ON A QUESTION OF KÜLSHAMMER FOR REDUCTIVE ALGEBRAIC GROUPS

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Abstract.

1. Introduction

Let G and H be linear algebraic groups over an algebraically closed field of characteristic $p > 0$. Pick a maximal unipotent subgroup U of H [this is unique up to conjugacy]. By a *representation* of H in G we mean a homomorphism of algebraic groups from H to G. The group G acts on the set of representations $\mathrm{Hom}(H, G)$ by $(g \cdot \rho)(h) = g\rho(h)g^{-1}$ for $h \in H$ and $g \in G$; we call the orbits *conjugacy classes*. We consider the following question.

Question 1.1. *Let $\sigma\colon U \to G$ be a representation. Are there only finitely many conjugacy classes of representation $\rho\colon H \to G$ such that $\rho|_U$ is G-conjugate to σ?*

Külshammer raised this question for finite H [2]; in this case the maximal unipotent subgroups are the Sylow p-subgroups. An example of Cram shows that the answer to Question 1.1 is no for $H = S_3$ and G a certain 3-dimensional non-connected group with G^0 unipotent [4, Appendix]. If $G = \mathrm{GL}_n$ and p does not divide $|H|$ then the answer to Question 1.1 is yes, by Maschke's Theorem. It follows from a well-known geometric argument of Richardson that if p does not divide $|H|$ then the answer to Question 1.1 is yes for any connected reductive G: one embeds G in some GL_n and studies the behavior of the induced map $\mathrm{Hom}(H, G) \to \mathrm{Hom}(H, \mathrm{GL}_n)$. In fact, standard representation-theoretic results imply that the answer is yes for any finite group when $G = \mathrm{GL}_n$, and Slodowy used Richardson's argument to show the answer is yes for arbitrary finite H and connected reductive G under mild hypotheses on $\mathrm{char}(p)$. [General case of finite H: there is now a counterexample in type G_2 [ref].]

In this paper we instead consider the case when H is connected and semisimple. We settle Question 1.1 as follows.

Theorem 1.2. *The answer to Question 1.1 is yes if H is connected and semisimple.*

Note that we allow G to be non-connected, but it is clear we can reduce immediately to the case when G is connected and $Z(G)^0$ is trivial. Also, it is clear that the answer to Question 1.1 is no for arbitrary connected reductive H (e.g., just take H to be a torus, $U = 1$ and G^0 non-unipotent).

An important tool in the work of [above] is nonabelian 1-cohomology. Let $\rho \in \mathrm{Hom}(H, G)$ and let P be a parabolic subgroup of G such that $\rho(H) \subseteq P$. Then H acts on $V := R_u(P)$ via $h \cdot u = \rho(h)u\rho(h)^{-1}$, and representations near ρ in an appropriate sense can be understood

1991 *Mathematics Subject Classification.* ??? (???)
Key words and phrases. ???

in terms of the nonabelian 1-cohomology $H^1(H,\rho,V)$. In Section [below] we discuss this cohomological approach and study its behavior with respect to restriction of representations from H to U. [Further discussion.]

[Based on D's PhD thesis. Related work by David Stewart: see below. He uses high-powered cohomology techniques; our methods are more elementary.]

[Acknowledgements: Marsden grants; Günter.]

2. Preliminaries

[Some useful results from ''On unipotent algebraic groups and 1-cohomology'', David Stewart:

Cor. 3.4.3: Let $B = TU$ be connected solvable group with unipotent radical U and maximal torus T. Let Q be a unipotent group on which B acts. Then the restriction map $H^1(B,Q) \to H^1(U,Q)$ is injective.

Thm 3.5.2: Let G be connected reductive and let Q be a unipotent group on which G acts. Then for any parabolic subgroup P of G, the restriction map $H^1(G,Q) \to H^1(P,Q)$ is an isomorphism of pointed sets.

Corollary 3.6.2. Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = LR_u(P)$ of G and let P_1 be any parabolic subgroup of H. Then H is G-conjugate to a subgroup of L if and only if P_1 is as well.]

Definition 2.1. Let X, Y be (algebraic) groups. Then we denote by $\mathrm{Hom}(X,Y)$ the set of (algebraic) group homomorphisms from X to Y.

Lemma 2.2. *Let $R \subset \mathrm{Hom}(K,P)$. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Let $\rho_1, \rho_2 \in R$ such that ρ_1 and ρ_2 lie in the same G-conjugacy class of R. Then there exists $g \in G$ such that

$$g\rho_1(x)g^{-1} = \rho_2(x),$$

for all $x \in K$.

