^{p^r}v_1v_2\end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \to V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^*, and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k.

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \tag{6.13}$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \tag{6.14}$$

Applying σ to both sides of (6.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma\left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \tag{6.15}$$

$$p_2(t^2u) = t^{p^r} p_2(u) \tag{6.16}$$

$$p_3(t^2u) = p_3(u). \tag{6.17}$$

From (6.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k. From (6.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (6.14):

$$\begin{aligned}
\sigma\left(\begin{pmatrix}1 & u_1+u_2\\ 0 & 1\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix}\right) * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix} \cdot \sigma\left(\begin{pmatrix}1 & u_2\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}1 & u_1\\ 0 & 1\end{pmatrix} \cdot \begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)\\ \lambda\end{pmatrix} * \begin{pmatrix}0\\ p_2(u_2)\\ \lambda\end{pmatrix}\\
&= \begin{pmatrix}0\\ p_2(u_1)+p_2(u_2)\\ 2\lambda\end{pmatrix}
\end{aligned}$$

That is,

$$p_2(u_1+u_2) = p_2(u_1)+p_2(u_2) \tag{6.18}$$

$$\lambda = 2\lambda. \tag{6.19}$$

By (6.19) we see that p_3 is in fact the zero polynomial, and (6.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^{N} \mu_i x^{p^i}, \tag{6.20}$$

for some μ_i in k.

Now combining (6.16) and (6.20) yields

$$\sum_{i=0}^{N} \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^{N} \mu_i u^{p^i}. \tag{6.21}$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l. By equating coefficients of u^{p^i} in (6.21) we get

$$\begin{aligned}
\mu_l t^{2p^l} &= \mu_l t^{p^r}\\
\implies \quad 2p^l &= p^r.
\end{aligned}$$

Thus 2 divides p^r, and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r, and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma\left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k.

We are now interested in the value of

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right) &= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right) + \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\right)\\
&= \begin{pmatrix}0\\ \mu\\ 0\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\begin{pmatrix}0\\ \mu\\ 0\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ \mu\\ 0\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\begin{pmatrix}\mu'\\ 0\\ 0\end{pmatrix} + \begin{pmatrix}\mu\\ \mu\\ \mu^2\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ \mu\\ 0\end{pmatrix} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\begin{pmatrix}\mu'+\mu\\ \mu\\ \mu^2\end{pmatrix}\\
&= \begin{pmatrix}0\\ \mu\\ 0\end{pmatrix} + \begin{pmatrix}\mu'+\mu\\ \mu'\\ \mu'^2\end{pmatrix}\\
&= \begin{pmatrix}\mu'+\mu\\ \mu'+\mu\\ \mu'^2\end{pmatrix}.
\end{aligned}$$

Since $\sigma\left(\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right)$ is fixed under the action of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$ for all t in k^* we must have $\mu' = \mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right)\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r}\mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{p^r}\left(\mu(cd)^{2^{r-1}}\right)^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ac^{-1} + a^2c^{-1}d\right)^{2^{r-1}} \\ \mu^2\left(1 + ad\right)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\left(ab\right)^{2^{r-1}} \\ \mu^2\left(bc\right)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a $\mathbf{v}$ in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
\tau\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &= \mathbf{v} + \sigma\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix},
\end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \to V$, where

$$\sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu\,(ab)^{2^{r-1}} \\ \mu^2\,(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
(\mathbf{s} \cdot \sigma_\mu)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (2\alpha+\beta)^\vee(s) \cdot \sigma_\mu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \\
&= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)\,(ab)^{2^{r-1}} \\ (s\mu)^2\,(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

our infinitely many V-conjugacy classes collapse to just two P_β-conjugacy classes:

$$\begin{aligned}
[\sigma_0] &= \{\sigma_0\}, \\
[\sigma_1] &= \{\sigma_\mu \,|\, \mu \in k^*\}.
\end{aligned}$$

