Develop and Test of an Improved 4-node Axisymmetric (ring) Element for Nearly Incompressible Materials

PART I
Formulation

AFEM Term Project

Reza Behrou, Farhad Shahabi

Contents

Problem Statement

✓Axisymmetric Problems

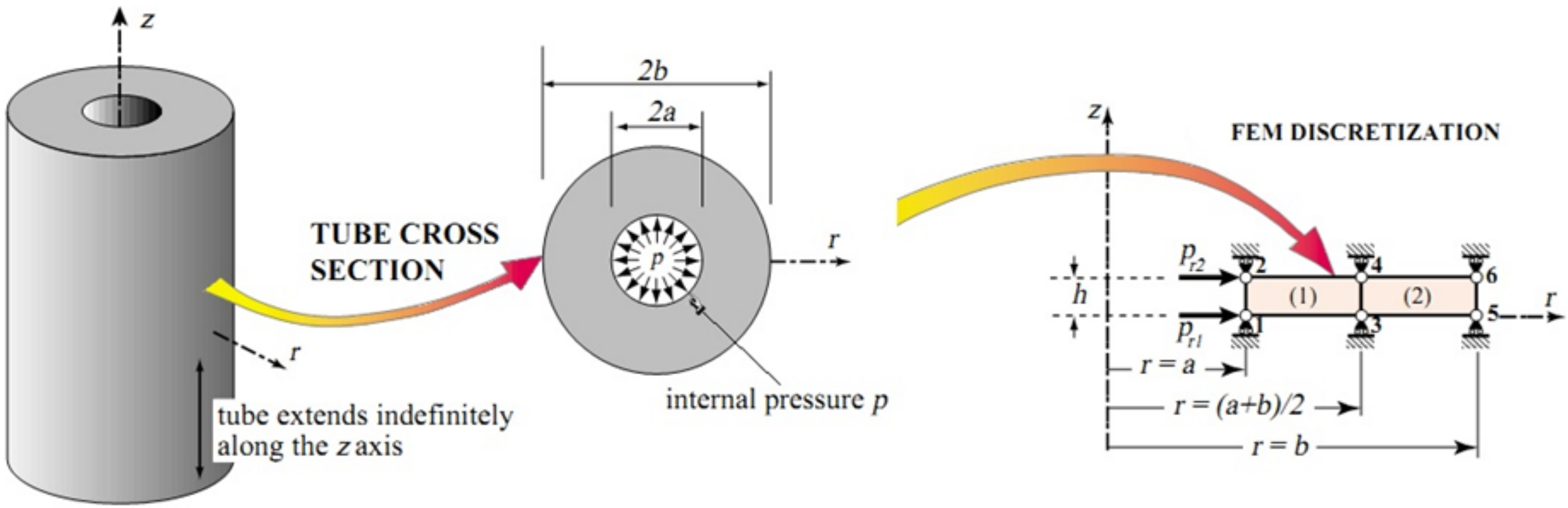
The axisymmetric problem deals with the analysis of structures of revolution under axisymmetric loading

Fabrication: Axisymmetric bodies are usually easier to manufacture than bodies with more complex geometries.

Strength: axisymmetric configurations are often optimal in terms of strength to weight ratio