Let $Q = gPg^{-1}$, hence $\rho_2(K) \subset P \cap Q$. Let T be a maximal torus of G contained in $P \cap Q$. Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

Then there exists $r \in P$ such that $q = grg^{-1}$, so

$$\begin{aligned} grg^{-1}Tgr^{-1}g^{-1} &= gTg^{-1} \\ \Rightarrow rg^{-1}Tgr^{-1} &= T. \end{aligned}$$

Therefore $gr^{-1} \in N_G(T)$.

Fix a finite set $N \subset N_G(T)$ of coset representatives for the Weyl group $W = N_G(T)/T$ and let $n \in N, t \in T$ such that

$$gr^{-1} = nt.$$

Let $q' = r^{-1}t^{-1}$ so $q' \in P$. Then

$$\begin{aligned}\rho_1(x) &= g^{-1}\rho_2(x)g \\ &= (q'n^{-1})\rho_2(x)nq'^{-1},\end{aligned}$$

for all $x \in K$. Hence $\rho_1 \in P\cdot(n^{-1}\cdot\rho_2)$. This shows that a G-conjugacy class of R is contained in a union of at most $|N| = |W|$ P-conjugacy classes.

Therefore, if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes.

The converse is trivial. □

We assume G is a possibly non-connected algebraic group over k and H is a linear algebraic group over k. We fix a maximal unipotent subgroup U of H.

Suppose H is connected, let B be a Borel subgroup of H, let X be an affine variety and let $f\colon H \to X$ be a morphism such that $f(hb) = f(h)$ for all $h \in H$ and all $b \in B$. Then f gives rise to a morphism $\overline{f}$ from the projective variety H/B to X. Since H/B is connected and X is affine, $\overline{f}$ must be constant, so f is constant. In particular, if V is an affine H-variety, $v \in V$ and the stabiliser H_v contains B then $H_v = H$.

Lemma 2.3. *Suppose G is connected and reductive. Let H_1, H_2 be connected reductive subgroups of G. Suppose B is a common Borel subgroup of both H_1 and H_2. Then $H_1 = H_2$.*

Proof. The quotient variety G/H_1 is affine since H_1 is reductive [ref], and H_2 acts on G/H_1 by left multiplication. The stabiliser in H_2 of the coset H_1 contains B, so it must equal the whole of H_2. Hence $H_2 \subseteq H_1$. The reverse inequality follows similarly, so $H_1 = H_2$. □

Lemma 2.4. *Let B be a Borel subgroup of H. Let $\rho_1, \rho_2 \in \mathrm{Hom}(H, G)$ such that $\rho_1|_B = \rho_2|_B$. Then $\rho_1 = \rho_2$.*

Proof. Define $f\colon H \to G$ by $f(h) = \rho_1(h)\rho_2(h)^{-1}$. For any $h \in H$, $b \in B$, $f(hb) = \rho_1(hb)\rho_2(hb)^{-1} = \rho_1(h)\rho_1(b)\rho_2(b)^{-1}\rho_2(h)^{-1} = \rho_1(h)\rho_2(h)^{-1} = f(h)$. So f gives rise to a morphism $\overline{f}$ from the connected projective variety H/B to G given by $\overline{f}(hB) = f(h)$. Then $\overline{f}$ is constant, so f is constant with value $f(1) = 1$, and the result follows. □

3. Proof of Theorem 1.2 for G reductive

In this section we assume G and H are connected and semisimple.

Lemma 3.1. *Let B be the Borel subgroup of H that contains U, and let T be a maximal torus of B. Let $\rho_1, \rho_2\colon H \to G$ be representations such that $\rho_1(B) = \rho_2(B)$ and $\rho_1(T) = \rho_2(T)$. Set $U' = \rho_1(U) = \rho_2(U)$. Suppose that for all $\alpha \in \Phi_T(B)$, $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ and there exists $\phi_\alpha \in \mathrm{Aut}(\rho_1(U_\alpha))$ such that for all $u \in U_\alpha$, $\rho_2(u) = \phi_\alpha(\rho_1(u))$. Then there exists $t' \in T$ such that $t' \cdot \rho_2 = \rho_1$.*

Proof. Since B is a Borel subgroup of H, $\rho_i(B)$ is a Borel subgroup of $\rho_i(H)$ for $i = 1, 2$. Lemma 2.3 implies that $\rho_1(H) = \rho_2(H)$. Hence we can assume that ρ_1 and ρ_2 are surjective. Set $T' = \rho_1(T) = \rho_2(T)$.

