ON A QUESTION OF KÜLSHAMMER FOR REDUCTIVE ALGEBRAIC GROUPS

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Abstract.

1. Introduction

Let G and H be linear algebraic groups over an algebraically closed field of characteristic $p > 0$. Pick a maximal unipotent subgroup U of H [this is unique up to conjugacy]. By a *representation* of H in G we mean a homomorphism of algebraic groups from H to G. The group G acts on the set of representations $\mathrm{Hom}(H, G)$ by $(g \cdot \rho)(h) = g\rho(h)g^{-1}$ for $h \in H$ and $g \in G$; we call the orbits *conjugacy classes.* We consider the following question.

Question 1.1. *Let $\sigma\colon U \to G$ be a representation. Are there only finitely many conjugacy classes of representation $\rho\colon H \to G$ such that $\rho|_U$ is G-conjugate to σ?*

Külshammer raised this question for finite H [2]; in this case the maximal unipotent subgroups are the Sylow p-subgroups. An example of Cram shows that the answer to Question 1.1 is no for $H = S_3$ and G a certain 3-dimensional non-connected group with G^0 unipotent [4, Appendix]. If $G = \mathrm{GL}_n$ and p does not divide $|H|$ then the answer to Question 1.1 is yes, by Maschke's Theorem. It follows from a well-known geometric argument of Richardson that if p does not divide $|H|$ then the answer to Question 1.1 is yes for any connected reductive G: one embeds G in some GL_n and studies the behavior of the induced map $\mathrm{Hom}(H, G) \to \mathrm{Hom}(H, \mathrm{GL}_n)$. In fact, standard representation-theoretic results imply that the answer is yes for any finite group when $G = \mathrm{GL}_n$, and Slodowy used Richardson's argument to show the answer is yes for arbitrary finite H and connected reductive G under mild hypotheses on char(p). [General case of finite H: there is now a counterexample in type G_2 [ref].]

In this paper we instead consider the case when H is connected and semisimple. We settle Question 1.1 as follows.

Theorem 1.2. *The answer to Question 1.1 is yes if H is connected and semisimple.*

Note that we allow G to be non-connected, but it is clear we can reduce immediately to the case when G is connected and $Z(G)^0$ is trivial. Also, it is clear that the answer to Question 1.1 is no for arbitrary connected reductive H (e.g., just take H to be a torus, $U = 1$ and G^0 non-unipotent).

An important tool in the work of [above] is nonabelian 1-cohomology. Let $\rho \in \mathrm{Hom}(H, G)$ and let P be a parabolic subgroup of G such that $\rho(H) \subseteq P$. Then H acts on $V := R_u(P)$ via $h \cdot u = \rho(h)u\rho(h)^{-1}$, and representations near ρ in an appropriate sense can be understood

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in terms of the nonabelian 1-cohomology $H^1(H, \rho, V)$. In Section [below] we discuss this cohomological approach and study its behavior with respect to restriction of representations from H to U. [Further discussion.]

[Based on D's PhD thesis. Related work by David Stewart: see below. He uses high-powered cohomology techniques; our methods are more elementary.]

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2. Preliminaries

[Some useful results from ''On unipotent algebraic groups and 1-cohomology'', David Stewart:

Cor. 3.4.3: Let $B = TU$ be connected solvable group with unipotent radical U and maximal torus T. Let Q be a unipotent group on which B acts. Then the restriction map $H^1(B,Q) \to H^1(U,Q)$ is injective.

Thm 3.5.2: Let G be connected reductive and let Q be a unipotent group on which G acts. Then for any parabolic subgroup P of G, the restriction map $H^1(G,Q) \to H^1(P,Q)$ is an isomorphism of pointed sets.

Corollary 3.6.2. Let H be a closed, connected, reductive subgroup of G contained in a parabolic $P = LR_u(P)$ of G and let P_1 be any parabolic subgroup of H. Then H is G-conjugate to a subgroup of L if and only if P_1 is as well.]

Definition 2.1. Let X, Y be (algebraic) groups. Then we denote by $\mathrm{Hom}(X, Y)$ the set of (algebraic) group homomorphisms from X to Y.

Lemma 2.2. *Let $R \subset \mathrm{Hom}(K, P)$. Then R is contained in a finite union of G-conjugacy classes if and only if it is contained in a finite union of P-conjugacy classes.*

Proof. Let $\rho_1, \rho_2 \in R$ such that ρ_1 and ρ_2 lie in the same G-conjugacy class of R. Then there exists $g \in G$ such that

$$g\rho_1(x)g^{-1} = \rho_2(x),$$

for all $x \in K$.

