Develop and Test of an Improved 4-node Axisymmetric (ring) Element for Nearly Incompressible Materials

PART II
Formulation & Results

AFEM Term Project

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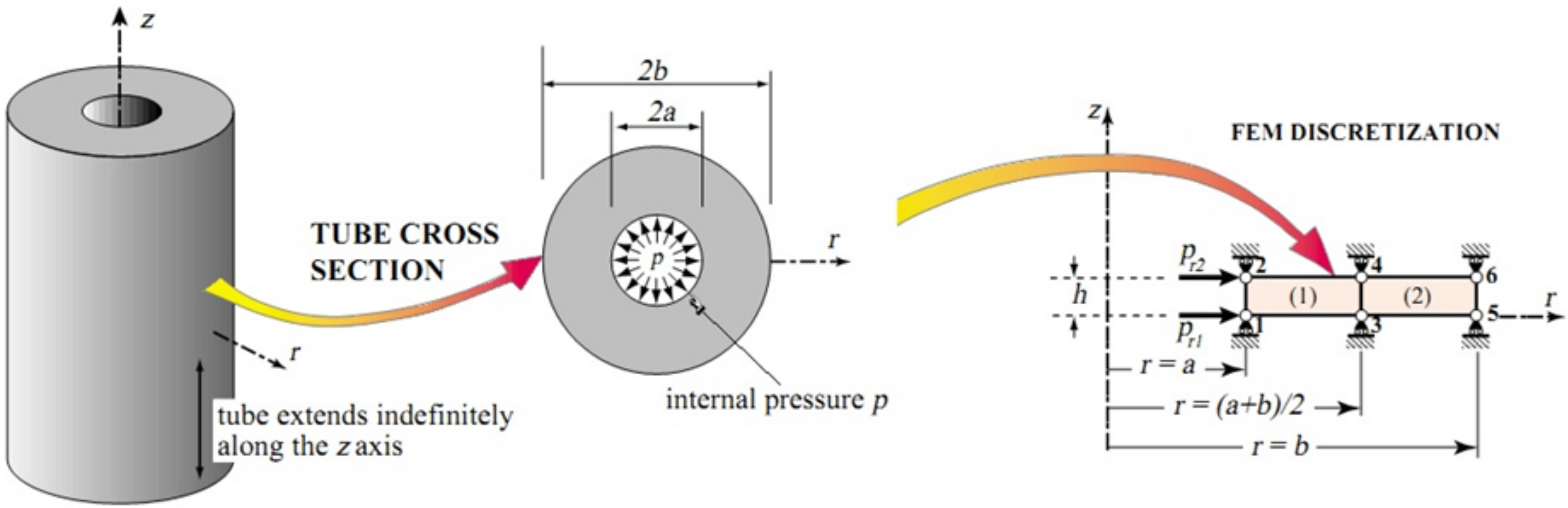
Contents

Problem Statement

✓Benchmark problem: a thick cylinder under internal pressure

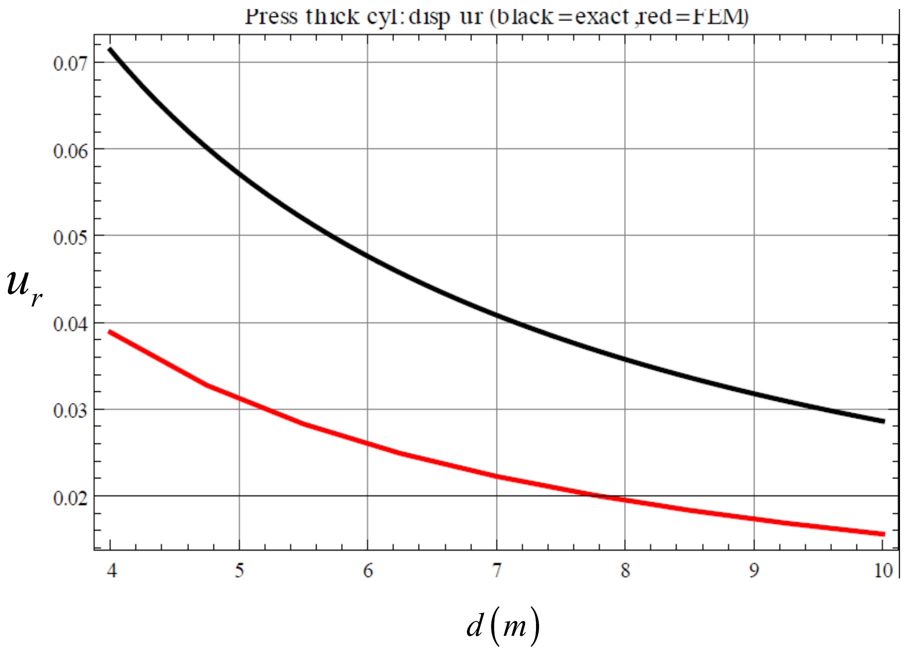


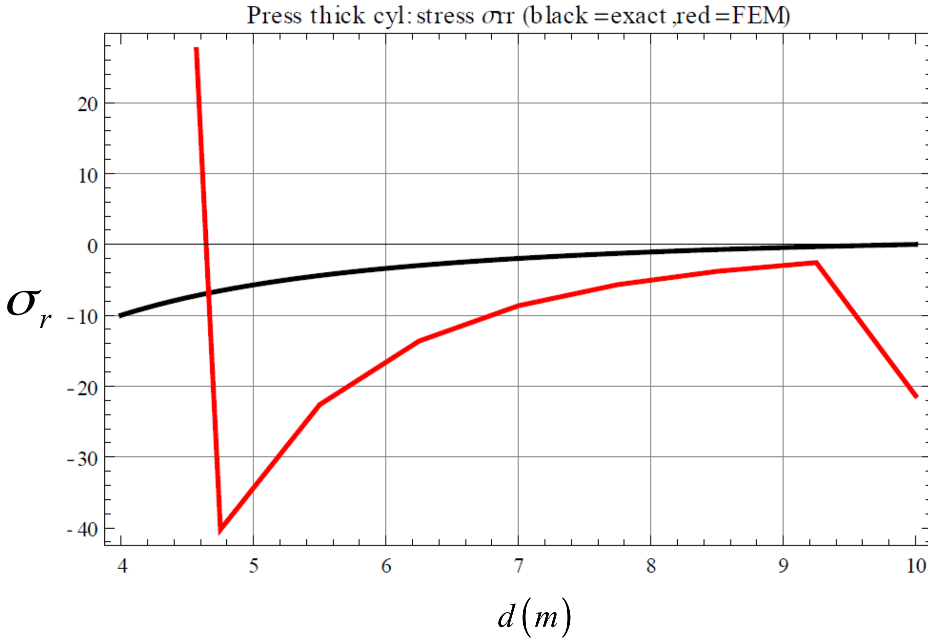
A thick cylinder under internal pressure dicretized by the 4-node quadrilateral axisymmetric elements

Motivation

✓The performance of standard 4-node isoparametric quadrilateral Element

$$\upsilon = 0.499$$





The accuracy of displacement and stress fields predicted by the standard 4-node quadrilateral for the near incompressible materials

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} C_{ijkl} \sigma_{ij} \sigma_{kl} dv - \int_s T_i u_i ds + \int_{s_T} \overline{T}_i u_i ds \right)$$

Assumed Stress Field

$$\{\sigma\} = [P]\{\beta\}$$

$$\{T\} = [R]\{\beta\}$$

Assumed Traction

Traction

$$\{T\} = [\sigma]\{n\}$$

$$\sigma_{ij,j} = 0$$

Balance of momentum

$$\{U\} = L\{u_i^e\}$$

Assumed Displacement Field

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl}\, dv - \int_s T_i u_i\, ds + \int_{s_T} \overline{T}_i u_i\, ds \right)$$

$$\int_s T_i u_i\, ds = \int_s \left(\{\beta\}^T [R]^T L \right) \{u_i^e\}\, ds$$

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} \left(\{\beta\}^T [H]\{\beta\}\right) dv \right) - \sum_n \left(\int_s \left(\{\beta\}^T [P]^T [B]\right)\{u_i^e\}\, ds + \int_{s_T} \overline{T}_i L \{u_i^e\}\, ds \right)$$

$$H = \int_{v_n} [P]^T [C][P]\, dV$$

$$G = \int_{v_n} [P]^T [B]\, dV$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation for the Axisymmetric Problem

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & 0 & \frac{\partial N_2^e}{\partial r} & 0 & \frac{\partial N_3^e}{\partial r} & 0 & \frac{\partial N_4^e}{\partial r} & 0 \\ 0 & \frac{\partial N_1^e}{\partial z} & 0 & \frac{\partial N_2^e}{\partial z} & 0 & \frac{\partial N_3^e}{\partial z} & 0 & \frac{\partial N_4^e}{\partial z} \\ \frac{N_1^e}{r} & 0 & \frac{N_2^e}{r} & 0 & \frac{N_3^e}{r} & 0 & \frac{N_4^e}{r} & 0 \\ \frac{\partial N_1^e}{\partial z} & \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_2^e}{\partial r} & \frac{\partial N_3^e}{\partial z} & \frac{\partial N_3^e}{\partial r} & \frac{\partial N_4^e}{\partial z} & \frac{\partial N_4^e}{\partial r} \end{bmatrix} \{u_i^e\} = [B]\{u_i^e\}$$

$$[P] = \begin{bmatrix} 1 & z & r & {}^{1}\!/_{r} & {}^{z}\!/_{r} & 0 & 0 \\ 0 & 0 & -3r & -{}^{1}\!/_{r} & -{}^{z}\!/_{r} & 1 & z \\ 1 & z & 2r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -{}^{1}\!/_{2r} \end{bmatrix}$$

