

# ON A QUESTION OF KÜLSHAMMER FOR REDUCTIVE ALGEBRAIC GROUPS

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ABSTRACT.

## 1. INTRODUCTION

Let  $G$  and  $H$  be linear algebraic groups over an algebraically closed field of characteristic  $p > 0$ . Pick a maximal unipotent subgroup  $U$  of  $H$  [this is unique up to conjugacy]. By a *representation* of  $H$  in  $G$  we mean a homomorphism of algebraic groups from  $H$  to  $G$ . The group  $G$  acts on the set of representations  $\text{Hom}(H, G)$  by  $(g \cdot \rho)(h) = g\rho(h)g^{-1}$  for  $h \in H$  and  $g \in G$ ; we call the orbits *conjugacy classes*. We consider the following question.

**Question 1.1.** *Let  $\sigma: U \rightarrow G$  be a representation. Are there only finitely many conjugacy classes of representation  $\rho: H \rightarrow G$  such that  $\rho|_U$  is  $G$ -conjugate to  $\sigma$ ?*

Külshammer raised this question for finite  $H$  [2]; in this case the maximal unipotent subgroups are the Sylow  $p$ -subgroups. An example of Cram shows that the answer to Question 1.1 is no for  $H = S_3$  and  $G$  a certain 3-dimensional non-connected group with  $G^0$  unipotent [4, Appendix]. If  $G = \text{GL}_n$  and  $p$  does not divide  $|H|$  then the answer to Question 1.1 is yes, by Maschke's Theorem. It follows from a well-known geometric argument of Richardson that if  $p$  does not divide  $|H|$  then the answer to Question 1.1 is yes for any connected reductive  $G$ : one embeds  $G$  in some  $\text{GL}_n$  and studies the behavior of the induced map  $\text{Hom}(H, G) \rightarrow \text{Hom}(H, \text{GL}_n)$ . In fact, standard representation-theoretic results imply that the answer is yes for any finite group when  $G = \text{GL}_n$ , and Slodowy used Richardson's argument to show the answer is yes for arbitrary finite  $H$  and connected reductive  $G$  under mild hypotheses on  $\text{char}(p)$ . [General case of finite  $H$ : there is now a counterexample in type  $G_2$  [ref].]

In this paper we instead consider the case when  $H$  is connected and semisimple. We settle Question 1.1 as follows.

**Theorem 1.2.** *The answer to Question 1.1 is yes if  $H$  is connected and semisimple.*

Note that we allow  $G$  to be non-connected, but it is clear we can reduce immediately to the case when  $G$  is connected and  $Z(G)^0$  is trivial. Also, it is clear that the answer to Question 1.1 is no for arbitrary connected reductive  $H$  (e.g., just take  $H$  to be a torus,  $U = 1$  and  $G^0$  non-unipotent).

An important tool in the work of [above] is nonabelian 1-cohomology. Let  $\rho \in \text{Hom}(H, G)$  and let  $P$  be a parabolic subgroup of  $G$  such that  $\rho(H) \subseteq P$ . Then  $H$  acts on  $V := R_u(P)$  via  $h \cdot u = \rho(h)u\rho(h)^{-1}$ , and representations near  $\rho$  in an appropriate sense can be understood

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in terms of the nonabelian 1-cohomology  $H^1(H, \rho, V)$ . In Section [below] we discuss this cohomological approach and study its behavior with respect to restriction of representations from  $H$  to  $U$ . [Further discussion.]

[Based on D's PhD thesis. Related work by David Stewart: see below. He uses high-powered cohomology techniques; our methods are more elementary.]

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## 2. PRELIMINARIES

[Some useful results from ‘‘On unipotent algebraic groups and 1-cohomology’’, David Stewart:

Cor. 3.4.3: Let  $B = TU$  be connected solvable group with unipotent radical  $U$  and maximal torus  $T$ . Let  $Q$  be a unipotent group on which  $B$  acts. Then the restriction map  $H^1(B, Q) \rightarrow H^1(U, Q)$  is injective.

Thm 3.5.2: Let  $G$  be connected reductive and let  $Q$  be a unipotent group on which  $G$  acts. Then for any parabolic subgroup  $P$  of  $G$ , the restriction map  $H^1(G, Q) \rightarrow H^1(P, Q)$  is an isomorphism of pointed sets.

Corollary 3.6.2. Let  $H$  be a closed, connected, reductive subgroup of  $G$  contained in a parabolic  $P = LR_u(P)$  of  $G$  and let  $P_1$  be any parabolic subgroup of  $H$ . Then  $H$  is  $G$ -conjugate to a subgroup of  $L$  if and only if  $P_1$  is as well.]

**Definition 2.1.** Let  $X, Y$  be (algebraic) groups. Then we denote by  $\text{Hom}(X, Y)$  the set of (algebraic) group homomorphisms from  $X$  to  $Y$ .

**Lemma 2.2.** *Let  $R \subset \text{Hom}(K, P)$ . Then  $R$  is contained in a finite union of  $G$ -conjugacy classes if and only if it is contained in a finite union of  $P$ -conjugacy classes.*

*Proof.* Let  $\rho_1, \rho_2 \in R$  such that  $\rho_1$  and  $\rho_2$  lie in the same  $G$ -conjugacy class of  $R$ . Then there exists  $g \in G$  such that

$$g\rho_1(x)g^{-1} = \rho_2(x),$$

for all  $x \in K$ .