Pick a base $\{\alpha'_1, \ldots, \alpha'_m\}$ for $\Phi_{T'}(B')$. Since ρ_1 and ρ_2 map a given root group of U with respect to T to a root group of U' with respect to T', there exist $\alpha_1, \ldots, \alpha_m \in \Phi_T(B)$ such that $\rho_1(U_{\alpha_i}) = \rho_2(U_{\alpha_i}) = U_{\alpha'_i}$ and $\alpha'_i(\rho_1(t)) = \alpha'_i(\rho_2(t))$ for all i. [This is not quite true as stated: some of the U_{α_i} may be mapped by ρ_1 and ρ_2 to 1; we're only interested in the case when this doesn't happen. A bit more argument is needed here.] As H is semisimple, this implies that $\rho_1|_T = \rho_2|_T$.

By hypothesis, each ϕ_{α_i} is an automorphism of $U_{\alpha'_i}$, so there exist $b_1, \ldots, b_m \in k^*$ such that $\phi_{\alpha_i}(\epsilon_{\alpha'_i}(x)) = \epsilon_{\alpha'_i}(b_i x)$ for each i and for all $x \in k$; so there exist $a_1, \ldots, a_m \in k^*$ such that $\rho_2(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(a_i x))$ for each i and for all $x \in k$ [check!].

The weights $\alpha'_1, \ldots, \alpha'_m$ are linearly independent as H is semisimple, so there exists $t' \in T'$ such that $\alpha_i(t') = a_i^{-1}$ for $1 \leq i \leq m$. We then have $(t' \cdot \rho_2)(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(x))$ for each i and for all $x \in k$. It follows that $\rho_1|_{\widetilde{U}} = (t' \cdot \rho_2)|_{\widetilde{U}}$, where $\widetilde{U}$ is the subgroup of U generated by the U_{α_i}. But $\rho_1(\widetilde{U}) = \rho_2(\widetilde{U}) = (t' \cdot \rho_2)(\widetilde{U})$ since the $U_{\alpha'_i}$ generate U' [ref: Humphreys?], and $\ker(\rho_1|_U) = \ker(\rho_2|_U) = \ker((t' \cdot \rho_2)|_U)$, so $\rho_1|_U = (t' \cdot \rho_2)|_U$.

To complete the proof, it is enough by Lemma 2.4 to show that $\rho_1|_B = (t' \cdot \rho_2)|_B$. But $(t' \cdot \rho_2)|_T = \rho_2|_T = \rho_1|_T$ from the discussion above, so we are done. $\square$

[Note: In the situation of the above lemma (and with ρ_1, ρ_2 assumed surjective), suppose we don't assume that $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for every α. Then ρ_1 and ρ_2 need no longer be G-conjugate. I suspect, however, that in this case they are $\mathrm{Aut}(G)$-conjugate. If this is the case then we should get a reasonable bound in Theorem 1.2 involving $|\mathrm{Out}(G)|$ and the size of the Weyl group of T' in $N_G(U')$ (cf. below).]

Theorem 3.2. *The answer to Külshammer's question is yes for H and G connected and semisimple.*

Proof. Let $\sigma\colon U \to G$ and let $U' = \sigma(U)$. Let B be the Borel subgroup of H that contains U and fix a maximal torus T of B. Fix a maximal torus T' of $N_G(U')$. Let $C = \{\rho \in \mathrm{Hom}(H, G) \mid \rho|_U = \sigma\}$. If $\rho \in C$ then $\rho(T)$ normalizes $\rho(U) = U'$, so $\rho(T)$ is a torus of $N_G(U')$. By conjugacy of maximal tori of $N_G(U')$, there exists $g \in N_G(U')$ such that $(g \cdot \rho)(T) \subseteq T'$. If $h \in N_G(U')$ and $(h \cdot \rho)(T) \subseteq T'$ then there exists $n \in N_{N_G(U')}(T')$ such that $(h \cdot \rho)(T) = ((ng) \cdot \rho)(T)$. The group $N_{N_G(U')}(T')/C_{N_G(U')}(T')$ is finite, so there is a finite set of subtori $S_1, \ldots, S_r$ of T' such that $\rho(T)$ is $N_G(U')$-conjugate to one of $S_1, \ldots, S_r$.