6.2 A rank 2 calculation

Let $G = B_4$. Let $\mathrm{char}(k) = 2$ and set $V := \langle U_\phi \,|\, \phi \in \Phi^+, \phi \neq \gamma+\delta, \phi \neq \gamma+2\delta\}\rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)$ $\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} \quad = \quad \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2v_1 \\ 0 \\ u_4v_1 \\ 0 \\ u_6v_1 \\ 0 \\ u_8v_1 \\ 0 \\ u_{10}v_1 \\ u_{10}v_1v_2 + u_8v_1v_4 + u_6^2v_1 + u_{11}v_2 + u_{10}v_3 + u_9v_4 + u_8v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \to \widetilde{A}_1\widetilde{A}_1 < L_{\{\gamma,\delta\}}$ defined by

$$\begin{aligned}
\rho_{r,s}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s})\\
\rho_{r,s}\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma+\delta)^\vee(t^{2^s})\\
\rho_{r,s}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta}
\end{aligned}$$

from which we obtain an action of SL_2 on V:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix}
v_1\\
c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2\\
c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3\\
c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4\\
c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5\\
v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10}\\
v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11}\\
a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4\\
a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5\\
a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2\\
a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3\\
v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8)\\
+ (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11})
\end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$ should be a polynomial function of u, so we let

$$\sigma\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix} = \begin{pmatrix}p_1(u)\\ \vdots\\ p_{12}(u)\end{pmatrix},$$

where each p_i $(1 \leq i \leq 12)$ is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \tag{6.22}$$

It is clear that for $i = 1, 6, 7, 12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t, p_i must be the zero polynomial for $i = 2, 3, 4, 5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2\, p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \to k$. That is for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^{N} w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^{N} x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^{N} y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^{N} z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (6.22)

$$\begin{aligned} \sum_{j=0}^{N} w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^{N} w_j u^{2^j} \\ \Rightarrow \quad w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow \quad l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) = w u^{2^r}, &\quad p_9(u) = x u^{2^r}, \\ p_{10}(u) = y u^{2^s}, &\quad p_{11}(u) = z u^{2^s}, \end{aligned}$$

for some w, x, y, z in k.

So, we have

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1\end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

We apply the same argument using the fact that each component of $\sigma\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}$ is a polynomial function, say $p_i'(u)$ for all u in k, to get

$$\sigma\begin{pmatrix} d^{-1} & 0 \\ c & d\end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k.

From this we deduce that

$$\begin{aligned}
\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} &= \sigma\left(\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} * \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\sigma\left(\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} * \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\cdot\left(\sigma\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} * \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\cdot\sigma\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right)\\
&= \begin{pmatrix}0\\ y+y'\\ z+z'\\ w+w'\\ x+x'\\ w'+y'\\ x'+z'\\ w+w'\\ x+x'\\ y+y'\\ z+z'\\ w'x'+y'z'\end{pmatrix}.
\end{aligned}$$

Furthermore, since $\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ is fixed under the action of $\begin{pmatrix}t & 0\\ 0 & t^{-1}\end{pmatrix}$, we have

$$\sigma\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}n_1\\ 0\\ 0\\ 0\\ 0\\ n_6\\ n_7\\ 0\\ 0\\ 0\\ 0\\ n_{12}\end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k. So in fact

$$\begin{aligned}
w' &= w\\
x' &= x\\
y' &= y\\
z' &= z\\
n_1 &= 0\\
n_6 &= w+y\\
n_7 &= x+z\\
n_{12} &= wx+yz.
\end{aligned}$$

Consider $\sigma\begin{pmatrix} a & b \\ c & d\end{pmatrix}$. If $c=0$ then we already have

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} &= \sigma\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} \cdot \sigma\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
\end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix},$$

and so

$$\begin{aligned}
\sigma\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma\left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \sigma\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Again, it can be shown that this is a sufficient condition for a 1-cocycle by applying [14, Proposition 2].

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same 1-cohomology class as σ if there is a $\mathbf{v}$ in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g.\mathbf{v}^{-1}$$

for all g in SL_2. Furthermore, τ' is conjugate to some 1-cocycle τ, where τ has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some $\mathbf{v}$ in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned}
\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\
&= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix}
\end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \quad \sim \quad (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P-conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta+\gamma+\delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s}\cdot\sigma)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ s^{-1}t^2 y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2 w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2 w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2 y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

It remains to show that there are infinitely many P-conjugacy classes by applying Lemma 4.6.

Chapter 7

Future work

Two major open questions remain: the first is an extension of Lemma 3.4

> Is it true that $H^1(SL_2(k), V) \to H^1(U(k), V)$ is injective, where U is the unipotent radical of $SL_2(k)$ and V is an algebraic group on which $SL_2(k)$ acts?