Let $Q = gPg^{-1}$, hence $\rho_2(K) \subset P \cap Q$. Let T be a maximal torus of G contained in $P \cap Q$. Since T and gTg^{-1} are maximal tori of Q they must be Q-conjugate, so there exists $q \in Q$ such that

$$qTq^{-1} = gTg^{-1}.$$

Then there exists $r \in P$ such that $q = grg^{-1}$, so

$$\begin{aligned} grg^{-1}Tgr^{-1}g^{-1} &= gTg^{-1} \\ \Rightarrow rg^{-1}Tgr^{-1} &= T. \end{aligned}$$

Therefore $gr^{-1} \in N_G(T)$.

Fix a finite set $N \subset N_G(T)$ of coset representatives for the Weyl group $W = N_G(T)/T$ and let $n \in N, t \in T$ such that

$$gr^{-1} = nt.$$

Let $q' = r^{-1}t^{-1}$ so $q' \in P$. Then

$$\begin{aligned}\rho_1(x) &= g^{-1}\rho_2(x)g \\ &= (q'n^{-1})\rho_2(x)nq'^{-1},\end{aligned}$$

for all $x \in K$. Hence $\rho_1 \in P\cdot(n^{-1}\cdot\rho_2)$. This shows that a G-conjugacy class of R is contained in a union of at most $|N| = |W|$ P-conjugacy classes.

Therefore, if R is contained in a finite union of G-conjugacy classes then it is contained in a finite union of P-conjugacy classes.

The converse is trivial. □

We assume G is a possibly non-connected algebraic group over k and H is a linear algebraic group over k. We fix a maximal unipotent subgroup U of H.

Suppose H is connected, let B be a Borel subgroup of H, let X be an affine variety and let $f\colon H \to X$ be a morphism such that $f(hb) = f(h)$ for all $h \in H$ and all $b \in B$. Then f gives rise to a morphism $\overline{f}$ from the projective variety H/B to X. Since H/B is connected and X is affine, $\overline{f}$ must be constant, so f is constant. In particular, if V is an affine H-variety, $v \in V$ and the stabiliser H_v contains B then $H_v = H$.

Lemma 2.3. *Suppose G is connected and reductive. Let H_1, H_2 be connected reductive subgroups of G. Suppose B is a common Borel subgroup of both H_1 and H_2. Then $H_1 = H_2$.*

Proof. The quotient variety G/H_1 is affine since H_1 is reductive [ref], and H_2 acts on G/H_1 by left multiplication. The stabiliser in H_2 of the coset H_1 contains B, so it must equal the whole of H_2. Hence $H_2 \subseteq H_1$. The reverse inequality follows similarly, so $H_1 = H_2$. □

Lemma 2.4. *Let B be a Borel subgroup of H. Let $\rho_1, \rho_2 \in \mathrm{Hom}(H, G)$ such that $\rho_1|_B = \rho_2|_B$. Then $\rho_1 = \rho_2$.*

Proof. Define $f\colon H \to G$ by $f(h) = \rho_1(h)\rho_2(h)^{-1}$. For any $h \in H$, $b \in B$, $f(hb) = \rho_1(hb)\rho_2(hb)^{-1} = \rho_1(h)\rho_1(b)\rho_2(b)^{-1}\rho_2(h)^{-1} = \rho_1(h)\rho_2(h)^{-1} = f(h)$. So f gives rise to a morphism $\overline{f}$ from the connected projective variety H/B to G given by $\overline{f}(hB) = f(h)$. Then $\overline{f}$ is constant, so f is constant with value $f(1) = 1$, and the result follows. □

3. Proof of Theorem 1.2 for G reductive

In this section we assume G and H are connected and semisimple.

Lemma 3.1. *Let B be the Borel subgroup of H that contains U, and let T be a maximal torus of B. Let $\rho_1, \rho_2\colon H \to G$ be representations such that $\rho_1(B) = \rho_2(B)$ and $\rho_1(T) = \rho_2(T)$. Set $U' = \rho_1(U) = \rho_2(U)$. Suppose that for all $\alpha \in \Phi_T(B)$, $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ and there exists $\phi_\alpha \in \mathrm{Aut}(\rho_1(U_\alpha))$ such that for all $u \in U_\alpha$, $\rho_2(u) = \phi_\alpha(\rho_1(u))$. Then there exists $t' \in T$ such that $t' \cdot \rho_2 = \rho_1$.*

Proof. Since B is a Borel subgroup of H, $\rho_i(B)$ is a Borel subgroup of $\rho_i(H)$ for $i = 1, 2$. Lemma 2.3 implies that $\rho_1(H) = \rho_2(H)$. Hence we can assume that ρ_1 and ρ_2 are surjective. Set $T' = \rho_1(T) = \rho_2(T)$.

