1/12/16

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CARLOS FELIPA'S

WHAT IS FEM?
A NUMERICAL
METHOD OF SOLVING PDE'S

WHAT IS DIFFERENCE BETWEEN FEM & FINITE DIFFERENCE?

NOTES.1

ODE:

$$\int_{x=0}^{x=L} \left[w(x) \left[\frac{d}{dx}\left(a \frac{du}{dx} \right) - C(x)\, u(x) + q(x) = 0 \right] \right] dx \qquad x \in (0, L)$$

$u(0) = 0$ ← DIRICHLET BOUNDARY CONDITION — "ESSENTIAL B.C."

$$a \left. \frac{du}{dx} \right|_{x=L} = Q_0$$ ← "VAN NEUMANN" NATURAL B.C.

↓ CONVERT TO WEAK FORM

① MULTIPLY BY WEIGHT FCN $w(x)$
② INTEGRATE BY PARTS

$$\int_{x=0}^{x=L} \left(\frac{d}{dx}\left(a \frac{du}{dx} \right) w(x) - C(x)\, u(x)\, w(x) + q(x)\, w(x) = 0 \right) dx$$

INTEGRATE BY PARTS

$$0 = -\int_{x=0}^{x=L} a \frac{du}{dx} \frac{dw}{dx}\, dx + \int_{x=0}^{x=L} q w - C w u \, dx + \underbrace{\left(a w \frac{du}{dx} \right)_{x=0}^{x=L}}_{\text{BOUNDARY TERM}} \qquad \forall w$$

← WEAK FORM

WHY STRONG/WEAK CONTINUITY REQ ON u.

$$+ w \boxed{a \left. \frac{du}{dx} \right|_{x=L}} - w a \left. \frac{du}{dx} \right|_{x=0}$$

NATURAL

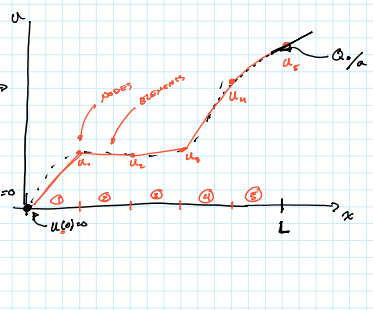
WHAT IS A NATURAL BOUNDARY CONDITION

NATURAL B.C. IS WHATEVER IS BEING MULTIPLIED BY THE WEIGHT FCN OR DERIV

↓ GUESS A SOL FOR u "APPROXIMATE THE SOL^"

$ax + b$

THIS GIVES CLUES ON HOW TO GET TO APPROX SOL →



$$\sum_{el.} \left[-\int_{x_a^{el}}^{x_b^{el}} \frac{du}{dx} \frac{dw}{dx}\, dx + \int_{x_a^{el}}^{x_b^{el}} q w - C w u \, dx + w a \left. \frac{du}{dx} \right|_{x_b^{el}} - w a \left. \frac{du}{dx} \right|_{x_a^{el}} \right] = 0$$

APPROX OF u → $u = ax + b$

NOT THE SAME a

LOOK @ ELEMENT e, LOOK NODES 1, 2

$$u^{(e)}(x_1^{(e)}) = u_1^{(e)} = a x_1 + b$$
$$u^{(e)}(x_2^{(e)}) = u_2^{(e)} = a x_2 + b$$

$$\begin{Bmatrix} u_1^{(e)} \\ u_2^{(e)} \end{Bmatrix} = \begin{bmatrix} 1 & x_1^{(e)} \\ 1 & x_2^{(e)} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix}$$

$$a = \frac{u_1^e x_2^e - u_2^e x_1^e}{x_2^e - x_1^e}$$

ELEMENT LENGTH

$$b = \frac{u_2^e - u_1^e}{x_2^e - x_1^e}$$

$$\hat{u}^{(e)} = N_1^{(e)} u_1^{(e)} + N_2^{(e)} u_2^{(e)}$$

FE SHAPE FCNS

$$\begin{cases} N_1^{(e)} = \dfrac{x_2^{(e)} - x}{h_e} \\ N_2^{(e)} = \dfrac{x - x_1^{(e)}}{h_e} \end{cases}$$

N_i — FINITE ELEMENT BASIS FCNS $\quad \sum N_i = 1 \quad N_i(x_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

PLUG IN WHERE YOU SEE A u INTO WK FORM

$$\boxed{u^{(e)} = \sum_{i=1}^{2} u_i^{(e)} N_i^{(e)}}$$

$$\int_{x_a}^{x_b} \sum_{j=1}^{2} \frac{du_j N_j}{dx} \frac{dw}{dx}\, dx + \int_{x_a}^{x_b} q w - C w \sum_{j=1}^{2} u_j N_j = 0 \qquad \forall w_i$$

$\sum_{i=1}^{2} \frac{dw_i N_i}{dx}$ $\quad \sum_{i=1}^{2} w_i N_i$ $\quad \sum_{i=1}^{2} w_i N_i$

GALERKIN METHOD

$$w = \sum_{i=1}^{2} w_i N_i$$

WHEN WE MAKE A WEIGHT FCN WITH SAME BASIS EXPANSION.

$\sum w_i f = 0 \quad \forall w_i$

$\Rightarrow f = 0$

$$\sum_{i=1}^{2} w_i \left[\int_{x_a}^{x_b} \sum_{j=1}^{2} \frac{dN_j}{dx} \frac{dN_i}{dx} u_i + \int_{x_a}^{x_b} q N_i - C \sum N_j N_i u_j \right] = 0$$

$$\boxed{K_{ij} u_j = F_i}$$

$$K_{ij} = \sum_i \left(\int \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx - \int^{x_b} C N_i N_j \, dx \right)$$

$$\boxed{K_{ij}\, u_j = F_i}$$

$$K_{ij} = \sum_j \left(\int \frac{dN_i}{dx} \frac{dN_j}{dx}\, dx - \int_{x_a}^{x_b} c\, N_i N_j\, dx \right)$$

$$F_i = \int_{x_a}^{x_b} q\, N_i\, dx$$

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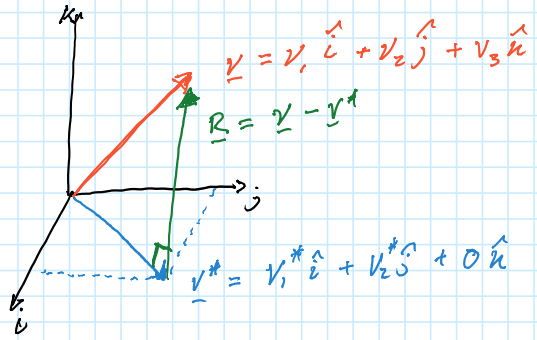
THE GOAL FORM:



BEST APPROX OF CLOSELY REPRESENTING $\underline{v}$ W/ $\underline{v}^*$ IS BY MINIMIZING $\|\underline{R}\|$

↳ INTUITIVELY

$\underline{v}^* = v_1\hat{i} + v_2\hat{j} + 0\hat{k}$

$\underline{R} \cdot \underline{v}^* = 0$

DOT PRODUCT = "INNER PRODUCT"

NOW, CONSIDER n DIM VECTORS

$$\boxed{\sum_i^n R_i v_i^* = 0}$$

WEIGHTED RESIDUAL METHODS:

$$\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - q = 0$$

MAKE A GUESS "APPROXIMATE SOLUTION"

$$\frac{d}{dx}\left(a\frac{d\hat{u}}{dx}\right) + c\hat{u} - q = R$$

← RESIDUAL

WANT TO MINIMIZE RESIDUAL.

SIMILAR? YES!

$$\int_0^L \underbrace{\left[\frac{d}{dx}\left(a\frac{d\hat{u}}{dx}\right) + c\hat{u} - q\right]}_{R} w\,dx = 0$$

$$\boxed{\int_0^L R\,w\,dx = 0}$$

INNER PRODUCT, SCALAR PRODUCT OF TWO FUNCTIONS

WEIGHT FXN ORTHOGONAL TO THE RESIDUAL

$$0 = \int_0^L \left(a\frac{dw}{dx}\frac{du}{dx} + cwu\right)dx - \int_0^L wq\,dx - \left[w\frac{du}{dx}\right]_0^L$$

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$$\hat{u} = \sum_{j=1}^{N} u_j \phi_j(x)$$

$$w = \sum_{i=1}^{N} w_i \phi_i(x)$$

$$\int_0^L a \frac{\sum w_i \phi_i}{dx} \frac{\sum u_j \phi_j}{dx} + c \sum w_i \phi_i \sum u_j \phi_j \Big) dx - \int_0^L \sum w_i \phi_i q \, dx - \left[\sum w_i \phi_i \frac{d \sum u_j \phi_j}{dx} \right]_0^L = 0$$

CAN CANCEL OUT W FROM EACH TERM

$$K_{ij} u_j - f_i - Q_i = 0$$

$$K_{ij} = \int a \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + c \phi_i \phi_j \, dx$$

$$f_i = \int q \phi_i \, dx$$

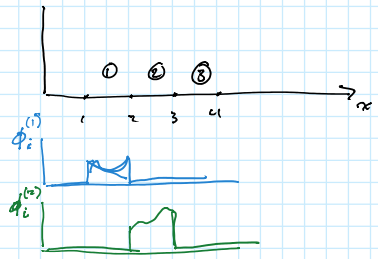
$$Q_i = a \frac{d\phi_i}{dx}(x_i)$$

HAVENT SAID ANYTHING?

WHAT KIND OF SHAPE FXNS ALLOW BEING VALID OVER DOMAIN & EACH ELEMENT?

- SHAPE FXN IS ZERO OUTSIDE ELEMENT BOUNDARY
- WHAT ARE SHAPE FXNS?

$$\int_0^L q \phi_i \, dx$$



ϕ ARE SHAPE FXNS.

SOLID MECHANICS

$a = AE$

$$K_{ij} = AE \int \phi_i' \phi_j' \, dx$$

$$K_{11} = AE \int_{x_1}^{x_2} \left(-\frac{1}{l^e}\right)\left(-\frac{1}{l^e}\right) dx$$

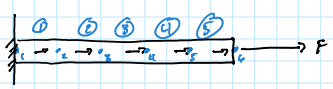
$$K_{ij} = \frac{AE}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

SHAPE FXNS FOR SOLID MECH

$$\phi_i(x_j) = \delta_{ij}$$

LINEAR ELEMENT

$$\phi_1 = \frac{x_2 - x}{x_2 - x_1}$$



$$\frac{AE}{l^e} = 1$$

$u(0) = 0$, u_0; $AE \frac{du}{dx} = F$

$$K_{ij} u_j = F_i$$

NODES →

$$\begin{bmatrix} \cancel{1} & 0 & 0 & 0 & 0 & 0 \\ -1 & 1+1 & -1 & & & \\ & -1 & 1+1 & -1 & & \\ & & -1 & 1+1 & -1 & \\ & & & -1 & 1+1 & -1 \\ & & & & -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} \cancel{F_1} \\ F_2 \\ F_3 \\ \vdots \\ F_6 \end{Bmatrix}$$

CONTINUITY REQ

$$u_2^{(1)} = u_1^{(2)}$$

$$u_2^{(e)} = u_1^{(e+1)}$$

MATLAB

$K^{-1} F \rightarrow K \backslash f$

NEED B.C.'S

IMPOSE B.C.'S ALONG FIRST ROW BY $\boxed{1 \; 0 \ldots 0}$

WANT TO KEEP THINGS SYMMETRIC TOO

PENALTY METHOD $\boxed{K_{11} = \text{MASSIVE } \# = X}$

$$\frac{X u_1 - u_2 + \ldots}{X} = \frac{X u_0}{X}$$

$$u_1 - \frac{u_2}{X} = u_0$$

EASY TO IMPLEMENT

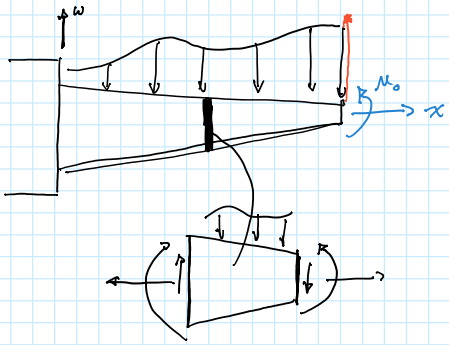
EASY TO IMPLEMENT

$$u_1 - \frac{u_2}{\cancel{X}} = u_0 \qquad \cancel{X}$$

$$u_1 \approx u_0$$

BEAMS

J.N. REDDY "INTRODUCTION TO FINITE ELEMENT METHOD"



$$\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) = f(x) \qquad 0 < x < L$$

STEPS:

1) MULTIPLY BY WEIGHT FXN — NOT W NOW
2) INTEGRATE BY PARTS
3) IMPOSE B.C.

$$M = b\frac{d^2w}{dx^2}$$

$$V = \frac{dM}{dx}$$

$$\frac{dV}{dx} = f$$

STEP 1

$$0 = \int_{x_e}^{x_{e+1}} v\left[\frac{d^2}{dx^2}\left(b\frac{d^2w}{dx^2}\right) - f\right]dx$$

WEIGHT FXN

STEP 2

INTEGRATE BY PARTS

WEIGHT

$$0 = \int_{x_e}^{x_{e+1}} \left[-\underbrace{\frac{dv}{dx}}_{1\text{ deriv}}\underbrace{\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)}_{3\text{ deriv}} - vf\right]dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right)\right]_{x_e}^{x_{e+1}}$$

$\rightarrow k_{ij} \simeq \int \phi_i' \phi_j'''$ → NOT SYMMETRIC

INTEGRATE BY PARTS AGAIN

$$0 = \int_{x_e}^{x_{e+1}} \left(b\underbrace{\frac{d^2v}{dx^2}}_{2\text{ deriv}}\underbrace{\frac{d^2w}{dx^2}}_{2\text{ deriv}} - vf\right)dx + \left[v\frac{d}{dx}\left(b\frac{d^2w}{dx^2}\right) - \frac{dv}{dx}b\frac{d^2w}{dx^2}\right]_{x_e}^{x_{e+1}}$$

ASSUME b CONST

$$v\boxed{b\frac{d^3w}{dx^3}}$$

NATURAL

SHEAR FORCE

BECOMES $\frac{dw}{dx}$

NBC → $b\frac{d^2w}{dx^2}$

EBC → $\frac{dw}{dx}$

BECOMES w

EBC → w

NBC → $b\frac{d^3w}{dx^3}$

WHY IS W AN EBD BUT dw/dx

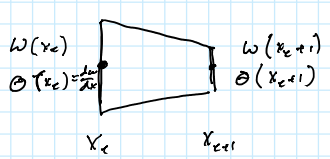
WHY DO INTEGRATION BY PARTS?

- TRANSFERS DERIV

→ • WEAK FORM, WEAKEN CONTINUITY REQUIREMENTS.

WHAT IS NATURAL, WHAT IS ESSENTIAL?

FIND WF & ANTHING MULTIPLIED BY



$$\tilde{w}(x) = C_1 + C_2x + C_3x^2 + C_4x^3$$

INTRODUCE

GENERALIZED DISPLAC...

$$u = \underbrace{\begin{Bmatrix} w(x_e) \\ -\theta(x_e) \\ w(x_{e+1}) \\ -\theta(x_{e+1}) \end{Bmatrix}}_{\text{KEEP ROTATION DOWN}} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

WILL ALSO HAVE GENERALIZED FORCE ARRAY

DISP

ROT

$$\tilde{w}(x_e) = C_1 + C_2 x_e + C_3 x_e^2 + C_4 x_e^3 = U_1$$
$$\tilde{w}'(x_e) = C_2 + 2C_3 x_e + 3C_4 x_e^2 = U_2$$
$$\tilde{w}(x_{e+1}) = \vdots = U_3$$
$$\tilde{w}'(x_{e+1}) = \vdots = U_4$$

$$\Rightarrow \tilde{w}(x) = U_1^e \phi_1^e(x) + U_2^e \phi_2^e(x) + U_3^e \phi_3^e + U_4 \phi_4^e = \sum_1^4 U_i \phi_i^e$$

ϕ_1

ϕ_2

IN NOTES...

SEE READING ASSIGNMENT

PP...

$\vdots$

Hermite cubic spline polynomials.

Cool properties

$$\phi_1^e(x_e) = 1$$
$$\phi_2^e(x_e) = 0$$
$$\phi_3^e(x_{e+1}) = 1$$
$$\phi_4^e(x_{e+1}) = 0$$

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ASSIGNMENT 2 P

GOAL OF FEM: SOLVE THE PDE.

$$\mathcal{L}(u) = f$$

THERE EXISTS SOME APPROXIMATE SOLN $\tilde{u}$

$$\mathcal{L}(\tilde{u}) - f \neq 0$$
$$= R$$

RESIDUAL <MINIMIZE THE RESIDUAL>

$$\int_\Omega R\, w\, d\Omega = 0$$

RESID. WEIGHT FXN

for "METHOD OF WEIGHTED RESIDUALS" → WHY? WILL BE ON TEST

THINK OF FORM OF APPROX SOL.

ALL WELL BEHAVED FXNS CAN BE

$$\tilde{u} = \sum u_i \phi_i$$

basis expansion of u

GO TO "GALERKIN"
"RALEIGH-RITZ"
"Collocation"
ONLY DIFFERENCE IS IN WEIGHT FXNS

WHAT DO THEY EVEN GIVE TO US?

$$\mathcal{L}(\tilde{u}) - f = R$$

$$\int_\Omega R\, w\, d\Omega = 0$$ ← IN TEGRATE THIS BY PARTS.

"DEGREE OF FREEDOM"
↳ DEPENDENT VAR
UNKNOWN u_i
UNKNOWN COEFFICIENTS
UNITS OF u_i ARE NOT ALL CONSISTENT.

