Quantized fields and temperature in charged dilatonic black hole spacetimes

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The stress-energy tensor of a quantized scalar field is computed in the reduced two-dimensional charged dilatonic black hole spacetime of Garfinkle, Horowitz, and Strominger. In order for the stress energy of quantized fields to be regular on the event horizon in both the extreme string metric and the conformally associated physical metric, it is necessary to assign a nonzero temperature $T = (8\pi e^{\phi_0} M)^{-1}$ to the extreme string metric, contrary to the expectation that this horizonless spacetime would have a natural temperature of zero. [S0556-2821(97)01406-9]

Garfinkle, Horowitz, and Strominger (GHS) [1], found static spherical charged black hole solutions in the low-energy approximation to string theory. These solutions possess nonconstant dilaton fields in addition to the electromagnetic field $F_{\mu\nu}$. The GHS black hole solutions have substantially different properties compared to the analogous Reissner-Nordström black hole solutions in general relativity. In particular, the GHS black hole lacks inner horizons, and while a maximal value of the charge exists which, in both sets of solutions, separates the black hole solutions from naked singularities, these extreme solutions have quite different properties in the two theories. The event horizon of the extreme GHS solution in what we call the physical metric, $g_{\mu\nu}$, is singular, whereas the extreme Reissner-Nordström black hole has a nonsingular event horizon with well-defined stress energy [6]. However, as noted by Garfinkle, Horowitz, and Strominger, the singular nature of the event horizon in the physical metric is irrelevant. Strings do not couple to the physical metric $g_{\mu\nu}$ but to the conformally related metric $e^{2\phi} g_{\mu\nu}$, which we will call the string metric, where ϕ is the dilaton field. In fact, in the string GHS metric, the extreme spacetime has no event horizon and is geodesically complete.

The thermodynamics of the Reissner-Nordström and GHS spacetimes also differs. In the Reissner-Nordström spacetime the black hole temperature decreases steadily from the Schwarzschild value to zero as the extreme limit is approached. In comparison the string GHS metric has a nonzero temperature independent of the charge, until the extreme solution is reached. Then the temperature apparently shifts discontinuously to zero since the extreme spacetime has no horizon.

In this paper, working in two-dimensional black hole metrics obtained by discarding the angular portions of the GHS metrics, we calculate the expectation value of the stress-energy tensor for a quantized conformally coupled massless scalar field in the Hartle-Hawking state. We find that by assigning the extreme string spacetime of GHS a nonzero temperature equal to that of the nonextreme string spacetime, it is possible to render $\langle T_{\mu\nu}\rangle$ regular everywhere outside and on the horizon for both the string and physical metrics.

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The effective Lagrangian for low-energy string theory used by Garfinkle, Horowitz, and Strominger is

$$L = \int d^4x \sqrt{-g}[-R + 2(\nabla\phi)^2 + e^{-2\phi}F^2], \qquad (1)$$

where ϕ is the dilaton field and $F_{\mu\nu}$ is the Maxwell field. Static, spherically symmetric black hole solutions to the field equations obtained from Eq. (1) are described by [1]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2$$
$$+ r\left(r - \frac{Q^2 e^{-2\phi_0}}{M}\right)d\Omega^2, \qquad (2)$$

$$e^{-2\phi} = e^{-2\phi_0}\left(1 - \frac{Q^2 e^{-2\phi_0}}{Mr}\right), \qquad (3)$$

and

$$F = Q \sin\theta d\theta \wedge d\phi. \qquad (4)$$

Here $d\Omega^2$ is the metric of the two-sphere, ϕ_0 is the asymptotic value of the dilaton field, and we have corrected the errors in the powers of e^{ϕ_0} in accordance with the erratum to Ref. [1]. The surface

$$r = \frac{Q^2 e^{-2\phi_0}}{M} \qquad (5)$$

is singular, and the extreme limit occurs when the charge is increased to a value sufficient to bring this surface into coincidence with the horizon at $r = 2M$, namely, when

$$Q^2 = 2M^2 e^{2\phi_0}. \qquad (6)$$

The strings do not couple to the physical metric $g_{\mu\nu}$ of Eq. (2) but rather to the conformally related string metric $e^{2\phi} g_{\mu\nu}$. Applying this conformal transformation to Eq. (2) yields