Pick a base $\{\alpha_1', \ldots, \alpha_m'\}$ for $\Phi_{T'}(B')$. Since ρ_1 and ρ_2 map a given root group of U with respect to T to a root group of U' with respect to T', there exist $\alpha_1, \ldots, \alpha_m \in \Phi_T(B)$ such that $\rho_1(U_{\alpha_i}) = \rho_2(U_{\alpha_i}) = U_{\alpha_i'}$ and $\alpha_i'(\rho_1(t)) = \alpha_i'(\rho_2(t))$ for all i. [This is not quite true as stated: some of the U_{α_i} may be mapped by ρ_1 and ρ_2 to 1; we're only interested in the case when this doesn't happen. A bit more argument is needed here.] As H is semisimple, this implies that $\rho_1|_T = \rho_2|_T$.

By hypothesis, each ϕ_{α_i} is an automorphism of $U_{\alpha'_i}$, so there exist $b_1, \ldots, b_m \in k^*$ such that $\phi_{\alpha_i}(\epsilon_{\alpha'_i}(x)) = \epsilon_{\alpha'_i}(b_i x)$ for each i and for all $x \in k$; so there exist $a_1, \ldots, a_m \in k^*$ such that $\rho_2(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(a_i x))$ for each i and for all $x \in k$ [check!].

The weights $\alpha'_1, \ldots, \alpha'_m$ are linearly independent as H is semisimple, so there exists $t' \in T'$ such that $\alpha_i(t') = a_i^{-1}$ for $1 \leq i \leq m$. We then have $(t' \cdot \rho_2)(\epsilon_{\alpha_i}(x)) = \rho_1(\epsilon_{\alpha_i}(x))$ for each i and for all $x \in k$. It follows that $\rho_1|_{\widetilde{U}} = (t' \cdot \rho_2)|_{\widetilde{U}}$, where $\widetilde{U}$ is the subgroup of U generated by the U_{α_i}. But $\rho_1(\widetilde{U}) = \rho_2(\widetilde{U}) = (t' \cdot \rho_2)(\widetilde{U})$ since the $U_{\alpha'_i}$ generate U' [ref: Humphreys?], and $\ker(\rho_1|_U) = \ker(\rho_2|_U) = \ker((t' \cdot \rho_2)|_U)$, so $\rho_1|_U = (t' \cdot \rho_2)|_U$.

To complete the proof, it is enough by Lemma 2.4 to show that $\rho_1|_B = (t' \cdot \rho_2)|_B$. But $(t' \cdot \rho_2)|_T = \rho_2|_T = \rho_1|_T$ from the discussion above, so we are done. $\square$

[Note: In the situation of the above lemma (and with ρ_1, ρ_2 assumed surjective), suppose we don't assume that $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for every α. Then ρ_1 and ρ_2 need no longer be G-conjugate. I suspect, however, that in this case they are $\mathrm{Aut}(G)$-conjugate. If this is the case then we should get a reasonable bound in Theorem 1.2 involving $|\mathrm{Out}(G)|$ and the size of the Weyl group of T' in $N_G(U')$ (cf. below).]

Theorem 3.2. *The answer to Külshammer's question is yes for H and G connected and semisimple.*

Proof. Let $\sigma\colon U \to G$ and let $U' = \sigma(U)$. Let B be the Borel subgroup of H that contains U and fix a maximal torus T of B. Fix a maximal torus T' of $N_G(U')$. Let $C = \{\rho \in \mathrm{Hom}(H, G) \mid \rho|_U = \sigma\}$. If $\rho \in C$ then $\rho(T)$ normalizes $\rho(U) = U'$, so $\rho(T)$ is a torus of $N_G(U')$. By conjugacy of maximal tori of $N_G(U')$, there exists $g \in N_G(U')$ such that $(g \cdot \rho)(T) \subseteq T'$. If $h \in N_G(U')$ and $(h \cdot \rho)(T) \subseteq T'$ then there exists $n \in N_{N_G(U')}(T')$ such that $(h \cdot \rho)(T) = ((ng) \cdot \rho)(T)$. The group $N_{N_G(U')}(T')/C_{N_G(U')}(T')$ is finite, so there is a finite set of subtori $S_1, \ldots, S_r$ of T' such that $\rho(T)$ is $N_G(U')$-conjugate to one of $S_1, \ldots, S_r$.

