

UNIVERSITY OF CANTERBURY

A Geometric Approach to Complete Reducibility

by

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Declaration of Authorship

I, DANIEL LOND, declare that this thesis titled, ‘A GEOMETRIC APPROACH TO COMPLETE REDUCIBILITY’ and the work presented in it are my own. I confirm that:

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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Abstract

College of Engineering
Department of Mathematics and Statistics

Doctor of Philosophy

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The Thesis Abstract ...

Acknowledgements

The acknowledgements and the people to thank ...

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Symbols

a	distance	m
P	power	W (Js^{-1})
ω	angular frequency	rads^{-1}
\vdots		

Dedication . . .

Chapter 1

Introduction

- What the thesis is about
- Motivation - link with other problems
- Highlight results - lead up to highlights
- Similar to abstract but less formal
- Outline of the contents, chapter by chapter

Chapter 2

Mathematical Preliminaries

Chapter 3

Külshammer's Second Problem

- Külshammer's First Problem
- Külshammer's Second Problem
- Counter example for non-reductive G
- Overture to Ch. 3-5

3.1 Külshammer's First Problem

3.2 Külshammer's Second Problem

3.3 A non-reductive counterexample

3) Look at the nonreductive counterexample in Slodowy's paper on Külshammer's problem. What is special about the 3-dimensional U that makes this counterexample work? Can you find similar structure in the unipotent radical of a reductive group?

Chapter 4

The 1-Cohomology

4.1 Abelian 1-Cohomology

4.1.1 Definitions

Let H be a group and V an abelian group (vector space) on which H acts homomorphically (linearly). We call a map σ from $H \rightarrow V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) + h_1 \cdot \sigma(h_2), \quad (4.1)$$

for all h_1, h_2 in H . Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \rightarrow V$.

We call the (4.1) the *1-cocycle condition*.

For any σ_1, σ_2 in $Z^1(H, V)$

$$\begin{aligned} (\sigma_1 + \sigma_2)(h_1 h_2) &= \sigma_1(h_1 h_2) + \sigma_2(h_1 h_2) \\ &= \sigma_1(h_1) + h_1 \cdot \sigma_1(h_2) + \sigma_2(h_1) + h_1 \cdot \sigma_2(h_2) \\ &= (\sigma_1(h_1) + \sigma_2(h_1)) + h_1 \cdot (\sigma_1(h_2) + \sigma_2(h_2)) \\ &= (\sigma_1 + \sigma_2)(h_1) + h_1 \cdot (\sigma_1 + \sigma_2)(h_2), \end{aligned}$$

so $Z^1(H, V)$ is closed under pointwise addition.

The trivial map from $H \rightarrow V$ that sends every h in H to the identity 0 in V is a 1-cocycle. Furthermore for any σ in $Z^1(H, V)$ we have

$$\begin{aligned}\sigma(1) = \sigma(1 \cdot 1) &= \sigma(1) + 1 \cdot \sigma(1) \\ &= \sigma(1) + \sigma(1) \\ &= 2\sigma(1),\end{aligned}$$

which implies that

$$\sigma(1) = 0.$$

From this we deduce that

$$\begin{aligned}\sigma(hh^{-1}) = \sigma(1) &= 0 \\ &= \sigma(h) + h \cdot \sigma(h^{-1}),\end{aligned}$$

and so each σ has an inverse defined by

$$-\sigma(h) = h \cdot \sigma(h^{-1}).$$

Therefore $Z^1(H, V)$ is a \mathbb{Z} -module under pointwise addition.

Given a v in V we define a 1-coboundary $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v - h \cdot v,$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}\chi_v^H(h_1 h_2) &= v - (h_1 h_2) \cdot v \\ &= v - h_1 \cdot (h_2 \cdot v) \\ &= v - h_1 \cdot (v - v + h_2 \cdot v) \\ &= v - h_1 \cdot v + h_1 \cdot (v - h_2 \cdot v) \\ &= \chi_v^H(h_1) + h_1 \cdot \chi_v^H(h_2),\end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

For any u, v in V and all h in H

$$\begin{aligned}
 (\chi_u^H + \chi_v^H)(h) &= \chi_u^H(h) + \chi_v^H(h) \\
 &= u - h \cdot u + v - h \cdot v \\
 &= (u + v) - h \cdot (u + v) \\
 &= \chi_{u+v}^H(h)
 \end{aligned}$$

is a 1-coboundary, and hence $B^1(H, V)$ is also closed under pointwise addition.

We see that $B^1(H, V)$ is a subgroup of $Z^1(H, V)$ via the two-step subgroup test. In fact it is easy to show that $B^1(H, V)$ is a \mathbb{Z} -submodule of $Z^1(H, V)$, so we may form the quotient module

$$H^1(H, V) = Z^1(H, V) / B^1(H, V),$$

called the *1-cohomology*.

Lemma 4.1. *Suppose H is linearly reductive. Then $H^1(H, V)$ is trivial [1].*

4.1.2 Maps between 1-cohomologies

Let ϕ be a homomorphism from $\tilde{H} \rightarrow H$, \tilde{H} being another group that acts on V . Suppose that for every h in H , ϕ satisfies

$$\phi(h) \cdot v = h \cdot v,$$

for all v in V . If σ is a 1-cocycle from $H \rightarrow V$ then we will show that the map denoted $Z^1(\phi)(\sigma)$ defined by

$$Z^1(\phi)(\sigma) = \sigma \circ \phi,$$

is a 1-cocycle from $\tilde{H} \rightarrow V$.

Take h_1, h_2 in H . We have

$$\begin{aligned}
 Z^1(\phi)(\sigma)(h_1 h_2) &= (\sigma \circ \phi)(h_1 h_2) \\
 &= \sigma(\phi(h_1 h_2)) \\
 &= \sigma(\phi(h_1)\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + \phi(h_1) \cdot \sigma(\phi(h_2)) \\
 &= \sigma(\phi(h_1)) + h_1 \cdot \sigma(\phi(h_2)) \\
 &= (\sigma \circ \phi)(h_1) + (\sigma \circ \phi)(h_2) \\
 &= Z^1(\phi)(\sigma)(h_1) + h_1 \cdot Z^1(\phi)(\sigma)(h_2).
 \end{aligned}$$

Moreover, it can be shown that $Z^1(\phi)$ maps $B^1(H, V)$ into $B^1(\tilde{H}, V)$. This leads us to define a map of 1-cohomologies,

$$H^1(\phi) : H^1(H, V) \rightarrow H^1(\tilde{H}, V),$$

defined by

$$\begin{array}{ccc}
 Z^1(H, V) & \xrightarrow{Z^1(\phi)} & Z^1(H, V) \\
 \pi \downarrow & & \downarrow \pi \\
 H^1(H, V) & \xrightarrow{H^1(\phi)} & H^1(\tilde{H}, V)
 \end{array}$$

where π and $\tilde{\pi}$ are the respective canonical projections of $Z^1(H, V)$ onto $H^1(H, V)$ and $Z^1(\tilde{H}, V)$ onto $H^1(\tilde{H}, V)$. To show that the map $H^1(\phi)$ is well-defined it is sufficient to notice that $Z^1(\phi)$ is a homomorphism.

Example 4.1. Let \tilde{H} be a subgroup of H and $i : \tilde{H} \rightarrow H$ the inclusion map. Then i gives rise to a well defined map

$$H^1(i) : H^1(H, V) \rightarrow H^1(\tilde{H}, V).$$

Lemma 4.2. Let H be a finite group and $\tilde{H} = H_p$ a Sylow p -subgroup of H . If V is a vector space then the map

$$H^1(i) : H^1(H, V) \rightarrow H^1(H_p, V)$$

is injective.

Proof. Let x be an element of $H^1(H, V)$ such that $H^1(i)(x) = 0$. Now choose a 1-cocycle σ in $Z^1(H, V)$ such that $\pi(\sigma) = x$. Hence $Z^1(i)(\sigma)$ is a 1-coboundary as its image under $\tilde{\pi}$ is 0. That is to say σ restricted to H_p is equal to a 1-coboundary, say $\chi_v^{H_p}$. But since

$\chi_v^{H_p}$ can be trivially extended to a 1-coboundary χ_v^H from $H \rightarrow V$, and

$$\pi(\sigma - \chi_v^H) = x,$$

we could well have chosen the 1-cocycle $(\sigma - \chi_v^H)$ as a representative for x . Hence there is no harm in assuming that σ is 0 when restricted to H_p . Now choose a set of representatives h_1, \dots, h_l in H for the coset space H/H_p and set

$$v^* = \sum_{i=1}^l \sigma(h_i).$$

Consider the 1-coboundary $\chi_{v^*}^H$ defined by v^*

$$\begin{aligned} \chi_{v^*}^H(h) &= v^* - h \cdot v^* \\ &= \sum_{i=1}^l \sigma(h_i) - h \cdot \sum_{i=1}^l \sigma(h_i) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i). \end{aligned}$$

By the 1-cocycle condition we have

$$\sigma(hh_i) = \sigma(h) + h \cdot \sigma(h_i),$$

from which we obtain

$$\begin{aligned} \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l h \cdot \sigma(h_i) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l (\sigma(hh_i) - \sigma(h)) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h). \end{aligned}$$

Now as the value of σ at a fixed h depends only on the value of σ at the representative h_j of the coset containing h we can collapse the middle term to yield

$$\begin{aligned} \chi_{v^*}^H(h) &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(hh_i) + \sum_{i=1}^l \sigma(h) \\ &= \sum_{i=1}^l \sigma(h_i) - \sum_{i=1}^l \sigma(h_i) + \sum_{i=1}^l \sigma(h) \\ &= l\sigma(h). \end{aligned}$$

Since $\gcd([H : H_p], p) = \gcd(l, p) = 1$, l is invertible and so

$$l^{-1}\chi_{v^*}^H(h) = \sigma(h).$$

Therefore σ is a 1-coboundary and so the kernel of $H(i)$ is trivial. \square

Example 4.2. Let

$$k = \bar{\mathbb{F}}_p = \bigcup_r \mathbb{F}_{p^r},$$

V a vector space on which $SL_2(k)$ acts, and $U(k)$ the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Then $U(\mathbb{F}_{p^r})$ is a Sylow p -subgroup of $SL_2(\mathbb{F}_{p^r})$ for each r , and the map

$$H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$$

is injective.