DISPLACEMENTS
ROTATION
TEMPERATURE
⋮

THE BEAM PROBLEM

1. WEAK FORM

$$\int_0^L \left[EI \frac{d^4 w}{dx^4} - f(x)\right] v\, dx = 0$$

w = transverse disp
v = weight fxn.

$$\int_0^L EI \frac{d^2 v}{dx^2}\frac{d^2 w}{dx^2} - v f\, dx - \left[\underbrace{v}_{w\ \text{EBC}} \underbrace{\frac{d}{dx}\left(\frac{d^2 w}{dx^2}\right)}_{\text{N.B.C}} - \underbrace{\frac{dv}{dx}}_{\frac{dw}{dx}\ \text{EBC}} \underbrace{EI \frac{d^2 w}{dx^2}}_{\text{N.B.C.}}\right]_0^L$$

WHAT DO WE WANT TO INTERPOLATE?
↳ DISPLACEMENT & ROTATION!
← HERMITIAN FE / POLYNOMIAL

FINITE ELEMENT MODEL OVER THE ELEMENT

$$\sum_{j=1}^{4} K_{ij}^{(e)} u_j^{(e)} - F_i^{(e)} = 0$$

INTERPOLATE THE PRIMARY VARIABLE, THE EBC

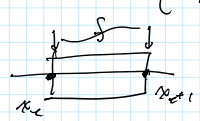
$$u^{(e)} = \begin{Bmatrix} w_1^e \\ \theta_1^e \\ w_2^e \\ \theta_2^e \end{Bmatrix}$$

$$K_{ij} = \int_{x_{el1}}^{x_{el2}} EI \frac{d^2 N_i^e}{dx} \frac{d^2 N_j}{dx^2}\, dx$$

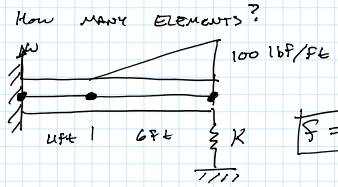
$$F_i^e = \int_{x_{el1}}^{x_{el2}} f N_i\, dx + Q_i$$

ϕ_i

$$Q_i = \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} V_1^e \\ M_1^e \\ V_2^e \\ M_2^e \end{Bmatrix}$$



HOW MANY ELEMENTS?



$\boxed{F = K\, w(L)}$ NOW A B.C.

$$u = \begin{Bmatrix} 0 \\ 0 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

$w(0) = 0$

$\left.\frac{dw}{dx}\right|_0 = 0$

$w(L) = ?$

$$\left.-\frac{d}{dx}\left(EI \frac{d^2 w}{dx^2}\right)\right|_L = -K\, w(L)$$

$$Q = \begin{Bmatrix} Q_1 \\ Q_2 \\ 0 \\ 0 \\ -k\,w(L) \\ Q_4 \end{Bmatrix}$$

u_5

$$K_{ij} u_{ij} = F_i + Q_i$$

START WITH F

$$F_i^{(1)} = \int 0\, N_i\, dx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$F_i^{(2)} = \int -\frac{100}{6} x\, N_i\, dx$$

$$\boxed{\int_4^{10} -\frac{100}{4} x\, N_1(x)\, dx = F_1^2 = F_3}$$

$$= F_2^2 = F_4$$

HOW TO SOLVE HARD MATH
1 WRITE BIGGER
2 START MULTIPLYING BY 1

$$F_i^{(2)} = \int -\frac{100}{6} x \, N_i \, dx$$

$$\int_4 -\frac{100}{4} x \, N_1(x) \, dx = F_1 = u_3$$

$$= F_2^2 = F_4$$

$$\vdots$$

$$K'_{14} = \int_1^4 EI \frac{d^2 N_1}{dx^2} \frac{d^2 N_4}{dx^2} \, dx =$$

36 INTEGRALS + 4 INTEGRALS

1 WRITE BIGGER

2 START MULTIPLYING BY 1

3 USE $\emptyset$

$$K'_{ij} = \begin{bmatrix} K'_{11} & \cdots & K'_{14} \\ & \ddots & \vdots \\ & & K'_{44} \end{bmatrix} \qquad F' = \begin{Bmatrix} F_1 \\ \vdots \\ F_4 \end{Bmatrix}$$

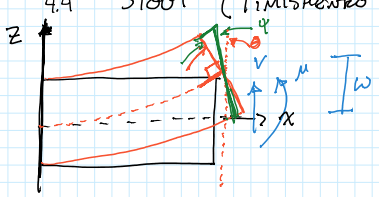
$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ kU_5 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & & & & \\ K_{31} & K_{32} & K_{33} & \cdots & & K_{36} \\ & & & \ddots & & \vdots \\ & & & & & K_{66} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3 \\ F_4 \\ F_5 - kU_5 \\ F_6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ +kU_5 \\ 0 \end{Bmatrix}$$

GLOBAL

$$F = \begin{Bmatrix} F_1' \\ F_2' \\ F_3' + F_1^2 \\ F_4' + F_2^2 \\ F_3^2 \\ F_4^2 \end{Bmatrix} \overset{\text{SATISFIED}}{=} \begin{Bmatrix} 0 \\ 0 \\ F_1^2 \\ F_2^2 \\ F_3^3 \\ F_4^4 \end{Bmatrix}$$

4.4 STOUT (TIMISHENKO BEAMS)

BELIEVE THE BOOK!



$$\theta = \frac{dw}{dx}$$

1ST RELAX RESTRICTION OF $\perp$ TO NEUTRAL AXIS

$$\psi \neq \frac{dw}{dx}$$

$$u_x = z\,\psi(x)$$

$$u_y = 0$$

$$u_z = w(x)$$

$$\varepsilon_{xz} = \frac{1}{2}\left(\frac{du_x}{dz} + \frac{du_z}{dx}\right) = \frac{1}{2}\left(\psi(x) + \frac{dw}{dx}\right) K_s$$

depends on GEOM OF BEAM (CONSTANT)

TIMOSHENKO BEAM THEORY

$$\left\{ \frac{d}{dx}\left(EI \frac{d\psi}{dx}\right) - K_s AG\left(\frac{dw}{dx} + \psi\right) \right\} w_1 \, dx = 0$$

$$\left\{ \left(\frac{d}{dx}\left(K_s AG\left(\frac{dw}{dx} + \psi\right)\right) + q \right) w_2 \, dx = 0 \right\}$$

COUPLED ODE'S

INTEGRATE BY PARTS.

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LAST TIME, QUICK & DIRTY TIMOSHENKO ROUNDUP.

TIMOSHENKO SET OF DIFF. EQ.

$$\frac{d}{dx}\left[GAk_s\left(\psi + \frac{dw}{dx}\right)\right] + f = 0$$

STRONG FORM
WK FORM

$$\frac{d}{dx}\left(EI\frac{d\psi}{dx}\right) - GAk_s\left(\psi + \frac{dw}{dx}\right) = 0$$

IF ASSUME $\psi = -\frac{dw}{dx}$ THEN GET EULER BEAM THEORY

WK FORM

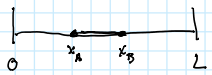
$$\int_0^L \left(\frac{d}{dx}\left[GAk_s\left(\psi + \frac{dw}{dx}\right)\right] w_1\, dx = 0 \qquad \forall\, w_1$$

$$\int_0^L \left(\frac{d}{dx}\left[EI\frac{d\psi}{dx}\right] - GAk_s\left(\psi + \frac{dw}{dx}\right)\right) w_2\, dx = 0 \qquad \forall\, w_2$$

INTEGRATE BY PARTS

$$\int_{x_A}^{x_B}\left[\frac{dw_1}{dx} GAk_s\left(\psi + \frac{dw}{dx}\right) - w_1 f\right] dx - \Big[w_1 \underbrace{GAk_s\left(\psi + \frac{dw}{dx}\right)}_{N.V.}\Big]_{x_A}^{x_B} = 0$$

$$\int_{x_A}^{x_B}\left[\frac{dw_2}{dx} EI\frac{d\psi}{dx} + w_2 GAk_s\left(\psi + \frac{dw}{dx}\right)\right] dx - \Big[w_2 \underbrace{EI\frac{d\psi}{dx}}_{N.V.}\Big]_{x_A}^{x_B} = 0$$



$w_1 \rightsquigarrow w$ PRIMARY VARIABLE

$w_2 \rightsquigarrow \psi$ PRIMARY VARIABLES

NATURAL VARIABLES — NV

$$Q_1^e = -GAk_s\left(\psi + \frac{dw}{dx}\right)_{x_A}$$

$$Q_2^e = -\left(EI\frac{d\psi}{dx}\right)_{x_A}$$

$$Q_3^e = (\qquad)_{x_B}$$

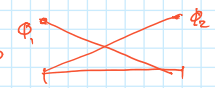
$$Q_4^e = (\qquad)_{x_B}$$

$$w = \sum_{j=1}^{m} w_j \phi_j^{(1)}, \qquad \psi = \sum_{j=1}^{n} s_i \phi_j^{(2)}$$

~ ROTATION

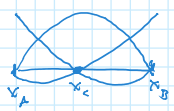
$$\left(\frac{dw}{dx}\right)^e = \frac{w_2^e - w_1^e}{h^e}$$

GOING TO USE A DIFFERENT ORDER FOR EACH.



CONSISTENT INTERPOLATION

$$w = \sum_{j=1}^{m} w_j \phi_j^{(1)} \qquad \psi = \sum_{j=1}^{n} s_j \phi_j^{(2)}$$



$w = a + bx + cx^2$ WHAT MUST THIS FORM INTERPOLATE? PRIMARY VARIABLE.

$$\boxed{m = n+1}$$

$m=3$	$n=2$
QUADRATIC	LINEAR

BUT THERE'S A BETTER WAY.

REDUCED INTEGRATION

$m = n$ BUT, ENERGY IS NEEDING DIFFERENT ORDER

$$\int_{x_A}^{x_B} GAk_s\left(\frac{dw}{dx}\right)^2 dx$$

$$0 = \sum_{j=1}^{m} K_{ij}^{11} w_j + \sum_{j=1}^{n} K_{ij}^{12} s_j - F_i^1$$

$$0 = \sum_{j=1}^{m} K_{ij}^{21} w_i + \sum_{j=1}^{n} K_{ij}^{22} s_i - F_i^2$$

$$\begin{bmatrix} [K^{11}] & [K^{12}] \\ [K^{21}] & [K^{22}] \end{bmatrix} \begin{Bmatrix} \{w\} \\ \{s\} \end{Bmatrix} = \begin{Bmatrix} \{F^1\} \\ \{F^2\} \end{Bmatrix}$$

$$K_{ij}^{11} = \int_{x_a}^{x_b} GAk_s \frac{d\phi_i^{(1)}}{dx} \frac{d\phi_j^{(1)}}{dx} \, dx$$

$$K_{ij}^{12} = \int_{x_a}^{x_b} GAk_s \frac{d\phi_i^{(1)}}{dx} \phi_j^{(2)} \, dx$$

SYMMETRIC

$$K_{12}^{12} = \int GAk_s \frac{d\phi_1}{dx} \phi_2 \, dx$$

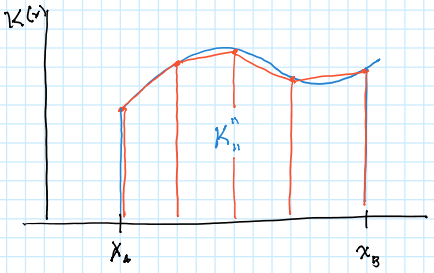
$$K_{21}^{12} = \int GAk_s \frac{d\phi_2}{dx} \phi_1 \, dx$$

GENERALLY NOT SYMMETRIC BECAUSE $\frac{d\phi_2}{dx} \rightarrow$ MODE $= \phi_1$

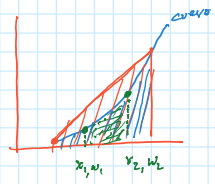
NUMERICAL INTEGRATION.

"QUADRATURE"

WHAT IS THE GOAL?



TRAPEZOID



$$A = \frac{f(x_1) + f(x_2)}{2}(x_2 - x_1)$$

$$A = w_1 f(x_1) + w_2 f(x_2)$$

$$A = \int f(x)\,dx \equiv \sum w_i f(x_i)$$

CAN I CHOOSE x_1, x_2 w_1, w_2 SUCH THAT $A_g = A_b$

4 UNKNOWNS, NEED 4 EQ

$f = x^0, x^1, x^2, x^3$

$$\int_{-1}^{1} x^0\,dx = w_1 \cancel{f(x_1)} + w_2 \cancel{f(x_2)} =$$

$$\int_{-1}^{1} x^1\,dx = w_1 \cancel{f(x_1)}\,x_1 + w_2 \cancel{f(x_2)}\,x_1 =$$

$$\int_{-1}^{1} x^2\,dx = w_1 x_1^2 + w_2 x_2^2 =$$

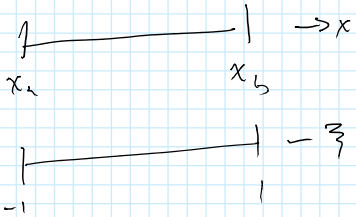
$$\int_{-1}^{1} x^3\,dx = w_1 x_1^3 + w_2 x_2^3 =$$

4 EQ
4 UNKNOWN

GAUSS QUADRATURE OF ORDER N

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{p} w_i f(\xi_i)$$

$p = 2$, $w_1 = w_2 = 1$, $-\xi_1 = \xi_2 = \frac{1}{\sqrt{3}}$



ISO PARAMETRIC MAPPING

$$\xi = a + bx$$

$$\xi(x_a) = a + b(x_a) = -1$$

$$\xi(x_b) = a + b(x_b) = 1$$

$$x = x_a \frac{1-\xi}{2} + x_b \frac{1+\xi}{2}$$

DON'T FORGET CHAIN RULE.

$$\int_{x_a}^{x_b} f(x)\,dx = \int_{-1}^{1} f(x(\xi)) \left(\frac{dx}{d\xi}\right) d\xi$$

JACOBIAN OF

$$= \frac{h_e}{2} \int_{-1}^{1} f(x(\xi))\,d\xi$$

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8.12 NEXT TIME 8.4

$$I = \int_a^b f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

RECAST IN NATURAL COORDS

$$= \int_{-1}^{1} f(x(\xi))J\,d\xi = \sum_{j=1}^{p} f(\xi_i)\,w_i$$

$\underline{w}\cdot\underline{v} = 0$ ORTHOGONAL VECT.

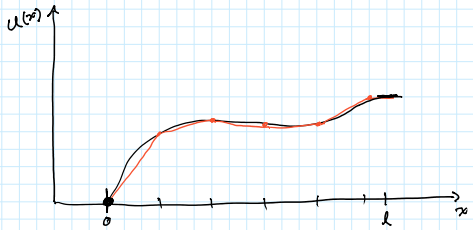
$\underline{v}_i\cdot\underline{v}_i = 1$

ORTHOGONAL POLYNOMIALS.

$$\int_{-1}^{1} L_i L_j\,d\xi = \begin{cases} 0 & i=j \\ 1 & i\neq j \end{cases}$$

GAUSS QUADRATURE IS EXACT FOR POLYNOMIALS OF DEGREE $2P-1$

2D FINITE ELEMENTS.



$$\frac{d}{dx}\left(a\frac{du}{dx}\right) + f = 0$$

$$u(0) = 0$$
$$a u'(L) = 0$$

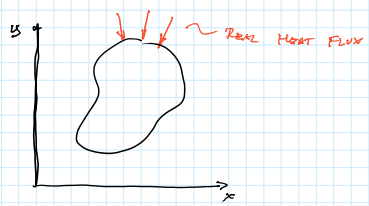
HOW TO APPROXIMATE ERROR?
LOTS OF LINES, AREA UNDER CURVE...

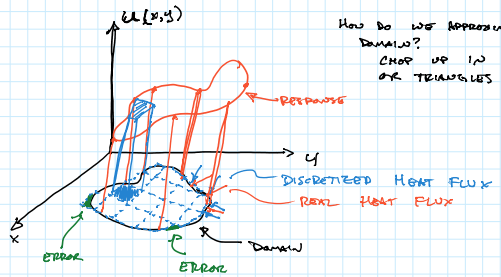
NO REAL DISCRETIZATION ERROR

2 LETTER OF REC FROM ACADEMIA

GRAD INFO REQUEST

2D FE DISCRETIZATION ERROR POPS UP





HOW DO WE APPROXIMATE THE DOMAIN?
CHOP UP IN RECTANGLES OR TRIANGLES

GOAL SOLVE PDE AT DISCRETE NODES

REDDY HAS LIST OF SINGLE DEGREE OF FREEDOM PROBLEMS.

$$\int_\Omega \left(-\frac{d}{dx}\left(a_{11}\frac{du}{dx} + a_{12}\frac{du}{dy}\right) - \frac{\partial}{dy}\left(a_{21}\frac{du}{dx} + a_{22}\frac{du}{dy}\right) + a_{00}u - f\right) w\,d\Omega = 0$$

ANISOTROPIC $a_{11} \neq a_{22} \neq a_{21} \neq a_{12}$

① MOVE TO LHS, INTEGRATE, MULTIPLY BY WEIGHT FCN

② INTEGRATE BY PARTS

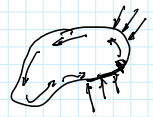
$$\Rightarrow 0 = \int_\Omega \underbrace{\left[\frac{dw}{dx}\left(a_{11}\frac{du}{dx} + a_{12}\frac{du}{dy}\right) + \frac{dw}{dy}\left(a_{21}\frac{du}{dx} + a_{22}\frac{du}{dy}\right) + a_{00}wu + wf\right]}_{R} dx\,dy$$

BOUNDARY TERM

$$-\oint_{\Gamma_e} w\underbrace{\left[n_x\left(a_{11}\frac{du}{dx} + a_{12}\frac{du}{dy}\right) + n_y\left(a_{21}\frac{du}{dx} + a_{22}\frac{du}{dy}\right)\right]}_{q_n} ds$$

q_n — NORMAL NORMAL TO BOUNDARY

INSTEAD OF GETTING B.C. ON POINTS WE'RE DOING A LINE INTEGRAL ON THE SURFACE.



