Topic 3:
Thick walled pressure vessels

3.1 The basic problem and its solution

The analysis of thick walled cylinders is often treated as a two-dimensional "plane stress" problem[1]. That is, the longitudinal stress $\sigma_{zz} = 0$. This assumption is not correct. A complete and correct solution can be obtained by superposing a uniform longitudinal stress, and will be completed later in these notes.

Figure 1 shows a circumferential slice through a thick walled pressure vessel and the associated co-ordinate system. It also shows a semicircular element cut from that vessel.

The symmetry of axisymmetric vessels indicates that the principal stress directions are radial, circumferential and longitudinal.

There cannot be any shear stress on the surfaces r = "constant" or z = "constant".

Normal radial stress, σ_{rr} and normal circumferential (hoop) stress, $\sigma_{\theta\theta}$ are independent of circumferential angle ϕ.

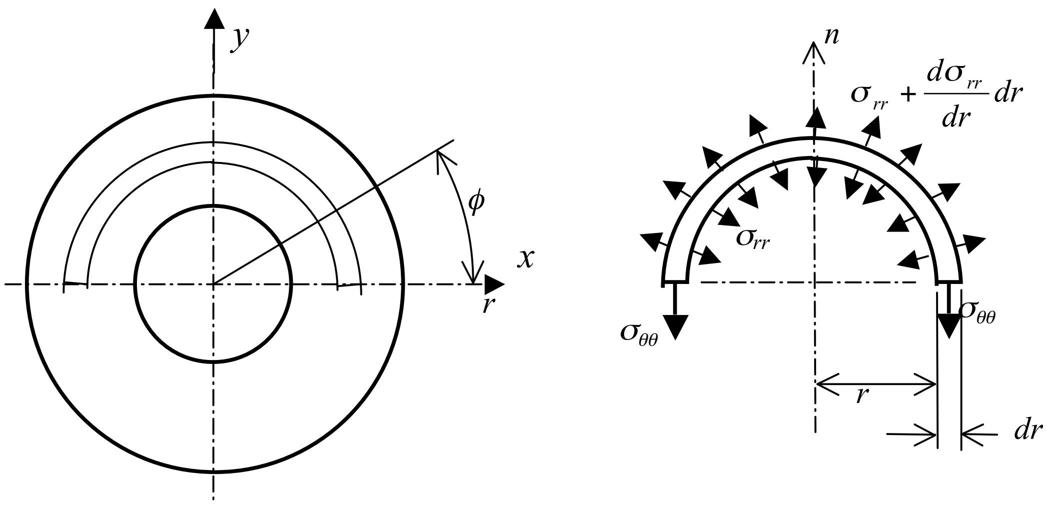


Figure 1. Left: circumferential slice of a thick walled pressure vessel.
Right: semicircular element cut from the vessel.
Cylindrical coordinates used. z-axis is normal to the page

[1] In real engineering components, stress (and strain) are three dimensional (3D) tensors. However, when one of the dimensions of the component is much smaller than the other two, the associated stress can be neglected and the resulting state of stress becomes bi-dimensional. This state is known as "plane stress" because the normal and shear stresses associated with the thin surface are zero. Examples include: thin-walled structures such as plates subject to in-plane loading or thin cylinders subject to pressure loading. The stresses are negligible with respect to the smaller dimension because significant stresses are not able to develop within the material and are small compared to the in-plane stresses.

Stress varies with radius, r and changes by an amount $\frac{d\sigma_{rr}}{dr}dr$ in the radial distance dr. Normal radial stress varies from σ_{rr} to $\sigma_{rr} + \frac{d\sigma_{rr}}{dr}dr$.

Equilibrium in the n-direction (i.e. $\sum F_{\text{vertical}} = 0$) results in:

$$2\sigma_{rr}\, r + 2\left(\sigma_{\theta\theta} + \frac{1}{2}\frac{d\sigma_{\theta\theta}}{dr}dr\right)dr - 2\left(\sigma_{rr} + \frac{d\sigma_{rr}}{dr}dr\right)(r + dr) = 0$$

where

- $2\sigma_{rr}\, r$ corresponds to forces associated with internal stress, and
- $\sigma_{\theta\theta}$ if a function of radius, r, averaged over the element thickness, dr.

Expanding this equation algebraically (i.e. multiplying out the terms) and neglecting higher order terms (i.e. higher order terms will have less impact on the final result) results in:

$$r\frac{d\sigma_{rr}}{dr} + (\sigma_{rr} - \sigma_{\theta\theta}) = 0 \tag{1}$$

Equation (1) is the equilibrium equation for a thick pressure vessel, and contains two unknowns, stresses σ_{rr} and $\sigma_{\theta\theta}$.

Some references develop equation (1) using an incremental circumferential angle, $d\phi$ rather than $\phi = 180°$ as was used here. The development using $d\phi$ is no more rigorous and offers the same (correct) outcome.

To determine a second independent equation so as to solve σ_{rr} and $\sigma_{\theta\theta}$, consider the deformation of the cylindrical pressure vessel (figure 2).

The displacement is constant in the circumferential direction (i.e. axisymmetric geometry) but varies along the radius (i.e. element deformation, and therefore vessel deformation, is a function of radius, only – $\frac{d}{d\theta} = 0$).

The radial displacement of a cylindrical surface of radius r is represented by u.

Then, at radius $(r + dr)$ the displacement of a cylindrical surface is $\left(u + \frac{du}{dr}dr\right)$.

Hence, a unit element, dr undergoes a total elongation in the radial direction of $\left(\frac{du}{dr}dr\right)$, with an associated radial strain of $\varepsilon_{rr} = \frac{du}{dr}$.

To find the circumferential strain, consider the circumferential circle of radius, r with circumference, $2\pi r$.

When an internal pressure, p_i is applied, that circle expands to radius $(r+u)$ with circumference $2\pi(r+u)$.

Thus, the unit elongation of an element in the circumferential direction is equal to the unit elongation of the corresponding radius. The circumferential strain is $\varepsilon_{\theta\theta} = \dfrac{u}{r}$ (i.e. $e = \Delta l / l_0$).

The shear strain $\gamma_{r\theta} = 0$ because there is no associated shear stress, $\sigma_{r\theta} = 0$ (i.e. Hooke's law and symmetry of geometry).

Using Hooke's law, assuming $\sigma_{zz} = 0$:

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) \text{ and } \varepsilon_{\theta\theta} = \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right) \tag{2}$$

Solve simultaneously to derive equations for stress:

$$\sigma_{rr} = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) \text{ and } \sigma_{\theta\theta} = \frac{E}{\left(1-\nu^2\right)}\left(\varepsilon_{\theta\theta} + \nu\varepsilon_{rr}\right) \tag{3}$$

The negative terms associated with Poisson's ratio in the equations for strain revert to positive terms in the stress equations.

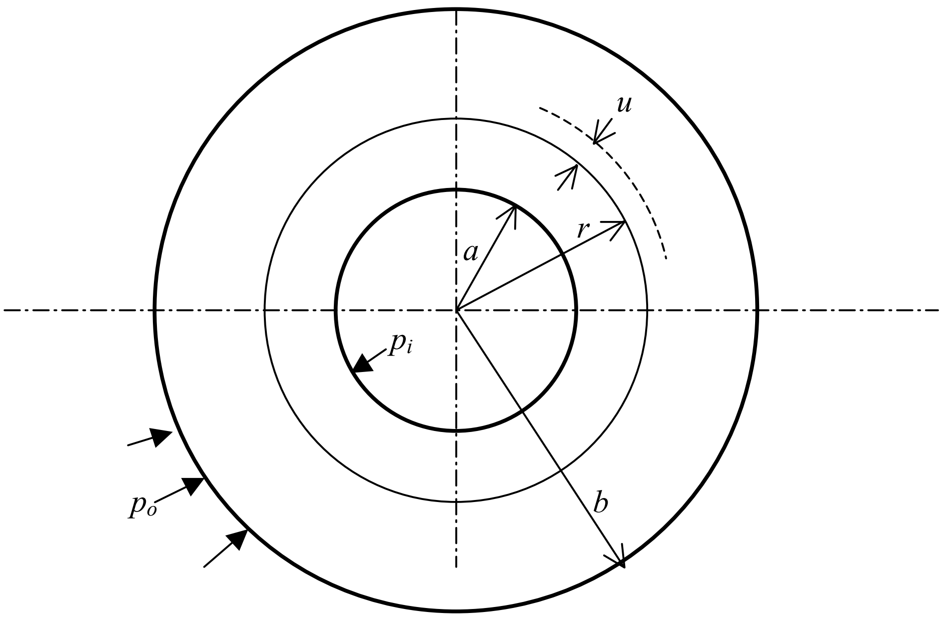


Figure 2. Deformation of the cylindrical pressure vessel.
At radius, r the vessel has a displacement, u (in the radial direction) when loaded.

The stress equations (3) can be substituted into the equilibrium equation (1) to produce:

$$\frac{E}{\left(1-\nu^2\right)}\left[\frac{d^2u}{dr^2}+\frac{\nu}{r}\frac{du}{dr}-\frac{\nu u}{r^2}+\frac{1}{r}\left(\frac{du}{dr}+\nu\frac{u}{r}-\frac{u}{r}-\nu\frac{du}{dr}\right)\right]=0$$

This equation simplifies to:

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}-\frac{u}{r^2}=0 \quad \text{or} \quad \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r\,u)\right)=0$$

The left hand equation on the left is the same equation as that derived for flat circular plates. The right hand equation is sometimes referred to as the homogeneous equation.

Integrating, multiplying by throughout by radius, *r* and integrating again produces the general solution with two constants of integration, C_1 and C_2:

$$u=C_1 r+\frac{C_2}{r} \tag{4}$$

This equation will produce a singularity error when $r = 0$ (i.e. the C_2/r term).