Proof. The group $GL_2(\mathbb{F}_{p^r})$ has order $(p^{2r} - 1)(p^{2r} - p^r)$ since there are $p^{2r} - 1$ choices of vectors for the first column (all choices excluding the zero vector), and $p^{2r} - p^r$ choices of vectors for the second column (all choices excluding multiples of the first vector). The determinant is a homomorphism of groups

$$\det : GL_2(\mathbb{F}_{p^r}) \rightarrow \mathbb{F}_{p^r}^*,$$

with kernel $SL_2(\mathbb{F}_{p^r})$. Therefore, by the First homomorphism theorem for groups

$$GL_2(\mathbb{F}_{p^r}) / SL_2(\mathbb{F}_{p^r}) \sim \det(GL_2(\mathbb{F}_{p^r})) = \mathbb{F}_{p^r}^*,$$

and so

$$\begin{aligned} |SL_2(\mathbb{F}_{p^r})| &= |GL_2(\mathbb{F}_{p^r})| / |\mathbb{F}_{p^r}^*| \\ &= (p^{2r} - 1)(p^{2r} - p^r) / (p^r - 1) \\ &= p^r(p^{2r} - 1). \end{aligned}$$

Since $|U(\mathbb{F}_{p^r})| = p^r$, $U(\mathbb{F}_{p^r})$ is a Sylow p -subgroup of $SL_2(\mathbb{F}_{p^r})$.

Fix a non-trivial $y \in H^1(SL_2(k), V)$ and choose a representative $\tau \in Z^1(SL_2(k), V)$ for y . For each $g \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_g^{(r)} : V \rightarrow V$ by

$$f_g^{(r)}(v) = \tau(g) - \chi_v(g) = \tau(g) - v + g \cdot v.$$

Consider the sequence of subsets of V defined by

$$C_r = \{v \in V \mid f_g^{(r)}(v) = 0\}.$$

Each subset C_r is closed and the inclusion $\mathbb{F}_{p^r} \subset \mathbb{F}_{p^{r+1}}$ induces the reverse inclusion $C_r \supset C_{r+1}$. The Noetherian property for V requires that the sequence becomes constant.

However, $y \neq 0$ so τ is not a 1-coboundary on $SL_2(k)$, which means the C_r 's are eventually empty. That is, there exists an integer s such that for any v in V

$$(\tau - \chi_v)|_{SL_2(\mathbb{F}_{p^s})} \neq 0.$$

Equivalently, if $y|_{SL_2(\mathbb{F}_{p^r})} = 0$ for all r then $y = 0$.

Take x in the kernel of the map $H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$. Then for each r , $x|_{U(\mathbb{F}_{p^r})} = 0$ so by (4.2) $x|_{SL_2(\mathbb{F}_{p^r})} = 0$. Therefore $x = 0$ and so $H^1(SL_2(k), V) \rightarrow H^1(U(k), V)$ is injective. \square

We could also consider appropriate maps $f : V \rightarrow \tilde{V}$ and following a similar chain of arguments as before we can define

$$H^1(f) : H^1(H, V) \rightarrow H^1(H, \tilde{V}),$$

or even

$$H^1(\phi, f) : H^1(H, V) \rightarrow H^1(\tilde{H}, \tilde{V}).$$

4.2 Non-abelian 1-Cohomology

4.2.1 The non-abelian setting

We will be interested in H, V algebraic groups, where we require that 1-cocycles be morphisms of varieties.

4.2.2 Definitions

Let H, V be algebraic groups, H acting on V . We call a map σ from $H \rightarrow V$ a *1-cocycle* if it satisfies

$$\sigma(h_1 h_2) = \sigma(h_1) * h_1 \cdot \sigma(h_2), \tag{4.2}$$

for all h_1, h_2 in H . Denote by $Z^1(H, V)$ the collection of all 1-cocycles from $H \rightarrow V$.

We call the (4.2) the *1-cocycle condition*.

Given a v in V we define a *1-coboundary* $\chi_v^H : H \rightarrow V$ to be

$$\chi_v^H(h) = v * h \cdot v^{-1},$$

and denote by $B^1(H, V)$ the collection of all 1-coboundaries.

For any v in V and any h_1, h_2 in H

$$\begin{aligned}
 \chi_v^H(h_1 h_2) &= v * (h_1 h_2) \cdot v^{-1} \\
 &= v * h_1 \cdot (h_2 \cdot v^{-1}) \\
 &= v * h_1 \cdot (v v^{-1} h_2 \cdot v) \\
 &= v * h_1 \cdot v * h_2 \cdot (v * h_2 \cdot v^{-1}) \\
 &= \chi_v^H(h_1) * h_2 \cdot \chi_v^H(h_2),
 \end{aligned}$$

so every 1-coboundary is also a 1-cocycle.

We say σ_1, σ_2 in $Z^1(H, V)$ are *equivalent* if there exists a v in V such that

$$\sigma_1(h) = v * \sigma_2(h) * h \cdot v^{-1}, \quad (4.3)$$

for all h in H . We call the set of equivalence classes of $Z^1(H, V)$ under the equivalence relation defined by (4.3) the *1-cohomology*, denoted $H^1(H, V)$.

4.2.3 Maps between 1-cohomologies

Lemma 4.3. *Let B be a Borel subgroup of SL_2 acting on an algebraic group V . Then $H^1(i) : H^1(SL_2, V) \rightarrow H^1(B, V)$ is injective.*

Proof. Let x be in the kernel of $H^1(i)$ and σ an element of $Z^1(SL_2, V)$ that projects onto the class x . Since $Z^1(i)(\sigma)$ projects to the trivial 1-cohomology class we may as well assume that $\sigma|_B = 1$. For there exists some v in V such that for all b in B

$$Z^1(i)(\sigma)(b) = v * b \cdot v^{-1}.$$

Consider the 1-cocycle $\hat{\sigma} : SL_2 \rightarrow V$ defined by

$$\hat{\sigma}(h) = v^{-1} * \sigma(h) * h \cdot v.$$

Then by construction $\hat{\sigma}$ also projects to the class x , and for all b in B

$$\begin{aligned}
 \hat{\sigma}(b) &= v^{-1} * \sigma(b) * b \cdot v \\
 &= v^{-1} * (v * b \cdot v^{-1}) * b \cdot v \\
 &= v^{-1} * v * b \cdot (v^{-1} * v) \\
 &= 1,
 \end{aligned}$$

so we may as well have chosen $\tilde{\sigma}$ instead as a representative for x .

Now consider the *homogeneous space* SL_2/B [2] and take the map

$$\tilde{\sigma} : SL_2/B \rightarrow V,$$

defined in the usual way under the canonical projection $\pi : SL_2 \rightarrow SL_2/B$:

$$\begin{array}{ccc} SL_2 & \xrightarrow{\sigma} & V \\ \pi \downarrow & \nearrow \tilde{\sigma} & \\ SL_2/B & & \end{array}$$

This map is well defined and is a morphism [3]. Now since SL_2/B is an irreducible projective variety [2], $\tilde{\sigma}$ must be constant [3]. Hence, as σ takes the value 1 for any b in B , $\tilde{\sigma}(hB) = 1$ for all cosets hB . Therefore, for all h in SL_2

$$\sigma(h) = \tilde{\sigma}(hB) = 1.$$

We have shown that σ is the 1-coboundary χ_1 which means that the kernel of $H^1(i)$ is trivial. \square

Lemma 4.4. *Let B be a Borel subgroup of SL_2 and U be the unipotent radical of B . Then $H^1(B, V) \rightarrow H^1(U, V)$ is injective. Moreover*

$$H^1(SL_2, V) \rightarrow H^1(U, V)$$

is injective.

Proof. As in the previous example, let x be an element of the kernel of $H^1(i) : H^1(B, V) \rightarrow H^1(U, V)$ and let σ in $Z^1(B, V)$ be a representative for x such that $\sigma|_U = 1$. Let T be a maximal torus for B . For any u in U and t in T there is a u' in U such that

$$ut = tu'.$$

Hence U acts trivially on $\sigma(T)$:

$$\begin{aligned} \sigma(ut) &= \sigma(tu') \\ \sigma(u) * u \cdot \sigma(t) &= \sigma(t) * t \cdot \sigma(u') \\ u \cdot \sigma(t) &= \sigma(t). \end{aligned}$$

Since T is linearly reductive, $H^1(T, V)$ is trivial [prove or reference], so that there is a v in V such that for all t in T

$$\sigma(t) = \chi_v(t) = v * t \cdot v^{-1}.$$

Consider the 1-cocycle τ in $Z^1(B, V)$ defined by

$$\tau(b) = v^{-1} * \sigma(b) * b \cdot v.$$

$$v * t \cdot v^{-1} = w * b \cdot w^{-1}$$

$$v * t \cdot v^{-1} = u \cdot v * b \cdot v^{-1}$$

□

Chapter 5

1-Cohomology Calculation

In this chapter we present a method of calculating the 1-cohomology $H^1(SL_2(k), V)$ where $V = R_u(P)$ is the unipotent radical of a parabolic subgroup P of a reductive group G . The motivation for this is to look for infinitely many conjugacy classes of representations of $SL_2(k)$ into G in the hope of finding a finite subgroup H of $SL_2(k)$ as a counterexample for Külshammer's Second Problem.

5.1 The method

Let G be a reductive group over an algebraically closed field k of characteristic p . Let Φ be the roots for G with $\Delta \subset \Phi^+ \subset \Phi$ the simple and positive roots, respectively, associated to a fixed maximal torus T of G .

[I want to see if this works for arbitrary rank] Let $P_\alpha < G$ be the parabolic subgroup of G corresponding to the simple root $\alpha \in \Delta$, with Levi subgroup L_α and unipotent radical V_α :

$$\begin{aligned} V_\alpha = R_u(P_\alpha) &= \langle U_\delta \in \Phi^+ \mid \delta \neq \alpha \rangle, \\ P_\alpha &= L_\alpha \ltimes V_\alpha. \end{aligned}$$

By [reference] there exists a homomorphism ρ_0 from $SL_2(k)$ into L_α under which

$$\begin{aligned} \rho_0 \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u) \\ \rho_0 \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u) \end{aligned}$$

We fix an integer $r > 0$ and define ρ_r to be the homomorphism from $SL_2(k)$ into L_α composed of ρ_0 and the Frobenius map,

$$\begin{aligned} F_r &: SL_2(k) \rightarrow SL_2(k) \\ (A_{ij}) &\mapsto (A_{ij})^{p^r}. \end{aligned}$$

That is

$$\rho_r = \rho_0 \circ F_r,$$

and satisfies

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} &= \epsilon_{-\alpha}(u^{p^r}). \end{aligned}$$

We let $SL_2(k)$ act on V_α via ρ_r and we consider 1-cocycles $\sigma \in Z^1(SL_2(k), V_\alpha)$. As we are interested in 1-cohomology classes, we may as well only consider those 1-cocycles that are zero on a maximal torus of $SL_2(k)$ [reference], so let $\sigma \in Z^1(SL_2(k), V_\alpha)$ such that

$$\sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) = 0,$$

for all $t \in k^*$. We can say a few things about these particular 1-cocycles which help us calculate the 1-cohomology. We refer to the results in [reference]:

$$\begin{aligned} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} \epsilon_{\delta}((t^{p^r})^{\langle \delta, \alpha \rangle} \lambda_{\delta}) \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \prod_{\delta} \epsilon_{\delta}(\lambda_{\delta}) &= \prod_{\delta} n_{\alpha} \epsilon_{\delta}(\lambda_{\delta}) n_{\alpha}^{-1}, \end{aligned}$$

where $n_{\alpha} = \epsilon_{\alpha}(1)\epsilon_{-\alpha}(-1)\epsilon_{\alpha}(1)$ and λ_{δ} are elements of the underlying field k .

