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Robotics & Automation

L4 Maths for EEE Handbook

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Abstract

This *Maths Handbook* summarises the key theories and tools you will need to successfully complete the Level 4 Mathematics for Electronic & Electrical Engineering module. It covers the maths you will need for your level 4 modules and many later modules. A companion textbook will be available for the Level 5 Advanced Mathematics for Electronic & Electrical Engineering module

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PART A

FOUNDATIONS

Basics of Numbers

Required Background: §C.1, Calculator Skills (Page 333)To solve engineering problems using mathematics, an understanding of numbers and basic arithmetic is required, along with knowledge about notation and symbols that can be used to be used to represent physical quantities.

This chapter starts with the basics of arithmetic, looking at the notation used in mathematics and how to formulate expressions to represent engineering problems. It then takes a look at numbers and what they are before going on to examine powers (or indices) and the use of powers for polynomials followed by looking at scientific & engineering notation.

1.1 Arithmetic

Basic operations

What do the notation symbols used in mathematics actually mean? Well you should all know the basic operations of addition, subtraction, multiplication and division. These are all binary operators - that is they operate on two numbers/variables which are sometimes referred to as operands. If we take x and y as our two variables we can say that:

- **Addition** is simply the **sum** of x and y and is written as $x + y$. This can be written as $y + x$ and still yield the same answer as addition is **commutative**. In fact we can add any number of variables together in any order and get the same answer.
- **Subtraction** is the representation of the **difference** between x and y — subtracting y from x is written as $x - y$. However this does not yield the same answer $y - x$ just as $6 - 4 = 2$ does not give same answer as $4 - 6 = -2$.
- **Multiplication** of x and y is written as $x \times y$ although usually we discard the multiplication sign to give simply xy or sometimes replace it with a simple dot $x \cdot y$. All of these donate the **product** of the variables. Multiplication is **associative** so with $x \times y \times z$ it doesn't matter if we evaluate $x \times y$ first then multiply result by z, or evaluate $y \times z$ first and multiply result by x. It is worth noting that if a product involves a symbol and a number then the accepted notation is number followed by symbol so $x \times 4$ is written simply as $4x$.
- **Division** is written as $x \div y$ for dividing x by y. This is also written as x/y or $\frac{x}{y}$ where the top line is known as the **numerator** and the bottom line is the **denominator**. As with subtraction, the quantity y/x is not the same as x/y and dividing any variable by 1 leaves the quantity unchanged while dividing by 0 results in the quantity ∞ (infinity).

Other notations

There are a number of other mathematical terms and notation that are commonly used in electrical & electronic engineering including:

- **Plus or minus sign** $\pm$ which means answer is either plus or minus the number after the sign. So if we say the answer lies in range 4 ± 2 then the answer is somewhere between 2 $(4 - 2)$ and 6 $(4 + 2)$.
- **Reciprocal of a number** which is found by **inverting** the value - that is making the old denominator the new numerator and vice-versa. As an example, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$. With a variable x the reciprocal is $\frac{1}{x}$ as we know that $x = \frac{x}{1}$.
- **Modulus** $|x|$ which is effectively the **absolute** value of x — so the size of the variable regardless of its sign. For example, $|2|$ and $|-2|$ are both equal to 2 as the modulus of both 2 and -2 is 2.

Relational operators

There are a number of operators that can be used to indicate the relationship between two sides of an expression:

- **Equals sign** $=$ which is used in a number of ways. Firstly in **equations**, where the **solutions** to the equation are the values of the variable where the left-hand side of the equation is equal to the right-hand side of the equation. So, for instance, for equation $x - 3 = 0$, the solution is $x = 3$.

  The second use is in **formulae** where physical quantities are related to each other through a formula. We can see this in the formula for the area of a circle (A) which is given by the formula $A = \pi r^2$.

  The final use of an equals sign is in **identities** which on first glance may appear like equations. In fact identities are where the expression is true for *all* values of the variable. An example of an identity is $(x - a)(x + a) = x^2 - a^2$ which is true for all values of x and a. Technically this should be written as $(x - a)(x + a) \equiv x^2 - a^2$ where $\equiv$ means 'is equivalent to' but in practice the equals sign is often used.
- **Not equals sign** $\neq$ which is used to indicate a expression on left-side is not equal to expression on right side. So $3 \neq 9$ is correct and we can say that x does not take the value of 5 by writing $x \neq 5$.
- **Relational operators** — the following operators all relate the left-hand side of the expression to the right-hand side.
  - **Less than and less than or equal to** $<$ and $\leqslant$. The less than ($<$) operator is used to indicate that the left-hand operand is smaller than the right-hand operand. The less than or equal to operator ($\leqslant$) indicates that the left hand operand is smaller than or equal to the right hand operand. So with numbers we can say that $3 < 9$ is true, as is $9 \leqslant 9$. In algebraic terms, the expression $x < 9$ indicates that the value of x is

below 9 while the expression $x \leqslant 9$ indicates that the maximum value x can take is 9.

- **Greater than and greater than or equal to** $>$ and $\geqslant$. The greater than ($>$) operator is used to indicate that the left-hand operand is larger than the right-hand operand. The greater than or equal to operator ($\geqslant$) indicates that the left hand operand is larger than or equal to the right hand operand. So with numbers we can say that $10 > 9$ is true, as is $9 \geqslant 9$. In algebraic terms, the expression $x > 9$ indicates that the value of x is above 9 while the expression $x \geqslant 9$ indicates that the minimum value x can take is 9.

The most likely place that you as engineers are going to meet expressions like these is in limits which is discussed later in this chapter.

Rules of Arithmetic

There are a number of rules for evaluating arithmetic expressions as laid out in the next few sections

Rule 1: Commutativity

Addition and multiplication are both commutative operations — that is any two numbers can be added or multiplied in any order without effecting the result:

$$3 + 4 = 4 + 3 = 7$$
$$3 \times 4 = 4 \times 3 = 12$$

However for subtraction and division the order of the two numbers does effect the result except when both numbers are equal to each other and not zero in case of division — so subtraction and multiplication are not commutative operations

$$4 - 3[= 1] \neq 3 - 4[= -1]$$
$$4 \div 2[= 2] \neq 2 \div 4[= 0.5]$$

Rule 2: Associativity

Addition and multiplication are both associative operations — that is any number of numbers can be added or multiplied in any order without effecting the result. This is true for any number of variables:

$$3 + (4 + 5) = (3 + 4) + 5 = 3 + 4 + 5 = 12$$
$$3 \times (4 \times 5) = (3 \times 4) \times 5 = 3 \times 4 \times 5 = 60$$

However for subtraction and division the order of the numbers does effect the result — so subtraction and multiplication are not associative operations

$$3 - (4 - 5)[= 4] \neq (3 - 4) - 5[= -6]$$
$$8 \div (4 \div 2)[= 4] \neq (8 \div 4) \div 2[= 1]$$

Rules of Precedence

These rules along with the use of brackets can be used to remove any ambiguity from an expression or calculation. So for example using numbers $20 - 5 \times 3$ could be either:

$$\begin{aligned} 20 - 5 \times 3 &= 15 \times 3 &&= 45 \quad \text{or} \\ 20 - 5 \times 3 &= 20 - 15 &&= 5 \end{aligned}$$

To remove this ambiguity we use the rules of precedence which state that for any calculation involving multiple arithmetic operators (so more than one of addition, subtraction, multiplication or division) the process is:

1. Evaluate terms in brackets first (see Section 3.3 in Chapter A.3 for nested brackets where you start with innermost brackets)
2. Working from the left evaluate divisions and multiplications as they are encountered. This will leave a calculation (eventually as may need more than one pass to remove all multiplications & divisions) involving just addition and subtraction.
3. Working from the left evaluate additions and subtractions as they are encountered

The following examples show these rules in operation - firstly without any brackets in Example A.1.1 and then with a set of brackets in Example A.1.2:

Example A.1.1

$$\begin{aligned} 4 + 3 \times 8 \div 4 - 9 \div 3 \times 2 + 1 \\ 4 + 24 \div 4 - 3 \times 2 + 1 \\ 4 + 6 - 6 + 1 \\ 10 - 5 = 5 \end{aligned}$$

Example A.1.2

$$\begin{aligned} (4 + 3 \times 8) \div 4 - 9 \div 3 \times 2 + 1 \\ (4 + 24) \div 4 - 9 \div 3 \times 2 + 1 \\ 28 \div 4 - 3 \times 2 + 1 \\ 7 - 6 + 1 \\ 1 + 1 = 2 \end{aligned}$$

Questions

Try not to use a calculator for these questions.

1. a) $3 + 4 =$
   b) $8 + (-3) =$
   c) $9 - 6 =$
   d) $5 - (-3) =$
   e) $(-4) + (-8) =$
   f) $(-14) - (-7) =$
   g) $5 \times 3 =$
   h) $9 \times (-4) =$
   i) $(-2) \times (-3) =$
   j) $12 \div 6 =$
   k) $15 \div (-5) =$
   l) $-16 \div 4 =$
   m) $(-18) \div (-9) =$
2. a) $20 + 10 \div 2 - 3 \times 2$
   b) $(5 + 7) \div (6 - 3) \times 2$
   c) $34 + 10 \div (2 - 3) \times 5$
3. State whether the following expressions are true or false:
   a) $4 < 8$
   b) $3 > 5$
   c) $-7 \geqslant -4$
   d) $-3 \leqslant 2$
   e) $(4 + 2) \neq (3 + 3)$
   f) $(12 - 3) = (4 + 5)$
   g) $(11 \times 5) > (7 \times 8)$
   h) $(48 \div 8) \leqslant (72 \div 6)$
   i) $(6 \times 2) = (121 \div 11)$

1.2 Numbers

Having looked at the basics of arithmetic lets take a deeper look at numbers. What do we actually mean by a real number? In essence we mean a number that can be placed somewhere on a straight line between $\pm\infty$. Figure A.1.1 shows a basic number line that displays the integer (whole numbers) numbers between ± 6 as well as two decimal numbers at $-2.5 = -2\frac{1}{2}$ and $0.75 = \frac{3}{4}$ and one fractional number at $3\frac{1}{3}$.

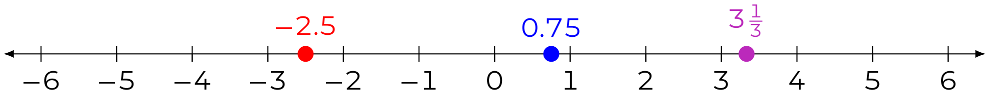


Figure A.1.1: Number Line

Prime numbers and factors

A prime number is defined as a positive number, greater than 1, which cannot be written as the product of two smaller integers — so the only numbers that divide exactly into a prime number (division results in an integer) are 1 and itself. Examples of the first few prime numbers are 2,3,5,7,11,13,17 and 19. Other than 2 all prime numbers are odd numbers as all even numbers are divisible by 2.

The *factors* of an integer n are the integer values that when n is divided by the factor, the result is another integer. For instance, the factors of 12 are $1, 2, 3, 4, 6, 12$ as dividing 12 by these numbers results in an integer number. We can factorise 12 in a number of ways:

$$12 = 2 \times 6, 12 = 3 \times 4, 12 = 2 \times 2 \times 3$$

The last factorisation above is a *prime factorisation* where all the factors are prime numbers. Prime numbers have been studied for a very long time (the ancient Greek mathematician Pythgoras studied them amongst others) and in more recent times the interest in them has been increased as they have important applications in Internet security and cryptography.

Highest common factor

Suppose we prime factorise 12 and 52:

$$12 = 2 \times 2 \times 3 \text{ and } 52 = 2 \times 2 \times 13$$

Some factors are common to both numbers — that is $2 \times 2 = 4$ so we can say the 4 is the *Highest Common Factor (HCF)* of 12 and 52.

> The highest common factor(HCF) of a set of numbers is the largest number that is a factor of all given numbers.

Lowest common multiple

Suppose we are given two or more numbers and we want to find the numbers that can be exactly divided by all the numbers. For example, given 3 and 6 we can see that 12, 24,36,48 etc. can all be divided by both 3 and 6. The smallest number which is divisible by both is 12 so we say that 12 is the *Lowest Common Multiple (LCM)* of 3 and 4.

> The lowest common multiple(LCM) of a set of numbers is the smallest number which can be divided by all the given numbers.

Fractions

As shown above we can represent non-integer numbers on a number line either as fractions or decimals. A fraction is simply one integer (the *numerator*) divided by another integer (the *denominator*). So for example, $\frac{1}{2}$ is fraction with a numerator of 1 and a denominator of 2. Fractions can be *proper*, *improper* or *mixed*:

- in a *proper* fraction the numerator is always less than the denominator e.g. $\dfrac{2}{5}$
- in an *improper* fraction the numerator is always greater than the denominator e.g. $\dfrac{7}{5}$
- a *mixed* fraction consist of an integer and a fraction e.g. $1\dfrac{2}{5}$

Every fraction has a different forms it can be expressed in — so for instance $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{3}{6}$ all have the same value. These are examples of *equivalent fractions* with $\frac{1}{2}$ being the *simplest form* as its numerator and denominator have no common factors. For example, the simplest form of $\frac{18}{48}$ is $\frac{3}{8}$ as 6 is a common factor of both 18 (3) and 48 (8×6).

Fractions can be positive or negative values, as with all real numbers. They can also be added, subtracted, multiplied and divided with the same rules as discussed in 1.1.

Addition & subtraction

To add or subtract fractions each fraction needs to be written in an equivalent form so that they all have the same denominator (the *common denominator*). You can then perform the operation on the numerators before expressing the result in its simplest form. Process is:

1. Find LCM of the denominators — this is common denominator.
2. Express each fraction in its equivalent form with the common denominator.
3. Perform the addition/subtraction operation with the numerators and then express resulting fraction in its simplest form.

Example A.1.3: Addition & Subtraction with fractions

Calculate answers to the following problems

1. $\dfrac{5}{8} + \dfrac{1}{6}$
2. $\dfrac{2}{3} - \dfrac{1}{4}$

Solution:

1. Original denominators are 6 and 8 — the LCM of these numbers is 24 ($6 \times 4 =$

24 and 8 × 3 = 24). So we convert the fractions to their equivalent fractions with a denominator of 24:

$$\frac{5}{8} = \frac{5 \times 3}{8 \times 3} = \frac{15}{24}$$
$$\frac{1}{6} = \frac{1 \times 4}{4 \times 4} = \frac{6}{24}$$

Now we can perform the addition:

$$\begin{aligned}\frac{5}{8} + \frac{1}{6} &= \frac{15}{24} + \frac{4}{24} \\ &= \frac{15+4}{24} \\ &= \frac{19}{24} \quad \text{19 is a prime number so this is simplest fraction}\end{aligned}$$

2. Original denominators are 3 and 4 — the LCM of these numbers is 12 (3 × 4 = 12 and 4 × 3 = 12). So we convert the fractions to their equivalent fractions with a denominator of 24:

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$
$$\frac{1}{4} = \frac{1 \times 3}{4 \times 3} = \frac{3}{12}$$

Now we can perform the subtraction:

$$\begin{aligned}\frac{2}{3} - \frac{1}{4} &= \frac{8}{12} - \frac{3}{12} \\ &= \frac{8-3}{12} \\ &= \frac{5}{12} \quad \text{5 is a prime number so this is simplest fraction}\end{aligned}$$

Multiplication

To multiply two fractions together we simply multiply their numerators to form the numerator of the answer and their denominators to form the denominator of the answer. Sometimes we then need to find the simplest fraction of the results.

