ON A QUESTION OF KÜLSHAMMER FOR REDUCTIVE ALGEBRAIC GROUPS

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Abstract.

1. Introduction

Let G and H be linear algebraic groups over an algebraically closed field of characteristic $p > 0$. Pick a maximal unipotent subgroup U of H [this is unique up to conjugacy]. By a *representation* of H in G we mean a homomorphism of algebraic groups from H to G. The group G acts on the set of representations $\mathrm{Hom}(H, G)$ by $(g \cdot \rho)(h) = g\rho(h)g^{-1}$ for $h \in H$ and $g \in G$; we call the orbits *conjugacy classes*. We consider the following question.

Question 1.1. *Let $\sigma\colon U \to G$ be a representation. Are there only finitely many conjugacy classes of representations $\rho\colon H \to G$ such that $\rho|_U$ is G-conjugate to σ?*

Külshammer raised this question for finite H [2]; in this case the maximal unipotent subgroups are the Sylow p-subgroups. An example of Cram shows that the answer to Question 1.1 is no for $H = S_3$ and G a certain 3-dimensional non-connected group with G^0 unipotent [4, Appendix]. If $G = \mathrm{GL}_n$ and p does not divide $|H|$ then the answer to Question 1.1 is yes, by Maschke's Theorem. It follows from a well-known geometric argument of Richardson that if p does not divide $|H|$ then the answer to Question 1.1 is yes for any connected reductive G: one embeds G in some GL_n and studies the behavior of the induced map $\mathrm{Hom}(H, G) \to \mathrm{Hom}(H, \mathrm{GL}_n)$. In fact, standard representation-theoretic results imply that the answer is yes for any finite group when $G = \mathrm{GL}_n$, and Slodowy used Richardson's argument to show the answer is yes for arbitrary finite H and connected reductive G under mild hypotheses on char(p). [General case of finite H: there is now a counterexample in type G_2 [ref].]

In this paper we instead consider the case when H is connected and semisimple. We settle Question 1.1 as follows.

Theorem 1.2. *The answer to Question 1.1 is yes if H is connected and semisimple.*

Note that we allow G to be non-connected, but it is clear we can reduce immediately to the case when G is connected and $Z(G)^0$ is trivial. Also, it is clear that the answer to Question 1.1 is no for arbitrary connected reductive H (e.g., just take H to be a torus, $U = 1$ and G^0 non-unipotent).

An important tool in the work of [above] is nonabelian 1-cohomology. Let $\rho \in \mathrm{Hom}(H, G)$ and let P be a parabolic subgroup of G such that $\rho(H) \subseteq P$. Then H acts on $V := R_u(P)$ via $h \cdot u = \rho(h)u\rho(h)^{-1}$, and representations near ρ in an appropriate sense can be understood

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in terms of the nonabelian 1-cohomology $H^1(H, \rho, V)$. In Section [below] we discuss this cohomological approach and study its behavior with respect to restriction of representations from H to U. [Further discussion.]

[Based on D's PhD thesis. Related work by David Stewart: see below. He uses high-powered cohomology techniques; our methods are more elementary.]

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2. Preliminaries

[Some useful results from ''On unipotent algebraic groups and 1-cohomology'', David Stewart:

Cor. 3.4.3: Let $B = TU$ be connected solvable group with unipotent radical U and maximal torus T. Let Q be a unipotent group on which B acts. Then the restriction map $H^1(B, Q) \to H^1(U, Q)$ is injective.

Thm 3.5.2: Let G be connected reductive and let Q be a unipotent group on which G acts. Then for any parabolic subgroup P of G, the restriction map $H^1(G, Q) \to H^1(P, Q)$ is an isomorphism of pointed sets.

Corollary 3.6.2. Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = LR_u(P)$ of G and let P_1 be any parabolic subgroup of H. Then H is G-conjugate to a subgroup of L if and only if P_1 is as well.]

Definition 2.1. Let X, Y be (algebraic) groups. Then we denote by $\mathrm{Hom}(X, Y)$ the set of (algebraic) group homomorphisms from X to Y.

Lemma 2.2. *Let $R \subset \mathrm{Hom}(K, P)$. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Let $\rho_1, \rho_2 \in R$ such that ρ_1 and ρ_2 lie in the same G-conjugacy class of R. Then there exists $g \in G$ such that

$$g\rho_1(x)g^{-1} = \rho_2(x),$$

for all $x \in K$.