$$H = 2\pi \int_{-1}^{1}\int_{-1}^{1} P^T C P \, r \det|J| \, d\xi d\eta$$

$$G = 2\pi \int_{-1}^{1}\int_{-1}^{1} P^T B \, r \det|J| \, d\xi d\eta$$

$$K = [G]^T [H]^{-T} [G]$$

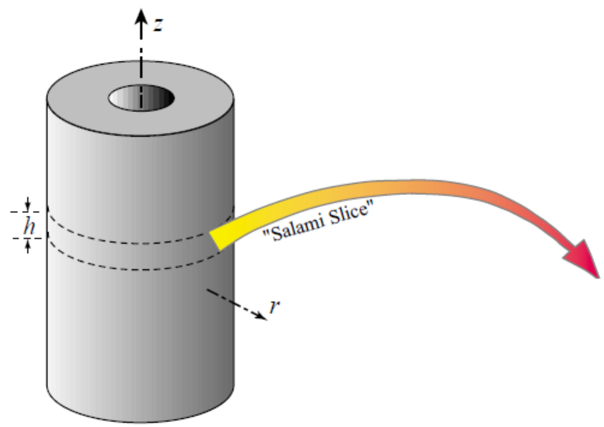
$$[\beta] = [H]^{-1} [G] \{u_i^e\}$$

$$\{\sigma\} = [P]\{\beta\}$$

The Benchmark Problems

Example No. 1

✓Thick wall cylinder with internal pressure



$$Input\ Parameters:$$

$$r_{\text{int}} = 4\,m$$

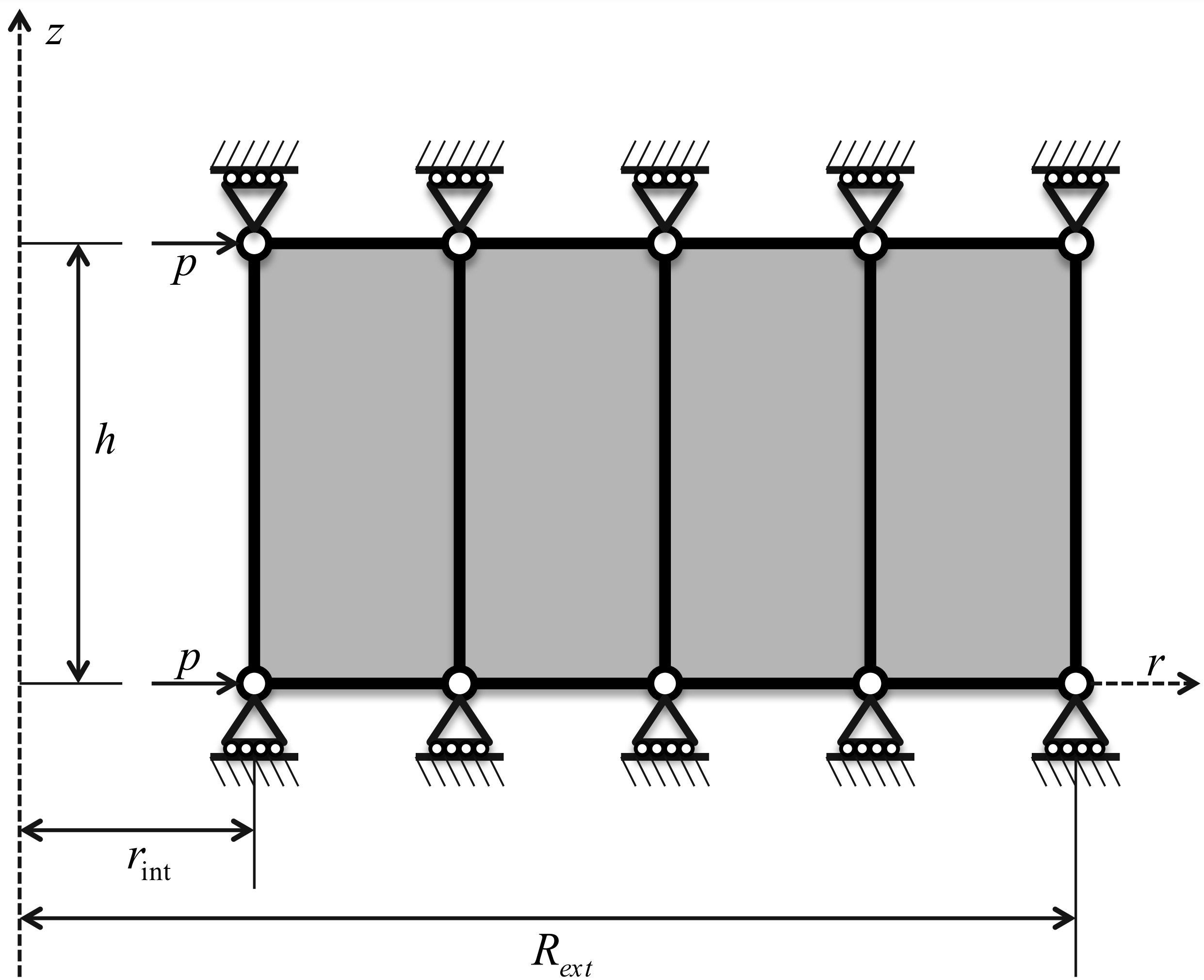
$$R_{ext} = 10\,m$$

$$h = 0.5\,m$$

$$E = 1.0e+05\ N/m^2$$

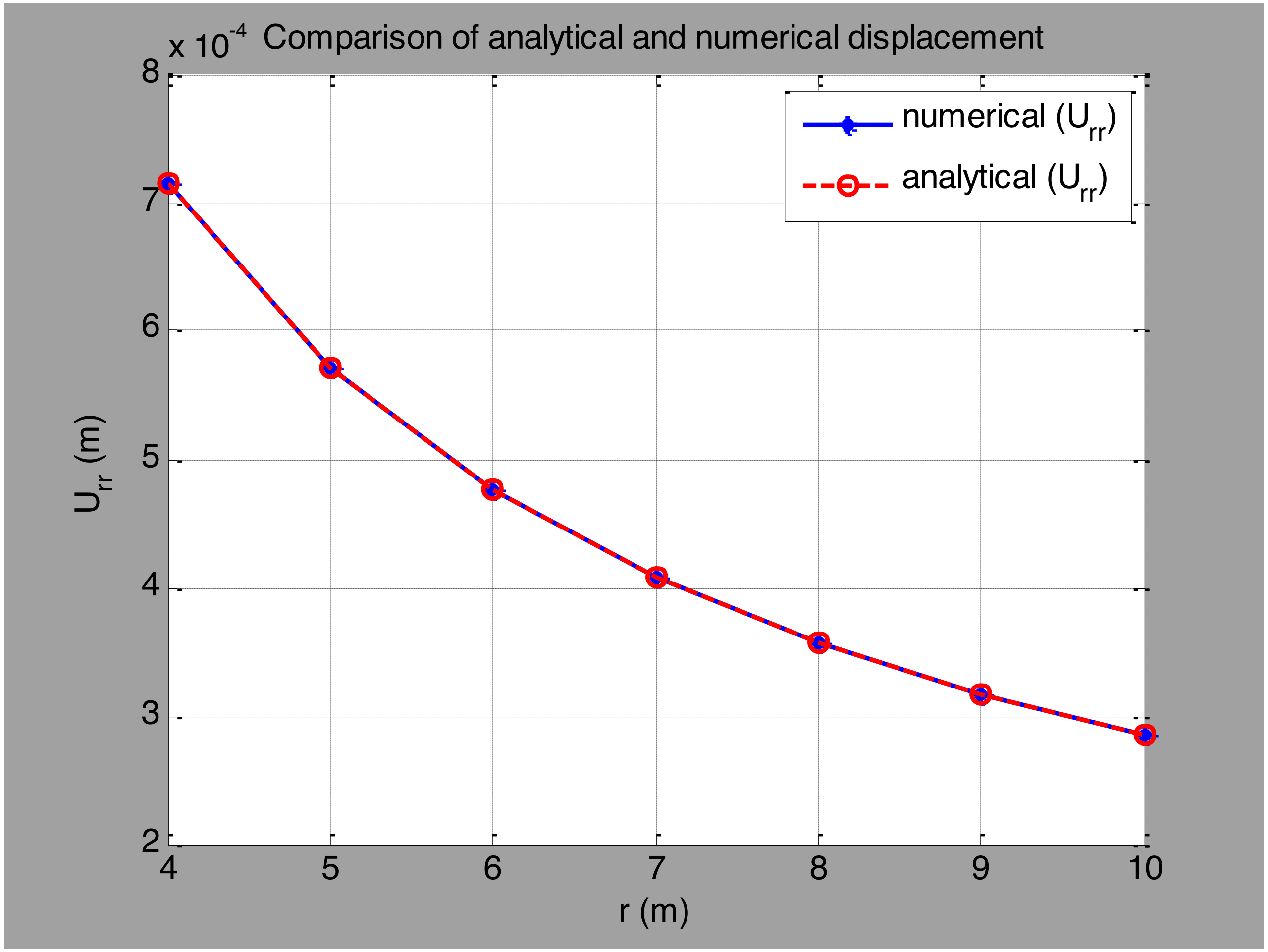
$$\nu = 0.4999999$$

$$p = 10\,N$$



Example No. 1

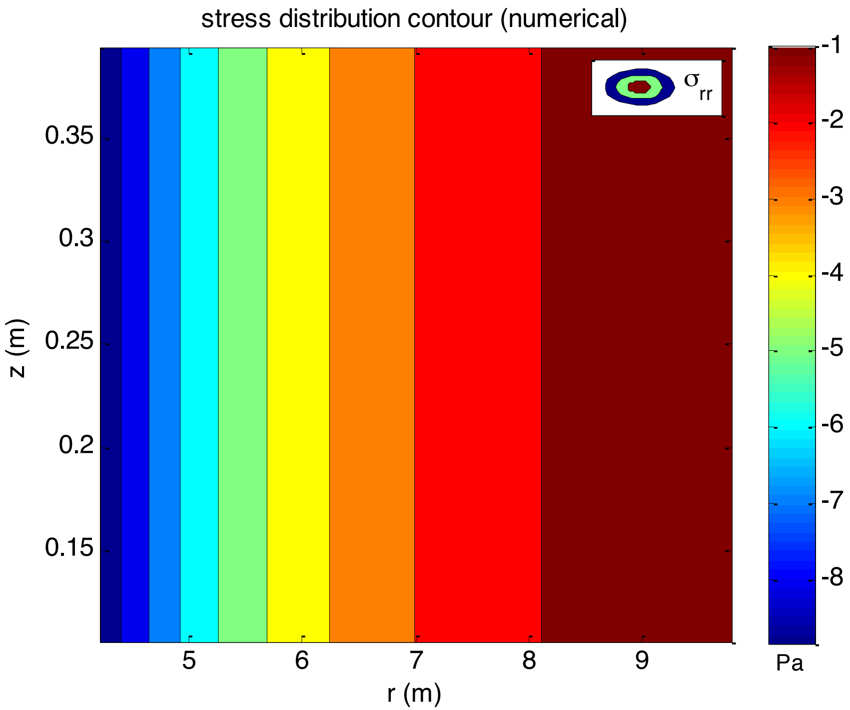
✓Thick wall cylinder with internal pressure