Substituting equation (4) into equation (2) yields:

$$\varepsilon_{rr}=\frac{du}{dr}=C_1-\frac{C_2}{r^2} \quad \text{and} \quad \varepsilon_{\theta\theta}=\frac{u}{r}=C_1+\frac{C_2}{r^2} \tag{5}$$

Substituting into equation (3) yields:

$$\sigma_{rr}=\frac{E}{\left(1-\nu^2\right)}\left[(1+\nu)C_1-(1-\nu)\frac{C_2}{r^2}\right] \quad \text{and} \quad \sigma_{\theta\theta}=\frac{E}{\left(1-\nu^2\right)}\left[(1+\nu)C_1+(1-\nu)\frac{C_2}{r^2}\right] \tag{6}$$

In this analysis, the following are constants:

- $\varepsilon_{rr}+\varepsilon_{\theta\theta}=2C_1$ (i.e. add the two equations for ε_{rr} and $\varepsilon_{\theta\theta}$)
- $\sigma_{rr}+\sigma_{\theta\theta}=\frac{2E}{1-\nu}C_1$ (i.e. add the two equations for σ_{rr} and $\sigma_{\theta\theta}$)

This is due to the opposite signs for the C_1 and C_2 terms.

3.2 Thick pressure vessels

Section 3.1 provides a general solution for thick pressure vessels. The constants C_1 and C_2 for any specific vessel must be found using the specific vessel's boundary conditions.

For a simple pressure vessel, the radial stress at the surfaces must be equal to the applied pressure (with a change of sign). As such, the appropriate boundary conditions can be expressed as:

$$\sigma_{rr} = -p_i \text{ at } r = a \quad \text{and} \quad \sigma_{rr} = -p_o \text{ at } r = b \tag{7}$$

where p_i = internal pressure and p_o = external pressure.

Substituting (7) into equations (6) yields:

$$C_1 = \frac{1-\nu}{E}\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} \quad \text{and} \quad C_2 = \frac{1+\nu}{E}(p_i - p_o)\frac{a^2 b^2}{b^2 - a^2}$$

Substituting these equations for constants C_1 and C_2 into equations (6) produces the radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress distribution equations:

$$\sigma_{rr} = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{(p_i - p_o)}{r^2}\frac{a^2 b^2}{b^2 - a^2} \tag{8}$$

$$\sigma_{\theta\theta} = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{(p_i - p_o)}{r^2}\frac{a^2 b^2}{b^2 - a^2} \tag{9}$$

The radial displacement of a cylindrical surface of radius r is:

$$u = \frac{1-\nu}{E}\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} r + \frac{1+\nu}{E}(p_i - p_o)\frac{a^2 b^2}{b^2 - a^2}\frac{1}{r} \tag{10}$$

These results were obtained assumed that the longitudinal stress, $\sigma_{zz} = 0$ (i.e. ends are unconstrained and open) and are referred to as Lame's equations. Hooke's law for the strain in the z-direction is:

$$\varepsilon_{zz} = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{rr} - \nu\sigma_{\theta\theta}\right)$$

Substituting (8) and (9) into this equation:

$$\varepsilon_{zz} = \frac{-2\nu}{E}\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} \tag{11}$$

Equation 11 identifies how much a vessel shrinks (i.e. negative ε_{zz}) due to internal pressure, p_i and expands (i.e. positive ε_{zz}) due to external pressure, p_o in the z-direction.

It shows that longitudinal strain, ε_{zz} is a constant (i.e. the radial variable, r is not present) and does not vary with radius[2].

Consequently, plane sections through the cylinder will remain plane.

The solution for any desired longitudinal tension can be obtained by superposition of a uniform longitudinal tensile force.

If the vessel has ends that are constrained and closed, a finite longitudinal stresses, σ_{zz} is present, with $\varepsilon_{zz} = 0$. Hooke's law for the strain in the z-direction becomes:

$$\varepsilon_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu\sigma_{rr} - \nu\sigma_{\theta\theta}\right)$$

Rearranging this equation:

$$\sigma_{zz} = \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)$$

Combining $\sigma_{rr} + \sigma_{\theta\theta}$ from equation (6):

$$\sigma_{rr} + \sigma_{\theta\theta} = \frac{2E}{\left(1-\nu^2\right)}\left[(1+\nu)C_1\right]$$

Substituting into Hooke's law equation for σ_{zz}:

$$\sigma_{zz} = 2EC_1\frac{\nu(1+\nu)}{\left(1-\nu^2\right)} = 2EC_1\frac{\nu}{(1-\nu)}$$

Knowing that $C_1 = \frac{1-\nu}{E}\frac{a^2p_i - b^2p_o}{b^2-a^2}$,

$$\sigma_{zz} = 2\nu\left(\frac{a^2p_i - b^2p_o}{b^2-a^2}\right)$$

If the vessel has ends that are unconstrained and closed, with internal and external pressures acting on the internal and external surfaces of the end, respectively, the axisymmetric force equilibrium becomes:

$$\sigma_{zz}\pi\left(b^2-a^2\right) + p_o\pi b^2 - p_i\pi a^2 = 0$$

Rearranging for σ_{zz}:

$$\sigma_{zz} = \frac{a^2p_i - b^2p_o}{b^2-a^2}$$

[2] If the cylinder is long in the z-direction, a "plane strain" state is assumed, where the strains associated with length, i.e the strains in the z-direction, $\varepsilon_{zz} = \gamma_{xy} = \gamma_{yx} = 0$.

3.3 Examples – case studies

3.3.1 Internal pressure only, $p_o = 0$

This is the most common circumstance, where equations (8) and (9) become:

$$\sigma_{rr} = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \tag{12}$$

The ratio of internal radius, a to external radius, b is the usual method by which information is presented. The stress distribution for $a/b = 0.5$ is shown in figure 3.

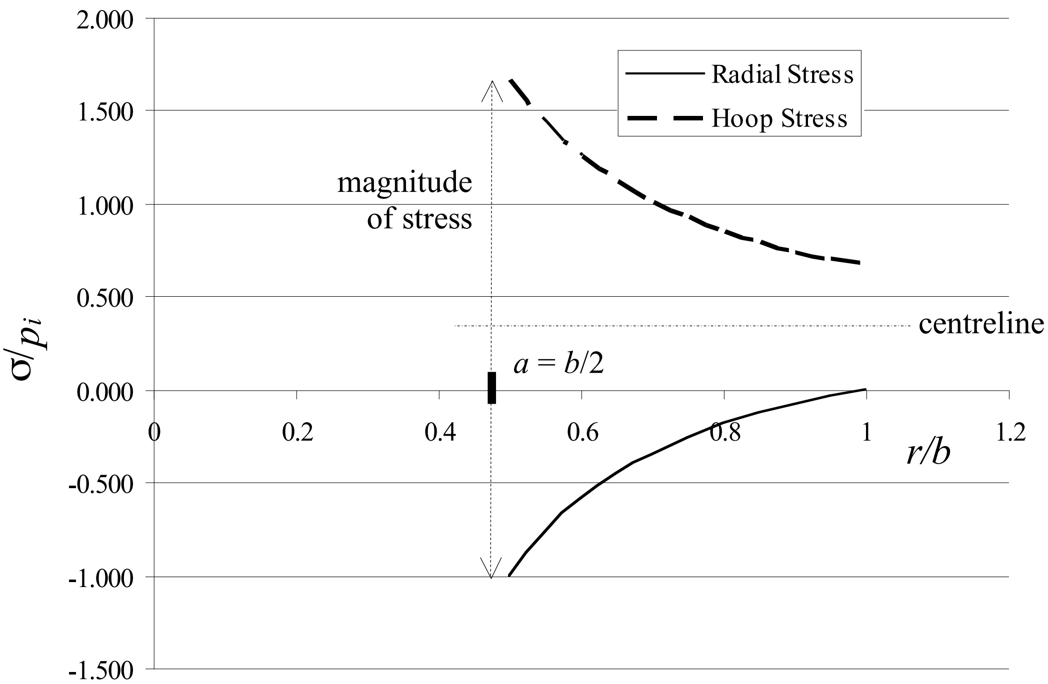


Figure 3. Radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress profiles for $a/b = 0.5$.
Both axes are dimensionless.
Curves are symmetric about a central horizontal line.

Suppose that a pressure vessel with inside radius, *a* is to be manufactured from a material with an allowable stress, *S*.

The designer needs to know:

- how do the allowable pressure and
- overall vessel mass

vary with outside radius, *b*?

Initial yielding will occur at the inside radius. Select a value for inside radius, *a* and then change b to then accommodate the required pressure.

Substituting $r = a$: into radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress equations:

$$\sigma_{rr\,\max} = -p_i \quad \text{and} \quad \sigma_{\theta\theta\,\max} = p_i \frac{b^2 + a^2}{b^2 - a^2}$$

The magnitude of the longitudinal stress will be between these two stress levels. Using Tresca's maximum shear stress criterion (i.e. $S_y = \sigma_1 - \sigma_3$):

$$p_i \left(\frac{b^2 + a^2}{b^2 - a^2} + 1 \right) = S \quad \text{or} \quad \frac{p_i}{S} = \frac{1}{2}\left(1 - \left(\frac{a}{b} \right)^2 \right)$$

The volume of material used per unit length of the vessel is $V = \pi\left(b^2 - a^2\right)$. Rearranging terms:

$$\frac{V}{a^2} = \pi\left(\left(\frac{b}{a} \right)^2 - 1 \right)$$

Figure 4 shows how the two variables, $\frac{p_i}{S}$ and $\frac{V}{a^2}$ vary with $\frac{b}{a}$.

Since *a* and *S* are fixed, figure 4 is effectively a graph of pressure versus outside radius, *b*.

Increasing *b*/*a* beyond a value of around 3 provides little increase in the allowable internal pressure, p_i.

Figure 5 is a graph of cylinder mass (proportional to the volume of material used) versus the contained pressure. The weight (and associated cost) of the vessel increases dramatically if the internal pressure is to be above approximately 40% of the allowable stress. The graph shows that simply increasing the wall thickness of a pressure vessel is not an effective way of achieving high internal pressures.