Let $Q = gPg^{-1}$, hence $\rho_2(K) \subset P \cap Q$. Let T be a maximal torus of G contained in $P \cap Q$. Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

Then there exists $r \in P$ such that $q = grg^{-1}$, so

$$\begin{gathered} grg^{-1}Tgr^{-1}g^{-1} = gTg^{-1} \\ \Rightarrow rg^{-1}Tgr^{-1} = T. \end{gathered}$$

Therefore $gr^{-1} \in N_G(T)$.

Fix a finite set $N \subset N_G(T)$ of coset representatives for the Weyl group $W = N_G(T)/T$ and let $n \in N, t \in T$ such that

$$gr^{-1} = nt.$$

Let $q' = r^{-1}t^{-1}$ so $q' \in P$. Then

$$\begin{aligned}\rho_1(x) &= g^{-1}\rho_2(x)g \\ &= (q'n^{-1})\rho_2(x)nq'^{-1},\end{aligned}$$

for all $x \in K$. Hence $\rho_1 \in P\cdot(n^{-1}\cdot\rho_2)$. This shows that a G-conjugacy class of R is contained in a union of at most $|N| = |W|$ P-conjugacy classes.

Therefore, if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes.

The converse is trivial. $\square$

We assume G is a possibly non-connected algebraic group over k and H is a linear algebraic group over k. We fix a maximal unipotent subgroup U of H.

Suppose H is connected, let B be a Borel subgroup of H, let X be an affine variety and let $f\colon H \to X$ be a morphism such that $f(hb) = f(h)$ for all $h \in H$ and all $b \in B$. Then f gives rise to a morphism $\overline{f}$ from the projective variety H/B to X. Since H/B is connected and X is affine, $\overline{f}$ must be constant, so f is constant. In particular, if V is an affine H-variety, $v \in V$ and the stabiliser H_v contains B then $H_v = H$.

Lemma 2.3. *Suppose G is connected and reductive. Let H_1, H_2 be connected reductive subgroups of G. Suppose B is a common Borel subgroup of both H_1 and H_2. Then $H_1 = H_2$.*

Proof. The quotient variety G/H_1 is affine since H_1 is reductive [ref], and H_2 acts on G/H_1 by left multiplication. The stabiliser in H_2 of the coset H_1 contains B, so it must equal the whole of H_2. Hence $H_2 \subseteq H_1$. The reverse inequality follows similarly, so $H_1 = H_2$. $\square$

Lemma 2.4. *Let B be a Borel subgroup of H. Let $\rho_1, \rho_2 \in \operatorname{Hom}(H,G)$ such that $\rho_1|_B = \rho_2|_B$. Then $\rho_1 = \rho_2$.*

Proof. Define $f\colon H \to G$ by $f(h) = \rho_1(h)\rho_2(h)^{-1}$. For any $h \in H$, $b \in B$, $f(hb) = \rho_1(hb)\rho_2(hb)^{-1} = \rho_1(h)\rho_1(b)\rho_2(b)^{-1}\rho_2(h)^{-1} = \rho_1(h)\rho_2(h)^{-1} = f(h)$. So f gives rise to a morphism $\overline{f}$ from the connected projective variety H/B to G given by $\overline{f}(hB) = f(h)$. Then $\overline{f}$ is constant, so f is constant with value $f(1) = 1$, and the result follows. $\square$

3. Proof of Theorem 1.2 for G reductive

In this section we assume G and H are connected and semisimple.

Lemma 3.1. *Let B be the Borel subgroup of H that contains U, and let T be a maximal torus of B. Let $\rho_1, \rho_2\colon H \to G$ be representations such that $\rho_1(B) = \rho_2(B)$ and $\rho_1(T) = \rho_2(T)$. Set $U' = \rho_1(U) = \rho_2(U)$. Suppose that for all $\alpha \in \Phi_T(B)$, $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ and there exists $\phi_\alpha \in \operatorname{Aut}(\rho_1(U_\alpha))$ such that for all $u \in U_\alpha$, $\rho_2(u) = \phi_\alpha(\rho_1(u))$. Then there exists $t' \in T$ such that $t'\cdot\rho_2 = \rho_1$.*

Proof. Since B is a Borel subgroup of H, $\rho_i(B)$ is a Borel subgroup of $\rho_i(H)$ for $i = 1, 2$. Lemma 2.3 implies that $\rho_1(H) = \rho_2(H)$. Hence we can assume that ρ_1 and ρ_2 are surjective. Set $T' = \rho_1(T) = \rho_2(T)$.

