Thermal divergences on the event horizons of two-dimensional black holes

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The expectation value of the stress-energy tensor $\langle T_{\mu\nu}\rangle$ of a free conformally invariant scalar field is computed in a general static two-dimensional black hole spacetime when the field is in either a zero temperature vacuum state or a thermal state at a nonzero temperature. It is found that for every static two-dimensional black hole the stress-energy diverges strongly on the event horizon unless the field is in a state at the natural black hole temperature which is defined by the surface gravity of the event horizon. This implies that both extreme and nonextreme two-dimensional black holes can only be in equilibrium with radiation at the natural black hole temperature.

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I. INTRODUCTION

Since the discovery of black hole radiance two decades ago, the study of quantum aspects of black holes has been regarded as one of the most likely areas in which to gain further insight into the nature of quantum gravitational processes. Recent work in several distinct areas has led to a common central concern: Does the stress-energy tensor of a quantized field diverge on the event horizon? If so, is the divergence weak and essentially ignorable, or is it strong, calling into question the very existence of a meaningful semiclassical black hole solution?

It is well known that Schwarzschild and Reissner-Nordström black holes have a well-defined temperature which is related to their surface gravity. If quantized fields are in a thermal state at this temperature (this is the Hartle-Hawking state), then one expects that the stress-energy is finite on the past and future event horizons.[1] This has been confirmed by numerical calculations in four dimensions [1,2]. If the fields are not in the Hartle-Hawking state, even if they are in a thermal state at some other temperature, then the stress-energy diverges severely on the event horizon of the black hole. This strong divergence indicates that no solution to the quantum or semiclassical theory would be "near" the classical solution in these cases; the back reaction to the quantized fields would profoundly affect the geometry in a nonperturbative fashion if the fields are not in the Hartle-Hawking state.

Despite the classic black hole uniqueness theorems, there are several reasons to be interested in quantum effects in more general black hole spacetimes. For example, back reaction effects due to the nonzero stress-energy of the quantum fields will alter the spacetime geometry near the event horizon of a Schwarzschild or Reissner-Nordström black hole [3–5]. Thus self-consistent solutions to the semiclassical back reaction equations will not be described by the exact classical geometries. Inclusion of additional fields, such as the dilaton suggested by superstring theories, may also necessitate examining a larger class of black hole metrics. Within the larger, general class of black hole metrics, those metrics which represent extreme black holes (usually defined as having a degenerate horizon structure and zero surface gravity) are of particular interest due to their possible stability against evaporation.

Extreme black holes play a very important role in certain contemporary investigations. One such area is the study of information loss due to the evaporation process [6]. By studying the absorption and reemission of radiation by an initially extreme black hole, the issues of Planck scale physics may be avoided, while capturing the essence of information loss. The simplicity of semiclassical theories in two dimensions allows for the explicit solution of such models [7].

A second area involves the investigation of pair creation of magnetically charged Reissner-Nordström black holes by an external magnetic field [8–10]. A curious discrepancy has been found between the pair creation rate for extreme and nonextreme black holes. This discrepancy can be understood simply if extreme Reissner-Nordström black holes are assigned zero entropy (notwithstanding their nonzero horizon area). Hawking, Horowitz, and Ross [11] have recently pointed out that in the Euclidean sector, the distance to the horizon of an extreme black hole is infinite in all directions (as opposed to merely in spacelike directions in the Lorentzian metric). They use this fact to argue that the entropy will formally be zero, explaining the pair creation rate discrepancy. Since the horizon is at an infinite distance, the

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[1]In this paper, we shall be concerned only with possible equilibrium states of the black hole–quantized field system. We shall not consider states such as the Unruh vacuum state, which is appropriate to a black hole formed by collapse, and which would be regular on the future event horizon but would diverge on the past event horizon.

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Euclidean geometry may then be identified with an arbitrary period without the penalty of introducing a conical singularity at the horizon. This fact leads them to conclude that extreme black holes can be in equilibrium with radiation at an arbitrary temperature.

Extreme dilaton black holes [12,13] may play an important role in superstring theories as representations of massive single string states [14,15]. These black holes are extreme in the sense that any increase in the charge of the hole would result in a nakedly singular spacetime. However, unlike the ordinary Kerr-Newman extreme black hole metrics of general relativity, these black holes may have zero, finite, or infinite surface gravities (and hence temperatures), depending on the value of a, the dilaton coupling. However, the assignment of thermodynamic properties such as temperature and entropy to elementary particles (strings) is problematic, particularly when infinite temperatures are contemplated. In order to avoid confusion, we shall hereafter reserve the use of the adjective "extreme" for those black holes (with or without dilaton fields) which have zero surface gravity, unless otherwise explicitly stated.

