Semiclassical Stability of the Extreme Reissner-Nordström Black Hole

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The stress-energy tensor of a free quantized scalar field is calculated in the extreme Reissner-Nordström black hole spacetime in the zero-temperature vacuum state. The stress-energy appears to be regular on the event horizon, contrary to the suggestion provided by two-dimensional calculations. An analytic calculation on the event horizon for a thermal state shows that if the temperature is nonzero then the stress-energy diverges strongly there.

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Extreme black holes have played an important role in recent investigations of information loss in black-hole evaporation, in studies of black-hole thermodynamics, and in superstring theory. In the first case models involving an extreme black hole have been studied [1] to avoid the difficulties associated with quantum gravity at the endpoint of evaporation. If a zero-temperature extreme black hole absorbs a small amount of radiation its temperature will increase, and its newly acquired mass will radiate away via the Hawking effect, leaving a zero-temperature extreme black hole again in the final state. This process may be completely described within the context of a semiclassical theory, avoiding the complications of Planck scale physics. A black hole is traditionally defined to be "extreme" if it has zero surface gravity; usually this is taken to imply zero temperature. However, in the context of black-hole thermodynamics, Hawking, Horowitz, and Ross [2] have recently suggested that extreme black holes can be in thermal equilibrium with radiation at an arbitrary temperature and may have zero entropy despite having nonzero horizon area. Finally, extreme black holes may play a crucial role in string theory, where they have been identified with massive single string states [3,4].

There is a potential problem with studying the properties of extreme black holes because there are predictions that the stress-energy tensor for quantized fields may diverge on the event horizons of such black holes. In two dimensions, Trivedi [5] has demonstrated that the trace anomaly will cause a divergence of the stress-energy of a quantized field on the horizon of a two-dimensional extreme black hole. In four dimensions the analytic approximations of Frolov and Zel'nikov [6] and Anderson, Hiscock, and Samuel [7] predict that the stress-energy tensors for conformally invariant fields and for arbitrarily coupled massless scalar fields, respectively, diverge on the event horizon of an extreme Reissner-Nordström black hole, for which the magnitude of the charge is equal to the mass. If the actual four-dimensional stress-energy tensors for these fields (rather than their two-dimensional conterparts or their analytical approximations) to diverge on the event horizons of extreme black holes, this would imply that quantum effects substantially alter the classical spacetime geometry around such black holes.

Such divergences in the stress-energy tensor would imply that extreme black holes are a third "venue" for quantum gravity (in addition to the very early Universe and the endpoint of black-hole evaporation), despite the fact that the spacetime curvatures near the event horizons of extreme black holes with large masses are arbitrarily small. While the precisely extreme case may be physically inaccessible [8], processes such as black-hole evaporation can yield states arbitrarily close to the extreme state [9]. If there is a physical divergence in the stress-energy of a quantized field for the extreme case, then presumably there must be an arbitrarily large stress-energy on the horizon of a nearly extreme black hole [10].

In this Letter we present an analytic calculation which shows that the stress-energy tensor of a free conformally invariant scalar field diverges on the event horizon of an extreme Reissner-Nordström black hole if the field is in a thermal state at any nonzero temperature. We also present the results of numerical computation of the stress-energy tensor for a free quantized massless scalar field with arbitrary curvature coupling in the extreme Reissner-Nordström spacetime with the field in the zero-temperature Euclidean vacuum state. These results provide strong evidence that the stress-energy is finite on the event horizon if the field is in the zero-temperature Euclidean vacuum state. Thus we find for the case of free quantized scalar fields that extreme Reissner-Nordström black holes can exist if the fields are in a zero-temperature vacuum state, but that they cannot exist in thermal equilibrium with radiation at a nonzero temperature.

The extreme Reissner-Nordström spacetime in four dimensions may be described by the metric [11]

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \left(1 - \frac{M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2, \tag{1}$$

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where the horizon is located at $r = M$. The coordinate system of Eq. (1) is singular at the horizon; in order to determine whether a stress-energy tensor is regular on the horizon it is necessary to set up a frame which is regular there. Evaluating the stress-energy tensor components in an orthonormal frame associated with a freely falling observer, one finds that the stress-energy will be regular at the horizon if and only if the components T_t^t, T_r^r, and T_θ^θ (note that $T_\phi^\phi = T_\theta^\theta$) and the quantity

$$F = \left(T_r^r - T_t^t\right)\left(1 - \frac{M}{r}\right)^{-2} \tag{2}$$

are regular as $r \to M$. Both the energy density and radial pressure as measured by an infalling observer will be proportional to F near the horizon.