Pick a base $\{\alpha_1', \ldots, \alpha_m'\}$ for $\Phi_{T'}(B')$. Since ρ_1 and ρ_2 map a given root group of U with respect to T to a root group of U' with respect to T', there exist $\alpha_1, \ldots, \alpha_m \in \Phi_T(B)$ such that $\rho_1(U_{\alpha_i}) = \rho_2(U_{\alpha_i}) = U_{\alpha_i'}$ and $\alpha_i'(\rho_1(t)) = \alpha_i'(\rho_2(t))$ for all i. [This is not quite true as stated: some of the U_{α_i} may be mapped by ρ_1 and ρ_2 to 1; we're only interested in the case when this doesn't happen. A bit more argument is needed here.] As H is semisimple, this implies that $\rho_1|_T = \rho_2|_T$.

By hypothesis, each ϕ_{α_i} is an automorphism of $U_{\alpha_i'}$, so there exist $b_1, \ldots, b_m \in k^*$ such that $\phi_{\alpha_i}(\epsilon_{\alpha_i'}(x)) = \epsilon_{\alpha_i'}(b_i x)$ for each i and for all $x \in k$; so there exist $a_1, \ldots, a_m \in k^*$ such that $\rho_2(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(a_i x))$ for each i and for all $x \in k$ [check!].

The weights $\alpha_1', \ldots, \alpha_m'$ are linearly independent as H is semisimple, so there exists $t' \in T'$ such that $\alpha_i(t') = a_i^{-1}$ for $1 \leq i \leq m$. We then have $(t' \cdot \rho_2)(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(x))$ for each i and for all $x \in k$. It follows that $\rho_1|_{\widetilde{U}} = (t' \cdot \rho_2)|_{\widetilde{U}}$, where $\widetilde{U}$ is the subgroup of U generated by the U_{α_i}. But $\rho_1(\widetilde{U}) = \rho_2(\widetilde{U}) = (t' \cdot \rho_2)(\widetilde{U})$ since the $U_{\alpha_i'}$ generate U' [ref: Humphreys?], and $\ker(\rho_1|_U) = \ker(\rho_2|_U) = \ker((t' \cdot \rho_2)|_U)$, so $\rho_1|_U = (t' \cdot \rho_2)|_U$.

To complete the proof, it is enough by Lemma 2.4 to show that $\rho_1|_B = (t' \cdot \rho_2)|_B$. But $(t' \cdot \rho_2)|_T = \rho_2|_T = \rho_1|_T$ from the discussion above, so we are done. $\square$

[Note: In the situation of the above lemma (and with ρ_1, ρ_2 assumed surjective), suppose we don't assume that $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for every α. Then ρ_1 and ρ_2 need no longer be G-conjugate. I suspect, however, that in this case they are $\mathrm{Aut}(G)$-conjugate. If this is the case then we should get a reasonable bound in Theorem 1.2 involving $|\mathrm{Out}(G)|$ and the size of the Weyl group of T' in $N_G(U')$ (cf. below).]

Theorem 3.2. *The answer to Külshammer's question is yes for H and G connected and semisimple.*

Proof. Let $\sigma \colon U \to G$ and let $U' = \sigma(U)$. Let B be the Borel subgroup of H that contains U and fix a maximal torus T of B. Fix a maximal torus T' of $N_G(U')$. Let $C = \{\rho \in \mathrm{Hom}(H, G) \mid \rho|_U = \sigma\}$. If $\rho \in C$ then $\rho(T)$ normalizes $\rho(U) = U'$, so $\rho(T)$ is a torus of $N_G(U')$. By conjugacy of maximal tori of $N_G(U')$, there exists $g \in N_G(U')$ such that $(g \cdot \rho)(T) \subseteq T'$. If $h \in N_G(U')$ and $(h \cdot \rho)(T) \subseteq T'$ then there exists $n \in N_{N_G(U')}(T')$ such that $(h \cdot \rho)(T) = ((ng) \cdot \rho)(T)$. The group $N_{N_G(U')}(T')/C_{N_G(U')}(T')$ is finite, so there is a finite set of subtori $S_1, \ldots, S_r$ of T' such that $\rho(T)$ is $N_G(U')$-conjugate to one of $S_1, \ldots, S_r$.