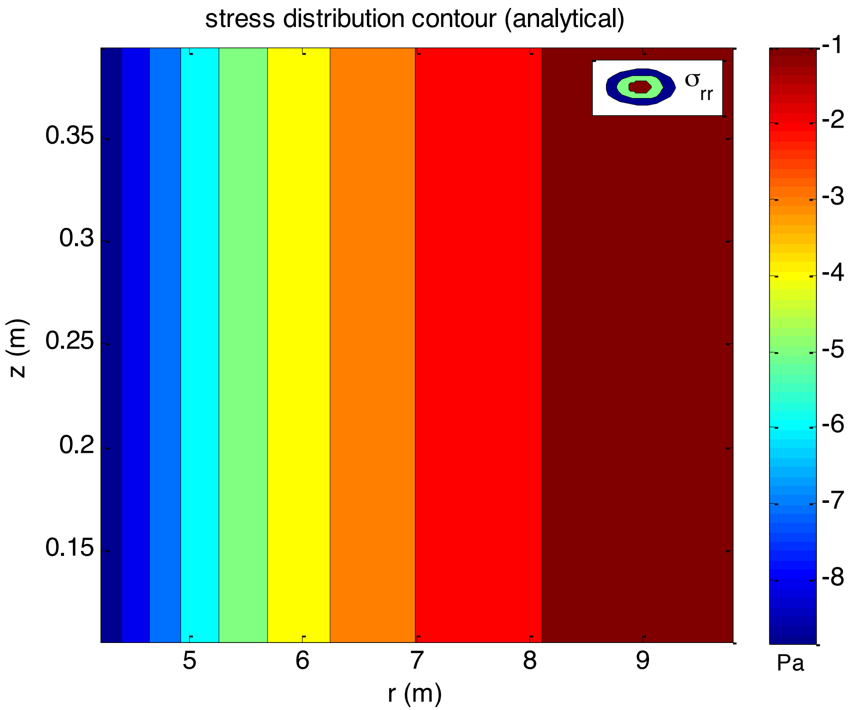


Comparison of analytical and numerical results - U_{rr}

Example No. 1

✓Thick wall cylinder with internal pressure

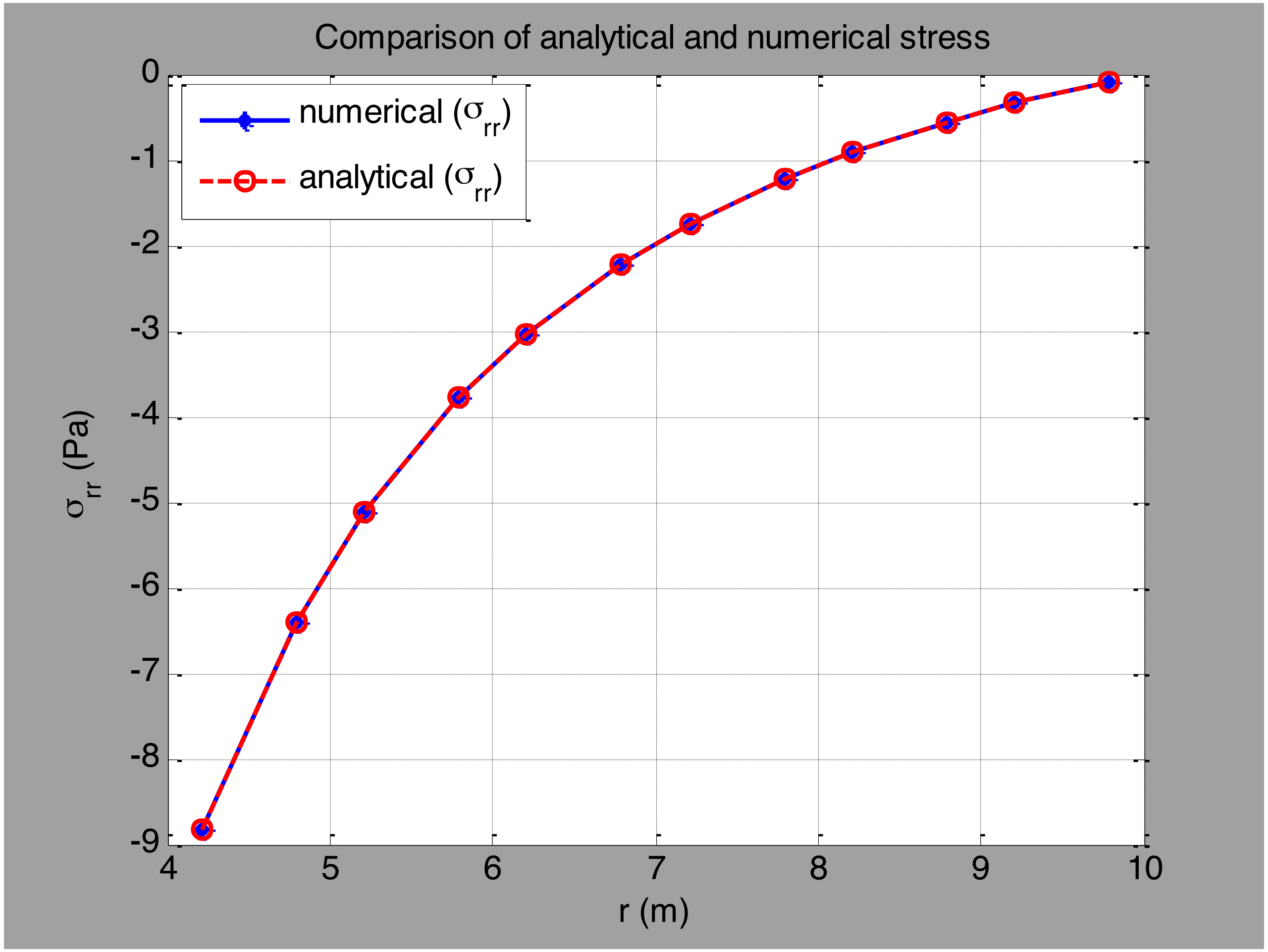




Radial stress distribution contour (σ_{rr}) – numerical & analytical results

Example No. 1

✓Thick wall cylinder with internal pressure



Comparison of analytical and numerical results (σ_{rr})

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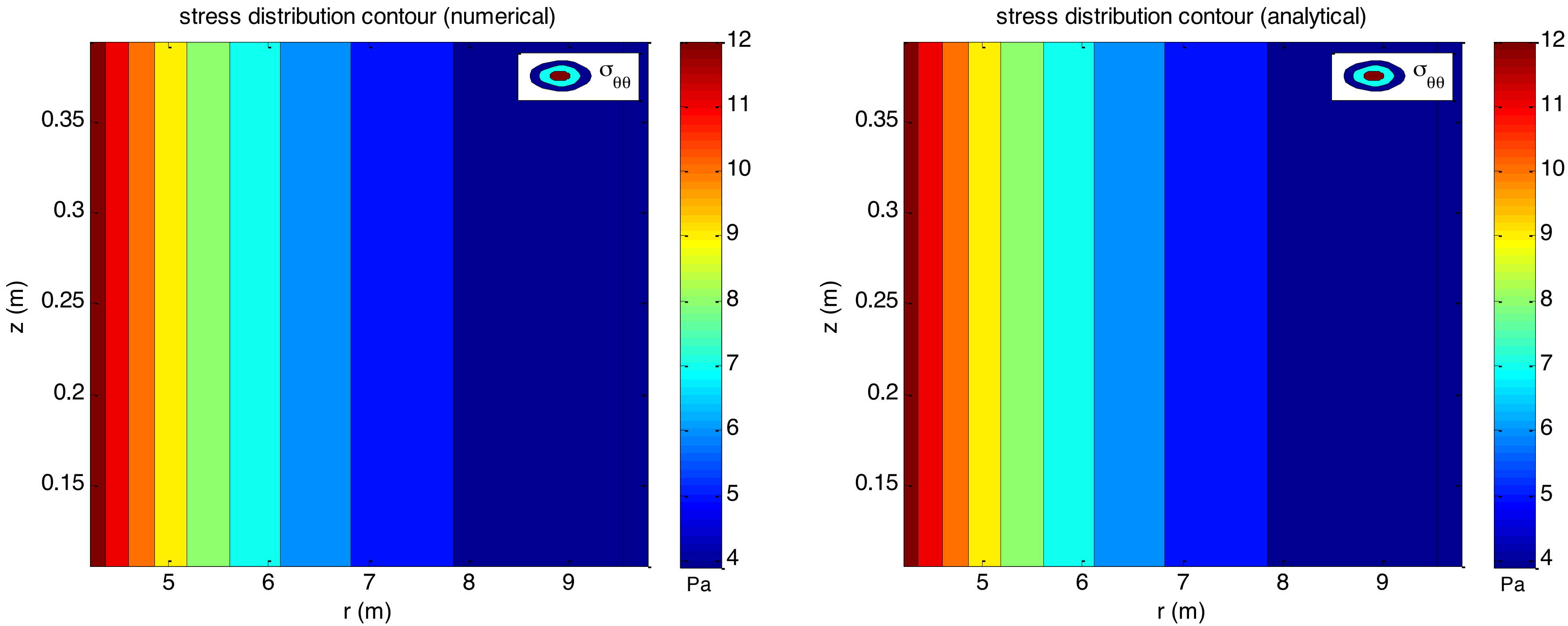
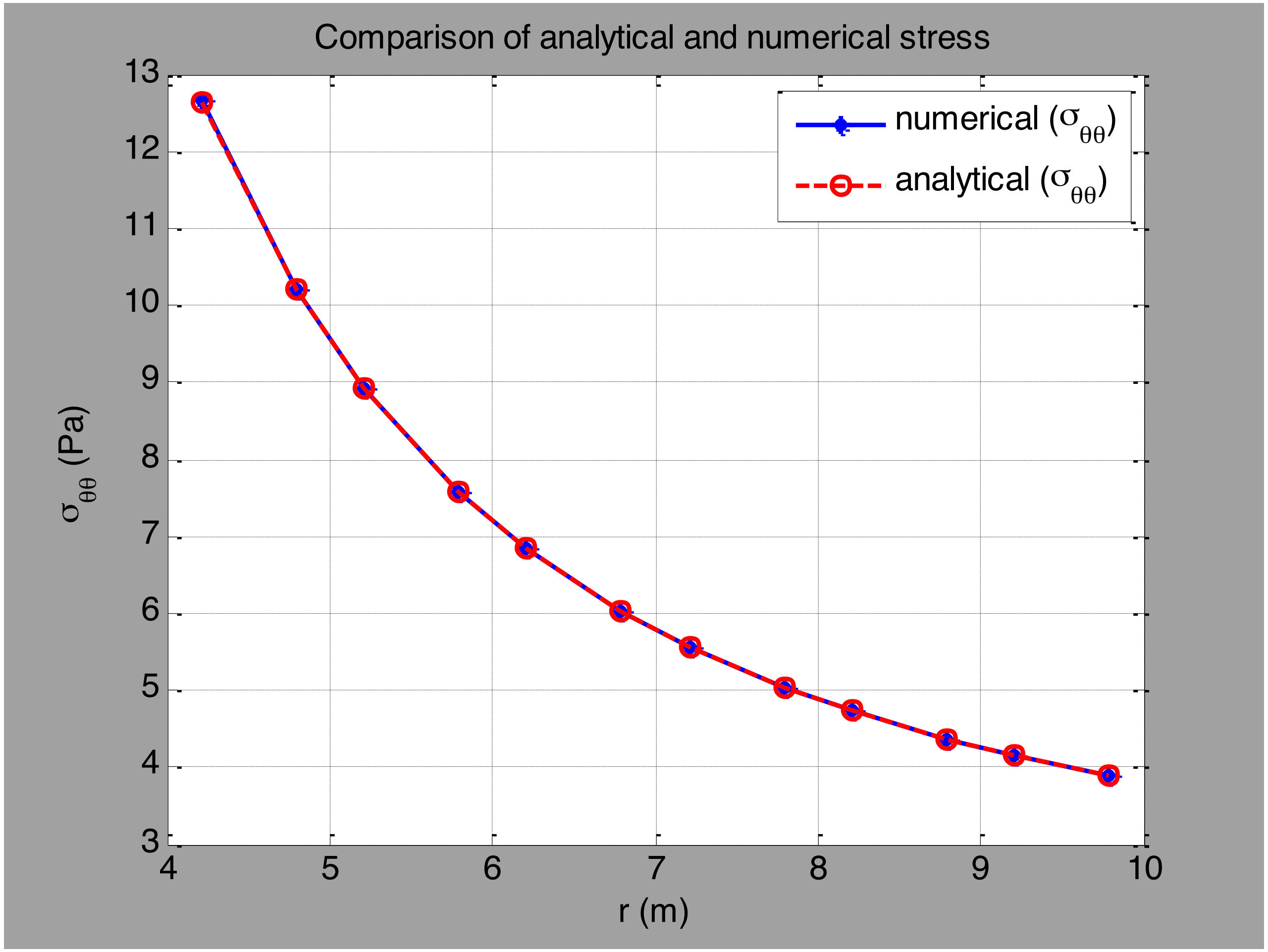