Lemma 5.1.

$$\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_{\delta}(x_{\delta}(u)),$$

where δ ranges $\Phi^+ - \{\alpha\}$ such that $\langle \delta, \alpha \rangle > 0$, and $x_{\delta} \in k[T]$ are polynomials in one variable.

Proof. We have the chain of morphisms

$$k \cong \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \xrightarrow{i} SL_2(k) \xrightarrow{\sigma} V_\alpha \xrightarrow{\pi_\delta} k$$

where i is the inclusion map and π_δ the projection onto the root subgroup V_δ . Hence, by the definition

$$x_\delta = \pi_\delta \circ \sigma \circ i$$

is a morphism from $k \rightarrow k$.

Now since

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

we use the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

Therefore

$$x_\delta(t^2 u) = (t^{p^r})^{\langle \delta, \alpha \rangle} x_\delta(u).$$

Since x_δ is a polynomial function there can only be non-negative powers of t on the left-hand side of the equality which forces $\langle \delta, \alpha \rangle \geq 0$. However, if $\langle \delta, \alpha \rangle = 0$ then x_δ is constant and hence zero, as σ is zero on $\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$. Therefore the non-zero x_δ occur precisely when $\langle \delta, \alpha \rangle > 0$. \square

Next we prove a couple of useful facts about root systems not containing G_2 or C_3 .

Lemma 5.2. *Suppose Φ is not of type G_2 and let $\alpha, \beta \in \Phi$. If $\alpha + \beta \in \Phi$ then $\langle \alpha, \beta \rangle \leq 0$.*

Proof.

$$\langle \alpha, \beta \rangle > 0 \iff (\alpha, \beta) > 0 \iff \cos(\theta) > 0,$$

where θ is the angle between α and β . Hence acute angles correspond to positive pairs. Referring to the A_2 and B_2 root system diagrams we find that no two roots meeting at an acute angle add to give another root. Therefore if $\langle \alpha, \beta \rangle > 0$ then $\alpha + \beta \notin \Phi$. \square

We must exclude the case $\Phi = G_2$ here since $\alpha, 2\alpha + \beta$ and $3\alpha + \beta$ are all roots (α short) but $\langle \alpha, 2\alpha + \beta \rangle = 1$.

Lemma 5.3. *Suppose Φ does not contain G_2 or C_3 . Let $\delta_1, \delta_2 \in \Phi$ and $\gamma \in \Delta$ be roots such that $\langle \delta_i, \gamma \rangle > 0$ ($i = 1, 2$). If $\delta_1 + \delta_2$ is a root, then δ_1 and δ_2 are of opposite sign.*

Proof. Suppose $\delta_1 + \delta_2 \in \Phi$. Let θ_i be the absolute value of the angle between δ_i and γ , ($i = 1, 2$) and let θ_3 be the absolute value of the angle between δ_1 and δ_2 . Then

$$\begin{aligned} \langle \delta_i, \gamma \rangle &> 0 & (i = 1, 2) \\ \implies (\delta_i, \gamma) &> 0 \\ \implies \cos(\theta_i) &> 0 \\ \implies \theta_i &< \pi/2, \end{aligned}$$

and similarly, using 5.2

$$\begin{aligned} \langle \delta_1, \delta_2 \rangle &\leq 0 \\ \implies \theta_3 &\geq \pi/2. \end{aligned}$$

So, without loss of generality, this leads to consider four cases:

- 1:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = 2\pi/3;$
- 2:** $\theta_1 = \pi/3, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 3:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/3, \quad \theta_3 = \pi/2;$
- 4:** $\theta_1 = \pi/4, \quad \theta_2 = \pi/4, \quad \theta_3 = \pi/2.$

[Wow, probably need more explanation there]

For the cases in which $\theta_3 = \pi/2$ we can reason from the root system diagrams that δ_1 and δ_2 lie in a B_2 subsystem of Φ , and they have the same length. Since $\delta_1 + \delta_2$ is a root it must be that δ_1 and δ_2 are short roots and their sum is a long root. However we must rule out the third case. For if $\theta_1 = \pi/4$ then δ_1 and γ are roots of different length

in a B_2 subsystem, but $\theta_2 = \pi/3$ implies that δ_2 and γ are roots of the same length in an A_2 subsystem, which is absurd.

The three roots must lie in a plane for cases one and four. That is they lie in some rank 2 subsystem; A_2 and B_2 respectively. Consulting the root system diagrams yields $\gamma = \delta_1 + \delta_2$ and the result holds.

In the second case we see that δ_1, δ_2 and γ do not lie together in a rank 2 subsystem, and that these roots are the same length which implies that γ is a short root. In fact, since a pair short roots lie in subsystems of type A_2 it must be that the rank 3 subsystem in which the four roots lie is of type C_3 . [Picture?][Wow, is that right? Maybe just say ‘we will show that they lie in a C_3 subsystem’.] \square

We return to the 1-cohomology calculation but assume that G does not contain G_2 or C_3 .

Corollary 5.4. *For any $u_1, u_2 \in k$*

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right).$$

Furthermore, the x_δ are homomorphisms.

Proof. We have

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) = \epsilon_\alpha(u_1^{p^r}) \prod_{\delta} \epsilon_\delta(x_\delta(u_2)) \epsilon_\alpha(-u_1^{p^r}),$$

with $\langle \delta, \alpha \rangle > 0$. By 5.2 $\alpha + \delta \notin \Phi$ so each ϵ_δ commutes with the ϵ_α . \square

Corollary 5.5. *The image of the group of upper triangular matrices of $SL_2(k)$ under σ lies in a product of commuting root groups of V_α .*

Proof. First consider

$$\sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \prod_{\delta} \epsilon_\delta(x_\delta(b)).$$

Suppose the roots δ_1 and δ_2 appear on the right hand side. By 5.1 $\delta_i \in \Phi^+ - \{\alpha\}$ and $\langle \delta_i, \alpha \rangle > 0$ ($i = 1, 2$), so 5.3 asserts that $\delta_1 + \delta_2$ is no root, hence, ϵ_{δ_1} and ϵ_{δ_2} commute.

Therefore, for any $a, b \in k$ with $a \neq 0$

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & ab \\ 0 & a^{-1} \end{pmatrix} \right) &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) \\ &= \prod_{\delta} \epsilon_{\delta} \left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b) \right). \end{aligned}$$

□

Since the x_{δ} are homomorphisms from $k \rightarrow k$ they must take the form

$$T \mapsto \sum_i \mu_i T^{p^i},$$

for some μ_i in k . Furthermore, combining the calculation in the proof of 5.1 with the result 5.4 we get that

$$\prod_{\delta} \epsilon_{\delta} (x_{\delta}(a^2 b)) = \prod_{\delta} \epsilon_{\delta} \left(a^{\langle \delta, \alpha \rangle p^r} x_{\delta}(b) \right),$$

severely restricting the possible polynomials x_{δ} . In fact, they are confined to be polynomials involving just one term, and the degree has already been decided upon fixing the integer r in the definition of ρ_r . For suppose x_{δ} and hence some μ_j is non-zero. Then equating the coefficients of b in the equality directly above yields

$$\begin{aligned} \mu_j a^{2p^j} &= \mu_j a^{\langle \delta, \alpha \rangle p^r} \\ \implies 2p^j &= \langle \delta, \alpha \rangle p^r. \end{aligned}$$

In [4] it is shown that the possible pairings of any two roots are bounded by ± 3 . Hence by 5.1 $\langle \delta, \alpha \rangle = 1, 2$ or 3 . It is now clear that if $\langle \delta, \alpha \rangle = 3$ then $x_{\delta} = 0$.

If $\langle \delta, \alpha \rangle = 1$ the characteristic of k must be 2 and $j = r - 1$. Otherwise $\langle \delta, \alpha \rangle = 2$ and $j = r$, but the characteristic of k is so far unrestricted.

Example 5.1. Let $G = G_2$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta\}$ with β being the long root. Let

$$V_{\alpha} = R_u(P_{\alpha}) = \langle U_{\beta}, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle.$$

We will write v in V_{α} in angled brackets for compactness:

$$\langle v_1, v_2, v_3, v_4, v_5 \rangle := \epsilon_{\beta}(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_{3\alpha+\beta}(v_4) \epsilon_{3\alpha+2\beta}(v_5) \in V_{\alpha}$$

The group law for V_α is

$$u * v = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4, u_5 + v_5 + 3u_3v_2 - u_4v_1, \rangle.$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-3p^r}v_1, a^{-p^r}v_2, a^{p^r}v_3, a^{3p^r}v_4, v_5 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = 0.$$

By 5.1

$$\sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \langle 0, 0, x_3(b), x_4(b), 0 \rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r}x_3(b) \\ x_4(a^2b) &= a^{3p^r}x_4(b). \end{aligned}$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & b_1 + b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

yields

$$\begin{aligned} x_3(b_1 + b_2) &= x_3(b_1) + x_3(b_2) \\ x_4(b_1 + b_2) &= x_4(b_1) + x_4(b_2) - 3b_1^{p^r}x_3(b_2). \end{aligned}$$

We see that x_3 is a homomorphism, so it is of the form

$$x_3(b) = \sum_i \mu_i b^{p^i}.$$

Suppose $x_3 \neq 0$. Then some $\mu_j \neq 0$ and

$$\begin{aligned}\mu_j(a^2b)^{p^j} &= a^{p^r} \mu_j b^{p^j} \\ \implies a^{2p^j} &= a^{p^r} \\ \implies p &= 2.\end{aligned}$$

But then

$$\begin{aligned}x_4(0) = x_4(b+b) &= x_4(b) + x_4(b) - 3b^{2^r} x_3(b) \\ &= b^{2^r} x_3(b),\end{aligned}$$

implies that x_3 is constant, hence zero.