Let  $Q = gPg^{-1}$ , hence  $\rho_2(K) \subset P \cap Q$ . Let  $T$  be a maximal torus of  $G$  contained in  $P \cap Q$ . Since  $T$  and  $gTg^{-1}$  are maximal tori of  $Q$  they must be  $Q$ -conjugate, so there exists  $q \in Q$  such that

$$qTq^{-1} = gTg^{-1}.$$

Then there exists  $r \in P$  such that  $q = grg^{-1}$ , so

$$\begin{aligned} grg^{-1}Tgr^{-1}g^{-1} &= gTg^{-1} \\ \Rightarrow rg^{-1}Tgr^{-1} &= T. \end{aligned}$$

Therefore  $gr^{-1} \in N_G(T)$ .

Fix a finite set  $N \subset N_G(T)$  of coset representatives for the Weyl group  $W = N_G(T)/T$  and let  $n \in N, t \in T$  such that

$$gr^{-1} = nt.$$

Let  $q' = r^{-1}t^{-1}$  so  $q' \in P$ . Then

$$\begin{aligned}\rho_1(x) &= g^{-1}\rho_2(x)g \\ &= (q'n^{-1})\rho_2(x)nq'^{-1},\end{aligned}$$

for all  $x \in K$ . Hence  $\rho_1 \in P \cdot (n^{-1} \cdot \rho_2)$ . This shows that a  $G$ -conjugacy class of  $R$  is contained in a union of at most  $|N| = |W|$   $P$ -conjugacy classes.

Therefore, if  $R$  is contained in a finite union of  $G$ -conjugacy classes then it is contained in a finite union of  $P$ -conjugacy classes.

The converse is trivial.  $\square$

We assume  $G$  is a possibly non-connected algebraic group over  $k$  and  $H$  is a linear algebraic group over  $k$ . We fix a maximal unipotent subgroup  $U$  of  $H$ .

Suppose  $H$  is connected, let  $B$  be a Borel subgroup of  $H$ , let  $X$  be an affine variety and let  $f: H \rightarrow X$  be a morphism such that  $f(hb) = f(h)$  for all  $h \in H$  and all  $b \in B$ . Then  $f$  gives rise to a morphism  $\bar{f}$  from the projective variety  $H/B$  to  $X$ . Since  $H/B$  is connected and  $X$  is affine,  $\bar{f}$  must be constant, so  $f$  is constant. In particular, if  $V$  is an affine  $H$ -variety,  $v \in V$  and the stabiliser  $H_v$  contains  $B$  then  $H_v = H$ .

**Lemma 2.3.** *Suppose  $G$  is connected and reductive. Let  $H_1, H_2$  be connected reductive subgroups of  $G$ . Suppose  $B$  is a common Borel subgroup of both  $H_1$  and  $H_2$ . Then  $H_1 = H_2$ .*

*Proof.* The quotient variety  $G/H_1$  is affine since  $H_1$  is reductive [ref], and  $H_2$  acts on  $G/H_1$  by left multiplication. The stabiliser in  $H_2$  of the coset  $H_1$  contains  $B$ , so it must equal the whole of  $H_2$ . Hence  $H_2 \subseteq H_1$ . The reverse inequality follows similarly, so  $H_1 = H_2$ .  $\square$

**Lemma 2.4.** *Let  $B$  be a Borel subgroup of  $H$ . Let  $\rho_1, \rho_2 \in \text{Hom}(H, G)$  such that  $\rho_1|_B = \rho_2|_B$ . Then  $\rho_1 = \rho_2$ .*

*Proof.* Define  $f: H \rightarrow G$  by  $f(h) = \rho_1(h)\rho_2(h)^{-1}$ . For any  $h \in H$ ,  $b \in B$ ,  $f(hb) = \rho_1(hb)\rho_2(hb)^{-1} = \rho_1(h)\rho_1(b)\rho_2(b)^{-1}\rho_2(h)^{-1} = \rho_1(h)\rho_2(h)^{-1} = f(h)$ . So  $f$  gives rise to a morphism  $\bar{f}$  from the connected projective variety  $H/B$  to  $G$  given by  $\bar{f}(hB) = f(h)$ . Then  $\bar{f}$  is constant, so  $f$  is constant with value  $f(1) = 1$ , and the result follows.  $\square$

### 3. PROOF OF THEOREM 1.2 FOR $G$ REDUCTIVE

In this section we assume  $G$  and  $H$  are connected and semisimple.