Circumferential stress distribution contour ($\sigma_{\theta\theta}$) – numerical & analytical results

Example No. 1

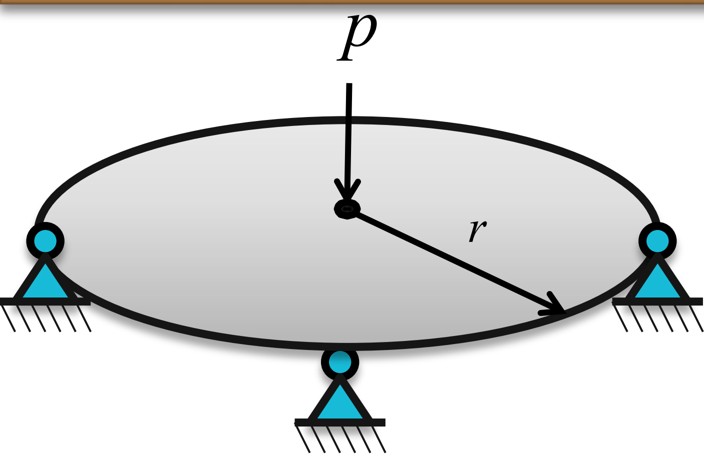
✓Thick wall cylinder with internal pressure

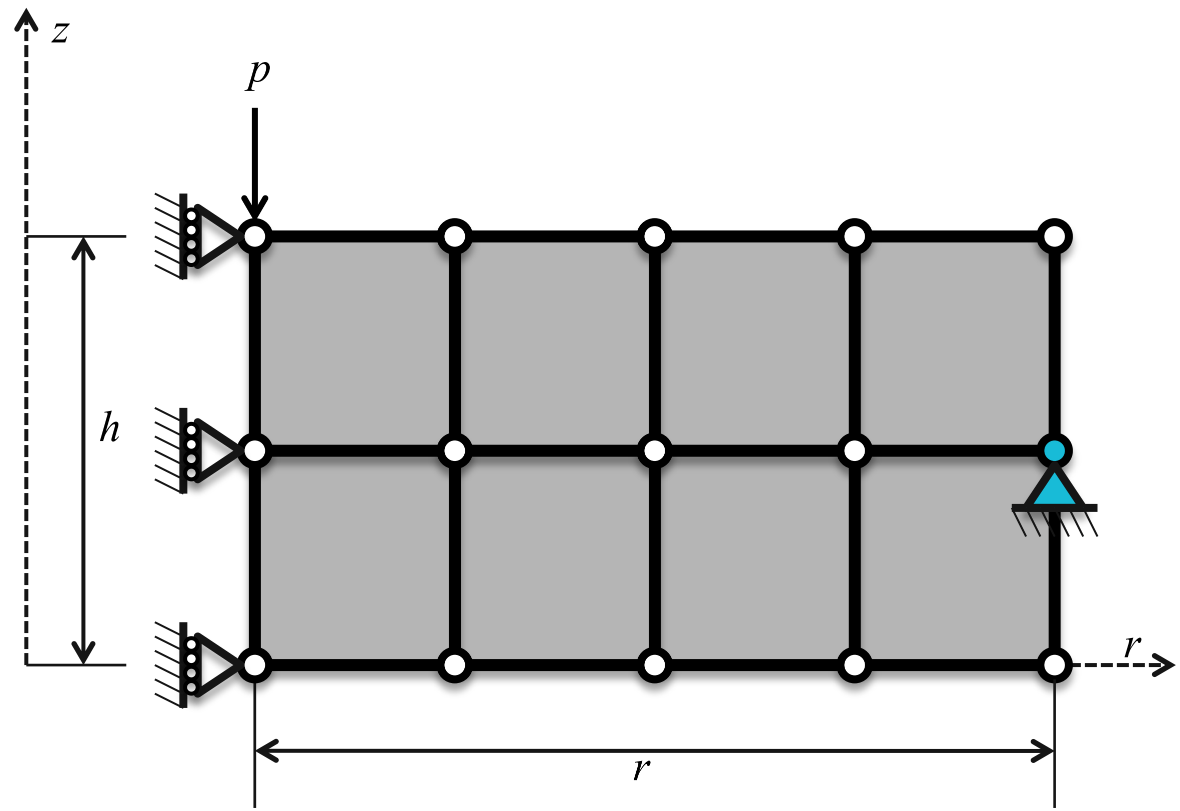


Comparison of analytical and numerical results ($\sigma_{\theta\theta}$)

Example No. 2

✓ **Simply supported circular plate with point load**





Input Parameters :