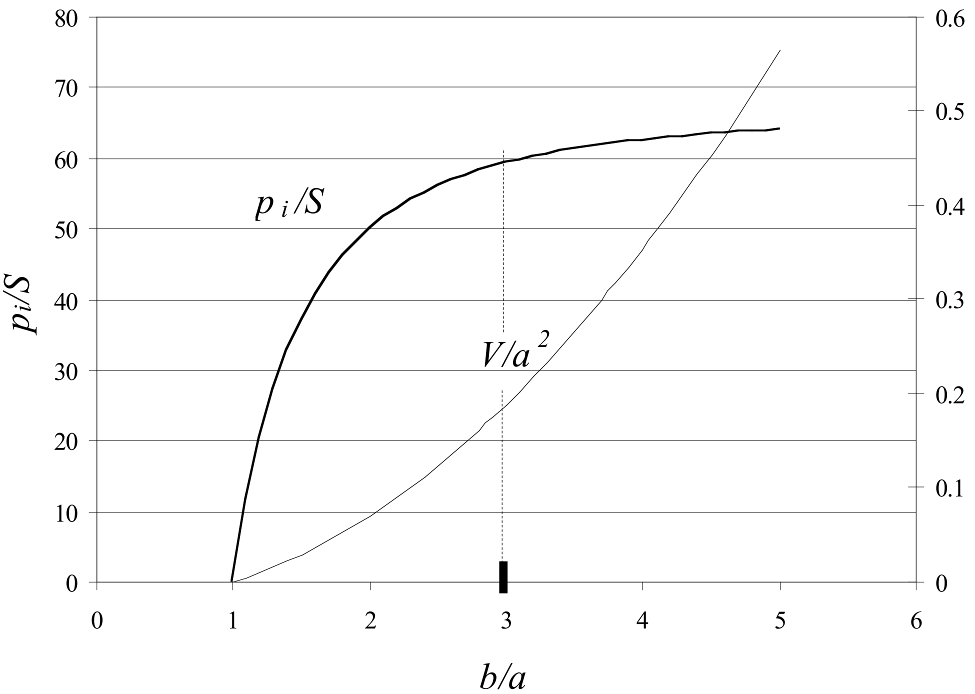


Figure 4. Variables, $\frac{p_i}{S}$ (internal pressure/stress) and $\frac{V}{a^2}$ (volume measure) versus diametral ratio b/a.

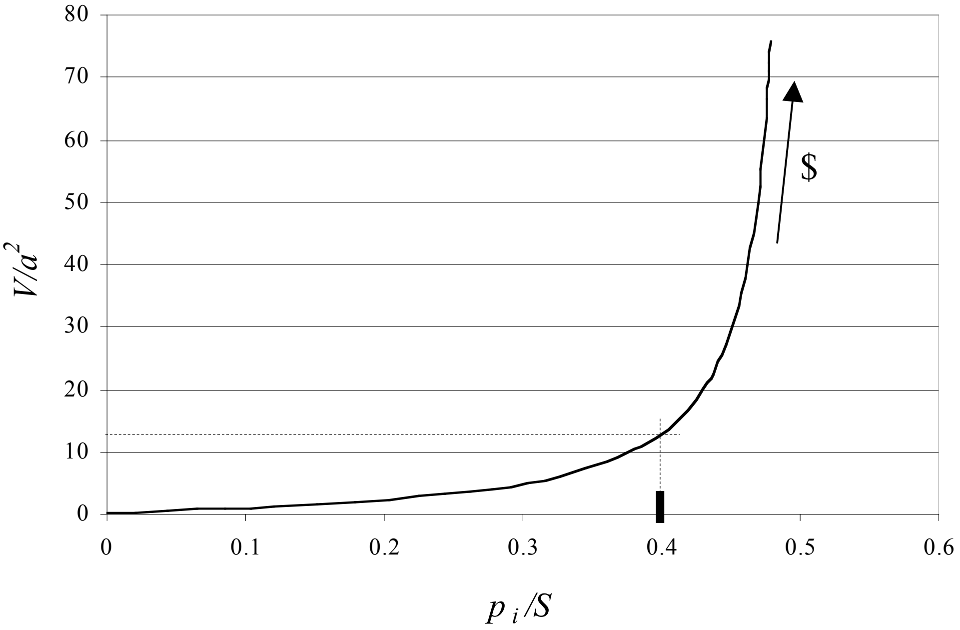


Figure 5. $\frac{V}{a^2}$ (volume measure) versus $\frac{p_i}{S}$ (internal pressure/stress).

When internal pressure, p_i is greater than 40% of allowable stress, S volume of material required, and associated cost (cost $\propto$ mass of material) increases rapidly.

Considerations:

- exotic materials with higher allowable stress, S
  (i.e. higher yield, S_{yld} and ultimate, S_{ut} strengths) are more expensive,
- within the chemical industry, vessel material selection are often limited by the properties of the materials being contained (e.g. corrosive),
- high pressure containment is expensive with a non-linear relationship between increasing pressure and associated cost.

3.3.2 External pressure only, $p_i = 0$ and $p_o \neq 0$

For example, submarine or vacuum chamber design.

Following the method used in example 1, equations (8) and (9) become:

$$\sigma_{rr} = \frac{-b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{-b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right) \tag{13}$$

At all radii, the hoop stress is greater than the external pressure.
Initial yield will occur at the inside, not the outside.

Vessels with external pressure may fail by buckling rather than yield.
This mode of failure is not considered here.

If $b >> a$ (i.e. an extremely small hole relative to the outside radius),

$$\sigma_{rr} = -p_o\left(1 - \frac{a^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = -p_o\left(1 + \frac{a^2}{r^2}\right)$$

Then, if $b > r >> a$ these equations for radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress reduce to $\sigma_{rr} = \sigma_{\theta\theta} = -p_o$. As such, the outside surface of the vessel is equally stressed in all directions. This state is referred to as "biaxial compression".

At the inside of the vessel, on the surface of the hole, $r = a$,

- $\sigma_{rr} = 0$
- $\sigma_{\theta\theta} = -2p_o$

This corresponds to a stress concentration factor of 2.0.

Setting the external pressure, p_o to an equivalent applied stress, σ facilitates the solution of a plate in biaxial tension with a circular hole, i.e. $p_o = -\sigma$ (figure 6).

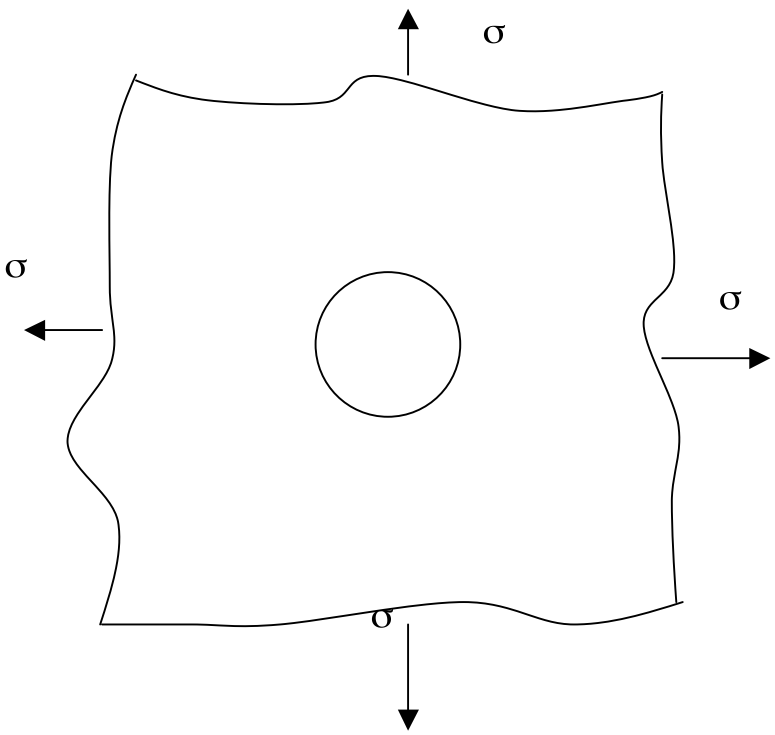


Figure 6. General plate with a circular hole in biaxial tension
(a large flat plate with the same stresses in both directions).

3.3.3 When can the thin shell theory be used?

The formulae for the stresses at the inside surface (i.e. $r = a$) of a thick walled vessel with internal pressure only are:

$$\sigma_{rr\,\max} = -p_i \quad \text{and} \quad \sigma_{\theta\theta\,\max} = p_i \frac{b^2 + a^2}{b^2 - a^2}$$

Using Tresca's maximum shear stress criterion (i.e. $S_y = \sigma_1 - \sigma_3$),

$\sigma_{\theta\theta} - \sigma_{rr} = \dfrac{2b^2}{b^2 - a^2} p_i$ is compared with the allowable stress, *S*.

For a vessel with a thin shell[3], Tresca is again used to compare the hoop stress $\sigma_h = p_i R/t$ with the allowable stress, *S*.

Substitute $t = b - a$ and $R = (b + a)/2$ results in $\sigma_h = \dfrac{p_i(b + a)}{2(b - a)}$.

The ratio of stress predicted by the thick and thin shell wall theories is:

$$\frac{\sigma_{\theta\theta} - \sigma_{rr}}{\sigma_h} = \frac{4b^2}{(b + a)^2} \tag{14}$$

[3] For thin walled pressure vessel design, hoop stress, σ_h is the only significant stress in the sectional plane, with radial pressure stress of far lesser magnitude and longitudinal stress, σ_l acting along the longitudinal axis of the vessel.

Figure 7 plots the ratio of $\sigma_{\text{thick}} = \sigma_{\theta\theta} - \sigma_{rr}$ and $\sigma_{\text{thin}} = \sigma_h$ versus the ratio of outer and inner vessel radius, b/a.
As a rule of thumb, if $b/a < 1.1$ then the thin shell theory can be used for engineering designs when an error of up to about 10% is tolerable.

More precise calculations should use the thick shell theory equations.