Define a relation $\equiv$ on C by $\rho_1 \equiv \rho_2$ if there exist $i \in \{1, \ldots, r\}$ and $g_1, g_2 \in N_G(U')$ such that $(g_1 \cdot \rho_1)(T) = (g_2 \cdot \rho_1)(T) = S_i$ and $\rho_1(U_\alpha) = \rho_2(U_\alpha)$ for all $\alpha \in \Phi_T(U)$. It is clear that $\equiv$ is an equivalence relation. Define $C_i = \{\rho \in G \mid (g \cdot \rho)(T) = S_i \text{ for some } g \in N_G(U')\}$. If $(g \cdot \rho)(T) = S_i$ then $(g \cdot \rho)$ must map any root group U_α of U with respect to T to a root group of U' with respect to S_i, so there are only finitely many possibilities for $(g \cdot \rho)(U_\alpha)$. It follows that $\equiv$ has only finitely many equivalence classes.

To complete the proof it is enough, therefore, to show that if $\rho_1 \equiv \rho_2$ then ρ_1 and ρ_2 are G-conjugate. So suppose g_1 and g_2 are [as in the defn of $\equiv$]. Set $U_{\alpha'} = \rho_1(U_\alpha) = \rho_2(\alpha)$ if this is nontrivial [recall that $\rho_1(U_\alpha)$ might be just 1]. Then the map $\phi\colon G \to G$ given by $\phi(g) = g_2 g_1^{-1} g g_1 g_2^{-1}$ gives rise to an automorphism of U'_α for all $\alpha' \in \Phi_{S_i}(U')$. It follows by Lemma 3.1 that $g_1 \cdot \rho_1$ and $g_2 \cdot \rho_2$ are T-conjugate, so we are done. $\square$

[Consequence: Look at **images** of representations and prove an analogous finiteness result, for connected reductive H as well?]

4. 1-cohomology

[Describe Daniel's approach using 1-cohomology.]

We recall Richardson's nonabelian 1-cohomology ([3]) below.

Definition 4.1. We call a morphism $\sigma : K \to V$ a *1-cocycle* if it satisfies

$$\sigma(xy) = \sigma(x)(x \cdot \sigma(y)),$$

for all $x, y \in K$. Denote by $Z^1(K, V)$ the collection of all 1-cocycles from K to V, and $H^1(K, V)$ the set of equivalence classes of $Z^1(K, V)$ under the relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \exists v \in V, \forall x \in K, \sigma_1(x) = v\sigma_2(x)(x \cdot v^{-1}).$$

We call $H^1(K, V)$ the *1-cohomology.*

Lemma 4.2. *Suppose K is linearly reductive and V is unipotent. Then $H^1(K, V)$ is trivial.* [3, Lemma 6.2.6].

Lemma 4.3 (Map of 1-Cohomologies)**.** *Let K', V' be algebraic groups such that K' acts on V' by group automorphisms. Let $\zeta : K' \to K$ be a homomorphism and let $\xi : V \to V'$ be a K'-equivariant homomorphism. Then the function $Z^1(\zeta, \xi)$ defined by*

$$Z^1(\zeta, \xi)(\sigma) = \xi \circ \sigma \circ \zeta,$$

maps $Z^1(K, V)$ to $Z^1(K', V')$. Furthermore, $Z^1(\zeta, \xi)$ descends to give a unique map

$$H^1(\zeta, \xi) : H^1(K, V) \to H^1(K', V'),$$

that makes the following diagram commute:

$$\begin{tikzcd} Z^1(K, V) \arrow[r, "Z^1(\zeta,\xi)"] \arrow[d] & Z^1(K', V') \arrow[d] \\ H^1(K, V) \arrow[r, "H^1(\zeta,\xi)"] & H^1(K', V'). \end{tikzcd}$$

For our applications, it is often the case that $V = V'$ and $\xi = \mathrm{id}_V$. Then we just write $Z^1(\zeta)$ and $H^1(\zeta)$.

Lemma 4.4. *Let V be a vector space over k,* $\mathrm{char}(k) = p$. *Let Γ be a finite group that acts linearly on V, and let Γ_p be a* Sylow p-subgroup *of Γ. Let ζ be the inclusion of Γ_p in Γ. Then the map*

$$H^1(\zeta) : H^1(\Gamma, V) \to H^1(\Gamma_p, V)$$

is injective. [1, III.10.4 Prop.].

Example 4.5. TODO: replace SL_2 with H.

Let $k = \overline{\mathbb{F}_p} = \bigcup_{r \in \mathbb{N}} \mathbb{F}_{p^r}$. Let V a be vector space over k on which $SL_2(k)$ acts linearly, and let $U_2(k)$ be the subgroup of $SL_2(k)$ consisting of upper unitriangular matrices. Let ζ be the inclusion of $U_2(k)$ in $SL_2(k)$.