Define a relation $\equiv$ on C by $\rho_1 \equiv \rho_2$ if there exist $i \in \{1, \ldots, r\}$ and $g_1, g_2 \in N_G(U')$ such that $(g_1 \cdot \rho_1)(T) = (g_2 \cdot \rho_1)(T) = S_i$ and $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for all $\alpha \in \Phi_T(U)$. It is clear that $\equiv$ is an equivalence relation. Define $C_i = \{\rho \in G \mid (g \cdot \rho)(T) = S_i \text{ for some } g \in N_G(U')\}$. If $(g \cdot \rho)(T) = S_i$ then $(g \cdot \rho)$ must map any root group U_α of U with respect to T to a root group of U' with respect to S_i, so there are only finitely many possibilities for $(g \cdot \rho)(U_\alpha)$. It follows that $\equiv$ has only finitely many equivalence classes.

To complete the proof it is enough, therefore, to show that if $\rho_1 \equiv \rho_2$ then ρ_1 and ρ_2 are G-conjugate. So suppose g_1 and g_2 are [as in the defn of $\equiv$]. Set $U_{\alpha'} = \rho_1(U_\alpha) = \rho_2(\alpha)$ if this is nontrivial [recall that $\rho_1(U_\alpha)$ might be just 1]. Then the map $\phi \colon G \to G$ given by $\phi(g) = g_2 g_1^{-1} g g_1 g_2^{-1}$ gives rise to an automorphism of U_α' for all $\alpha' \in \Phi_{S_i}(U')$. It follows by Lemma 3.1 that $g_1 \cdot \rho_1$ and $g_2 \cdot \rho_2$ are T-conjugate, so we are done. $\square$

[Consequence: Look at **images** of representations and prove an analogous finiteness result, for connected reductive H as well?]

4. 1-cohomology

[Describe Daniel's approach using 1-cohomology.]

We recall Richardson's nonabelian 1-cohomology ([3]) below.

Definition 4.1. We call a morphism $\sigma : K \to V$ a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x)(x \cdot \sigma(y)),$$

for all $x, y \in K$. Denote by $Z^1(K,V)$ the collection of all 1-cocycles from K to V, and $H^1(K,V)$ the set of equivalence classes of $Z^1(K,V)$ under the relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \exists v \in V, \forall x \in K, \sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}).$$

We call $H^1(K,V)$ the *1-cohomology*.

Note that if V is abelian then $Z^1(K,V)$ is a group under pointwise addition and $H^1(K,V) = Z^1(K,V)/B^1(K,V)$, where $B^1(K,V)$ is the subgroup of 1-cocycles of the form $\chi_v(x) = v\,(x^{-1} \cdot v)$ for some $v \in V$.

Lemma 4.2. *Suppose K is linearly reductive and V is unipotent. Then $H^1(K,V)$ is trivial.* [3, Lemma 6.2.6].

Lemma 4.3 (Map of 1-Cohomologies)**.** *Let K', V' be algebraic groups such that K' acts on V' by group automorphisms. Let $\zeta : K' \to K$ be a homomorphism and let $\xi : V \to V'$ be a K'-equivariant homomorphism. Then the function $Z^1(\zeta, \xi)$ defined by*

$$Z^1(\zeta,\xi)(\sigma) = \xi \circ \sigma \circ \zeta,$$

maps $Z^1(K,V)$ to $Z^1(K',V')$. Furthermore, $Z^1(\zeta,\xi)$ descends to give a unique map

$$H^1(\zeta,\xi) : H^1(K,V) \to H^1(K',V'),$$

that makes the following diagram commute:

$$\begin{tikzcd} Z^1(K,V) \arrow[r, "Z^1(\zeta,\xi)"] \arrow[d] & Z^1(K',V') \arrow[d] \\ H^1(K,V) \arrow[r, "H^1(\zeta,\xi)"] & H^1(K',V'). \end{tikzcd}$$

For our applications, it is often the case that $V = V'$ and $\xi = \mathrm{id}_V$. Then we just write $Z^1(\zeta)$ and $H^1(\zeta)$.

Lemma 4.4. *Let V be a vector space over k,* $\mathrm{char}(k) = p$. *Let Γ be a finite group that acts linearly on V, and let Γ_p be a* Sylow p-subgroup *of Γ. Let ζ be the inclusion of Γ_p in Γ. Then the map*

$$H^1(\zeta) : H^1(\Gamma,V) \to H^1(\Gamma_p,V)$$

is injective. [1, III.10.4 Prop.].