Example A.1.4: Multiplication with fractions

Calculate answers to the following problems:

1. $\frac{1}{3} \times \frac{5}{7}$
2. $\frac{2}{3} \times \frac{3}{4}$

Solution:

1. $\frac{1}{3} \times \frac{5}{7} = \frac{1 \times 5}{4 \times 7} = \frac{5}{21}$
2. $\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 4} = \frac{6}{12} = \frac{1}{2}$

An alternate answer would involve cancelling common factors:
$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times \cancel{3}}{\cancel{3} \times 4} = \frac{2}{4} = \frac{1}{2}$

Sometimes you can be asked to calculate quantities such as '$\frac{1}{3}$ of 45' or '$\frac{2}{5}$ of 70 — in these calculations we treat 'of' as a multiplications, so for example:

$$\frac{1}{3} \times 45 = 15$$
$$\frac{2}{5} \times 70 = 2 \times 14 = 28$$

Division

Division of one fraction by another fraction involves inverting the second fraction and then multiplying the result by the first fraction. To *invert* a fraction we simply swap the numerator and denominator so denominator becomes new numerator and numerator becomes new denominator.

Example A.1.5: Division with fractions

Calculate answers to the following problems:

1. $\frac{2}{3} \div \frac{5}{6}$
2. $2\frac{1}{2} \div \frac{3}{4}$

Solution:

1. $\frac{2}{3} \div \frac{5}{6} = \frac{2}{3} \times \frac{6}{5} = \frac{2 \times 6}{3 \times 5} = \frac{12}{15} = \frac{4}{5}$
2. $2\frac{1}{2} \div \frac{3}{4} = \frac{5}{2} \times \frac{4}{3} = \frac{5 \times 4}{2 \times 3} = \frac{20}{6} = \frac{10}{3} = 3\frac{1}{3}$

Decimals

Decimals are simply the result of dividing one number by another number where the second number is not a factor of the first number. So for instance, $3 \div 2 = 1.5$ as 2 is not a factor of 3. Fractions and decimals are interchangeable in that a fraction results in a decimal number when the division is done so $3 \div 2 = 1\frac{1}{2}$. A fraction may be an exact decimal (as for $\frac{1}{2} = 0.5$) or may end up as a recurring pattern as a decimal such as $\frac{1}{3} = 0.33\dot{3}$ and $\frac{1}{7} = 0.\dot{1}4285\dot{7}$.

All the mathematical operations we saw in Section 1.1 for integer numbers can be done using decimal numbers but they can result in an answer with a large number of digits after the decimal point

$$12.28 \div 4.64 = 2.646551724137931$$

To make these numbers more manageable they can be rounded to a number of *significant figures* or *decimal places*

Significant figures

The number of significant figures is counted from the first non-zero digit from the left of the number. Once the requisite number of Significant Figures (S.F.) has been reached the rest of the digits are discarded but with regard to the following condition:

> If the first digit in the group of digits to be discarded is greater than or equal to 5, 1 is added to the last significant numeral.
> If the value of the last significant digit is 9 and the last first discarded digit is 5 or more then the last significant digit becomes 0 and the value of the previous digit is incremented by 1.

Example A.1.6

1. Round the result of $12.28 \div 4.64$ to the following number of significant figures: 2, 3, 5 and 6
2. Round 0.0629543678 to the following number of significant figures: 2, 3, 5 and 6

Solution

1. $12.28 \div 4.64 = 2.646551724137931$
   Rounding the result:

2 S.F.	2.6	final significant digit remains same as first digit discarded is 4 (< 5)
3 S.F.	2.65	incremented final significant digit as first number discarded is 6 $(\geqslant 5)$
5 S.F.	2.6466	incremented final significant digit as first number discarded is 5 $(\geqslant 5)$
6 S.F.	2.64655	final significant digit remains same as first number discarded is 1 (< 5)

2. Rounding 0.0629543678

2 S.F.	0.062	First significant digit is 6. Final significant digit remains same as number digit discarded is 4 (< 5)
3 S.F.	0.0630	as final significant digit is 9 and the first number discarded is 5 $(\geqslant 5)$
5 S.F.	0.062954	final significant digit remains same as first number discarded is 3 (< 5)
6 S.F.	0.06295440	incremented final significant digit as first number discarded is 6 $(\geqslant 5)$

Decimal places

When we are rounding to a number of Decimal Places (D.P.) then we count the number of digits after the decimal point. Once the requisite number of digits has been reached the rest of the digits are discarded but with the same condition on the value of last digit as for significant figures in terms of incrementing it by 1 if the first discarded digit is greater than or equal to 5.

Example A.1.7

1. Round the result of $12.28 \div 4.64$ to the following number of decimal places: 2, 3, 5 and 6
2. Round 0.0629543678 to the following number of decimal places: 2, 3, 5 and 6

Solution:

1. $12.28 \div 4.64 = 2.646551724137931$
   Rounding the result:

2 D.P.	2.65	incremented final digit as first number discarded is 6 ($\geqslant 5$)
3 D.P.	2.647	incremented final digit as first number discarded is 5 ($\geqslant 5$)
5 D.P.	2.64655	final digit remains same as first number discarded is 1 (< 5)
6 D.P.	2.646552	incremented final digit as first number discarded is 7 ($\geqslant 5$)

2. Rounding 0.0629543678

2 D.P.	0.06	final digit remains same as first number discarded is 2 (< 5)
3 D.P.	0.063	incremented final digit as first number discarded is 9 ($\geqslant 5$)
5 D.P.	0.06295	final digit remains same as first number discarded is 4 (< 5)
6 D.P.	0.062954	final digit remains same as first number discarded is 3 (< 5)

Comparing the results in Examples A.1.6 and A.1.7 we can see that with numbers that have a value that is greater than or equal to 1 then decimal places gives longer answers whereas for values less than 0.1 significant figures will give longer answers.

Rounding up or down

Above we always applied rounding to the nearest value - that is we incremented the last digit if the first discarded digit was greater than or equal to 5. However there are situations when we want to round down (so towards 0) or up (away from zero). So if we want to round down to 2 D.P. 2.646551724137931 then the answer is 2.64 whereas rounding up to 2 D.P. the result is 2.65. If you are rounding a negative number it is important to note the rounding down is towards 0 and rounding up is away from 0 so the resulting absolute values (no sign) are the same whether number is positive or negative.

Example A.1.8

1. Round 0.0629543678 down to the following number of decimal places: 2, 3 and 5
2. Round 0.0629543678 up to the following number of decimal places: 2, 3 and 5
3. Round -0.0629543678 down to the following number of decimal places: 2, 3 and 5
4. Round -0.0629543678 up to the following number of decimal places: 2, 3 and 5

Solution:

1. 0.0629543678 down
   - 2 D.P. 0.06
   - 3 D.P. 0.062
   - 5 D.P. 0.06295
2. 0.0629543678 up
   - 2 D.P. 0.07
   - 3 D.P. 0.063
   - 5 D.P. 0.06296
3. -0.0629543678 down
   - 2 D.P. -0.06
   - 3 D.P. -0.062
   - 5 D.P. -0.06295
4. -0.0629543678 up
   - 2 D.P. -0.07
   - 3 D.P. -0.063
   - 5 D.P. -0.06296

There are three important functions to note here that we use to round decimal or fractional numbers :

- `round(x, pre)` rounds to the nearest value, the value of `x` to the number of decimal places given by `pre`
- `floor(x)` performs the floor operation on `x` - that is rounds down to the nearest integer
- `ceil(x)` performs the ceiling operation on `x` - that is rounds up to the nearest integer

Number types

We can define the set of real numbers as being *rational* or *irrational*:

- *Rational* numbers can be expressed as fractions and/or decimals — if expressed as a decimal, the value is either an exact value ($\frac{1}{4} = 0.25$) or a recurring pattern ($\frac{1}{7} = 0.\dot{1}4285\dot{7}$).
- *Irrational* numbers cannot be expressed as fractions and in decimal form have an infinite

string of numerals that have no recurring pattern. Examples of known irrational numbers are π, $\sqrt{2}$ and e.

Questions

1. Write each of the following a product of prime factors:

   a) 924 b) 825 c) 2310 d) 3155

2. Find the HCF and LCM of each pair of numbers

   a) 9 & 21 b) 15 & 85 c) 42 & 66 d) 64 & 360

3. Calculate each of the following sums to

   i) 5 significant figures
   ii) 4 decimal places

   a) $\dfrac{3.21^2 + (5.77 - 3.11)}{8.32 - 2.64 \times \sqrt{2.56}}$

   b) $\dfrac{3.1242 \times 1.95}{6}(3 \times 5.44^2 + 1.95^2)$

   c) $\sqrt{\dfrac{2 \times 0.577}{3.142 \times 2.64} + \dfrac{2.64^2}{3}}$

4. Reduce each of the following fractions to its simplest mixed form:

   a) $\dfrac{6}{24}$ b) $\dfrac{104}{48}$ c) $-\dfrac{120}{15}$ d) $-\dfrac{51}{7}$

5. Evaluate the following sums - giving answer in fractional form:

   a) $\dfrac{9}{2} - \dfrac{4}{5} \div \dfrac{4}{9} \times \dfrac{3}{11}$ b) $\dfrac{\frac{2}{3} + \frac{7}{5} \div \frac{2}{9} \times \frac{1}{3}}{\frac{7}{3} - \frac{11}{2} \times \frac{2}{5} + \frac{4}{9}}$

6. Write each of the following decimals in its simplest mixed fraction form:

   a) 0.12 b) 5.25 c) $5.\dot{3}0\dot{6}$ d) $-9.\dot{6}$

7. State whether the following numbers are rational or irrational:

   a) 2.12523 b) $\dfrac{1}{\sqrt{2}}$ c) $\dfrac{2}{7}$ d) 0.5e

   e) $\dfrac{3}{\sqrt{9}}$ f) $5.\dot{3}0\dot{6}$ g) $\dfrac{\pi}{6}$ h) $\dfrac{\sqrt{3}}{2}$

1.3 Powers or Indices

Consider the product $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 256$. This can be more simply written as $2^8 = 256$. The 2 is known as the *base* number and 8 is known as either the *power* or *index*. Similarly $a \times a \times a = a^3$ where a is the base and 3 is the index/power.

Standard Power Values

Unity power	Any base raised to the power of 1 will equal the base number. So $3^1 = 3$ & $20^1 = 20$.
Zero power	Any base raised to the power of 0 will equal 1. So $3^0 = 1$ & $20^0 = 1$.

These facts are important for understanding number bases and how we use them.

Multiplication with Powers

Consider the following calculation:

$$\begin{aligned} 9 = 3^2 &\text{ and } 81 = 3^4 \\ 9 \times 81 = 3^2 \times 3^4 &= (3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3) \\ &= (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^6 = 3^{2+4} \end{aligned}$$

This is the *first law of indices* where multiplying two numbers together which can be written as powers of the same base number is equivalent to raising the same base number to the sum of the powers

$$x^m \cdot x^n = x^{m+n} \tag{A.1.1}$$

Division with Powers

Consider the following calculation:

$$\begin{aligned} \frac{81}{9} &= \frac{3^4}{3^2} \\ &= \frac{(3 \cdot 3 \cdot 3 \cdot 3)}{(3 \cdot 3)} \\ &= (3 \cdot 3) = 3^2 = 3^{4-2} \end{aligned}$$

This is the *second law of indices* where dividing one number by another number where both numbers can be written as powers of the same base number is equivalent to raising the same base number to the difference of the powers

$$\frac{x^m}{x^n} = x^{m-n} \tag{A.1.2}$$

Negative Powers

Consider the following calculation:

$$\begin{aligned}\frac{9}{81} &= \frac{3^2}{3^4} \\ &= \frac{(3 \cdot 3)}{(3 \cdot 3 \cdot 3 \cdot 3)} \\ &= \frac{1}{(3.3)} = \frac{1}{3^2}\end{aligned}$$

Division of powers means *subtraction* of powers

$$\frac{9}{81} = \frac{3^2}{3^4} = 3^{2-4} = 3^{-2}$$

Thus we can interpret the changing of the sign of a power as the inversion of the expression

$$x^{-m} = \frac{1}{x^m} \ \& \ x^m = \frac{1}{x^{-m}} \tag{A.1.3}$$

Powers of powers

Consider the following calculation:

$$(3^4)^2 = (3^4) \cdot (3^4) = 3^{4+4} = 3^8$$

This is the *third law of indices* which states that the result of a base number raised to a power being raised to a further power is equivalent to the base number raised the the product of the powers

$$\left(x^m\right)^n = x^{m \cdot n} \tag{A.1.4}$$

Fractional powers

Taking equation A.1.4 and making $x = 4$, $m = \frac{1}{2}$ and $n = 2$

$$\left(4^{1/2}\right)^2 = 4^{1/2 \cdot 2} = 4^1 = 4$$

So we can say that $4^{1/2} = \sqrt{4}$. In general, a fractional power $1/n$ represents the n^th^ root of the base number.

Summary of laws of indices

The laws of indices can be summarised as:

$$\begin{aligned} a^m \cdot a^n &= a^{m+n} \\ \frac{a^m}{a^n} &= a^{m-n} \\ (a^m)^n &= a^{m \cdot n} \\ a^{-m} &= \frac{1}{a^m} \\ a^0 &= 1 \end{aligned} \tag{A.1.5}$$

Questions

Write the following as single expressions of the form a^m:

1. $8^5 \times 8^3$
2. $5^4 \times 5^{-2}$
3. $4^6 \div 4^4$
4. $3^3 \div 3^{-2}$
5. $(2^2)^3$
6. $(7^6)^{1/3}$
7. $(11^9)^0$
8. $(9^0)^{-8}$

1.4 Scientific notation

In science, we use scientific notation as an easy way to represent very small or very large numbers. For example instead of writing 0.0000000039 we can write 3.9×10^{-9} and instead of writing 2340000 we can write 2.34×10^6. But what does the notation mean?

Well think of 3.9×10^{-9} as the product of two terms: 3.9 known as the *digit term* and 10^{-9} known as the *exponent term*. The digit term is just the digits with a decimal point after the first digit and the exponent term is the power of 10 that puts the decimal point in the right place. Table A.1.1 shows typical representations in scientific notation with the same digit term

All the examples in Table A.1.1 have three significant figures — significant figures are the number of digits in the digit term. The leading/trailing zeros do not count as significant figures. So if you are asked to give an answer to 2 significant figures (sometimes written as 2 s.f.) this means an answer where there would be 2 digits in the digit term of the scientific notation of the answer.

10000	10^{4}	34500	3.45×10^{4}	
1000	10^{3}	3450	3.45×10^{3}	
100	10^{2}	345	3.45×10^{2}	*Not often used*
10	10^{1}	34.5	3.45×10^{1}	*Not usually used*
1	10^{0}	3.45	3.45×10^{0}	*Not usually used*
0.1	10^{-1}	0.345	3.45×10^{-1}	*Not usually used*
0.01	10^{-2}	0.0345	3.45×10^{-2}	*Not often used*
0.001	10^{-3}	0.00345	3.45×10^{-3}	
0.0001	10^{-4}	0.000345	3.45×10^{-4}	

Table A.1.1: Example Scientific Notation

Engineering notation

Prefix	Symbol	Base 10	Decimal
yotta	Y	10^{24}	1 000 000 000 000 000 000 000 000
zetta	Z	10^{21}	1 000 000 000 000 000 000 000
exa	E	10^{18}	1 000 000 000 000 000 000
peta	P	10^{15}	1 000 000 000 000 000
tera	T	10^{12}	1 000 000 000 000
giga	G	10^{9}	1 000 000 000
mega	M	10^{6}	1 000 000
kilo	k	10^{3}	1 000
hecto	*h*	10^{2}	100
deca	*da*	10^{1}	10
		10^{1}	1
deci	*d*	10^{-1}	0.1
centi	*c*	10^{-2}	0.01
milli	m	10^{-3}	0.001
micro	μ	10^{-6}	0.000 001
nano	n	10^{-9}	0.000 000 001
pico	p	10^{-12}	0.000 000 000 001
femto	f	10^{-15}	0.000 000 000 000 001
atto	a	10^{-18}	0.000 000 000 000 000 001
zepto	z	10^{-21}	0.000 000 000 000 000 000 001
yocto	y	10^{-24}	0.000 000 000 000 000 000 000 001

Table A.1.2: SI prefixes

In engineering, we often use a specialised version of scientific notation where the exponent terms are limited to specific powers of ten. If the general term for a scientific expression is $d \times 10^{p}$, then, in engineering notation, the exponent p is always chosen to be divisible by three. These powers are then given defined prefixes in the International System of Units (SI) as defined in Table A.1.2 which also includes 4 prefixes that are not part of the engineering notation (hecto/deca/deci/centi) as they represents powers of 10 which are not divisible by three.