Therefore $x_3 = 0$, so x_4 is a homomorphism:

$$x_4(b) = \sum_i \nu_i b^{p^i}.$$

If $x_4 \neq 0$ then there is a $\nu_j \neq 0$ and we get

$$\begin{aligned}\nu_j(a^2b)^{p^j} &= a^{3p^r} \nu_j b^{p^j} \\ \implies a^{2p^j} &= a^{3p^r} \\ \implies 2p^j &= 3p^r,\end{aligned}$$

which implies that 2 divides p and 3 divides p , a contradiction. Hence $x_4 = 0$ and

$$\sigma\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = 0.$$

Example 5.2. Let $G = C_3$. Fix a maximal torus, labeling the positive simple roots $\Delta = \{\alpha, \beta, \gamma\}$ with γ being the long root and connected to β . Let

$$V_\alpha = R_u(P_\alpha) = \langle U_\beta, U_\gamma, U_{\alpha+\beta}, U_{\beta+\gamma}, U_{\alpha+\beta+\gamma}, U_{2\beta+\gamma}, U_{\alpha+2\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle.$$

Again we will write v in V_α in angled brackets for ease of notation:

$$\begin{aligned}\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8 \rangle &:= \\ \epsilon_\beta(v_1)\epsilon_\gamma(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{2\beta+\gamma}(v_6)\epsilon_{\alpha+2\beta+\gamma}(v_7)\epsilon_{2\alpha+2\beta+\gamma}(v_8) &\in V_\alpha\end{aligned}$$

The group law for V_α is

$$\begin{aligned} u * v = & \\ \langle u_1 + v_1, u_2 + v_2, u_3 + v_3, u_4 + v_4 + u_2 + v_1, u_5 + v_5 - u_3v_2, u_6 + v_6 + u_2v_1^2 + 2u_4v_1, \\ & u_7 + v_7 + u_2u_3v_1 + u_2v_1v_3 + u_5v_1 + u_4v_3, u_8 + v_8 - u_3^2v_2 - 2u_3v_2v_3 + 2u_5v_3 \rangle. \end{aligned}$$

We compute the action

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot v = \langle a^{-p^r}v_1, v_2, a^{p^r}v_3, a^{-p^r}v_4, a^{p^r}v_5, a^{-2p^r}v_6, v_7, a^{2p^r}v_8 \rangle.$$

Let σ be in $Z^1(SL_2, V_\alpha)$ such that

$$\sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) = 0.$$

By 5.1

$$\sigma \left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right) = \langle 0, 0, x_3(b), 0, x_5(b), 0, 0, x_8(b) \rangle.$$

Applying σ to both sides of the identity

$$\begin{pmatrix} 1 & a^2b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

yields

$$\begin{aligned} x_3(a^2b) &= a^{p^r} x_3(b) \\ x_5(a^2b) &= a^{p^r} x_5(b) \\ x_8(a^2b) &= a^{2p^r} x_8(b). \end{aligned}$$

Since the polynomials x_3, x_5, x_8 are homomorphisms (5.2) we get

$$\begin{aligned} \sum_i \lambda_i (a^2b)^{p^i} &= a^{p^r} \sum_i \lambda_i b^{p^i} \\ \sum_i \mu_i (a^2b)^{p^i} &= a^{p^r} \sum_i \mu_i b^{p^i} \\ \sum_i \nu_i (a^2b)^{p^i} &= a^{2p^r} \sum_i \nu_i b^{p^i}, \end{aligned}$$

from which we can deduce

$$\begin{aligned} x_3 \neq 0 &\implies x_3(b) = \lambda b^{p^{r+1}}, p = 2 \\ x_5 \neq 0 &\implies x_5(b) = \mu b^{p^{r+1}}, p = 2 \\ x_8 \neq 0 &\implies x_8(b) = \nu b^{p^r}. \end{aligned}$$

Therefore, if the image of the group of upper (uni-)triangular matrices of SL_2 under σ is $\langle U_{\alpha+\beta}, U_{\alpha+\beta+\gamma}, U_{2\alpha+2\beta+\gamma} \rangle$ then the characteristic of k must be 2, and so the image is a product of commuting root groups.

We may now state and prove the main result.

[would like]

Theorem 5.6. *Let G be a reductive linear algebraic group over a closed field of positive characteristic p and let $\Gamma = SL_2(k)$. Then the answer to the algebraic interpretation of Külshammer's Second Problem [ref] is "yes".*

Proof. Need to:

- handle arguments above with G possibly containing G_2 and C_3 .
- drop the restriction of rank-1 parabolics
- now we have abelian 1-cohomology and can apply result from previous chapter

□

5.2 A rank 1 calculation

[INCLUDE G_2 OR B_2 CALCULATIONS]

Let T be a maximal torus of B_2 over an algebraically closed field k of characteristic p . We label the positive roots for B_2 as $\alpha, \beta, \alpha + \beta, 2\alpha + \beta$. We have from [2, §33.4]:

$$\begin{aligned} \epsilon_\beta(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_\beta(y)\epsilon_{\alpha+\beta}(xy)\epsilon_{2\alpha+\beta}(x^2y) \\ \epsilon_{\alpha+\beta}(y)\epsilon_\alpha(x) &= \epsilon_\alpha(x)\epsilon_{\alpha+\beta}(y)\epsilon_{2\alpha+\beta}(2xy), \end{aligned}$$

and

$$\begin{aligned}
n_\alpha \epsilon_\beta(x) n_\alpha^{-1} &= \epsilon_{2\alpha+\beta}(x) \\
n_\alpha \epsilon_{\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_{\alpha+\beta}(-x) \\
n_\alpha \epsilon_{2\alpha+\beta}(x) n_\alpha^{-1} &= \epsilon_\beta(x) \\
n_\beta \epsilon_\alpha(x) n_\beta^{-1} &= \epsilon_{\alpha+\beta}(x) \\
n_\beta \epsilon_{\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_\alpha(-x) \\
n_\beta \epsilon_{2\alpha+\beta}(x) n_\beta^{-1} &= \epsilon_{2\alpha+\beta}(x)
\end{aligned}$$

A proper parabolic subgroup of B_2 is conjugate to one of

$$\begin{aligned}
P_\alpha &= \langle B, U_{-\alpha} \rangle \\
P_\beta &= \langle B, U_{-\beta} \rangle,
\end{aligned}$$

where B is the Borel subgroup of B_2 containing T

$$B = \langle T, U_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle.$$

The two parabolic subgroups have the Levi decompositions

$$\begin{aligned}
P_\alpha &= L_\alpha \ltimes R_u(P_\alpha) \\
&= \langle T, U_\alpha, U_{-\alpha} \rangle \ltimes \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle \\
P_\beta &= L_\beta \ltimes R_u(P_\beta) \\
&= \langle T, U_\beta, U_{-\beta} \rangle \ltimes \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle
\end{aligned}$$

5.2.1 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (short) root α :

$$V = R_u(P_\alpha) = \langle U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\alpha$ defined by

$$\begin{aligned}\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\alpha(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \alpha^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\alpha,\end{aligned}$$

where r is some non-negative integer.

Note that V is abelian. Now SL_2 acts on V via ρ_r : write $\mathbf{v} = \epsilon_\beta(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)$ in V as a column vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\alpha(-u^{p^r}) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_\alpha(-u^{p^r}) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\beta(v_1) \epsilon_\alpha(-u^{p^r}) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(-2u^{p^r} v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(u^{p^r}) \epsilon_\alpha(-u^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(-u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{2p^r} v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2) \\
&= \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2 - u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1) \\
&= \begin{pmatrix} v_1 \\ v_2 - u^{p^r} v_1 \\ v_3 - 2u^{p^r} v_2 + u^{2p^r} v_1 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \alpha^\vee(t^{p^r}) \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\alpha^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\beta(\beta(\alpha^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\alpha^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\alpha^\vee(t^{p^r})) v_3) \\
&= \epsilon_\beta((t^{p^r})^{\langle \beta, \alpha \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha + \beta, \alpha \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha + \beta, \alpha \rangle} v_3) \\
&= \begin{pmatrix} t^{-2p^r} v_1 \\ v_2 \\ t^{2p^r} v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= n_\alpha \epsilon_\beta(v_1) n_\alpha^{-1} n_\alpha \epsilon_{\alpha+\beta}(v_2) n_\alpha^{-1} n_\alpha \epsilon_{2\alpha+\beta}(v_3) n_\alpha^{-1} \\
&= \epsilon_{2\alpha+\beta}(v_1) \epsilon_{\alpha+\beta}(-v_2) \epsilon_\beta(v_3) \\
&= \epsilon_\beta(v_3) \epsilon_{\alpha+\beta}(-v_2) \epsilon_{2\alpha+\beta}(v_1) \\
&= \begin{pmatrix} v_3 \\ -v_2 \\ v_1 \end{pmatrix}.
\end{aligned}$$

We can combine the above calculations to get an explicit formula for the action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} d^{2p^r} v_1 - 2(cd)^{p^r} v_2 + c^{2p^r} v_3 \\ (ad + bc)^{p^r} v_2 - (bd)^{p^r} v_1 - (ac)^{p^r} v_3 \\ b^{2p^r} v_1 - 2(ab)^{p^r} v_2 + a^{2p^r} v_3 \end{pmatrix}$$

Now let σ' in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$. By [some reference] σ' is conjugate to a 1-cocycle σ that has the additional property that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* . Since we are ultimately concerned with the 1-cohomology, that is, conjugacy classes of 1-cocycles, we may proceed with σ instead.

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix}.$$

Now we make use of the very simple relations

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.1)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.2)$$

to get further information on the polynomials p_i ($i = 1, 2, 3$).

If we apply σ to both sides of (5.1), using the 1-cocycle condition on the right hand side, then we get

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) + \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_1(t^2u) = t^{-2p^r} p_1(u) \quad (5.3)$$

$$p_2(t^2u) = p_2(u) \quad (5.4)$$

$$p_3(t^2u) = t^{2p^r} p_3(u). \quad (5.5)$$

From (5.4) it is clear that p_2 is constant, so there is a λ in k such that $p_2(x) = \lambda$ for all x in k . Now notice that on the left hand side of (5.3) there are only non-negative powers of t , and on the right hand side there are only non-positive powers of t . This equality is only satisfied if $p_1(x) = 0$ for all x in k , so p_1 is the zero polynomial.

We apply σ to (5.2) and using the 1-cocycle condition to obtain

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right). \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.6)$$

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2) - 2u_1^{p^r} p_2(u_2). \quad (5.7)$$

Since p_2 is constant, (5.6) implies that p_2 is the zero polynomial, which means (5.7) becomes

$$p_3(u_1 + u_2) = p_3(u_1) + p_3(u_2).$$

Hence p_3 is a homomorphism, that is, of the form

$$p_3(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.8)$$

for some μ_i in k .