Then the map

$$H^1(\zeta) : H^1(SL_2(k), V) \to H^1(U_2(k), V) \tag{4.6}$$

is injective.

Proof. Let $r \in \mathbb{N}$ and denote the inclusion maps

$$\begin{aligned} \zeta_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(\mathbb{F}_{p^r}), \\ \iota_r &: SL_2(\mathbb{F}_{p^r}) \hookrightarrow SL_2(k), \\ \iota'_r &: U_2(\mathbb{F}_{p^r}) \hookrightarrow U_2(k). \end{aligned}$$

By Lemma 4.3 we get the following commutative diagram,

$$\begin{tikzcd} H^1(SL_2(k), V) \arrow[r, "H^1(\zeta)"] \arrow[d, "H^1(\iota_r)"'] & H^1(U_2(k), V) \arrow[d, "H^1(\iota'_r)"] \\ H^1(SL_2(\mathbb{F}_{p^r}), V) \arrow[r, "H^1(\zeta_r)"] & H^1(U_2(\mathbb{F}_{p^r}), V). \end{tikzcd} \tag{4.7}$$

It is elementary to show that $U(\mathbb{F}_{p^r})$ is a Sylow p-subgroup of $SL_2(\mathbb{F}_{p^r})$, so by Lemma 4.4, $H^1(\zeta_r)$ is injective for all $r \in \mathbb{N}$.

Let $\sigma \in Z^1(SL_2(k), V)$ such that $\sigma \notin B^1(SL_2(k), V)$, that is,

$$\sigma \neq \chi_v^{SL_2(k)}, \tag{4.8}$$

for any $v \in V$. For each $x \in SL_2(\mathbb{F}_{p^r})$ define the morphism $f_x : V \to V$ by

$$f_x(v) = \sigma(x) - \chi_v^{SL_2(k)}(x).$$

Since $\mathbb{F}_{p^{r!}} \subset \mathbb{F}_{p^{(r+1)!}}$ we have $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$. Consider the sequence $\{C_r\}_{r\in\mathbb{N}}$ defined by

$$C_r = \{v \in V \,|\, \forall x \in SL_2(\mathbb{F}_{p^r}),\ f_x(v) = 0\}.$$

Then

$$\begin{aligned} \bigcap_{r\in\mathbb{N}} C_{r!} &= \{v \in V \,|\, \forall x \in SL_2(k),\ f_x(v) = 0\} \\ &= \emptyset \quad \text{(Equation 4.8)}. \end{aligned}$$

Each C_r is closed, and the inclusion $SL_2(\mathbb{F}_{p^{r!}}) \subset SL_2(\mathbb{F}_{p^{(r+1)!}})$ induces the reverse inclusion for the subsequence $C_{r!} \supset C_{(r+1)!}$. Then the Noetherian property for V requires that the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ becomes constant, and since $\cap_{r\in\mathbb{N}} C_{r!} = \emptyset$, the subsequence $\{C_{r!}\}_{r\in\mathbb{N}}$ is eventually empty. That is, there exists $s \in \mathbb{N}$ such that

$$Z^1(\iota_s)(\sigma) \neq \chi_v^{SL_2(\mathbb{F}_{p^s})},$$

for any $v \in V$. We have shown that if $\sigma \in Z^1(SL_2(k), V)$ such that $Z^1(\iota_{r!})(\sigma) \in B^1(SL_2(\mathbb{F}_{p^{r!}}), V)$ for all $r \in \mathbb{N}$, then $\sigma \in B^1(SL_2(k), V)$.

So, let $\sigma \in Z^1(SL_2(k), V)$ such that $\psi(\sigma) \in \mathrm{Ker}\,(H^1(\zeta))$. Then, consulting the commutative diagram in Equation 4.7,

$$\begin{aligned}
&\psi(\sigma) \in \mathrm{Ker}\left(H^1(\iota_r') \circ H^1(\zeta)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &\psi(\sigma) \in \mathrm{Ker}\left(H^1(\zeta_r) \circ H^1(\iota_r)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \in \mathrm{Ker}\left(H^1(\zeta_r)\right), \forall r \in \mathbb{N}\\
\Rightarrow\, &H^1(\iota_r)(\psi(\sigma)) \text{ is trivial }, \forall r \in \mathbb{N}\\
\Rightarrow\, &Z^1(\iota_r)(\sigma) \in B^1(SL_2(\mathbb{F}_{p^r}), V), \forall r \in \mathbb{N}\\
\Rightarrow\, &\sigma \in B^1(SL_2(k), V)\\
\Rightarrow\, &\psi(\sigma) \in H^1(SL_2(k), V) \text{ is trivial.}
\end{aligned}$$

This shows $H^1(\zeta)$ is injective. □

Definition 4.9. Let $\rho \in \mathrm{Hom}(K, P)$. We associate with ρ the map $\rho^L \in \mathrm{Hom}(K, L)$ defined by $\rho^L = \rho|_L$, and the 1-cocycle $\sigma_\rho \in Z^1(K, V)$ defined by $\sigma_\rho(x) = \rho(x)\rho^L(x^{-1})$.

