Pressure loading of a thick walled cylinder is a relatively simple example where we can demonstrate explicitly how we can satisfy all the governing equations and boundary conditions to arrive at a complete solution of a stress analysis problem. Figures 1(a), (b) show the geometry and loading of the problem.
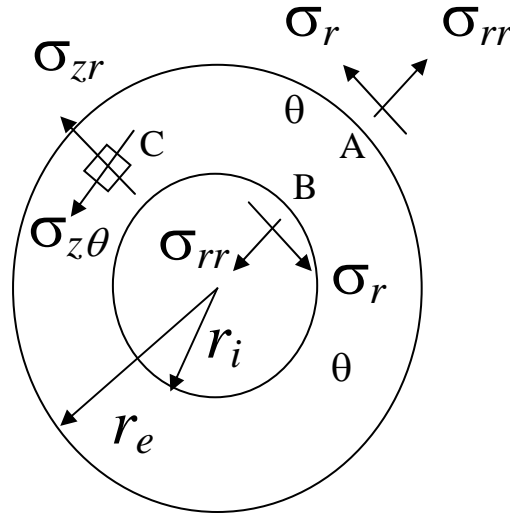

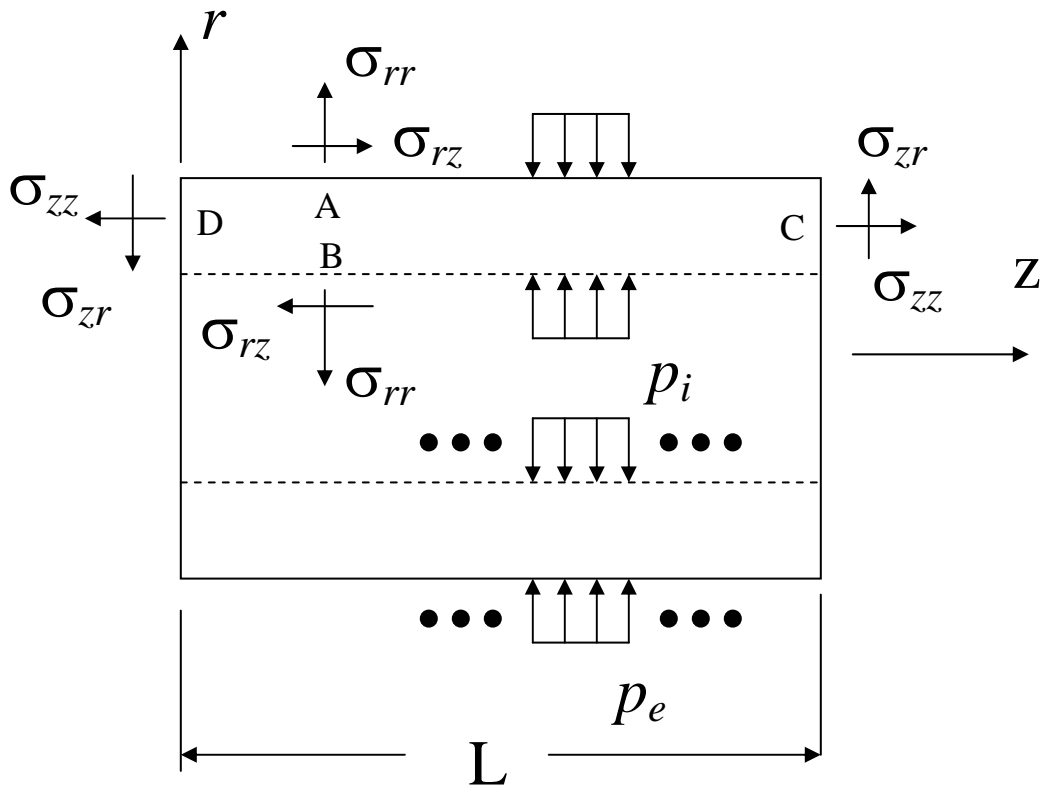Fig 1(a) Stresses on the cylinder surfaces (the z-axis is outwards from this planar view)

Fig. 1 (b) Stresses on the surfaces of the cylinder and the pressure loading.

It is assumed that the internal pressure, p_i , and the external pressure, p_e , are uniform along the entire inner and outer surfaces as shown in Fig. 1(b) and that all the other surfaces of the cylinder are unloaded. In this case the boundary conditions for this problem are:

for all $0 \le \theta \le 2\pi$, $0 \le z \le L$ on the inner and outer surfaces we have

$$\sigma_{rr}\left(r,\theta,z\right)\Big|_{r=r_i} = -p_i$$
$$\sigma_{r\theta}\left(r,\theta,z\right)\Big|_{r=r_i} = \sigma_{rz}\left(r,\theta,z\right)\Big|_{r=r_i} = 0$$
$$\sigma_{rr}\left(r,\theta,z\right)\Big|_{r=r_e} = -p_e \tag{1}$$
$$\sigma_{r\theta}\left(r,\theta,z\right)\Big|_{r=r_e} = \sigma_{rz}\left(r,\theta,z\right)\Big|_{r=r_e} = 0$$