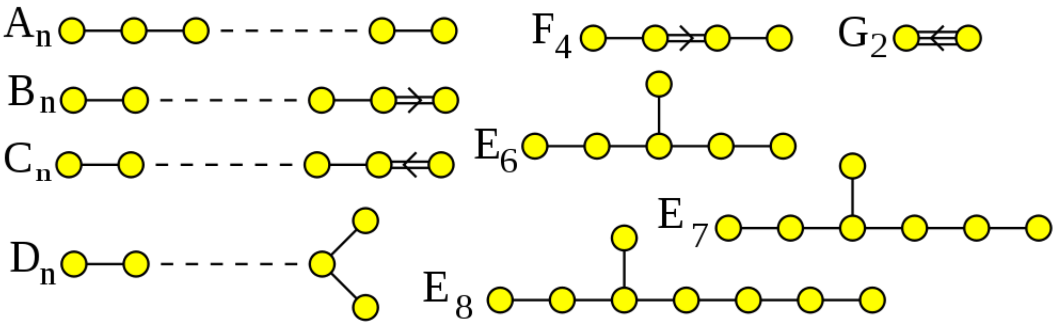
In view of Lemma 3.4 we need only consider whether $H^1(B, V) \to H^1(U, V)$ is injective, B a Borel subgroup of $SL_2(k)$. There is evidence for this conjecture in our calculations in Chapter 6.

If the first were true, we would further investigate the results in Chapter 5 which culminate in verifying that under certain conditions 1-cocycles from $Z^1(SL_2(k), V)$ which are trivial on a maximal torus T have image lying in a product of commuting root groups of V. Although V is rarely abelian, could we use the fact that their image is abelian to show that $H^1(SL_2(k), V) \to H^1(U, V)$ is injective?

If the answer to these two open questions turns out to be true, then constructing an argument similar to Theorem 4.8 for the algebraic analogue of Külshammer's second question would show that the answer is positive for $SL_2(k)$ and reductive G.