**Lemma 3.1.** *Let  $B$  be the Borel subgroup of  $H$  that contains  $U$ , and let  $T$  be a maximal torus of  $B$ . Let  $\rho_1, \rho_2: H \rightarrow G$  be representations such that  $\rho_1(B) = \rho_2(B)$  and  $\rho_1(T) = \rho_2(T)$ . Set  $U' = \rho_1(U) = \rho_2(U)$ . Suppose that for all  $\alpha \in \Phi_T(B)$ ,  $\rho_1(U_\alpha) = \rho_2(U_\alpha)$  and there exists  $\phi_\alpha \in \text{Aut}(\rho_1(U_\alpha))$  such that for all  $u \in U_\alpha$ ,  $\rho_2(u) = \phi_\alpha(\rho_1(u))$ . Then there exists  $t' \in T$  such that  $t' \cdot \rho_2 = \rho_1$ .*

*Proof.* Since  $B$  is a Borel subgroup of  $H$ ,  $\rho_i(B)$  is a Borel subgroup of  $\rho_i(H)$  for  $i = 1, 2$ . Lemma 2.3 implies that  $\rho_1(H) = \rho_2(H)$ . Hence we can assume that  $\rho_1$  and  $\rho_2$  are surjective. Set  $T' = \rho_1(T) = \rho_2(T)$ .

Pick a base  $\{\alpha'_1, \dots, \alpha'_m\}$  for  $\Phi_{T'}(B')$ . Since  $\rho_1$  and  $\rho_2$  map a given root group of  $U$  with respect to  $T$  to a root group of  $U'$  with respect to  $T'$ , there exist  $\alpha_1, \dots, \alpha_m \in \Phi_T(B)$  such that  $\rho_1(U_{\alpha_i}) = \rho_2(U_{\alpha_i}) = U_{\alpha'_i}$  and  $\alpha'_i(\rho_1(t)) = \alpha'_i(\rho_2(t))$  for all  $i$ . [This is not quite true as stated: some of the  $U_{\alpha_i}$  may be mapped by  $\rho_1$  and  $\rho_2$  to 1; we're only interested in the case

when this doesn't happen. A bit more argument is needed here.] As  $H$  is semisimple, this implies that  $\rho_1|_T = \rho_2|_T$ .

By hypothesis, each  $\phi_{\alpha_i}$  is an automorphism of  $U_{\alpha'_i}$ , so there exist  $b_1, \dots, b_m \in k^*$  such that  $\phi_{\alpha_i}(\epsilon_{\alpha'_i}(x)) = \epsilon_{\alpha'_i}(b_i x)$  for each  $i$  and for all  $x \in k$ ; so there exist  $a_1, \dots, a_m \in k^*$  such that  $\rho_2(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(a_i x))$  for each  $i$  and for all  $x \in k$  [check!].

The weights  $\alpha'_1, \dots, \alpha'_m$  are linearly independent as  $H$  is semisimple, so there exists  $t' \in T'$  such that  $\alpha_i(t') = a_i^{-1}$  for  $1 \leq i \leq m$ . We then have  $(t' \cdot \rho_2)(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(x))$  for each  $i$  and for all  $x \in k$ . It follows that  $\rho_1|_{\tilde{U}} = (t' \cdot \rho_2)|_{\tilde{U}}$ , where  $\tilde{U}$  is the subgroup of  $U$  generated by the  $U_{\alpha_i}$ . But  $\rho_1(\tilde{U}) = \rho_2(\tilde{U}) = (t' \cdot \rho_2)(\tilde{U})$  since the  $U_{\alpha'_i}$  generate  $U'$  [ref: Humphreys?], and  $\ker(\rho_1|_U) = \ker(\rho_2|_U) = \ker((t' \cdot \rho_2)|_U)$ , so  $\rho_1|_U = (t' \cdot \rho_2)|_U$ .

To complete the proof, it is enough by Lemma 2.4 to show that  $\rho_1|_B = (t' \cdot \rho_2)|_B$ . But  $(t' \cdot \rho_2)|_T = \rho_2|_T = \rho_1|_T$  from the discussion above, so we are done.  $\square$

[Note: In the situation of the above lemma (and with  $\rho_1, \rho_2$  assumed surjective), suppose we don't assume that  $\rho_1(U_\alpha) = \rho_2(U_\alpha)$  for every  $\alpha$ . Then  $\rho_1$  and  $\rho_2$  need no longer be  $G$ -conjugate. I suspect, however, that in this case they are  $\text{Aut}(G)$ -conjugate. If this is the case then we should get a reasonable bound in Theorem 1.2 involving  $|\text{Out}(G)|$  and the size of the Weyl group of  $T'$  in  $N_G(U')$  (cf. below).]

**Theorem 3.2.** *The answer to Külshammer's question is yes for  $H$  and  $G$  connected and semisimple.*

*Proof.* Let  $\sigma: U \rightarrow G$  and let  $U' = \sigma(U)$ . Let  $B$  be the Borel subgroup of  $H$  that contains  $U$  and fix a maximal torus  $T$  of  $B$ . Fix a maximal torus  $T'$  of  $N_G(U')$ . Let  $C = \{\rho \in \text{Hom}(H, G) \mid \rho|_U = \sigma\}$ . If  $\rho \in C$  then  $\rho(T)$  normalizes  $\rho(U) = U'$ , so  $\rho(T)$  is a torus of  $N_G(U')$ . By conjugacy of maximal tori of  $N_G(U')$ , there exists  $g \in N_G(U')$  such that  $(g \cdot \rho)(T) \subseteq T'$ . If  $h \in N_G(U')$  and  $(h \cdot \rho)(T) \subseteq T'$  then there exists  $n \in N_{N_G(U')}(T')$  such that  $(h \cdot \rho)(T) = ((ng) \cdot \rho)(T)$ . The group  $N_{N_G(U')}(T')/C_{N_G(U')}(T')$  is finite, so there is a finite set of subtori  $S_1, \dots, S_r$  of  $T'$  such that  $\rho(T)$  is  $N_G(U')$ -conjugate to one of  $S_1, \dots, S_r$ .