Example 4.5. Let $k = \overline{\mathbb{F}}_p = \bigcup_{r\in\mathbb{N}} \mathbb{F}_{p^r}$. Let V a be vector space over k on which H acts linearly, and let U be a maximal unipotent subgroup of H. Let ζ be the inclusion of U in H.

Then the map

$$H^1(\zeta) : H^1(H,V) \to H^1(U,V) \tag{4.6}$$

is injective.

Proof. For each $r \in \mathbb{N}$, consider the following maps between algebraic groups $H = H(k)$, $H(\mathbb{F}_{p^r})$, $U = U(k)$, and $U(\mathbb{F}_{p^r})$:

$$\begin{aligned}
\zeta_r &: U(\mathbb{F}_{p^r}) \hookrightarrow H(\mathbb{F}_{p^r}),\\
\iota_r &: H(\mathbb{F}_{p^r}) \hookrightarrow H,\\
\iota_r' &: U(\mathbb{F}_{p^r}) \hookrightarrow U.
\end{aligned}$$

We have a family of commutative diagrams (Lemma 4.3), with $H^1(\zeta_r)$ injective (Lemma 4.4):

$$\begin{tikzcd}
H^1(H,V) \arrow[r, "H^1(\zeta)"] \arrow[d, "H^1(\iota_r)"'] & H^1(U,V) \arrow[d, "H^1(\iota_r')"] \\
H^1(H(\mathbb{F}_{p^r}),V) \arrow[r, "H^1(\zeta_r)"] & H^1(U(\mathbb{F}_{p^r}),V).
\end{tikzcd} \tag{4.7}$$

Let $\sigma \in Z^1(H,V)$ such that $\sigma \notin B^1(H,V)$, and for each $x \in H(\mathbb{F}_{p^r})$ define the morphism $f_x : V \to V$ by $f_x(v) = \sigma(x)(\chi_v(x))^{-1}$.

Now consider the sequence $\{C_r\}_{r\in\mathbb{N}}$ defined by $C_r = \{v \in V \,|\, \forall x \in H(\mathbb{F}_{p^r}),\ f_x(v) = 0\}$. By the choice of σ, the following set is empty:

$$\bigcap_{r\in\mathbb{N}} C_{r!} = \{v \in V \,|\, \forall x \in H,\ f_x(v) = 0\}$$

Each C_r is closed, and the inclusion $H(\mathbb{F}_{p^{r!}}) \subset H(\mathbb{F}_{p^{(r+1)!}})$ (since $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$) induces the reverse inclusion for the subsequence $C_{r!} \supset C_{(r+1)!}$. The Noetherian property for V requires that the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ becomes constant, and since $\cap_{r\in\mathbb{N}} C_{r!} = \emptyset$, this means the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ is eventually empty. That is, there exists $s \in \mathbb{N}$ such that

$$Z^1(\iota_s)(\sigma) \neq \chi_v,$$

for any $v \in V$.

We have shown that if $\sigma \in Z^1(H,V)$ such that $Z^1(\iota_{r!})(\sigma) \in B^1(H(\mathbb{F}_{p^{r!}}),V)$ for all $r \in \mathbb{N}$, then $\sigma \in B^1(H,V)$.

So, let ψ be the canonical projection $Z^1(H,V) \to H^1(H,V)$, and let $\sigma \in Z^1(H,V)$ such that $\psi(\sigma) \in \operatorname{Ker}(H^1(\zeta))$. Then, consulting the commutative diagram in Equation 4.7,

$$\begin{aligned}
&\psi(\sigma) \in \operatorname{Ker}\left(H^1(\iota_r') \circ H^1(\zeta)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &\psi(\sigma) \in \operatorname{Ker}\left(H^1(\zeta_r) \circ H^1(\iota_r)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \in \operatorname{Ker}\left(H^1(\zeta_r)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \text{ is trivial }, \forall r \in \mathbb{N}\\
\Rightarrow\, &Z^1(\iota_r)(\sigma) \in B^1(SL_2(\mathbb{F}_{p^r}),V), \forall r \in \mathbb{N}\\
\Rightarrow\, &\sigma \in B^1(SL_2(k),V)\\
\Rightarrow\, &\psi(\sigma) \in H^1(SL_2(k),V) \text{ is trivial.}
\end{aligned}$$

This shows $H^1(\zeta)$ is injective. ☐

Definition 4.8. Let $\rho \in \operatorname{Hom}(K,P)$. We associate with ρ the map $\rho^L \in \operatorname{Hom}(K,L)$ defined by $\rho^L = \rho|_L$, and the 1-cocycle $\sigma_\rho \in Z^1(K,V)$ defined by $\sigma_\rho(x) = \rho(x)\rho^L(x^{-1})$.