Some of these will be familiar — for instance 'cm' means centimetre which is one hundredth of a metre. Similarly 'km' is kilometre which is one thousand metres. Others will be used extensively throughout your studies for defining values of components such as 8.2 kΩ which is a resistor of value 8200 ohms; or 1 μF which donates a capacitor with a capacitance of

0.000001 Farads.

In most cases you will see the units written ‘as normal‘ with the unit value (magnitude) followed by the appropriate unit symbol. However in some cases we use a more compact form for common units: especially for resistance. Thus you will see the value of an 8.2 kΩ resistor also written as ‘8k2 Ω’, or even just ‘8k2‘. These are accepted alternatives for resistor values — but in most cases you are encouraged to follow the standard form outlined above. Examples of components that use the small values include capacitors where a 1.5 nF capacitor can also be written as ‘1n5 F’ and inductors where a 3.3 μH inductor can also be written as ‘3μ3 H’.

Units are *not* there for decoration. Units are an integral part of the quantity you are defining. Without using the correct units, in the correct form, you are making it very hard for someone else to follow your reasoning.

Questions

1. Write each of the following as a standard decimal number
   - a) 3.2044×10^3
   - b) 16.1105×10^{-2}
   - c) 0.01254×10^5
   - d) 123.7895×10^{-6}
   - e) 1.2345×10^4
   - f) 26.789×10^{-1}
   - g) 0.005678×10^7
   - h) 234.7896×10^{-4}
2. Write each of the following in scientific notation
   - a) 918273.465
   - b) 134.65
   - c) 4563.23
   - d) 4567891.23
   - e) 0.012345
   - f) 0.002401
   - g) 0.00005678
   - h) 0.000006785
3. Write each of the following in engineering notation
   - a) 6700 Ω
   - b) 1.5×10^4 Ω
   - c) 5.8×10^5 Ω
   - d) 2.2^2 Ω
   - e) 1.5^{-11} F
   - f) 4.7×10^{-8} F
   - g) 6.8×10^{-4} H
   - h) 3.3×10^{-5} H
4. Write each of the following in scientific notation
   - a) 4k7 Ω
   - b) 22 kΩ
   - c) 820 kΩ
   - d) 10 MΩ
   - e) 330 μH
   - f) 8μ2 H
   - g) 2p2 F
   - h) 680 nF

1.5 Summary

This chapter has taken a look at the basics of numbers, arithmetic and notation that are the technical foundation for much of engineering mathematics.

Number Bases

Required Background: §A.1.3, Powers and Indices (Page 15); §C.1, Calculator Skills (Page 333)

2.1 Introduction

What do we mean by a number base? Effectively, it is just a way of representing numbers. We can chose to represent numbers in many ways — in essence what we think of as a number is just a symbol representing a number, much as writing down our name represents us, but it is still us if we chose to type it, use a different alphabet or colour.

Number bases are just different ways of representing the same number. What you are used to is base 10 or denary numbers - which probably arose because we have 10 fingers to count on. In electronics, counting is much of the time either a current or voltage is present or not, so binary (base 2) numbers are often used. Also hexadecimal (base 16) is common as a way of representing larger numbers which can still be interpreted as presence or not. Occasionally octal (base 8) numbers are used and there is a special format known as Binary Coded Decimal (BCD) which is used for particular applications where we want to represent denary numbers easily in binary.

The following sections go into more detail about all of these bases starting with denary as the familiar representation to you. This enables us to look at the principles of number bases with a familiar set of numbers.

2.2 Denary

Basics

You will be very familiar with denary numbers - that is 1, 20, 345, 6789 etc. But what do they actually represent? Well lets take 20 - this represents 2 ‘tens’ and no ‘units’ (or ‘ones’). What about 6789: this is 6 ‘thousands’, 7 ‘hundreds’, 8 ‘tens’ and 9 ‘units’. You should be able to see a pattern where each column is 10 times the column to its right. In effect, this is the principle of a number base where numbers increase from the right by multiplying the previous column by the base. So the number six hundred and forty-eight thousand, five hundred and ninety-two is written as 648592 which is created as shown in Table A.2.1:

The column values can be calculated from the base raised to an appropriate power — in other words for base 10 numbers each column is 10^x where $x = n + 1$ with n being the power (index) of the column to the right.

10^5	10^4	10^3	10^2	10^1	10^0
100000	10000	1000	100	10	1
6	4	8	5	9	2

Table A.2.1: Denary Number Representation for ‘648592’

What about decimal numbers rather than integers — well the same principles apply as the columns after the decimal point represent 10^{-1}, 10^{-2}, 10^{-3} etc. So the number 23.456 represents two ‘tens’, three ‘units’, four ‘one-tenths’, five ‘one hundredths’ and six ‘one thousandths’.

10^1	10^0	.	10^{-1}	10^{-2}	10^{-3}
10	1	.	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$
10	1	.	0.1	0.01	0.001
2	3	.	4	5	6

Table A.2.2: Decimal Number Representation for ‘23.456’

Denary addition & subtraction

Having explained what denary numbers represent how to the basic arithmetic operations of addition & subtraction actually work? Let’s look at basic example of addition where we add 532 and 679 together.

Example A.2.1

$$\begin{array}{cccc} & 5 & 3 & 2 \\ + & 6 & 7 & 9 \\ {\scriptstyle 1} & {\scriptstyle 1} & {\scriptstyle 1} & \\ \hline 1 & 2 & 1 & 1 \end{array}$$

As you should know, we start addition with the units column — $9 + 2 = 11$ so we put 1 in the units column of answer and carry 1 to the tens column which now becomes $7 + 3 + 1 = 11$ so again 1 in tens column of answer and carry 1 to hundreds column which is now $5 + 6 + 1 = 12$ so 2 in hundreds column of answer and carry 1 to thousands column giving the answer of 1211.

What about subtraction? Well here we borrow 1 from next column to left if we will get a negative number (number less than 0) by doing subtraction in the current column — this reduces number in column to left by 1. As an example we can do the sum $632 - 457 = 175$.

Example A.2.2

$$\begin{array}{cccc} & \overset{5}{\cancel{6}} & \overset{\overset{1}{2}}{\cancel{3}} & \overset{1}{2} \\ - & 4 & 5 & 7 \\ \cline{2-4} & 1 & 7 & 5 \end{array}$$

In this case in the units column $2 - 7 < 0$ so we borrow 1 from the tens column making that now 2 and write 5 in units column of answer as $12 - 7 = 5$. Now looking at tens column $2 - 5 < 0$ so we borrow 1 from the hundreds column making that now 5 and the tens column 12 so answer tens column is 7 as $12 - 5 = 7$. Finally the hundreds column of answer is 1, as $5 - 4 = 1$.

What happens if the immediate column to the left has 0 in it. Well in this case we borrow from the preceding column and ripple the borrow through. So for example the sum $602 - 427 = 175$ as shown in Example A.2.3. In this case in the units column $2 - 7 < 0$ so we try to borrow 1 from the tens column however this has a value of 0 so we borrow 1 from the hundreds column making that now 5. The tens column now becomes 10 so we borrow 1 from it making it now 9 and the units column 12. So we can now write 5 in units column of answer as $12 - 7 = 5$. Now looking at tens column we have $9 - 7 = 2$ so we write 2 in the answer tens column. Finally the hundreds column of answer is 1, as $5 - 4 = 1$.

Example A.2.3

$$\begin{array}{cccc} & \overset{5}{\cancel{6}} & \overset{\overset{9}{\cancel{1}}}{\cancel{0}} & \overset{1}{2} \\ - & 4 & 2 & 7 \\ \hline & 1 & 7 & 5 \end{array}$$

These concepts of carry for addition and borrow for subtraction are common to all number bases.

Denary multiplication & division

Here we consider long multiplication - i.e. multiplication/division of two numbers where each number contains at least 2 digits. The procedure for multiplication uses partial products that are successively shifted left by one space and added together. We multiply each column in the top number by a single column in the bottom shifting to the left so that the last digit of result is in the column of the number we are multiplying by. We then add the resulting numbers together to give us the result. So for instance the sum $612 \times 24 = 14688$ can be calculated as shown in example A.2.4:

Example A.2.4: Long multiplication

$$\begin{array}{cccccccl} & & & 6 & 1 & 2 & & \\ & & \times & & 2 & 4 & & \\ \cline{3-6} & & 2 & 4 & 4 & 8 & & \text{Multiply 612 by 4} \\ + & 1 & 2 & 2 & 4 & & & \text{Multiply 612 by 2 shifting answer left} \\ \cline{2-6} & 1 & 4 & 6 & 8 & 8 & & \text{by 1 column} \end{array}$$

Example A.2.5: Long division

$25\overline{)425}$	$4 \div 25 = 0$ remainder 4	The first digit of the dividend (4) divided by the divisor
$\begin{array}{r} 0 \\ 25\overline{)425} \end{array}$		The whole number result is placed at the top. Ignore any remainders at this point
$\begin{array}{r} 0 \\ 25\overline{)425} \\ 0 \end{array}$	$25 \times 0 = 0$	The answer from the first operation is multiplied by the divisor. The result is placed under the number divided into.
$\begin{array}{r} 0 \\ 25\overline{)425} \\ \underline{0} \\ 4 \end{array}$	$4 - 0 = 4$	Now we subtract bottom number from top number.
$\begin{array}{r} 0 \\ 25\overline{)425} \\ \underline{0} \\ 42 \end{array}$		Now we bring down next number (2) in dividend.
$\begin{array}{r} 1 \\ 25\overline{)425} \\ \underline{0} \\ 42 \\ -25 \\ 125 \end{array}$	$42 \div 25 = 1$ reminder 12	Divide new number (42) by 25 put result at top and subtract result × divisor from bottom which gives remainder value. Bring next number down from dividend.
$\begin{array}{r} 15 \\ 25\overline{)425} \\ \underline{0} \\ 42 \\ -25 \\ 125 \\ \underline{-125} \\ 000 \end{array}$	$125 \div 25 = 5$ reminder 0	Divide new number (125) by 25 put result at top and subtract result × divisor from bottom

Long division involves working out the number of multiples of the divisor (number dividing by) that goes into each part of the dividend (number you are dividing). It is easier to explain using an example sum $425 \div 125 = 17$ as shown in Example A.2.5.

Questions

Calculate the answers to the following problems showing all working (so avoiding using a calculator if possible)

1. $234 + 567$
2. $256 + 1024$

3. 903 – 645
4. 4096 – 512
5. 345 × 15
6. 1024 × 32
7. 675 ÷ 15
8. 2048 ÷ 16

2.3 Binary

Basics

One of the key number bases within electronics is base 2 or binary — simply as for digital systems the basic principle is there are two states. These can be donated by on/off, true/false or high/low — all of which can effectively be represented by binary numbers where each column can either be 0 or 1. The term 'bit' is shorthand for '*bi*nary digi*t*' and is the smallest unit of storage in a computer. As you should know, we tend to measure computer storage in *bytes* where 1 byte is a group of 8 bits. It is use of binary as the basis for computing that has lead to the fact that computer memory (Random Access Memory (RAM)) is always a multiple of two — ie. 4GB, 8GB, 16GB rather than 10GB, 20GB etc. It also explains why in a kilobyte of memory there is actually 1024 bytes ($2^{10} = 1024$ rather than 1000 bytes.

Binary uses powers of 2 for its columns so the number 101011_2 represents the number as shown in Table A.2.3.

2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	1

Table A.2.3: Binary Number Representation for '101011_2'

As with denary numbers we can have fractional points of number with the same principle that the columns after the decimal point represent $2^{-1} = 0.5_{10}$, $2^{-2} = 0.25_{10}$, $2^{-3} = 0.125_{10}$, $2^{-4} = 0.0625_{10}$, etc. This is much less common than decimal but is used in fixed point systems. However most modern 32 bit+ processors implement floating point numbers as discussed in Chapter A.8. Fixed point is more likely to be found in 8 bit devices. *Note*: the number of bits in a processor represents the width of the data bus in the processor. It is also worth pointing out terminology at this point:

- Least Significant Bit (LSB) indicates the right hand most bit or last bit in a binary number.
- Most Significant Bit (MSB) indicates the left hand most bit or first bit in a binary number.

Converting Binary to Decimal

Binary uses powers of 2 for its columns so the number 101011_2 represents the number as shown in Table A.2.3. So to convert to decimal:

$$\begin{aligned} 101011_2 &= (1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (1 \times 2^0) \\ &= 32 + 0 + 8 + 0 + 2 + 1 \\ &= 43_{10} \end{aligned}$$

In general to convert a binary number to a decimal number we need to know the powers of 2 as shown in Table A.2.4 which can be extended further for higher powers of 2.

2^0	1	2^4	16	2^8	256
2^1	2	2^5	32	2^9	512
2^2	4	2^6	64	2^{10}	1024
2^3	8	2^7	128	2^{11}	2048

Table A.2.4: Powers of 2

Converting Decimal to Binary

This is slightly more complicated than converting binary to decimal and is easiest explained through examples. There are two methods - one using Table A.2.4 to work out the sum of powers of 2 that add up to the decimal number. The other is to repeatedly divide by 2.