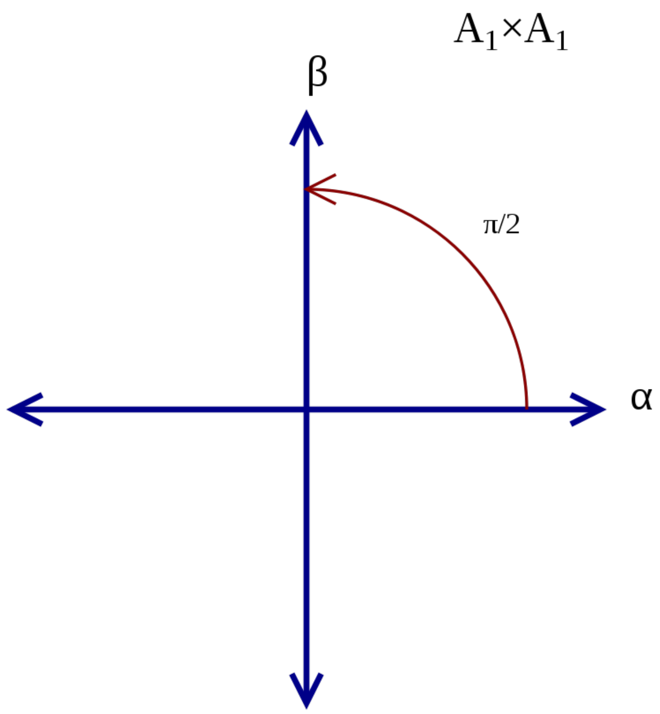
Appendix A

Finite Dynkin Diagrams

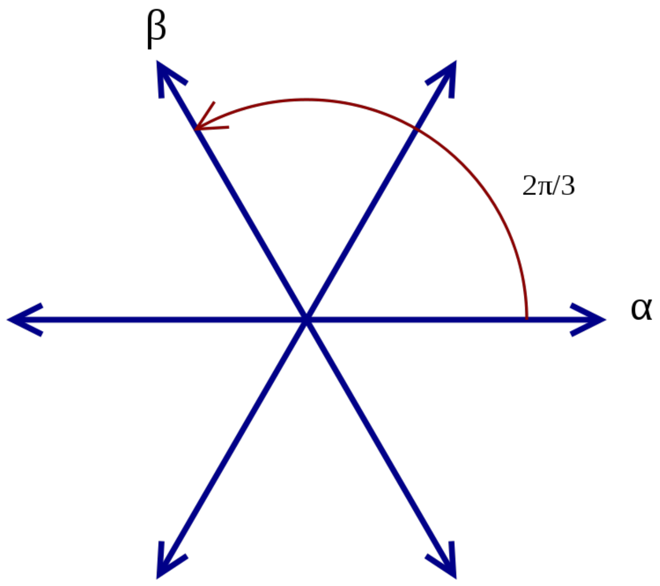


Appendix B

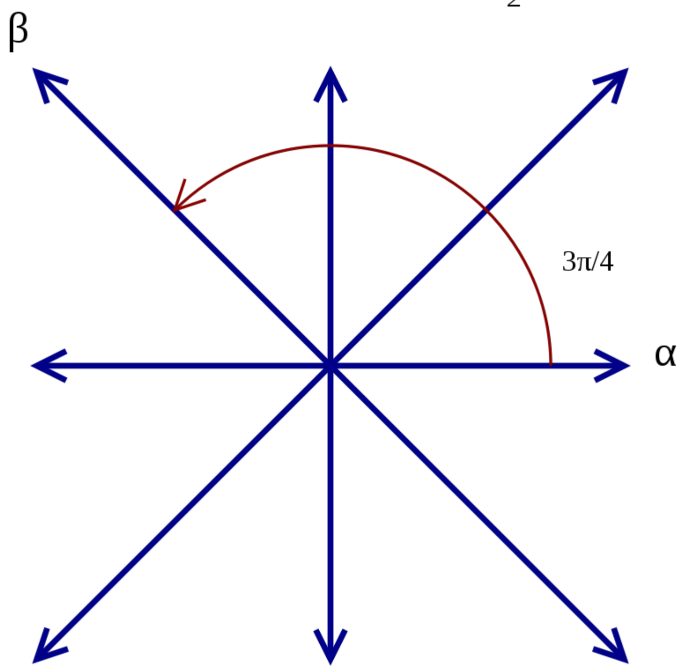
Rank 2 Root System Diagrams

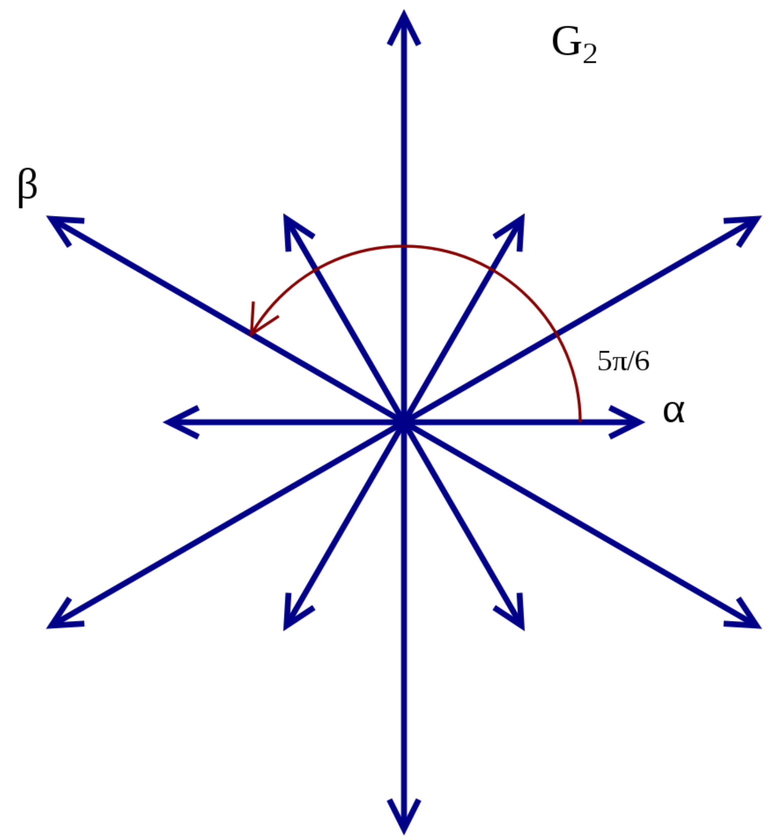


A$_2$



B$_2$




G2
β
5π/6
α

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