The previously predicted divergences occur for the quantity F in Eq. (2) when the fields are in the zero-temperature vacuum state. For a conformally invariant scalar field in a two-dimensional extreme Reissner-Nordström spacetime [5,12]

$$F = -\frac{M}{6\pi r^2(r - M)}. \tag{3}$$

For an arbitrarily coupled massless scalar field in a four-dimensional extreme Reissner-Nordström spacetime the analytic approximations for the stress-energy tensor give [6,7]

$$F = \frac{14M^3 - 11M^2 r}{360\pi^2 r^6(r - M)} - \frac{M^2}{60\pi^2 r^6}\ln\left[\frac{\mu}{2r}(r - M)\right], \tag{4}$$

where μ is an arbitrary positive constant.

At the event horizon of the extreme Reissner-Nordström black hole the (t, r, θ, ϕ) components of the stress-energy tensor for a conformally invariant field can be computed analytically. This is because the geometry at the horizon is asymptotically congruent to that of the conformally flat Bertotti-Robinson electromagnetic spacetime [13–15]. The Bertotti-Robinson metric may be written as

$$ds^2 = M^2\bar{r}^{-2}(-dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2). \tag{5}$$

The manner in which the extreme Reissner-Nordström metric approaches that of Eq. (5) as $r \to M$ may be seen by expanding the metric coefficients in Eq. (1) in a power series about $r = M$ and then setting $\bar{r} = M^2/(r - M)$. Brown and Cassidy [16] and Bunch [17] have derived the following expression for the stress-energy tensor of a conformally invariant quantized field in a conformally flat spacetime:

$$\begin{aligned}\langle T_{\mu\nu}\rangle_{\rm ren} = {}& (\tilde{g}/g)^{1/2}\langle T_{\mu\nu}[\tilde{g}_{\kappa\lambda}]\rangle_{\rm ren} + \frac{\alpha}{3}(g_{\mu\nu}R^{;\kappa}{}_{;\kappa} - R_{;\mu\nu} + RR_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}R^2) \\ &+ \beta(\tfrac{2}{3}RR_{\mu\nu} - R_\mu{}^\kappa R_{\kappa\nu} + \tfrac{1}{2}g_{\mu\nu}R_{\kappa\lambda}R^{\kappa\lambda} - \tfrac{1}{4}g_{\mu\nu}R^2).\end{aligned} \tag{6}$$

Here $\tilde{g}$ is the flat space metric and $\langle T_{\mu\nu}[\tilde{g}]\rangle$ is the stress-energy tensor of the quantized field in the flat space; for the conformally invariant scalar field $\alpha = \beta = (2880\pi^2)^{-1}$. Combining Eqs. (5) and (6) with a thermal state in Minkowski space, we find the stress-energy tensor of a conformally invariant scalar field in the Bertotti-Robinson spacetime at temperature T is [18]

$$\begin{aligned}\langle T_\mu^\nu\rangle_{\rm ren} = {}& \frac{1}{2880\pi^2 M^4}\,{\rm diag}(1,1,1,1) \\ &+ \frac{M^4}{(r - M)^4}\frac{\pi^2 T^4}{30}\,{\rm diag}(-1,\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{3}).\end{aligned} \tag{7}$$

This is clearly divergent in the limit $r \to M$ if $T \neq 0$.

If $T = 0$, then the stress-energy tensor in the Bertotti-Robinson spacetime is regular everywhere. However, this does not guarantee that the stress-energy is finite on the event horizon in the extreme Reissner-Nordström spacetime. This is because the Reissner-Nordström metric only asymptotically approaches the Bertotti-Robinson one; the quantity F in Eq. (2) may be nonzero on the event horizon of the extreme black hole, although it is precisely zero in the Bertotti-Robinson spacetime.

To investigate whether there is a divergence on the event horizon in the $T = 0$ case we have applied the method developed by Anderson, Hiscock, and Samuel [7] for calculating the stress-energy tensor of quantized scalar field in static spherical spacetimes to the case of a massless arbitrarily coupled field in the extreme Reissner-Nordström spacetime. Using these techniques, one may compute the stress-energy tensor components at any value of r outside the horizon [19].