Define a relation  $\equiv$  on  $C$  by  $\rho_1 \equiv \rho_2$  if there exist  $i \in \{1, \dots, r\}$  and  $g_1, g_2 \in N_G(U')$  such that  $(g_1 \cdot \rho_1)(T) = (g_2 \cdot \rho_2)(T) = S_i$  and  $\rho_1(U_\alpha) = \rho_2(U_\alpha)$  for all  $\alpha \in \Phi_T(U)$ . It is clear that  $\equiv$  is an equivalence relation. Define  $C_i = \{\rho \in G \mid (g \cdot \rho)(T) = S_i \text{ for some } g \in N_G(U')\}$ . If  $(g \cdot \rho)(T) = S_i$  then  $(g \cdot \rho)$  must map any root group  $U_\alpha$  of  $U$  with respect to  $T$  to a root group of  $U'$  with respect to  $S_i$ , so there are only finitely many possibilities for  $(g \cdot \rho)(U_\alpha)$ . It follows that  $\equiv$  has only finitely many equivalence classes.

To complete the proof it is enough, therefore, to show that if  $\rho_1 \equiv \rho_2$  then  $\rho_1$  and  $\rho_2$  are  $G$ -conjugate. So suppose  $g_1$  and  $g_2$  are [as in the defn of  $\equiv$ ]. Set  $U_{\alpha'} = \rho_1(U_\alpha) = \rho_2(U_\alpha)$  if this is nontrivial [recall that  $\rho_1(U_\alpha)$  might be just 1]. Then the map  $\phi: G \rightarrow G$  given by  $\phi(g) = g_2 g_1^{-1} g g_1 g_2^{-1}$  gives rise to an automorphism of  $U'_{\alpha'}$  for all  $\alpha' \in \Phi_{S_i}(U')$ . It follows by Lemma 3.1 that  $g_1 \cdot \rho_1$  and  $g_2 \cdot \rho_2$  are  $T$ -conjugate, so we are done.  $\square$

[Consequence: Look at **images** of representations and prove an analogous finiteness result, for connected reductive  $H$  as well?]

#### 4. 1-COHOMOLOGY

[Describe Daniel's approach using 1-cohomology.]

Richardson introduces the nonabelian 1-cohomology in [3], defined below.

**Definition 4.1.** We call a morphism  $\sigma : K \rightarrow V$  a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x)(x \cdot \sigma(y)),$$

for all  $x, y \in K$ . Denote by  $Z^1(K, V)$  the collection of all 1-cocycles from  $K$  to  $V$ , and  $H^1(K, V)$  the set of equivalence classes of  $Z^1(K, V)$  under the relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \exists v \in V, \forall x \in K, \sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}).$$

We call  $H^1(K, V)$  the *1-cohomology*.

**Lemma 4.2.** *Suppose  $K$  is linearly reductive and  $V$  is unipotent. Then  $H^1(K, V)$  is trivial. [3, Lemma 6.2.6].*

**Lemma 4.3** (Map of 1-Cohomologies). *Let  $K', V'$  be algebraic groups such that  $K'$  acts on  $V'$  by group automorphisms. Let  $\zeta : K' \rightarrow K$  be a homomorphism and let  $\xi : V \rightarrow V'$  be a  $K'$ -equivariant homomorphism. Then the function  $Z^1(\zeta, \xi)$  defined by*

$$Z^1(\zeta, \xi)(\sigma) = \xi \circ \sigma \circ \zeta,$$

*maps  $Z^1(K, V)$  to  $Z^1(K', V')$ . Furthermore,  $Z^1(\zeta, \xi)$  descends to give a unique map*

$$H^1(\zeta, \xi) : H^1(K, V) \rightarrow H^1(K', V'),$$

*that makes the following diagram commute:*

$$\begin{array}{ccc} Z^1(K, V) & \longrightarrow & Z^1(K', V') \\ \downarrow & & \downarrow \\ H^1(K, V) & \longrightarrow & H^1(K', V'). \end{array}$$

For our applications, it is often the case that  $V = V'$  and  $\xi = \text{id}_V$ . Then we just write  $Z^1(\zeta)$  and  $H^1(\zeta)$ .

**Lemma 4.4.** *Let  $V$  be a vector space over  $k$ ,  $\text{char}(k) = p$ . Let  $\Gamma$  be a finite group that acts linearly on  $V$ , and let  $\Gamma_p$  be a Sylow  $p$ -subgroup of  $\Gamma$ . Let  $\zeta$  be the inclusion of  $\Gamma_p$  in  $\Gamma$ . Then the map*

$$H^1(\zeta) : H^1(\Gamma, V) \rightarrow H^1(\Gamma_p, V)$$

*is injective. [1, III.10.4 Prop.].*

**Example 4.5. TODO: replace  $SL_2$  with  $H$ .**

Let  $k = \overline{\mathbb{F}}_p = \bigcup_{r \in \mathbb{N}} \mathbb{F}_{p^r}$ . Let  $V$  be a vector space over  $k$  on which  $SL_2(k)$  acts linearly, and let  $U_2(k)$  be the subgroup of  $SL_2(k)$  consisting of upper unitriangular matrices. Let  $\zeta$  be the inclusion of  $U_2(k)$  in  $SL_2(k)$ .