Define a relation $\equiv$ on C by $\rho_1 \equiv \rho_2$ if there exist $i \in \{1, \ldots, r\}$ and $g_1, g_2 \in N_G(U')$ such that $(g_1 \cdot \rho_1)(T) = (g_2 \cdot \rho_1)(T) = S_i$ and $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for all $\alpha \in \Phi_T(U)$. It is clear that $\equiv$ is an equivalence relation. Define $C_i = \{\rho \in G \mid (g \cdot \rho)(T) = S_i \text{ for some } g \in N_G(U')\}$. If $(g \cdot \rho)(T) = S_i$ then $(g \cdot \rho)$ must map any root group U_α of U with respect to T to a root group of U' with respect to S_i, so there are only finitely many possibilities for $(g \cdot \rho)(U_\alpha)$. It follows that $\equiv$ has only finitely many equivalence classes.

To complete the proof it is enough, therefore, to show that if $\rho_1 \equiv \rho_2$ then ρ_1 and ρ_2 are G-conjugate. So suppose g_1 and g_2 are [as in the defn of $\equiv$]. Set $U_{\alpha'} = \rho_1(U_\alpha) = \rho_2(\alpha)$ if this is nontrivial [recall that $\rho_1(U_\alpha)$ might be just 1]. Then the map $\phi\colon G \to G$ given by $\phi(g) = g_2 g_1^{-1} g g_1 g_2^{-1}$ gives rise to an automorphism of U'_α for all $\alpha' \in \Phi_{S_i}(U')$. It follows by Lemma 3.1 that $g_1 \cdot \rho_1$ and $g_2 \cdot \rho_2$ are T-conjugate, so we are done. $\square$

[Consequence: Look at **images** of representations and prove an analogous finiteness result, for connected reductive H as well?]

4. 1-cohomology

[Describe Daniel's approach using 1-cohomology.]

Richardson introduces the nonabelian 1-cohomology in [3], defined below.

Definition 4.1. We call a morphism $\sigma : K \to V$ a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x)(x \cdot \sigma(y)),$$

for all $x, y \in K$. Denote by $Z^1(K, V)$ the collection of all 1-cocycles from K to V, and $H^1(K, V)$ the set of equivalence classes of $Z^1(K, V)$ under the relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \exists v \in V, \forall x \in K, \sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}).$$

We call $H^1(K, V)$ the *1-cohomology.*

Lemma 4.2. *Suppose K is linearly reductive and V is unipotent. Then $H^1(K, V)$ is trivial.* [3, Lemma 6.2.6].

Lemma 4.3. *Let V be a vector space over k,* $\mathrm{char}(k) = p$. *Let Γ be a finite group that acts linearly on V, and let Γ_p be a* Sylow p-subgroup *of Γ. Let ζ be the inclusion of Γ_p in Γ. Then the map*

$$H^1(\zeta) : H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

is injective. [1, III.10.4 Prop.].

Example 4.4. TODO: replace SL_2 with H.

Let $k = \overline{\mathbb{F}_p} = \bigcup_{r \in \mathbb{N}} \mathbb{F}_{p^r}$. Let V a be vector space over k on which $SL_2(k)$ acts linearly, and let $U_2(k)$ be the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Let ζ be the inclusion of $U_2(k)$ in $SL_2(k)$.

Then the map

$$H^1(\zeta) : H^1(SL_2(k), V) \to H^1(U_2(k), V) \tag{4.5}$$

is injective.

Proof. Let $r \in \mathbb{N}$ and denote the inclusion maps

$$\zeta_r : U_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(\mathbb{F}_{p^r}),$$
$$\iota_r : SL_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(k),$$
$$\iota'_r : U_2(\mathbb{F}_{p^r}) \hookrightarrow U_2(k).$$

By Lemma 4.8 (Remark **??**) we get the following commutative diagram,

$$\begin{tikzcd} H^1(SL_2(k), V) \arrow[r, "H^1(\zeta)"] \arrow[d, "H^1(\iota_r)"'] & H^1(U_2(k), V) \arrow[d, "H^1(\iota'_r)"] \\ H^1(SL_2(\mathbb{F}_{p^r}), V) \arrow[r, "H^1(\zeta_r)"'] & H^1(U_2(\mathbb{F}_{p^r}), V). \end{tikzcd} \tag{4.6}$$

It is elementary to show that $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$ (Appendix **??**), so by Lemma 4.3, $H^1(\zeta_r)$ is injective for all $r \in \mathbb{N}$.