Now combining (5.5) and (5.8) yields

$$\sum_{i=0}^N \mu_i (t^2u)^{p^i} = t^{2p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.9)$$

If p_3 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating the coefficients of u in (5.9) we get

$$\begin{aligned}\mu_l t^{2p^l} &= \mu_l t^{2p^r} \\ \implies p^l &= p^r.\end{aligned}$$

Therefore $l = r$. This means that the only non-zero μ_i is already specified by the choice of r in defining ρ_r .

Letting $\mu_l = \mu$ in k , we have

$$\begin{aligned}\sigma\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \sigma\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma\left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(a^{-1}b)^{p^r} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix}.\end{aligned}$$

If we are to find a non-trivial 1-cohomology $H^1(SL_2, V)$ then σ cannot be a 1-coboundary. But if the characteristic of k , p , is not equal to 2 then by setting \mathbf{v} in V as

$$\mathbf{v} = \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix},$$

we get for all a in k^* and all b in k

$$\begin{aligned}
 \chi_v \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \mathbf{v} - \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ \mu 2^{-1} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \mu 2^{-1} \\ -\mu(ab)^{p^r} \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{p^r} \end{pmatrix} \\
 &= \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right).
 \end{aligned}$$

That is, σ takes the value of a 1-coboundary on the subgroup of upper triangular matrices of SL_2 . By [some reference], this means that σ is a 1-coboundary from the whole of $SL_2 \rightarrow V$, and hence the 1-cohomology $H^1(SL_2, V)$ is trivial. Therefore it is necessary to proceed with $p = 2$:

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}. \quad (5.10)$$

We can use an entirely similar argument to the one in calculating (5.10) to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

remembering that k now has characteristic 2. On the one hand

$$\begin{aligned}
\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu + \mu' \\ \mu \\ \mu \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \mu \end{pmatrix} + \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu' \end{pmatrix} = \begin{pmatrix} \mu + \mu' \\ \mu' \\ \mu + \mu' \end{pmatrix}.
\end{aligned}$$

On the other hand, by applying σ to both sides of the equality

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix},$$

we get

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

Therefore $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is an element of V that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$. Referring to the formula for the action of SL_2 on V we see that such an element of V is of the form

$$\begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix},$$

which implies that $\mu = \mu'$.

Finally, consider

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right).$$

If $c = 0$ then we already have

$$\sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 0 \\ \mu(ab)^{2^r} \end{pmatrix}.$$

Otherwise c^{-1} exists and we can compute

$$\begin{aligned} \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\ &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ \mu(cd)^{2^r} \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^r} \\ \mu \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \mu(ac^{-1})^{2^r} \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu + (ac^{-1})^{2^r} \mu(cd)^{2^r} \\ (ac^{-1})^{2^r+1} \mu(cd)^{2^r} \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(1 + ad)^{2^r} \\ \mu(ac^{-1})^{2^r} (1 + ad)^{2^r} \end{pmatrix} = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

In fact, we see that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

holds in either case.

[Show converse - Steinberg relations]

Now if σ is in the same conjugacy class as τ then by [some reference]

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As before, we consider 1-cocycles that are zero on $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, so this means considering

\mathbf{v} that is fixed by the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

Therefore each μ in k corresponds to a conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$ where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^r} \\ \mu(bc)^{2^r} \\ \mu(ab)^{2^r} \end{pmatrix},$$

and the 1-cocycle τ is in the class $[\sigma_\mu]$ if there is a \mathbf{v} in V such that

$$\tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \mathbf{v} + \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}.$$

As discussed in [ref previous section] we can use this result to find the 1-cocycles from $SL_2 \rightarrow P_\alpha$ by considering the action of $Z(L_\alpha)^\circ$, the connected centre of the Levi subgroup L_α . Now,

$$Z(L_\alpha)^\circ = \langle \gamma^\vee(x) \mid x \in k \rangle$$

where γ is a root in $\Phi_{\alpha,\beta}$ such that

$$\langle \alpha, \gamma \rangle = 0. \quad (5.11)$$

Since $\gamma = m\alpha + n\beta$ for some integers m, n , we have

$$\langle \alpha, \gamma \rangle = \langle \alpha, m\alpha + n\beta \rangle \quad (5.12)$$

and so

$$\begin{aligned} \langle \alpha, m\alpha + n\beta \rangle &= 0 \\ \iff \langle m\alpha + n\beta, \alpha \rangle &= 0 \\ \iff m\langle \alpha, \alpha \rangle + n\langle \beta, \alpha \rangle &= 0 \\ \iff 2m - 2n &= 0 \\ \iff m &= n. \end{aligned}$$

Therefore $Z(L_\alpha)^\circ = \langle (\alpha + \beta)^\vee(x) \mid x \in k \rangle$. Taking an element $\mathbf{s} = (\alpha + \beta)^\vee(s)$ of $Z(L_\alpha)^\circ$ we compute the action of \mathbf{s} on the 1-cocycle σ_μ as follows:

$$\begin{aligned} (\mathbf{s} \cdot \sigma_\mu) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (\alpha + \beta)^\vee(s) \epsilon_\beta (\mu(cd)^{2^r}) \epsilon_{\alpha+\beta} (\mu(bc)^{2^r}) \epsilon_{2\alpha+\beta} (\mu(ab)^{2^r}) (\alpha + \beta)^\vee(s)^{-1} \\ &= \epsilon_\beta \left(s^{\langle \beta, \alpha+\beta \rangle} \mu(cd)^{2^r} \right) \epsilon_{\alpha+\beta} \left(s^{\langle \alpha+\beta, \alpha+\beta \rangle} \mu(bc)^{2^r} \right) \epsilon_{2\alpha+\beta} \left(s^{\langle 2\alpha+\beta, \alpha+\beta \rangle} \mu(ab)^{2^r} \right) \\ &= \begin{pmatrix} (s^2 \mu)(cd)^{2^r} \\ (s^2 \mu)(bc)^{2^r} \\ (s^2 \mu)(ab)^{2^r} \end{pmatrix}. \end{aligned}$$

So we see that the infinitely many conjugacy classes of 1-cocycles from $SL_2 \rightarrow V$ collapse

to just two classes when we consider the action of $Z(L_\alpha)^\circ$, that is, moving from V -conjugacy to P_α -conjugacy:

$$\begin{aligned} [\sigma_0] &= \{\sigma_0\} \\ [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}. \end{aligned}$$

5.2.2 Example

Let V be the unipotent radical of the parabolic subgroup of B_2 defined by the (long) root β :

$$V = R_u(P_\beta) = \langle U_\alpha, U_{\alpha+\beta}, U_{2\alpha+\beta} \rangle,$$

and let ρ_r be the homomorphism from $SL_2 \rightarrow L_\beta$ defined by

$$\begin{aligned} \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\beta(u^{p^r}) \\ \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \beta^\vee(t^{p^r}) \\ \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} &= n_\beta, \end{aligned}$$

where r is some non-negative integer.

Note that V is not abelian in general. The Group Law for V can be computed as follows.

Let \mathbf{v}, \mathbf{w} in V . We have, using notation similar to the previous example

$$\begin{aligned} \mathbf{v} * \mathbf{w} &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1)\epsilon_\alpha(w_1)\epsilon_{\alpha+\beta}(v_2)\epsilon_{2\alpha+\beta}(2v_2w_1)\epsilon_{\alpha+\beta}(w_2)\epsilon_{2\alpha+\beta}(v_3)\epsilon_{2\alpha+\beta}(w_3) \\ &= \epsilon_\alpha(v_1 + w_1)\epsilon_{\alpha+\beta}(v_2 + w_2)\epsilon_{2\alpha+\beta}(v_3 + w_3 + 2v_2w_1) \\ &= \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 + 2v_2w_1 \end{pmatrix}. \end{aligned}$$

Now we compute the action of SL_2 on V via ρ_r . Let \mathbf{v} be an element of V :

$$\begin{aligned}
\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right)^{-1} \\
&= \epsilon_\beta(u^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_\beta(u^{p^r}) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(u^{p^r} v_1) \epsilon_{2\alpha+\beta}(u^{p^r} v_1^2) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) \epsilon_\beta(u^{p^r}) \epsilon_\beta(-u^{p^r}) \\
&= \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2 + u^{p^r} v_1) \epsilon_{2\alpha+\beta}(v_3 + u^{p^r} v_1^2) \\
&= \begin{pmatrix} v_1 \\ v_2 + u^{p^r} v_1 \\ v_3 + u^{p^r} v_1^2 \end{pmatrix} \\
\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right)^{-1} \\
&= \beta^\vee(t^{p^r}) \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) (\beta^\vee(t^{p^r}))^{-1} \\
&= \epsilon_\alpha(\alpha(\beta^\vee(t^{p^r})) v_1) \epsilon_{\alpha+\beta}((\alpha + \beta)(\beta^\vee(t^{p^r})) v_2) \epsilon_{2\alpha+\beta}((2\alpha + \beta)(\beta^\vee(t^{p^r})) v_3) \\
&= \epsilon_\alpha((t^{p^r})^{\langle \alpha, \beta \rangle} v_1) \epsilon_{\alpha+\beta}((t^{p^r})^{\langle \alpha+\beta, \beta \rangle} v_2) \epsilon_{2\alpha+\beta}((t^{p^r})^{\langle 2\alpha+\beta, \beta \rangle} v_3) \\
&= \begin{pmatrix} t^{-p^r} v_1 \\ t^{p^r} v_2 \\ v_3 \end{pmatrix} \\
\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \mathbf{v} &= \rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{v} \left(\rho_r \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) \epsilon_{\alpha+\beta}(v_2) \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= n_\beta \epsilon_\alpha(v_1) n_\beta^{-1} n_\beta \epsilon_{\alpha+\beta}(v_2) n_\beta^{-1} n_\beta \epsilon_{2\alpha+\beta}(v_3) n_\beta^{-1} \\
&= \epsilon_{\alpha+\beta}(v_1) \epsilon_\alpha(-v_2) \epsilon_{2\alpha+\beta}(v_3) \\
&= \epsilon_\alpha(-v_2) \epsilon_{\alpha+\beta}(v_1) \epsilon_{2\alpha+\beta}(v_3 - 2v_1 v_2) \\
&= \begin{pmatrix} -v_2 \\ v_1 \\ v_3 - 2v_1 v_2 \end{pmatrix}.
\end{aligned}$$

Or, more explicitly

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} c^{p^r} v_2 + d^{p^r} v_1 \\ a^{p^r} v_2 + b^{p^r} v_1 \\ v_3 + (ac)^{p^r} v_2^2 + (bd)^{p^r} v_1^2 + 2(bc)^{p^r} v_1 v_2 \end{pmatrix}.$$

As in the previous example we let σ in $Z^1(SL_2, V)$ be a 1-cocycle from $SL_2 \rightarrow V$ such that

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

for all t in k^* , and

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ p_2(u) \\ p_3(u) \end{pmatrix},$$

for all u in k .