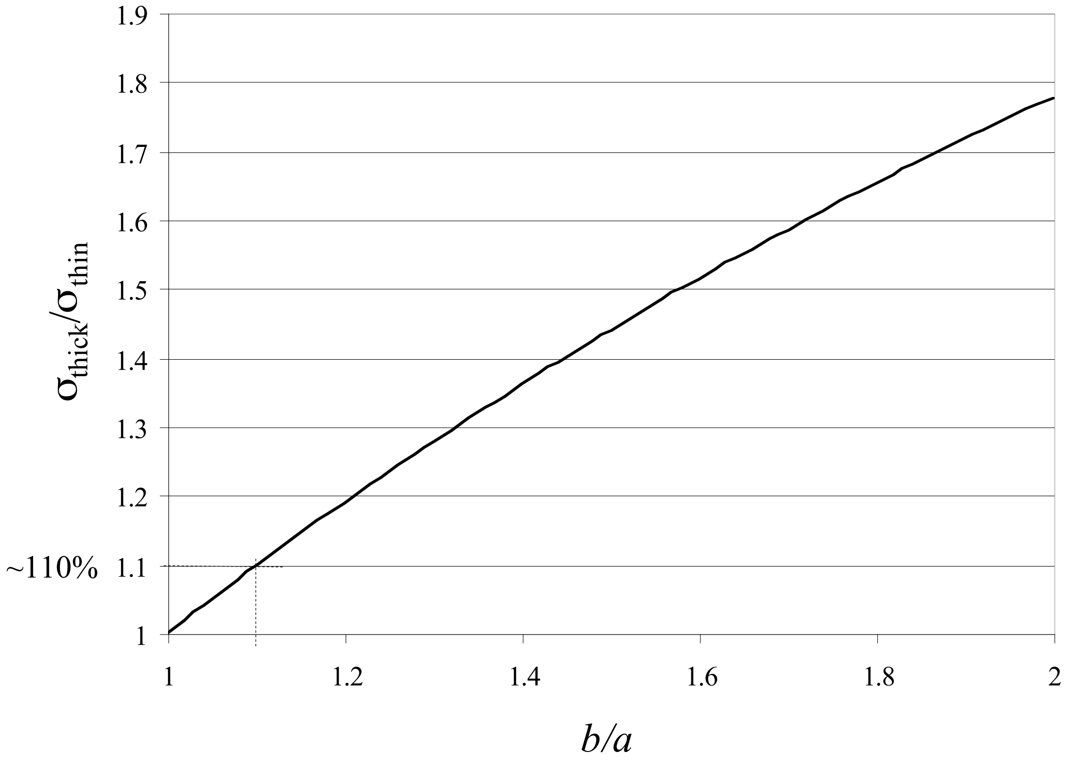


Figure 7. Ratio of $\sigma_{\text{thick}} = \sigma_{\theta\theta} - \sigma_{rr}$ and $\sigma_{\text{thin}} = \sigma_h$ versus the geometric ratio of outer and inner radii, b/a ($b/a \approx 1.1$ is identified).

3.3.4 Press and shrink fits

Knowing radial displacements precisely are important to enable precise fits between components, in this work, concentric cylinders (shells).

For thick pressure vessel design, the constants C_1 and C_2 were found to be:

$$C_1 = \frac{1-\nu}{E}\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} \quad \text{and} \quad C_2 = \frac{1+\nu}{E}(p_i - p_o)\frac{a^2 b^2}{b^2 - a^2}$$

Substitution into $u = C_1 r + \frac{C_2}{r}$ provides equations for the radial displacement, u of a cylindrical surface of radius, r (equation 10):

$$u = \frac{1-\nu}{E}\frac{a^2 p_i - b^2 p_o}{b^2 - a^2} r + \frac{1+\nu}{E}(p_i - p_o)\frac{a^2 b^2}{b^2 - a^2}\frac{1}{r}$$

At the inner surface, $r = a$:

$$u_a = \frac{a}{E\left(b^2 - a^2\right)}\left[a^2 p_i - b^2(2p_o - p_i) + \nu\, p_i\left(b^2 - a^2\right)\right] \tag{15}$$

If, in addition, $p_o = 0$ as would often be true for the outer shell of a shrink fit:

$$u_a = \frac{a\, p_i}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} + \nu\right) \tag{16}$$

At the outer surface $r = b$

$$u_b = \frac{b}{E\left(b^2 - a^2\right)}\left[a^2(2p_i - p_o) - b^2 p_o + \nu\, p_o\left(b^2 - a^2\right)\right] \tag{17}$$

If, in addition, $p_i = 0$ as would often be true for the inner shell of a shrink fit:

$$u_b = \frac{-b\, p_o}{E}\left(\frac{a^2 + b^2}{b^2 - a^2} - \nu\right) \tag{18}$$

These formulae are used in the design of duplex pressure vessels (section 3.4).

3.4 Duplex[4] Pressure Vessels

The preceding analysis of pressure vessels with an internal pressure demonstrated that adding more material to the outside of the vessel is ineffective in achieving a high allowable pressure.

If the vessel is made from two thick concentric tubes, fitted together with an interference fit, then:

- the stress in the outer shell is increased while
- the stress in the inner shell is decreased.

In this way, the vessel material is stressed more uniformly with an increased efficiency of material use.

In principle, more than two concentric shells would offer an even better result, but the extra complications in manufacture and assembly is not usually justified by the resulting vessel performance.

This method has an additional advantage that the inner shell can be made, for example, from an expensive corrosion resistant material (e.g. stainless steel, sometimes with a rubber or polymer inner lining), while the outer shell is made from a less exotic material (e.g. carbon steel) that may offer increased strength (figure 10) and reduced cost.

Superposition is used in the analysis of this vessel design. Figure 8 shows a series of schematic images associated with the analysis of a duplex pressure vessel.

The complete vessel (figure 8, top left) has:

- internal pressure, p_i
- external pressure, p_o and
- interference pressure (not the interference fit pressure, p_f) at the interface of the two shells.

The stress distribution in the overall vessel is obtained by superposition of the three simple cases: A, B and C (figure 8).

The following discussion considers the assembly and loading of the vessel.

Components A and B are assembled without any external pressure, p_i or p_o applied. Because of the interference there will be an interference fit pressure p_f on the inside of the outer shell and an equal pressure on the outside of the inner shell. Both components can be solved using the analysis already derived (section 3.3) and the stress distributions can be found.

[4] Duplex (*adjective*): consisting of two parts, especially two identical or equivalent parts.

The usual assembly method is to heat the outer shell, causing it to expand. The two shells can then be assembled with minimal sliding (frictional) force. When the outer shell cools and contracts the interference pressure can be quite high. Accurate machining is essential to control the dimensions to provide the required interference fit (contact) pressure, p_f.

If a working pressure load is then applied, the whole vessel deflects as if a single component. In particular, the radial displacement of both the inner and outer shells is the same. Consequently, the elastic behaviour and change in stress distribution due to the external pressure load is exactly as it would be for a single shell, when both shells are made from the same material. The advantage of a duplex vessel is associated with the pre-stress resulting from p_f.

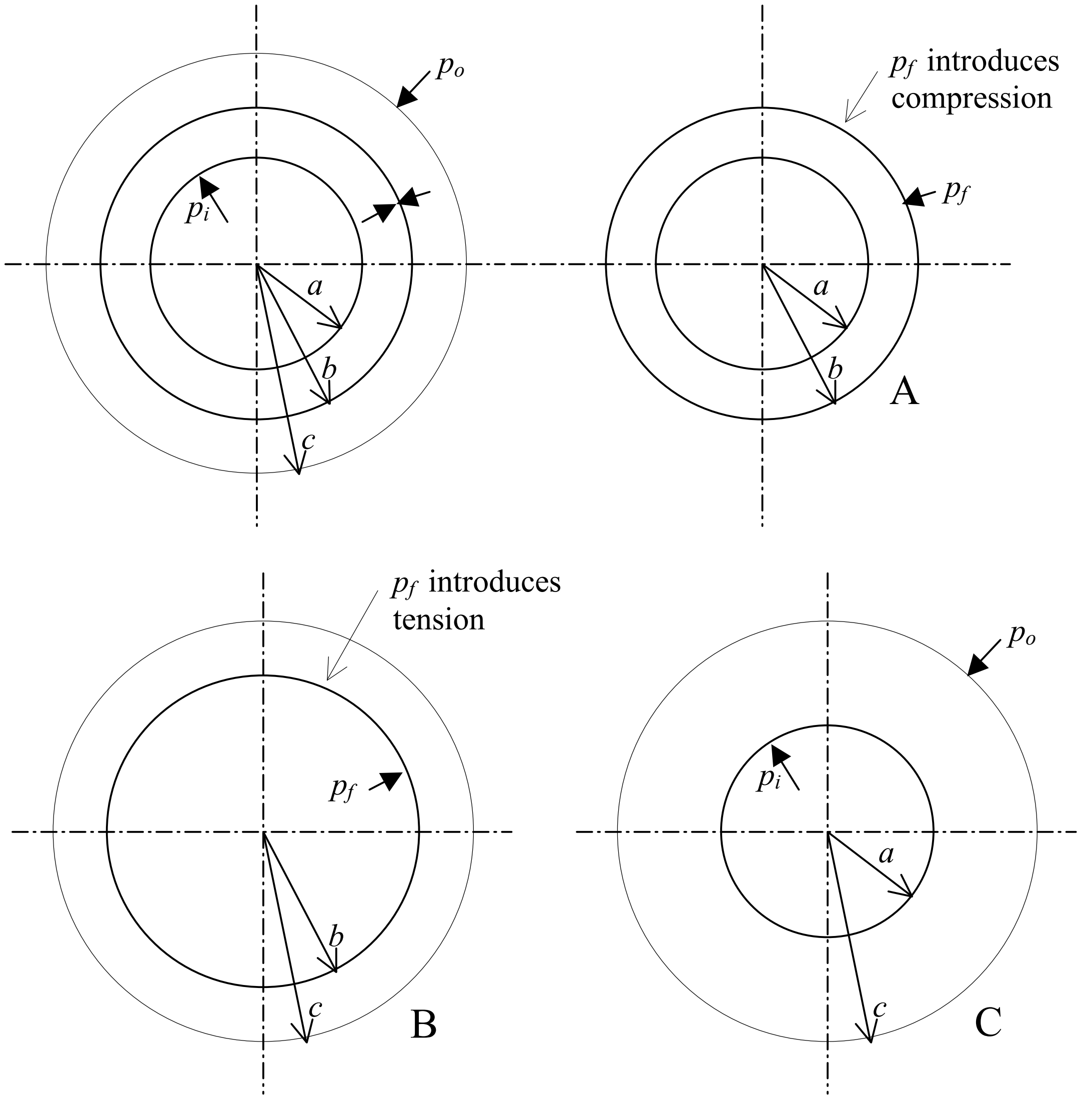


Figure 8. Top left: complete duplex complete vessel schematic.
Top right: A = inner vessel with interference fit pressure, p_f .
Bottom left: B = outer vessel with interference fit pressure, p_f .
Bottom right: C = equivalent vessel with external loads internal pressure, p_i and external pressure, p_o .