Problem Statement

✓ **Benchmark problem: a thick cylinder under internal pressure**

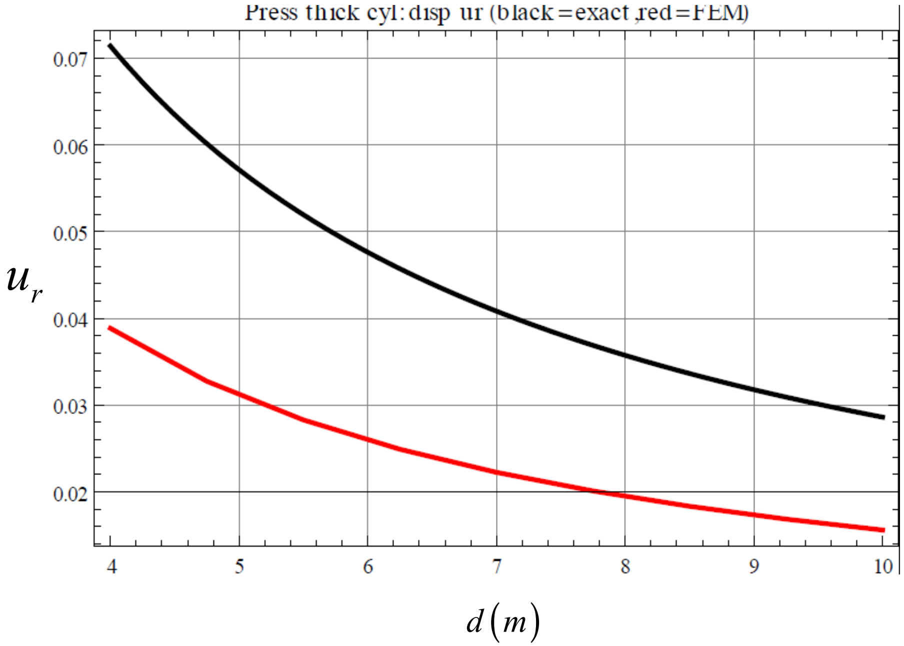


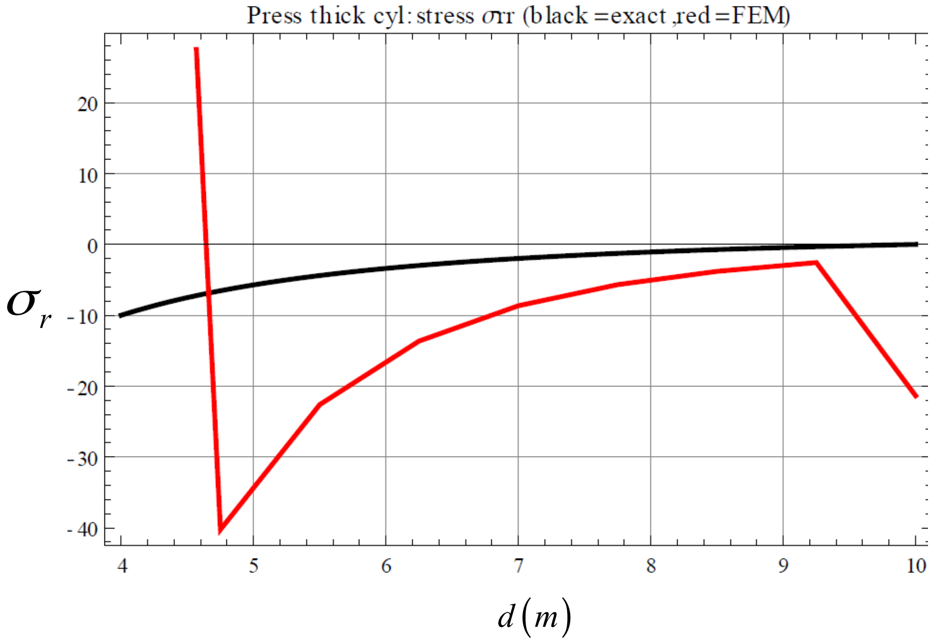
A thick cylinder under internal pressure discretized by the 4-node quadrilateral axisymmetric elements