Starting with the first method let us express 93_{10} in binary. Looking at Table A.2.4 the highest number in table which does not exceed 93 is 64 so we can write:

$$93 = 64 + 29$$

Now we take 29 and see that 16 is highest number that does not exceed it giving us:

$$93 = 64 + 16 + 13$$

Focussing on 13 we can see that 8 is the highest number that does not exceed it leading to:

$$93 = 64 + 16 + 8 + 5$$

We now focus on 5 and can see this is $4 + 1$ so this gives us:

Example A.2.6

$$\begin{aligned} 93_{10} &= 64 + 16 + 8 + 4 + 1 \\ &= 2^6 + 2^4 + 2^3 + 2^2 + 2^0 \\ &= 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 1(2^2) + 0(2^1) + 1(2^0) \\ &= 1011101_2 \end{aligned}$$

Now let us express 800_{10} in binary using the same method of using Table A.2.4 - noting as a start that 512 is highest number which does not exceed 800:

Example A.2.7

$$\begin{aligned}800_{10} &= 512 + 288\\ &= 512 + 256 + 32\\ &= 2^9 + 2^8 + 2^5\\ &= 1(2^9) + 1(2^8) + 0(2^7) + 0(2^6) + 1(2^5) + 0(2^4) + 0(2^3) + 0(2^2)\\ &+ 0(2^1) + 0(2^0)\\ &= 1100100000_2\end{aligned}$$

The other way to convert decimal numbers to binary numbers is to divide by 2 repeatedly and note the remainder until you get an answer of $0r1$. So for 93_{10}:

Example A.2.8

	Remainder
$93 \div 2 = 46r1$	1
$46 \div 2 = 23r0$	0
$23 \div 2 = 11r1$	1
$11 \div 2 = 5r1$	1
$5 \div 2 = 2r1$	1
$2 \div 2 = 1r0$	0
$1 \div 2 = 0r1$	1

We now read the remainder column form the bottom to the top to give us the binary number:

$$93_{10} = 1011101_2$$

Doing the same for the second number 800_{10}:

Example A.2.9

	Remainder
$800 \div 2 = 400r0$	0
$400 \div 2 = 200r0$	0
$200 \div 2 = 100r0$	0
$100 \div 2 = 50r0$	0
$50 \div 2 = 25r0$	0
$25 \div 2 = 12r1$	1
$12 \div 2 = 6r0$	0
$6 \div 2 = 3r0$	0
$3 \div 2 = 1r1$	1
$1 \div 2 = 0r1$	1

We now read the remainder column form the bottom to the top to give us the binary number:

$$800_{10} = 1100100000_2$$

Binary Addition

Binary addition is basically similar to denary addition but remembering that there are only two valid numbers - that is 1 and 0. Starting off with simple sums so $3 + 2 = 5$ which in binary becomes:

Example A.2.10

$$\begin{array}{ccc} & 1 & 1 \\ \underset{1}{+} & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}$$

You can see in this when we add two 1's in binary we put 0 in that column and carry 1 as $1 + 1 = 2_{10} = 10_2$. There are in fact four basic rules for binary addition:

$$\begin{array}{ll} 0 + 0 = 0_{10} = 00 & \text{Sum of 0 with a carry of 0} \\ 0 + 1 = 1_{10} = 01 & \text{Sum of 1 with a carry of 0} \\ 1 + 0 = 1_{10} = 01 & \text{Sum of 1 with a carry of 0} \\ 1 + 1 = 2_{10} = 10 & \text{Sum of 0 with a carry of 1} \end{array} \tag{A.2.1}$$

When you have a carry of 1 (highlighted below in A.2.2) you end up with a situation when three bits are being added together. it is worth noting that a carry of 0 results in the same rules as in A.2.1:

$$\begin{array}{ll} 1 + 0 + 0 = 01 & \text{Sum of 1 with a carry of 0} \\ 1 + 0 + 1 = 10 & \text{Sum of 0 with a carry of 1} \\ 1 + 1 + 0 = 10 & \text{Sum of 0 with a carry of 1} \\ 1 + 1 + 1 = 11 & \text{Sum of 1 with a carry of 1} \end{array} \tag{A.2.2}$$

For instance $3 + 3 = 6$:

Example A.2.11

$$\begin{array}{ccc} & 1 & 1 \\ \underset{1}{+} & \underset{1}{1} & 1 \\ \hline 1 & 1 & 0 \end{array}$$

This is the basic principle of binary addition so if we ended up with adding four 1's together then we would put 0 in current column and carrying 0 to the next column to left and carrying 1 two columns to left. This situation could arise in adding three binary numbers together where in addition the four rules shown in A.2.1 and A.2.2 there are another two situations that can arise:

$$\begin{aligned} 1+1+1+1 &= 100 \quad \text{Sum of 0 with a } 1^{st} \text{ carry of 0 \& } 2^{nd} \text{ carry of 1} \\ 1+1+1+1+1 &= 101 \quad \text{Sum of 1 with a } 1^{st} \text{ carry of 0 \& } 2^{nd} \text{ carry of 1} \end{aligned} \tag{A.2.3}$$

As an example consider $15 + 14 + 15 = 44_{10} = 101100_2$ which demonstrates how these situations can arise:

Example A.2.12

$$\begin{array}{cccccc} & & 1 & 1 & 1 & 1 \\ + & & 1 & 1 & 1 & 0 \\ + & & 1 & 1 & 1 & 1 \\ {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & {\scriptstyle 0} & {\scriptstyle 1} & \\ & & {\scriptstyle 1} & & & \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \end{array}$$

The basic principle of binary addition is to write the number of ones in the answer in binary with the LSB being in the current column and the remaining bits being in columns to left as appropriate - so for $7(111_2)$ we would put 1 as answer in current column, 1 carried one column to left and 1 carried 2 columns to left.*R25a

Binary Subtraction

Let us take the simple example of $3 - 2 = 1$ which in binary becomes:

Example A.2.13

$$\begin{array}{ccc} & 1 & 1 \\ - & 1 & 0 \\ \hline & 0 & 1 \end{array}$$

As with addition there are four basic rules:

$$\begin{aligned} &0 - 0 = 0 \\ &1 - 1 = 0 \\ &1 - 0 = 1 \\ &1011 = 1 \quad\; 0 - 1 \text{ with a borrow of 1} \end{aligned} \tag{A.2.4}$$

As with denary subtraction you sometimes have to borrow from the next column to the left. This is only required in binary when trying to subtract a 1 from a 0 as illustrated below for $5 - 3 = 2$ in binary:

Example A.2.14

$$\begin{array}{cccc} & \overset{0}{\cancel{1}} & \overset{1}{0} & 1 \\ - & 0 & 1 & 1 \\ \hline & 0 & 1 & 0 \end{array}$$

Binary Multiplication & Division

For multiplication and division of binary numbers they work in a similar fashion to denary long multiplication and division,but remembering that any multiplication with 0 in it results in a 0 i.e. the four basic rules for multiplying two bits are:.

$$\begin{aligned} 0 \times 0 &= 0 \\ 0 \times 1 &= 0 \\ 1 \times 0 &= 0 \\ 1 \times 1 &= 1 \end{aligned} \tag{A.2.5}$$

Binary multiplication example $7 \times 5 = 35$:

Example A.2.15

$$\begin{array}{cccccc l} & & & 1 & 1 & 1 & \\ & & \times & 1 & 0 & 1 & \\ \cline{3-6} & & & 1 & 1 & 1 & \text{Multiply 111 by 1} \\ + & & 0 & 0 & 0 & & \text{Multiply 111 by 0 shifting answer left by 1} \\ + & \underset{1}{1} & \underset{1}{1} & 1 & & & \text{Multiply 111 by 1 shifting answer left by 2} \\ \hline 1 & 0 & 0 & 0 & 1 & 1 & 100011_2 = 2^5 + 2^1 + 2^0 = 32 + 2 + 1 = 35_{10} \end{array}$$

Binary division example $6 \div 2 = 3$:

Example A.2.16

$$\begin{array}{r r l} & 11 & 11_2 = 2^1 + 2^0 = 3_{10} \\ 10 & \big) 110 & \\ & -10 & \text{First result is 1 — subtract 10 from first 2 digits of dividend} \\ \cline{2-2} & 010 & \text{Bring down third digit of dividend} \\ & -\ \ 10 & \text{Again result of 1 — subtract 10} \\ \cline{2-2} & 00 & \end{array}$$

Complements of binary numbers

There are two forms of binary complement that can be used to represent negative numbers:

- 1’s complement: change (invert) all bits (so 0 becomes 1 and 1 becomes 0).
- 2’s complement: find the 1’s complement and add 1 to it.

Most computer systems handle negative numbers using 2’s complement notation so that is the form we will concentrate on in Chapter A.8 on Number Representation. However as an example, let’s find the 1’s & 2’s complement of $184_{10} = 10111000_2$:

Example A.2.17

$$\begin{array}{ccccccccccl}
 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & & \textbf{1's complement} \\
+ & & & & & & & & 1 & & \\
 & & & & & {}^{1} & {}^{1} & {}^{1} & & & \\
\hline
 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & & \textbf{2's complement}
\end{array}$$

The reasons computers prefer 2s complement over 1s complement is that in 1s complement you have 2 representations for zero (+0 0000 and -0 1111 for 4 bits) and also that sometimes you end up having to do a second addition of a carry bit when using it to do a subtraction. In 2s complement 0 is only 0000 (4 bit). Figure A.2.1 shows all the binary representations using 4 bits and how they relate to signed numbers in 2s complement form.

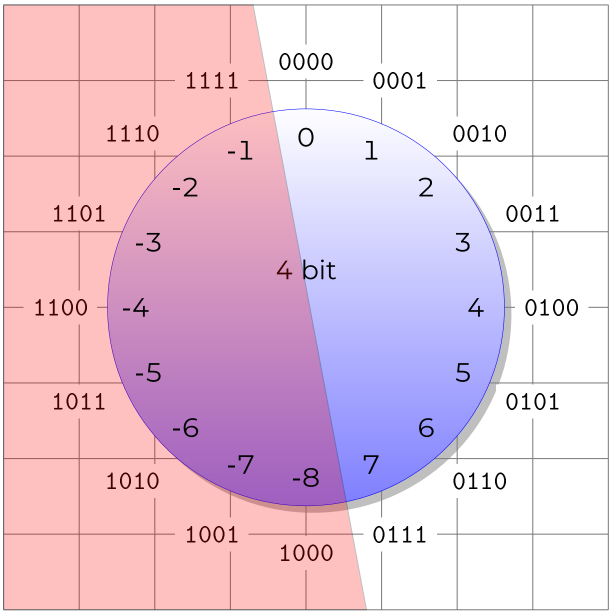


Figure A.2.1: 2s complement wheel for 4 bits

Also it is easy to detect an overflow (when answer to a problem is greater than the number of bits allowed) — this occurs when you cross boundary between -8 and 7 in Figure A.2.1. So for instance

$$\begin{array}{ccccc}
 & 0 & 1 & 0 & 1 \\
+ & 0 & 0 & 1 & 1 \\
 & {}^{1} & {}^{1} & {}^{1} & \\
\hline
 & 1 & 0 & 0 & 0
\end{array}$$

Clearly the addition of 2 positive numbers does not result in a negative number (MSB of 1) so the result has overflowed available bits.

There is an alternative method of finding the 2's complement of a binary number which is:

1. Start at the right with the LSB and write the bits as they are up to and including the first 1.
2. Take the 1's complement of the remaining bits

Using the same example as above let's use the alternative method to find the 2's complement of $184_{10} = 10111000_2$:

Example A.2.18

$$\begin{array}{ll} 10111000 & \text{Binary number} \\ \overbrace{0100}^{\text{step 2}}\underbrace{1000}_{\text{step 1}} & \text{2's complement by alternative method} \end{array}$$

To reverse the complement of a number you just do the same process again.

The major use of 2's complement that you will come across initially is as an alternative to binary subtraction where you replace the number being subtracted by it's 2's complement and add this to the first number - discarding any bits above the number of bits in the largest number in expression. Lets take $5 - 3 = 2$ in binary — first off we need to find the 2's complement of $011_2 = 101_2$ and then add it to the binary representation of $5_{10} = 101_2$ as shown below:

Example A.2.19

$$\begin{array}{cccc} & 1 & 0 & 1 \\ + & 1 & 0 & 1 \\ \scriptstyle 1 & & \scriptstyle 1 & \\ \hline \cancel{1} & 0 & 1 & 0 \end{array}$$

In the above example, we discard the 1 in MSB of the answer as the largest original number is 3 bits (for 5) and the answer is 4 bits in size. These concepts are explored further in Chapter A.8.

Questions

1. Convert the following binary numbers to decimal:

a) 1011_2	b) 1101_2	c) 10110_2
d) 100010_2	e) 1100101_2	f) 10101001_2
g) 11001100_2	h) 111000111_2	i) 1010100101_2

2. Convert the following decimal numbers to binary:

a) 13_{10}	b) 24_{10}	c) 45_{10}
d) 56_{10}	e) 89_{10}	f) 177_{10}
g) 359_{10}	h) 765_{10}	i) 1678_{10}

3. Do the following binary sums:

a) $1110_2 + 100_2$	b) $1010_2 + 1100_2$	c) $1001_2 + 1010_2$
d) $1110_2 - 100_2$	e) $1100_2 - 1010_2$	f) $1010_2 - 101_2$

4. Do the following binary sums:

a) $11_2 \times 10_2$	b) $101_2 \times 10_2$	c) $100_2 \times 11_2$
d) $1100_2 \div 10_2$	e) $110_2 \div 11_2$	f) $1111_2 \div 11_2$

5. Find the 2's complement of the following numbers - number of bits for result in brackets):

a) 01110_2 (5 bit)	b) 010101_2 (6 bit)	c) 01101001_2 (8 bit)
d) 011001100_2 (9 bit)	e) 13_{10} (8 bit)	f) 24_{10} (8 bit)
g) 56_{10} (8 bit)	h) 89_{10} (8 bit)	i) 110_{10} (8 bit)

2.4 Hexadecimal

Basics

Another key number base system you will come across is base 16 or hexadecimal — this is used as a compact way of writing or displaying binary numbers as the conversion between binary & hexadecimal numbers is easy. Computers and microprocessors only understand 1s and 0s but with long binary numbers it is very easy to make a mistake and either transpose a bit or drop it entirely. To program a processor you need to use "machine language" which is in effect binary, but in practice we write the machine code in hexadecimal. Programming IDEs convert whatever code we write (in whatever language) to machine code to programme the device where each pair of hexadecimal digits represents a byte of data/code.

Hexadecimal is, as states above, base sixteen so consists of 16 numeric and alphabetic characters — the familiar 0-9 and then the letters A-F. Each hexadecimal character represents a 4-bit binary number. Table A.2.5 lists the equivalent values in decimal, binary and hexadecimal for the decimal numbers 0-19.

Decimal	Binary	Hexadecimal	Decimal	Binary	Hexadecimal
0_{10}	00000_2	0_{16}	10_{10}	01010_2	A_{16}
1_{10}	00001_2	1_{16}	11_{10}	01011_2	B_{16}
2_{10}	00010_2	2_{16}	12_{10}	01100_2	C_{16}
3_{10}	00011_2	3_{16}	13_{10}	01101_2	D_{16}
4_{10}	00100_2	4_{16}	14_{10}	01110_2	E_{16}
5_{10}	00101_2	5_{16}	15_{10}	01111_2	F_{16}
6_{10}	00110_2	6_{16}	16_{10}	10000_2	10_{16}
7_{10}	00111_2	7_{16}	17_{10}	10001_2	11_{16}
8_{10}	01000_2	8_{16}	18_{10}	10010_2	12_{16}
9_{10}	01001_2	9_{16}	19_{10}	10011_2	13_{16}

Table A.2.5: Decimal, Binary & Hexadecimal Numbers

As can be seen, hexadecimal works the same as binary and decimal in that when you reach the maximum value of a column (F) you start over with another column. For 2 digits the maximum value is FF_{16} which is 255_{10} ($2^8 - 1$) so to count beyond this value you need three hexadecimal digits. SO $256_{10} = 100_{16}$. For a three digit hexadecimal number the maximum value is $FFF_{16} = 4095_{10} = 2^{12} - 1$ and for a four digit hexadecimal number the maximum value is $FFFF_{16} = 65535_{10} = 2^{16} - 1$. You should be able to see a pattern in that for an k digit hexadecimal number the maximum value can be calculated as $16^k - 1 = 2^{4k} - 1$ as $16 = 2^4$.

Engineering application A.2.1

In a computer system with 4GB of memory (that is 2^{32}bytes), you need 32 bits to uniquely address each location in memory. This is much easier written as an 8 digit hexadecimal number than 32 bits.

Converting binary to hexadecimal

Decimal	Binary	Hexadecimal	Decimal	Binary	Hexadecimal
0_{10}	0000_2	0_{16}	8_{10}	1000_2	8_{16}
1_{10}	0001_2	1_{16}	9_{10}	1001_2	9_{16}
2_{10}	0010_2	2_{16}	10_{10}	1010_2	A_{16}
3_{10}	0011_2	3_{16}	11_{10}	1011_2	B_{16}
4_{10}	0100_2	4_{16}	12_{10}	1100_2	C_{16}
5_{10}	0101_2	5_{16}	13_{10}	1101_2	D_{16}
6_{10}	0110_2	6_{16}	14_{10}	1110_2	E_{16}
7_{10}	0111_2	7_{16}	15_{10}	1111_2	F_{16}

Table A.2.6: Decimal, Binary & Hexadecimal Numbers 0-15

This is a simple procedure where you break the binary number into 4-bit blocks starting from the LSB (right-most bit) and replace each 4-bit binary block by the corresponding hexadecimal digit as defined in Table A.2.6. As an example let us convert 111101101101000111_2 to hexadecimal:

Example A.2.20

$$\underbrace{0011}_{\downarrow \atop 3}\underbrace{1101}_{\downarrow \atop D}\underbrace{1011}_{\downarrow \atop B}\underbrace{0100}_{\downarrow \atop 4}\underbrace{0111}_{\downarrow \atop 7} \qquad = 3DB47_{16}$$

Note that we add two zeros to create a 4-bit group for the left-hand most group.