COME UP WITH SOME GUESS FOR u

APPROX BY SMALLEST ORDER POLYNOMIAL

$$u^e = \sum_{j=1}^{n} u_j \phi_j(x,y)$$

GALERKIN METHOD

n EQUAL TO GET SYSTEM OF EQNS.

$$w^e = \sum_{j=1}^{n} w_i \phi_i(x,y)$$

$\sum_{j=1}^{n}$

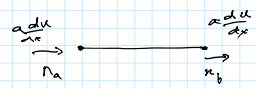
WHY DO ALL THIS WHEN THINGS ARE MORE REALISTIC?
SO WE CAN FORCE IT TO BE LOOKING LIKE AND THROW IT INTO A SYSTEM OF EQNS.

$$\sum_{j=1}^{n} K_{ij}^e u_j^e = f_i^e + Q_i^e$$

w 1-D

$$\int_{\Omega_e} -a\frac{d^2u}{dx^2}w\,dx + \ldots$$

$\downarrow$ I.B.P.



$$\int_{\Omega_e} a\frac{dw}{dx}\frac{du}{dx} + \ldots - \left[a w \frac{du}{dx}\right]_{x_a}^{x_b}$$

$$= a w \left.\frac{du}{dx}\right|_{x_a} + a w \left.\frac{du}{dx}\right|_{x_b}$$

$$- a n_a w \left.\frac{du}{dx}\right|_{x_a} + a n_b w \left.\frac{du}{dx}\right|_{x_b}$$

$\sum_{j=1}^{n}$ SO WE CAN FORCE IT TO BE LOOKING LINEAR AND THROW IT INTO A SYSTEM OF EQNS.

$$\sum_{j=1}^{n} K_{ij}^{e} u_j^{e} = f_i^{e} + Q_i^{e}$$

$$K_{ij}^{e} = \int_{\Omega_e} \left[\frac{d\phi_i}{dx}\left(a_{11}\frac{d\phi_j}{dx} + a_{12}\frac{d\phi_j}{dy}\right) + \frac{d\phi_i}{dy}\left(a_{21}\frac{d\phi_j}{dx} + a_{22}\frac{d\phi_j}{dy}\right) + a_{00}\phi_i\phi_j \right] dx\,dy$$

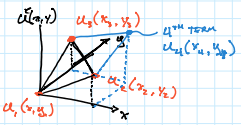
IT'S JUST POLYNOMIAL INTEGRATION

$$f_i^{e} = \int_{\Omega_e} f\,\phi_i^{e}\,dx\,dy \qquad Q_i^{e} = \oint_{\Gamma_e} q_n\,\phi_i^{e}\,ds$$

NOW WE NEED SHAPE FXN

$$u^{e}(x,y) = \sum u_j\,\phi_j(x,y)$$

SIMPLEST: POLYNOMIAL



$$= c_1 + c_2 x + c_3 y + c_4 xy$$

$$\left[\begin{array}{l} u^{e}(x_1,y_1) = c_1 + c_2 x_1 + c_3 y_1 + c_4 x_1 y_1 \\ u^{e}(x_2,y_2) = c_1 + c_2 x_2 + c_3 y_2 + c_4 x_2 y_2 \\ u^{e}(x_3,y_3) = c_1 + c_2 x_3 + c_3 y_3 + c_4 x_3 y_3 \\ u^{e}(x_4,y_4) = c_1 + c_2 x_4 + c_3 y_4 + c_4 x_4 y_4 \end{array}\right]$$

QUADRILATERALS

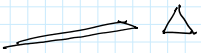
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} \Rightarrow \{u\} = [A]\{c\} \Rightarrow \{c\} = [A]^{-1}\{c\}$$

$$[A]^{-1} = \frac{1}{2A_e}\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix}$$

$$2A_e = \alpha_1 + \alpha_2 + \alpha_3$$

WHY IS BAD ELEMENT SHAPES GIVE BAD ANS.

IT'S ABOUT THE AREA!

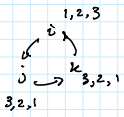


$$\alpha_i = x_j y_k - x_k y_j \qquad i \neq j \neq k$$

$$\beta_i = y_j - y_k$$

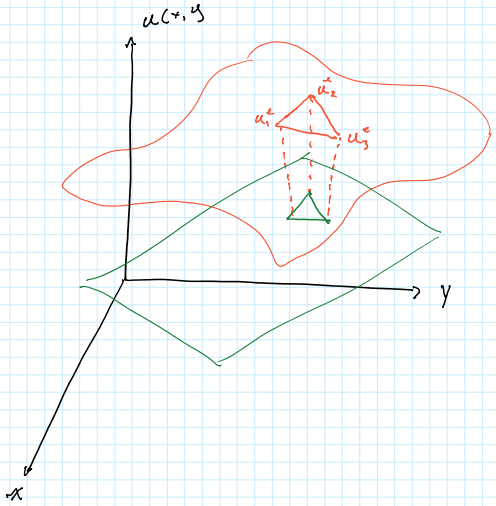
$$\gamma_i = -(x_j - x_k)$$

i, j, k PERMUTE IN THE NATURAL ORDER



$$\alpha_1 = x_2 y_3 - x_3 y_2$$

$$\phi_j^{e} = \frac{1}{2A_e}\left(\alpha_j^{e} + \beta_j^{e} x + \gamma_j^{e} y\right), \quad j = 1,2,3$$



$$\left.\begin{array}{l} u^{e}(x_r,y_r) = u_1\phi_1(x_r,y_r) \\ \quad + u_2\phi_2(x_r,y_r) \\ \quad + u_3\phi_3(x_r,y_r) \end{array}\right\}$$ THIS IS JUST 2D INTERPOLATION

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Thursday, January 28, 2016 3:40 PM

LEFT OFF LOOKING AT BOUNDARY INT $\int$

$$K^e_{ij} u_j = F^e_i + Q^e_i$$

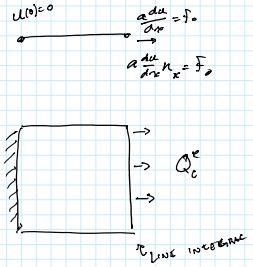
STIFFNESS — UNKNOWN DEGREES OF FREEDOM

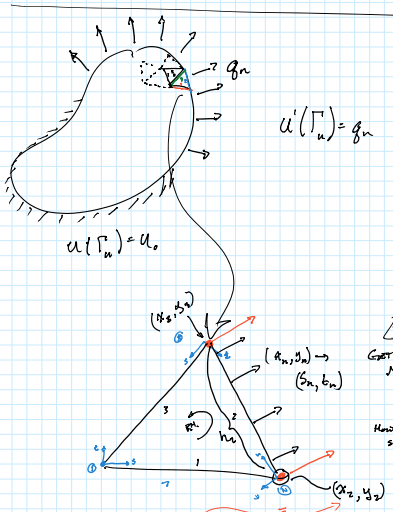
VOL INTEG. — BODY FORCE OR TRACTION

$$F_i = \int_\Omega f_0 \phi_i \, d\Omega = \text{APPLIED LOAD}$$

$$Q^e_i = \int_{\Gamma^e} q_n \phi_i \, ds$$

SURF INTEGRAL





$$u'(\Gamma_n) = q_n$$

LINE

$$T \frac{d^2u}{dx^2} + f_0 = 0$$

h_2

WE NEED TO GET TO PULLING FORCES AT NODES

$f_0 h_2/2$ $\quad f_0 h_2/2$

NOW 2D.

GET IT TO THE NODES

HOW? SETUP A COMPONENT BASIS

$$x = a_1 + b_1 s + c_1 t$$
$$y = a_2 + b_2 s + c_2 t$$

$x, y \to s, t$

Trying to get linear mapping

$t=0$, $s=0$: $x = x_1 = a_1$, $y = y_1 = a_2$

$s = h_2$, $t=0$: $x = x_2 + b_1 h_2 + 0 = x_3$, $y = y_2 + b_2 h_2 + 0 = y_3$

$x =$
$y =$

Don't need t in the end.

EDGE 1

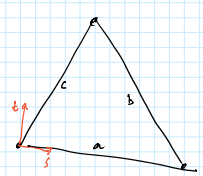
$$x(s) = x_1 + (x_2 - x_1)\frac{s}{h_1}$$
$$y(s) = y_1 + (y_2 - y_1)\frac{s}{h_1}$$

EDGE 2

$$x(s) = x_2 + (x_3 - x_2)\frac{s}{h_2}$$
$$y(s) = y_2 + (y_3 - y_2)\frac{s}{h_2}$$

EDGE 3

$$x(s) = x_3 + (x_1 - x_3)\frac{s}{h_3}$$
$$y(s) = y_3 + (y_1 - y_3)\frac{s}{h_3}$$



$s=0$ $t=0$
$s=a$, $t=0$
$s=c$, $t=b$

PBLD NOTES

$$Q^e_i = \int_{\Gamma_e} q_n \phi_i \, d\Gamma$$

SHAPE FCNS FOR TRIANGLE ELEMENT EVALUATED ON THIS SIDE.

$$= \int_{\text{Side 1}} q_n \phi_i^{(1)} ds_1 + \int_{\text{Side 1}} q_n \phi_i^{(2)} ds_2 + \int_{\text{Side 2}} q_n \phi_i^{(3)} ds_3$$

$$= \int_0^{h_1} + \int_0^{h_2} + \int_0^{h_3}$$

$$\phi_1 = \frac{1}{2A}\left(\alpha_1 + \beta_1\left(\left(1 - \frac{s}{a}\right)x_1 + \frac{s}{a}x_2\right) + \gamma_1\left(\left(1 - \frac{s}{a}\right)y_1 + \frac{s}{a}y_2\right)\right)$$

AROUND EQ 8.47

$a = h_1$ * MISTAKEN INTERCHANGE

$$\phi_1^{(1)} = 1 - \frac{s}{a} = 1 - \frac{s}{h_1}$$
$$\phi_2^{(1)} = \frac{s}{h_1}$$

$\phi_3^{(1)} = 0$ NODE 3 DOES NOT LIE ON LINE.

EDGE 1

$\phi_1 = 1 - s/h_1$
$\phi_2 = s/h_1$
$\phi_3 = 0$

EDGE 2

$\phi_1 = 0$
$\phi_2 = 1 - s/h_2$
$\phi_3 = s/h_2$

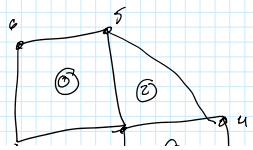
EDGE 3

$\phi_1 = s/h_3$
$\phi_2 = 0$
$\phi_3 = 1 - s/h_3$

vs

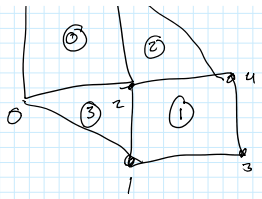
CONNECTIVITY TABLE

Element #	Nodes



CONNECTIVITY TABLE

Element #	NODES
0	0, 2, 5, 6
1	1, 3, 4, 2
2	2, 4, 5, -1
3	1, 2, 0, -1



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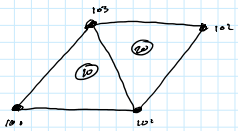
EXAMPLES OF HEAT TRANSFER



$$-\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) - f = 0$$

$$T_{\Gamma_u} = \bar{T} \qquad \left(\frac{\partial T}{\partial x} n_x + \frac{\partial T}{\partial y} n_y\right)_{\Gamma_q} = \bar{q}$$

4 NODES, 1 DOF EACH

4x4

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array}$$
$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix}$$

CONNECTIVITY TABLE W/ LABELS

EL	NODE #		
10	100,	101,	, 103
20	101,	102,	103

TRANSLATE THIS INTO ELEMENT & NODE #'S

1	1	2	3
2	2	4	3

HOW DO WE POPULATE MATRIX

EL_1

$$\begin{bmatrix} k^1_{11} & k^1_{12} & k^1_{13} \\ & k^1_{22} & k^1_{23} \\ & & k^1_{33} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^1_2 \\ u^1_3 \end{bmatrix} = \{f^{(1)}\}$$

REPLACE LOCAL W/ GLOBAL

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \xrightarrow{\text{GO TO } 4\times 4} \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} =$$

COLS CORRESPOND TO THE LOCAL NODE.

$$(4\times 3)(3\times 3)(3\times 4)$$

HAVE

THESE GUYS WORK W/ CONNECTIVITY TABLE

ROWS CORRESPOND WITH GLOBAL NODE #

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{L} \begin{bmatrix} & K' & \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{L^T}$$

$$[L][k^1][L]^T = [k^1_G]$$

$$\boxed{\sum_{e=1}^{n} [L^{e\times 3}][k^{e\times 3}][L^{e\times 3}]^T = K_G}$$

$$\sum_e L^{e\times 3} f^{e\times 3} = F_G$$

$$\sum_e L^{e\times 3} u^{e\times 3} = U_G$$

LOCAL GLOBAL

NOW LOOK AT L FOR ELEMENT 2

$[L^2]$

$$\begin{array}{ccc} K & U & f \end{array}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & k^2_{11} & & \\ 0 & & & \\ 0 & & & \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

3x3 MATRIX MATH

$$[L^2] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

PLATE SYSTEMS:

$$\int \left[-\frac{\partial}{\partial y}\left(\kappa \frac{\partial T}{\partial y}\right) - \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) - f(x,y)\right] w(x,y)\, dxdy = 0 \qquad \forall w$$

PLANE SYSTEMS:

$$\int_\Omega \left[-\frac{\partial}{\partial x}\left(K_x \frac{\partial T}{\partial x}\right) - \frac{\partial}{\partial y}\left(K_y \frac{\partial T}{\partial y}\right) - f(x,y) \right] w(x,y)\, dx\, dy = 0 \qquad \forall w$$

$$K_x \frac{\partial T}{\partial x} n_x + K_y \frac{\partial T}{\partial y} n_y + \underbrace{\beta (T - T_\infty)} = \hat{q}_n$$

$$0 = \underbrace{\int_\Omega \left(K_x \frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + K_y \frac{\partial T}{\partial y}\frac{\partial w}{\partial y} - w f \right) dx\, dy}_{\text{Same}} - \int_\Gamma w \left(K_x \frac{\partial T}{\partial x} n_x + K_y \frac{\partial T}{\partial y} n_y \right) ds$$

$$0 = \int_\Gamma w \left(q_n - \beta (T - T_\infty) \right) ds$$

$$T = \sum T_i \phi_i(x,y)$$

$$K_{ij}^e = \int_\Omega \left(K_x \frac{\partial T}{\partial x}\frac{\partial w}{\partial x} + K_y \frac{\partial T}{\partial y}\frac{\partial w}{\partial y} \right) dx\, dy \qquad F_i^e = \int_\Omega w f\, dx\, dy + \int \beta T_\infty w\, ds + \int w q_n\, ds$$

EXTRA CONVECTION TERM STIFF.

$$H_{ij}^e = \beta \int_\Gamma T w\, ds \qquad \left[K_{ij} + H_{ij} \right] T_j = F_i$$

$$T = \sum T_j \phi_j$$

$$w = \sum w_i \phi_i$$

$$K_{ij} = \int_\Omega K \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x} + K_y \frac{\partial \phi_i}{\partial y}\frac{\partial \phi_j}{\partial y}\, dx\, dy$$

$$H_{ij} = \beta \int_\Gamma \phi_i \phi_j\, ds$$

ACTUAL HW $\longrightarrow$ HW #1

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + c u = 0$$

YOU'LL HAVE AN EXTRA CONTRIBUTION ABOVE ? BEYOND WHAT'S IN BOOK

$$c \phi_i \phi_j$$

HOW TO USE THESE TERMS IN COMPUTATION.

```
k = lambda a, b: array([[        ]]
```

```
Kbc = K.copy()
```

```
u3 = u6 = 0
Kbc = copy
```

Ex 8.6a

*5 $u_1 = .37411345 5$

$u_2 = 0.25578172$

$u_3 = 0.2357271$

NORMAL WEIGHT TXN

$$g_i = \int_0^h g(x)\, \phi_i(x)\, dx$$

$$= \frac{g_0}{n} \int_0^h s \left\{ \begin{matrix} \frac{h-s}{h} \\ s/n \end{matrix} \right\} ds$$

SHAPE FCNS I'M INTERESTED IN!

2/4/16

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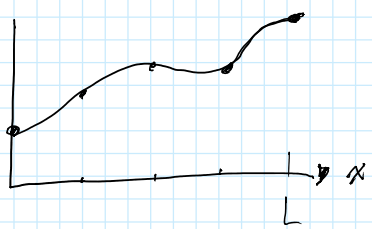
ODE

$$\frac{d}{dx}\left(a\frac{du}{dx}\right) + f = 0$$

BCS $u(0) = u_0,\ a u'(L) = t,\quad x \in (0, L)$



PRIMARY PURPOSE IS FINDING (u^e)

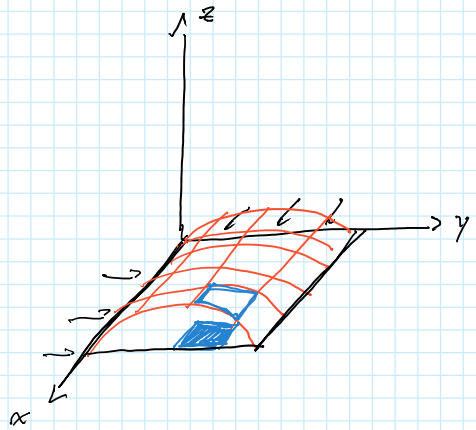
THEN STRESSES / STRAINS IS POST-PROCESSING.