Motivation

✓The performance of standard 4-node isoparametric quadrilateral Element

$$\upsilon = 0.499$$

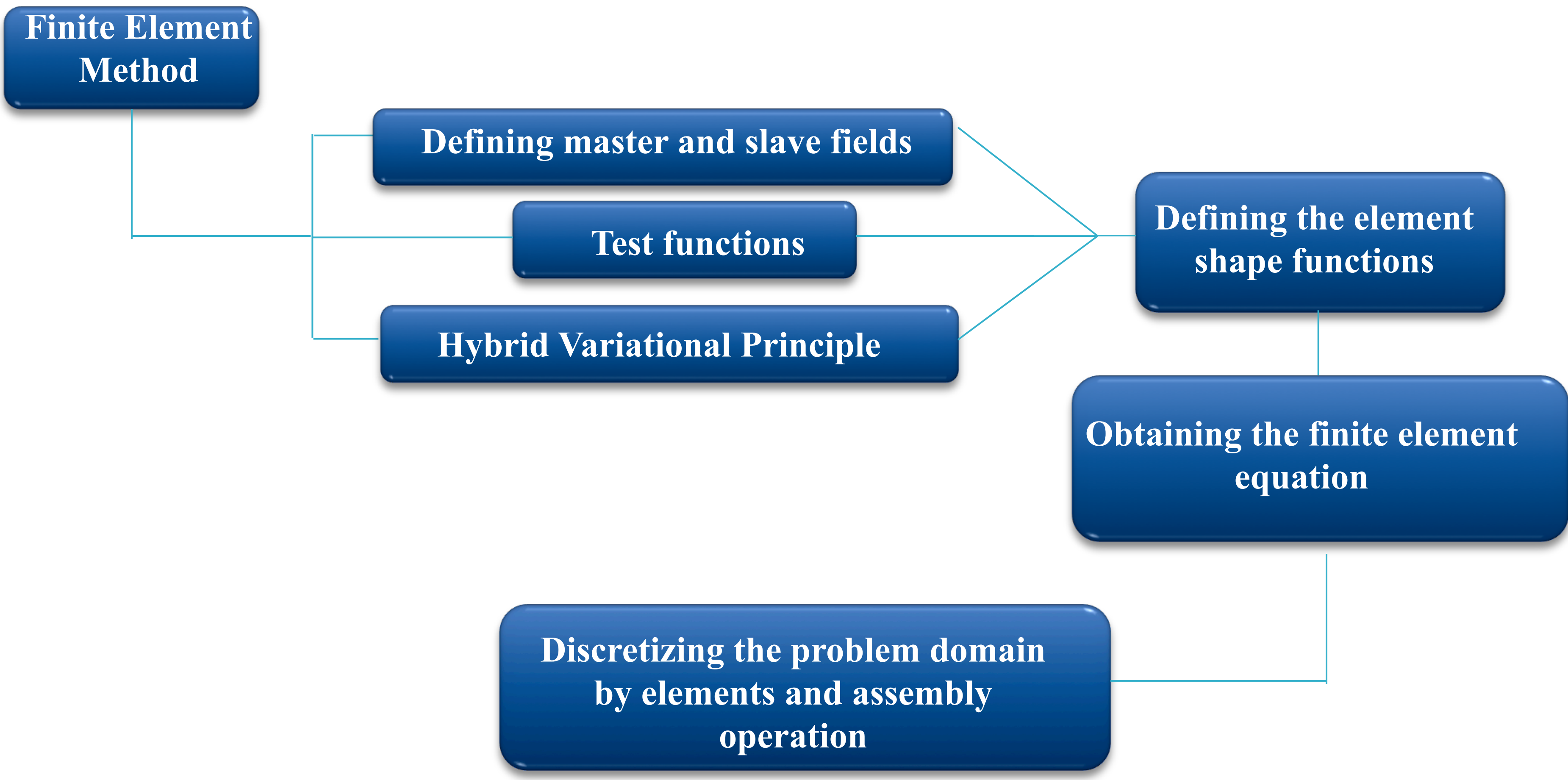




The accuracy of displacement and stress fields predicted by the standard 4-node quadrilateral for the near incompressible materials