and for all $0 \le \theta \le 2\pi$, $r_i \le r \le r_e$ on the cylinder ends

$$\sigma_{zz}\left(r,\theta,z\right)\Big|_{z=0} = \sigma_{zr}\left(r,\theta,z\right)\Big|_{z=0} = \sigma_{z\theta}\left(r,\theta,z\right)\Big|_{z=0} = 0$$
$$\sigma_{zz}\left(r,\theta,z\right)\Big|_{z=L} = \sigma_{zr}\left(r,\theta,z\right)\Big|_{z=L} = \sigma_{z\theta}\left(r,\theta,z\right)\Big|_{z=L} = 0 \tag{2}$$

Given that the pressure loading is all radially directed, it is reasonable to assume that the only stresses developed in the cylinder are the normal stresses $\sigma_{rr}, \sigma_{\theta\theta}$ and to assume that these stresses are only a function of r, i.e.

$$\sigma_{rr} = \sigma_{rr}\left(r\right), \sigma_{\theta\theta} = \sigma_{\theta\theta}\left(r\right)$$
$$\sigma_{r\theta} = \sigma_{rz} = \sigma_{z\theta} = \sigma_{zz} = 0 \tag{3}$$

In this case the boundary conditions of Eq. (2) are satisfied identically for the cylinder ends and the only boundary conditions in Eq. (1) that are not satisfied identically are

$$\sigma_{rr}\left(r\right)\Big|_{r=r_i} = -p_i$$
$$\sigma_{rr}\left(r\right)\Big|_{r=r_e} = -p_e \tag{4}$$

Under these assumptions, the equations of equilibrium, which in cylindrical coordinates are

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + f_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + f_\theta = 0 \qquad (5)$$

$$\frac{\partial \sigma_{zr}}{\partial r} + \frac{\sigma_{zr}}{r} + \frac{1}{r}\frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + f_z = 0$$

reduce to only one non-trivial equation

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \qquad (6)$$

The stress state given by Eq. (3) is one of plane stress so that the 3-D stress-strain relations for a homogeneous, isotropic elastic solid reduce to

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - v\sigma_{\theta\theta}\right)$$

$$\varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - v\sigma_{rr}\right)$$

$$\varepsilon_{zz} = \frac{-v}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

$$\varepsilon_{r\theta} = \frac{\sigma_{r\theta}}{G} = 0 \qquad (7)$$

$$\varepsilon_{z\theta} = \frac{\sigma_{z\theta}}{G} = 0$$

$$\varepsilon_{rz} = \frac{\sigma_{rz}}{G} = 0$$

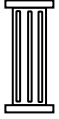or, inverting these relations, we have the usual plane stress equations

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\varepsilon_{rr} + v\varepsilon_{\theta\theta}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\varepsilon_{\theta\theta} + v\varepsilon_{rr}\right) \qquad (8)$$

$$\varepsilon_{zz} = \frac{-v}{E}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

To guarantee that we automatically satisfy compatibility, we will work directly with the displacements as the fundamental unknowns for this problem. Because of the symmetry we expect that there will be no displacement component, u_θ, in the cylinder and that the other displacements will also be independent of θ. Also, because the pressure loading is uniform in z, we expect that the radial displacement is only a function of r, i.e.

$u_r = u_r(r)$. In this case the strains in the cylinder, which are given in cylindrical coordinates as

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \; \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \; \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

(9)

reduce to

$$\varepsilon_{rr} = \frac{du_r}{dr}, \; \varepsilon_{\theta\theta} = \frac{u_r}{r}, \; \varepsilon_{zz} = \frac{du_z}{dz}$$

$$\varepsilon_{r\theta} = \varepsilon_{\theta z} = \varepsilon_{rz} = 0$$

(10)

Placing the strain-displacement relations in Eq.(10) into the stress-strain relations of Eq. (8) gives

$$\sigma_{rr} = \frac{E}{1-v^2}\left(\frac{du_r}{dr} + v\frac{u_r}{r}\right)$$

$$\sigma_{\theta\theta} = \frac{E}{1-v^2}\left(\frac{u_r}{r} + v\frac{du_r}{dr}\right)$$

$$\varepsilon_{zz} = \frac{du_z}{dz} = \frac{-v}{E}(\sigma_{rr} + \sigma_{\theta\theta})$$

(11)

so that when the stresses of Eq. (11) are substituted into the equilibrium equation (Eq. (6)), we find

$$\frac{d^2 u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

(12)

It is easy to solve this equation for the displacement since we can rewrite it equivalently in the form

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru_r)\right] = 0$$

(13)