$$ds^2_{\text{string}} = -\frac{(1-2Me^{\phi_0}/\rho)}{(1-Q^2e^{-\phi_0}/M\rho)}d\tau^2$$
$$+\frac{d\rho^2}{(1-2Me^{\phi_0}/\rho)(1-Q^2e^{-\phi_0}/M\rho)}+\rho^2 d\Omega^2, \quad (7)$$

where the coordinates (τ,ρ) are defined by $\tau=e^{\phi_0}t$, $\rho=e^{\phi_0}r$. In the extreme limit, when $Q^2=2M^2e^{\phi_0}$, Eq. (7) reduces to

$$ds^2 = -d\tau^2 + \left(1-\frac{2Me^{\phi_0}}{\rho}\right)^{-2}d\rho^2 + \rho^2 d\Omega^2. \quad (8)$$

This extreme geometry has no horizon, and the spatial geometry of constant τ hypersurfaces is geometrically identical to that of $t=$ const surfaces in the extreme Reissner-Nordström metric. As a result, the $\rho=2e^{\phi_0}M$ surface is at an infinite proper distance from any point in the manifold with $\rho>2e^{\phi_0}M$. In the GHS extreme spacetime, the distance is infinite in any direction, whereas in the extreme Reissner-Nordström case, the distance to $r=M$ is infinite only in spacelike directions.

Since the GHS metric is the low-energy approximation to full string theory black hole solutions, it is sensible to examine the physics of ordinary quantized free fields (as opposed to quantized strings) in the GHS black hole background. While obtaining values for tensor objects such as $\langle\psi^2\rangle$ or $\langle T_{\mu\nu}\rangle$ for a quantized field ψ can be quite difficult in curved four-dimensional spacetimes, calculating $\langle T_{\mu\nu}\rangle$ for a conformally coupled field in a two-dimensional spacetime is both a straightforward and often valuable exercise [2–4]. In this paper we restrict our attention to two dimensions, computing $\langle T_{\mu\nu}\rangle$ for a conformally coupled quantized scalar field in the two-dimensional metrics obtained from Eqs. (2) and (7) by setting the angular coordinates θ and ϕ to constant values. We examine the behavior of a quantized field in both the string and physical metrics.

The stress-energy tensor of a quantized scalar field in a two-dimensional black hole spacetime for the Hartle-Hawking state may be easily calculated; for convenience, in this paper we will use the approach of Ref. [4]. The spatial coordinate of the black hole metric must first be transformed in order to place the metric in ''Schwarzschild gauge''

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}. \quad (9)$$

The temperature defined by the geometry is then given by

$$T = \frac{f'|_{r_0}}{4\pi}, \quad (10)$$

where r_0 is the radius of the event horizon and a prime denotes differentiation with respect to r. The trace of the stress-energy tensor is given by the conformal anomaly,

$$\langle T_\alpha{}^\alpha\rangle = -\frac{f''}{24\pi}. \quad (11)$$

The components may then be completely determined by quadrature of the conservation equation, with regularity of the stress-energy tensor on the black hole horizon fixing the integration constant. Specifically,

$$\langle T_r{}^r\rangle = -\frac{f'^2}{96\pi f} + \frac{\pi T^2}{6f}, \quad (12)$$

$$\langle T_t{}^t\rangle = \langle T_\alpha{}^\alpha\rangle - \langle T_r{}^r\rangle. \quad (13)$$

The physical GHS metric of Eq. (2), when restricted to two dimensions, becomes simply the two-dimensional Schwarzschild metric for all values of Q. No transformation is needed and one can quickly calculate that the spacetime has a temperature of

$$T_{\text{Sch}} = \frac{1}{8\pi M}, \quad (14)$$

while the components of the stress-energy tensor for the quantized field are found to be

$$\langle T_t{}^t\rangle = \frac{56M^3 - 4M^2r - 2Mr^2 - r^3}{384\pi M^2 r^3}, \quad (15)$$

$$\langle T_r{}^r\rangle = \frac{1}{384\pi M^2}\left(1+\frac{2M}{r}\right)\left(1+\frac{4M^2}{r^2}\right), \quad (16)$$