Then the map

$$(4.6) \quad H^1(\zeta) : H^1(SL_2(k), V) \rightarrow H^1(U_2(k), V)$$

is injective.

*Proof.* Let  $r \in \mathbb{N}$  and denote the inclusion maps

$$\begin{aligned}\zeta_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(\mathbb{F}_{p^r}), \\ \iota_r &: SL_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(k), \\ \iota'_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow U_2(k).\end{aligned}$$

By Lemma 4.3 we get the following commutative diagram,

$$(4.7) \quad \begin{array}{ccc} H^1(SL_2(k), V) & \xrightarrow{H^1(\zeta)} & H^1(U_2(k), V) \\ \downarrow H^1(\iota_r) & & \downarrow H^1(\iota'_r) \\ H^1(SL_2(\mathbb{F}_{p^r}), V) & \xrightarrow{H^1(\zeta_r)} & H^1(U_2(\mathbb{F}_{p^r}), V). \end{array}$$

It is elementary to show that  $U(\mathbb{F}_{p^r})$  is a Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_{p^r})$ , so by Lemma 4.4,  $H^1(\zeta_r)$  is injective for all  $r \in \mathbb{N}$ .

Let  $\sigma \in Z^1(SL_2(k), V)$  such that  $\sigma \notin B^1(SL_2(k), V)$ , that is,

$$(4.8) \quad \sigma \neq \chi_v^{SL_2(k)},$$

for any  $v \in V$ . For each  $x \in SL_2(\mathbb{F}_{p^r})$  define the morphism  $f_x : V \rightarrow V$  by

$$f_x(v) = \sigma(x) - \chi_v^{SL_2(k)}(x).$$

Since  $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$  we have  $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$ . Consider the sequence  $\{C_r\}_{r \in \mathbb{N}}$  defined by

$$C_r = \{v \in V \mid \forall x \in SL_2(\mathbb{F}_{p^r}), f_x(v) = 0\}.$$

Then

$$\begin{aligned} \bigcap_{r \in \mathbb{N}} C_{r!} &= \{v \in V \mid \forall x \in SL_2(k), f_x(v) = 0\} \\ &= \emptyset \quad (\text{Equation 4.8}). \end{aligned}$$

Each  $C_r$  is closed, and the inclusion  $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$  induces the reverse inclusion for the subsequence  $C_{r!} \supset C_{(r+1)!}$ . Then the Noetherian property for  $V$  requires that the subsequence  $\{C_{r!}\}_{r \in \mathbb{N}}$  becomes constant, and since  $\bigcap_{r \in \mathbb{N}} C_{r!} = \emptyset$ , the subsequence  $\{C_{r!}\}_{r \in \mathbb{N}}$  is eventually empty. That is, there exists  $s \in \mathbb{N}$  such that

$$Z^1(\iota_s)(\sigma) \neq \chi_v^{SL_2(\mathbb{F}_{p^s})},$$

for any  $v \in V$ . We have shown that if  $\sigma \in Z^1(SL_2(k), V)$  such that  $Z^1(\iota_{r!})(\sigma) \in B^1(SL_2(\mathbb{F}_{p^{r!}}), V)$  for all  $r \in \mathbb{N}$ , then  $\sigma \in B^1(SL_2(k), V)$ .

So, let  $\sigma \in Z^1(SL_2(k), V)$  such that  $\psi(\sigma) \in \text{Ker}(H^1(\zeta))$ . Then, consulting the commutative diagram in Equation 4.7,

$$\begin{aligned}
& \psi(\sigma) \in \text{Ker}(H^1(\iota'_r) \circ H^1(\zeta)), \forall r \in \mathbb{N} \\
& \Rightarrow \psi(\sigma) \in \text{Ker}(H^1(\zeta_r) \circ H^1(\iota_r)), \forall r \in \mathbb{N} \\
& \Rightarrow H^1(\iota_r)(\psi(\sigma)) \in \text{Ker}(H^1(\zeta_r)), \forall r \in \mathbb{N} \\
& \Rightarrow H^1(\iota_r)(\psi(\sigma)) \text{ is trivial}, \forall r \in \mathbb{N} \\
& \Rightarrow Z^1(\iota_r)(\sigma) \in B^1(SL_2(\mathbb{F}_{p^r}), V), \forall r \in \mathbb{N} \\
& \Rightarrow \sigma \in B^1(SL_2(k), V) \\
& \Rightarrow \psi(\sigma) \in H^1(SL_2(k), V) \text{ is trivial.}
\end{aligned}$$

This shows  $H^1(\zeta)$  is injective. □

**Definition 4.9.** Let  $\rho \in \text{Hom}(K, P)$ . We associate with  $\rho$  the map  $\rho^L \in \text{Hom}(K, L)$  defined by  $\rho^L = \rho|_L$ , and the 1-cocycle  $\sigma_\rho \in Z^1(K, V)$  defined by  $\sigma_\rho(x) = \rho(x)\rho^L(x^{-1})$ .