NOW 2D

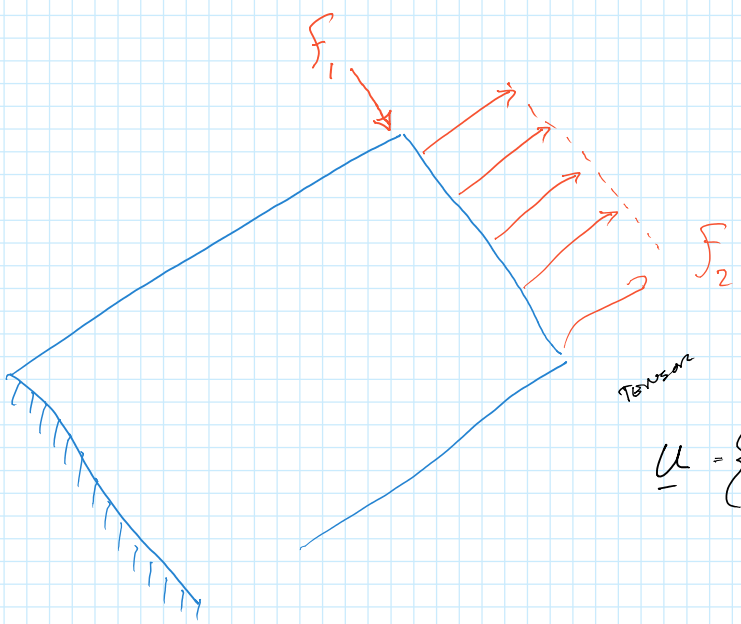
$$k\left(\frac{\partial^2 u}{\partial x} + \frac{\partial^2 u}{\partial y^2}\right) + f = 0$$

$$u(x_0, y_0) = u_0$$

$$\frac{\partial u}{\partial n} + \beta u = g$$



STRESS - DISPLACEMENT

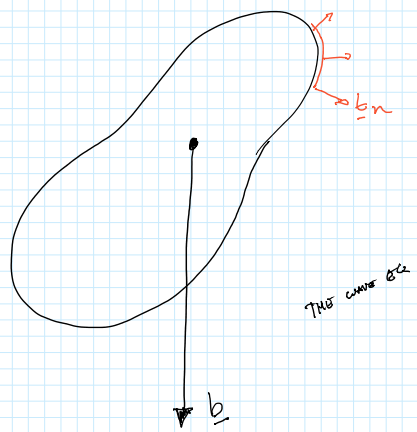


TENSOR NOTATION INDICIAL.

$$\underline{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = u_i$$

2/9/15

Tuesday, February 9, 2016 3:38 PM



THE CHOICE OF

$$\int \underline{b}\, dv + \oint \underline{t}_n\, d\Gamma = \underline{0}$$

$$\int \underline{b}\, dv + \oint \underline{\underline{\sigma}} \cdot \underline{n}\, d\Gamma = \underline{0}$$

$$\int \underline{b} + \nabla \cdot \underline{\underline{\sigma}}\, dv = \underline{0}$$

$$\underline{b} + \nabla \cdot \underline{\underline{\sigma}} = 0 \qquad \forall \text{ DOMAINS. IN STATIC EQ.}$$

$$\underline{b} + \nabla \cdot \underline{\underline{\sigma}} = \rho \underline{a}$$

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{b} = \underline{a}$$

$$\frac{\partial \sigma_{ij}}{\partial x_i} + b_j = a_j \qquad i, j = 1, 2, 3$$

$$\frac{\partial}{\partial x_i}\left(C_{ijkl}\, \varepsilon_{kl}\right) + b_j = 0$$

$$\frac{1}{2}\frac{\partial}{\partial x_i}\left[C_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)\right] + b_j = 0$$

$$\frac{1}{2} C_{ijkl}\left[\frac{\partial^2 u_k}{\partial x_i \partial x_l} + \frac{\partial^2 u_l}{\partial x_i \partial x_k}\right] + b_j = 0$$

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

$$\int \left[\nabla \cdot \underline{\underline{\sigma}} + \underline{b}\right] \underline{w}\, dV = 0 \qquad \forall \underline{w}$$

INTEGRATE BY PARTS

$$\int \nabla \cdot \underline{\underline{\sigma}} \cdot \underline{w} + \underline{b} \cdot \underline{w}\, dv = 0 \qquad \forall \underline{w}$$

$$\int \underline{\underline{\sigma}} : \nabla \underline{w} + \underline{b} \cdot \underline{w}\, dv - \int \underline{\underline{\sigma}} \cdot \underline{n} \cdot \underline{w}\, dA = 0$$

$$\int \sigma_{ij} \frac{\partial w_i}{\partial x_j}$$

WEAK FORM:

$$\frac{1}{2}\int \left[C_{ijkl}\underbrace{\left[\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right]}_{\text{STRAIN}} \frac{\partial w_i}{\partial x_j} + 2 b_i w_i\right] dv - \int \left(\sigma_{ij} n_j\right) w_i\, dA = 0 \qquad \forall w_i$$

NOW, PLUG IN APPROXIMATE SOLUTION (NOTATION MESS)

$$u_i = \sum_{a=1}^{n} u_i^a N^a(x) \qquad w_i = \sum_{a=1}^{n} w_i^a N^a(x)$$

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = \frac{1}{2}\sum_{a=1}^{n}\left(u_k^a \frac{\partial N^a}{\partial x_l} + u_l^a \frac{\partial N^a}{\partial x_k}\right)$$

LOTS OF KRR & KIR LATER: LOOK AT T.J.R. HUGHES.

LOOK BACK TO EASY CASES

$$K_{ij} + F_i = 0$$

$$K_{ij} = \int_{x_a}^{x_b} K \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\, dx \qquad \text{1-D CASE}$$

$$= \int_{x_a}^{x_b} \frac{\partial N_i}{\partial x} K \frac{\partial N_j}{\partial x}\, dx$$

$$= \int \overset{\text{col}}{\{N\}}\, k\, \overset{\text{row}}{[N]}\, dx \qquad \frac{\partial N}{\partial x} = B$$

$$= \int_{x_a}^{x_b} [N]^T k [N]\, dx = \int_{x_a}^{x_b} [B]^T k [B]\, dx$$

$$[K]\{u\} + \{F\} = 0 \qquad B_{aj} = \frac{\partial N^a}{\partial x_j}$$

$$[K] = \int [B]^T [C][B]\, dV$$

$$\{F\} = \int b_i N^a\, dv + \int t_i N^a\, dA$$

PLANE STRESS & PLANE STRAIN

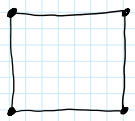
10.20's

PLANE STRESS

IN 2D

$$[K]^{2D} = \int [B]^T [C][B]\, \partial x\, \partial y\, \partial z = t \int [B]^T [C][B]\, \partial x\, \partial y$$

thickness



$$[C] = ? = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \quad - \; 3\times 3$$

$$\underline{\underline{\sigma}} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zz} \end{Bmatrix} \xrightarrow{\text{PLANE STRESS}} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{xy} \\ \sigma_{xy} \end{Bmatrix}$$

$$u_i = C_1 + C_2 x + C_3 y + C_4 xy$$

$$u_i = \sum_a u_i^a N^a(x,y)$$

$$\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix} \begin{matrix} \to x \text{ disp} \\ \to y \text{ disp} \end{matrix} = \begin{Bmatrix} \sum_{i=1}^{n} u_i N_i \\ \sum_{i=1}^{n} v_i N_i \end{Bmatrix} = \cancel{\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

n = number of nodes in element.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [\Psi]\{\Delta\}$$

$$= [\Psi]\{\Delta\}$$

$$\{\varepsilon\} = [B]\{\Delta\}$$

$$[B] = [T][\Psi] \qquad [T] = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial x & \partial/\partial y \end{bmatrix}$$

differential matrix operator.

$$\cancel{\partial/\partial x\, N_1 \quad 0 \quad 0}$$

$$= \begin{bmatrix} \partial/\partial x\, N_1 & 0 & \partial/\partial x\, N_2 & \cdots \\ 0 & \partial/\partial y\, N_1 & 0 & \cdots \\ \partial/\partial x\, N_1 & \partial/\partial y\, N_1 & & \cdots \end{bmatrix}$$

2/11/2016

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$$\nabla \cdot \underline{\underline{\sigma}} + \underline{b} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}$$

$$\downarrow$$

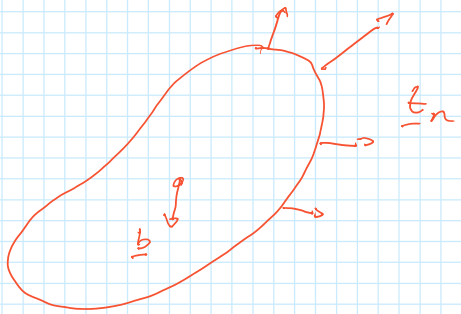
$$\underline{\sigma} = \underline{\underline{C}} : \underline{\varepsilon}$$

$$\{\underline{\sigma}\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}$$

$$\{\underline{\varepsilon}\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\varepsilon_{xy} = 2\varepsilon_{12}$$

$\uparrow$ engineering



$$\int \underline{b}\, dv + \int \underline{t}_n\, ds = \underline{0}$$

$$\hookrightarrow \int \underline{\underline{\sigma}} \cdot \underline{n}\, ds$$

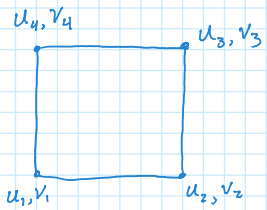
$$\int (\underline{b} + \underline{\underline{\sigma}} \cdot n)\, dv = \underline{0}$$

$$\cancel{\underline{b} + \underline{\underline{\sigma}} \cdot \underline{n} = 0}$$

$$\int (b + \nabla \cdot \underline{\underline{\sigma}}) \cdot w\, dv = 0$$

FOR 2D:

$$t \int (\underline{b} + \nabla \cdot \underline{\underline{\sigma}}) \cdot \underline{w}\, dx\, dy = 0$$



$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \sum u_i N_i \\ \sum v_i N_i \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

$$= [\Psi] \qquad \{\Delta\}$$

$$\{\varepsilon_{ij}\} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\{\varepsilon\} = [T] \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$[K^e]\{\Delta\} = \{F^e\}$$

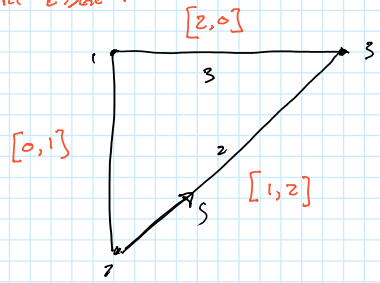
$$\Psi$$

$$[k^e] = h \int [B]^T [C] [B]\, dx\, dy$$

$$[B] = [T][\Psi]$$

$$\{F^e\} = h \int [\Psi]\{b\}\, dx\, dy + h \int [\Psi][q]\, ds$$

RIGHT TRI EDGE FORCE



$$\{F_{s12}\} = \begin{Bmatrix} 0 \\ 0 \\ \int_0^{h_{s12}} N_1 q_1\, ds \\ \int_0^{h_{s12}} N_1 q_2\, ds \\ \int_0^{h_{s12}} N_2 q_1\, ds \\ \int_0^{h_{s12}} N_2 q_2\, ds \end{Bmatrix}$$

$$\{F_{s13}\} = \begin{Bmatrix} \int q_1 N_2\, ds \\ \int q_2 N_2\, ds \\ 0 \end{Bmatrix} \qquad N = \begin{Bmatrix} 1 - \frac{x}{h} \\ \frac{x}{h} \end{Bmatrix}$$

$$\left\{ f_{s_3} \right\} = \begin{Bmatrix} \int q_z N_z \, ds \\ \vdots \\ \int q_1 N_1 \, ds \\ \int q_2 N_1 \, ds \end{Bmatrix}$$

2/16/16

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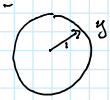
$$K^e_{ij} = \int N_i' N_j' \, dA$$

$$K^e_{ij} = N_i' N_j' \int dA = N_i' N_j' A$$

ISOPARAMETRIC REPRESENTATION

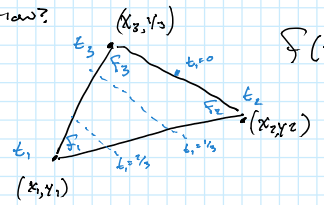
WHAT IS A PARAMETRIC REPRESENTATION?



$y = \sqrt{1-x^2}$

$x = \cos\theta$

$y = \sin\theta$

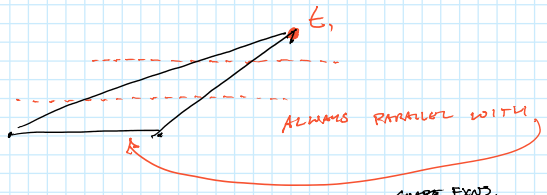


$$f(x;y) = a_0 + a_1 x + a_2 y$$

$$f(t_1, t_2, t_3) = f_1 t_1 + f_2 t_2 + f_3 t_3 = f_1 N_1 + f_2 N_2 + f_3 N_3$$

ANOTHER WAY TO DO SHAPE FCNS

TRIANGULAR COORDINATES t_i MOST TEXTS USE ξ "EXCEL"



SHAPE FCNS.

THIS IS GENERAL.

$$u_x = u_{1x} N_1 + u_{2x} N_2 + u_{3x} N_3$$

GEOM

DISP

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ u_{1x} & u_{2x} & u_{3x} \\ u_{1y} & u_{2y} & u_{3y} \\ h_1 & h_2 & h_3 \\ T_1 & T_2 & T_3 \end{bmatrix} \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \\ h \\ T \end{Bmatrix}$$

ISO PARAMETRIC

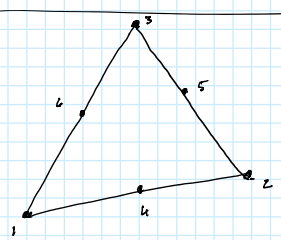
SAME

APPROXIMATION USED FOR GEOM, DISP, ETC...

$$1 = \sum_i N_i \qquad x = \sum_i x_i N_i \qquad y = \sum_i y_i N_i$$

ISOPARAMETRIC

$$u_x = \sum_i u_{xi} N_i \qquad u_y = \sum_i u_{yi} N_i$$

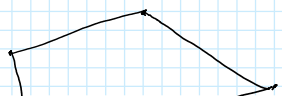


Hey

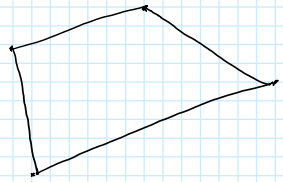
$$\begin{Bmatrix} 1 \\ x \\ y \\ u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ u_{x1} & \cdots & & & & \\ u_{y1} & \cdots & & & & \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{Bmatrix}$$

$$N_1 = t_1(2t_1 - 1)$$

$$N_2 = t_2(2t_2 - 1)$$

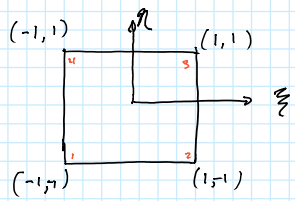


$$\begin{matrix} 1 & 1 & 1 & 1 & 1 & N_1 \\ x & x_1 & x_2 & x_3 & x_4 & N_2 \end{matrix}$$



$$\begin{matrix} 1 \\ x \\ y \\ u_{x_1} \\ u_x \end{matrix} = \begin{matrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ & & \vdots & \\ & & & \end{matrix} \begin{matrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{matrix}$$

ANOTHER WAY: NATURAL COORDS.



NEED A MAPPING LIKE 1-D

NOT A LINEAR MAPPING

$C_1 + C_2 x + C_3 y + C_{34} xy$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$

WHY -1 TO 1? GAUSS QUADRATURE!

$$N_i = \frac{(x-x_1)(x-x_2)\cdots(x-x_j)\cdots(x-x_n)}{C_i}$$

$n \neq i$

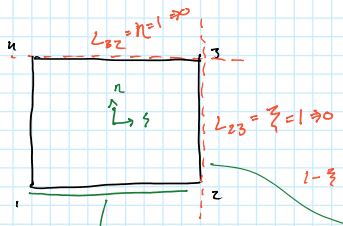
$j \neq i$

LAGRANGE INTERPOLATED POLYNOMIALS

N_i

$N_1 = C_1(\xi-1)(\eta-1)$

$N_i = C_i \prod L_j \quad j=1\ldots n, \quad i \neq j$



$N_1 = C_1 L_{23} L_{34}$

$L_{34} = 1-\eta$

$L_{23} = 1-\xi$

$1-\xi \quad = L_{23} = 0$

$0 = L_{23} = 1-\xi$

WHAT IS THE EQ OF THIS LINE

$\eta = -1$ OR $\eta + 1 = 0$

NANSON'S RELATION

$$\int N_x' N_y' \, dx\, dy$$

$$\int N_\xi' N_\xi' \frac{\partial x}{\partial \xi} \partial \xi \frac{\partial y}{\partial \xi} \partial \xi$$

$$\det(J) \int N_\xi' N_\xi' \, \partial\xi\, \partial\eta$$

$$f_i = \int_\Omega S N_i \, d\Omega + \int_\Gamma (\qquad) \, d\Gamma$$

$$A = \begin{matrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \end{matrix}$$

$$N = A^{-1} P$$

$$P = 1, x, y$$

QUAD STRENGTH
LOOKS LIKE TRI STRIP

Feb 18, 2016

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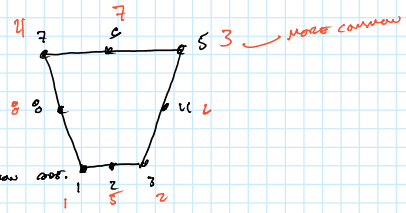
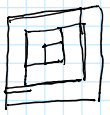
A05 DUE NEXT THUR

EXERCISE 03

- PERFORM COMPUTATIONAL PART IN THE CODE OF YOUR CHOICE.
- COMPARE MY ANSWER TO THE PROBLEM 005.
- WHY ARE THEY DIFFERENT.





Keywords

- FULL INTEGRATION
- B-BAR METHOD
- SELECTIVELY REDUCED INTEGRATION

→ DESCRIBE WHAT PROGRAM CODE IS DOING

MEET IN CADE LAB NEXT TUE.

$$K^e_{ij} = \int \frac{\partial N^e_i}{\partial x_k} D_{km} \frac{\partial N^e_m}{\partial x_j} \, dV$$

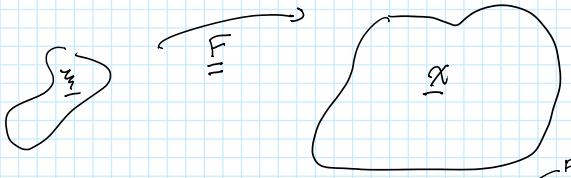
SAME THING

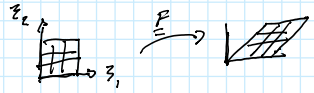
$$[K^e] = \int_{\Omega^e} [B]^T [D] [B] \, dV$$

TRUE FOR EVERY CONTINUUM ELEMENT.

$$x = \sum x_i N_i$$

RECALL FROM CONTINUUM MECHANICS





DEFORMATION GRADIENT.