Symbolically, the stress-energy tensor components may be divided into conformal and nonconformal contributions in terms of two tensors, C_μ^ν and D_μ^ν:

$$\langle T_\mu^\nu\rangle = C_\mu^\nu + (\xi - \tfrac{1}{6})D_\mu^\nu, \tag{8}$$

where ξ is the curvature coupling constant for the scalar field. The values of C_t^t, C_r^r, and C_θ^θ extrapolated to the horizon are the same, within the accuracy of our computations (approximately five digits near the horizon for these components), as those obtained from the analytic calculation for the Bertotti-Robinson spacetime, Eq. (7). The behavior of the components of C_μ^ν is illustrated in Fig. 1. The values of the components are clearly finite as $r \to M$ and they smoothly approach the Bertotti-Robinson values on the horizon. The behavior of the corresponding components of D_μ^ν is illustrated in Fig. 2. They approach zero as $r \to M$ and are thus finite also.

In order to determine the regularity of the stress-energy tensor on the horizon, we must also examine the quantity F defined in Eq. (2). This quantity is plotted in Figs. 3 and 4 for the contributions of C_μ^ν and D_μ^ν, respectively. In Fig. 3 we also show F for the conformally invariant scalar

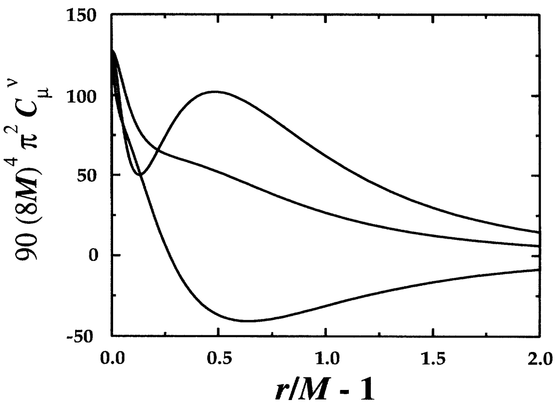

FIG. 1. The curves in this figure display the values of C_t^t, C_r^r, and C_θ^θ (from top to bottom at $r = 2M$) for massless scalar fields around an extreme Reissner-Nordström black hole.

field for the two-dimensional Reissner-Nordström black hole and the four-dimensional analytic approximation with $\mu = 2$. Our data points are explicitly displayed for the numerically computed curves to give the reader a better idea of the quality of the information; we have, however, suppressed the dots representing our calculated points at $r = 1.002M$, $1.003M$, and $1.004M$ for clarity. The numerically computed curves stop at our innermost data point $r = 1.001M$ since we cannot directly compute the value of F on the horizon.

Inspection of these figures clearly shows that no divergence at the horizon is to be expected. This may be made quantitative by fitting our data points to a series in the dimensionless quantity $s = (r - M)/M$ and comparing the magnitude of the coefficient of hypothetical s^{-1} and $\ln(s)$ terms with other finite terms. Fits may be tried with different sets of points and different number of terms in the series to obtain a robust estimate of the power in an s^{-1} term. Such calculations all yield a coefficient for the s^{-1} term which is at most of order 10^{-4} times the coefficient

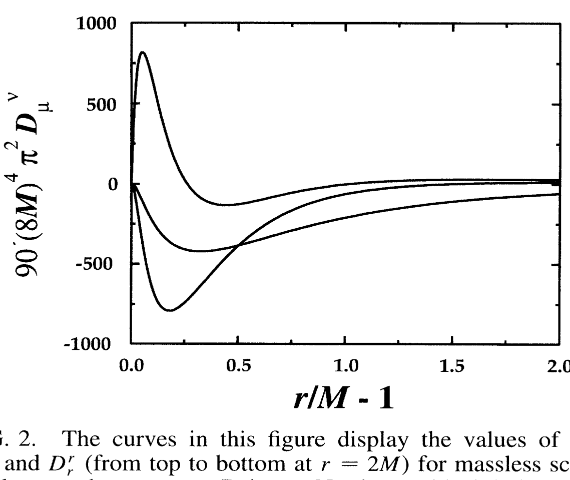

FIG. 2. The curves in this figure display the values of D_θ^θ, D_t^t, and D_r^r (from top to bottom at $r = 2M$) for massless scalar fields around an extreme Reissner-Nordström black hole.