**Definition 4.10.** Let  $\omega \in \text{Hom}(K, L)$ . We denote by  $Z^1(K, V)_\omega$  the set of 1-cocycles from  $K$  to  $V$  where  $K$  acts on  $V$  via  $\omega$ ; that is,  $x \cdot v = \omega(x) \cdot v$ . Likewise, denote by  $H^1(K, V)_\omega$  the 1-cohomology obtained from  $Z^1(K, V)_\omega$ .

Define  $\text{Hom}(K, P)_\omega = \{\rho \in \text{Hom}(K, P) \mid \rho^L = \omega\}$ . More generally, if  $R \subset \text{Hom}(K, P)$  define  $R_\omega = \{\rho \in R \mid \rho^L = \omega\}$ .

**Lemma 4.11.** Let  $\omega \in \text{Hom}(K, L)$ . The map  $z : \text{Hom}(K, P)_\omega \rightarrow Z^1(K, V)_\omega$  defined by

$$z(\rho)(x) = \rho(x)\omega(x^{-1}),$$

for all  $\rho \in \text{Hom}(K, P)_\omega$  and all  $x \in K$ , is a bijection. Furthermore,  $z$  descends to a bijection  $h : \text{Hom}(K, P)_\omega/V \rightarrow H^1(K, V)_\omega$  that makes the following diagram commute:

$$\begin{array}{ccc}
\text{Hom}(K, P)_\omega & \longrightarrow & Z^1(K, V)_\omega \\
\downarrow & & \downarrow \\
\text{Hom}(K, P)_\omega/V & \longrightarrow & H^1(K, V)_\omega.
\end{array}$$

Since  $C_L(\omega(K))$  normalizes  $V$ ,  $C_L(\omega(K))$  acts on  $\text{Hom}(K, P)_\omega/V$ . Notice that

$(\text{Hom}(K, P)_\omega/V)/C_L(\omega(K))$ , is canonically isomorphic to  $\text{Hom}(K, P)_\omega/V C_L(\omega(K))$ . Let  $\zeta = \text{id}_K$  and  $\xi_c : V \rightarrow V$  be defined  $\xi_c(v) = cvc^{-1}$  for all  $v \in V$ . We have an action of  $C_L(\omega(K))$  on  $H^1(K, V)_\omega$  defined by

$$c \cdot \bar{\sigma} = H^1(\zeta, \xi_c)(\bar{\sigma}),$$

for  $c \in C_L(\omega(K))$ ,  $\bar{\sigma} \in H^1(K, V)_\omega$ .

It is easy to check that the map  $h : \text{Hom}(K, P)_\omega/V \rightarrow H^1(K, V)_\omega$  in Lemma 4.11 is equivariant, so there exists a unique map  $\tilde{h} : \text{Hom}(K, P)_\omega/V C_L(\omega) \rightarrow H^1(K, V)_\omega/C_L(\omega)$  such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}(K, P)_\omega/V & \longrightarrow & H^1(K, V)_\omega \\
\downarrow & & \downarrow \\
\mathrm{Hom}(K, P)_\omega/VC_L(\omega) & \longrightarrow & H^1(K, V)_\omega/C_L(\omega).
\end{array}$$

**Definition 4.12.** Let  $K' < K$ , let  $\zeta$  the inclusion of  $K'$  in  $K$ , and let  $\xi = \mathrm{id}_V$ . Then the map  $H^1(\zeta) : H^1(K, V)_\omega \rightarrow H^1(K', V)_{\omega \circ \zeta}$  descends to give a unique map

$$H^1(K, V)_\omega/C_L(\omega) \rightarrow H^1(K', V)_{\omega \circ \zeta}/C_L(\omega \circ \zeta),$$

such that the following diagram commutes:

$$\begin{array}{ccc}
H^1(K, V)_\omega & \longrightarrow & H^1(K', V)_{\omega \circ \zeta} \\
\downarrow & & \downarrow \\
H^1(K, V)_\omega/C_L(\omega) & \longrightarrow & H^1(K', V)_{\omega \circ \zeta}/C_L(\omega \circ \zeta).
\end{array}$$

**Definition 4.13.** We define  $\mathrm{Hom}(K, P)^L = \{\rho^L \mid \rho \in \mathrm{Hom}(K, P)\}$ . More generally, when  $R \subset \mathrm{Hom}(K, P)$  we define  $R^L = \{\rho^L \mid \rho \in R\}$ .

**Lemma 4.14.** Let  $R \subset \mathrm{Hom}(K, P)$ . Suppose  $R = P \cdot \rho$  for some  $\rho \in R$ . Then  $R^L = L \cdot \rho^L$ . More generally, if  $R = P \cdot R$  then  $R^L = L \cdot R^L$ .

**Lemma 4.15.** Let  $R \subset \mathrm{Hom}(K, P)$  and suppose that  $R = P \cdot R$ . Then for all  $\omega \in \mathrm{Hom}(K, L)$ , all  $\rho \in R_\omega$ ,  $R_\omega \cap P \cdot \rho = (VC_L(\omega)) \cdot \rho$ .