FCN OF

$$\underline{x} = \underline{x}(\underline{\xi})$$

$$\partial \underline{x} = \frac{\partial \underline{x}}{\partial \underline{\xi}} \cdot \partial \underline{\xi}$$

$$= \underline{\underline{F}} \cdot \partial \underline{\xi}$$

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{\xi}} = \underline{\underline{J}}^T$$

$$\underline{x} = \begin{Bmatrix} x \\ y \end{Bmatrix} \qquad \underline{\xi} = \begin{Bmatrix} \xi \\ \eta \end{Bmatrix}$$

$$F_{ij} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix} = J^T \qquad F_{ij} = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} \end{bmatrix} = \frac{\partial x_i}{\partial X_j}$$

$$\partial \underline{\xi} = \underline{\underline{F}}^{-1} \cdot \partial \underline{x} = \mathrm{adj}(\underline{\underline{F}}) \cdot \partial \underline{x} = \frac{1}{J} \underline{\underline{F}}^{cT} \cdot \partial \underline{x}$$

$$= \frac{\partial \underline{\xi}}{\partial \underline{x}} \cdot \partial \underline{x}$$

COFACTOR TRANSPOSE

$$J = \det(\underline{\underline{F}}) \leftarrow \text{SCALAR!}$$

$$\boxed{\begin{aligned} \underline{\underline{F}} &= \frac{\partial \underline{x}}{\partial \underline{\xi}} = \underline{\underline{J}}^T \\ J &= \det(\underline{\underline{F}}) \\ \underline{\underline{F}}^{-1} &= \frac{1}{J} \underline{\underline{F}}^{cT} \\ \partial \underline{x} &= \underline{\underline{F}} \cdot \partial \underline{\xi} \end{aligned}}$$

$$\frac{\partial N^e_i}{\partial \underline{x}} = \frac{\partial N^e_i}{\partial \underline{\xi}} \cdot \frac{\partial \underline{\xi}}{\partial \underline{x}}$$

$$= \frac{\partial N^e_i}{\partial \underline{\xi}} \cdot \underline{\underline{F}}^{-1}$$

$$= \underline{\underline{F}}^{-T} \cdot \frac{\partial N_i}{\partial \underline{x}} = \boxed{\underline{\underline{J}}^{-1} \cdot \frac{\partial N_i}{\partial \underline{\xi}} = \frac{\partial N^e_i}{\partial \underline{x}}}$$

$$\partial \underline{x} = \underline{\underline{F}} \cdot \partial \underline{\xi}$$

$$K_{ij}^e = \int \frac{\partial N_i^e}{\partial x_k} D_{km} \frac{\partial N_m^e}{\partial x_j} \, \partial V$$

$$\underline{\xi} = \begin{Bmatrix} \xi \\ \eta \end{Bmatrix} \qquad \underline{x} = \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$\underline{\underline{F}}^{-1} = \frac{\partial \underline{\xi}}{\partial \underline{x}}$$

$$\underline{\underline{F}} = [P][X]$$

$$\partial \underline{\xi} =$$

$$= \underline{\underline{F}}^{-T} \cdot \frac{\partial N_i}{\partial \underline{\xi}} = \boxed{\underline{\underline{J}}^{-1} \cdot \frac{\partial N_i}{\partial \underline{\xi}} = \frac{\partial N_i^e}{\partial \underline{x}}}$$

THIS DOESNT CHANGE W/R/T GEOM.

THIS DOES.

$$F_{ij} = \frac{\partial x^i}{\partial \xi_j} = \frac{\partial}{\partial \xi_j} \sum_m x_m^i N_m = x_m^i \underbrace{\frac{\partial}{\partial \xi_j} N_m}_{\text{CAN DO THIS NO PROB}}$$

$$x^i = \sum x_j N_j$$

KNOWN NODAL COORD

$$= \underbrace{\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ y_1 & y_2 & y_3 & \cdots & y_n \end{bmatrix}}_{[X]^T} \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial \xi} & \frac{\partial N_1}{\partial \eta} \\ \frac{\partial N_2}{\partial \xi} & \frac{\partial N_2}{\partial \eta} \\ \vdots & \vdots \\ \frac{\partial N_n}{\partial \xi} & \frac{\partial N_n}{\partial \eta} \end{bmatrix}}_{[P]^T} = [P][X]$$

THEN THE GRAND WIZARD:

$$K_{ij}^e = \int f \, d\nu$$

$$\partial \nu = J \, \partial V$$

PHYSICAL COORD — NATURAL COORD

$$= h \int f J \, \partial \xi \, \partial \eta$$

SUBST ONE JACOBIAN!

ALL FROM -1 TO 1 IN NATURAL COORD
USE GAUSS QUADRATURE!

$$K_{ij}^e = \int_{-1}^{1} \int_{-1}^{1} f J \, \partial \xi \, \partial \eta$$

$$= \sum_{m=1}^{p} \int_{-1}^{1} f[\xi_j, gp_1] J \, \partial \xi \, \omega_m$$

$$= \sum_{m=1}^{p} \sum_{n=1}^{p} f(gp_n, gp_m) J \, \omega_n \omega_m$$

gp = gauss point

LINEAR 2 GP
EIGHT NODE → 3 GP., $p=3$

FOR TRIANGLES

$$K_{ij}^e = \sum_{m=1}^{p} f(gp_m) J \, \omega_m$$

SPECIAL GAUSS POINT FOR TRIANGLES.

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MIDTERM 1 WK FROM TUE.

3/8/16

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NONLINEAR MATERIALS $\nabla \cdot \underline{\underline{\sigma}} + b = 0$

$$\frac{\partial}{\partial x_i} \sigma_{ji} + b_j = 0$$

1 MULTIPLY BY WEIGHT FUN
2 INTEGRATE OVER DOMAIN

$$\int \underbrace{\left(\frac{\partial}{\partial x_i}\sigma_{ji} + b_j\right)}_{u \to \tilde{u}} w_j \, dV = 0 \quad , \forall w_i \in \Gamma_u$$

GOOD FORM FINITE ELEMENT EQN

$$\int R_j w_j \, dV$$

$$\int \left(\sigma_{ij} \frac{\partial w_i}{\partial x_j} - w_j b_j\right) dV - \int t_j w_j \, dA = 0 \qquad w_i = \sum_{n=1}^{N} N^n(x) w_i = [P]\{w\}$$

$$\int \left(\sigma_{ij} \frac{\partial N^n}{\partial x_j} - N^n b_j\right) dV \cdot w_i^n - \int t_j N^n w_j^n \, dA$$

SUPPOSE NOW WE DON'T KNOW STRESS IS LINEAR WITH STRAIN

STILL KEEP SMALL STRAIN ASSUMPTION

$$\sigma_{ij} = \sigma_{ij}\left[\varepsilon_{kl}(u_i^n)\right]$$

$$\sigma_{ij} = D_{ijkl}\,\varepsilon_{kl}$$

$$\varepsilon_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = \frac{1}{2}\left(\frac{\partial N^n}{\partial x_l} u_k^n + \frac{\partial N^n}{\partial x_k} u_l^n\right)$$

$$[K]\{u\} = \{F\}$$

$$[K^e] = \int [B]^T [D^e][B] \, \cancel{dV} \; dA$$

$$\{F^e\} = \int [P^e]\{t^e\} \, \cancel{dA} \; dS + \int [P^e]\{b^e\} \, \cancel{dV} \; dA$$

2D CASE

$$\sum_i \int_{S_i} [P]_{S_i} \{t\} \, \partial S_i$$

$$\left[\int \left(\sigma_{ij}\left[\varepsilon_{kl}(u_i^n)\right] \frac{\partial N^n}{\partial x_j} - b_i N^n\right) dV - \int t_i N^n \, dA\right] w_i^n = 0$$

USE NEWTON SOLVER $\quad v_i^n \leftarrow v_i^n + \partial v_i^n$

$$\boxed{f(x + dx) = f(x) + \frac{\partial f}{\partial x} dx} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dx)^2 + \dots$$

$$\int \sigma_{ij}\left[\varepsilon_{kl}(v_i^b + dv_i^b)\right] \frac{\partial N^n}{\partial x_j} dV - \underbrace{\int b_i N^n \, dV - \int t_i N^n \, dA}_{\text{GOING ALONG FOR RIDE}} = 0$$

$$\int \left[\sigma_{ij}\left[\varepsilon_{kl}(v_i^b)\right] + \frac{\partial \sigma_{ij}}{\partial \varepsilon_{lm}} \frac{\partial \varepsilon_{lm}}{\partial v_k^b} \partial v_k^b\right] \frac{\partial N^n}{\partial x_j} dV - \int b_i N^n dV - \int t_i N^n dA = 0$$

SMALL STRAIN APPROX

$$\varepsilon_{lm} = \frac{1}{2}\sum_{a=1}^{N}\left(\frac{\partial N^a}{\partial x_m} v_l^a + \frac{\partial N^a}{\partial x_l} v_m^a\right)$$

$$\frac{\partial \varepsilon_{lm}}{\partial v_k^b} = \frac{1}{2}\sum_{a=1}^{N}\left(\frac{\partial N^a}{\partial x_m}\frac{\partial v_l^a}{\partial v_k^b} + \frac{\partial N^a}{\partial x_l}\frac{\partial v_m^a}{\partial v_k^b}\right)$$

$$= \frac{1}{2}\sum_{a=1}^{N}\left(\frac{\partial N^a}{\partial x_m}\delta_{lk}\delta_{ab} + \frac{\partial N^a}{\partial x_l}\delta_{ab}\delta_{mk}\right)$$

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{lm}} = \frac{\partial \sigma_{ji}}{\partial \varepsilon_{ml}}$$

A BUNCH OF K.R.R.'S LATER

WHAT IS DIFF BETWEEN DIRAC DELTA & KRONECKER DELTA

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{km}} \frac{\partial \varepsilon_{km}}{\partial v_k^b} = \frac{\partial \sigma_{ij}}{\partial \varepsilon_{km}} \frac{\partial N^b}{\partial x_m}$$

$$\int \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} \frac{\partial N^a}{\partial x_j} \frac{\partial N^b}{\partial x_l} dV \cdot dv_k^b + \int \sigma_{ij} \frac{\partial N^a}{\partial x_j} dV - \int b_i N^a dV - \int t_i N^a dA = 0$$

$$\frac{\partial N^a}{\partial x_i} \Rightarrow [B] \qquad \frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = D_{ijkl}$$

"STIFFNESS"

"CONSISTENT JACOBIAN"

$$N^a \Rightarrow [P]$$

$$[K]\{dv\} = \{F\} - \{R\}$$

THE RESIDUAL

dV IS JUST PERTURBATION OF u

$$[K] = \int [B]^T [D][B]$$

$$\{F\} = \int [P][B]\, dV + \int [P]\{t\}\, dA$$

$$\{R\} = \int \{\sigma\}[B]\, dV$$

HOW TO SOLVE:

1) INITIAL GUESS FOR v_i^a
2) FOR CURRENT GUESS, COMPUTE $[K], \{R\}, \{F\}$
3) APPLY B.C.'S
4) SOLVE THE GLOBAL SYSTEM $[K^{**}]\{dv\} = \{F^*\} - \{R^*\}$ FOR $\{dv\}$
5) LET $v_i^a \leftarrow v_i^a + dv_i^a$
6) CHECK CONVERGENCE

$$e_1 = \frac{1}{\sqrt{n}} \left\| (\{R\} - \{F\}) \right\|$$

$$e_2 = \left\| \{dv\} \right\|$$

$$e = \sqrt{e_1^2 + e_2^2}$$

Midterm

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~~UP TO P~~ 2 PARTS CLOSED BOOK

1/3 CONCEPTUAL

2/3 PARAL 2/3 TOTAL

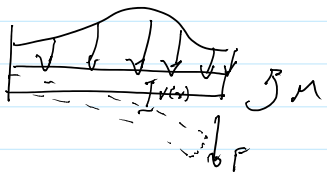
1 1-D PROBLEM

1 2-D PROBLEM PLANE STRESS/STRAIN

HEAT TRANSFER*

NEUMAN B.C.'S DIRICHLET

$$\frac{\partial}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right) + q(x) = 0$$



$$\int EI \frac{d^2N}{dx_i^2}\frac{d^2v}{dx_j^2}\,dx\, v_j + \int q(x) N_i\, dx$$

$$-\underbrace{EI\,w}_{\boxed{V}}\,\frac{d^3v}{dx^3} - EI\frac{\partial w}{\partial x}\frac{\partial^2 v}{\partial x^2}$$

PRIMARY VAR

LOOK FOR

SECONDARY VAR

- ALWAYS HAS PHYSICAL MEANING

WHAT ARE ESSENTIAL/PRIMARY/DIRICHLE

NEUMANN/NATURAL/SECONDARY

LOOK FOR W

REPLACE W W/ PRIMARY VARIABLE

$$-EI\,\boxed{\frac{\partial w}{\partial x}}\,\frac{\partial^2 v}{\partial x^2}$$

SECONDARY NATURAL

$$\frac{\partial v}{\partial x}$$

PRIMARY

IBP

$$\int u v'\, dx = uv - \int du\, v\, dx$$

3/22/2016

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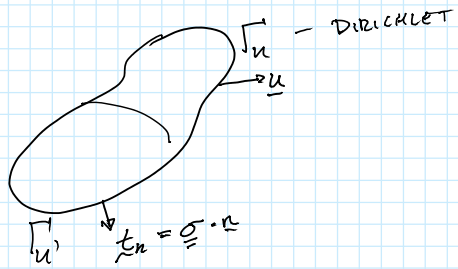
10 MORE CLASS PERIODS

PROJECT IS A BIG DEAL.

NOW ON 2D → 3D

CHAPTER 10 - VARIATIONAL METHODS

3D ELASTOSTATICS



NEUMANN

STRAIN ENERGY

$$e = \sigma_{ij}\varepsilon_{ij} = \underline{\underline{\sigma}} : \underline{\underline{\varepsilon}}$$

INNER PRODUCT $\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \sigma_{ij}\varepsilon_{ij}$

MATRIX PRODUCT $\underline{\underline{\sigma}} \cdot \underline{\underline{\varepsilon}} = \sigma_{ij} \cdot \varepsilon_{jk}$

DYADIC PRODUCT $\underline{\underline{\sigma}}\,\underline{\underline{\varepsilon}} = \sigma_{ij}\varepsilon_{kl}$

NOTATIONS

$\underline{u}$ disp

$\underline{b}$ BOD FORCE

$\underline{t}$ TRACTION

$$\underline{\underline{\underline{\underline{C}}}} = C_{ijkl}$$

GRADIENT

$$\underline{\nabla}\phi = \phi_{,i} = \frac{\partial \phi}{\partial x_i} = \begin{Bmatrix} \partial\phi/\partial x_1 \\ \partial\phi/\partial x_2 \\ \partial\phi/\partial x_3 \end{Bmatrix}$$

DIV

$$\nabla \cdot \underline{u} = u_{i,i} = \frac{\partial u_i}{\partial x_i} = \text{SCALAR}$$

$$\nabla \cdot \underline{\underline{\sigma}} = \text{div}\,\underline{\underline{\sigma}}$$

$$= \sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j} = \begin{Bmatrix} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} \\ \vdots \\ + \frac{\partial \sigma_{33}}{\partial x_3} \end{Bmatrix}$$

* WATCH OFF DIAG TERM

$$\underline{\underline{\sigma}} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}$$

FELIPA USES ANOTHER CONVENTION.

$$u = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}[\sigma]^T[\varepsilon]$$

$$= \frac{1}{2}\left[\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + 2\sigma_{31}\varepsilon_{31} + 2\sigma_{32}\varepsilon_{32} + 2\sigma_{12}\varepsilon_{12}\right]$$

2 BECAUSE $\underline{\underline{\sigma}}$ & $\underline{\underline{\varepsilon}}$ IS SYM

$$\underline{\underline{\varepsilon}} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \end{Bmatrix}$$

USING VECTOR NOTATION

$$\underline{\underline{\varepsilon}} = \begin{Bmatrix} \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix}$$

2 BECAUSE $\underline{\underline{\sigma}}$ & $\underline{\underline{\varepsilon}}$ IS SYM

USING VECTOR NOTATION

$$\frac{1}{2}\underline{\underline{\sigma}} : \underline{\underline{\varepsilon}} = \frac{1}{2}\left[\sigma_{11}\varepsilon_{11} + \sigma_{22}\varepsilon_{22} + \sigma_{33}\varepsilon_{33} + \sigma_{12}\varepsilon_{12} + \sigma_{23}\varepsilon_{23} + \sigma_{13}\varepsilon_{13}\right]$$

OFTEN STRAIN HAS 2 HERE

VOIGHT NOTATION

$$= \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}$$

ENGINEERING SHEAR STRAIN

CONSTITUTIVE RELATIONSHIP

$$\underline{\underline{\sigma}} = f(\underline{u}) \quad \text{OR } \underline{\underline{F}} \text{ OR } \underline{\underline{\varepsilon}} \ldots$$

DEFORMATION

$$\underline{\underline{\varepsilon}} = \begin{matrix} \varepsilon_{11} & \frac{\partial u_1}{\partial x_1} \\ \varepsilon_{22} & \partial u_2/\partial x_2 \\ \varepsilon_{33} & \partial u_3/\partial x_3 \\ \gamma_{12} & \neq \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \gamma_{23} & \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \gamma_{13} & \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \end{matrix}$$

$$\begin{bmatrix} \partial/\partial x_1 & 0 & 0 \\ 0 & \partial/\partial x_2 & 0 \\ 0 & 0 & \partial/\partial x_3 \\ \partial/\partial x_2 & \frac{\partial}{\partial x_1} & 0 \\ 0 & \partial/\partial x_3 & \frac{\partial}{\partial x_2} \\ \partial/\partial x_3 & 0 & \partial/\partial x_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \overset{\text{REDDY}}{[T]} \{u\}$$

$$\boxed{\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\nabla}\underline{u} + (\underline{\nabla}\underline{u})^T\right)}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial x_j}{\partial x_i}\right)$$

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} + \underline{b} = 0$$

$$[T]^T\{\sigma\} + \{b\} = 0$$

$$\Gamma_u \quad - \underline{u} = \underline{u}_0$$

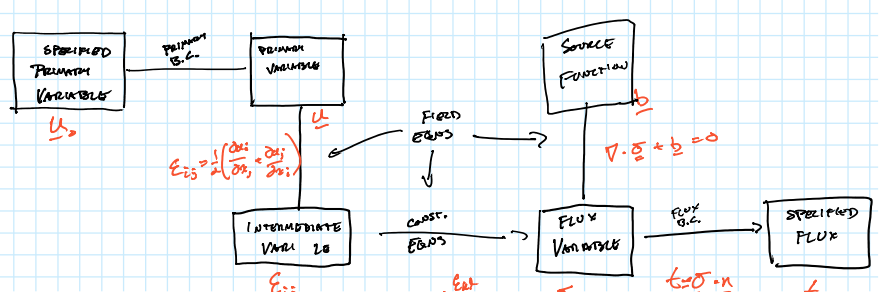
$$\Gamma_{u'} \quad - \underline{t} = \sigma_n = \underline{\underline{\sigma}} \cdot \underline{n}$$

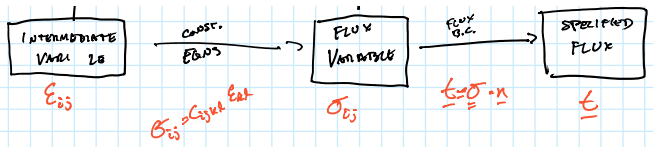
$$\underline{t} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \begin{bmatrix} n_1 & 0 & 0 & n_1 & n_2 & 0 \\ 0 & n_2 & 0 & 0 & n_2 & n_3 \\ 0 & 0 & n_3 & n_1 & 0 & n_3 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma \end{Bmatrix}$$

$$= [P_n]\{\sigma\}$$

VARIATIONAL METHODS — MINIMIZING ENERGY

~~TAUTY DIAGRAM~~ TONTI DIAGRAM





EXERCISE 10.4 10.5 DUE NEXT THUR.