$r = 10\,m$

$h = 0.5\,m$

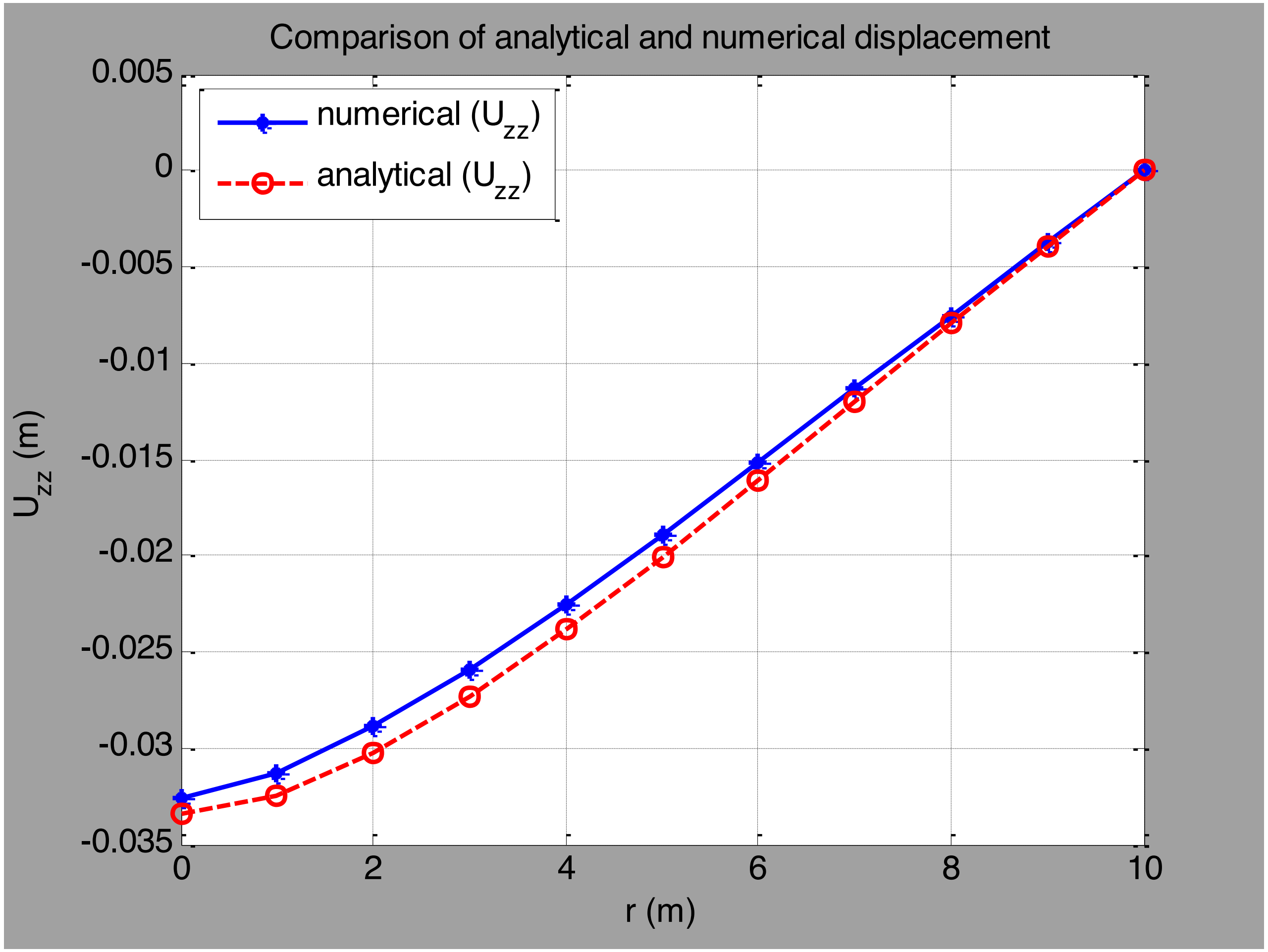
$E = 1.0e+05\ N/m^2$

$\nu = 0.4999999$

$p = 10\,N$

Example No. 2

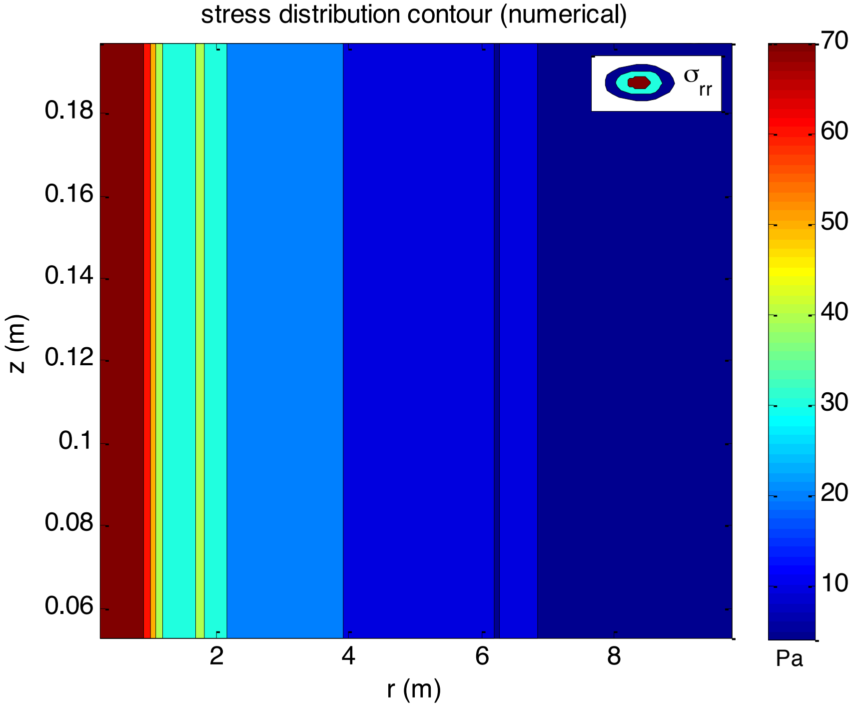
✓Simply supported circular plate with point load