**Theorem 4.16.** Let  $\Phi$  be the composition of the canonical projections  $\mathrm{Hom}(K, P)_\omega \rightarrow \mathrm{Hom}(K, P)_\omega/V$  and  $\mathrm{Hom}(K, P)_\omega/V \rightarrow \mathrm{Hom}(K, P)_\omega/VC_L(\omega)$ . Let  $R \subset \mathrm{Hom}(K, P)$  and suppose  $P \cdot R = R$ . Then  $R$  is a finite union of  $P$ -conjugacy classes if and only if

- (i)  $R^L$  is a finite union of  $L$ -conjugacy classes, and
- (ii) for each  $\omega \in \mathrm{Hom}(K, L)$ ,  $(\tilde{h} \circ \Phi)(R_\omega)$  is finite in  $H^1(K, V)_\omega/C_L(\omega)$ .

*Remark.* By Lemma 4.14, if  $R$  is a finite union of  $P$ -conjugacy classes then  $R^L$  is a finite union of  $L$ -conjugacy classes. Furthermore, conditions (i) and (ii) are equivalent to

- (i')  $R_\omega = \emptyset$  for all but finitely many  $L$ -conjugacy classes of  $\omega \in \mathrm{Hom}(K, L)$ , and
- (ii') for each  $\omega \in \mathrm{Hom}(\Gamma, L)$ ,  $R_\omega$  is a finite union of  $VC_L(\omega)$ -conjugacy classes,

respectively. We obtain (ii)  $\Leftrightarrow$  (ii') by appealing to the bijection  $\tilde{h}$ , while (i)  $\Leftrightarrow$  (i') is self-evident.

**Definition 4.17.** Let  $K' < K$  and let  $\zeta$  be the inclusion of  $K'$  in  $K$ . Define the maps

$$\begin{aligned}
\mathcal{Z}(\zeta) : \mathrm{Hom}(K, P)_\omega &\rightarrow \mathrm{Hom}(K', P)_{\omega \circ \zeta}, \\
\mathcal{H}(\zeta) : \mathrm{Hom}(K, P)_\omega/V &\rightarrow \mathrm{Hom}(K', P)_{\omega \circ \zeta}/V, \\
\tilde{\mathcal{H}}(\zeta) : \mathrm{Hom}(K, P)_\omega/VC_L(\omega) &\rightarrow \mathrm{Hom}(K', P)_{\omega \circ \zeta}/VC_L(\omega \circ \zeta),
\end{aligned}$$



by

$$\begin{aligned}\mathcal{Z}(\zeta)(\rho) &= \rho \circ \zeta, \\ \mathcal{H}(\zeta)(\phi(\rho)) &= \phi'(\rho \circ \zeta), \\ \tilde{\mathcal{H}}(\zeta)(\Phi(\rho)) &= \Phi'(\rho \circ \zeta).\end{aligned}$$

**Proposition 4.18.** *The following diagram commutes:*

$$\begin{array}{ccccc}\mathrm{Hom}(K, P)_\omega & \xrightarrow{z} & Z^1(K, V)_\omega & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ \mathrm{Hom}(K, P)_\omega/V & \xrightarrow{h} & H^1(K, P)_\omega & & \\ \downarrow & \searrow \mathcal{Z}(\zeta) & \downarrow & \searrow Z^1(\zeta) & \\ \mathrm{Hom}(K, P)_\omega/VC_L(\omega) & \xrightarrow{\tilde{h}} & H^1(K, P)_\omega/C_L(\omega) & & \\ \downarrow & \searrow \mathcal{H}(\zeta) & \downarrow & \searrow H^1(\zeta) & \\ & & \mathrm{Hom}(K', P)_{\omega \circ \zeta} & \xrightarrow{z'} & Z^1(K', V)_{\omega \circ \zeta} \\ & \searrow \tilde{\mathcal{H}}(\zeta) & \downarrow & \searrow \tilde{H}^1(\zeta) & \downarrow \\ & & \mathrm{Hom}(K', P)_{\omega \circ \zeta}/V & \xrightarrow{h'} & H^1(K', V)_{\omega \circ \zeta} \\ & & \downarrow & & \downarrow \\ & & \mathrm{Hom}(K', P)_{\omega \circ \zeta}/VC_L(\omega \circ \zeta) & \xrightarrow{\tilde{h}'} & H^1(K', V)_{\omega \circ \zeta}/C_L(\omega \circ \zeta)\end{array}$$

**Theorem 4.19.** *Let  $K' < K$ , let  $\zeta$  be the inclusion of  $K'$  in  $K$ , and let  $\xi = \mathrm{id}_V$ . Let  $R \subset \mathrm{Hom}(K, P)$  such that  $R = P \cdot R$ , and let  $S = \mathcal{Z}(\zeta)(R)$ . Suppose*

- (i)  $R^L$  is a finite union of  $L$ -conjugacy classes,
- (ii) for all  $\omega \in \mathrm{Hom}(K, L)$  such that  $R_\omega \neq \emptyset$ , the map

$$\tilde{H}^1(\zeta) : H^1(K, V)_\omega/C_L(\omega) \rightarrow H^1(K', V)_{\omega \circ \zeta}/C_L(\omega \circ \zeta),$$

*has finite fibres, and*

- (iii)  $S$  is a finite union of  $P$ -conjugacy classes.