EACH ONE- LINERS!

PROJECT

FEMIX (THERMAL + SOLID)

3/24/16

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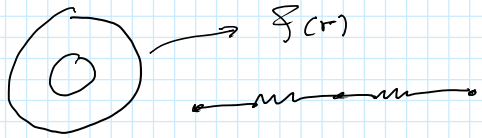
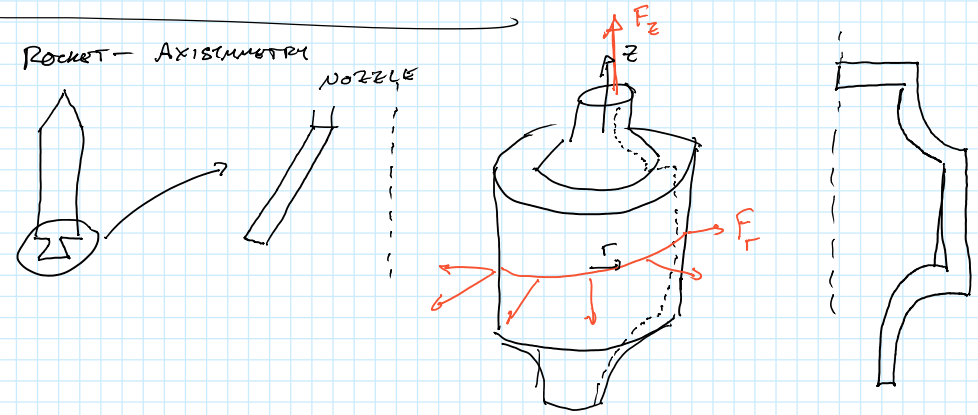
PROJECT PROPOSALS DUE TODAY
1-2 SENTENCES.

REDUCED DIMENSION
CYLINDRICAL COORDS



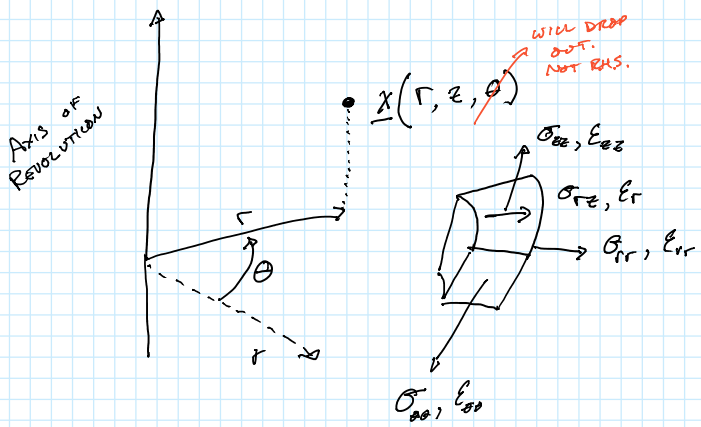
ROCKET – AXISYMMETRY



CRITERIA

- GEOM IS AXISYMMETRIC
- LOADING IS "

DEFINE TERMS



$$\underline{\underline{\varepsilon}} = \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{zz} \\ \varepsilon_{\theta\theta} \\ \gamma_{rz} \end{Bmatrix}$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_{rr} & \sigma_{rz} & \cancel{\sigma_{r\theta}}^{\,0} \\ \sigma_{rz} & \sigma_{zz} & \cancel{\sigma_{z\theta}}^{\,0} \\ \cancel{\sigma_{r\theta}}^{\,0} & \cancel{\sigma_{z\theta}}^{\,0} & \sigma_{\theta\theta} \end{bmatrix} = \begin{Bmatrix} \sigma_{rr} \\ \sigma_{zz} \\ \sigma_{\theta\theta} \\ \sigma_{rz} \end{Bmatrix}$$

WE SEEK A SOLUTION TO:

$$\sum_e \left[\int_{V_e} (\nabla \cdot \underline{\underline{\sigma}} + \rho b) \cdot \underline{w} \, dv \right] = 0 \quad \forall w$$

BECOMES $\left[\frac{1}{r}\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial z} \right]$

I.B.P. & ASSUME CONSTANT σ

$$\sum_e \left[\int_{S_e} \underline{w} \cdot \underline{\underline{\sigma}} \cdot \underline{n} \, dS - \int_{A_e} \underline{\underline{\sigma}} : \nabla \underline{w} \, dA + \int_{A_e} \rho \underline{b} \cdot \underline{w} \, dA \right] = 0 \quad \forall \underline{w}$$

$K_{ij} u_j = F_i$ $\qquad \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}} : \underline{\underline{\varepsilon}}, \quad \underline{\underline{\varepsilon}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)$

$$u = \sum u_j N_j$$

COORD SYS.

$$\hat{\underline{r}} = \begin{Bmatrix} r \\ z \\ \theta \end{Bmatrix}$$

$$\hat{\underline{r}} = \cos\theta\, \hat{\underline{i}} + \sin\theta\, \hat{\underline{j}}$$

$$\underline{\theta} = -r\sin\theta\, \hat{\underline{i}} + r\cos\theta\, \hat{\underline{j}}$$

$$\underline{z} = \hat{\underline{k}}$$

$$\partial V = 2\pi r\, dA$$

$$\sum_e \left(\int_{V_e} (\nabla\cdot\underline{\underline{\sigma}} + \rho \underline{b})\cdot \underline{\omega}\; 2\pi r\, dA \right) = 0 \quad \forall \omega$$

$\boxed{\text{You can get weird behavior at center because of } \underline{r}.}$

$$\sum_e \left[\underbrace{\frac{1}{2\pi} \int_{V_e} \frac{1}{r} \nabla\cdot \underline{\underline{\sigma}}\cdot \underline{\underline{\omega}}\, dV}_{\text{IBP}} + \int \rho \underline{b}\cdot \underline{\omega}\, dA \right] = 0 \quad \forall \omega$$

NOT THIS

AREA WEIGHTED RESIDUAL

"PETROV-GALERKIN"

KEY STEP!

$$\sum_e \left[\int_{S_e} \underbrace{\underline{\omega}\cdot \underline{\underline{\sigma}}\cdot \underline{n}}_{\underline{t}_n}\, dS - \int_{A_e} \underline{\underline{\sigma}} : \nabla\left(\frac{1}{r}\underline{\omega}\right) r\, dA + \int_{A_e} \rho \underline{b}\cdot \underline{\omega}\, dA \right] = 0 \quad \forall \underline{\omega}$$

$$\nabla\left(\frac{1}{r}\underline{\omega}\right) = \frac{1}{r}\frac{\partial \omega_\alpha}{\partial r_\beta} - \underbrace{\frac{1}{r}\Gamma^{\gamma}_{\alpha\beta}\, \omega_\gamma}_{\text{CHRISTOFFEL SYMBOL}} - \frac{1}{r^2}\delta_{1\beta}\, \omega_\alpha$$

REMEMBER CYL COORDS

α, β ARE TAKING PLACE OF i, j AS IN CARTESIAN COORDS.

LINKS DERIVS OF BASIS FCNS W/ BASIS

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \qquad \Gamma^{r}_{\theta\theta} = -r$$

PLUG INTO WK FORM

$$\sum \left[\int_{S_e} \omega_\alpha \sigma_{\alpha\beta} n_\beta\, dS - \int_{A_e} \left(\sigma_{\alpha\beta} \frac{\partial \omega_\alpha}{\partial r_\beta} + \underbrace{r\sigma^{\theta\theta}\omega_1 - \frac{1}{r}\sigma_{\alpha 1}\omega_\alpha}_{\text{ALL THAT IS NEW.}} \right) dA + \int_{A_e} \rho b_\alpha \omega_\alpha\, dA \right] = 0 \quad \forall \omega_\alpha$$

FELLIPA HAS AN r IN HERE

$$\sigma_{\theta\theta} = r^2 \sigma^{\theta\theta}$$

KRONECKER INSERTION

$$\sum \left[\int_{S_e} \omega_\alpha \sigma_{\alpha\beta} n_\beta\, dS - \int_{A_e} \left(\sigma_{\alpha\beta} \frac{\partial \omega_\alpha}{\partial r_\beta} + \underbrace{\frac{1}{r}\left(\sigma_{\theta\theta}\delta_{\alpha 1} - \sigma_{\alpha 1}\right)}_{\text{"STRESS STIFFENING"}} \omega_\alpha \right) dA + \int_{A_e} \rho b_\alpha \omega_\alpha\, dA \right] = 0$$

$$\omega_\alpha = \sum \omega_{\alpha I} N_I$$

$$f^I_\alpha = \int \sigma_{\alpha\beta} \frac{\partial N_I}{\partial r_\beta}\, dA + \int \left(\sigma_{\theta\theta}\delta_{\alpha 1} - \sigma_{\alpha 1}\right) \frac{1}{r} N_I\, dA$$

$$f_\alpha^I = \int_{A_e} \sigma_{\alpha\beta} \frac{\partial N_I}{\partial r_\beta} \, \partial A + \underbrace{\int_{A_e} \left(\sigma_{\theta\theta} \, \delta_{\alpha 1} - \sigma_{\alpha 1} \right) \frac{1}{r} \, N_I \, \partial A}_{\text{"CONVECTED STRESS"}}$$

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Tuesday, March 29, 2016 3:44 PM

CURVILINEAR COORDS

$$\underline{v} = v_1 \underline{e}_1 + v_2 \underline{e}_2 + v_3 \underline{e}_3 \qquad \underline{e}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \quad \underline{e}_2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix} \quad \underline{e}_3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

ORTHO-NORMAL

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij} \qquad \|\underline{e}_i\| = 1$$

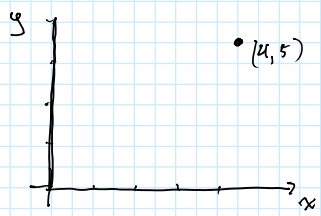
CYL COORDS

$$\underline{v} = v_r \underline{e}_r + v_\theta \underline{e}_\theta + v_z \underline{e}_z$$

ACTUALLY SHOULD BE:

$$= v^r \underline{e}_r + v^\theta \underline{e}_\theta + v^z \underline{e}_z$$

CO GO BELOW



$$\underline{x} = 4\underline{e}_1 + 5\underline{e}_2 + 0\underline{e}_3$$

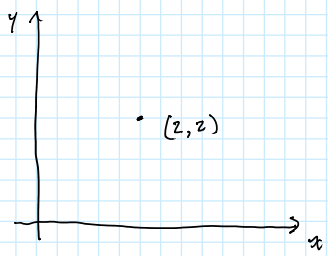
SCALE $\underline{e}_1$ SUCH THAT $\hat{\underline{e}}_1 = \begin{Bmatrix} 100 \\ 0 \end{Bmatrix}$ $\hat{\underline{e}}_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

NOW WRITE DOWN AN EXPRESSION FOR $\underline{x}$ IN THE NEW BASIS.

$$\hat{\underline{e}}_1 = 100\,\underline{e}_1$$

$$\underline{x} = \frac{4}{100}\hat{\underline{e}}_1 + 5\underline{e}_2 + 0\underline{e}_3$$

CONTRAVARIANT



$$\underline{E} = \frac{\partial \phi}{\partial \underline{x}}$$

$$\hat{\underline{E}} = 200\,\hat{\underline{e}}_1 + 2\underline{e}_2$$

COVARIANT COMPONENT

① $$\sum_e \left[\int_{V_e} \nabla \cdot \underline{\underline{\sigma}} \cdot \underline{\omega} - \rho \underline{b} \cdot \underline{\omega} \, \partial V \right] = 0 \quad \forall \underline{\omega} \qquad \partial V = 2\pi r \, dA$$

② $$\sum_e \left[\frac{1}{2\pi} \int \nabla \cdot \underline{\underline{\sigma}} \cdot \left(\frac{1}{r} \hat{\underline{\omega}} \right) - \frac{\rho}{r} \underline{b} \cdot \underline{\omega} \, dV \right] = 0 \quad \forall \omega$$

THIS IS THE TRPF. USING AREA WEIGHTS

③ $\partial V = 2\pi r \, dA$

$$\sum \left[\frac{1}{2\pi} \int_{A_e} \nabla \cdot \underline{\underline{\sigma}} \left(\frac{1}{r} \underline{\omega} \right) r \, dA - \int_{A_e} \rho \underline{b} \cdot \underline{\omega} \, dA \right] = 0 \quad \forall \underline{\omega}$$

④ IBP

$$\sum_e \left[\int \underline{\underline{\sigma}} : \nabla \left(\frac{1}{r} \underline{\omega} \right) r \, dA + \int_{A_e} \rho \underline{\omega} \cdot \underline{b} \, dA - \int_{S_e} \underline{\omega} \cdot \underline{t} \, ds \right] = 0 \quad \forall \underline{\omega}$$

SKETCH OF HOW TO EVALUATE $\nabla \left(\frac{1}{r} \underline{\omega} \right)$

$$\underline{\omega} = \omega^r \underline{e}_r + \omega^\theta \underline{e}_\theta + \omega^z \underline{e}_z$$

$$= \sum \omega^\alpha \underline{e}_\alpha \qquad \alpha \rightarrow (r, \theta, z)$$

$$\underset{\sim}{\omega} = \omega^r \underline{e}_r + \omega^\theta \underline{e}_\theta + \omega^z \underline{e}_z$$

$$= \sum \omega^\alpha \underline{e}_\alpha \qquad \alpha \to (r, \theta, z)$$

β

$\uparrow$ ← USED INSTEAD OF i, j, k

SINCE ω_α AND $\underline{e}_\alpha$ VARY IN SPACE

$$\partial \underline{\omega} = \partial \omega^\alpha \underline{e}_\alpha + \omega^\alpha \partial \underline{e}_\alpha$$

$$\boxed{\partial \omega^\alpha = \frac{\partial \omega^\alpha}{\partial x^\beta} \partial x^\beta = \frac{\partial \omega^\alpha}{\partial x^\beta} \underline{e}^\beta \cdot \partial \underline{x}}$$

INCREMENT IN CARTESIAN x

$$\partial \underline{e}^\alpha = \frac{\partial \underline{e}^\alpha}{\partial x^\beta} \underline{e}^\beta \cdot \partial \underline{x}$$

$$\Gamma_{\alpha\beta} = \frac{\partial \underline{e}^\alpha}{\partial x^\beta}$$

CHRISTOFFEL SYMBOL OF THE 2ND KIND.

$$\Gamma^\gamma_{\alpha\beta} = \underline{e}^\gamma \cdot \Gamma_{\alpha\beta}$$

$$\boxed{\partial \underline{e}_\alpha = \Gamma^\gamma_{\alpha\beta} \underline{e}_\gamma \partial x^\beta}$$

FOR CYL COORDS

$$\underline{e}_r = \cos\theta\, \underline{e}_1 + \sin\theta\, \underline{e}_2$$

$$\underline{e}_\theta = -r \sin\theta\, \underline{e}_1 + r\cos\theta\, \underline{e}_2$$

$$\underline{e}_z = e_3$$

$$\Gamma^\theta_{r\theta} = \underline{e}_\theta \cdot \Gamma_{r\theta} = \frac{1}{r}$$

$$\Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^r_{zz} = -r$$

ONLY NON-ZERO COMP.

$$\Gamma_{rr} = \frac{\partial \underline{e}_r}{\partial x^r} = 0 \qquad \Gamma_{r\theta} = \frac{\partial \underline{e}^r}{\partial x^\theta} = -\sin\theta\, \underline{e}_1 + \cos\theta\, \underline{e}_2 = \frac{1}{r} \underline{e}_\theta$$

SO

$$\nabla\left(\frac{1}{r} \underline{\omega}\right) = \frac{\partial}{\partial x^\beta}\left(\frac{1}{r} \omega^\alpha \underline{e}_\alpha\right)$$

$$= \frac{1}{r} \frac{\partial \omega_\alpha}{\partial x^\beta} - \frac{1}{r} \Gamma^\gamma_{\alpha\beta} \cdot \omega_\alpha - \frac{1}{r^2} \delta_{1\beta} \omega_\alpha$$

5)

$$\sum_e \left[\left(\sigma_{ij} \frac{\partial \omega_i}{\partial x_j} - r\, \sigma^{33} \omega_1 + \frac{1}{r} \sigma_{i1} \omega_i\right) dA - \int \rho\, b_i \omega_i\, dA - \int t_i \omega_i\, ds\right] = 0$$

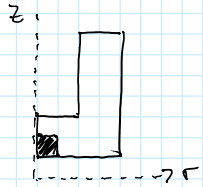
6) LET $\sigma^{33} = \frac{1}{r^2} \sigma_{33}$

$$\int_{A_e} \sigma_{ij} \frac{\partial \omega_i}{\partial x_j} dA - \int \frac{1}{r}\left(\sigma_{33} \delta_{i1} - \sigma_{i1}\right) \omega_i\, dA$$

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$$

Z:

PLUG IN CONST. MODEL



PLUG IN CONST. MODEL

1st TERM OF

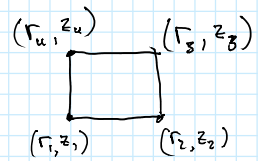
$$\int \underbrace{\frac{\partial w_i}{\partial x_j}}_{[B]^T} \underbrace{C_{ijkl}}_{[D]} \underbrace{\frac{\partial u_k}{\partial x_l}}_{[B]}$$

WHAT DO WE DO TO GET B?