Finally, Trivedi [16] has shown that the stress-energy of a quantized conformally coupled massless field has a weak divergence on the event horizon of (almost) any extreme two-dimensional black hole if the field is in a zero temperature vacuum state. We have recently shown [17] that this divergence does not occur for the four-dimensional extreme Reissner-Nordström black hole and that no divergence in the stress-energy occurs on the horizon if the field is in a zero temperature vacuum state. However, if the field is in a thermal state at any nonzero temperature a severe divergence in the stress-energy does occur on the horizon. This means that there is a well-defined temperature (zero) for the extreme Reissner-Nordström black hole and it cannot be coupled to radiation at an arbitrary temperature as suggested by Hawking, Horowitz, and Ross [11].

These studies raise several related natural questions concerning the stress-energy of quantized fields in black hole spacetimes. Does the divergence discovered by Trivedi exist in four-dimensional extreme black hole spacetimes other than the extreme Reissner-Nordström spacetime? Is the stress-energy of quantized fields in nonextreme black hole spacetimes always finite on the event horizon if the fields are in the Hartle-Hawking state and does it always diverge otherwise? Can other extreme black holes be coupled to radiation at an arbitrary temperature even if the extreme Reissner-Nordström black hole cannot?

The answers to these questions will have important implications. For example, if the divergence found by Trivedi for extreme black holes existed for some four-dimensional extreme black holes, then quantum effects would greatly alter the spacetime geometry near the horizons of such black holes, even though all curvatures in that region may be far smaller than the Planck scale. On the other hand, if one could assign an extreme black hole a temperature other than that defined by the surface gravity, this would possibly allow one to avoid infinite temperatures in dilatonic extreme black holes.

The recent development of a method of numerically computing the stress-energy of quantized scalar fields in an arbitrary static spherically symmetric four-dimensional spacetime [2] has made it possible to address these questions directly in four dimensions; as noted above they have already been answered for Schwarzschild, Reissner-Nordström, and extreme Reissner-Nordström black holes. However, the numerical methods do not allow one to examine arbitrary black holes in four dimensions. Instead one must numerically compute the stress-energy for one black hole geometry at a time.

For this reason it is of interest to look at the two dimensional case. Here the stress-energy can be computed analytically for a conformally invariant field in an arbitrary two dimensional spacetime. Thus one can examine all static two-dimensional black hole geometries. In this paper we compute the stress-energy for a conformally invariant field in a general two-dimensional static black hole spacetime by directly integrating the conservation equation. This method is well known; the integration of the conservation equation on a specific [18,19] or general (extreme) [16] black hole background has been previously carried out. The main new feature of our work is the interpretation of the resulting expressions in terms of boundary conditions (e.g., temperature) on the quantized field. We investigate under what conditions the stress-energy is finite or, at most, weakly divergent on the event horizon. We find in all cases that the stress-energy tensor diverges strongly on the horizon unless the fields are in a thermal state with a temperature equal to the natural temperature defined by the surface gravity of the black hole. Hence, the divergence we previously found in the four-dimensional extreme Reissner-Nordström spacetime is of precisely the same form as that which occurs when any two-dimensional black hole, extreme or nonextreme, is assigned an "unnatural" temperature.

For two-dimensional extreme dilaton black holes, this implies that one cannot escape the divergent behavior associated with infinite temperatures by choosing to identify the Euclidean metric with a different period (and hence temperature). Any such attempt will result in a strongly divergent stress-energy tensor on the black hole's horizon.

Our sign conventions and notation follow Misner, Thorne, and Wheeler [20]; we also use natural units ($G = c = \hbar = k_B = 1$) throughout.

II. CALCULATION OF $\langle T_{\mu\nu} \rangle$

The most general two-dimensional static spacetime metric may be written in the form

$$ds^2 = -F(R)dt^2 + \frac{1}{H(R)}dR^2. \tag{1}$$

It is always possible to transform this metric to "Schwarzschild gauge" by defining a new spatial coordinate

$$r = \int \left(\frac{F}{H}\right)^{1/2} dR, \tag{2}$$

which yields, as the general metric form,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2, \tag{3}$$

where $f(r)$ is an arbitrary function of r. This metric possesses horizons at the locations r_n, where $f(r_n) = 0$, $(n = 0, 1, 2, ...)$. Markovic and Poisson [21] have recently discussed stress-energy divergences on Cauchy horizons in two-dimensional spacetimes. In this paper we consider black holes with an arbitrary number of horizons, but we restrict our attention to the outer event horizon at $r = r_0$.