We use the same two identities to further investigate the 1-cocycle:

$$\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} \quad (5.13)$$

$$\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix}, \quad (5.14)$$

Applying σ to both sides of (5.13), using the 1-cocycle condition on the right hand side, we get

$$\sigma \left(\begin{pmatrix} 1 & t^2u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right).$$

That is

$$p_1(t^2u) = t^{-p^r} p_1(u) \quad (5.15)$$

$$p_2(t^2u) = t^{p^r} p_2(u) \quad (5.16)$$

$$p_3(t^2u) = p_3(u). \quad (5.17)$$

From (5.17) we find that p_3 is constant-valued, say $p_3(x) = \lambda$ in k for all x in k . From (5.15) we see that there are only non-negative powers of t on the left hand side and only non-positive powers the right hand side. Therefore p_1 is the zero polynomial.

Now applying σ to both sides of (5.14):

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \right) * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) \\ \lambda \end{pmatrix} * \begin{pmatrix} 0 \\ p_2(u_2) \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ p_2(u_1) + p_2(u_2) \\ 2\lambda \end{pmatrix}
 \end{aligned}$$

That is,

$$p_2(u_1 + u_2) = p_2(u_1) + p_2(u_2) \quad (5.18)$$

$$\lambda = 2\lambda. \quad (5.19)$$

By (5.19) we see that p_3 is in fact the zero polynomial, and (5.18) implies that p_2 is a homomorphism, that is, of the form

$$p_2(x) = \sum_{i=0}^N \mu_i x^{p^i}, \quad (5.20)$$

for some μ_i in k .

Now combining (5.16) and (5.20) yields

$$\sum_{i=0}^N \mu_i (t^2 u)^{p^i} = t^{p^r} \sum_{i=0}^N \mu_i u^{p^i}. \quad (5.21)$$

If p_2 is not the zero polynomial then there is a non-zero μ_l for some index l . By equating coefficients of u^{p^i} in (5.21) we get

$$\begin{aligned}
 \mu_l t^{2p^l} &= \mu_l t^{p^r} \\
 \implies 2p^l &= p^r.
 \end{aligned}$$

Thus 2 divides p^r , and since p is a prime, $p = 2$. Furthermore $l = r - 1$. This means that the non-zero μ_l is already specified by the choice of r in defining ρ_r , and that r must be non-zero if p_2 is to be non-zero.

Referring to the Group Law we see that V is abelian in characteristic 2, so we will use the '+' symbol for combining elements of V from now on.

Proceeding with $p = 2$, $r > 0$ and letting $\mu_l = \mu$, we have

$$\begin{aligned}
 \sigma \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right) + \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(a^{-1}b)^{2^{r-1}} \\ 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ \mu(ab)^{2^{r-1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

We can use an entirely similar argument to show that

$$\sigma \left(\begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu'(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix},$$

for some μ' in k .

We are now interested in the value of

$$\sigma \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right).$$

We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} \mu' \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu \\ \mu \\ \mu^2 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu' + \mu \\ \mu \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu \\ 0 \end{pmatrix} + \begin{pmatrix} \mu' + \mu \\ \mu' \\ \mu'^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu' + \mu \\ \mu' + \mu \\ \mu'^2 \end{pmatrix}.
\end{aligned}$$

Since $\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ for all t in k^* we must have $\mu' = \mu$.

Suppose $c \neq 0$. We have

$$\begin{aligned}
\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \right) + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \mu(cd)^{2^{r-1}} \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \mu^2 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ 0 \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ 0 \\ \mu^2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \mu(ac^{-1})^{2^{r-1}} \\ 0 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ (ac^{-1})^{2^r} \mu(cd)^{2^{r-1}} \\ \mu^2 + (ac^{-1})^{2^r} (\mu(cd)^{2^{r-1}})^2 \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ac^{-1} + a^2c^{-1}d)^{2^{r-1}} \\ \mu^2(1 + ad)^{2^r} \end{pmatrix} \\
&= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.
\end{aligned}$$

But the above result holds when $c = 0$ too, so we conclude that

$$\sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

[Show converse is true]

As in the previous example, we choose a \mathbf{v} in V that is fixed by $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ and compute

$$\begin{aligned}
 \tau \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= \mathbf{v} + \sigma \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} + \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ v_3 \end{pmatrix} \\
 &= \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix},
 \end{aligned}$$

which tells us that for each μ in k we get a distinct conjugacy class of 1-cocycles $[\sigma_\mu]$ from $SL_2 \rightarrow V$, where

$$\sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} \mu(cd)^{2^{r-1}} \\ \mu(ab)^{2^{r-1}} \\ \mu^2(bc)^{2^r} \end{pmatrix}.$$

But as before if we consider the action of $Z(L_\beta)$ on our 1-cocycles

$$\begin{aligned}
 (\mathbf{s} \cdot \sigma_\mu) \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) &= (2\alpha + \beta)^\vee(s) \cdot \sigma_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) \\
 &= \begin{pmatrix} (s\mu)(cd)^{2^{r-1}} \\ (s\mu)(ab)^{2^{r-1}} \\ (s\mu)^2(bc)^{2^r} \end{pmatrix}.
 \end{aligned}$$

our infinitely many V -conjugacy classes collapse to just two P_β -conjugacy classes:

$$\begin{aligned}
 [\sigma_0] &= \{\sigma_0\}, \\
 [\sigma_1] &= \{\sigma_\mu \mid \mu \in k^*\}.
 \end{aligned}$$

5.3 A rank 2 calculation

Is $Im(\rho_{r,s})$ irred in $L_{\gamma,\delta}$?

No $\rightarrow \text{Im}(\rho_{r,s})$ inside (a conjugate of) $P_\gamma(B_2)$ or $P_\delta(B_2)$. Then it's inside $P_\gamma = L_\gamma \ltimes R_u(P_\gamma)$ or $P_\delta = L_\delta \ltimes R_u(P_\delta)$, so it's inside L_γ or L_δ .

1) Know about non G-cr in B_2 , can I put them in an $A_1 A_1$?

1a) Can this sit inside a rank 1 Levi?

2) Use $B_2 = SO_5$.

3) Take $\text{Im}(\rho_{r,s})$, can we conjugate it into P_γ or P_δ ?

Let $\text{char}(k) = 2$ and set $V := \langle U_\phi \mid \phi \in \Phi^+, \phi \neq \gamma + \delta, \phi \neq \gamma + 2\delta \rangle$. We will write $\mathbf{v} = \epsilon_\alpha(v_1)\epsilon_\beta(v_2)\epsilon_{\alpha+\beta}(v_3)\epsilon_{\beta+\gamma}(v_4)\epsilon_{\alpha+\beta+\gamma}(v_5)\epsilon_{\beta+\gamma+\delta}(v_6)\epsilon_{\alpha+\beta+\gamma+\delta}(v_7)\epsilon_{\beta+\gamma+2\delta}(v_8)\epsilon_{\alpha+\beta+\gamma+2\delta}(v_9)\epsilon_{\beta+2\gamma+2\delta}(v_{10})\epsilon_{\alpha+\beta+2\gamma+2\delta}(v_{11})\epsilon_{\alpha+2\beta+2\gamma+2\delta}(v_{12}) \in V$ as a column vector:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \\ v_{10} \\ v_{11} \\ v_{12} \end{pmatrix}.$$

The Group Law on V is

$$\mathbf{u} * \mathbf{v} = \mathbf{u} + \mathbf{v} + \begin{pmatrix} 0 \\ 0 \\ u_2 v_1 \\ 0 \\ u_4 v_1 \\ 0 \\ u_6 v_1 \\ 0 \\ u_8 v_1 \\ 0 \\ u_{10} v_1 \\ u_{10} v_1 v_2 + u_8 v_1 v_4 + u_6^2 v_1 + u_{11} v_2 + u_{10} v_3 + u_9 v_4 + u_8 v_5 \end{pmatrix}.$$

For integers $r, s \geq 0$ we have a homomorphism $\rho_{r,s} : SL_2 \rightarrow \tilde{A}_1 \tilde{A}_1 < L_{\{\gamma, \delta\}}$ defined by

$$\begin{aligned} \rho_{r,s} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} &= \epsilon_\delta(u^{2^r}) \cdot \epsilon_{\gamma+\delta}(u^{2^s}) \\ \rho_{r,s} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} &= \delta^\vee(t^{2^r}) \cdot (\gamma + \delta)^\vee(t^{2^s}) \\ \rho_{r,s} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= n_\delta \cdot n_{\gamma+\delta} \end{aligned}$$

from which we obtain an action of SL_2 on V :

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v} = \begin{pmatrix} v_1 \\ c^{2^{s+1}}v_{10} + d^{2^{s+1}}v_2 \\ c^{2^{s+1}}v_{11} + d^{2^{s+1}}v_3 \\ c^{2^{r+1}}v_8 + d^{2^{r+1}}v_4 \\ c^{2^{r+1}}v_9 + d^{2^{r+1}}v_5 \\ v_6 + (bd)^{2^r}v_4 + (bd)^{2^s}v_2 + (ac)^{2^r}v_8 + (ac)^{2^s}v_{10} \\ v_7 + (bd)^{2^r}v_5 + (bd)^{2^s}v_3 + (ac)^{2^r}v_9 + (ac)^{2^s}v_{11} \\ a^{2^{r+1}}v_8 + b^{2^{r+1}}v_4 \\ a^{2^{r+1}}v_9 + b^{2^{r+1}}v_5 \\ a^{2^{s+1}}v_{10} + b^{2^{s+1}}v_2 \\ a^{2^{s+1}}v_{11} + b^{2^{s+1}}v_3 \\ v_{12} + (bd)^{2^{r+1}}v_4v_5 + (bd)^{2^{s+1}}v_2v_3 + (bc)^{2^{r+1}}(v_4v_9 + v_5v_8) \\ + (bc)^{2^{s+1}}(v_2v_{11} + v_3v_{10}) + (ac)^{2^{r+1}}(v_8v_9) + (ac)^{2^{s+1}}(v_{10}v_{11}) \end{pmatrix}$$