SHAPE FUNS

$$\begin{Bmatrix} 1 \\ r \\ z \\ u_r \\ u_z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ z_1 & z_2 & z_3 & z_4 \\ u_{r1} & u_{r2} & u_{r3} & u_{r4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{Bmatrix}$$

~~SHAPES~~ "RING ELEMENT NOTES" 3.3?



$$\begin{Bmatrix} u_r \\ u_z \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_{r1} \\ u_{z1} \\ u_{r2} \\ u_{z2} \\ u_{r3} \\ u_{z3} \\ u_{r4} \\ u_{z4} \end{Bmatrix} = [P]\{u\}$$

STRAIN VECTOR

$$\{\varepsilon\} = \begin{Bmatrix} e_{rr} \\ e_{zz} \\ e_{\theta\theta} \\ 2e_{rz} \end{Bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \cdots \\ \vdots & & \vdots & & \vdots \\ N_1/r & 0 & N_2/r & 0 & N_3/r \;\; 0 \;\; N_4/r \;\; 0 \end{bmatrix} \{u\}$$

$$[B] = [T][P]$$

$$\{\varepsilon\} = [B]\{d\}$$

$$[K]\{u\} = \{f\}$$

$$[K] = \int [B]^T [D][B]\, dA - \underbrace{\int \frac{1}{r}\left(\sigma_{33}\delta_{1i} - \sigma_{i1}\right) w_i \, dr}$$

THIS WINDS UP BEING SAME AS PLANE STRAIN.

HW NEXT WK RE-CAST THIS AS MAT FORM.

flow to not let go to infinity

$$\{f\} = \int \rho \{b\}[P]\, dA + \int \{t\}[P]\, ds$$

GAUSS QUAD NOT

$$\{f\} = \int_{A_e} p\{b\}[P]\,dA + \int_{S_t} \{t\}[P]\,ds$$

INFINITE

GAUSS QUAD NOT @ NODES.

HOMEWORK

SINGLE ELEMENT FELLIPA 3.16

EVALUATE PROBLEM OF YOUR CHOICE (SINGLE ELEM)

USING FELLIPAS & SANTOS. BODY FORCE

RUN IN FE PROGRAM

LOOK FOR DIFFERENCES.

START W/ PLANE STRAIN

CONTENT IS TO ASSUM.

ATHIS WK ATVM

10.4, 10.5

I am a

3/31/16

Thursday, March 31, 2016 3:47 PM

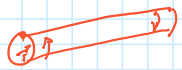
- ERROR ANALYSIS
- VALIDATION & VERIFICATION

1) COMPUTATIONAL — finite ~~arithmetic~~ SMALL, quadrature error
2) DOMAIN —
3) APPROXIMATION ERROR

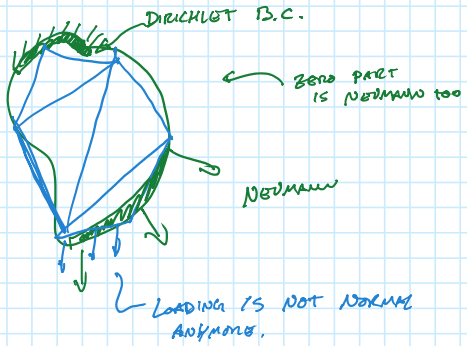
} LINKED VIA MESH

⋮

DOMAIN ERROR



$$G\frac{\partial^2 u}{\partial r^2} + b = 0$$



$$u = u_N = \sum_e \sum_{i=1}^{n} u_i^{(e)} N_i^{(e)} = \sum_{I=1}^{N} u_I \Phi_I$$

QUANTIFY DIFF BETWEEN u & u_n $\quad E = u - u_n$

ERROR METRICS

"NORMS" JUST 1MD VECTORS $\quad v = \|\underline{v}\| = \sqrt{\underline{v}\cdot\underline{v}}$

WHAT ABOUT FCN $\quad \bar{f} = \|f\| = \int f\cdot f \, dx$ ← INNER PRODUCT

HOW ABOUT MAT $[A]$ $\quad \sqrt{A:A}$ = FROBENIUS NORM

$\|[A]\{x\}\|$ $\quad$ LOTS OF NORMS!

"ENERGY NORM"

$$\|u - u_N\|_m = \left[\int_a^b \sum_{i=0}^{m} \left(\frac{\partial^i u}{\partial x^i} - \frac{\partial^i u_N}{\partial x^i}\right)^2 \partial x\right]^{1/2}$$

↑ ORDER OF DIFF EQ (MOST TIMES 2)

"P-NORM" SPECIALIZATION OF ↑

$$L_p = \left[\int_a^b (u - u_N)^p \, \partial x\right]^{1/p}$$

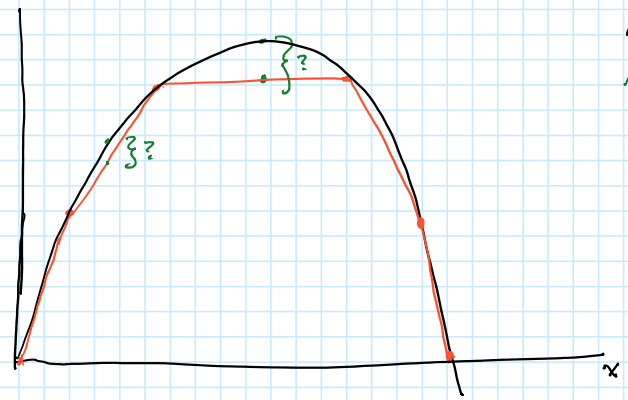
"L_2 NORM" SPECIALIZATION OF ↑

"L_2 NORM" SPECIALIZATION OF $\mathcal{J}$

$$\| u - u_N \|_0 = \left[\int_a^b (u - u_n)^2 \, dx \right]^{1/2}$$

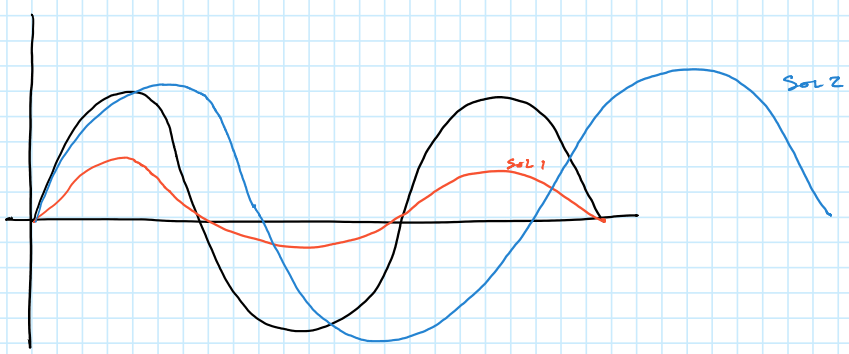
"MAX NORM"

$$\| u - u_N \|_\infty = \max_{a \le x \le b} \left| u(x) - u_N(x) \right|$$



L_2 = AREA BETWEEN CURVES.

MAX NORM



ERRORS L_2 COULD BE SAME!

MAY NOT GIVE PHYSICAL INSIGHT.

$$\| e \|_m = \| u - u_N \|_m \le c h^p \qquad p = K + 1 - m \qquad P > 0$$

m ← ORDER OF DERIV IN ERROR MEAS.

K ← ORDER OF APPROX

h ← CHARACTERISTIC ELEMENT L

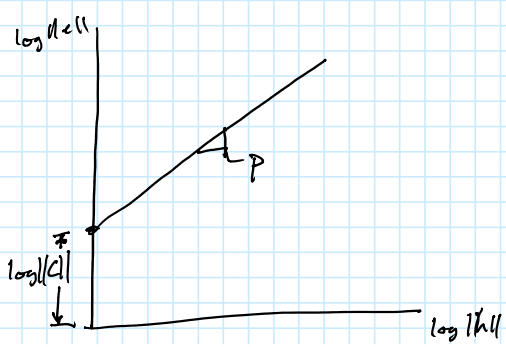
c ← SAME CONST

"P- REFINEMENT"

$$\log \| e \|_m = \log (c h^p)$$
$$= \log(c) + p \log(h)$$

P-REF INCREASES ORDER OF APPROX... MORE NODES

H-REF DROP ELEMENT SIZE.



RATE OF CONVERGENCE

WE DON'T KNOW u! THIS IS ALL QUALITATIVE!

1) RESIDUAL METHODS ~ RE-EVALUATE PDE HOW MUCH LEFT TO BALANCE.
2) FLUX - PROJECTION
3) EXTRAPOLATION
4) INTERPOLATION ERROR ESTIMATES.

RICHARDSON'S EXTRAPOLATION

$$u - u_h^p(x) = C_{p+1} h^{k+1} + \mathcal{O}(h^{k+1})$$

DON'T KNOW

FE SOL

RUN 1ST

RUN AGAIN WITH REDUCED h

$$u - u_{h/2}^p(x) = C_{k+1}\left(\frac{h}{2}\right)^{k+1} + \mathcal{O}(h^{k+2})$$

SUBTRACT 1ST ERN FROM 2ND

$$u_{h/2}^p(x) - u_h^p(x) = \underbrace{C_{k+1} h^{k+1}}_{\text{ERROR HIDDEN HERE}}\left(1 - \frac{1}{2^{k+1}}\right) + \mathcal{O}(h^{k+2})$$

$$C h^{k+1} \cong \frac{u_{h/2}^p - u_h^p}{1 - \frac{1}{2^{k+1}}}$$

THE ERROR FROM GOING FROM 1 MESH RELATIVE TO THE OTHER.

MMS

- VALIDATION - AM I SOLVING THE RIGHT PROBLEM?
  NEED DATA FROM LAB.
- VERIFICATION ~ AM I SOLVING THE EQUATIONS RIGHT?
  NEED ANALYTIC SOLUTION.

MMS

$$\frac{\partial \sigma}{\partial x} + b = 0$$

$$u(0) = 0$$

$$\sigma(L) = 0$$

$$E\frac{\partial^2 u}{\partial x^2} + b = 0$$

$u(x) = f(x)$

$f(x) = c_1\cos(x) + c_2\sin(x) + c_3 x^3 + c_4 x$

$$E\frac{\partial^2 f}{\partial x^2} + b = 0$$

SOLVE FOR b $\quad b = -E\frac{\partial^2 f}{\partial x^2}$

COOLER RIGHT

SOLVE FOR $b = -C \frac{\partial u}{\partial x}$

SATISFY B.C.'S

$u(0) = C_1 + \ldots = 0$ TO GET C_i

YOU CONSTRUCT YOUR OWN EXACT SOL FOR $\underline{b}$ LOADING

4/5/16

Tuesday, April 5, 2016 3:46 PM

HW DUE TUE:

SOLVE AXISYM USING FELLIPAS FORM
REDDY'S FORM
FEM OF CHOOSING

+ NEW PROBLEM

8.1 IN FELLIPA'S NOTES. # OF COMPUTATIONAL STEPS.
How much more time!

WHAT IS NUMBER OF MULTIPLICATION PER SECONDS
How many computations / second
RELATED TO FLOPS/SEC.

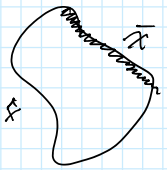
METHOD OF MANUFACTURED SOLUTIONS.

VERIFY NOT VALIDATE.

SKETCH.

$$L(u) - f(u) = 0$$

$$L = E\frac{\partial^2}{\partial x^2} \qquad u(\bar{x}) = \bar{u}, \quad u'(\hat{x}) = \hat{u}'$$



$g(x)$ IS A SOLUTION TO $L(u) - f(u) = 0$
($g(x)$ IS $u(x)$)
SUBJECT TO (MUST SATISFY BOUNDARY CONDITION)

$$g(\bar{x}) = \bar{g} = \bar{u}$$
$$g'(\hat{x}) = \hat{g} = \hat{u}$$

SHOULD BE!

1. CHOOSE $g(x)$
2. REQUIRE $g(x)$ TO SATISFY B.C.
3. SOLVE FOR $f(g) = L(g)$
4. IN THE MODEL PROBLEM
   - ENFORCE B.C's.
   - APPLY $f(g)$
5. COMPARE $\hat{g}$ TO g WHERE $\hat{g}$ IS NUMERICAL SOL.

REVIEW OF SOLID ELEMENTS

$$\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b} = 0$$

1-D:

$$E\frac{\partial^2 u}{\partial x^2} + q = 0 \qquad [K]^e = \int_{V_e} E\{N'\}^T[N']\,dx$$

$$\{u\} = \begin{Bmatrix} u_{1x} \\ u_{1y} \\ \vdots \\ u_{nx} \\ u_{ny} \end{Bmatrix}$$

BODY $\{f\}_\Omega = \int q\{N\}\,dx$

BOUNDARY $\{f\}_{\partial\Omega} = \{Q\} = EAN'\Big|_{x_a}^{x_b}$

2-D:

$$[K]^e = t\int [B]^T[D][B]\,dA$$

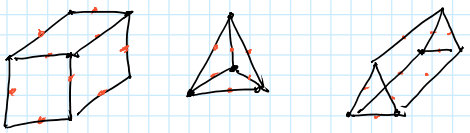
$$[K]^e = t \int [B]^T [D][B]\, dA$$

BODY $\{P\}_\Omega = t \int q \{P\}\, dA$

$$\{P\}_{d\Omega} = \int_S \{t\}\cdot\{N\}\, ds$$

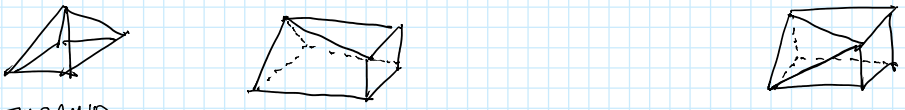
BOUNDARY

3D: GOOD FOR SOLID, IF THIN & COULD HAVE BENDING, NOT SO GREAT. SHELLS ARE BETTER.



"NORMAL 3-D ELEMS"

NON-STANDARD.



PYRAMID

"RICK"

$$\begin{array}{c} 1 \\ x \\ y \\ z \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots \\ x_1 & x_2 & x_3 & & \\ y_1 & y_2 & y_3 & & \\ z_1 & z_2 & z_3 & & \cdots \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ N_3 \\ \vdots \\ N_n \end{Bmatrix}$$

$$u = \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ u_{2x} \\ u_{2y} \\ u_{2z} \\ \vdots \end{Bmatrix}$$

$$\sum_i N_i = 1$$

$$x = \sum_i x_i N_i$$

$$y = \sum_i y_i N_i$$

$$z = \sum_i z_i N_i$$

ISOPARAMETRIC FORMULATION.

$$u_x = \sum_i u_{ix} N_i$$

$$u_y = \sum_i u_{iy} N_i$$

APPROXIMATION DISP WITH SAME SHAPE FCNS AS GEOM.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \vdots \end{Bmatrix} \qquad \gamma_{ij} = 2\varepsilon_{ij} \quad i \neq j$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \end{Bmatrix} \qquad \tau_{ij} = \sigma_{ij} \quad i \neq j$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \qquad \gamma_{ij} = 2\varepsilon_{ij} \quad i \neq j$$

$$\gamma_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} \qquad \tau_{ij} = \sigma_{ij} \quad i \neq j$$

FOR LINEAR PROB.

$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\varepsilon\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & 0 & \frac{\partial N_2}{\partial x} & 0 & 0 & \cdots \\ 0 & \frac{\partial N_1}{\partial y} & 0 & 0 & \frac{\partial N_2}{\partial y} & 0 & \cdots \\ 0 & 0 & \frac{\partial N_1}{\partial z} & 0 & 0 & \frac{\partial N_2}{\partial z} & \cdots \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & 0 & \cdots & & & \\ 0 & \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial y} & & & & \\ \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_1}{\partial x} & & & & \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ \vdots \end{Bmatrix}$$

$$= [B]\{u\}$$

$$[K] = \int_V [B]^T \underset{6\times 6}{[D]} [B]$$

$$= \sum W_i J_i [B_i]^T [D_i][B_i]$$

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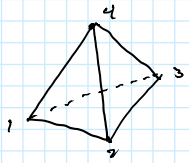
NEXT THUR 4/21

~1 PAGE STATUS REPORT ON PROJECT.

1/ GROUP.

LINEAR TETRAHEDRON RELIRA 5/6?



- LIN, SHAPE FXN LINEAR
- STRAIGHT SIDES
- PLANAR FACES

} NORMALLY WORTHLESS. FOR STRESS/ DISP.

B/C STRAIN IS CONST OVER WHOLE ELEM.

CAN HAVE NEGATIVELY EXTRAPOLATED STRESS.

WORKS PRETTY WELL FOR HEAT TRANSFER.

VOLUME:

$$V = \int_{\Omega^e} d\Omega^e = \frac{1}{6}\det\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

$$= \frac{1}{6}\det[J] = \frac{1}{6}J$$

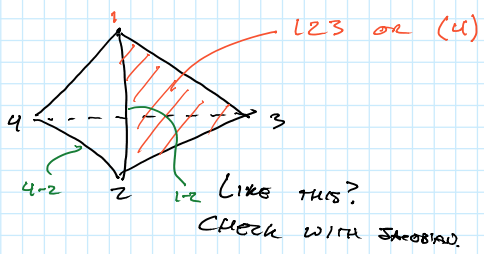
$$\boxed{V = \frac{J}{6}}$$

V MUST BE POSITIVE.

1) NODES MUST NOT BE CO-PLANAR

2) NODES NEED NODE ORDERING

- PICK A NODE TO BE #1
- FORM A FACE THAT INCLUDES #1
- LABEL OPPOSITE CORNER #4
- LABEL PICKED FACE NODES C.C.W.



CORNERS

DESCRIBED BY LABELS 1, 2, 3, 4

EDGES

INTERSECTION OF TWO PLANES
DESCRIBED BY CORNER-NODES 1-2 4-2 OR 2-4

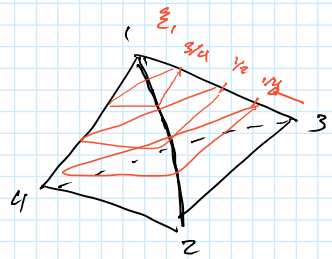
FACES

SHADED FACE COULD BE 1, 2, 3 OR 123
FACE OPPOSITE OF NODE.

TET COORDS

POSITION IN CARTESIAN (x, y, z)

TET COORDS $(\xi_1, \xi_2, \xi_3, \xi_4)$



$$\xi_i = \begin{cases} 1 & @ \text{ CORNER } i \\ 0 & @ \text{ OTHER CORNERS} \end{cases}$$

$$\sum_i \xi_i = 1$$

WHY TET COORDS?