The conservation of stress-energy in the metric of Eq.(3) yields the differential equations

$$T^r_{t\ ,r} = 0 \tag{4}$$

and

$$T^r_{r\ ,r} + \frac{1}{2}\frac{f'}{f}T^r_r - \frac{1}{2}\frac{f'}{f}T^t_t = 0, \tag{5}$$

where a prime denotes differentiation with respect to r. Hereafter we will generally suppress the expectation value brackets for simplicity of notation. Equation (5) may be simplified by rewriting T^t_t in terms of the trace of the stress-energy tensor: $T^t_t = T^\alpha_\alpha - T^r_r$. The conservation equations are then easily integrated:

$$T^r_t = C_1 \tag{6}$$

and

$$T^r_r = \frac{C_2}{f} + \frac{1}{2f}\int_{r_0}^{r} f' T^\alpha_\alpha dr. \tag{7}$$

Equations (6) and (7) are the complete solution to the conservation of stress-energy equations in two dimensions. The components of the stress-energy tensor in a given spacetime then depend on one function T^α_α and two integration constants C_1 and C_2.

If the field is chosen to be conformally invariant, then the trace is given by the conformal anomaly $T^\alpha_\alpha = \frac{R}{24\pi}$, where R is the Ricci scalar, which becomes

$$T^\alpha_\alpha = -\frac{f''}{24\pi} \tag{8}$$

for the metric of Eq. (3). In the case of a conformally invariant field, all information concerning the quantum state of the field is then encoded in the two integration constants. Substituting the trace anomaly from Eq. (8) into Eq. (7), one can explicitly perform the integration to find

$$T^r_r = \frac{C_2}{f} - \frac{f'^2}{96\pi f} + \frac{\pi}{6f}\left(T_{\text{Hawking}}\right)^2, \tag{9}$$

where we have defined T_{Hawking} in terms of the surface gravity κ of the horizon at r_0:

$$T_{\text{Hawking}} = \frac{\kappa}{2\pi} = \frac{f'|_{r_0}}{4\pi}. \tag{10}$$

Note we have not assumed that T_{Hawking} represents the physical temperature of the black hole; rather we have simply used a familiar definition to simplify a collection of terms involving $f'(r_0)$.

The integration constants are fixed by choosing a particular quantum state for the field. We will consider a state in which the black hole is in thermal equilibrium with a surrounding heat bath. The requirement of thermal equilibrium implies that T^ν_μ must be invariant under time reversal, and thus $T^r_t = C_1 = 0$.

The remaining integration constant C_2 is now determined by fixing the form of T^r_r in an asymptotically flat region far from the horizon, where $f \rightarrow$ const. (The apparently more general asymptotically flat form $f' \rightarrow$ const simply amounts to choosing asymptotically Rindler coordinates rather than Minkowski.) We assume that the stress-energy approaches the form appropriate to a two-dimensional gas of massless scalar bosons at temperature T far from the horizon,

$$T^r_r \rightarrow \frac{\pi}{6}T^2, \tag{11}$$

as the metric becomes asymptotically flat. Evaluating Eq. (9) in the asymptotically flat region and using Eq. (11), we find

$$C_2 = \frac{\pi}{6}T^2 - \frac{\pi}{6}\left(T_{\text{Hawking}}\right)^2. \tag{12}$$

III. $\langle T_{\mu\nu}\rangle$ ON THE HORIZON

Having completely integrated the conservation equation and solved for the stress-energy tensor, we turn to the issue of its regularity on the event horizon. Since the coordinate system used in the metric of Eq. (3) is singular on the event horizon at r_0, we will evaluate the stress-energy tensor components in an orthonormal frame attached to a freely falling observer. The basis vectors of the frame are chosen to be the two-velocity $e^\alpha_0 = u^\alpha$ and a unit length spacelike vector $e^\alpha_1 = n^\alpha$ orthogonal to u^α, so that $n^\alpha u_\alpha = 0$ and $n^\alpha n_\alpha = +1$. Using the timelike Killing vector field to define a conserved energy, the geodesic equation may then be solved to find

$$u^t = \gamma/f, \quad u^r = -\sqrt{\gamma^2 - f}\ , \tag{13}$$

and

$$n^t = -\frac{1}{f}\sqrt{\gamma^2 - f}, \quad n^r = \gamma\ , \tag{14}$$

where γ is the energy per unit mass along the geodesic. The components of the stress-energy tensor in the freely falling orthonormal frame are then given in terms of the coordinate components by

$$T_{00} = \frac{\gamma^2\left(T^r_r - T^t_t\right)}{f} - T^r_r, \tag{15}$$

$$T_{11} = \frac{\gamma^2 \left(T_r^r - T_t^t\right)}{f} + T_t^t, \tag{16}$$

and

$$T_{01} = -\frac{\gamma\sqrt{\gamma^2 - f}\left(T_r^r - T_t^t\right)}{f}. \tag{17}$$