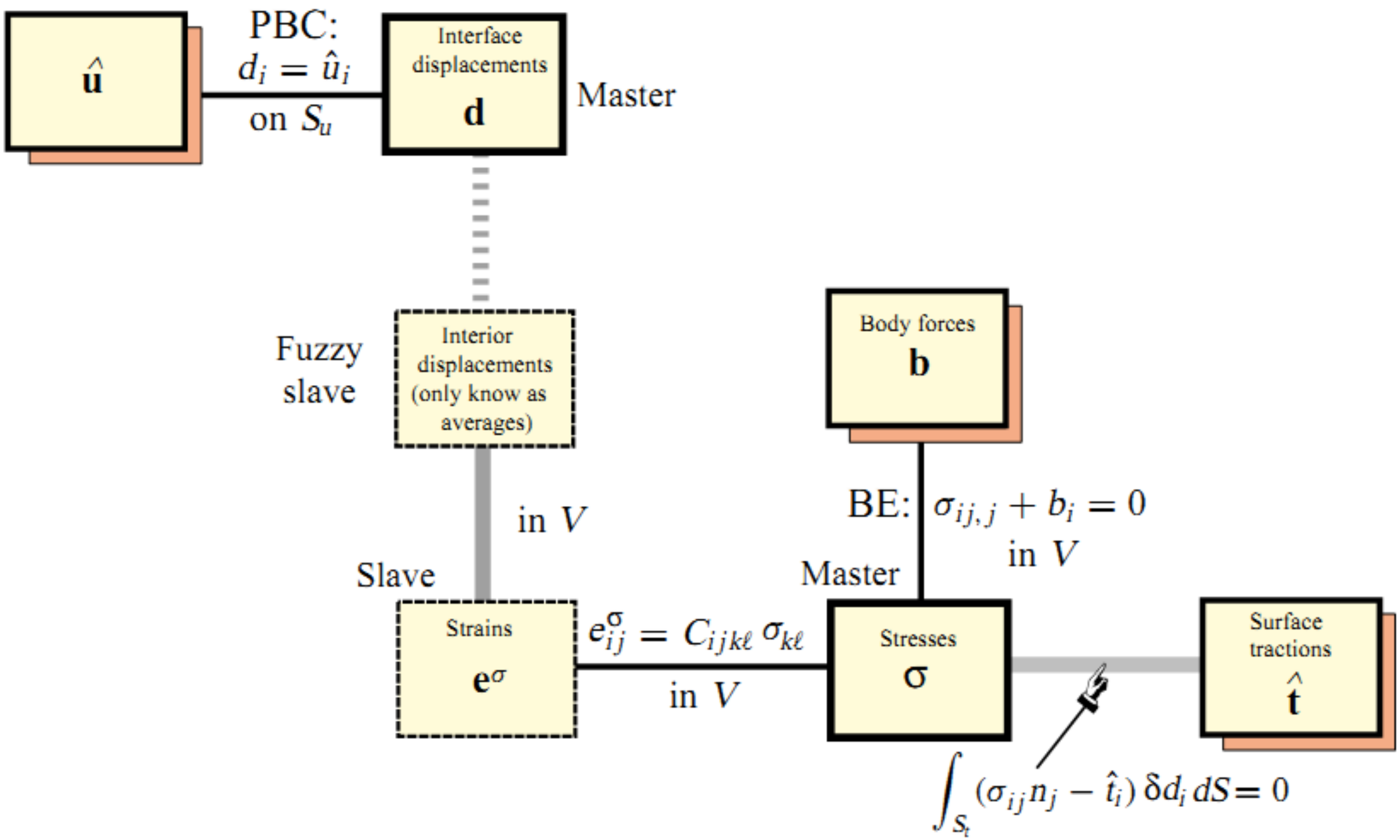
Finite Element Analysis

✓The Finite Element Solution Chart



Hybrid Variational Principle

✓Tonti Diagram



Schematics of Weak Form of the equilibrium-stress-hybrid principle.