*Then  $R$  is a finite union of  $P$ -conjugacy classes.*

*Remark.* Since  $R = P \cdot R$ , then by Lemma 4.14  $R^L$  is already a union of  $L$ -conjugacy classes. The point of (i) is that the union is finite.

*Proof.* Since  $P \cdot R = R$ , it follows that  $P \cdot S = S$ . By definition  $S \subset \mathrm{Hom}(K', V)$  and by (iii)  $S$  is a finite union of  $P$ -conjugacy classes. Therefore by Theorem 4.16

- (iv)  $S^L$  is a finite union of  $L$ -conjugacy classes, and

(v) for each  $\omega \in \text{Hom}(K', L)$ ,  $(\tilde{h}' \circ \Phi')(S_\omega)$  is finite in  $H^1(K', V)_{\omega \circ \zeta} / C_L(\omega \circ \zeta)$ .

Let  $\omega \in \text{Hom}(K, L)$ . Clearly  $(\tilde{h} \circ \Phi)(R_\omega)$  is finite if  $R_\omega = \emptyset$ , so suppose  $R_\omega \neq \emptyset$ . We have  $\mathcal{Z}(\zeta)(R_\omega) \subset S_{\omega \circ \zeta}$ , and by the commutative diagram in Definition 4.17,

$$\tilde{h}' \circ \Phi' \circ \mathcal{Z}(\zeta) = \widetilde{H^1}(\zeta) \circ \tilde{h} \circ \Phi.$$

Therefore

$$\begin{aligned} \left( \widetilde{H^1}(\zeta) \circ \tilde{h} \circ \Phi \right) (R_\omega) &= \left( \tilde{h}' \circ \Phi' \circ \mathcal{Z}(\zeta) \right) (R_\omega) \\ &= (\tilde{h}' \circ \Phi') (\mathcal{Z}(\zeta)(R_\omega)) \\ &\subset (\tilde{h}' \circ \Phi')(S_{\omega \circ \zeta}). \end{aligned}$$

Then  $(\tilde{h}' \circ \Phi')(S_{\omega \circ \zeta})$  is finite by (v), and

$$(\tilde{h} \circ \Phi)(R_\omega) \subset \widetilde{H^1}(\zeta)^{-1} \left( \left( \widetilde{H^1}(\zeta) \circ \tilde{h} \circ \Phi \right) (R_\omega) \right),$$

is finite by (ii).

Therefore  $(\tilde{h} \circ \Phi)(R_\omega)$  is finite in any case. Together with (i) we may apply Theorem 4.16, so  $R$  is a finite union of  $P$ -conjugacy classes.  $\square$

*Remark.* It is straightforward to show that  $H^1(K, V)_\omega / C_L(\omega)$  is finite if and only if  $H^1(K, V)_w / C_L(\omega)^\circ$  is finite.

[We've proved Theorem 1.2 for reductive  $G$  (it's enough to take  $G$  to be connected and semisimple, since  $H$  is connected and semisimple). I think the theorem follows for arbitrary non-reductive  $G$  by taking the result for the special case of reductive groups and applying it to the reductive group  $G/V$ , then using David Stewart's result that the restriction map  $H^1(B, V) \rightarrow H^1(U, V)$  is injective. Here  $V = R_u(G)$  and  $B$  is a Borel subgroup of  $G$  with unipotent radical  $U$ .]

[An application:

Theorem: Let  $G$  be a reductive group such that the simple components of  $G$  are all of type  $A_n$  or  $B_2$ . Then the answer to K's question is yes for any  $H$  with  $H^0$  semisimple.

Proof (sketch): We can approximate  $H$  with a sufficiently large finite subgroup, so assume without loss that  $H$  is finite. It's enough to study the simple components of  $G$  separately, so without loss we assume  $G$  is simple. If  $G$  is of type  $B_2$  and  $p \neq 2$  then we're done by Slodowy's paper, so without loss we assume that either  $G$  is of type  $A_n$ , or  $G$  is of type  $B_2$  and  $p = 2$ . Note that maximal parabolics of  $G$  have abelian unipotent radicals under this hypothesis.

We now use an inductive proof. If  $\rho$  is  $G$ -irreducible then we're done. Otherwise put  $\rho$  into a maximal parabolic  $P$ . The key point is that we can transfer the problem from  $P$  into a Levi of  $P$  by the usual arguments, because the restriction map  $H^1(H, V) \rightarrow H^1(U, V)$  is injective for any finite  $H$  if  $V = R_u(P)$  is abelian. (Here  $U$  is a Sylow  $p$ -subgroup of  $H$ .) By induction on  $\dim(G)$ , the result is true for  $L$ —for the simple components of  $L$  are all of type  $A_n$  or  $B_2$ —and we get what we want.

This result shows that the  $G_2$  counterexample to K's question for finite  $H$  is in a sense the smallest possible: there is no such counterexample for any other  $G$  of rank 1 or 2.]

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