Converting hexadecimal to binary

This is simply reversing the procedure for converting from binary to hexadecimal — so convert each hexadecimal digit to its binary representation as in Table A.2.6 and discard any leading zeros (zeros to left of leftmost 1). As an example let us convert $6EAF3_{16}$ to binary

Example A.2.21

$$\overbrace{0110}^{6 \atop \downarrow}\overbrace{1110}^{E \atop \downarrow}\overbrace{1010}^{A \atop \downarrow}\overbrace{1111}^{F \atop \downarrow}\overbrace{0011}^{3 \atop \downarrow} \qquad = 1101110101011110011_2$$

It should be clear that hexadecimal numbers are much more compact than the equivalent binary number. Therefore, since the conversion is so easy, hexadecimal numbers are commonly used to represent binary numbers in displays and programming.

Hexadecimal to decimal conversion

There are basically two methods for doing this conversion

1. Convert hexadecimal number to binary number and then convert binary to decimal.
2. Directly convert hexadecimal to decimal using weights of digits in increasing powers of 16.

Let us use both methods to convert $53C_{16}$ to decimal.

Example A.2.22: Method 1

$$\begin{aligned}
\overbrace{0101}^{\overset{5}{\downarrow}}\overbrace{0011}^{\overset{3}{\downarrow}}\overbrace{1100}^{\overset{C}{\downarrow}} \quad &= 2^{10} + 2^{8} + 2^{5} + 2^{4} + 2^{3} + 2^{2} \\
&= 1024 + 256 + 32 + 16 + 8 + 4 \\
&= 1340_{10}
\end{aligned}$$

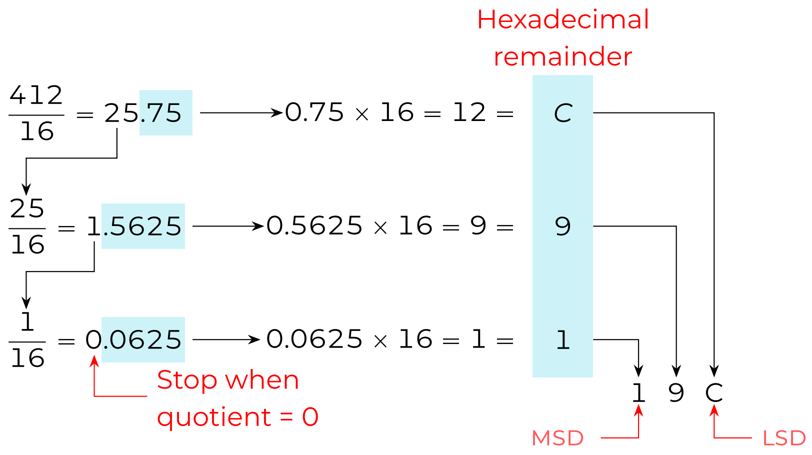
Example A.2.23: Method 2

$$\begin{aligned}
53C_{16} &= (5 \times 16^{2}) + (3 \times 16^{1}) + (12 \times 16^{0}) \\
&= (5 \times 256) + (3 \times 16) + (12 \times 1) \\
&= 1280 + 48 + 12 \\
&= 1340_{10}
\end{aligned}$$

Decimal to hexadecimal conversion

Similarly to the method described above for conversion of decimal to binary numbers by repeated division by 2, we can repeatedly divide a decimal number by 16 and use the remainders to form the hexadecimal number with the first remainder being the Least Significant Digit (LSD) and the last remainder being the Most Significant Digit (MSD). We stop dividing when the whole number quotient is 0. As an example let us convert the decimal number 412_{10} to hexadecimal:

Example A.2.24



Hexadecimal addition & subtraction

Addition

Addition of two hexadecimal numbers follows the same pattern as for denary & binary numbers — that is add the columns in turn from the LSD and carry any excess to the next column to the left. It is easiest to do hexadecimal addition by converting each set of digits to decimal, adding them together and then converting back to hexadecimal. As an example

Example A.2.25

Calculate $8CA_{16} + B49_{16}$.

Solution

Using rule of add the columns in turn from the LSD and carry any excess to the next column to the left by converting hexadecimal to decimal and converting answer back.

1. Add right hand column $A_{16} + 9_{16} = 10_{10} + 9_{10} = 19_{10} = 13_{16}$ so put 3 in answer and carry 1.
2. Add next column to left $C_{16} + 4_{16} + 1_{16} = 12_{10} + 4_{10} + 1_{10} = 17_{10} = 11_{16}$ so put 1 in answer and carry 1.
3. Add next column to left $8_{16} + B_{16} + 1_{16} = 8_{10} + 11_{10} + 1_{10} = 20_{10} = 14_{16}$ so put 4 in answer and carry 1.

$$\begin{array}{cccc} & 8 & C & A \\ \underset{1}{+} & \underset{1}{B} & \underset{1}{4} & 9 \\ \hline 1 & 4 & 1 & 3 \end{array}$$

You can also do hexadecimal addition by converting numbers in entirety to either denary or binary and using the rules in previous sections of this chapter to add the resulting numbers together before converting back to hexadecimal.

Subtraction

The subtraction of one hexadecimal number from another hexadecimal is usually done by finding the 2's complement of the number being subtracted and then adding this to the number being subtracted from using hexadecimal addition. There are three methods of finding the 2's complement of a hexadecimal number:

1. Convert hexadecimal number to binary then take the 2's complement of the binary number and convert the result back to hexadecimal.
2. Subtract the hexadecimal from the maximum hexadecimal number and add 1.
3. Write the sequence of single hexadecimal digits ($0 - F$) and underneath write the sequence in reverse ($F - 0$) — the 1's complement of a hex digit is the number below it. So write down the 1's complement of the number you want to find the 2's complement of and add 1 to it

Example A.2.26

Let us find the 2's complement of $4C$ using the three methods.
Solution

1. Subtract from maximum

   $$4C_{16} = 01001100_2$$

   2's complement of binary

   $$= 10110100_2$$

   Convert to hexadecimal

   $$10110100_2 = B4_{16}$$

2. Subtract from maximum hexadecimal value to give 1's complement

   $$FF_{16} - 4C_{16} = B3_{16}$$

   Add 1 to results

   $$B3_{16} + 1 = B4_{16}$$

3. Write down sequence & reversed sequence to find 1's complement

   $$\begin{array}{c} FED\boxed{C}BA98765\boxed{4}3210 \\ 012\boxed{3}456789A\boxed{B}CDEF \end{array}$$

   Read 1's complement from the sequences

   $$B3_{16}$$

   Add 1 to results

   $$B3_{16} + 1 = B4_{16}$$

Example A.2.27

Find the answer to question

$$A7_{16} - 4C_{16}$$

Solution

First step is to find the 2's complement of $4C_{16} = B4_{16}$. Now add this value to $A7_{16}$ discarding any digits above 2 in result:

$$\begin{array}{ccc} & A & 7 \\ + & B & 4 \\ \scriptstyle 1 & & \\ \hline \cancel{1} & 5 & \mathrm{B} \end{array}$$

$$A7_{16} - 4C_{16} = 5B_{16}$$

Questions

1. Convert the following binary numbers to hexadecimal:
   - a) 1011_2
   - b) 1101_2
   - c) 1000_2
   - d) 00100010_2
   - e) 1100101_2
   - f) 10101001_2
   - g) 11001100_2
   - h) 111000111_2
   - i) 1010100101_2
2. Convert the following hexadecimal numbers to binary:
   - a) 14_{16}
   - b) $7C_{16}$
   - c) 94_{16}
   - d) DB_{16}
   - e) $F5_{16}$
   - f) 269_{16}
   - g) $30A_{16}$
   - h) $E1B_{16}$
   - i) $7F3_{16}$
3. Convert the following decimal numbers to hexadecimal:
   - a) 13_{10}
   - b) 24_{10}
   - c) 45_{10}
   - d) 56_{10}
   - e) 89_{10}
   - f) 177_{10}
   - g) 359_{10}
   - h) 765_{10}
   - i) 1678_{10}
4. Convert the following hexadecimal numbers to decimal:
   - a) 12_{16}
   - b) $7A_{16}$
   - c) 92_{16}
   - d) $D9_{16}$
   - e) $F3_{16}$
   - f) 469_{16}
   - g) $80A_{16}$
   - h) $E5B_{16}$
   - i) $6CF_{16}$
5. Do the following hexadecimal sums:
   - a) $A_{16} + 4_{16}$
   - b) $A2_{16} + 5_{16}$
   - c) $F_{16} + 17_{16}$
   - d) $46_{16} + B0_{16}$
   - e) $A5_{16} + 4F_{16}$
   - f) $3CE_{16} + 8D5_{16}$
6. Find the 8 bit 2's complement of the following hexadecimal numbers:
   - a) 79_{16}
   - b) 70_{16}
   - c) $5C_{16}$
   - d) $1D_{16}$
   - e) 47_{16}
   - f) $2F_{16}$
   - g) 63_{16}
   - h) 45_{16}
   - i) 31_{16}

7. Do the following hexadecimal sums:

a) $CF_{16} - 79_{16}$	b) $F3_{16} - 5C_{16}$	c) $80_{16} - 5A_{16}$
d) $7A_{16} - 2F_{16}$	e) $D9_{16} - 63_{16}$	f) $FF_{16} - E5_{16}$

2.5 Octal

Basics

Like hexadecimal numbers octal numbers (base 8) are a convenient way of representing binary numbers as each octal digit represent 3 bits. However they are not used as much as computers and microprocessors tend to work in bytes (8 bits) and occasionally nibbles (4 bits) which are much more conveniently represented by hexadecimal numbers. However it is still useful to understand what an octal number is.

Octal numbers only use the decimal digits 0 – 7 as they are base 8 so each digit represents a power of 8. Each octal character represents a 3 bit binary number. Table A.2.7 shows the equivalent values in decimal, binary & octal numbers for the decimal numbers 0-11. It can be seen from this table that octal works the same as all other number bases in that when you reach the maximum value in a column (7 for octal) you start over with another column.

Decimal	Binary	Octal	Decimal	Binary	Octal
0_{10}	0000_2	0_8	6_{10}	0110_2	6_8
1_{10}	0001_2	1_8	7_{10}	0111_2	7_8
2_{10}	0010_2	2_8	8_{10}	1000_2	10_8
3_{10}	0011_2	3_8	9_{10}	1001_2	11_8
4_{10}	0100_2	4_8	10_{10}	1010_2	12_8
5_{10}	0101_2	5_8	11_{10}	1011_2	13_8

Table A.2.7: Decimal, Binary & Octal Numbers 0-11

Converting binary to octal

Decimal	Binary	Octal	Decimal	Binary	Octal
0_{10}	000_2	0_8	4_{10}	100_2	4_8
1_{10}	001_2	1_8	5_{10}	101_2	5_8
2_{10}	010_2	2_8	6_{10}	110_2	6_8
3_{10}	011_2	3_8	7_{10}	111_2	7_8

Table A.2.8: Decimal, Binary & Octal Numbers 0-7

This is a simple procedure where you break the binary number into 3-bit blocks starting from the LSB (right-most bit) and replace each 3-bit binary block by the corresponding octal digit as defined in Table A.2.8. As an example let us convert 110011101000111_2 to octal:

Example A.2.28

$$\underbrace{110}_{\substack{\downarrow\\ 6}}\underbrace{011}_{\substack{\downarrow\\ 3}}\underbrace{101}_{\substack{\downarrow\\ 5}}\underbrace{000}_{\substack{\downarrow\\ 0}}\underbrace{111}_{\substack{\downarrow\\ 7}} \quad = 63507_8$$

Converting octal to binary

This is simply reversing the procedure for converting from binary to octal — so convert each octal digit to its binary representation as in Table A.2.8 and discard any leading zeros (zeros to left of leftmost 1). As an example let us convert 67513_8 to binary

Example A.2.29

$$\overbrace{110}^{\substack{6\\ \downarrow}}\overbrace{111}^{\substack{7\\ \downarrow}}\overbrace{101}^{\substack{5\\ \downarrow}}\overbrace{001}^{\substack{1\\ \downarrow}}\overbrace{011}^{\substack{3\\ \downarrow}} \quad = 110111101001011_2$$

Octal to decimal conversion

There are basically two methods for doing this conversion

1. Convert octal number to binary number and then convert binary to decimal.
2. Directly convert octal to decimal using weights of digits in increasing powers of 8.

Let us use both methods to convert 534_8 to decimal.

Example A.2.30: Method 1

$$\begin{aligned}\overbrace{101}^{\substack{5\\ \downarrow}}\overbrace{011}^{\substack{3\\ \downarrow}}\overbrace{100}^{\substack{4\\ \downarrow}} \qquad &= 2^8 + 2^6 + 2^4 + 2^3 + 2^2\\ &= 256 + 64 + 16 + 8 + 4\\ &= 348_{10}\end{aligned}$$

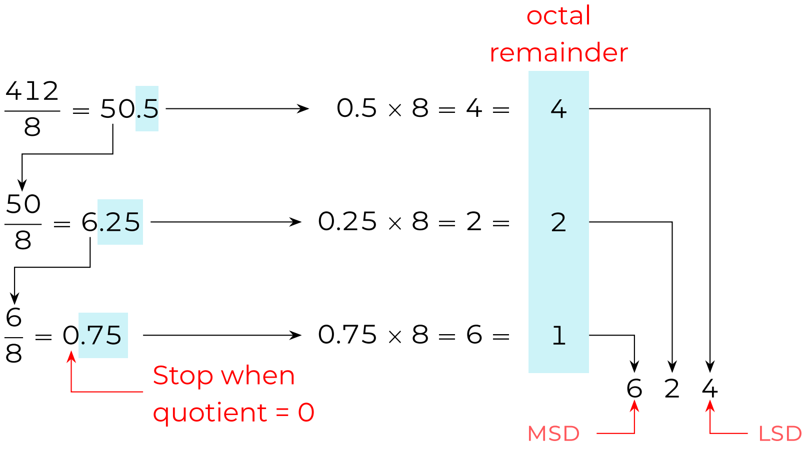
Example A.2.31: Method 2

$$\begin{aligned}534_8 &= (5 \times 8^2) + (3 \times 8^1) + (4 \times 8^0)\\ &= (5 \times 64) + (3 \times 8) + (4 \times 1)\\ &= 320 + 24 + 4\\ &= 348_{10}\end{aligned}$$

Decimal to octal conversion

Similarly to the method described above for conversion of decimal to binary numbers by repeated division by 2, we can repeatedly divide a decimal number by 16 and use the remainders to form the octal number with the first remainder being the LSD and the last remainder being the MSD. We stop dividing when the whole number quotient is 0. As an example let us convert the decimal number 412_{10} to octal:

Example A.2.32



Questions

1. Convert the following binary numbers to octal:

   a) 1011_2 b) 1000_2 c) 100010_2 d) 110101_2
   e) 10101001_2 f) 11001100_2 g) 111000111_2 h) 1010100101_2

2. Convert the following octal numbers to binary:

   a) 14_8 b) 70_8 c) 27_8 d) 65_8
   e) 267_8 f) 375_8 g) 413_8 h) 763_8

3. Convert the following decimal numbers to octal:

   a) 13_{10} b) 24_{10} c) 56_{10} d) 89_{10}
   e) 177_{10} f) 359_{10} g) 765_{10} h) 1678_{10}

4. Convert the following octal numbers to decimal:

   a) 12_8 b) 74_8 c) 45_8 d) 63_8
   e) 467_8 f) 201_8 g) 356_8 h) 543_8

2.6 Binary-Coded Decimal

This is a way of representing each decimal digit with a binary code. There are 10 groups used in BCD each of 4 bits as you need to be able to represent all digits between 0 and 9 (10 in total). The simplest and most common way of doing this is through what is known as the

8421 code - which is simply each decimal digit converted to its 4-bit representation in binary. In fact unless it is stated otherwise it is assumed that the term BCD means 8421 code.