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} dv - \int_s T_i u_i ds + \int_{s_T} \overline{T}_i u_i ds \right)$$

Assumed Stress Field

$$\{\sigma\} = [P]\{\beta\} + [P_F]\{\beta_F\}$$

Balance of Linear momentum

$$\sigma_{ij,j} + b_i = \rho \ddot{u}_i$$

Assumed Displacement Field

$$\{U\} = L\{u_i^e\}$$

Traction

$$\{T\} = [\sigma]\{n\}$$

Assumed Traction

$$\{T\} = [R]\{\beta\} + [R_F]\{\beta_F\}$$

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2}\left(\{\beta\}^T [H]\{\beta\} + \{\beta_F\}^T [H_{FF}]\{\beta_F\} \right) + \{\beta\}^T [H_F]\{\beta_F\} dv \right)$$
$$- \sum_n \left(-\int_s \left(\{\beta\}^T [R]^T L + \{\beta_F\}^T [R_F]^T L \right) \{u_i^e\} ds + \int_{s_T} \overline{T}_i L \{u_i^e\} ds \right)$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation

$$H = \int_{v_n} [P]^T [C][P] dV \qquad H_F = \int_{v_n} [P]^T [C][P_F] dV \qquad H_{FF} = \int_{v_n} [P_F]^T [C][P_F] dV$$

Simplifying the problem by solving for the static case with the negligible body force

$$\beta_F = P_F = R_F = 0$$

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} \left(\{\beta\}^T [H] \{\beta\} \right) dv \right) - \sum_n \left(\int_s \left(\{\beta\}^T [R]^T L \right) \{u_i^e\} ds + \int_{s_T} \overline{T_i} L \{u_i^e\} ds \right)$$

$$\int_s \left(\{\beta\}^T [R]^T L \right) \{u_i^e\} ds = \int_s T_i u_i ds \Rightarrow \int_s \sigma_{ij} u_i n_j ds$$

$$\Rightarrow \int_{v_n} \sigma_{ij,j} u_i dv + \int_{v_n} \sigma_{ij} u_{i,j} dv = \int_{v_n} \sigma_{ij} \varepsilon_{ij} dv = \int_{v_n} [\sigma]^T [\varepsilon] dv$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation

$$\Pi_{Hybrid} = \sum_n \left(\int_{v_n} \frac{1}{2} \left(\{\beta\}^T [H] \{\beta\} \right) dv \right) - \sum_n \left(\int_s \left(\{\beta\}^T \underbrace{[P]^T [B]}_{G} \right) \{u_i^e\} dv + \int_{s_T} \overline{T}_i L \{u_i^e\} ds \right)$$

$$G = \int_{v_n} [P]^T [B] dV$$

Taking the variation of the above functional WRT β

$$[H][\beta] - [G]\{u_i^e\} = 0 \Rightarrow [\beta] = [H]^{-1}[G]\{u_i^e\}$$

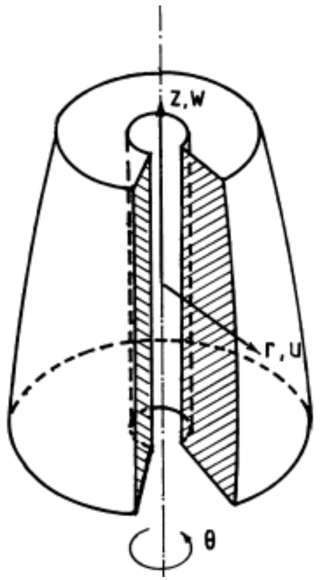
$$\Pi_{Hybrid} = -\sum_n \left(\int_{v_n} \frac{1}{2} \left(\{u_i^e\}^T [G]^T [H]^{-T} [G] \{u_i^e\} \right) dv \right) + \int_{s_T} \overline{T}_i L \{u_i^e\} ds$$