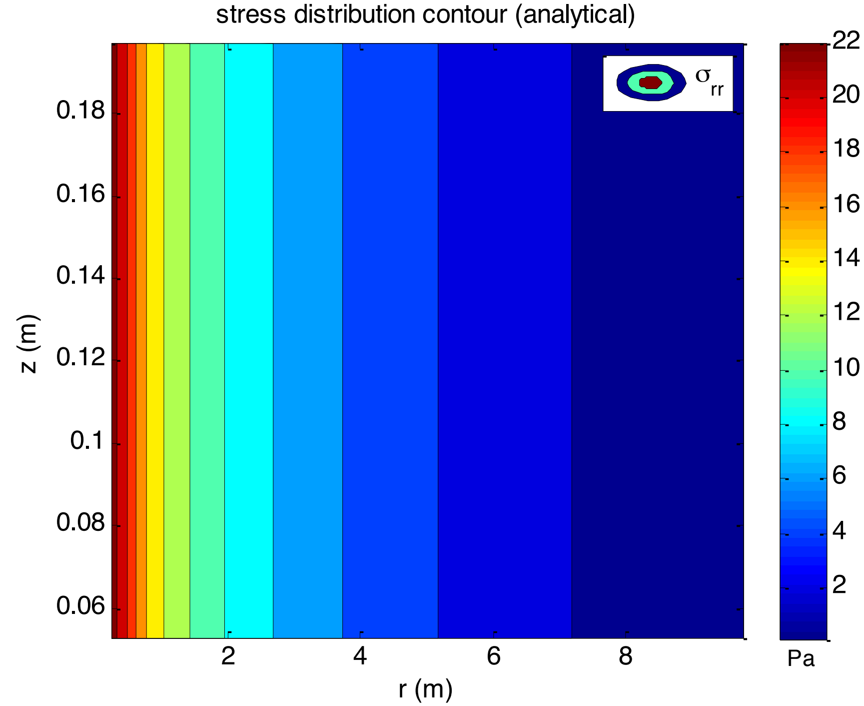


Comparison of analytical and numerical results - U_{zz}

Example No. 2

✓Simply supported circular plate with point load

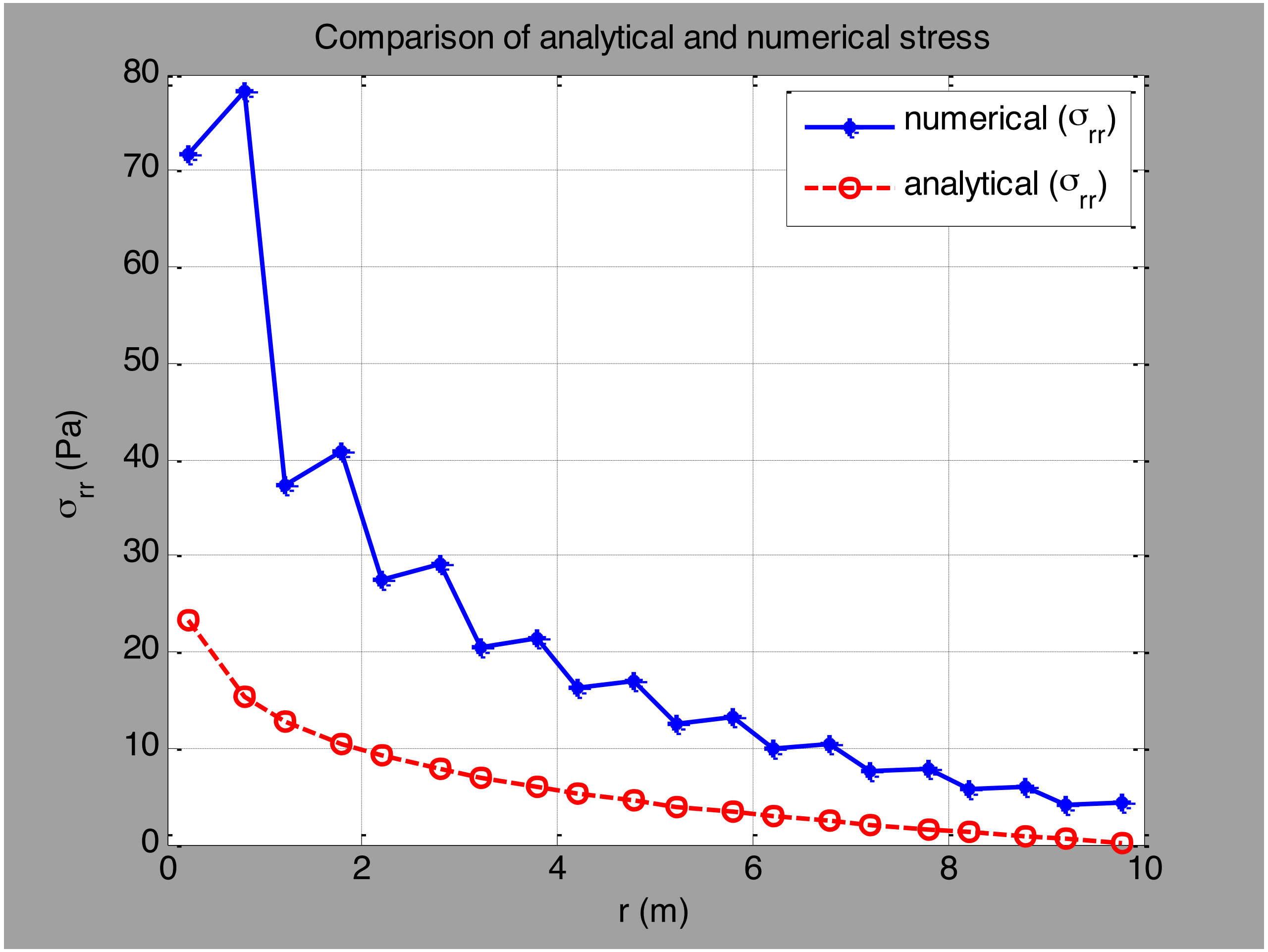




Radial stress distribution contour (σ_{rr}) – numerical & analytical results

Example No. 2

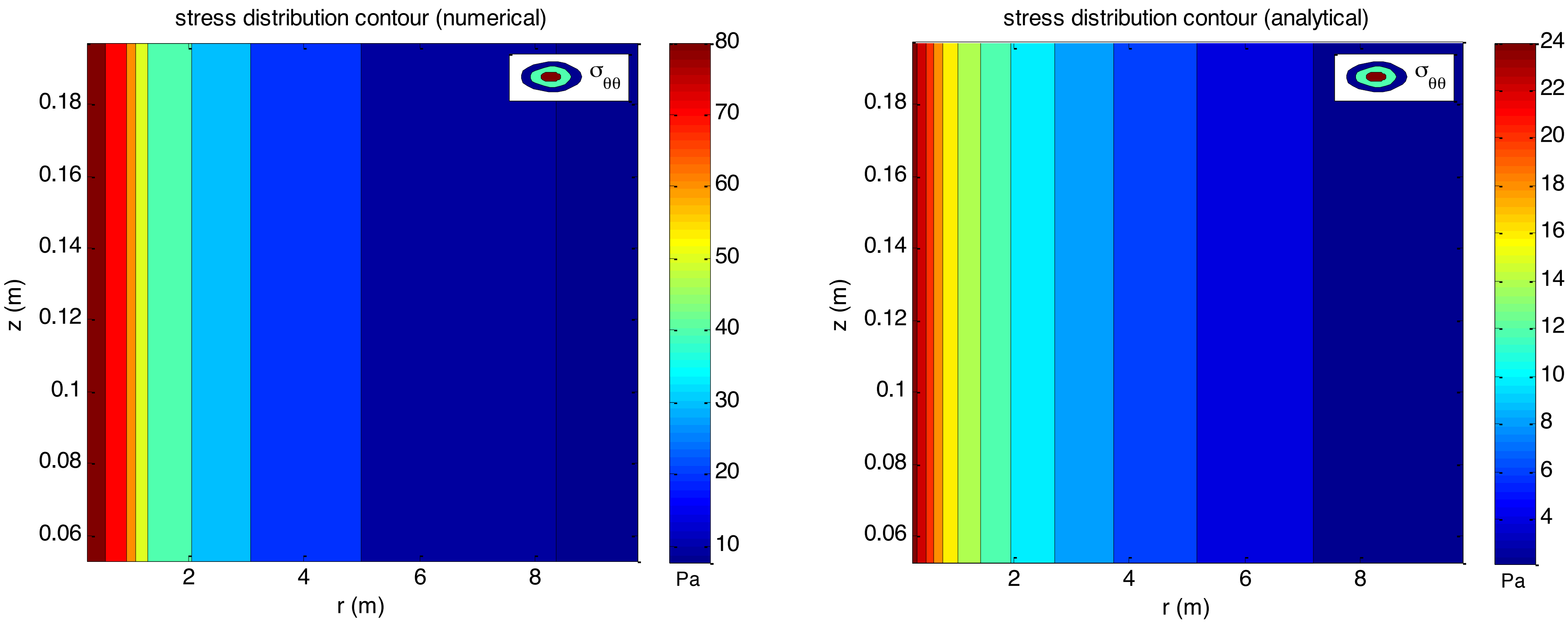
✓Simply supported circular plate with point load



Comparison of analytical and numerical results (σ_{rr})

Example No. 2

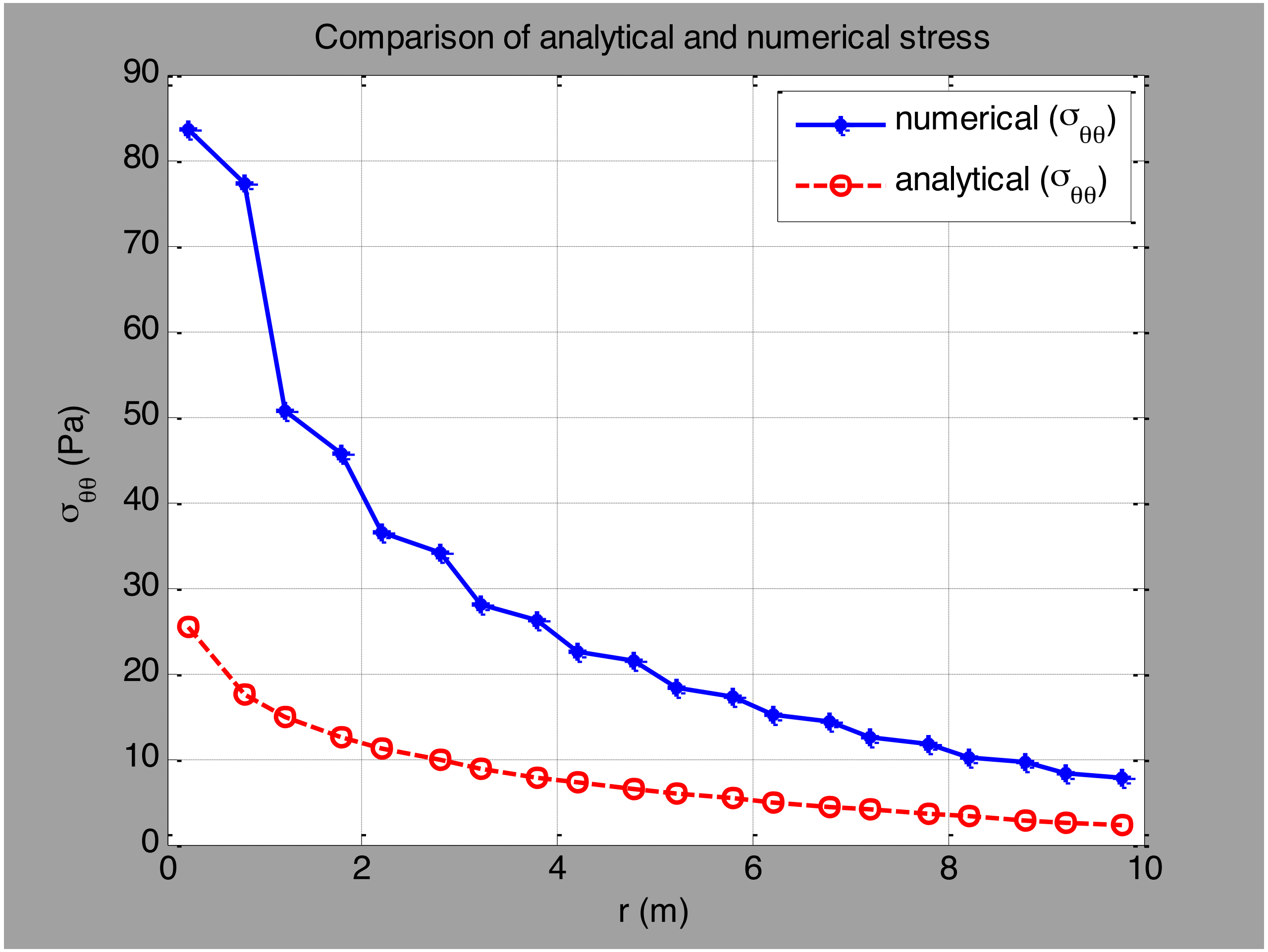
✓Simply supported circular plate with point load



Circumferential stress distribution contour ($\sigma_{\theta\theta}$) – numerical & analytical results

Example No. 2

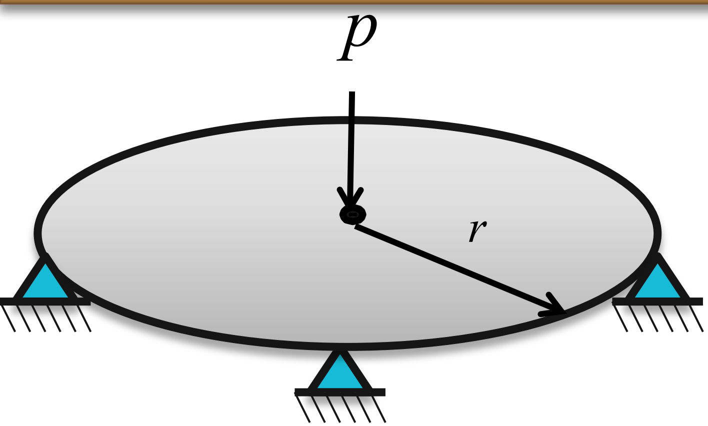
✓Simply supported circular plate with point load

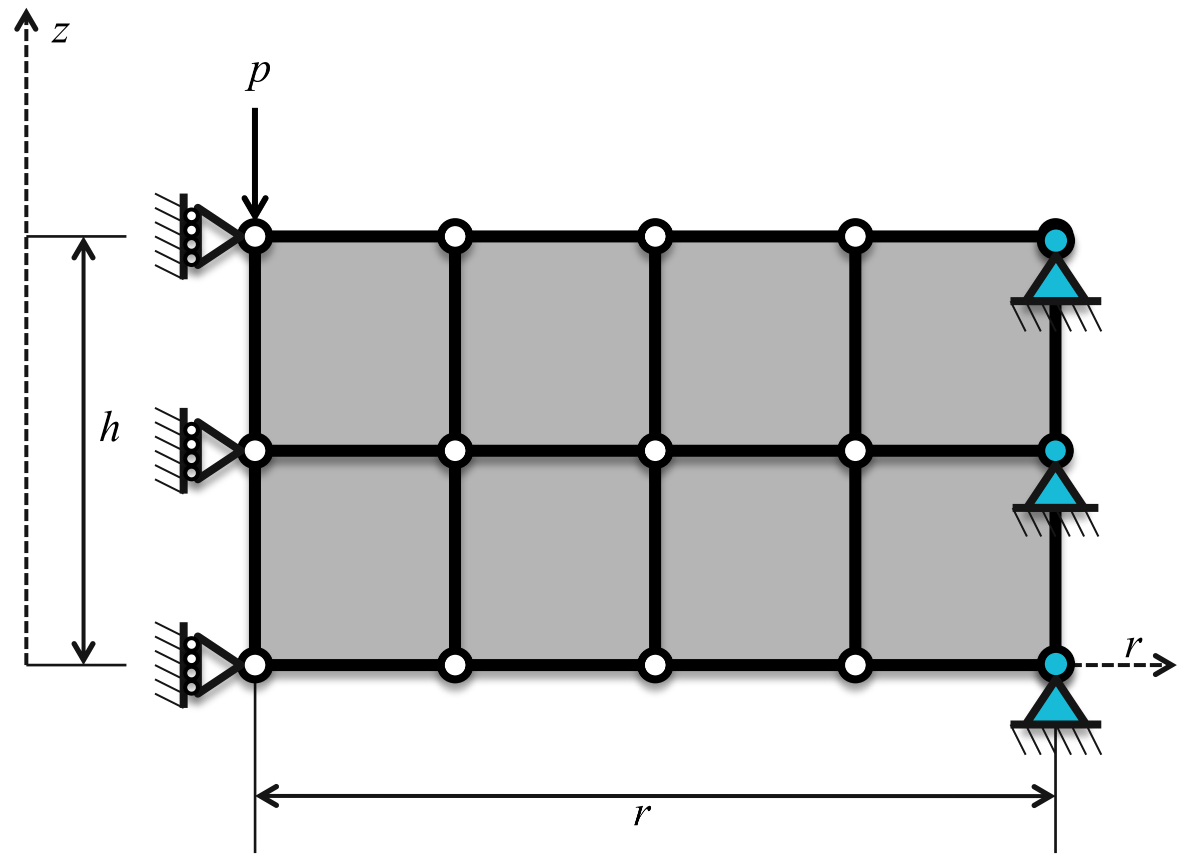


Comparison of analytical and numerical results ($\sigma_{\theta\theta}$)