Consider the first step of assembling the inner and outer shells, where:

- the inner radius of the outer shell is b_o and
- the outer radius of the inner shell be b_i .

If there is to be an interference fit, then $b_i > b_o$.

After the assembly process has been completed and material temperatures have equalised, both shells have the same radius, b. Figure 9 demonstrates the process.

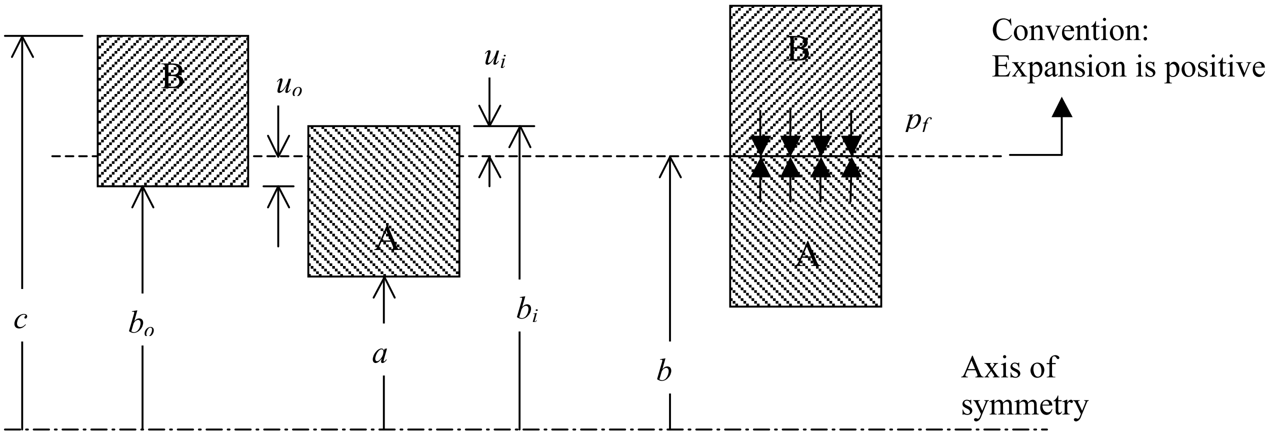


Figure 9. Sections of shells A and B (unit thickness), showing the deflection of both inner, u_i and outer, u_o shells and the associated interference fit pressure, p_f after assembly. Convention: shrinkage is negative, so u_i has a negative value.

Let $b_i - b_o = \delta = u_i + u_o$, where b_i and b_o are the precise dimensions of the shells that will create an interference (shrink) fit when assembled.

After assembling the two shells (sometimes referred to as "fitting"):

- For the inner shell, $p_o = p_f$ and $p_i = 0$
  Inner shell dimensions: inner radius = a, as assembled outer radius = b_A
  Substituting into equation (18):

$$u_i = \frac{-b_A p_f}{E}\left(\frac{b_A^2 + a^2}{b_A^2 - a^2} - \nu\right) \tag{19}$$

- For the outer shell $p_o = 0$ and $p_i = p_f$
  Outer shell dimensions: as assembled inner radius = b_B, outer radius = c
  Substituting into equation (16):

$$u_o = \frac{b_B p_f}{E}\left(\frac{b_B^2 + c^2}{c^2 - {b_B}^2} + \nu\right) \tag{20}$$

Deflection of the outer surface of the inner shell, u_i is negative, so
$\delta = u_o - u_i$ and $b_A \approx b_B = b$.

Substitute equations (19) and (20) into $\delta = u_o - u_i$ and rearrange to solve for the interference fit pressure, p_f :

$$p_f = \frac{E\delta}{b}\left[\frac{\left(b^2-a^2\right)\left(c^2-b^2\right)}{2b^2\left(c^2-a^2\right)}\right] \tag{21}$$

Having found the interference pressure, p_f the stress distribution in both the inner and outer shells can now be found using equations (12) and (13), modified here to accommodate duplex vessel terminology (i.e. surface radii *a*, *b* and *c*).

Outer shell stresses due to interference fit:

$$\sigma_{rr} = \frac{b^2 p_f}{c^2-b^2}\left(1-\frac{c^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{b^2 p_f}{c^2-b^2}\left(1+\frac{c^2}{r^2}\right) \tag{22}$$

Inner shell stresses due to interference fit:

$$\sigma_{rr} = \frac{-b^2 p_f}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{-b^2 p_f}{b^2-a^2}\left(1+\frac{a^2}{r^2}\right) \tag{23}$$

The stress distributions due to the external pressure loads are available from equations (8) and (9), modified to accommodate duplex vessel terminology (i.e. surface radii *a*, *b* and *c*):

$$\sigma_{rr} = \frac{a^2 p_i - c^2 p_o}{c^2-a^2} - \frac{\left(p_i - p_o\right)}{r^2}\frac{a^2 c^2}{c^2-a^2} \tag{24}$$

$$\sigma_{\theta\theta} = \frac{a^2 p_i - c^2 p_o}{c^2-a^2} + \frac{\left(p_i - p_o\right)}{r^2}\frac{a^2 c^2}{c^2-a^2} \tag{25}$$

Using Tresca's maximum shear stress criterion (i.e. $S_y = \sigma_1 - \sigma_3$), stress components σ_{rr} and $\sigma_{\theta\theta}$ are then simply added to assess the factor of safety of the duplex vessel being designed.

3.4.1 Duplex vessel example

A compound pressure vessel has both shells made from steel with:

- Young's modulus, E = 210 GN/m^2 and
- thermal expansion co-efficient, $\alpha = 11 \times 10^6$

The three surface radii are a = 800, b = 1,000 and c = 1,200 mm
Radial interference, $\delta = 0.154$ mm
Working internal pressure, p_i = 30 MN/m^2.

Solving the stress distribution for this vessel involves substituting values into the equations (21) to (25).

Equation (21) finds interference pressure, p_f = 3.202 MN/m^2.

The stress distributions are as shown in figure 10. The radial interference has been carefully chosen so that both shells have the same $(\sigma_{\theta\theta} - \sigma_{rr})$ and, consequently, the same design factor of safety given that the shells are made from the same material.

The temperature needed to expand the outer shell by the radial interference $\delta = 0.154$ mm is calculated using $\delta = \alpha\, r\, \Delta T$.
A mere $\Delta T = 14$ °C temperature difference is required to expand the outer shell by $\delta = 0.154$ mm at $r = b = 1000$ mm.

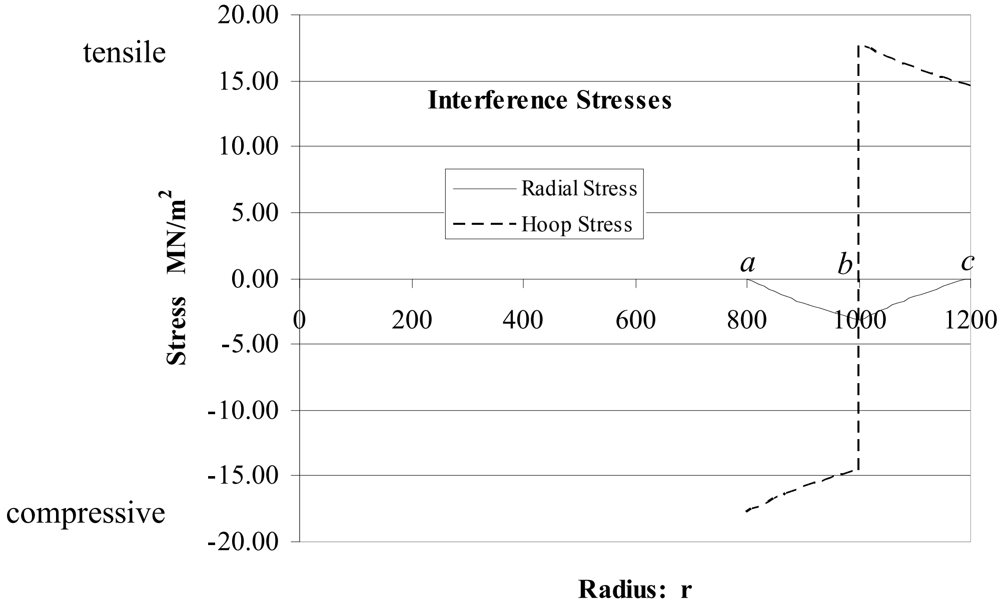
In practice, a much higher temperature difference would be used so that the shells could be easily assembled.