Definition 4.10. Let $\omega \in \mathrm{Hom}(K, L)$. We denote by $Z^1(K, V)_\omega$ the set of 1-cocycles from K to V where K acts on V via ω; that is, $x \cdot v = \omega(x) \cdot v$. Likewise, denote by $H^1(K, V)_\omega$ the 1-cohomology obtained from $Z^1(K, V)_\omega$.

Define $\mathrm{Hom}(K, P)_\omega = \{\rho \in \mathrm{Hom}(K, P) \,|\, \rho^L = \omega\}$. More generally, if $R \subset \mathrm{Hom}(K, P)$ define $R_\omega = \{\rho \in R \,|\, \rho^L = \omega\}$.

Lemma 4.11. *Let $\omega \in \mathrm{Hom}(K, L)$. The map $z : \mathrm{Hom}(K, P)_\omega \to Z^1(K, V)_\omega$ defined by*

$$z(\rho)(x) = \rho(x)\omega(x^{-1}),$$

for all $\rho \in \mathrm{Hom}(K, P)_\omega$ and all $x \in K$, is a bijection. Furthermore, z descends to a bijection $h : \mathrm{Hom}(K, P)_\omega/V \to H^1(K, V)_\omega$ that makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Hom}(K, P)_\omega \arrow[r, "z"] \arrow[d] & Z^1(K, V)_\omega \arrow[d] \\
\mathrm{Hom}(K, P)_\omega/V \arrow[r, "h"] & H^1(K, V)_\omega.
\end{tikzcd}$$

Since $C_L(\omega(K))$ normalizes V, $C_L(\omega(K))$ acts on $\mathrm{Hom}(K, P)_\omega/V$. Notice that $(\mathrm{Hom}(K, P)_\omega/V)/C_L(\omega(K))$, is canonically isomorphic to $\mathrm{Hom}(K, P)_\omega/VC_L(\omega(K))$. Let $\zeta = \mathrm{id}_K$ and $\xi_c : V \to V$ be defined $\xi_c(v) = cvc^{-1}$ for all $v \in V$. We have an action of $C_L(\omega(K))$ on $H^1(K, V)_\omega$ defined by

$$c \cdot \bar{\sigma} = H^1(\zeta, \xi_c)\,(\bar{\sigma}),$$

for $c \in C_L(\omega(K)), \bar{\sigma} \in H^1(K, V)_\omega$.

It is easy to check that the map $h : \mathrm{Hom}(K, P)_\omega/V \to H^1(K, v)_\omega$ in Lemma 4.11 is equivariant, so there exists a unique map $\tilde{h} : \mathrm{Hom}(K, P)_\omega/VC_L(\omega) \to H^1(K, V)_\omega/C_L(\omega)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] \arrow[d] & H^1(K,V)_\omega \arrow[d] \\
\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \arrow[r, "\tilde{h}"] & H^1(K,V)_\omega/C_L(\omega).
\end{tikzcd}$$

Definition 4.12. Let $K' < K$, let ζ the inclusion of K' in K, and let $\xi = \mathrm{id}_V$. Then the map $H^1(\zeta) : H^1(K,V)_\omega \to H^1(K',V)_{\omega\circ\zeta}$ descends to give a unique map

$$\tilde{H}^1(\zeta) : H^1(K,V)_\omega/C_L(\omega) \to H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta),$$

such that the following diagram commutes:

$$\begin{tikzcd}
H^1(K,V)_\omega \arrow[r, "H^1(\zeta)"] \arrow[d] & H^1(K',V)_{\omega\circ\zeta} \arrow[d] \\
H^1(K,V)_\omega/C_L(\omega) \arrow[r, "\tilde{H}^1(\zeta)"] & H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta).
\end{tikzcd}$$

Definition 4.13. We define $\mathrm{Hom}(K,P)^L = \{\rho^L \mid \rho \in \mathrm{Hom}(K,P)\}$. More generally, when $R \subset \mathrm{Hom}(K,P)$ we define $R^L = \{\rho^L \mid \rho \in R\}$.