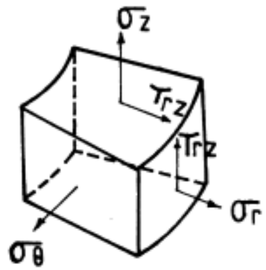
Taking the variation of the above functional WRT u_i^e

$$K = [G]^T [H]^{-T} [G]$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation for the Axisymmetric Problem





The schematic of axisymmetric element

$$\varepsilon_r = \frac{\partial u}{\partial r},\ \varepsilon_\theta = \frac{u}{r}\ and\ \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\varepsilon = C\,\sigma$$

$$\sigma = \{\sigma_z \quad \sigma_\theta \quad \sigma_r \quad \tau_{rz}\}^T \qquad \varepsilon = \{\varepsilon_z \quad \varepsilon_\theta \quad \varepsilon_r \quad \gamma_{rz}\}^T$$

$$u = \sum_{i=1}^{4} N_i(\xi,\eta)u_i \qquad r = \sum_{i=1}^{4} N_i(\xi,\eta)r_i$$

$$w = \sum_{i=1}^{4} N_i(\xi,\eta)w_i \qquad z = \sum_{i=1}^{4} N_i(\xi,\eta)z_i$$

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation for the Axisymmetric Problem

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{rz}}{\partial r} + \frac{\tau_{rz}}{r} = 0$$

$$\nabla^2 \left(\sigma_r + \sigma_\theta + \sigma_z\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\sigma_r + \sigma_\theta + \sigma_z\right) = 0$$

The Assumed stress functions

$$\sigma_z = \beta_6 + \beta_7 z - 3\beta_3 r - \beta_4 \frac{1}{r} - \beta_5 \frac{z}{r}$$

$$\sigma_\theta = \beta_1 + \beta_2 z + 2\beta_3 r$$

$$\sigma_r = \beta_1 + \beta_2 z + \beta_3 r + \beta_4 \frac{1}{r} + \beta_5 \frac{z}{r}$$

$$\tau_{rz} = \beta_5 - \frac{1}{2}\beta_7 r$$

The Number of Stress Parameters

$$n_\beta \geq n_{DOF} - n_{RB}$$

Hybrid Variational Principle

✓Hybrid Variational Principle Formulation for the Axisymmetric Problem

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1^e}{\partial r} & 0 & \frac{\partial N_2^e}{\partial r} & 0 & \frac{\partial N_3^e}{\partial r} & 0 & \frac{\partial N_4^e}{\partial r} & 0 \\ 0 & \frac{\partial N_1^e}{\partial z} & 0 & \frac{\partial N_2^e}{\partial z} & 0 & \frac{\partial N_3^e}{\partial z} & 0 & \frac{\partial N_4^e}{\partial z} \\ \frac{N_1^e}{r} & 0 & \frac{N_2^e}{r} & 0 & \frac{N_3^e}{r} & 0 & \frac{N_4^e}{r} & 0 \\ \frac{\partial N_1^e}{\partial z} & \frac{\partial N_1^e}{\partial r} & \frac{\partial N_2^e}{\partial z} & \frac{\partial N_2^e}{\partial r} & \frac{\partial N_3^e}{\partial z} & \frac{\partial N_3^e}{\partial r} & \frac{\partial N_4^e}{\partial z} & \frac{\partial N_4^e}{\partial r} \end{bmatrix} \{u_i^e\} = [B]\{u_i^e\}$$

$$[P] = \begin{bmatrix} 1 & z & r & 1/r & z/r & 0 & 0 \\ 0 & 0 & -3r & -1/r & -z/r & 1 & z \\ 1 & z & 2r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1/2r \end{bmatrix}$$

$$H = 2\pi \int_{-1}^{1}\int_{-1}^{1} P^T C P \, r \det|J| \, d\xi d\eta$$

$$G = 2\pi \int_{-1}^{1}\int_{-1}^{1} P^T B \, r \det|J| \, d\xi d\eta$$

$$K = [G]^T [H]^{-T} [G]$$

Thanks

For Your

Attention