Definition 4.9. Let $\omega \in \mathrm{Hom}(K, L)$. We denote by $Z^1(K, V)_\omega$ the set of 1-cocycles from K to V where K acts on V via ω; that is, $x \cdot v = \omega(x) \cdot v$. Likewise, denote by $H^1(K, V)_\omega$ the 1-cohomology obtained from $Z^1(K, V)_\omega$.

Define $\mathrm{Hom}(K, P)_\omega = \{\rho \in \mathrm{Hom}(K, P) \mid \rho^L = \omega\}$. More generally, if $R \subset \mathrm{Hom}(K, P)$ define $R_\omega = \{\rho \in R \mid \rho^L = \omega\}$.

Lemma 4.10. *Let $\omega \in \mathrm{Hom}(K, L)$. The map $z : \mathrm{Hom}(K, P)_\omega \to Z^1(K, V)_\omega$ defined by*

$$z(\rho)(x) = \rho(x)\omega(x^{-1}),$$

for all $\rho \in \mathrm{Hom}(K, P)_\omega$ and all $x \in K$, is a bijection. Furthermore, z descends to a bijection $h : \mathrm{Hom}(K, P)_\omega/V \to H^1(K, V)_\omega$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}(K, P)_\omega \arrow[r, "z"] \arrow[d] & Z^1(K, V)_\omega \arrow[d] \\
\mathrm{Hom}(K, P)_\omega/V \arrow[r, "h"] & H^1(K, V)_\omega.
\end{tikzcd}$$

Since $C_L(\omega(K))$ normalizes V, $C_L(\omega(K))$ acts on $\mathrm{Hom}(K, P)_\omega/V$. Notice that

$(\mathrm{Hom}(K, P)_\omega/V)/C_L(\omega(K))$, is canonically isomorphic to $\mathrm{Hom}(K, P)_\omega/VC_L(\omega(K))$. Let $\zeta = \mathrm{id}_K$ and $\xi_c : V \to V$ be defined $\xi_c(v) = cvc^{-1}$ for all $v \in V$. We have an action of $C_L(\omega(K))$ on $H^1(K, V)_\omega$ defined by

$$c \cdot \bar{\sigma} = H^1(\zeta, \xi_c)(\bar{\sigma}),$$

for $c \in C_L(\omega(K)), \bar{\sigma} \in H^1(K, V)_\omega$.

It is easy to check that the map $h : \mathrm{Hom}(K, P)_\omega/V \to H^1(K, v)_\omega$ in Lemma 4.10 is equivariant, so there exists a unique map $\tilde{h} : \mathrm{Hom}(K, P)_\omega/VC_L(\omega) \to H^1(K, V)_\omega/C_L(\omega)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(K, P)_\omega/V \arrow[r, "h"] \arrow[d] & H^1(K, V)_\omega \arrow[d] \\
\mathrm{Hom}(K, P)_\omega/VC_L(\omega) \arrow[r, "\tilde{h}"] & H^1(K, V)_\omega/C_L(\omega).
\end{tikzcd}$$

Definition 4.11. Let $K' < K$, let ζ the inclusion of K' in K, and let $\xi = \mathrm{id}_V$. Then the map $H^1(\zeta) : H^1(K, V)_\omega \to H^1(K', V)_{\omega\circ\zeta}$ descends to give a unique map

$$\tilde{H}^1(\zeta) : H^1(K, V)_\omega/C_L(\omega) \to H^1(K', V)_{\omega\circ\zeta}/C_L(\omega \circ \zeta),$$

such that the following diagram commutes:

$$\begin{tikzcd}
H^1(K, V)_\omega \arrow[r, "H^1(\zeta)"] \arrow[d] & H^1(K', V)_{\omega\circ\zeta} \arrow[d] \\
H^1(K, V)_\omega/C_L(\omega) \arrow[r, "\tilde{H}^1(\zeta)"] & H^1(K', V)_{\omega\circ\zeta}/C_L(\omega \circ \zeta).
\end{tikzcd}$$

Definition 4.12. We define $\mathrm{Hom}(K, P)^L = \{\rho^L \mid \rho \in \mathrm{Hom}(K, P)\}$. More generally, when $R \subset \mathrm{Hom}(K, P)$ we define $R^L = \{\rho^L \mid \rho \in R\}$.

