ON A QUESTION OF KÜLSHAMMER FOR REDUCTIVE ALGEBRAIC GROUPS

DANIEL LOND AND BENJAMIN MARTIN

Abstract.

1. Introduction

Let G and H be linear algebraic groups over an algebraically closed field of characteristic $p > 0$. Pick a maximal unipotent subgroup U of H [this is unique up to conjugacy]. By a *representation* of H in G we mean a homomorphism of algebraic groups from H to G. The group G acts on the set of representations $\mathrm{Hom}(H, G)$ by $(g \cdot \rho)(h) = g\rho(h)g^{-1}$ for $h \in H$ and $g \in G$; we call the orbits *conjugacy classes.* We consider the following question.

Question 1.1. *Let $\sigma\colon U \to G$ be a representation. Are there only finitely many conjugacy classes of representation $\rho\colon H \to G$ such that $\rho|_U$ is G-conjugate to σ?*

Külshammer raised this question for finite H [ref]; in this case the maximal unipotent subgroups are the Sylow p-subgroups. An example of Cram shows that the answer to Question 1.1 is no for $H = S_3$ and G a certain 3-dimensional non-connected group with G^0 unipotent [ref: Slodowy]. If $G = \mathrm{GL}_n$ and p does not divide $|H|$ then the answer to Question 1.1 is yes, by Maschke's Theorem. It follows from a well-known geometric argument of Richardson that if p does not divide $|H|$ then the answer to Question 1.1 is yes for any connected reductive G: one embeds G in some GL_n and studies the behavior of the induced map $\mathrm{Hom}(H, G) \to \mathrm{Hom}(H, \mathrm{GL}_n)$. In fact, standard representation-theoretic results imply that the answer is yes for any finite group when $G = \mathrm{GL}_n$, and Slodowy used Richardson's argument to show the answer is yes for arbitrary finite H and connected reductive G under mild hypotheses on char(p). [General case of finite H: there is now a counterexample in type G_2 [ref].]

In this paper we instead consider the case when H is connected and semisimple. We settle Question 1.1 as follows.

Theorem 1.2. *The answer to Question 1.1 is yes if H is connected and semisimple.*

Note that we allow G to be non-connected, but it is clear we can reduce immediately to the case when G is connected and $Z(G)^0$ is trivial. Also, it is clear that the answer to Question 1.1 is no for arbitrary connected reductive H (e.g., just take H to be a torus, $U = 1$ and G^0 non-unipotent).

An important tool in the work of [above] is nonabelian 1-cohomology. Let $\rho \in \mathrm{Hom}(H, G)$ and let P be a parabolic subgroup of G such that $\rho(H) \subseteq P$. Then H acts on $V := R_u(P)$ via $h \cdot u = \rho(h)u\rho(h)^{-1}$, and representations near ρ in an appropriate sense can be understood

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in terms of the nonabelian 1-cohomology $H^1(H, \rho, V)$. In Section [below] we discuss this cohomological approach and study its behavior with respect to restriction of representations from H to U. [Further discussion.]

[Based on D's PhD thesis. Related work by David Stewart: see below. He uses high-powered cohomology techniques; our methods are more elementary.]

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2. Preliminaries

[Some useful results from ''On unipotent algebraic groups and 1-cohomology'', David Stewart:

Cor. 3.4.3: Let $B = TU$ be connected solvable group with unipotent radical U and maximal torus T. Let Q be a unipotent group on which B acts. Then the restriction map $H^1(B, Q) \to H^1(U, Q)$ is injective.

Thm 3.5.2: Let G be connected reductive and let Q be a unipotent group on which G acts. Then for any parabolic subgroup P of G, the restriction map $H^1(G, Q) \to H^1(P, Q)$ is an isomorphism of pointed sets.

Corollary 3.6.2. Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = LR_u(P)$ of G and let P_1 be any parabolic subgroup of H. Then H is G-conjugate to a subgroup of L if and only if P_1 is as well.]

We assume G is a possibly non-connected algebraic group over k and H is a linear algebraic group over k. We fix a maximal unipotent subgroup U of H.

Suppose H is connected, let B be a Borel subgroup of H, let X be an affine variety and let $f\colon H \to X$ be a morphism such that $f(hb) = f(h)$ for all $h \in H$ and all $b \in B$. Then f gives rise to a morphism $\overline{f}$ from the projective variety H/B to X. Since H/B is connected and X is affine, $\overline{f}$ must be constant, so f is constant. In particular, if V is an affine H-variety, $v \in V$ and the stabiliser H_v contains B then $H_v = H$.