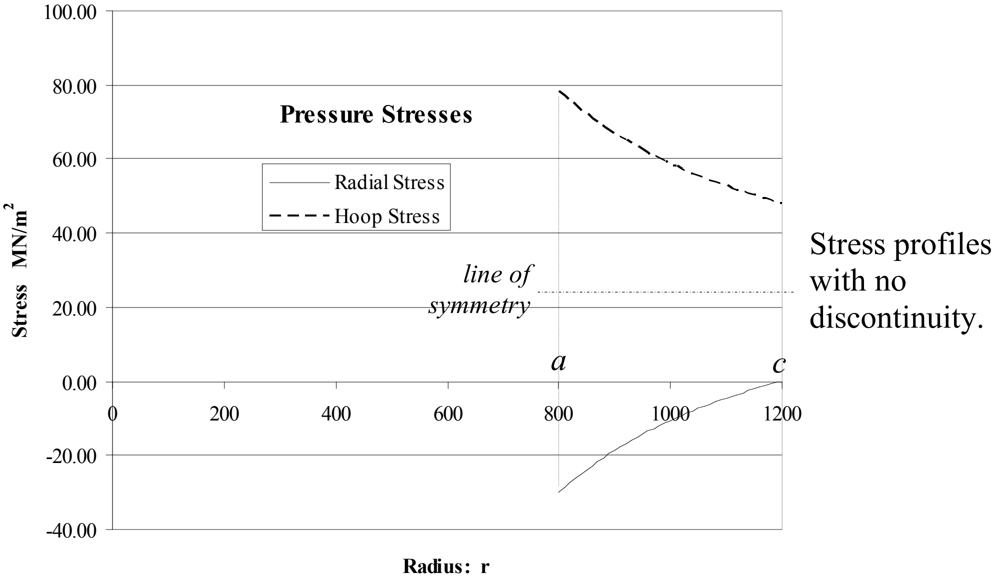
The radial interference chosen allows both shells to reach a maximum allowable stress simultaneously.

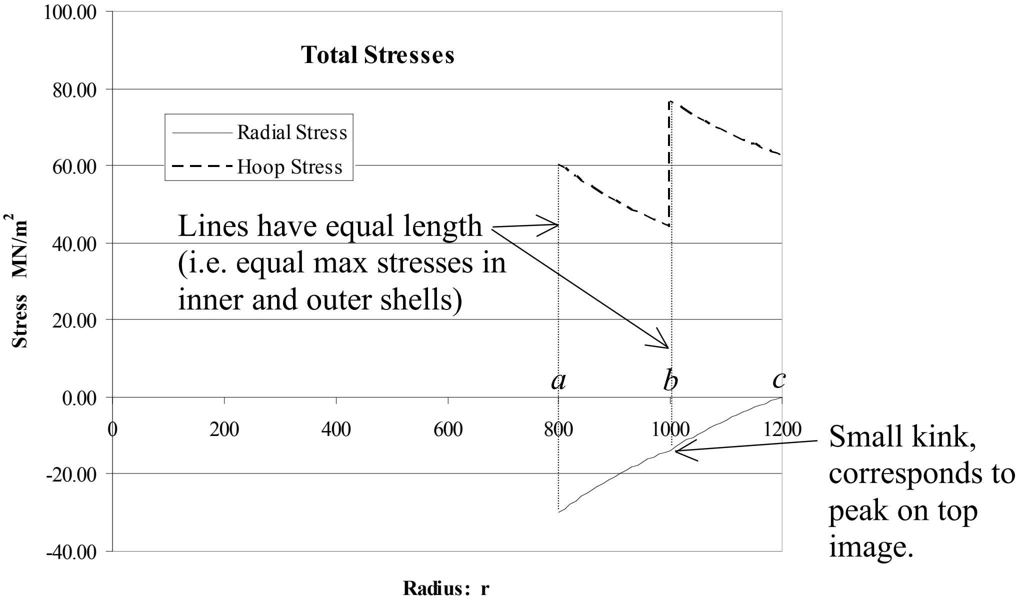
This example demonstrates, for internally applied pressure, that both shells would initially fail (i.e. stress is maximum) on their inner surfaces.

If equal maximum stresses are to be achieved in the inner and outer shells, the machining of the interfering surfaces must be extremely accurate to facilitate the required radial interference.

Figure 10 (following page):
Top: stresses from interference pressure, p_f (equations 22 and 23)
Middle: stresses from applied pressure, p_i (equations 24 and 25)
Bottom: Combined stresses resulting from p_f and p_i


tensile
compressive
Interference Stresses
Radial Stress
Hoop Stress
Stress MN/m2
Radius: r
a
b
c
20.00
15.00
10.00
5.00
0.00
-5.00
-10.00
-15.00
-20.00
0
200
400
600
800
1000
1200


Pressure Stresses
Radial Stress
Hoop Stress
Stress MN/m2
Radius: r
line of
symmetry
Stress profiles
with no
discontinuity.
a
c
100.00
80.00
60.00
40.00
20.00
0.00
-20.00
-40.00
0
200
400
600
800
1000
1200


Total Stresses
Radial Stress
Hoop Stress
Stress MN/m2
Radius: r
Lines have equal length
(i.e. equal max stresses in
inner and outer shells)
Small kink,
corresponds to
peak on top
image.
a
b
c
100.00
80.00
60.00
40.00
20.00
0.00
-20.00
-40.00
0
200
400
600
800
1000
1200

3.4.2 The use of different materials in duplex vessel design

Frequently it is convenient to use different materials for the inner and outer shells of a duplex vessel. For example, in corrosive environments.

The initial interference pressure, p_f is found by using equations (19) and (20):

$$u_i = \frac{-b_A p_f}{E}\left(\frac{b_A^2 + a^2}{b_A^2 - a^2} - \nu\right) \quad \text{and} \quad u_o = \frac{b_B p_f}{E}\left(\frac{b_B^2 + c^2}{c^2 - b_B{}^2} + \nu\right)$$

where $\delta = u_o - u_i$ and $b_A \approx b_B = b$. Substitute the radial interference, δ into:

$$p_f = \frac{\delta}{bK} \tag{26}$$

where $K = \frac{1}{E_o}\left(\frac{b^2 + c^2}{c^2 - b^2} + \nu_o\right) + \frac{1}{E_i}\left(\frac{b^2 + a^2}{b^2 - a^2} - \nu_i\right)$

and
E_o, E_i = Young's modulus of outer and inner shells, respectively
ν_o, ν_i = Poisson's ratio of outer and inner shells, respectively.

However, the stress superposition exercise completed in section 3.4.1 becomes more difficult with dissimilar material properties. The vessel would not behave as if a single shell, as the elastic behaviour and stress distribution due to the externally applied pressures, p_i or p_o will vary due to the dissimilar material properties.

Problems of this kind are best approached from first principles.

The inner shell has a known internal pressure, p_i
but the pressure on its external surface pressure, p_s and deflection, u_i are unknown.

The outer shell has a known external pressure, p_o and unknown pressure on its internal surface, p_s .

The deflection of the inside surface of the outer sell is $u_o = u_i + \delta$.
This relationship between the displacements of the shells and the radial interference, δ is sufficient to solve for the interference surface pressure, p_s .

The stress distribution in both shells can then be found.

3.4.3 Optimum radius for interface ($r = b$) for shells of same material

The best use of material arises when the two shells reach their yield stress, S_{yld} at the same externally applied pressure, p_i or p_o. Initial yield occurs on the inner surface of both inner and outer shells. If the two shells are made from the same material then the critical (design) stresses can be equated to derive an equation for initial interference pressure, p_f.

The algebra is substantial but facilitates the elimination of p_f from subsequent stress equations for either shell.

The following equation can then be obtained for the internal pressure at yield:

$$(p_i)_{\text{yld}} = S_{\text{yld}}\left[1 - \frac{1}{2}\left(\frac{a^2}{b^2} + \frac{b^2}{c^2}\right)\right] \tag{27}$$

Then, by solving $\dfrac{\partial (p_i)_{\text{yld}}}{\partial b} = 0$ a maximum can be found for $(p_i)_{\text{yld}}$ when $b = \sqrt{ac}$.

This demonstrates there is an optimum radius for the interface between shells (i.e. the interference radius, b) and it occurs when the interface radius is the geometric mean of the inside and outside radii of the vessel, a and c, respectively.

3.4.4 Interference fit of a wheel onto a shaft

It is common practice to use interference fits[5] to fix a wheel or ring onto a shaft. Examples include sliding and rolling element bearings, and turbine discs. Differential thermal expansion, hydraulic pressure and axial loading (pressing the components together) are among the methods used to achieve an interference fit between components.

The use of adhesives, particularly for applications where the components will never have to be disassembled, has resulted in press fitting being a less common method of assembly.

If the shaft is hollow the problem is exactly the same as assembling two halves of a duplex (compound) pressure vessel.

If the shaft is solid, its stress distribution is more easily solved.

From section 3.3.2:

> If $b > r >> a$ the equations for radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress reduce to $\sigma_{rr} = \sigma_{\theta\theta} = -p_o$. As such, the outside surface of the vessel is equally stressed in all directions. This state is referred to as "biaxial compression".

When considering duplex vessels: $\sigma_{rr} = \sigma_{\theta\theta} = p_f$.

The radial strain is found using Hooke's law:

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) = -\frac{p_f(1-\nu)}{E}$$

The radial displacement, u_i at the outer surface of the shaft, $r = b$ is:

$$u_i = b\,\varepsilon_{rr} = -\frac{b\,p_f(1-\nu)}{E} \tag{28}$$

The interference pressure, p_f. is calculated using:

$$p_f = \frac{E\delta\left(c^2 - b^2\right)}{2bc^2} \tag{29}$$

[5] Detailed information available in Australian Standard AS1654 – 1974 "Limits and fits for engineering"

3.5 Autofrettage of pressure vessels

Autofrettage is a metal fabrication technique in which a pressure vessel is subjected to very high internal pressure, causing internal portions of the vessel to yield, which introduces internal residual stresses (compressive around the inner surface and tensile around the outer surface) when the very high pressure is removed. This approach is an alternative to duplex vessel design, and is more likely to be used due to its lower manufacture cost.

Autofrettage is commonly used in the manufacture of high pressure pump cylinders, battleship and tank cannon barrels[6] and fuel injection systems for diesel engines. While some work hardening will occur duing the autofrettage process, work hardening is not the primary mechanism of strengthening.

Metals such as steel and titanium exhibit a distinct “yield point”. That is, at small strains, within their elastic zone, they deform without work hardening.
Most metals do not exhibit a yield point and continue to work-harden[7] through ongoing loading

This treatment of autofrettage assumes the material under consideration is “elastic – ideally plastic”, where no work hardening occurs (figure 11).