Now let σ be a 1-cocycle from SL_2 to V such that for all t in k^*

$$\sigma \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since σ is a morphism of varieties, each component of $\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ should be a polynomial function of u , so we let

$$\sigma \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_1(u) \\ \vdots \\ p_{12}(u) \end{pmatrix},$$

where each p_i ($1 \leq i \leq 12$) is as required. Applying σ to the identity

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & t^2 u \\ 0 & 1 \end{pmatrix},$$

gives rise to the following equations

$$p_i(t^2 u) = \begin{cases} p_i(u), & i = 1, 6, 7, 12 \\ t^{-2^{r+1}} p_i(u), & i = 4, 5 \\ t^{-2^{s+1}} p_i(u), & i = 2, 3 \\ t^{2^{r+1}} p_i(u), & i = 8, 9 \\ t^{2^{s+1}} p_i(u), & i = 10, 11 \end{cases} \quad (5.22)$$

It is clear that for $i = 1, 6, 7, 12$ the polynomials p_i must be constant-valued, say λ_i for some fixed λ_i in k (resp). Furthermore, since $p_i(t^2 u)$ involves only non-negative powers of t , p_i must be the zero polynomial for $i = 2, 3, 4, 5$. Now consider the identity

$$\begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u_1 + u_2 \\ 0 & 1 \end{pmatrix}.$$

Applying σ to both sides yields

$$\begin{aligned} p_1(u_1 + u_2) &= p_1(u_1) + p_1(u_2) \\ p_6(u_1 + u_2) &= p_6(u_1) + p_6(u_2) \\ p_7(u_1 + u_2) &= p_7(u_1) + p_7(u_2) + p_6(u_1)p_1(u_2) \\ p_8(u_1 + u_2) &= p_8(u_1) + p_8(u_2) \\ p_9(u_1 + u_2) &= p_9(u_1) + p_9(u_2) + p_8(u_1)p_1(u_2) \\ p_{10}(u_1 + u_2) &= p_{10}(u_1) + p_{10}(u_2) \\ p_{11}(u_1 + u_2) &= p_{11}(u_1) + p_{11}(u_2) + p_{10}(u_1)p_1(u_2) \\ p_{12}(u_1 + u_2) &= p_{12}(u_1) + p_{12}(u_2) + (p_6(u_1))^2 p_1(u_2). \end{aligned}$$

Now we see that the constant polynomials p_1, p_6, p_7, p_{12} must in fact be the zero polynomial and the remaining polynomials must be homomorphisms from $k \rightarrow k$. That is

for some w_j, x_j, y_j, z_j in k and all u in k

$$\begin{aligned} p_8(u) &= \sum_{j=0}^N w_j u^{2^j} \\ p_9(u) &= \sum_{j=0}^N x_j u^{2^j} \\ p_{10}(u) &= \sum_{j=0}^N y_j u^{2^j} \\ p_{11}(u) &= \sum_{j=0}^N z_j u^{2^j}, \end{aligned}$$

If σ is not the trivial 1-cocycle then one of the polynomials above is not the zero polynomial. Suppose for instance that p_8 is not the zero polynomial, so that $w_l \neq 0$ for some index $l \geq 0$. By (5.22)

$$\begin{aligned} \sum_{j=0}^N w_j (t^2 u)^{2^j} &= t^{2^{r+1}} \sum_{j=0}^N w_j u^{2^j} \\ \Rightarrow w_l (t^2 u)^{2^l} &= t^{2^{r+1}} w_l u^{2^l} \\ \Rightarrow l &= r. \end{aligned}$$

The same kind of calculation for the other polynomials shows that

$$\begin{aligned} p_8(u) &= w u^{2^r}, & p_9(u) &= x u^{2^r}, \\ p_{10}(u) &= y u^{2^s}, & p_{11}(u) &= z u^{2^s}, \end{aligned}$$

for some w, x, y, z in k .

So, we have

$$\begin{aligned} \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}. \end{aligned}$$

We apply the same argument using the fact that each component of $\sigma \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ is a polynomial function, say $p'_i(u)$ for all u in k , to get

$$\sigma \begin{pmatrix} d^{-1} & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y'(cd)^{2^s} \\ z'(cd)^{2^s} \\ w'(cd)^{2^r} \\ x'(cd)^{2^r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

for some w', x', y', z' in k .

From this we deduce that

$$\begin{aligned}
 \sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} * \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right) \\
 &= \begin{pmatrix} 0 \\ y + y' \\ z + z' \\ w + w' \\ x + x' \\ w' + y' \\ x' + z' \\ w + w' \\ x + x' \\ y + y' \\ z + z' \\ w'x' + y'z' \end{pmatrix}.
 \end{aligned}$$

Furthermore, since $\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$, we have

$$\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} n_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_6 \\ n_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ n_{12} \end{pmatrix},$$

for some n_1, n_6, n_7, n_{12} in k . So in fact

$$\begin{aligned}
 w' &= w \\
 x' &= x \\
 y' &= y \\
 z' &= z \\
 n_1 &= 0 \\
 n_6 &= w + y \\
 n_7 &= x + z \\
 n_{12} &= wx + yz.
 \end{aligned}$$

Consider $\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $c = 0$ then we already have

$$\begin{aligned}
 \sigma \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} &= \sigma \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} * \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \cdot \sigma \begin{pmatrix} 1 & a^{-1}b \\ 0 & 1 \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ w(ab)^{2^{r+1}} \\ x(ab)^{2^{r+1}} \\ y(ab)^{2^{s+1}} \\ z(ab)^{2^{s+1}} \\ 0 \end{pmatrix}.
 \end{aligned}$$

Otherwise, $c \neq 0$ and we can write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix},$$

and so

$$\begin{aligned}
\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \sigma \left(\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \sigma \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \sigma \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} * \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1 \end{pmatrix} \cdot \left(\sigma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \sigma \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ n_6 + w(ad)^{2^r} + y(ad)^{2^s} \\ n_7 + x(ad)^{2^r} + z(ad)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ n_{12} + wx(ad)^{2^{r+1}} + yz(ad)^{2^{s+1}} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.
\end{aligned}$$

We see that in any case

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

Conversely, suppose we have a map $\sigma : SL_2 \rightarrow V$ of the form

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix},$$

for some w, x, y, z in k and integers $r, s \geq 0$.

[Show σ is a 1-cocycle]

Next we shall describe $H^1(SL_2, V)$. Recall that a 1-cocycle τ' is in the same conjugacy class as σ if there is a \mathbf{v} in V such that

$$\tau'(g) = \mathbf{v} * \sigma(g) * g \cdot \mathbf{v}^{-1}$$

for all g in SL_2 . Furthermore, τ' is conjugate to some 1-cocycle τ , where τ has the added property that

$$\tau \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Thus σ is conjugate to τ by some \mathbf{v} in V that is fixed under the action of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$:

$$\begin{aligned} \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \mathbf{v} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \mathbf{v}^{-1} \\ &= \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} \end{pmatrix} * \sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_6 \\ v_7 + v_1 v_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_{12} + v_1 v_6^2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ (z + yv_1)(cd)^{2^s} \\ w(cd)^{2^r} \\ (x + wv_1)(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ (x + wv_1)(bc)^{2^r} + (z + yv_1)(bc)^{2^s} \\ w(ab)^{2^r} \\ (x + wv_1)(ab)^{2^r} \\ y(ab)^{2^s} \\ (z + yv_1)(ab)^{2^r} \\ w(x + wv_1)(bc)^{2^{r+1}} + y(z + yv_1)(bc)^{2^{s+1}} \end{pmatrix} \end{aligned}$$

We can denote this relationship by

$$(w, x, y, z) \sim (w, x + \lambda w, y, z + \lambda y),$$

where the 4-tuple (w, x, y, z) represents the 1-cocycle

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y(cd)^{2^s} \\ z(cd)^{2^s} \\ w(cd)^{2^r} \\ x(cd)^{2^r} \\ w(bc)^{2^r} + y(bc)^{2^s} \\ x(bc)^{2^r} + z(bc)^{2^s} \\ w(ab)^{2^r} \\ x(ab)^{2^r} \\ y(ab)^{2^s} \\ z(ab)^{2^r} \\ wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}} \end{pmatrix}.$$

We find infinitely many conjugacy classes, for instance for each x, z in k the family of classes of the form

$$[(0, x, 0, z)] = \{(0, x, 0, z)\}.$$

Now we consider P -conjugacy. An element $\mathbf{s} = \alpha^\vee(s)(\beta + \gamma + \delta)^\vee(t) \in Z(L)$ acts on the 1-cocycle σ by

$$(\mathbf{s} \cdot \sigma) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 \\ s^{-1}t^2y(cd)^{2^s} \\ sz(cd)^{2^s} \\ s^{-1}t^2w(cd)^{2^r} \\ sx(cd)^{2^r} \\ s^{-1}t^2(w(bc)^{2^r} + y(bc)^{2^s}) \\ sx(bc)^{2^r} + z(bc)^{2^s} \\ s^{-1}t^2w(ab)^{2^r} \\ sx(ab)^{2^r} \\ s^{-1}t^2y(ab)^{2^s} \\ sz(ab)^{2^r} \\ t^2(wx(bc)^{2^{r+1}} + yz(bc)^{2^{s+1}}) \end{pmatrix}$$

5.4 A Non-Reductive Counterexample

In [5] a counterexample to [ref KII] is presented for a closed field k of characteristic $p = 2$ and a non-reductive algebraic group G .

Example 5.3. Let Q be the algebraic group isomorphic to the affine space \mathbf{A}^3 with the group multiplication law:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 + u_1v_1 + u_2v_2 + u_1v_2 \end{pmatrix}.$$

Let $\Gamma = \langle \sigma, \tau | \sigma^3 = \tau^2 = 1, \tau\sigma\tau = \sigma^2 \rangle$ and $\Gamma_2 = \langle \tau \rangle$ the Sylow 2-subgroup of Γ . Γ acts on Q via

$$\begin{aligned} \tau \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1u_2 \end{pmatrix} \\ \sigma \cdot \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_3 \end{pmatrix}. \end{aligned}$$

Let $G = Q \rtimes \Gamma$. Then there are infinitely many pairwise G -conjugate classes of extensions to the representation $\rho : \Gamma_2 \rightarrow G$ defined by the natural inclusion $\Gamma_2 \rightarrow \Gamma \rightarrow G$ [5, Appendix].