Lemma 2.1. *Suppose G is connected and reductive. Let H_1, H_2 be connected reductive subgroups of G. Suppose B is a common Borel subgroup of both H_1 and H_2. Then $H_1 = H_2$.*

Proof. The quotient variety G/H_1 is affine since H_1 is reductive [ref], and H_2 acts on G/H_1 by left multiplication. The stabiliser in H_2 of the coset H_1 contains B, so it must equal the whole of H_2. Hence $H_2 \subseteq H_1$. The reverse inequality follows similarly, so $H_1 = H_2$. $\square$

Lemma 2.2. *Let B be a Borel subgroup of H. Let $\rho_1, \rho_2 \in \mathrm{Hom}(H, G)$ such that $\rho_1|_B = \rho_2|_B$. Then $\rho_1 = \rho_2$.*

Proof. Define $f\colon H \to G$ by $f(h) = \rho_1(h)\rho_2(h)^{-1}$. For any $h \in H$, $b \in B$, $f(hb) = \rho_1(hb)\rho_2(hb)^{-1} = \rho_1(h)\rho_1(b)\rho_2(b)^{-1}\rho_2(h)^{-1} = \rho_1(h)\rho_2(h)^{-1} = f(h)$. So f gives rise to a morphism $\overline{f}$ from the connected projective variety H/B to G given by $\overline{f}(hB) = f(h)$. Then $\overline{f}$ is constant, so f is constant with value $f(1) = 1$, and the result follows. $\square$

3. Proof of Theorem 1.2 for G reductive

In this section we assume G and H are connected and semisimple.

Lemma 3.1. *Let B be the Borel subgroup of H that contains U, and let T be a maximal torus of B. Let $\rho_1, \rho_2 \colon H \to G$ be representations such that $\rho_1(B) = \rho_2(B)$ and $\rho_1(T) = \rho_2(T)$. Set $U' = \rho_1(U) = \rho_2(U)$. Suppose that for all $\alpha \in \Phi_T(B)$, $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ and there exists $\phi_\alpha \in \mathrm{Aut}(\rho_1(U_\alpha))$ such that for all $u \in U_\alpha$, $\rho_2(u) = \phi_\alpha(\rho_1(u))$. Then there exists $t' \in T$ such that $t' \cdot \rho_2 = \rho_1$.*

Proof. Since B is a Borel subgroup of H, $\rho_i(B)$ is a Borel subgroup of $\rho_i(H)$ for $i = 1, 2$. Lemma 2.1 implies that $\rho_1(H) = \rho_2(H)$. Hence we can assume that ρ_1 and ρ_2 are surjective. Set $T' = \rho_1(T) = \rho_2(T)$.

Pick a base $\{\alpha'_1, \ldots, \alpha'_m\}$ for $\Phi_{T'}(B')$. Since ρ_1 and ρ_2 map a given root group of U with respect to T to a root group of U' with respect to T', there exist $\alpha_1, \ldots, \alpha_m \in \Phi_T(B)$ such that $\rho_1(U_{\alpha_i}) = \rho_2(U_{\alpha_i}) = U_{\alpha'_i}$ and $\alpha'_i(\rho_1(t)) = \alpha'_i(\rho_2(t))$ for all i. [This is not quite true as stated: some of the U_{α_i} may be mapped by ρ_1 and ρ_2 to 1; we're only interested in the case when this doesn't happen. A bit more argument is needed here.] As H is semisimple, this implies that $\rho_1|_T = \rho_2|_T$.

By hypothesis, each ϕ_{α_i} is an automorphism of $U_{\alpha'_i}$, so there exist $b_1, \ldots, b_m \in k^*$ such that $\phi_{\alpha_i}(\epsilon_{\alpha'_i}(x)) = \epsilon_{\alpha'_i}(b_i x)$ for each i and for all $x \in k$; so there exist $a_1, \ldots, a_m \in k^*$ such that $\rho_2(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(a_i x))$ for each i and for all $x \in k$ [check!].

The weights $\alpha'_1, \ldots, \alpha'_m$ are linearly independent as H is semisimple, so there exists $t' \in T'$ such that $\alpha_i(t') = a_i^{-1}$ for $1 \leq i \leq m$. We then have $(t' \cdot \rho_2)(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(x))$ for each i and for all $x \in k$. It follows that $\rho_1|_{\widetilde{U}} = (t' \cdot \rho_2)|_{\widetilde{U}}$, where $\widetilde{U}$ is the subgroup of U generated by the U_{α_i}. But $\rho_1(\widetilde{U}) = \rho_2(\widetilde{U}) = (t' \cdot \rho_2)(\widetilde{U})$ since the $U_{\alpha'_i}$ generate U' [ref: Humphreys?], and $\ker(\rho_1|_U) = \ker(\rho_2|_U) = \ker((t' \cdot \rho_2)|_U)$, so $\rho_1|_U = (t' \cdot \rho_2)|_U$.

To complete the proof, it is enough by Lemma 2.2 to show that $\rho_1|_B = (t' \cdot \rho_2)|_B$. But $(t' \cdot \rho_2)|_T = \rho_2|_T = \rho_1|_T$ from the discussion above, so we are done. ☐

[Note: In the situation of the above lemma (and with ρ_1, ρ_2 assumed surjective), suppose we don't assume that $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for every α. Then ρ_1 and ρ_2 need no longer be G-conjugate. I suspect, however, that in this case they are $\mathrm{Aut}(G)$-conjugate. If this is the case then we should get a reasonable bound in Theorem 1.2 involving $|\mathrm{Out}(G)|$ and the size of the Weyl group of T' in $N_G(U')$ (cf. below).]