Let $\sigma \in Z^1(SL_2(k), V)$ such that $\sigma \notin B^1(SL_2(k), V)$, that is,

$$\sigma \neq \chi_v^{SL_2(k)}, \tag{4.7}$$

for any $v \in V$. For each $x \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_x : V \to V$ by

$$f_x(v) = \sigma(x) - \chi_v^{SL_2(k)}(x).$$

Since $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$ we have $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$. Consider the sequence $\{C_r\}_{r\in\mathbb{N}}$ defined by

$$C_r = \{v \in V \mid \forall x \in SL_2(\mathbb{F}_{p^r}),\ f_x(v) = 0\}.$$

Then

$$\begin{aligned}\bigcap_{r\in\mathbb{N}} C_{r!} &= \{v \in V \mid \forall x \in SL_2(k),\ f_x(v) = 0\}\\ &= \emptyset \quad \text{(Equation 4.7)}.\end{aligned}$$

Each C_r is closed, and the inclusion $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$ induces the reverse inclusion for the subsequence $C_{r!} \supset C_{(r+1)!}$. Then the Noetherian property for V requires that the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ becomes constant, and since $\cap_{r\in\mathbb{N}} C_{r!} = \emptyset$, the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ is eventually empty. That is, there exists $s \in \mathbb{N}$ such that

$$Z^1(\iota_s)(\sigma) \neq \chi_v^{SL_2(\mathbb{F}_{p^s})},$$

for any $v \in V$. We have shown that if $\sigma \in Z^1(SL_2(k), V)$ such that $Z^1(\iota_{r!})(\sigma) \in B^1(SL_2(\mathbb{F}_{p^{r!}}), V)$ for all $r \in \mathbb{N}$, then $\sigma \in B^1(SL_2(k), V)$.

So, let $\sigma \in Z^1(SL_2(k), V)$ such that $\psi(\sigma) \in \mathrm{Ker}\,(H^1(\zeta))$. Then, consulting the commutative diagram in Equation 4.6,

$$\begin{aligned}&\psi(\sigma) \in \mathrm{Ker}\left(H^1(\iota'_r) \circ H^1(\zeta)\right), \forall r \in \mathbb{N}\\ \Rightarrow\, &\psi(\sigma) \in \mathrm{Ker}\left(H^1(\zeta_r) \circ H^1(\iota_r)\right), \forall r \in \mathbb{N}\\ \Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \in \mathrm{Ker}\left(H^1(\zeta_r)\right), \forall r \in \mathbb{N}\\ \Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \text{ is trivial }, \forall r \in \mathbb{N}\\ \Rightarrow\, &Z^1(\iota_r)(\sigma) \in B^1(SL_2(\mathbb{F}_{p^r}), V), \forall r \in \mathbb{N}\\ \Rightarrow\, &\sigma \in B^1(SL_2(k), V)\\ \Rightarrow\, &\psi(\sigma) \in H^1(SL_2(k), V) \text{ is trivial.}\end{aligned}$$

This shows $H^1(\zeta)$ is injective. □

Lemma 4.8 (Map of 1-Cohomologies)**.** ***TODO: be brief***

Let K', V' be algebraic groups such that K' acts on V' by group automorphisms. Let $\zeta : K' \to K$ be a homomorphism and let $\xi : V \to V'$ be a K'-equivariant homomorphism; that is, suppose that $\xi(\zeta(x) \cdot v) = x \cdot \xi(v)$ for all $x \in K', v \in V$.

Then the function $Z^1(\zeta, \xi)$ defined by

$$Z^1(\zeta, \xi)(\sigma) = \xi \circ \sigma \circ \zeta,$$

maps $Z^1(K, V)$ to $Z^1(K', V')$.

Furthermore, $Z^1(\zeta, \xi)$ descends to give a well-defined map

$$H^1(\zeta, \xi) : H^1(K, V) \to H^1(K', V'),$$

defined by

$$H^1(\zeta, \xi)(\psi(\sigma)) = \left(\psi' \circ Z^1(\zeta, \xi)\right)(\sigma),$$

for all $\sigma \in Z^1(K,V)$*, where* ψ' *is the canonical projection from* $Z^1(K',V')$ *to* $H^1(K',V')$*. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} Z^1(K,V) & \xrightarrow{Z^1(\zeta,\xi)} & Z^1(K',V') \\ \psi\downarrow & & \downarrow\psi' \\ H^1(K,V) & \xrightarrow{H^1(\zeta,\xi)} & H^1(K',V'). \end{array}$$

Definition 4.9. We associate with $\rho \in \operatorname{Hom}(K,P)$ the 1-cocycle $\sigma_\rho \in Z^1(K,V)$ defined by $\sigma_\rho(x) = \rho(x)\rho|_L(x^{-1})$.