Proof. Our proof will be way of a 1-cohomology calculation. Choose a 1-cocycle $\alpha \in Z^1(\Gamma, Q)$ such that $\alpha|_{\langle \sigma \rangle} = 1$. Let

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

for some $u_1, u_2, u_3 \in k$. Since τ is an involution we have

$$\begin{aligned}
 1 = \alpha(\tau^2) &= \alpha(\tau) \times \tau \cdot \alpha(\tau) \\
 &= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} u_2 \\ u_1 \\ u_3 + u_1^2 + u_2^2 + u_1 u_2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ 2u_3 + 2u_1^2 + u_2^2 + 3u_1 u_2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 + u_2 \\ u_1 + u_2 \\ u_2^2 + u_1 u_2 \end{pmatrix}.
 \end{aligned}$$

This shows $u_1 = u_2$, so

$$\alpha(\tau) = \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix}.$$

Furthermore, as $\tau\sigma\tau = \sigma^2$ we obtain

$$\begin{aligned}
 1 = \alpha(\sigma^2) &= \alpha(\tau\sigma\tau) \\
 &= \alpha(\tau) \times \tau \cdot \alpha(\sigma\tau) \\
 &= \alpha(\tau) \times \tau \cdot \alpha(\sigma) \times \tau\sigma \cdot \alpha(\tau) \\
 &= \alpha(\tau) \times \tau\sigma \cdot \alpha(\tau) \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau\sigma \cdot \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \tau \cdot \begin{pmatrix} u_1 \\ 0 \\ u_3 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ u_1 \\ u_3 \end{pmatrix} \times \begin{pmatrix} 0 \\ u_1 \\ u_3 + u_1^2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ 0 \\ 2u_3 + 3u_1^2 \end{pmatrix} \\
 &= \begin{pmatrix} u_1 \\ 0 \\ u_1^2 \end{pmatrix}.
 \end{aligned}$$

Therefore $u_1 = 0$. Hence a typical 1-cocycle that is trivial on $\langle \sigma \rangle$ satisfies

$$\alpha_u(\tau) = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \quad (u \in k).$$

Now we calculate the class $[\alpha_u] \in H^1(\Gamma, Q)$. Suppose $\alpha_v \sim \alpha_u$. Then there is a $q \in Q$ fixed under the action of σ , that is of the form

$$q = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix},$$

such that $\alpha_v(\gamma) = q \times \alpha_u(\gamma) \times \gamma \cdot q^{-1}$. In particular, for $\gamma = \tau$

$$\begin{aligned}
 \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \tau \cdot \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ u + \lambda \end{pmatrix} \\
 &= \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.
 \end{aligned}$$

Hence only if $u = v$ are two 1-cocycles of the particular form in the same class, and therefore $H^1(\Gamma, Q)$ is infinite. [to finish]. \square

[note how to make my example look like the Slodowy one]

It is natural to ask whether this leads to a reductive counterexample, although we can quickly verify that the answer is “not immediately”. For suppose there was a reductive group with unipotent radical *containing* the multiplication law:

$$\begin{aligned}
 &\dots \epsilon_\alpha(u_\alpha) \dots \epsilon_\beta(u_\beta) \dots \epsilon_\gamma(u_\gamma) \times \dots \epsilon_\alpha(v_\alpha) \dots \epsilon_\beta(v_\beta) \dots \epsilon_\gamma(v_\gamma) \\
 &= \dots \epsilon_\alpha(u_\alpha + v_\alpha) \dots \epsilon_\beta(u_\beta + v_\beta) \dots \epsilon_\gamma(u_\gamma + v_\gamma + u_\alpha v_\alpha + u_\beta v_\beta + u_\alpha v_\beta).
 \end{aligned}$$

Then setting $u_\delta = v_\delta = 0$ whenever $\delta \neq \alpha$ gives

$$\epsilon_\alpha(u_\alpha) \times \epsilon_\alpha(v_\alpha) = \epsilon_\alpha(u_\alpha + v_\alpha) \epsilon_\gamma(u_\alpha v_\alpha),$$

which is absurd. [try find more examples]

Chapter 6

Conclusion

Appendix A

Further Calculations

G	P	Z^1	H^1	$V\text{-conj}$	$P\text{-conj}$
B_2 (α short)	P_α	✓	✓	✓	✓
	P_β	✓	✓	✓	✓
G_2 (α short)	P_α	✓			
C_3 (γ long)	P_α	✓			
[5]	$Q \rtimes SL(2, 2)$	✓	✓	✓	

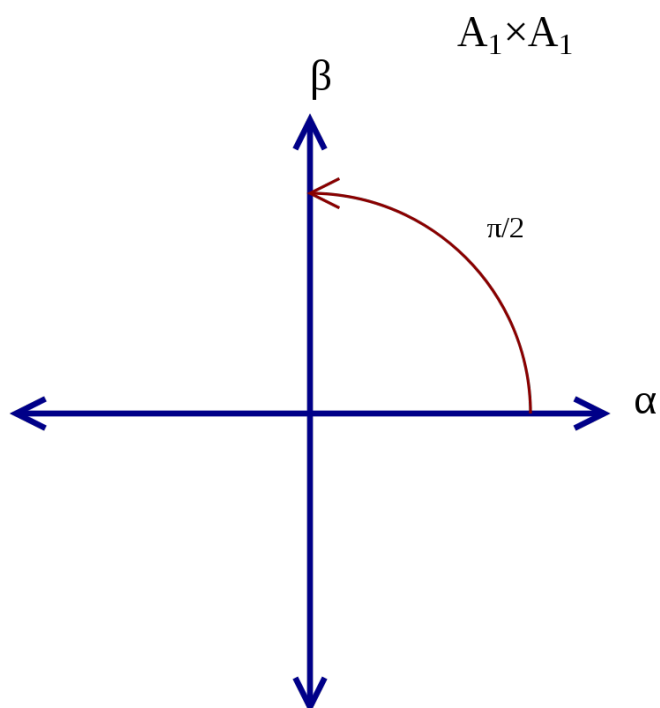
Appendix B

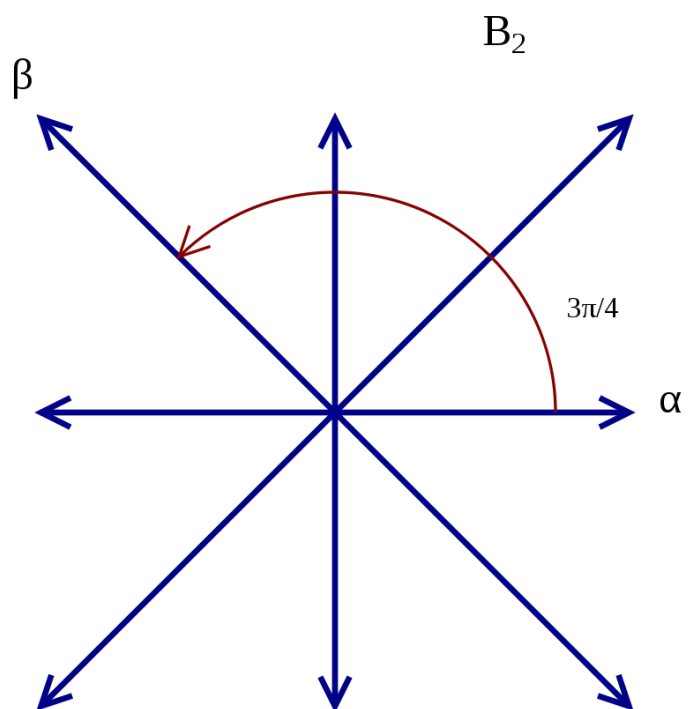
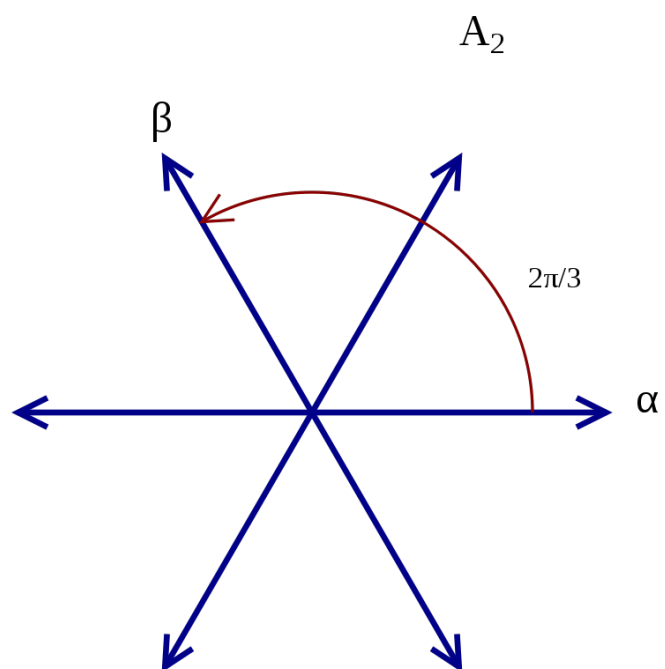
Source Code

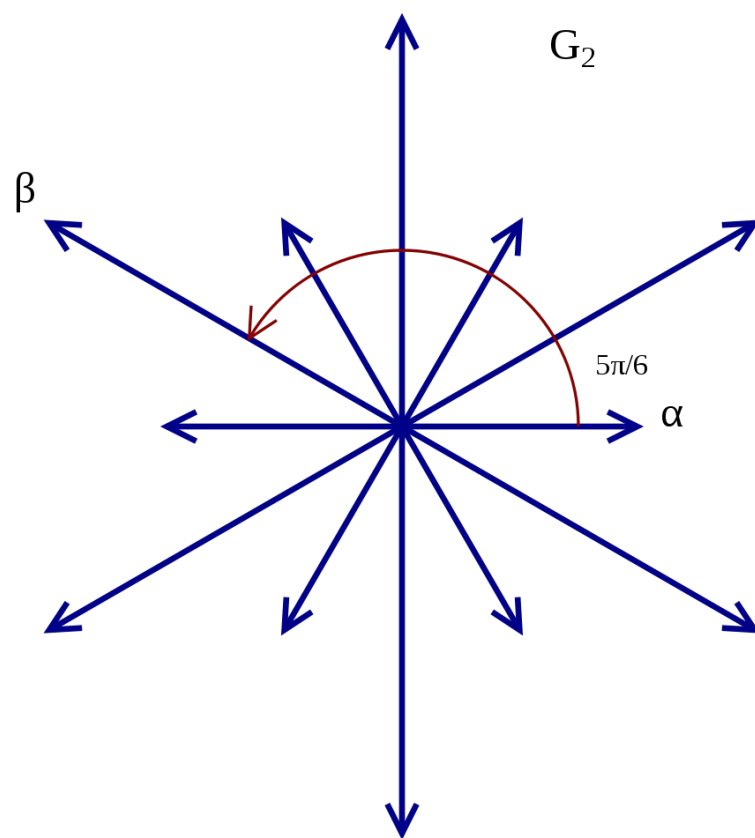
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Appendix C

Rank 2 Root System Diagrams







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