Lemma 4.14. *Let $R \subset \mathrm{Hom}(K,P)$. Suppose $R = P \cdot \rho$ for some $\rho \in R$. Then $R^L = L \cdot \rho^L$. More generally, if $R = P \cdot R$ then $R^L = L \cdot R^L$.*

Lemma 4.15. *Let $R \subset \mathrm{Hom}(K,P)$ and suppose that $R = P \cdot R$. Then for all $\omega \in \mathrm{Hom}(K,L)$, all $\rho \in R_\omega$, $R_\omega \cap P \cdot \rho = (VC_L(\omega)) \cdot \rho$.*

Theorem 4.16. *Let π be the canonical projection $\mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K,P)_\omega/VC_L(\omega)$. Let $R \subset \mathrm{Hom}(K,P)$ and suppose $P \cdot R = R$. Then R is a finite union of P-conjugacy classes if and only if*

(i) *R^L is a finite union of L-conjugacy classes, and*
(ii) *for each $\omega \in \mathrm{Hom}(K,L)$, $(\tilde{h} \circ \pi)(R_\omega)$ is finite in $H^1(K,V)_\omega/C_L(\omega)$.*

Remark. By Lemma 4.14, if R is a finite union of P-conjugacy classes then R^L is a finite union of L-conjugacy classes. Furthermore, conditions (i) and (ii) are equivalent to

(i′) $R_\omega = \emptyset$ for all but finitely many L-conjugacy classes of $\omega \in \mathrm{Hom}(K,L)$, and
(ii′) for each $\omega \in \mathrm{Hom}(\Gamma,L)$, R_ω is a finite union of $VC_L(\omega)$-conjugacy classes,

respectively.

Definition 4.17. Let $K' < K$ and let ζ be the inclusion of K' in K. Define

$$\mathcal{Z}(\zeta) : \mathrm{Hom}(K,P)_\omega \to \mathrm{Hom}(K',P)_{\omega\circ\zeta},$$

by $\mathcal{Z}(\zeta)(\rho) = \rho \circ \zeta$, for all ρ in $\mathrm{Hom}(K,P)_\omega$. Furthermore, define

$$\mathcal{H}(\zeta) : \mathrm{Hom}(K,P)_\omega/V \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V,$$
$$\tilde{\mathcal{H}}(\zeta) : \mathrm{Hom}(K,P)_\omega/VC_L(\omega) \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta),$$

by composing the image of a representative in $\mathrm{Hom}(K,P)_\omega$ under $\mathcal{Z}(\zeta)$ with the respective canonical projections

$$\begin{aligned}
&\mathrm{Hom}(K',P)_{\omega\circ\zeta} \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V,\\
&\mathrm{Hom}(K',P)_{\omega\circ\zeta} \to \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta).
\end{aligned}$$

Proposition 4.18. *The following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Hom}(K,P)_\omega \arrow[r, "z"] \arrow[d] \arrow[dddr, "\mathcal{Z}(\zeta)"] & Z^1(K,V)_\omega \arrow[d] \arrow[dddr, "Z^1(\zeta)"] & \\
\mathrm{Hom}(K,P)_\omega/V \arrow[r, "h"] \arrow[d] \arrow[dddr, "\mathcal{H}(\zeta)"] & H^1(K,P)_\omega \arrow[d] \arrow[dddr, "H^1(\zeta)"] & \\
\mathrm{Hom}(K,P)_\omega/VC_L(\omega) \arrow[r, "\tilde{h}"] \arrow[dddr, "\tilde{\mathcal{H}}(\zeta)"] & H^1(K,P)_\omega/C_L(\omega) \arrow[dddr, "\tilde{H}^1(\zeta)"] & \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta} \arrow[r, "z'"] \arrow[d] & Z^1(K',V)_{\omega\circ\zeta} \arrow[d] \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta}/V \arrow[r, "h'"] \arrow[d] & H^1(K',V)_{\omega\circ\zeta} \arrow[d] \\
& \mathrm{Hom}(K',P)_{\omega\circ\zeta}/VC_L(\omega\circ\zeta) \arrow[r, "\tilde{h}'"] & H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta)
\end{tikzcd}$$

Theorem 4.19. *Let $K' < K$, let ζ be the inclusion of K' in K, and let $\xi = \mathrm{id}_V$. Let $R \subset \mathrm{Hom}(K,P)$ such that $R = P\cdot R$, and let $S = \mathcal{Z}(\zeta)(R)$. Suppose*

(i) *R^L is a finite union of L-conjugacy classes,*
(ii) *for all $\omega \in \mathrm{Hom}(K,L)$ such that $R_\omega \neq \emptyset$, the map*

$$\tilde{H}^1(\zeta) : H^1(K,V)_\omega/C_L(\omega) \to H^1(K',V)_{\omega\circ\zeta}/C_L(\omega\circ\zeta),$$

has finite fibres, and
(iii) *S is a finite union of P-conjugacy classes.*

Then R is a finite union of P-conjugacy classes.