and the trace anomaly is

$$\langle T_\alpha{}^\alpha\rangle = \frac{M}{6\pi r^3}. \quad (17)$$

In order to calculate the stress-energy tensor of a quantized field in the GHS string metric described by Eq. (7) restricted to two dimensions, the metric must first be transformed to Schwarzschild gauge, which is accomplished by defining a new spatial coordinate x,

$$x = \int \left(1-\frac{e^{-\phi_0}Q^2}{M\rho}\right)^{-1}d\rho. \quad (18)$$

The derivatives in Eqs. (10)–(13) are now taken with respect to the new coordinate x. We shall, however, express the resulting stress-energy tensor components in terms of the original (τ,ρ) coordinates of Eq. (7). For all nonextreme values of Q, the string metric has temperature

$$T_{\text{string}} = \frac{1}{8\pi e^{\phi_0}M}. \quad (19)$$

To obtain the components of the stress-energy tensor one may either directly integrate the conservation equation as we did for the physical metric, or utilize the relation between stress-energy tensors of conformally invariant fields in conformally related spacetimes [5]. In two dimensions, if two geometries are related by a conformal factor $\Omega(x)$, such that $\bar{g}_{\mu\nu}=\Omega^2(x)g_{\mu\nu}$, then the vacuum stress-energy tensors of a conformally invariant scalar or spinor field are related by

$$\langle \bar{T}_\mu{}^\nu \rangle_{\text{ren}} = \left(\frac{g}{\bar{g}} \right)^{1/2} \langle T_\mu{}^\nu \rangle_{\text{ren}} + \left(\frac{1}{12\pi} \right) \Big[(\Omega^{-3}\Omega_{;\alpha\mu}$$

$$- 2\Omega^{-4}\Omega_{,\alpha}\Omega_{,\mu})g^{\alpha\nu}$$

$$+ \delta_\mu{}^\nu g^{\alpha\sigma} \left(\frac{3}{2}\Omega^{-4}\Omega_{,\alpha}\Omega_{,\sigma} - \Omega^{-3}\Omega_{;\alpha\sigma} \right) \Big], \qquad (20)$$

where all derivatives are taken with respect to the unbarred metric. The components of the stress-energy tensor for the quantized scalar field in the string spacetime are found to be

$$\langle T_r{}^r \rangle = [384\pi e^{3\phi_0}M^3\rho^2(e^{\phi_0}M\rho - Q^2)]^{-1}$$

$$\times [8e^{5\phi_0}M^5 - 8e^{3\phi_0}M^3Q^2 + 2e^{\phi_0}MQ^4$$

$$+ (4e^{4\phi_0}M^4 - 4e^{2\phi_0}M^2Q^2 + Q^4)\rho$$

$$+ (2e^{3\phi_0}M^3 - 2e^{\phi_0}MQ^2)\rho^2 + e^{2\phi_0}M^2\rho^3],$$

$$(21)$$

$$\langle T_\tau{}^\tau \rangle = [384\pi e^{3\phi_0}M^3\rho^3(e^{\phi_0}M\rho - Q^2)]^{-1}[-32e^{4\phi_0}M^4Q^2$$

$$+ 16e^{2\phi_0}M^2Q^4 + (56e^{5\phi_0}M^5 - 24e^{3\phi_0}M^3Q^2$$

$$- 2e^{\phi_0}MQ^4)\rho - (4e^{4\phi_0}M^4 - 4e^{2\phi_0}M^2Q^2 + Q^4)\rho^2$$

$$- (2e^{3\phi_0}M^3 - 2e^{\phi_0}MQ^2)\rho^3 - e^{2\phi_0}M^2\rho^4], \qquad (22)$$

and the trace anomaly is given by

$$\langle T_\alpha{}^\alpha \rangle = \frac{(Q^2 - e^{2\phi_0}M^2)(Q^2 - 2e^{\phi_0}M\rho)}{24\pi e^{\phi_0}M\rho^3(e^{\phi_0}M\rho - Q^2)}. \qquad (23)$$

It is interesting to compare the physical behavior of the stress energy of the quantized field in the GHS charged black hole string metric, given by Eqs. (21)–(23) with those computed for the usual Reissner-Nordström black hole (these may be obtained from the expressions for the Unruh vacuum components given in Ref. [3] by a simple transformation). For the GHS black hole, the energy density as seen by a static observer $\epsilon = -\langle T_\tau{}^\tau \rangle$ is always negative in the neighborhood of the horizon; the radial stress is similarly always positive, indicating a pressure. In comparison, the energy density of a quantized field near the horizon of a two-dimensional Reissner-Nordström black hole is negative for $Q^2 < 8M^2/9$, but is positive for larger values of Q^2. The radial stress is also positive for $Q^2 < 8M^2/9$, and negative for larger values of Q^2.