Integrating once on r gives

$$\frac{d}{dr}(ru_r) = C_1' r \tag{14}$$

where C_1' is a constant of integration. Integrating Eq.(14) once more then yields

$$u_r = \frac{C_1'}{2}r + \frac{C_2}{r} = C_1 r + \frac{C_2}{r} \tag{15}$$

where $C_1 = C_1'/2, C_2$ are both constants of integration. If the displacement expression in Eq. (15) is placed into the stress-strain relations of Eq. (11), one finds

$$\sigma_{rr} = \frac{E}{1-v}C_1 - \frac{E}{1+v}\frac{C_2}{r^2}$$
$$\sigma_{\theta\theta} = \frac{E}{1-v}C_1 + \frac{E}{1+v}\frac{C_2}{r^2} \tag{16}$$
$$\varepsilon_{zz} = \frac{-2v}{1-v}C_1 = constant$$

If we define two new constants, A, B as

$$A = \frac{E}{1-v}C_1, \quad B = \frac{E}{1+v}C_2 \tag{17}$$

then we can rewrite Eq.(16) as

$$\sigma_{rr} = A - \frac{B}{r^2}$$
$$\sigma_{\theta\theta} = A + \frac{B}{r^2} \tag{18}$$
$$\varepsilon_{zz} = \frac{du_z}{dz} = \frac{-2vA}{E} = constant$$

At this point our solution for the stresses (Eq. (18)) and the displacement (Eq. (15)) satisfy equilibrium, compatibility, and the stress-strain relations and are given in terms of two unknown constants (C_1 and C_2 or A and B). To find these constants we must satisfy the boundary conditions (Eq. (4)) which , together with Eq.(18) yield

$$A - \frac{B}{r_i^2} = -p_i$$

$$A - \frac{B}{r_e^2} = -p_e$$

(19)

whose solution is

$$A = \frac{r_i^2 p_i - r_e^2 p_e}{r_e^2 - r_i^2}, \quad B = \frac{r_i^2 r_e^2 (p_i - p_e)}{r_e^2 - r_i^2}$$

(20)

which gives the stresses and displacements as:

$$\sigma_{rr} = \frac{p_i r_i^2}{r_e^2 - r_i^2}\left(1 - \frac{r_e^2}{r^2}\right) - \frac{p_e r_e^2}{r_e^2 - r_i^2}\left(1 - \frac{r_i^2}{r^2}\right)$$

$$\sigma_{\theta\theta} = \frac{p_i r_i^2}{r_e^2 - r_i^2}\left(1 + \frac{r_e^2}{r^2}\right) - \frac{p_e r_e^2}{r_e^2 - r_i^2}\left(1 + \frac{r_i^2}{r^2}\right)$$

$$\varepsilon_{zz} = \frac{du_z}{dz} = \frac{-2v}{E}\left(\frac{r_i^2 p_i - r_e^2 p_e}{r_e^2 - r_i^2}\right) \Rightarrow u_z = \frac{-2v}{E}\left(\frac{r_i^2 p_i - r_e^2 p_e}{r_e^2 - r_i^2}\right)z + C$$

$$u_r = \frac{(1-v)}{E}\left(\frac{r_i^2 p_i - r_e^2 p_e}{r_e^2 - r_i^2}\right)r + \frac{(1+v)}{E}\frac{r_e^2 r_i^2 (p_i - p_e)}{r_e^2 - r_i^2}\frac{1}{r}$$

(21)

where C is an arbitrary constant displacement (translation) in the z-direction.

The solution of Eq. (21) is an exact solution to our pressurized cylinder problem since it satisfies all the governing equations and boundary conditions. One can show that for a pressurized cylinder subject to an internal pressure only, the highest stresses occur on the inner wall where we have

$$\sigma_{rr}\big|_{r=r_i} = \frac{p_i r_i^2}{r_e^2 - r_i^2}\left(1 - \frac{r_e^2}{r_i^2}\right) = -p_i$$

$$\sigma_{\theta\theta}\big|_{r=r_i} = \frac{p_i r_i^2}{r_e^2 - r_i^2}\left(1 + \frac{r_e^2}{r_i^2}\right)$$

(22)

For a thin wall cylinder these stresses become

$$\sigma_{rr}\big|_{r=r_i} = -p_i$$

$$\sigma_{\theta\theta}\big|_{r=r_i} = \frac{p_i r_i^2}{r_e^2 - r_i^2}\left(1 + \frac{r_e^2}{r_i^2}\right) = \frac{p_i}{t}\left(\frac{r_e^2 + r_i^2}{r_e + r_i}\right) = \frac{p_i}{t}\left(\frac{r_e + r_i}{2} + \frac{t^2}{2(r_e + r_i)}\right) \quad (23)$$

$$\approx \frac{p_i}{t}\left(\frac{r_e + r_i}{2}\right) = \frac{p_i r_m}{t}$$

in terms of the thickness $t = r_e - r_i$ and the mean radius $r_m = (r_e + r_i)/2$. This agrees with the result for the thin wall cylindrical pressure vessel hoop stress obtained in Strength of Materials (where the small radial stress component is normally ignored).