The assumption that no work hardening occurs applies well to steel and is an approximation for work hardening materials such as aluminium.

This treatment of autofrettage can be adapted to consider the contribution of work hardening, but will not be done here.

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[6] When used to increase the pressure carrying capacity (i.e. strengthen) cannon barrels, the internal barrel diameter is pre-bored slightly undersized, and then a slightly oversized die is pushed (pressed) through the barrel. The amount of initial undersized diameter in combination with the oversized diameter of the die are calculated to strain the barrel material past its elastic limit and into its plastic deformation zone, so that the final strained diameter is the final desired bore.

[7] Work hardening, or strain hardening, is an increase in the strength of a material due to plastic deformation. In solid metals, permanent change of shape is usually carried out on a microscopic scale by defects called dislocations, which are created by the stress associated with loading. At normal temperatures these defects do not anneal out of the material but build up as the material is loaded, where various dislocations interfere with each other’s motion, with a resulting increase in strength and reduced ductility.

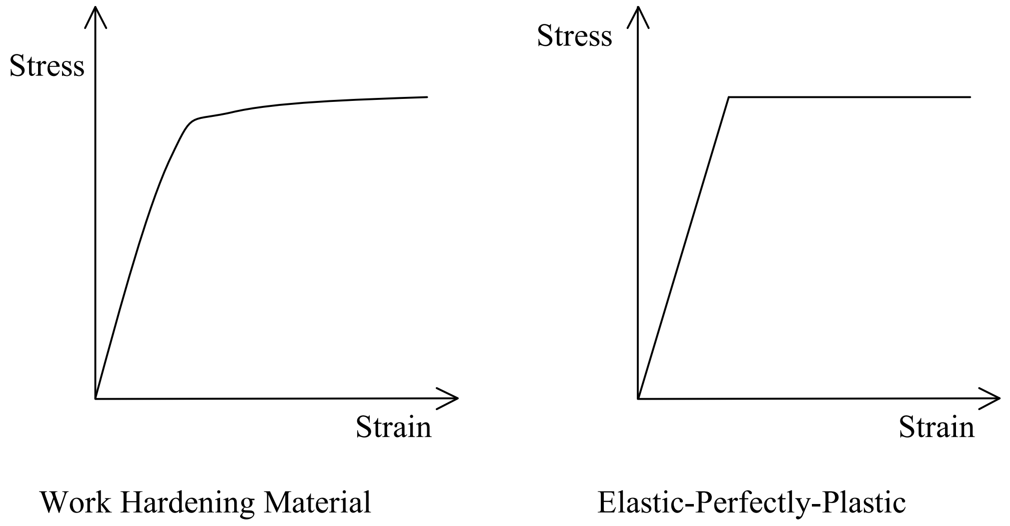

Figure 11. Idealised stress-strain curves for
"work hardening" and "elastic-perfectly-plastic" materials.

Autofrettage seeks to obtain a better stress distribution in a thick pressure vessel by deliberately yielding the inner zone. The vessel is usually filled with water and sufficient pressure, p_i is applied to plastically deform the inner part of the vessel from its inside radius, *a* to the intermediate radius, *c* (figure 12). This deformation will not usually occur throughout the entire vessel.

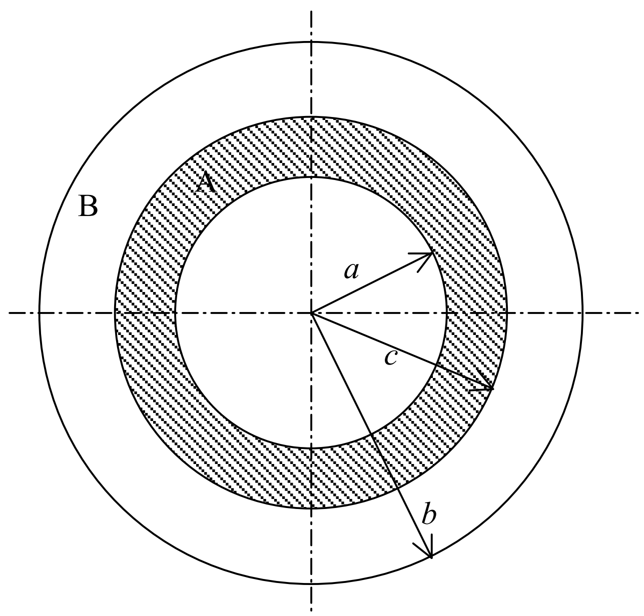

Figure 12. Plastic deformation of the inner part of a vessel from its inside radius, *a* to an intermediate radius, *c*. The resulting yielded zone, A (shaded) and the remaining elastic zone, B following an autofrettage process, are illustrated.

Equilibrium equation (1) can be applied in the plastic zone because no assumption about the nature of the material was made in its derivation, i.e.

$$r\frac{d\sigma_{rr}}{dr}+(\sigma_{rr}-\sigma_{\theta\theta})=0$$

Hooke's law doesn't apply at the interface between the yielded and non-yielded zones (i.e. where $r = c$) due to discontinuous material properties at the interface.

Using Tresca's maximum shear stress criterion (i.e. $S_y=\sigma_1-\sigma_3$) for the plastic material (yielded zone A in figure 12):

$$\sigma_{\theta\theta}-\sigma_{rr}=S_{\text{yld}} \tag{30}$$

where S_{yld} is the constant yield point stress.

Substituting equation (1) for the plastic zone into equation (30) results in:

$$\frac{d\sigma_{rr}}{dr}=\frac{S_{\text{yld}}}{r} \tag{31}$$

Integrating equation (31) results in:

$$\sigma_{rr}=S_{\text{yld}}\log_e r+C \tag{32}$$

where C is a constant of integration.

To find the constant of integration, recall the end condition reported in equation (7), i.e. $\sigma_{rr}=-p_i$ at $r = a$.

This results in $C=-p_i-S_{\text{yld}}\log_e a$ and equation (32) for radial stress becomes:

$$\sigma_{rr}=S_{\text{yld}}\log_e\left(\frac{r}{a}\right)-p_i \tag{33}$$

Substituting back into equation (30) provides an equation for hoop stress:

$$\sigma_{\theta\theta}=S_{\text{yld}}+\sigma_{rr}=S_{\text{yld}}\left(1+\log_e\left(\frac{r}{a}\right)\right)-p_i \tag{34}$$

At the intermediate radius, c the radial stress is:

$$\sigma_{rr,\,c}=S_{\text{yld}}\log_e\left(\frac{c}{a}\right)-p_i \tag{35}$$

Consider the remaining elastic zone, B as a thick shell with inside radius, c that is on the verge of yielding.

Initial yielding will occur at the inside radius of the elastic zone, B. Substituting $r = c$ and $a = c$ into equations (12) for the radial and hoop stresses in a pressure vessel with internal pressure but no external pressure results in:

$$\sigma_{rr\,\max} = -p_i \quad \text{and} \quad \sigma_{\theta\theta\,\max} = p_i \frac{b^2 + c^2}{b^2 - c^2} \tag{36}$$

Again using Tresca's criterion (equation 30), the pressure to cause yielding, p_c can be found using $\sigma_{\theta\theta\,\max} - \sigma_{rr\,\max} = S_{\text{yld}}$ and equation (36):

$$p_c\left(\frac{b^2 + c^2}{b^2 - c^2} + 1\right) = p_c\left(\frac{2b^2}{b^2 - c^2}\right) = S_{\text{yld}} \tag{37}$$

Rearranging and substituting $\sigma_{rr\,\max} = -p_i$ at the "inside" radius of the non-yielded zone, c (i.e. $\sigma_{rr,\,c} = -p_c$):

$$-\sigma_{rr,\,c} = p_c = S_{\text{yld}} \frac{\left(b^2 - c^2\right)}{2b^2} \tag{38}$$

Equating equations (35) and (38) for $\sigma_{rr,\,c}$ results in:

$$\frac{p_i}{S_{\text{yld}}} = \log_e\left(\frac{c}{a}\right) + \frac{1}{2}\left(1 - \frac{c^2}{b^2}\right) \tag{39}$$

Equation (39) can now be used to find the internal pressure, p_i needed to cause yielding in the vessel to a desired intermediate radius, c.

The stresses in the elastic zone of the vessel, B are those of an elastic pressure vessel with inner radius, c outer radius, b and internal pressure, p_c (based on equation 12):

$$\sigma_{rr} = \frac{-c^2\, \sigma_{rr,\,c}}{b^2 - c^2}\left(1 - \frac{b^2}{r^2}\right) \quad \text{and} \quad \sigma_{\theta\theta} = \frac{-c^2 \sigma_{rr,\,c}}{b^2 - c^2}\left(1 + \frac{b^2}{r^2}\right) \tag{40}$$

3.5.1 Autofrettage example

This example will compare the results from the duplex vessel example (section 3.4.1) with the autofrettage approach.

A thick walled pressure vessel is made from steel with:

- Young's modulus, $E = 210$ GN/m^2 and
- thermal expansion co-efficient, $\alpha = 11\times10^6$

Internal radius $a = 800$, and external radius $b = 1{,}200$ mm
Working internal pressure, $p_i = 30$ MN/m^2.

In section 3.4.1, an internal pressure of 30 MN/m^2 was used.
The radial interference was adjusted to make both shells yield simultaneously, and the associated maximum Tresca equivalent stress was 90 MN/m^2.

A yield stress of $S_{\text{yld}} = 90$ MN/m^2 will be used for this example, corresponding to the material property in section 3.4.1.

Equation (39) can be solved for different values of c, and the results graphed (figure 13). The vessel will initially yield at a pressure of 25 MN/m^2, so it appears that the duplex pressure vessel design has gained 20%[8] in allowable pressure.

Considering increased intermediate radius, c and the associated benefit (figure 13):

- Yielding the vessel material to a radius of 1,000 mm requires an increased internal pressure, $p_i = 33.8$ MN/m^2 (35.2% improvement).
- Yielding the vessel material to a plastic zone radius of 1,100 mm requires an internal pressure, $p_i = 35.9$ MN/m^2 (43.6% improvement).