Example No. 3

✓Clamped circular plate with point load





$Input\ Parameters:$

$r \quad = 10\,m$

$h \quad = 0.5\,m$

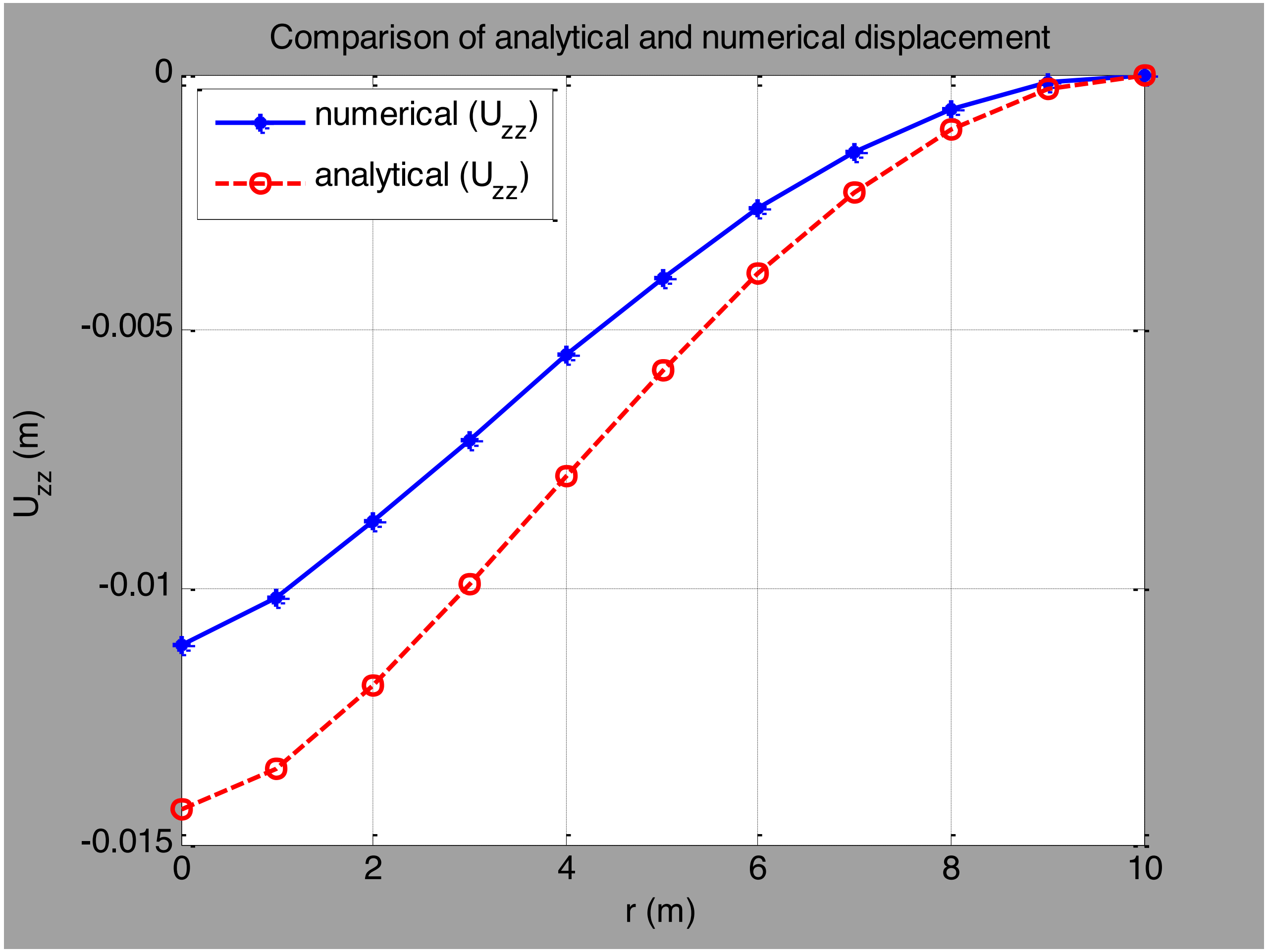
$E \quad = 1.0e+05\ N/m^2$

$\nu \quad = 0.4999999$

$p \quad = 10\,N$

Example No. 3

✓Clamped circular plate with point load



Comparison of analytical and numerical results - U_{zz}

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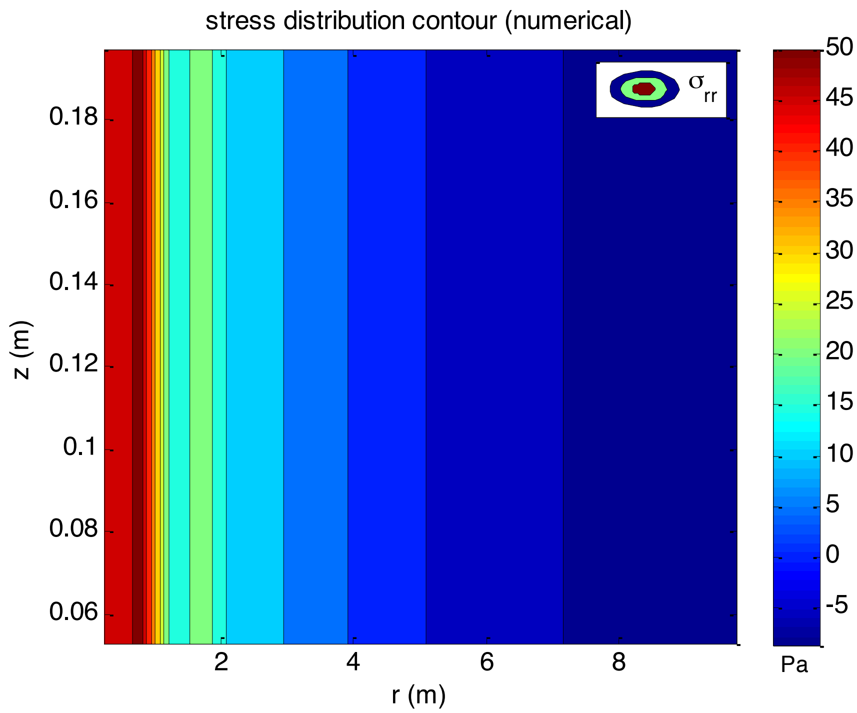
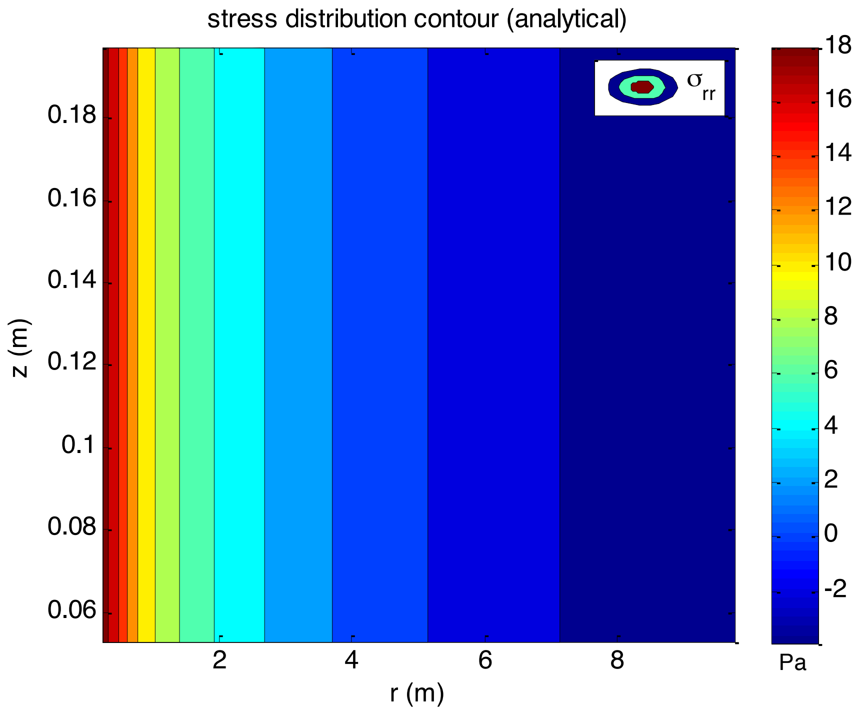
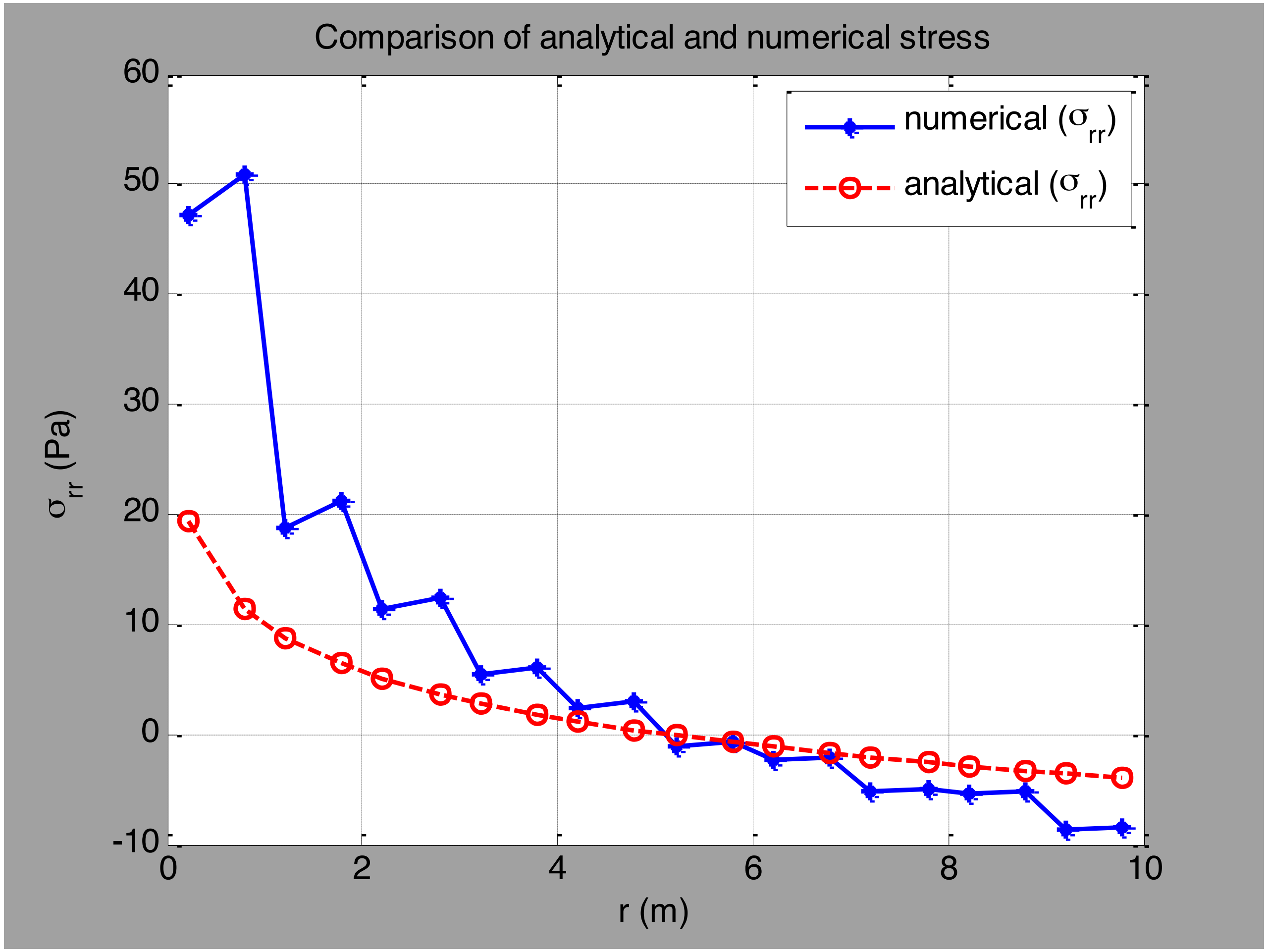