Remark. Since $R = P\cdot R$, then by Lemma 4.14 R^L is already a union of L-conjugacy classes. The point of (i) is that the union is finite.

Proof. Since $P \cdot R = R$, it follows that $P \cdot S = S$. By definition $S \subset \mathrm{Hom}(K', V)$ and by (iii) S is a finite union of P-conjugacy classes. Define the canonical projections

$$\pi : \mathrm{Hom}(K, P)_\omega \to \mathrm{Hom}(K, P)_\omega / VC_L(\omega)$$
$$\pi' : \mathrm{Hom}(K', P)_\omega \to \mathrm{Hom}(K', P)_\omega / VC_L(\omega).$$

By Theorem 4.16

(iv) S^L is a finite union of L-conjugacy classes, and

(v) for each $\omega \in \mathrm{Hom}(K', L)$, $(\tilde{h'} \circ \pi')\,(S_\omega)$ is finite in $H^1(K', V)_{\omega\circ\zeta}/C_L(\omega \circ \zeta)$.

Let $\omega \in \mathrm{Hom}(K, L)$. Clearly $(\tilde{h} \circ \pi)\,(R_\omega)$ is finite if $R_\omega = \emptyset$, so suppose $R_\omega \neq \emptyset$. We have $\mathcal{Z}(\zeta)(R_\omega) \subset S_{\omega\circ\zeta}$, and by the commutative diagram in Definition 4.17,

$$\tilde{h'} \circ \pi' \circ \mathcal{Z}(\zeta) = \tilde{H}^1(\zeta) \circ \tilde{h} \circ \pi.$$

Therefore

$$\begin{aligned}
(\tilde{H}^1(\zeta) \circ \tilde{h} \circ \pi)(R_\omega) &= (\tilde{h'} \circ \pi' \circ \mathcal{Z}(\zeta))(R_\omega) \\
&= (\tilde{h'} \circ \pi')\,(\mathcal{Z}(\zeta)(R_\omega)) \\
&\subset (\tilde{h'} \circ \pi')(S_{\omega\circ\zeta}).
\end{aligned}$$

Then $(\tilde{h'} \circ \pi')(S_{\omega\circ\zeta})$ is finite by (v), and

$$(\tilde{h} \circ \pi)(R_\omega) \subset \tilde{H}^1(\zeta)^{-1}((\tilde{H}^1(\zeta) \circ \tilde{h} \circ \pi)(R_\omega)),$$

is finite by (ii).

Therefore $(\tilde{h} \circ \pi)(R_\omega)$ is finite in any case. Together with (i) we may apply Theorem 4.16, so R is a finite union of P-conjugacy classes. $\square$

Remark. It is straightforward to show that $H^1(K, V)_\omega / C_L(\omega)$ is finite if and only if $H^1(K, V)_w / C_L(\omega)^\circ$ is finite.

[We've proved Theorem 1.2 for reductive G (it's enough to take G to be connected and semisimple, since H is connected and semisimple). I think the theorem follows for arbitrary non-reductive G by taking the result for the special case of reductive groups and applying it to the reductive group G/V, then using David Stewart's result that the restriction map $H^1(B, V) \to H^1(U, V)$ is injective. Here $V = R_u(G)$ and B is a Borel subgroup of G with unipotent radical U.]

[An application:

Theorem 4.20. *Let G be a reductive group such that the simple components of G are all of type A_n or B_2. Then the answer to K's question is yes for any H with H^0 semisimple.*

Proof. (TODO) We can approximate H with a sufficiently large finite subgroup, so assume without loss that H is finite. It's enough to study the simple components of G separately, so without loss we assume G is simple. If G is of type B_2 and $p \neq 2$ then we're done by Slodowy's paper, so without loss we assume that either G is of type A_n, or G is of type B_2 and $p = 2$. Note that maximal parabolics of G have abelian unipotent radicals under this hypothesis.

We now use an inductive proof. If ρ is G-irreducible then we're done. Otherwise put ρ into a maximal parabolic P. The key point is that we can transfer the problem from P into a Levi of P by the usual arguments, because the restriction map $H^1(H,V) \to H^1(U,V)$ is injective for any finite H if $V = R_u(P)$ is abelian. (Here U is a Sylow p-subgroup of H.) By induction on $\dim(G)$, the result is true for L—for the simple components of L are all of type A_n or B_2—and we get what we want. $\square$

This result shows that the G_2 counterexample to K's question for finite H is in a sense the smallest possible: there is no such counterexample for any other G of rank 1 or 2.]

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