Let $\omega \in \operatorname{Hom}(K,L)$.

Definition 4.10. We denote by $Z^1(K,V)_\omega$ the set of 1-cocycles from K to V where K acts on V via ω. Likewise, denote by $H^1(K,V)_\omega$ the 1-cohomology obtained from $Z^1(K,V)_\omega$.

Definition 4.11. Define $\operatorname{Hom}(K,P)_\omega = \{\rho \in \operatorname{Hom}(K,P) \mid \rho^L = \omega\}$. More generally, if $R \subset \operatorname{Hom}(K,P)$ define $R_\omega = \{\rho \in R \mid \rho^L = \omega\}$.

Lemma 4.12. *Let* $\omega \in \operatorname{Hom}(K,L)$*. The map*

$$z : \operatorname{Hom}(K,P)_\omega \to Z^1(K,V)_\omega,$$

defined by

$$z(\rho)(x) = \rho(x)\omega(x^{-1}),$$

for all $\rho \in \operatorname{Hom}(K,P)_\omega$ *and all* $x \in K$*, is a bijection.*

Lemma 4.13. *For* $\rho \in \operatorname{Hom}(K,P)_\omega$*, define*

$$h(\phi(\rho)) = \psi(z(\rho)).$$

Then h *is a well-defined bijection from* $\operatorname{Hom}(K,P)_\omega/V$ *to* $H^1(K,P)_\omega$*. Moreover, the following diagram commutes:*

$$\begin{array}{ccc} \operatorname{Hom}(K,P)_\omega & \xrightarrow{z} & Z^1(K,V)_\omega \\ \phi\downarrow & & \downarrow\psi \\ \operatorname{Hom}(K,P)_\omega/V & \xrightarrow{h} & H^1(K,V)_\omega. \end{array}$$

[We've proved Theorem 1.2 for reductive G (it's enough to take G to be connected and semisimple, since H is connected and semisimple). I think the theorem follows for arbitrary non-reductive G by taking the result for the special case of reductive groups and applying it to the reductive group G/V, then using David Stewart's result that the restriction map $H^1(B,V) \to H^1(U,V)$ is injective. Here $V = R_u(G)$ and B is a Borel subgroup of G with unipotent radical U.]

[An application:

Theorem: Let G be a reductive group such that the simple components of G are all of type A_n or B_2. Then the answer to K's question is yes for any H with H^0 semisimple.

Proof (sketch): We can approximate H with a sufficiently large finite subgroup, so assume without loss that H is finite. It's enough to study the simple components of G separately, so without loss we assume G is simple. If G is of type B_2 and $p \neq 2$ then we're done by Slodowy's paper, so without loss we assume that either G is of type A_n, or G is of type B_2 and $p = 2$. Note that maximal parabolics of G have abelian unipotent radicals under this hypothesis.

We now use an inductive proof. If ρ is G-irreducible then we're done. Otherwise put ρ into a maximal parabolic P. The key point is that we can transfer the problem from P into a Levi of P by the usual arguments, because the restriction map $H^1(H,V) \to H^1(U,V)$ is injective for any finite H if $V = R_u(P)$ is abelian. (Here U is a Sylow p-subgroup of H.) By induction on $\dim(G)$, the result is true for L—for the simple components of L are all of type A_n or B_2—and we get what we want.

This result shows that the G_2 counterexample to K's question for finite H is in a sense the smallest possible: there is no such counterexample for any other G of rank 1 or 2.]

References

[1] K. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer, 1982.

[2] B. Külshammer. Donovan's conjecture, crossed products and algebraic group actions. *Israel J. Math.*, 92(1–3):295–306, 1995.

[3] RW Richardson. On orbits of algebraic groups and Lie groups. *Bulletin of the Australian Mathematical Society*, 25(01):1–28, 1982.

[4] P. Slodowy. Two notes on a finiteness problem in the representation theory of finite groups. *Algebraic groups and Lie groups: a volume of papers in honour of the late RW Richardson*, page 331, 1997.

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