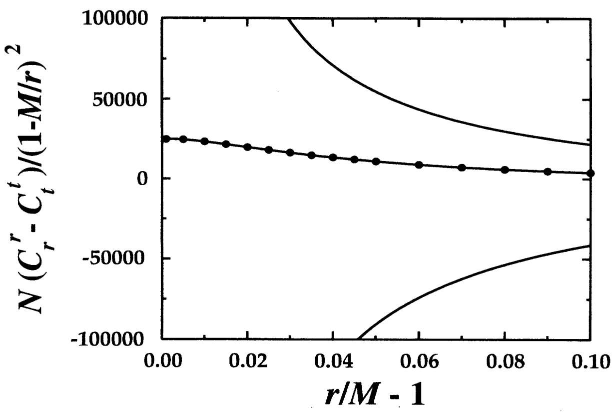

FIG. 3. These curves display the value of the quantity $N(C_r^r - C_t^t)/(1 - M/r)^2$ for a massless scalar field around an extreme Reissner-Nordström black hole. From top to bottom the curves represent the four-dimensional analytic approximation, the numerically computed four-dimensional values, and the two-dimensional black hole result. The constant N has the value $90(8M)^4\pi^2$ for the four-dimensional curves and $30\,000\pi M^2$ for two-dimensional curve. No divergence at the horizon ($r = M$) is apparent for the numerically computed curve.

of the constant term. In contrast from Eq. (3) the ratio of these coefficients for the two-dimensional stress-energy tensor is 1/2, while from Eq. (4) for the analytical approximation (with $\mu = 2$ chosen for simplicity) the ratio is 3/29. As an example of a typical fit, if our innermost data points at $s = 0.001$, 0.002, 0.003, and 0.004 are fitted to an expansion in powers of s, including divergent s^{-1} and $\ln(s)$ terms, we obtain for the tensor C_μ^ν the fit

$$90(8M)^4\pi^2(C_r^r - C_t^t)\left(1 - \frac{M}{r}\right)^{-2} = \frac{0.8122}{s} + 1333\ln(s) + 33\,600 - 435\,000s\,. \quad (9)$$

Since our numerical calculations of F near the event horizon are accurate to between two and three digits,

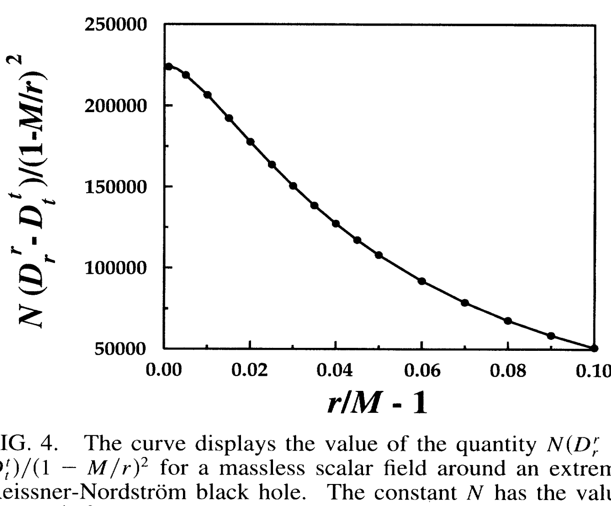

FIG. 4. The curve displays the value of the quantity $N(D_r^r - D_t^t)/(1 - M/r)^2$ for a massless scalar field around an extreme Reissner-Nordström black hole. The constant N has the value $90(8M)^4\pi^2$. No divergence at the horizon ($r = M$) is apparent.

it is clear that all the power in the s^{-1} term is simple "numerical noise," not to be taken seriously.

Given that no divergence of the form s^{-1} exists, one can try to fit a series with just a $\ln(s)$ term plus terms which are finite on the horizon. In these cases we find the coefficient of the ln term is generally at most 10^{-2} times the coefficient of the constant term. In contrast, from Eq. (4) this ratio is 6/29 in the analytic approximation.

Thus, numerical calculations strongly suggest that the divergence in the two-dimensional black-hole stress-energy tensor discovered by Trivedi and those in the four-dimensional analytic approximations to the stress-energy tensor do not exist in the actual four-dimensional stress-energy tensor, at least in the case of an extreme Reissner-Nordström spacetime. Thus real, physical extreme black holes will not have divergent vacuum stress-energy on their horizon if they are in the zero-temperature vacuum state. However, our analytic calculations also show that the stress-energy of free quantized fields does diverge on the event horizon of an extreme Reissner-Nordstrom black hole if the fields are in a nonzero temperature thermal state. Thus, in this case, the suggestion of Hawking, Horowitz, and Ross [2] that extreme black holes can be in thermal equilibrium with radiation at an arbitrary temperature is incorrect.

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