Since the value of γ is arbitrary, the stress-energy will be regular on the horizon only if T_t^t, T_r^r, and the combination $\left(T_r^r - T_t^t\right)/f$ are each separately finite at r_0. Because a possible divergence in either T_t^t or T_r^r will be made stronger by the extra f^{-1} in the combination $\left(T_r^r - T_t^t\right)/f$, we will focus on this combination as representing the strongest possible divergence in $\langle T_{\mu\nu}\rangle$. Using Eqs. (9),(10) and the trace anomaly, this combination of terms may be written as

$$\frac{T_r^r - T_t^t}{f} = \frac{2C_2}{f^2} - \frac{1}{48\pi}\frac{f'^2 - f'^2|_{r_0} - 2ff''}{f^2}. \tag{18}$$

The second term on the right hand side of Eq. (18) is 0/0 on the event horizon. Applying l'Hospital's rule to this term and rewriting $\left(T_r^r - T_t^t\right)/f$ in the limit as $r \to r_0$ gives

$$\lim_{r\to r_0} \frac{T_r^r - T_t^t}{f} = \left(\frac{2C_2}{f^2} + \frac{1}{48\pi}\frac{f'''}{f'}\right)\Bigg|_{r_0}. \tag{19}$$

This result was derived by Trivedi for the case of extreme black holes. For an extreme black hole (by the conventional definition, i.e., one with zero surface gravity), $f'|_{r_0} = 0$, and the second term of Eq. (19) diverges. This is the unavoidable divergence Trivedi previously discovered [16]. The only escape from this divergence would be if f''' approaches zero as fast or faster than f' does in the limit $r \to r_0$.[2] Of course two-dimensional extreme black hole metrics for which this occurs form a set of measure zero in the space of all extreme two-dimensional black hole metrics. However, it remains to be seen whether two-dimensional gravitational dynamics, when semiclassical backreaction is included, might cause extreme black hole solutions to evolve towards such a state.

If $C_2 \neq 0$, then there is a far more serious divergence of the stress-energy tensor on the event horizon. The energy density and pressure seen by an infalling observer will diverge as f^{-2}, irrespective of whether the black hole is extreme or not. Both Christensen and Fulling [18] and Trivedi [16] noted this and set $C_2 = 0$; here, we expand on the physical interpretation of the meaning of this restriction on C_2. We have previously seen that the integration constant C_2 may be expressed in terms of the difference between the square of the "natural" temperature of the black hole, T_{Hawking}, defined by the surface gravity, and the square of the asymptotic temperature assigned to the black hole, T, as shown in Eq. (12). We thus see that unless the temperature assigned to the black hole, T, is precisely equal to the natural temperature defined by the surface gravity, T_{Hawking}, the stress-energy of a quantized field will diverge strongly on the event horizon. Further, the form of the divergence is independent of whether the black hole is extreme or nonextreme[3]; extreme (zero surface gravity) black holes have a natural temperature, namely, zero, in precisely the same fashion as nonextreme holes, and may not be assigned arbitrary temperature without serious consequences. The divergence of $\langle T_{\mu\nu}\rangle$ on the event horizon of an extreme Reissner-Nordström black hole in four dimensions when the temperature is chosen to be other than zero [17] is thus seen to be simply an example of the general strong divergence which occurs for all black holes when assigned an inappropriate temperature.

In conclusion, the stress-energy tensor of a conformally coupled quantized field will diverge strongly on the event horizon of any two-dimensional black hole unless the temperature of the black hole is chosen to be equal to the natural temperature defined by the surface gravity of the horizon. If the temperature is chosen to be the natural value, then the stress-energy tensor will be regular on the horizon unless the black hole is extreme. If the black hole is extreme, then there will be a weak divergence of the stress-energy on the horizon, except in a set of metrics of measure zero. Whether two-dimensional extreme black hole metrics evolve naturally towards these nondivergent cases when semiclassical backreaction is included remains to be determined.

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[2]If one generalizes Trivedi's result to a nonconformally coupled field [so that the trace of the stress-energy tensor is not given by Eq. (8)], then the condition for regularity of the stress-energy on the event horizon is that the limit of $(T_\alpha{}^\alpha)'/f'$ as $r \to r_0$ must be finite.

[3]For the nonextreme case, in which the horizon is bifurcate, this is simply the two-dimensional version of the well-known result of Kay and Wald [22], although derived here by a different method.

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