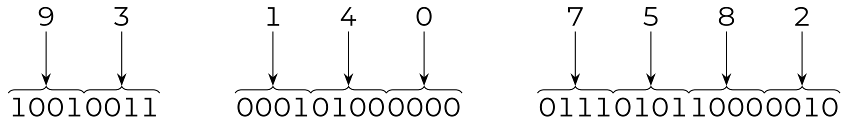
Decimal Digit	0	1	2	3	4
BCD	0000	0001	0010	0011	0100
Decimal Digit	5	6	7	8	9
BCD	0101	0110	0111	1000	1001

Table A.2.9: Decimal/BCD conversion

It should be obvious that there are six 4-bit code combinations that are invalid with the BCD - that is 1010, 1011, 1100, 1101, 1110 and 1111. Care is needed when designed systems using BCD that you do not somehow end up with one of these combinations.

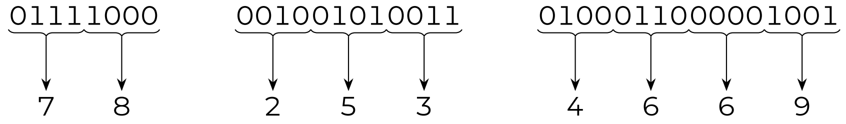
To express any decimal number in BCD you simply replace each digit by the 4 bit representation. Example A.2.33 shows how you convert 93_{10}, 140_{10} and 7582_{10} into BCD.

Example A.2.33



To convert from BCD to decimal is done by simply by starting at rightmost bit (LSB) and group the bits into groups of 4 and then write down the decimal digit represented by each group. Example A.2.34 shows the conversion of BCD codes 01111000, 001001010011 and 0100011000001001 to decimal.

Example A.2.34



You may be wandering what the use is of BCD code — it is in fact used for devices that display numbers using 7 segment LED displays such as digital clocks, digital thermometers and digital meters. The use of BCD simplifies the displaying of decimal numbers but it is only really useful when limited processing is required as performing even simple operations such as adding two BCD numbers together is complicated by the 6 invalid states.

Questions

1. Convert the following BCD numbers to decimal:

a) 1001 b) 10110 c) 100010 d) 1100101
e) 10001001 f) 10011000 g) 100000111 h) 1001000101

2. Convert the following decimal numbers to BCD:

a) 13_{10} b) 24_{10} c) 56_{10} d) 89_{10}
e) 177_{10} f) 359_{10} g) 765_{10} h) 1678_{10}

2.7 Gray Codes

The Gray Code is a way of representing numbers in binary where *each word is only 1 bit different the previous and next words in a sequence*. Taking the simplest example for 2 bits in normal binary the pattern is 00 → 01 → 10 → 11 and then back to beginning (00). However the 2 bit Gray code is 00 → 01 → 11 → 10 then back to beginning (00). Like a binary number a Gray code can have any number of bits — the four bit version is shown in Table A.2.10. Looking at the table you can see that between successive Gray code words there is only one bit change. So for instance going from 11_{10} to 12_{10}, the Gray code word goes from 1110 to 1010 (one bit change in third bit from right) whereas the binary code changes from 1011 to 1100 which is a change of three bits (all bits except the MSB).

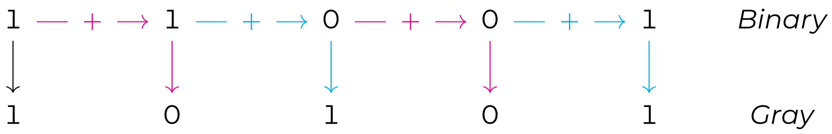
Decimal	Binary	Gray Code	Decimal	Binary	Gray Code
0	0000	0000	8	1000	1100
1	0001	0001	9	1001	1101
2	0010	0011	10	1010	1111
3	0011	0010	11	1011	1110
4	0100	0110	12	1100	1010
5	0101	0111	13	1101	1011
6	0110	0101	14	1110	1001
7	0111	0100	15	1111	1000

Table A.2.10: Four-bit Gray Code

In order to convert a binary number to the corresponding Gray code representation there are two rules to follow:

1. The MSB in the Gray code is the same as the MSB in the binary number.
2. Going from left to right add each adjacent pair of binary bits to get the next Gray code bit — discarding any carries.

As an example, let us convert $11001_2 = 25_{10}$ to its 5 bit Gray code representation:

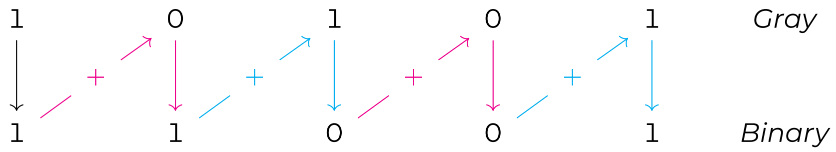


In order to convert from Gray code to the corresponding binary number again there are two rules to follow:

1. The MSB (left-most) in the binary number is the same as the MSB in the Gray code.

2. Add each binary number bit generated to the Gray code bit in the next adjacent position — discarding any carries.

As an example, let us convert the 5 bit Gray code 10101 to its corresponding binary number (reverse of above conversion):



Gray codes are often used to eliminate errors in systems where you move between states represented by binary numbers but the bits may change at slightly different points in time so potential for error is considerably reduced when only one bit changes. This can be found in digital systems designed using state machines.

Questions

1. Convert the following Gray code numbers to their decimal equivalent:

   a) 10111
   b) 10110
   c) 00001
   d) 01001
   e) 10101
   f) 11001
   g) 01100
   h) 01011

2. Convert the following decimal numbers to their 5 bit Gray code equivalent:

   a) 15_{10}
   b) 24_{10}
   c) 7_{10}
   d) 5_{10}
   e) 22_{10}
   f) 28_{10}
   g) 21_{10}
   h) 19_{10}

2.8 Summary

This chapter has looked at the various number bases that you are likely to come across in electronic & electrical engineering. We often find ourselves using binary or hexadecimal numbers particularly in digital-based systems and programming because it is easy to see which bits are on and off. So in a system where you are explicitly turning LEDs connected to the output of a 8 bit register in a sequence it is easier to do this using an 8 digit binary number of a 2 digit hexadecimal number than have to think in decimal.

Algebraic Manipulation

Required Background: §C.1, Calculator Skills (Page 333)To solve engineering problems using mathematics, symbols have to be used to represent physical quantities — the manipulation of these symbols is **algebra**. The ability to re-arrange algebraic expressions is crucial in many parts of electronic & electrical engineering and is a fundamental concept for all engineers. But what does algebra mean? To explain we can use the puzzle below:

Think of a number
Double it
Add 10 to the result
Double the result
Add this result to the number you first thought of
Divide the result by 5
Take away the number you first thought of
The answer is ***4***

Why? Well if we represent the unknown number by a then we can say:

Explanation

Think of a number	a
Double it	$(2 \times a)$
Add 10 to the result	$(2 \times a) + 10$
Double the result	$2 \times ((2 \times a) + 10)$ $= (2 \times (2 \times a)) + (2 \times 10)$ $= (4 \times a) + 20$
Add this result to the number you first thought of	$a + ((4 \times a) + 20) = (5 \times a) + 20$
Divide the result by 5	$[(5 \times a) + 20] \div 5 = a + 4$
Take away the number you first thought of	$a + 4 - a = 4$

The above puzzle has demonstrated the basic principle of algebraic expressions — that is *a letter or symbol can be used to represent an unknown number and can be manipulated exactly the same as an ordinary number can be.*

This chapter starts with the basics, looking at the notation used in mathematics and how to formulate algebraic expressions. The nest section is on powers (or indices) including scientific notation and the use of powers for polynomials. Following this is a section on the basics of algebraic manipulation and how an expression can be re-ordered so as to be useful or solving a particular problem. There is then a brief look at how to represent various specific types of expression followed be a section on how to solve simultaneous equations.

3.1 Basics of Algebra

Introduction

In this section we are going to look at the concepts of symbols and how we can use them to formulate algebraic expressions followed by the basic rules of algebra which in many ways are the same as the rules of arithmetic discussed in Chapter A.1.

Using symbols

Many concepts and ideas within electrical & electronic engineering can be succinctly communicated using mathematics which also provides a powerful tool-set for formulating and solving problems. In order to do this we have to be able to represent physical quantities using symbols. We can see above how we can use a single variable a to describe operations. The choice of what we use as symbols is largely up to us although it can be very helpful to choose letters that have some relation to what they are representing — e.g. T for temperature and t for time. You will find as you study within the field that many quantities have predefined letters such as resistors being donated by R, direct current (DC) voltages by V and DC currents by I. Where we have a number of quantities of the same type within a problem then we tend to use numbers to distinguish between them — either $R1, R2$ etc or as *subscripts* R_1, R_2 etc.

For particular, problems some symbols can represent fixed and unchanging values which are *constants*, although these can vary between calculations, and some may change value which are *variables*. Convention is that the letters x, y & z are used for variables and the earlier letters a, b & c are used for constants where there are no predefined naming conventions.

One constant that is worth mentioning is the Greek letter pi, written as π which represents the value 3.14159... and appears in the expression for the area of a circle (A): $A = \pi r^2$ (where r is radius of circle). Greek letters in fact are frequently used to represent constant values or variables so it is worth looking at the whole alphabet in Table A.3.1.

Using symbols

When we create a quantity consisting of symbols, numbers and basic maths operators (add, subtract, multiply & divide), then we have created an *algebraic expression*. So $x + 5$ and $x - 2$ are examples of algebraic expressions. If you divide one expression by another then you get an *algebraic fraction*. Two examples of algebraic fractions are given in equation A.3.1:

$$\frac{x+4}{x-2} \quad \text{and} \quad \frac{2x-y}{x+4z} \tag{A.3.1}$$

Α	α	alpha	Ι	ι	iota	Ρ	ρ	rho
Β	β	beta	Κ	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	Τ	τ	tau
Δ	δ	delta	Μ	μ	mu	Υ	υ	upsilon
Ε	ε	epsilon	Ν	ν	nu	Φ	ϕ	phi
Ζ	ζ	zeta	Ξ	ξ	xi	Χ	χ	chi
Η	η	eta	Ο	ο	omricon	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

Table A.3.1: Greek Alphabet

As with numbers, we can find the reciprocal of an algebraic fraction by inverting it so the reciprocals of the expressions in equation A.3.1 are given in equation A.3.2:

$$\frac{x-2}{x+4} \quad \text{and} \quad \frac{x+4z}{2x-y} \tag{A.3.2}$$

Rules of Algebra

The rules governing algebra are similar to those discussed in Chapter A.1 but based round algebraic expressions as opposed to numbers. There are three rules (commutativity, associativity and distributivity) but first off let's remind ourselves of the rules of precedence - which are the same in algebra as for numbers.

Precedence rules

These rules along with the use of brackets can be used to remove any ambiguity from an expression or calculation. So for example using symbols $x - y \times z$ could be either:

$$\begin{aligned} &(x-y)\times z = xz - yz \quad \text{or} \\ &x-(y\times z) = z - yz \end{aligned}$$

To remove this ambiguity we use the rules of precedence which state that for any calculation involving multiple arithmetic operators (so more than one of addition, subtraction, multiplication or division) the process is:

1. Evaluate terms in brackets first (see below for nested brackets where you start with innermost brackets)
2. Working from the left evaluate divisions and multiplications as they are encountered. This will leave a calculation (eventually as may need more than one pass to remove all multiplications & divisions) involving just addition and subtraction.
3. Working from the left evaluate additions and subtractions as they are encountered

Rule 1: Commutativity

Addition and multiplication are both commutative operations — that is any two numbers x and y can be added or multiplied in any order without effecting the result:

$$\begin{aligned} x + y &= y + x \\ xy &= yx \end{aligned}$$

However for subtraction and division the order of the two numbers does effect the result except for very special cases — so subtraction and multiplication are not commutative operations

$$\begin{aligned} x - y &\neq y - x \quad \text{unless } x = y \\ x \div y &\neq y \div x \quad \text{unless } x = y \text{ and } x \neq 0 \text{ and } y \neq 0 \end{aligned}$$

Rule 2: Associativity

Addition and multiplication are both associative operations — that is the numbers x, y and z can be added or multiplied in any order without effecting the result. This is true for any number of variables:

$$\begin{aligned} x + (y + z) &= (x + y) + z = x + y + z \\ x(yz) &= (xy)z = xyz \end{aligned}$$

However for subtraction and division the order of the numbers does effect the result except for very special cases — so subtraction and multiplication are not associative operations

$$\begin{aligned} x - (y - z) &\neq (x - y) - z \quad \text{unless } z = 0 \\ x \div (y \div z) &\neq (x \div y) \div z \quad \text{unless } z = 1 \text{ and } y \neq 0 \end{aligned}$$

Rule 3: Distributivity

Distributivity indicates whether in a mixed operation (multiplication or division with addition or subtraction) whether the result is the same if multiplication/division is on left or right of the addition/subtraction. In general for multiplication we get the same answer when we multiply a number by a group of numbers added together or do each multiplication separately then add them.

Multiplication is distributed over addition and subtraction from both the left and the right as:

$$\begin{aligned} &x(y + z) = xy + xz \text{ and } (x + y)z = xz + yz \\ &x(y - z) = xy - xz \text{ and } (x - y)z = xz - yz \end{aligned}$$

Division is only distributed over addition and subtraction from the right as:

$$(x+y) \div z = x \div z + y \div z \quad \text{but}$$
$$x \div (y+z) \neq x \div y + x \div z$$

or in fraction terms:

$$\frac{(x+y)}{z} = \frac{x}{z} + \frac{y}{z} \quad \text{but}$$
$$\frac{x}{y+z} \neq \frac{x}{y} + \frac{x}{z}$$

Questions

Using a, b and c as the variables as required describe the following rules:

1. Addition is a commutative operation
2. Multiplication is a commutative operation
3. Addition is an associative operation
4. Multiplication is an associative operation
5. Subtraction is not a commutative operation
6. Division is not a commutative operation
7. Subtraction is not an associative operation
8. Division is not an associative operation

3.2 Algebraic Multiplication and Division

This section is purely in here for reference as it is unlikely you will ever need to do this but it is useful to understand that what you know of long division and long multiplication can be applied to algebra as well.

Multiplication

A simple example of algebraic multiplication using just numbers and x is shown in Example A.3.1:

Example A.3.1

$$\begin{aligned}(x+3)(x+4) &= x(x+4) + 3(x+4)\\ &= x^2 + 4x + 3x + 12\\ &= x^2 + 7x + 12\end{aligned}$$

A more complex example is $(3x+4)(x^2+2x+5)$, as shown in Example A.3.2, in which we need to follow the principles of long multiplication and multiply each term in second expression by $3x$ and then by 4 and add the results together:

Example A.3.2

$$\begin{array}{lrcrcrcr}
 & & & x^2 & + & 2x & + & 5 \\
 & & & & & 3x & + & 4 \\
\cline{2-8}
\text{Multiply throughout by } 3x & 3x^3 & + & 6x^2 & + & 15x & & \\
\text{Multiply throughout by } 4 & & & 4x^2 & + & 8x & + & 20 \\
\cline{2-8}
\text{Add the two lines} & 3x^3 & + & 10x^2 & + & 23x & + & 20
\end{array}$$

So $(3x+4)(x^2+2x+5) = 3x^3+10x^2+23x+20$.