Theorem 3.2. *The answer to Külshammer's question is yes for H and G connected and semisimple.*

Proof. Let $\sigma \colon U \to G$ and let $U' = \sigma(U)$. Let B be the Borel subgroup of H that contains U and fix a maximal torus T of B. Fix a maximal torus T' of $N_G(U')$. Let $C = \{\rho \in \mathrm{Hom}(H, G) \mid \rho|_U = \sigma\}$. If $\rho \in C$ then $\rho(T)$ normalizes $\rho(U) = U'$, so $\rho(T)$ is a torus of $N_G(U')$. By conjugacy of maximal tori of $N_G(U')$, there exists $g \in N_G(U')$ such that $(g \cdot \rho)(T) \subseteq T'$. If $h \in N_G(U')$ and $(h \cdot \rho)(T) \subseteq T'$ then there exists $n \in N_{N_G(U')}(T')$ such that $(h \cdot \rho)(T) = ((ng) \cdot \rho)(T)$. The group $N_{N_G(U')}(T')/C_{N_G(U')}(T')$ is finite, so there is a finite set of subtori $S_1, \ldots, S_r$ of T' such that $\rho(T)$ is $N_G(U')$-conjugate to one of $S_1, \ldots, S_r$.

Define a relation $\equiv$ on C by $\rho_1 \equiv \rho_2$ if there exist $i \in \{1, \ldots, r\}$ and $g_1, g_2 \in N_G(U')$ such that $(g_1 \cdot \rho_1)(T) = (g_2 \cdot \rho_1)(T) = S_i$ and $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for all $\alpha \in \Phi_T(U)$. It is clear that $\equiv$ is an equivalence relation. Define $C_i = \{\rho \in G \mid (g \cdot \rho)(T) = S_i \text{ for some } g \in N_G(U')\}$. If $(g \cdot \rho)(T) = S_i$ then $(g \cdot \rho)$ must map any root group U_α of U with respect to T to a root group of U' with respect to S_i, so there are only finitely many possibilities for $(g \cdot \rho)(U_\alpha)$. It follows that $\equiv$ has only finitely many equivalence classes.

To complete the proof it is enough, therefore, to show that if $\rho_1 \equiv \rho_2$ then ρ_1 and ρ_2 are G-conjugate. So suppose g_1 and g_2 are [as in the defn of $\equiv$]. Set $U_{\alpha'} = \rho_1(U_\alpha) = \rho_2(\alpha)$ if this is nontrivial [recall that $\rho_1(U_\alpha)$ might be just 1]. Then the map $\phi\colon G \to G$ given by $\phi(g) = g_2 g_1^{-1} g g_1 g_2^{-1}$ gives rise to an automorphism of U'_α for all $\alpha' \in \Phi_{S_i}(U')$. It follows by Lemma 3.1 that $g_1 \cdot \rho_1$ and $g_2 \cdot \rho_2$ are T-conjugate, so we are done. $\square$

[Consequence: Look at **images** of representations and prove an analogous finiteness result, for connected reductive H as well?]

4. 1-cohomology

[Describe Daniel's approach using 1-cohomology.]

[We've proved Theorem 1.2 for reductive G (it's enough to take G to be connected and semisimple, since H is connected and semisimple). I think the theorem follows for arbitrary non-reductive G by taking the result for the special case of reductive groups and applying it to the reductive group G/V, then using David Stewart's result that the restriction map $H^1(B,V) \to H^1(U,V)$ is injective. Here $V = R_u(G)$ and B is a Borel subgroup of G with unipotent radical U.]

[An application:

Theorem: Let G be a reductive group such that the simple components of G are all of type A_n or B_2. Then the answer to K's question is yes for any H with H^0 semisimple.

Proof (sketch): We can approximate H with a sufficiently large finite subgroup, so assume without loss that H is finite. It's enough to study the simple components of G separately, so without loss we assume G is simple. If G is of type B_2 and $p \neq 2$ then we're done by Slodowy's paper, so without loss we assume that either G is of type A_n, or G is of type B_2 and $p = 2$. Note that maximal parabolics of G have abelian unipotent radicals under this hypothesis.

We now use an inductive proof. If ρ is G-irreducible then we're done. Otherwise put ρ into a maximal parabolic P. The key point is that we can transfer the problem from P into a Levi of P by the usual arguments, because the restriction map $H^1(H,V) \to H^1(U,V)$ is injective for any finite H if $V = R_u(P)$ is abelian. (Here U is a Sylow p-subgroup of H.) By induction on $\dim(G)$, the result is true for L—for the simple components of L are all of type A_n or B_2—and we get what we want.

This result shows that the G_2 counterexample to K's question for finite H is in a sense the smallest possible: there is no such counterexample for any other G of rank 1 or 2.]

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Department of Mathematics, University of Aberdeen, King's College, Fraser Noble Building, Aberdeen AB24 3UE, United Kingdom
E-mail address: B.Martin@abdn.ac.uk