Lemma 4.13. *Let $R \subset \mathrm{Hom}(K,P)$. Suppose $R = P \cdot \rho$ for some $\rho \in R$. Then $R^L = L \cdot \rho^L$. More generally, if $R = P \cdot R$ then $R^L = L \cdot R^L$.*

Lemma 4.14. *Let $R \subset \mathrm{Hom}(K,P)$ and suppose that $R = P \cdot R$. Then for all $\omega \in \mathrm{Hom}(K,L)$, all $\rho \in R_\omega$, $R_\omega \cap P \cdot \rho = (VC_L(\omega)) \cdot \rho$.*

Theorem 4.15. *Let π be the canonical projection $\mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K,P)_\omega / VC_L(\omega)$. Let $R \subset \mathrm{Hom}(K,P)$ and suppose $P \cdot R = R$. Then R is a finite union of P-conjugacy classes if and only if*

(i) *R^L is a finite union of L-conjugacy classes, and*
(ii) *for each $\omega \in \mathrm{Hom}(K,L)$, $(\tilde{h} \circ \pi)(R_\omega)$ is finite in $H^1(K,V)_\omega / C_L(\omega)$.*

Remark. By Lemma 4.13, if R is a finite union of P-conjugacy classes then R^L is a finite union of L-conjugacy classes. Furthermore, conditions (i) and (ii) are equivalent to

(i′) $R_\omega = \emptyset$ for all but finitely many L-conjugacy classes of $\omega \in \mathrm{Hom}(K,L)$, and
(ii′) for each $\omega \in \mathrm{Hom}(\Gamma, L)$, R_ω is a finite union of $VC_L(\omega)$-conjugacy classes,

respectively.

Definition 4.16. Let $K' < K$ and let ζ be the inclusion of K' in K. Define

$$\mathcal{Z}(\zeta) : \mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K',P)_{\omega\circ\zeta},$$

by $\mathcal{Z}(\zeta)(\rho) = \rho \circ \zeta$, for all ρ in $\mathrm{Hom}(K,P)_\omega$. Furthermore, define

$$\begin{aligned} \mathcal{H}(\zeta) &: \mathrm{Hom}(K,P)_\omega / V \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V, \\ \tilde{\mathcal{H}}(\zeta) &: \mathrm{Hom}(K,P)_\omega / VC_L(\omega) \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta), \end{aligned}$$

by composing the image of a representative in $\mathrm{Hom}(K,P)_\omega$ under $\mathcal{Z}(\zeta)$ with the respective canonical projections

$$\begin{aligned} \mathrm{Hom}(K',P)_{\omega\circ\zeta} &\to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V, \\ \mathrm{Hom}(K',P)_{\omega\circ\zeta} &\to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta). \end{aligned}$$

Proposition 4.17. *The following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega \arrow[r, "z"] \arrow[d] \arrow[dddr, "\mathcal{Z}(\zeta)"] & Z^1(K,V)_\omega \arrow[d] \arrow[dddr, "Z^1(\zeta)"] & \\
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] \arrow[d] \arrow[dddr, "\mathcal{H}(\zeta)"] & H^1(K,P)_\omega \arrow[d] \arrow[dddr, "H^1(\zeta)"] & \\
\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \arrow[r, "\tilde{h}"] \arrow[dddr, "\tilde{\mathcal{H}}(\zeta)"] & H^1(K,P)_\omega/C_L(\omega) \arrow[dddr, "\tilde{H}^1(\zeta)"] & \\
 & \mathrm{Hom}(K',P)_{\omega\circ\zeta} \arrow[r, "z'"] \arrow[d] & Z^1(K',V)_{\omega\circ\zeta} \arrow[d] \\
 & \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V \arrow[r, "h'"] \arrow[d] & H^1(K',V)_{\omega\circ\zeta} \arrow[d] \\
 & \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta) \arrow[r, "\tilde{h}'"] & H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta)
\end{tikzcd}
$$

Theorem 4.18. *Let $K' < K$, let ζ be the inclusion of K' in K, and let $\xi = \mathrm{id}_V$. Let $R \subset \mathrm{Hom}(K,P)$ such that $R = P \cdot R$, and let $S = \mathcal{Z}(\zeta)(R)$. Suppose*

(i) *R^L is a finite union of L-conjugacy classes,*
(ii) *for all $\omega \in \mathrm{Hom}(K,L)$ such that $R_\omega \neq \emptyset$, the map*

$$\tilde{H}^1(\zeta) : H^1(K,V)_\omega/C_L(\omega) \to H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta),$$

has finite fibres, and
(iii) *S is a finite union of P-conjugacy classes.*

Then R is a finite union of P-conjugacy classes.