What about when there isn't a term in one power. Well lets look at Example A.3.3 where we calculate $(2x+4)(3x^3-2x-5)$. You will note that there is no x^2 term in the second expression so conventionally we insert a term of $0x^2$ to make working out easier:

Example A.3.3

$$\begin{array}{rcrcrcrcr}
 & & 3x^3 & + & 0x^2 & - & 2x & - & 5 \\
 & & & & & & 2x & + & 4 \\
\hline
6x^4 & + & 0x^3 & - & 4x^2 & - & 10x & & \\
 & & 12x^3 & + & 0x^2 & - & 8x & - & 20 \\
\hline
6x^4 & + & 12x^3 & - & 4x^2 & - & 18x & - & 20
\end{array}$$

Division

Algebraic division follows the principles of long division of numbers where you find the value that makes the number you need and out it in result. For example consider $(4x^3-2x^2-6x+9) \div (2x+3)$. For the first term in result we need to multiply $2x$ by $2x^2$ to get a value of $4x^3$. So $2x^2$ goes in result and we multiply $2x+3$ by $2x^2$ and subtract answer from term we are dividing. We then bring the next term down, work out the value we need to remove the x^2 term and continue on in this vain as shown in Example A.3.4

Example A.3.4

$$\begin{array}{rrcrcrcr}
 & 2x^2 & - & 4x & + & 3 & & \\
\cline{2-8}
\multicolumn{1}{r|}{2x + 3} & 4x^3 & - & 2x^2 & - & 6x & + & 9 \\
 & 4x^3 & + & 6x^2 & & & & \\
\cline{2-4}
 & & - & 8x^2 & - & 6x & & \\
 & & - & 8x^2 & - & 12x & & \\
\cline{4-6}
 & & & & & 6x & + & 9 \\
 & & & & & 6x & + & 9 \\
\cline{6-8}
 & & & & & 0 & + & 0
\end{array}$$

Similar to algebraic multiplication if we have a missing power in the term being divided we add it in with a coefficient of 0. So $(4x^3 + 2x^2 + 9) \div (2x + 3)$ is worked out in Example A.3.5:

Example A.3.5

$$
\begin{array}{cccccccccc}
 & & & 2x^2 & - & 2x & + & 3 & & \\
\cline{4-10}
2x & + & 3 & \multicolumn{1}{|c}{4x^3} & + & 2x^2 & + & 0x & + & 9 \\
 & & & 4x^3 & + & 6x^2 & & & & \\
\cline{4-6}
 & & & & - & 4x^2 & + & 0x & & \\
 & & & & - & 4x^2 & - & 6x & & \\
\cline{6-8}
 & & & & & & & 6x & + & 9 \\
 & & & & & & & 6x & + & 9 \\
\cline{8-10}
 & & & & & & & 0 & + & 0
\end{array}
$$

Questions

1. Multiply $(x + 3)$ by $(x - 1)$
2. Multiply $(x^2 + 2x)$ by $(x + 1)$
3. Multiply $(x^3 - 2x + 1)$ by $(x^2 - 1)$
4. Divide $(3x^2 + 5x - 2)$ by $(x + 2)$
5. Divide $(10x^2 + 11x - 6)$ by $(2x + 3)$
6. Divide $(14x^2 - 19x - 3)$ by $(2x - 3)$

3.3 Re-Ordering Expressions

Introduction

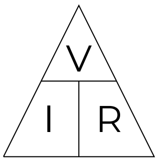
In this section we are going to look at how we can re-order algebraic expressions so as to solve engineering problems. There are a number of basic techniques that can be used to manipulate algebraic expressions which include gathering like terms, removing brackets and factorisation. There are also rules that need to be followed when handling algebraic fractions.

Engineering application A.3.1

A very common expression that we have to re-order in electronic & electrical engineering is Ohm's Law - that is $V = IR$ as we will often have two values and want to find the third so the law can be alternatively written as:

$$
\begin{aligned}
V &= IR \\
I &= \frac{V}{R} \\
R &= \frac{V}{I}
\end{aligned}
$$

These equations can also be represented as a triangle:



To use this triangle you take the value you want to find and then you can visually see how the other two quantities are used to find this quantity.

Gathering ‘like terms’

Like terms are expressions that are multiples of the same variable/quantity. So, for example, $5x, 2x$ & $\frac{2}{3}x$ are all multiples of x so are like terms. Other examples include:

- $y^2, 4y^2$ & $\frac{1}{2}y^2$ which are multiples of y^2.
- az, bz & cz which are multiples of z.
- $2x^2y, 7x^2y$ & $\frac{3}{4}x^2y$ which are multiples of x^2y.

The importance of like terms is that we can collect them together and add or subtract them to simplify expressions. Similarly any numbers without variables in an algebraic expression can be collected together and added or subtracted to get one number. Example A.3.6 shows an example of like term gathering including numbers.

Example A.3.6

$$\begin{aligned} 3x + 4y + 5z + 6 + x - 7y - 2z - 7 &= 0 \\ 3x + x + 4y - 7y + 5z - 2z + 6 - 7 &= 0 \\ 4x - 3y + 3z - 1 &= 0 \\ 4x - 3(y + z) &= 1 \end{aligned} \tag{A.3.3}$$

Expanding Brackets

Many algebraic expressions contain brackets and in order to simplify them is may be necessary to remove the brackets. There are rules associated with bracket removal (expansion of expressions) which are basically:

1. multiplying or dividing each term inside the bracket by the term outside the bracket, but
2. if the term outside the bracket is negative then eahc term insides the bracket changes sign.

Removing one set of brackets — $a(b+c)$

The expression $3(x+y)$ means that both x and y are multiplied by 3 to produce result $3x+3y$. Thus the expressions $3(x+y)$ and $3x+3y$ are equivalent. More generally for any three variables the distributive laws are:

$$\begin{aligned} a(b+c) &= ab+ac \\ a(b-c) &= ab-ac \end{aligned} \tag{A.3.4}$$

Sometimes bracketed variables are first but rules are the same where each variable inside bracket is multiplied by variable(s) outside the bracket:

$$\begin{aligned} (a+b+c)d &= ad+bd+cd \\ (a-b)cd &= acd-bcd \end{aligned} \tag{A.3.5}$$

Removing two sets of brackets — $(a+b)(c+d)$

We will need at times to consider two bracketed terms multiplied together such as $(a+b)(c+d)$. In this case we can consider the second bracket as a single variable and apply the distributive law to give $a(c+d)+b(c+d)$. We then can apply the distributive law to each of the two terms to give answer:

$$\begin{aligned} (a+b)(c+d) &= a(c+d)+b(c+d) \\ &= ac+ad+bc+bd \end{aligned} \tag{A.3.6}$$

The alternative way of looking at this expression is that each term in the first bracket multiplies each term in the second bracket.

There is a specific case for expression $(a+b)(a-b)$ which is a handy relationship to know:

$$\begin{aligned} (a+b)(a-b) &= a(a-b)+b(a-b) \\ &= a^2-ab+ba-b^2 \\ \therefore (a+b)(a-b) &= a^2-b^2 \end{aligned} \tag{A.3.7}$$

Nested brackets

Sometimes we need to expand expressions where we have brackets nested within other brackets. In this case, the innermost set of brackets is removed first. An example is shown in Example A.3.7. It is worth noting that in this example, in removing first set of brackets, we have $-2 \times -3b = 6b$ as a negative value multiplied by a negative value gives a positive answer.

Example A.3.7

$$\begin{aligned} 5(a+[2-2(a-3b)]) &= 5(a+[2-2a+6b]) \\ &= 5(a-2-2a+6b) \\ &= 5a-10-10a+12b \\ &= 12b-5a-10 \end{aligned}$$

Factorisation

This is the reverse of the process described above where we want to express a term using brackets or the *factors* of the expression. In terms of numbers, a number can be said to be *factorised* when it is written as a product. For instance 15 can be written as 5 × 3 so we can say the *factors* of 15 are 3 and 5. Remember that the *factors* of a number are always multiplied together.

Similarly we can factorise algebraic expressions by spotting the common symbols/numbers within an expression and taking them outside a set of brackets. For example in Example A.3.8, the first expression has a common factor of x, the second of 4, and the third of $2x$:

Example A.3.8

$$\begin{aligned} xy+xz &= x(y+z) \\ 4x-4z &= 4(x-z) \\ 2xy+4xz &= 2x(y+2z) \end{aligned}$$

The trick is to spot the *common factors* which is basically a matter of practice.

Quadratic expressions

This should be revision from GCSE Maths but it is useful to remind ourselves of how we can factorise quadratic expressions. First of the definition of a quadratic expression is:

A *quadratic expression* is an expression of the form ax^2+bx+c where a, b and c are numbers.

a which is known as the *coefficient* of x^2 cannot equal zero, but b (coefficient of x) and c (*constant term*) can be equal to zero.

Let us consider the product $(x+1)(x-2)$ and remove the brackets using Equation A.3.6 to give us the quadratic expression x^2-x-2. The factors of this expression are $(x+1)$ and $(x-2)$ which we know because that is the product we started with. However how do we handle factorising a quadratic expression. Well firstly for an expression where the coefficient of x^2 is 1, the following expansion is always true:

$$\begin{aligned}(x+m)(x+n) &= x^2+mx+nx+mn \\ &= x^2+(m+n)x+mn\end{aligned} \tag{A.3.8}$$

This means that the coefficient of x is $(m+n)$ and the constant term is mn - so if we can find the values of m and n we can find the factors of the expression.

When the coefficient of x^2 is not 1 it is sometimes possible to find a common factor to make it 1 and then use the above to find the factors. So for instance for the expression $4x^2+16x+12$ there is a common factor of 4 which makes expression $4(x^2+4x+3)$ which can be factored as $4(x+3)(x+1)$.

For a general quadratic expression $ax^2+bx+c=0$, it is often easiest to use the quadratic roots equation as given in Equation A.3.9, which allows you to calculate the two values of x (donate them as m and n) where this expression is true ($(x-m)(x-n)=0$). There are always two roots for a quadratic expression or in other words a quadratic expression always has two values of x where the expression equals zero.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{A.3.9}$$

For higher order expression the number of roots is equal to the highest order of x. So a cubic expression (where highest order of x is x^3) has three roots etc. These tend to be more trial and error to solve as you can find a value of x which makes expression equal to 0 then use algebraic division to remove the factor from the expression. This can be repeated until you are left with a quadratic expression. However we usually use Matlab or a similar programme to do the solution for us.

Algebraic Fractions

Similar to one number divided by another number being known as a *numeric fraction* one algebraic expression divided by another algebraic expression is called an *algebraic fraction*. The expression on the top line is called the *numerator* and the one on the bottom line is called the *denominator* Example A.3.9 shows examples of such fractions:

Example A.3.9

$$\frac{2x}{y},\quad \frac{1}{x+y} \quad \text{and} \quad \frac{x^2+2x-1}{x^3+x+1}$$

In this section we are going to look at how such fractions can be added, subtracted, multiplied, divided and first off simplified.

Common Factors

This is to do with using the principle that common factors in the numerator and denominator cancel each other out. Looking at a simple numeric example consider the expression $\frac{15}{35}$ - in this case we can factorise $15=3\times5$ and $35=7\times5$. So this results in:

$$\begin{aligned}\frac{15}{45} &= \frac{3 \times \cancel{5}}{7 \times \cancel{5}}\\ &= \frac{3}{7}\end{aligned}$$

It is important to remember that only *common factors* can be cancelled in this way - $\frac{10}{35}$ and $\frac{3}{7}$ have identical numerical values so they are equivalent fractions. It is just that $\frac{3}{7}$ is simplified compared to $\frac{10}{35}$. The same principle can be applied to algebraic fractions.

Example A.3.10

Simplify if possible a) $\frac{xy}{4y}$, b) $\frac{3x}{5xy}$ and c) $\frac{x}{x+4y}$.

Answers

a) In $\frac{xy}{4y}$ there is a common factor of y in the numerator and denominator which can be cancelled:

$$\frac{x\cancel{y}}{4\cancel{y}} = \frac{x}{4}$$

b) In $\frac{3x}{5xy}$, there is a common factor of x thus:

$$\frac{3\cancel{x}}{5\cancel{x}y} = \frac{3}{5y}$$

c) In $\frac{x}{x+4y}$, there is no common factor as it is addition not multiplication in the denominator and for factors it needs to be multiplication.

Addition & Subtraction

As with numerical fractions, the operations of addition & subtraction on algebraic fractions can only occur when the denominators are the same. This means to add two algebraic fractions together or subtract one from another you need to make the denominator of each fraction the LCM of the two denominators. Often it is easiest to multiply the numerator of each fraction by the denominator of the other fraction and then simplify later. This can be seen in second equation in Example A.3.11 where 2 is a common factor in the numerator and denominator

Example A.3.11

$$\frac{2x}{3} + \frac{y}{2} = \frac{2x \cdot 2}{3 \cdot 2} + \frac{3 \cdot y}{2 \cdot 3} = \frac{4x+3y}{6}$$
$$\frac{3x}{4} - \frac{y}{2} = \frac{3x \cdot 2}{4 \cdot 2} - \frac{4 \cdot y}{2 \cdot 4} = \frac{6x-4y}{8} = \frac{2(3x-2y)}{2 \cdot 4} = \frac{3x-2y}{4}$$

Multiplication & Division

Multiplication of algebraic fractions is the same as for numerical fractions - that is you multiply the numerators and the denominators separately.

$$\frac{2x}{3} \times \frac{y}{2} = \frac{2xy}{6}$$
$$\frac{3x}{4} \times \frac{y}{2} = \frac{3xy}{8}$$

To divide one fraction by another fraction we make use of the concept of *reciprocal* which is unity divided by the number. In the case of fractions the reciprocal of a fraction means you swap the numerator and denominator over. For instance the reciprocal of $\frac{x}{y}$ is given by:

$$\frac{1}{x/y} = 1 \div \frac{x}{y} = 1 \times \frac{y}{x} = \frac{y}{x}$$

So to divide by an algebraic fraction we simply multiply by its reciprocal as shown in Example A.3.12.

Example A.3.12

$$\frac{2x}{3} \div \frac{y}{2} = \frac{2x}{3} \times \frac{2}{y} = \frac{4x}{3y}$$
$$\frac{3x}{4} \div \frac{y}{2} = \frac{3x}{4} \times \frac{2}{y} = \frac{6x}{4y} = \frac{3x}{2y}$$

Questions

1. Simplify the following expressions:
    a) $4uv - 7uz - 6wz + 2uv + 3wz$
    b) $2u^2vw - 3wu^2v + 3vuw^2 + 2u^2wv + 2w^2vu - 3uvw^2$
    c) $x^6 \times x^{-2} \div x^3$
    d) $\frac{4xy^2}{8xy}$
2. Expand the following expressions by removing brackets
    a) $3x(2y - 4z)(y + 2z)$
    b) $(3a - 4b)(a + 4b)(3a - b)$
    c) $4(2x + 3(5 - 2(x - y)))$
    d) $3(5(u - 4(v - 1)) - 2(w - 3))$
3. Expand and simplify the following expressions
    a) $(3x^2y)^3$
    b) $(x^2 - 2x + 1)(x + 1)$
    c) $(5x^2 + 5x - 10)(x + 3)$
    d) $\left(5x^2\, y^{-3/2}\, z^{1/4}\right)^2 \times \left(4x^4y^2z\right)^{-1/2}$
4. Factorise the following expressions
    a) $x^2 - 9x + 20$
    b) $x^2 + 3x - 10$
    c) $x^2 - 2x - 24$
    d) $4x^2 + 16x + 15$
    e) $2x^2 - x - 15$
    f) $6x^2 - 13x + 6$
5. Express the following expressions as a single fraction in their simplest form

a) $\frac{3}{x+2}+\frac{2}{x+3}$ b) $\frac{3x}{x-1}-\frac{2}{x+1}$ c) $x+\frac{2}{x-5}$

d) $x-2-\frac{1}{x-1}+\frac{x}{x+1}$ e) $\frac{x+2}{x+3}\times\frac{x-2}{x+1}$ f) $\frac{2x-3}{3x+1}\times\frac{3x-2}{x+3}$

g) $\frac{x+2}{x+3}\div\frac{x-2}{x+1}$ h) $\frac{2x-3}{x+3}\div\frac{3x-2}{x-3}$ i) $\frac{2x+2}{x-2}\div\frac{5x+5}{2x-4}$

3.4 Representation

All of the above is an introduction to basic algebra which you need to understand in order to be able to manipulate equations in doing circuit analysis for instance. There are also a few mathematical symbols you may come across within your course that donate summation of a sequence of numbers and the product (multiplication) of a sequence of numbers.