- CORNERS

WHY TET COORDS:

- CORNERS

  $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$

- MID POINTS OF EDGES

  $\overline{12} \Rightarrow \left(\frac{1}{2}, \frac{1}{2}, 0, 0\right)$

- FACE CENTROID

  $124\ (3) \Rightarrow \left(\frac{1}{3}, \frac{1}{3}, 0, \frac{1}{3}\right)$

- ELEMENT CENTROID

  $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$

LINEAR INTERPOLATION

$F(x,y,z)$ WITH

$F_i\ (i=1,2,3,4)$ CAN BE EXPRESSED AS

$$F(\xi_1, \xi_2, \xi_3, \xi_4) = F_1\xi_1 + F_2\xi_2 + F_3\xi_3 + F_4\xi_4 = \sum F_i \xi_i$$

$$F(x,y,z) = 4x + 9y - 8z + 3$$

$(0,0,0)$ $(1,0,0)$ $(0,1,0)$, $(0,0,1)$

$F_1 = 3$ $F_2 = 7$ $F_3 = 12$ $F_4 = 5$ ← PLUG IN NODAL VALS INTO TO GET F_i

$$F_\xi = 3\xi_1 + 7\xi_2 + 12\xi_3 - 5\xi_4$$

$$F_2 = 3\xi_1 + 7\xi_2 + 12\xi_3 - 5\xi_4 = 2\ (1)$$

$f(x+y) = f(x) + f(y)$

$f(\alpha x) = \alpha f(x)$

$1 = \xi_1 + \xi_2 + \xi_3 + \xi_4$

How to Go from AFFINE to LINEAR! IN TET COORDS.

$$F_2 = \xi_1 + 5\xi_2 + 10\xi_3 - 7\xi_4$$

TO ξ, FROM DIFF SPACES?

COORD TRANSFORMATION

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}}_{\text{JACOBIAN}} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix}$$

$$\{\bar{X}\} = [J]\{\xi\}$$

$$\{\xi\} = [J]^{-1}\{x\}$$

$$[J]^{-1} = \frac{1}{6}\begin{bmatrix} 6V_{01} & a_1 & b_1 & c_1 \\ 6V_{02} & a_2 & b_2 & c_2 \\ 6V_{03} & a_3 & b_3 & c_3 \\ 6V_{04} & a_4 & b_4 & c_4 \end{bmatrix}$$

V_{01} = SUB-VOLUME ??

DERIV

$$\frac{\partial x}{\partial \xi_i} = \frac{\partial}{\partial \xi_i}\left(\sum x_i \xi_i\right) = x_i$$

$$\frac{\partial y}{\partial \xi_i} = y_i$$

$$\frac{\partial z}{\partial \xi_i} = z_i$$

$$\partial F = \frac{\partial F}{\partial \xi_i} \cdot \partial \xi_i = \frac{1}{6} V \left(\frac{\partial F}{\partial \xi} a_1 + \frac{\partial F}{\partial \xi} a_2 + \frac{\partial F}{\partial \xi} a_3 + \frac{\partial F}{\partial \xi} a_4\right)$$

$$\frac{\partial \xi}{\partial \xi_i} = \xi_i$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial \xi_i} \cdot \frac{\partial \xi_i}{\partial x} = \frac{1}{6} V \left(\frac{\partial F}{\partial \xi_1} a_1 + \frac{\partial F}{\partial \xi_2} a_2 + \frac{\partial F}{\partial \xi_3} a_3 + \frac{\partial F}{\partial \xi_4} a_4 \right)$$

$$= \frac{a_i}{6V} \frac{\partial F}{\partial \xi_i}$$

INTEGRATION

$$\int \xi_i \xi_j \xi_k \, \partial \Omega^e = (\qquad)$$ ROLE IN ELLIPSE

$$\int \xi_i \, \partial \Omega^e = \frac{1}{4} V$$

$$\int \xi_i \xi_j \, \partial \Omega^e = \begin{cases} V/10 & i = j \\ V/20 & i \neq j \end{cases}$$

CREATE AN ELEMENT.

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \\ u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x & & & \\ y & & & \\ z & & & \\ u_{x1} & u_{x2} & u_{x3} & u_{x4} \\ u_{y1} & u_{y2} & u_{y3} & u_{y4} \\ u_{z1} & u_{z2} & u_{z3} & u_{z4} \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}$$

$$\{u^e\} = \left[u_{x1} \; u_{y1} \; u_{z1} \; \ldots \; u_{z4} \right]^T$$

$$N_i = \xi_i$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \end{Bmatrix} = [T] \{u^e\}$$

$$[T] = \begin{bmatrix} \partial/\partial x & 0 & 0 \\ 0 & \partial/\partial y & 0 \\ 0 & 0 & \partial/\partial z \\ \partial/\partial y & \partial/\partial x & 0 \\ & \vdots & \end{bmatrix}$$

$$\{u^e\} = [P] \{u\}$$

$$[P] = \begin{bmatrix} N_1 & 0 & 0 & N_2 & 0 & 0 & \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ldots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \end{bmatrix}$$

$$[\varepsilon] = [T][P]\{u^e\}$$

$$= [B] \{u^e\}$$

$$\underset{6 \times 12}{[B]} = [T][P]$$

$$= [B]\{u^e\}$$

$$\underset{6\times 12}{[B]} = [T][P]$$

$$[K] = \int [B]^T [D][B]\, d\Omega$$

$$= V[B]^T[D][B]$$ USEFUL FOR SMALL STRAIN

$$\{f_n\} = \int [P]^T \{b\}\, d\Omega$$ BODY FORCE

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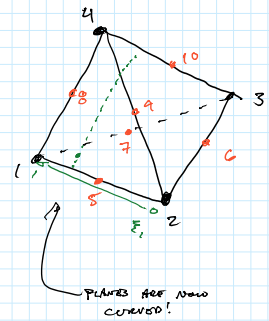
Quadratic Tet

Harder to mesh
Better results for stress-displacement

CCW looking from node 4.



Advantages: (over 4 node tet)

- Better stress-disp
- Geometry
  - Auto-meshing
  - Curved sides

Disadvantages: (relative to cubic)

- More complicated formulation
  - Non-constant Jacobian.
- Larger memory footprint (computationally)
  - ~3x storage!

Linear

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix}$$

Quad

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{10} \\ y_1 & \cdots & y_{10} \\ z_1 & \cdots & z_{10} \end{bmatrix} \begin{Bmatrix} N_1^e \\ \vdots \\ N_{10}^e \end{Bmatrix}$$

Isoparametric

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} \quad \begin{bmatrix} u_{x1} & & u_{x10} \\ u_{y1} & & u_{y10} \\ u_{z1} & & u_{z10} \end{bmatrix}$$

- Shape fxns

$N_i = \xi_i(2\xi_i - 1)$ $\quad i = 1,2,3,4$

$N_i^e = 4\xi_j\xi_k$ $\quad i = 5,6,7,8,9,10$

$j,k = 1,2$ if $i=5$
$\quad = 2,3$ if $i=6$
$\quad \vdots$

Midside node

$\bar{x}_k = (x_i + x_j)/2$ $\quad$ if in exact middle degenerates into linear

Jacobian is constant over element

$J = V/6$

Derivatives of a function

$F(x,y,z)$

$$= F(\xi_1, \xi_2, \xi_3, \xi_4) = F(\xi_i) = \sum_k F_k N_k$$

$F_k = F(\xi_1, \xi_2, \xi_3, \xi_4)$ @ node k

What kinds of fxns

$1, x, y, z, u_x, T \ldots$

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}(F_k N_k)$$

$$= F_k \frac{\partial N_k}{\partial x}$$

$$= F_k\left(\frac{\partial N_k}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial N_k}{\partial \xi_2}\frac{\partial \xi_2}{\partial x} + \cdots\right) \quad \text{or} \quad F_k \frac{\partial N_k}{\partial \xi_j}\frac{\partial \xi_j}{\partial x}$$

$$\frac{\partial F}{\partial x} = F_k \frac{\partial N_k}{\partial \xi_i}\frac{\partial \xi_i}{\partial x}$$

$$\frac{\partial F}{\partial y} = F_k \frac{\partial N_k}{\partial \xi_i}\frac{\partial \xi_i}{\partial y}$$

$$\frac{\partial F}{\partial z} = F_k \frac{\partial N_k}{\partial \xi_i}\frac{\partial \xi_i}{\partial z}$$

$$\begin{Bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \partial F/\partial z \end{Bmatrix} = \begin{bmatrix} \partial\xi_1/\partial x & \cdots & \partial\xi_4/\partial x \\ \partial\xi_1/\partial y & \cdots & \partial\xi_4/\partial y \\ \partial\xi_1/\partial z & \cdots & \partial\xi_4/\partial y \end{bmatrix} \begin{Bmatrix} F_k \frac{\partial N_k}{\partial \xi_1} \\ \vdots \\ F_k \frac{\partial N_k}{\partial \xi_4} \end{Bmatrix} \xrightarrow{\text{transpose}} \begin{bmatrix} F_k \partial N_k/\partial \xi_1 & \cdots & F_k \partial N_k/\partial \xi_1 \end{bmatrix} \begin{bmatrix} \partial\xi_1/\partial x & \partial\xi_1/\partial y & \partial\xi_1/\partial z \\ \vdots & \vdots & \vdots \\ \partial\xi_4/\partial y & \partial\xi_4/\partial y & \partial\xi_4/\partial z \end{bmatrix} = \begin{bmatrix} \partial F/\partial x & \partial F/\partial y & \partial F/\partial z \end{bmatrix}$$

trying to get Jacobian mapping

subs F_k for 1

same!

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_k \frac{\partial N_k}{\partial \xi_1} & \cdots & & x_k \frac{\partial N_k}{\partial \xi_1} \end{bmatrix} \quad \begin{bmatrix} \partial 1/\partial x \; 0 & \partial 1/\partial y \; 0 & \partial 1/\partial z \; 0 \\ \partial x/\partial x \; 1 & \partial x/\partial y \; 0 & \partial x/\partial z \; 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_k \frac{\partial N_k}{\partial \xi_1} & \cdots & & x_k \frac{\partial N_k}{\partial \xi_1} \\ y_k & \longrightarrow & & \\ z_k & \longrightarrow & & \end{bmatrix}}_{\text{JACOBIAN}} \begin{bmatrix} & & \\ & P & \\ & & \end{bmatrix} = \underbrace{\begin{bmatrix} \partial 1/\partial x \; 0 & \partial 1/\partial y \; 0 & \partial 1/\partial z \; 0 \\ \partial x/\partial x \; 1 & \partial x/\partial y \; 0 & \partial x/\partial z \; 0 \\ \partial y/\partial x \; 0 & \partial y/\partial y \; 1 & \partial y/\partial z \; 0 \\ \partial z/\partial x \; 0 & \partial z/\partial y \; 0 & \partial z/\partial z \; 1 \end{bmatrix}}_{I_{aug}}$$

— IDENTITY AUGMENTED

$$[P] = [J]^{-1}[I_{aug}]$$

$$J = \det[J]$$

$$[P] = \frac{1}{J}\begin{bmatrix} a_1 & b_1 & c_1 \\ \vdots & & \\ a_4 & b_4 & c_4 \end{bmatrix}$$

$$\frac{\partial \xi_i}{\partial x} = \frac{a_i}{J}$$

$$\frac{\partial \xi_i}{\partial y} = \frac{b_i}{J}$$

$$\frac{\partial \xi_i}{\partial z} = \frac{c_i}{J}$$

$$\boxed{\frac{\partial F}{\partial x} = F_k \frac{\partial N_k}{\partial \xi_i} \frac{a_i}{J}}$$

LINEAR CASE $J = 6V$

INTEGRALS:

$$\int_{\Omega^e} F(x,y,z)\, \partial\Omega^e = \int_{\Omega^e} F(x,y,z)\, \partial x\, \partial y\, \partial z$$

$$\boxed{= \int_{\Omega^e} F(\xi_1, \xi_2, \xi_3, \xi_4) \frac{J}{6}\, \partial\xi_1\, \partial\xi_2\, \partial\xi_3\, \partial\xi_4}$$

$$= \sum_p F(\xi_1^p, \xi_2^p, \xi_3^p, \xi_4^p)\, \omega_p \frac{J(\xi_1^p, \xi_2^p, \xi_3^p, \xi_4^p)}{6}$$

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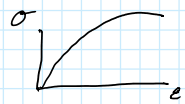
1) Nonlinear Geometry
2) Contact / Shell elements.

Group Nonlinearities

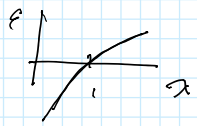
Material

Stress nonlinear in strain, deformation



Geometric

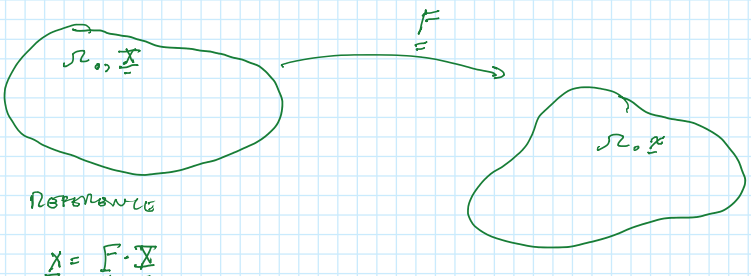
Strain is nonlinear with stretch



$$\int \left(\nabla \cdot \underline{\underline{\sigma}} + \rho \underline{b}\right) \cdot \omega \, dV = 0 \qquad \forall \omega$$

IBP

$$\int_{\Omega} \left(\underline{\underline{\sigma}} : \nabla \underline{\omega} + \rho \underline{b} \cdot \omega\right) dV - \int_{\Gamma} \underline{t} \cdot \underline{\omega} \, dA$$



Reference

$$\underline{x} = \underline{\underline{F}} \cdot \underline{X}$$

$$d\underline{x} = \underline{\underline{F}} \cdot d\underline{X}$$

Last integral in reference config.

Nanson's relation

$$\int_{\Omega_0} \left(\underline{\underline{\sigma}} : \nabla \underline{\omega} + \rho \underline{b} \cdot \omega\right) J \, dV_0 - \int_{\Gamma_0} \underline{t} \cdot \underline{\omega} \, \eta \, dA_0 = 0$$

$$\underline{n} \, dA = J \underline{N} \cdot \underline{\underline{F}}^{-1} dA_0$$

$$\eta = J \sqrt{\underline{N} \cdot \underline{\underline{F}}^{-1} \cdot \underline{\underline{F}}^{-T} \cdot \underline{N}}$$

$$\rho_0 = \rho J$$

$$\int_{\Omega_0} \left(J \underline{\underline{\sigma}} : \nabla \underline{\omega} + \rho_0 \underline{b} \cdot \omega\right) dV_0 - \int_{\Gamma_0} \underline{t} \, \underline{\omega} \, \eta \, dA_0$$

$J \underline{\underline{\sigma}} = \underline{\underline{\tau}}$ Kirchoff stress

$$\int_{\Omega_0} \left(\underline{\underline{\tau}} : \nabla \underline{\omega} + \rho_0 \underline{b} \cdot \omega\right) dV_0 - \int_{\Gamma_0} \underline{t} \cdot \underline{\omega} \, \eta \, dA_0$$

FE Eqns

- Displacements

$$u_i(\underline{X}) = \sum_{a=1}^{n} N^a(\underline{X}) \, u_i^a$$

Analogous to $[P]\{u^e\}$

DEF GRAD

$$\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{X}} = \underline{\underline{I}} + \nabla_0 \underline{u}$$

$$F_{ij} = \delta_{ij} + \sum_{a=1}^{n} \frac{\partial N^a}{\partial X_j} u_i^a$$

· DERIVATIVES

$$X_i = \sum_{a=1}^{n} N^a X_i^a$$

$$J_{ij} = \frac{\partial X_i}{\partial \xi_j} = \sum_{a=1}^{n} \frac{\partial N^a}{\partial \xi_j} X_i^a$$

$$\boxed{\frac{\partial N}{\partial X_i} = \frac{\partial N^a}{\partial \xi_k} J^{-1}_{kj}} = \frac{\partial N^a}{\partial \xi_k} \frac{\partial \xi_k}{\partial X_j}$$

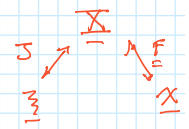
$$\nabla \underline{w} = \frac{\partial w_i}{\partial X_j} = \sum_{a=1}^{n} \frac{\partial N^a}{\partial X_k} F^{-1}_{kj} w_i^a$$

3 COORDINATE SYSTEMS

$\underline{X}$ REFERENCE

x CURRENT

ξ NATURAL



$$\left\{ \int \tau_{ij}\left[F_{pq}(u_k^b)\right] \frac{\partial N^a}{\partial X_m} F^{-1}_{mj}\, dV_0 - \int \rho_0 b_i N^a\, dV_0 - \int t_i N^a \eta\, dA_0 \right\} w_i^a = 0$$

KINDA LIKE $\sigma[B]$

I WANT GUESS

$$u_k^b = V_k^b$$

$$V_k^b \longrightarrow V_k^b + dV_k^b$$

$$\int \tau_{ij}\left[F_{pq}(V_k^b + dV_k^b)\right] \frac{\partial N^a}{\partial X_m} (F + dF)^{-1}_{mj}\, dV_0 + \int \rho_0 b_i N^a\, dV_0 - \int t_i N^a (\eta + d\eta)\, dA_0 = 0$$

LINEARIZE THIS
THINK
TAYLOR SERIES

$$\tau + \frac{\partial \tau}{\partial F}\frac{\partial F}{\partial V}\partial V$$

$\sim$
$[C]$

$$K_{aibk}\, dV_k^b + R_i^a - F_i^a = 0$$

SYMMETRIC

NOT SYMMETRIC

$$K_{aibk} = \int_{V_0} \frac{d\tau_{ij}}{dF_{kl}} \frac{\partial N^b}{\partial X_l} \frac{\partial N^a}{\partial X_m} F^{-1}_{mj}\, dV_0 \underbrace{- \int \tau_{ij} \frac{\partial N^a}{\partial X_m} F^{-1}_{mk} \frac{\partial N^a}{\partial X_p} F^{-1}_{pj}\, dV_0 - \int t_i N^a \frac{\partial N}{\partial V_k^b}\, dA_0}_{\text{NOW TO NONLINEAR GEOMETRY}}$$

a, b ARE NODES

"GEOMETRIC STIFFENING"

CODE TAKES LONGER BECAUSE

1) MORE THINGS TO COMPUTE

2) SOLVERS FOR NON-SYMMETRIC