Remark. Since $R = P \cdot R$, then by Lemma 4.13 R^L is already a union of L-conjugacy classes. The point of (i) is that the union is finite.

Proof. Since $P \cdot R = R$, it follows that $P \cdot S = S$. By definition $S \subset \mathrm{Hom}(K',V)$ and by (iii) S is a finite union of P-conjugacy classes. Define the canonical projections

$$\pi : \mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K,P)_\omega/VC_L(\omega)$$
$$\pi' : \mathrm{Hom}(K',P)_\omega \to \mathrm{Hom}(K',P)_\omega/VC_L(\omega).$$

By Theorem 4.15

(iv) S^L is a finite union of L-conjugacy classes, and
(v) for each $\omega \in \mathrm{Hom}(K',L)$, $(\tilde{h}' \circ \pi')(S_\omega)$ is finite in $H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta)$.

Let $\omega \in \text{Hom}(K, L)$. Clearly $(\tilde{h} \circ \pi)(R_\omega)$ is finite if $R_\omega = \emptyset$, so suppose $R_\omega \neq \emptyset$. We have $\mathcal{Z}(\zeta)(R_\omega) \subset S_{\omega\circ\zeta}$, and by the commutative diagram in Definition 4.16,

$$\tilde{h'} \circ \pi' \circ \mathcal{Z}(\zeta) = \tilde{H^1}(\zeta) \circ \tilde{h} \circ \pi.$$

Therefore

$$\begin{aligned}(\tilde{H^1}(\zeta) \circ \tilde{h} \circ \pi)(R_\omega) &= (\tilde{h'} \circ \pi' \circ \mathcal{Z}(\zeta))(R_\omega) \\ &= (\tilde{h'} \circ \pi')\,(\mathcal{Z}(\zeta)(R_\omega)) \\ &\subset (\tilde{h'} \circ \pi')(S_{\omega\circ\zeta}).\end{aligned}$$

Then $(\tilde{h'} \circ \pi')(S_{\omega\circ\zeta})$ is finite by (v), and

$$(\tilde{h} \circ \pi)(R_\omega) \subset \tilde{H^1}(\zeta)^{-1}((\tilde{H^1}(\zeta) \circ \tilde{h} \circ \pi)(R_\omega)),$$

is finite by (ii).

Therefore $(\tilde{h} \circ \pi)(R_\omega)$ is finite in any case. Together with (i) we may apply Theorem 4.15, so R is a finite union of P-conjugacy classes. $\square$

Remark. It is straightforward to show that $H^1(K, V)_\omega / C_L(\omega)$ is finite if and only if $H^1(K, V)_w / C_L(\omega)^\circ$ is finite.

In [ref] Bate, Martin, and Röhrle provide a finite subgroup H of G_2 for which the answer to Question 1.1 is "no". We show that this is the smallest example possible, in the sense that there are no other such examples for any G of rank 1 or 2.

Theorem 4.19. *Let G be a reductive group such that the simple components of G are all of type A_n or B_2. Then the answer to Question 1.1 is "yes" for any H with H^0 semisimple.*

Proof. We can approximate H with a sufficiently large finite subgroup, so assume without loss that H is finite. It's enough to study the simple components of G separately, so without loss we assume G is simple. If G is of type B_2 and $p \neq 2$ then we're done by [4], so without loss we assume that either G is of type A_n, or G is of type B_2 and $p = 2$. Note that maximal parabolics of G have abelian unipotent radicals under this hypothesis.

We now use an inductive proof. If ρ is G-irreducible then we're done. Otherwise put ρ into a maximal parabolic P. The key point is that we can transfer the problem from P into a Levi of P by the usual arguments, because the restriction map $H^1(H, V) \to H^1(U, V)$ is injective for any finite H if $V = R_u(P)$ is abelian. (Here U is a Sylow p-subgroup of H.) By induction on $\dim(G)$, the result is true for L—for the simple components of L are all of type A_n or B_2—and we get what we want. $\square$

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