Extending the plastic zone to larger radii offers little additional gain
(i.e. the curve in figure 13 asymptotes to a horizontal line).

This example shows that autofrettage is a more effective way of achieving high pressures than duplex vessels, especially when the precise manufacturing requirements for duplex designs are considered.

The radial, σ_{rr} and hoop, $\sigma_{\theta\theta}$ stress distributions in the vessel when the intermediate radius, $c = 1{,}000$ mm are found using equations (33), (34) and (40). The resulting stresses are graphed in figure 14, showing a constant yield stress of $S_{\text{yld}} = 90$ MN/m^2 in the yielded zone, i.e. from $r = a$ until $r = c$.

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[8] i.e. $(30-25)/25\times100 = 20\%$

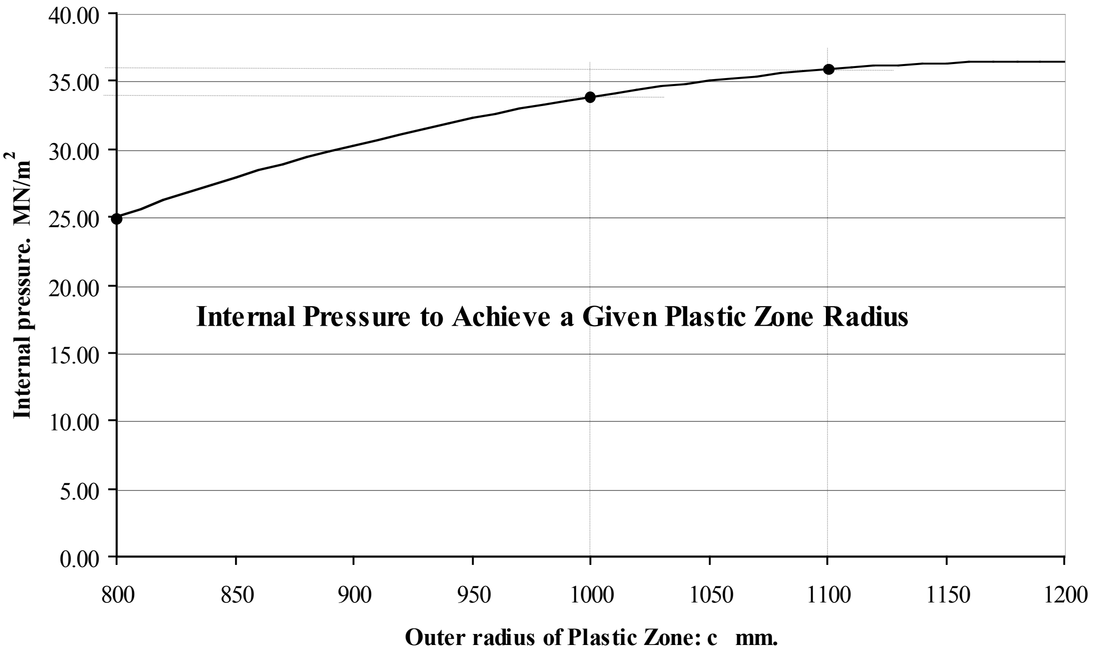

Figure 13. Internal pressure, p_i required to achieve plastic zone radius, c. Valid for example 3.5.1 conditions only, using equation (39).

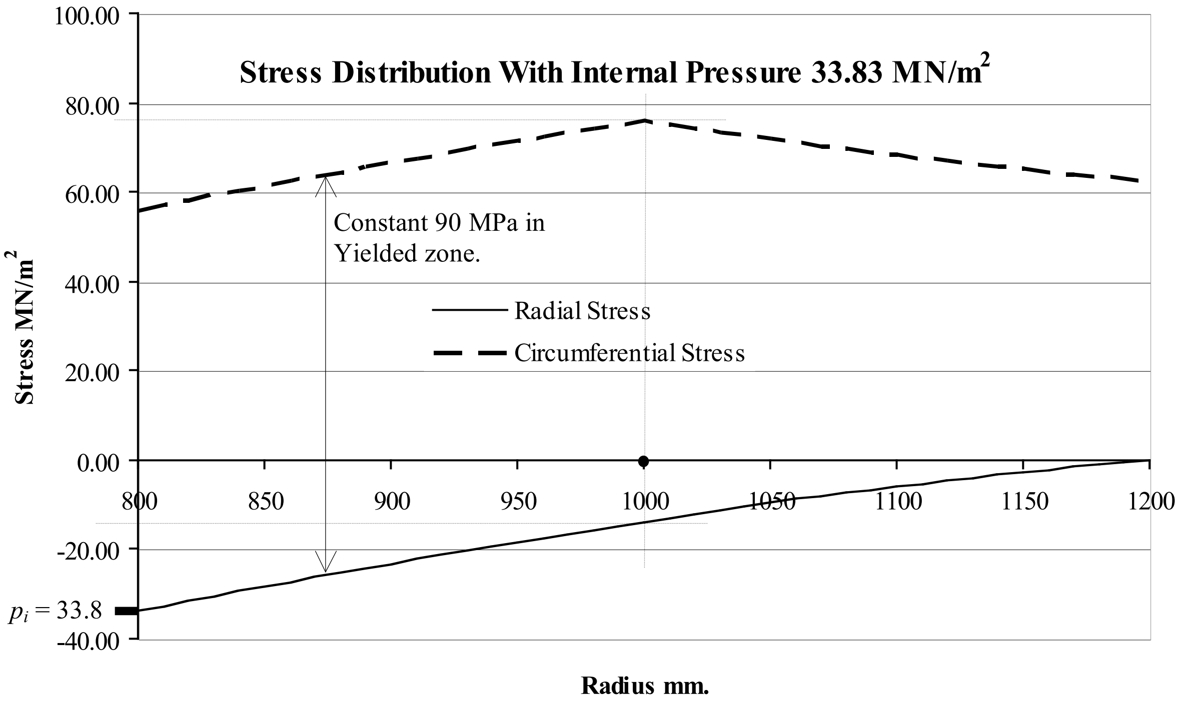

Figure 14. Radial, σ_{rr} and circumferential (hoop), $\sigma_{\theta\theta}$ stresses in the vessel when intermediate radius, c = 1,000 mm
(i.e. internal autofrettage pressure, p_i = 33.8 MN/m^2).
To the left of c = 1,000 mm, use equations (33) and (34)
To the right of c = 1,000 mm, use equation (40)

When the internal autofrettage pressure, p_i is released, the vessel again behaves elastically, although with residual stresses and associated permanent deformations.

Therefore, the stress distribution when the internal pressure, p_i returns to zero can be found by subtracting the elastic stress distribution for the same internal pressure, as shown in figure 15.

The residual stresses do not approach the yield stress on this occasion. They usually never approach the yield stress of the material of manufacture.

The stress distribution at the working pressure can be found using the same approach to that used to find the residual stress profile (i.e. find the elastic stress graphs at a design pressure and superimpose with the residual stress profile).

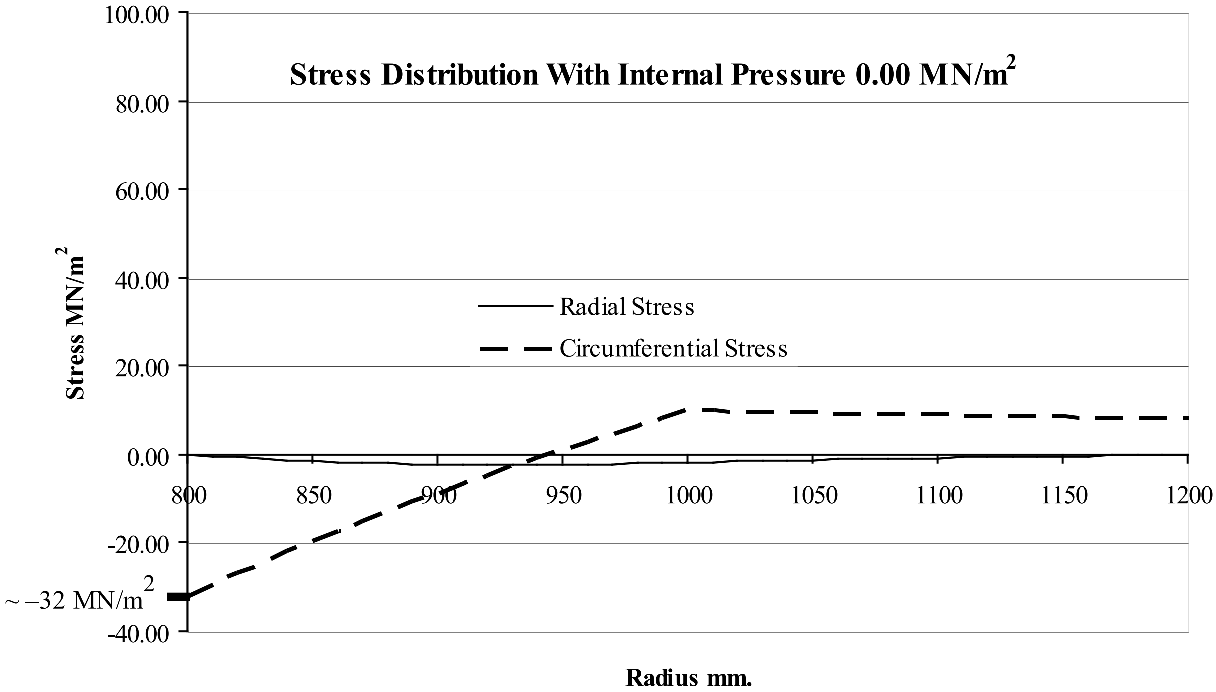

Figure 15. Radial, σ_{rr} and circumferential (hoop), $\sigma_{\theta\theta}$ stresses in the vessel when intermediate radius, c = 1,000 mm.
Internal autofrettage pressure, p_i = 33.8 MN/m^2 has been reduced to 0.