The case of the extreme GHS black hole, with $Q^2 = 2e^{2\phi_0}M^2$, must be treated separately, as it possesses no horizon. In fact, once the metric of Eq. (8) is reduced to two-dimensional form, it is flat, and hence has no trace anomaly. The lack of a horizon means no geometrically defined temperature exists in this case. Garfinkle, Horowitz, and Strominger suggest zero temperature as the natural state for this spacetime, even though the temperature is then a discontinuous function of Q^2. Alternatively, one could choose the temperature to be continuous by assigning a value $T_{\text{extreme}} = (8\pi e^{\phi_0}M)^{-1}$. If an arbitrary temperature T_{extreme} is temporarily assigned, then the most general time-reversal symmetric (i.e., equilibrium) solution of the stress-energy conservation equation is that of a simple boson gas:

$$\langle T_\rho{}^\rho \rangle = -\langle T_\tau{}^\tau \rangle = \frac{\pi}{6}T_{\text{extreme}}^2. \qquad (24)$$

Consider now the temperatures associated with both the physical and string metrics. The physical metric is always precisely Schwarzschild (independent of Q) in two dimensions, and hence must always have $T = (8\pi M)^{-1}$; assignment of any other temperature would lead to a divergent stress energy for the quantized field on the horizon [4]. Similarly, the string metric's natural temperature is $T = (8\pi e^{\phi_0}M)^{-1}$ for all values $Q^2 < e^{2\phi_0}M^2$. Again, assignment of any other temperature would cause a strong divergence in the stress energy on the horizon. These natural, geometrically defined temperatures and the associated stress-energy tensors for quantized fields are also related directly through the conformal transformation of Eq. (20).

In the extreme case, the physical metric is still precisely Schwarzschild, and thus the spacetime must either be assigned the usual temperature or suffer divergent stress energy on the horizon. In the extreme string metric, though, there is no horizon, and hence any temperature (including zero) can now be assigned to the spacetime without causing singular behavior in the quantized fields in this metric. However, the conformal relation of Eq. (20) picks out a unique temperature for the extreme string metric. If the quantized fields are to be regular in both the physical and string metrics, one must choose $T_{\text{extreme}} = (8\pi e^{\phi_0}M)^{-1}$. Any other choice in the string metric will lead to divergences on the horizon of the physical metric by Eq. (20). Thus, it appears that the natural choice for the temperature of the extreme string metric is not zero, despite the lack of horizon, but rather $T_{\text{extreme}} = (8\pi e^{\phi_0}M)^{-1}$. This choice preserves continuity of the temperature in the string metric, and is the only choice which will allow quantized fields in both the string and physical metrics to be regular outside and on all horizons.

It is interesting to compare and contrast this result to the case of the extreme Reissner-Nordström black hole [6] in four dimensions. Hawking, Horowitz, and Ross [7] noted that the Euclidean section of the extreme Reissner-Nordström metric allowed one to identify the geometry with arbitrary period in Euclidean time (and hence arbitrary temperature) without creating a conical singularity in the Euclidean spacetime. Loranz, Hiscock, and Anderson [4] demonstrated, however, that the stress-energy tensor of a quantized field would diverge strongly on the event horizon in the Lorentzian spacetime unless the temperature was chosen to be zero, the value obtained by extrapolating the form of the temperature function $T(M,Q)$ from the nonextreme Reissner-Nordström black hole. In the present case, we again find that there will be a divergence of the stress energy of a quantized field (in this case, for a field which takes values on the conformally related physical metric) if the temperature of

the extreme GHS string metric is taken to be a value other than that extrapolated from the nonextreme GHS black hole temperature function. On the other hand, in this case, it means the extreme GHS metric must be assigned a nonzero temperature, whereas in the extreme Reissner-Nordström case the natural temperature was zero.

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