Radial stress distribution contour (σ_{rr}) – numerical & analytical results

Example No. 3

✓Clamped circular plate with point load



Comparison of analytical and numerical results (σ_{rr})

Example No. 3

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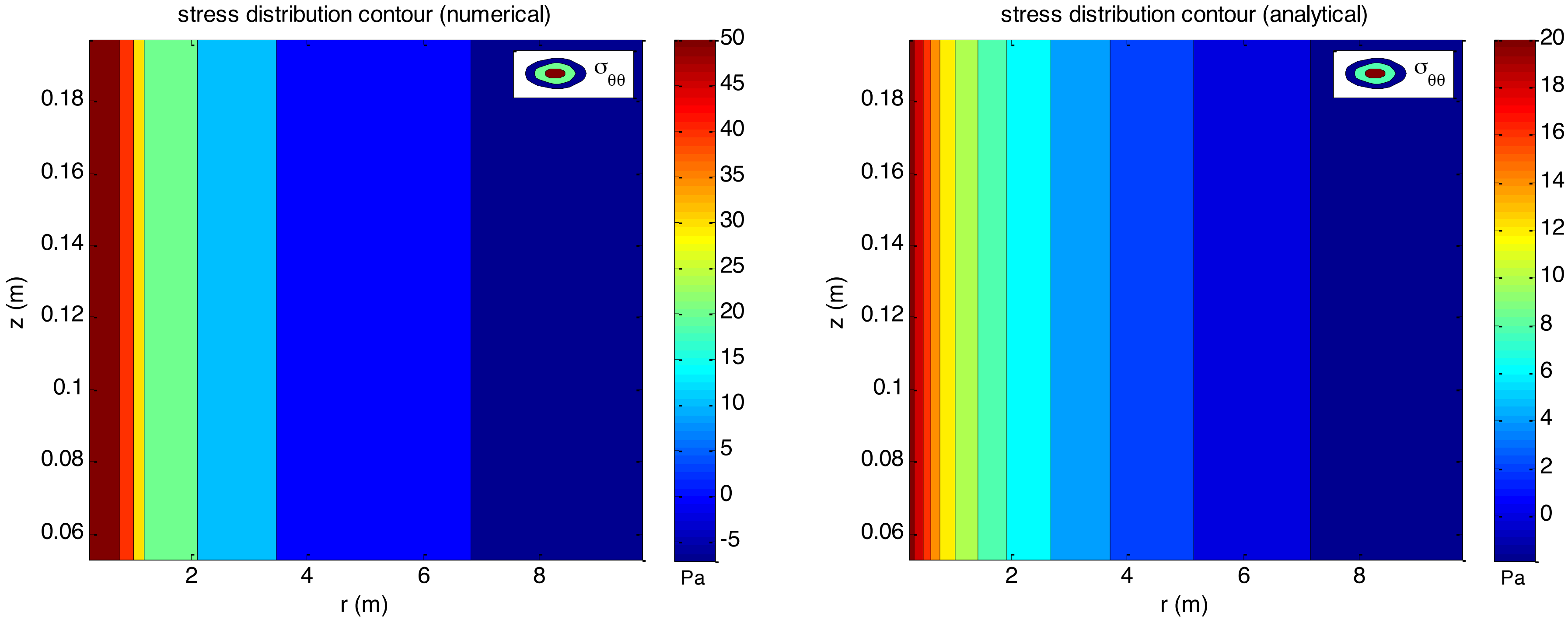
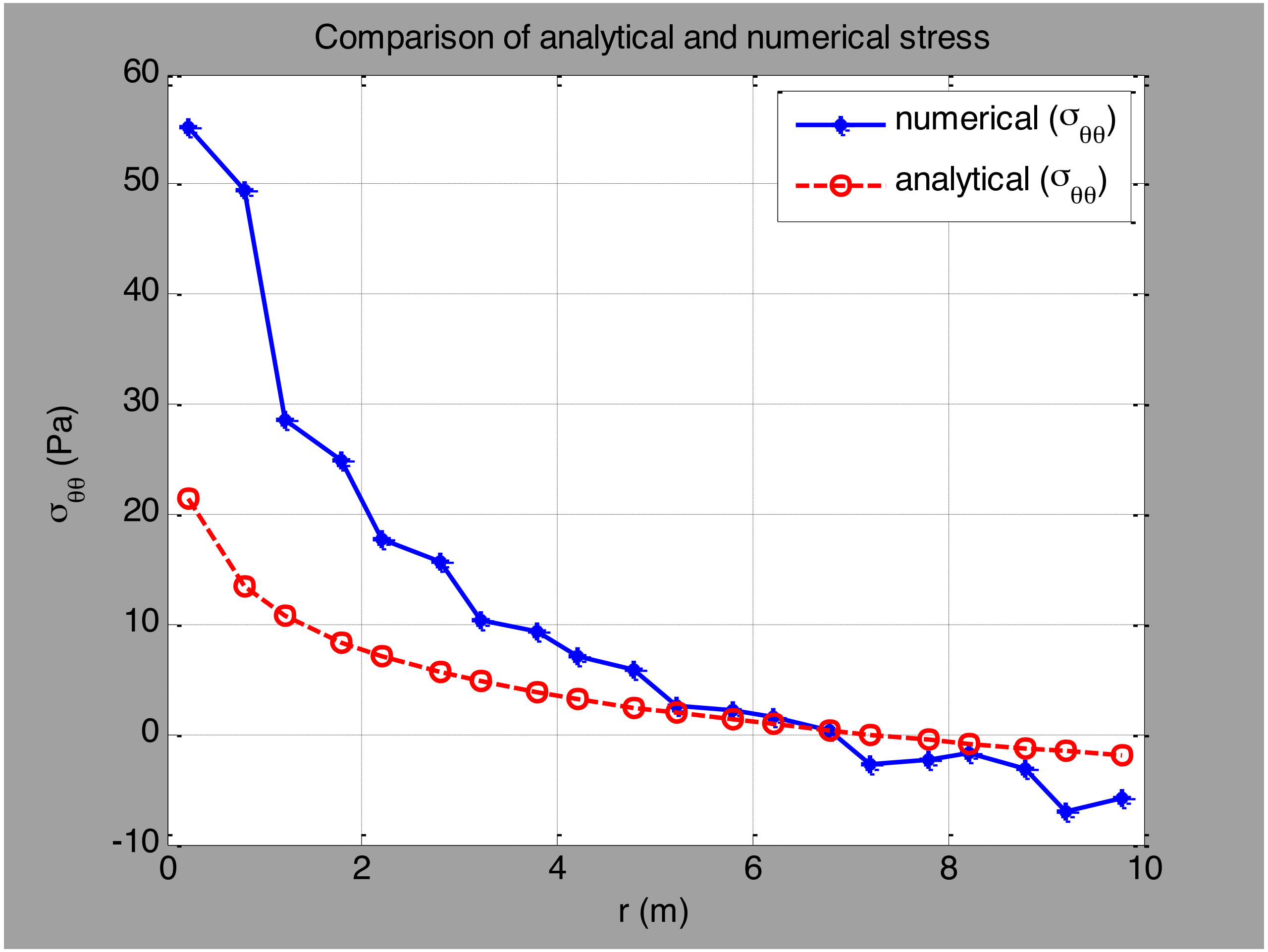


Circumferential stress distribution contour ($\sigma_{\theta\theta}$) – numerical & analytical results

Example No. 3

✓Clamped circular plate with point load



Comparison of analytical and numerical results ($\sigma_{\theta\theta}$)

Conclusion

- ❑ It is difficult to come up with an appropriate stress function which can handle all axisymmetric problems with hybrid formulation.

- ❑ The hybrid finite element formulation would enhance the performance of the element in capturing the mechanical response of the near incompressible materials.

- ❑ As the stress function can be chosen by some complicated functions, the numerical integration might have lost their accuracy.

Thanks

For Your

Attention