For instance suppose, we want to donate the sum of the first 5 even numbers (that is $2+4+6+8+10$). Every even number is divisible by 2 so each number can be represented as $2r$ where r is some integer number. Equation A.3.10 shows how we represent this mathematically where the numbers above and below the Σ sign (known as *limits*) represent the range of terms we want with the number below being minimum value and number above being maximum value. Note that we define which variable we are counting over in the minimum limit. Equation A.3.11 shows the representation of the sum of the first n odd numbers using the fact that an odd number is always and even number minus 1 (so can be represented by $2r-1$):

$$\sum_{r=1}^{5}2r=2+4+6+8+10 \tag{A.3.10}$$

$$\sum_{r=1}^{n}(2r-1)=1+3+5+7+...+(2n-1) \tag{A.3.11}$$

Engineering application A.3.2: Kirchoff's Current Law

Kirchoff's current law [a] states that the sum of the currents flowing into a junction/node must equal the sum of the currents flowing out of it. If we consider the currents flowing into a junction as positive and the current flowing out of a junction as negative, and say they are N currents at a junction, denoted by $I_1, I_2, I_3, ..., I_N$ then:

$$I_1+I_2+I_3+...+I_N=0$$

$$\sum_{k=1}^{N}I_k=0$$

---

[a] You will learn about this in EEP1

Similarly the product of a sequence of numbers can be represented by Π within appropriate limits. So the product of the first 5 odd numbers is represented as shown in equation A.3.12 and the product of the first n even numbers is represented as shown in equation A.3.13.

$$\prod_{r=1}^{5}(2r-1) = 1 \times 3 \times 5 \times 7 \times 9 \tag{A.3.12}$$

$$\prod_{r=1}^{n} 2r = 2 \times 4 \times 6 \times 8 \times ... \times 2n \tag{A.3.13}$$

With limits we can set the upper limit to be ∞ so basically keep going forever - this is more common in integration and for certain transforms as we shall see later in this book in Chapters B.2 and Advanced Engineering Mathematics handbook (Wyatt-Millington & Love 2022)

Questions

1. Write out fully what is meant by each of the following expressions:
   a) $\sum_{k=1}^{8} x_k$
   b) $\sum_{j=1}^{7} (x_j - 1)^2$
   c) $\sum_{k=1}^{4} (-1)^k k$
   d) $\sum_{k=1}^{6} (-1)^{k-1} k^2$
   e) $\prod_{n=1}^{5} x_n$
   f) $\prod_{m=1}^{2} x_m^2$
2. Write the following more concisely using sigma($\sum$) or pi($\prod$) notation as appropriate:
   a) $2^4 + 3^4 + 4^4 + ... + 10^4$
   b) $2 - 1 + \frac{2}{3} - \frac{1}{2} ... + \frac{2}{9}$
   c) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10}$
   d) $1 \times x^2 \times x^4 \times x^6$
   e) $(-1) \times x^2 \times (-x^3) \times x^4 \times (-x^5)$

3.5 Simultaneous Equations

We can find ourselves in a situation where there is more than one unknown quantity involved. In this case, usually, there will be more than one equation. This is particularly the case in circuit analysis using Kirchoff's current and voltage laws so being able to solve equations like these is important.

Example A.3.13

$$\begin{aligned} 2x + 3y &= 12 \\ -x + 2y &= 1 \end{aligned} \tag{A.3.14}$$

Example A.3.13 shows two equations with two unknowns, x and y. These equations are known as *simultaneous equations* As a general rule of thumb the number of equations you have should be at least the same as the number of unknowns you have to solve for.

There are a number of methods that can be sued to solve simultaneous equations - we are going to look at two *elimination (or balancing)* and *substitution*. Which method you chose to use depends on you but it is worth understanding both methods.

Elimination

This method involves removing (eliminating) one of the unknowns. To do this we use that fact that if we multiply or divide both sides of an equation by a non-zero number an equivalent equation is the result. For example given the equation:

$$2x + y = 3$$

If we multiply both sides by say 3 we find

$$6x + 3y = 9$$

which is equivalent to the original equation.

So for two simultaneous equations we can eliminate one unknown variable by modifying the equations so that the coefficients of that unknown are the same in both equations. we can then subtract one equation from the other to eliminate that unknown quantity. Using this the solution to A.3.13 is given in Example A.3.14

Example A.3.14: Solution by elimination method

Taking the equations in A.3.13, if we multiply the first equation by 2 and the second equation by 3 this gives:

$$\begin{aligned} 4x + 6y &= 24 \\ -3x + 6y &= 3 \end{aligned}$$

Now we subtract the second equation from the first to eliminate y:

$$\begin{aligned} (4x + 6y &= 24) \\ -\underline{(-3x + 6y} &= 3) \\ 7x &= 21 \\ \therefore\ x &= 3 \end{aligned}$$

Having calculated x we can now get y by substituting for x in one of the equations:

$$\begin{aligned} -3 + 2y &= 1 \\ 2y &= 4 \\ \therefore\ y &= 2 \end{aligned}$$

It is now worth checking your answers by substituting for x and y in both equations to make sure they are true.

$$\begin{aligned} 2x + 3y &= 2 \cdot 3 + 3 \cdot 2 = 6 + 6 = \mathbf{12} \\ -x + 2y &= -3 + 2 \cdot 2 = -3 + 4 = \mathbf{1} \end{aligned} \tag{A.3.15}$$

Substitution

This method involves expressing one unknown variable in terms of the other from one equation and substituting into the other equation to solve for one unknown.

Example A.3.15: Solution by substitution method

Taking the equations in A.3.13, we rearrange the second equation to give us x in terms of y:

$$\begin{aligned} -x + 2y &= 1 \\ 2y &= 1 + x \\ 2y - 1 &= x \end{aligned} \tag{A.3.16}$$

Then we can substitute for x in the first equation:

$$\begin{aligned} 2(2y - 1) + 3y &= 12 \\ 4y - 2 + 3y &= 12 \\ 7y &= 12 + 2 = 14 \\ \therefore\ y &= 2 \end{aligned}$$

Having found the value of y we can substitute back into A.3.16 to give us the value of x

$$\begin{aligned} 2(2) - 1 &= x \\ 4 - 1 &= x \\ \therefore\ x &= 3 \end{aligned}$$

As shown above in A.3.15, you should check your calculated values are correct - in this case we know they are as they are the same as calculated above using the elimination method in Example A.3.14.

Questions

Use either method - or ideally try both initially to work out which you prefer.

1. Solve $4x + 2y = 0$ and $3x - y = 10$
2. Solve $2x + 3y = 12$ and $3x + 2y = 13$
3. Solve $2x - 7y = 9$ and $x + 6y = -5$
4. Solve $2x - 2y = 8$ and $9x + 7y = 4$
5. Solve $y - 7x = 1$ and $6x + 4y = 4$
6. Solve $7x + y = -5$ and $5x + 6y = 7$
7. Solve $7x + 3y = 6$ and $5x + 5y = 6$
8. Solve $12x - 8y = -6$ and $7x + 2y = 4$
9. Solve $-5x - 3y = 2$ and $7y - 7x = 7$
10. Solve $y = x + 1$ and $y = 4x - 2$
11. Solve $y = 2x + 3$ and $y = 5x - 3$
12. Solve $y = 3x - 1$ and $2x + 4y = 10$
13. Solve $6x + y = 4$ and $5x + 2y = 1$
14. Solve $x - 3y = 1$ and $2x + 5y = 35$
15. Solve $2x + \frac{1}{3}y = 1$ and $3x + 5y = 6$
16. Solve $4x + 3y = 5$ and $2x - \frac{3}{4}y = 1$
17. Solve $5x + 3y = 9$ and $2x - 3y = 12$
18. Solve $2x - 3y = 9$ and $2x + y = 13$
19. Solve $x + 7y = 10$ and $3x - 2y = 7$
20. Solve $5x + y = 10$ and $7x - 3y = 14$
21. Solve $\frac{1}{3}x + y = \frac{10}{3}$ and $2x + \frac{1}{4}y = \frac{11}{4}$
22. Solve $3x - 2y = \frac{5}{2}$ and $\frac{1}{3}x + 3y = -\frac{4}{3}$
23. Solve $x = 3y$ and $4x - 5y = 35$
24. Solve $x = \frac{1}{3}y$ and $2y - 6x = 9$
25. Solve $7x + 3y = -15$ and $12y - 5x = 39$

3.6 Summary

This chapter has taken a look at the basics of algebra and how we can re-arrange algebraic expressions to get the required results. This is used a lot in circuit analysis and is fundamental to understanding this area of electronic & electrical engineering. Alongside the manipulation of algebraic expressions and solution of simultaneous equations, another key element of the chapter that you will be using time and time again is the scientific notation in section 1.4 as this crops up everywhere in engineering - particularly for component values etc.

Functions

Required Background: §A.3, Algebra (Page 45); §C.1, Calculator Skills (Page 333)

4.1 Introduction

Within engineering we use the concept of mathematical models a lot. Functions are central to this — they can be used to describe how quantities change such as the variation in strength of a signal with both time and distance. Functions are in essence a rule that when given an input produce a known output. They can often be plotted on a graph to gain greater understanding of its behaviour.

In engineering when we talk about mathematical models which are on the whole ideal representations of a physical situation or a system. Often we need to use abstraction to simplify the model and to ignore unnecessary complications. This may mean accuracy is to a level compromised, but as with everything we come across as engineers there is a judgement call to be made about the compromise. As a simple example we use Ohm's Law $V = IR$ to model the behaviour of a resistor when a voltage is applied to it. This ignores any variation in current density across the resistor and assumes that a single current value is acceptable/ It also ignores that fact that with a large enough voltage the resistor will breakdown.

Using functions to model a system enables engineers to predict what will happen when changes occur in the actual system. It is often more convenient and safer to model the system rather than experiment on the actual system. In order to understand the plots within this chapter you may need to refer to the Chapter A.6 on graphs.

4.2 Basics of Functions

Functions are simply algebraic expressions that model a system in terms of input variables (arguments) and output variables. The arguments to a function are usually *independent variables* whereas the outputs are usually *dependent variables*. Taking a simple example $y = f(x) = 2x + 1$, x is the argument and is independent, whereas y depends on what value x is given.

Often inputs will be within given limits or we may specify limits in order to model the system. The set of allowed inputs for an argument is known as the *domain* of a function. For example the domain of equation A.4.1 is the closed *interval* $[-3, 5]$ (both end points in the interval), whereas the domain of equation A.4.2 is the semi-open interval $(-3, 5]$ (one end point included donated by ']') and the domain of equation A.4.3 is the open interval $(-3, 5)$ (both end points outside the interval). For functions with no defined limits as in equation A.4.4 then the domain is $(-\infty, \infty)$.

$$f(x) = x + 1 \qquad -3 \leqslant x \leqslant 5 \tag{A.4.1}$$

$$f(x) = x + 1 \qquad -3 < x \leqslant 5 \tag{A.4.2}$$

$$f(x) = x + 1 \qquad -3 < x < 5 \tag{A.4.3}$$

$$f(x) = x + 1 \tag{A.4.4}$$

We can illustrate the first three domains on number lines with filled in circles for closed ends and open circles for open ends as shown in Figure A.4.1.

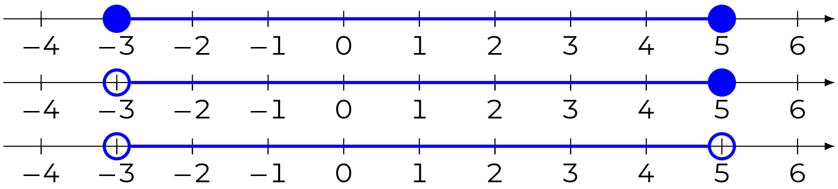


Figure A.4.1: Representations of domains for equations A.4.1 to A.4.2

The *range* of a function is the set of allowed output values — so the range for $f(x)$ in equation A.4.1 is $[-9, 21]$ and range in equation A.4.3 is $(-9, 21)$.

It is worth considering what is a function and what is not. In a function there should be one output value per input - that is a single input cannot produce more than one output. This means $y = \pm\sqrt{x}$ is not a function as you get two outputs per one input. For instance if $x = 4$, $y = -2$ or $y = +2$. However the expression $y = +\sqrt{x}$ is a function as it explicitly says take the positive square root value.

Although functions cannot be *one-to-many* values, they can be *one-to-one* or *many-to-one*. In the first case one input produces one output, whereas for the second many inputs could generate one output value. We will see examples of many-to-one functions with the cosine, sine and tangent functions in Chapter A.5 where due to periodicity of the functions many inputs gave the same output value.

Questions

1. State the domain and range of the following functions. Draw the domain on a number line where possible:

    a) $f(x) = x^2$
    b) $h(x) = 2x^2 - 1$ $\qquad 0 \leqslant x$
    c) $g(t) = 4t - 1$ $\qquad 0 \leqslant t$
    d) $y(x) = x^3$
    e) $f(t) = 0.4t + 3$ $\qquad -2 \leqslant t \leqslant 10$
    f) $g(x) = 2x - 5$ $\qquad 0 \leqslant t \leqslant 8$

2. If $f(x) = 3x + 5$, find:

a) $f(3)$	b) $f(-3)$
c) $f(y)$	d) $f(x-1)$
e) $f(2\alpha)$	f) $f(x^2)$

3. Given $X_L(f) = 2\pi f L$, if $L = 10\,\mu\text{H}$ find X_L when $f = 10\text{kHz}$.
4. Given $X_C(f) = \dfrac{1}{2\pi f C}$, if $C = 10\,\text{pF}$ find X_C when $f = 50\text{Hz}$.

4.3 Polynomials

These are functions that use one input variable raised to the power. We met quadratic expressions in Chapter A.3, but in general a polynomial function has form:

Given n is a non negative integer number and $a_0, a_1 ..., a_{n-1}, a_n$ are constants a polynomial function has form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2 x^2 + a_1 x + a_0 \tag{A.4.5}$$

The *degree* or *order* of a polynomial function is the value of the highest power —- for polynomials up to a degree of 4 there are specific names given as defined in Table A.4.1

Polynomial	*Order/Degree*	*Name*
$ax^4 + bx^3 + cx^2 + dx + e$	4	Quartic
$ax^3 + bx^2 + cx + d$	3	Cubic
$ax^2 + bx + c$	2	Quadratic
$ax + b$	1	Linear
a	0	Constant

Table A.4.1: Polynomials

Engineering application A.4.1: Engineering examples

Where do we find polynomial functions in engineering?

Ohm's Law

Ohm's Law states that

$$V = IR$$

This is an example of a linear equation in that for any resistor with a resistance R (which is constant for a given temperature) it defines the relationship between the current I through the resistor and the voltage V across the resistor.

Power Law

The power law states that the power dissipated in a device is given by:

$$P = IV$$

which again is a linear relationship on appearance. However it is often the case that both the voltage V and the current I are variables rather than constants. So in this case if we know the resistance of the device R we can replace either V or I using Ohm's Law.

$$P = (IR)I = I^2R$$

$$P = V\left(\frac{V}{R}\right) = \frac{V^2}{R}$$

Both of these are quadratic functions as the input variable is squared.

Wind power turbines

For the standard wind turbines found in the UK, the wind that drives the turbine blades basically consists of many air molecules each having a small mass. As air passes the blade it carries kinetic energy — the resulting wind power P is given by formula

$$P = \frac{1}{2}Mv^2$$

where M is total mass of air per second passing the blade (in kg/s) and v is the velocity of the air (in m/s).

The mass per second can be calculated by considering the are of the blade A, the density of air ρ and the velocity of the air:

$$M - \rho Av$$

Combining the two equations together leads to a cubic expression for P in terms of the independent variable v:

$$P = \frac{1}{2}(\rho Av)v^2 = \frac{1}{2}\rho Av^3$$

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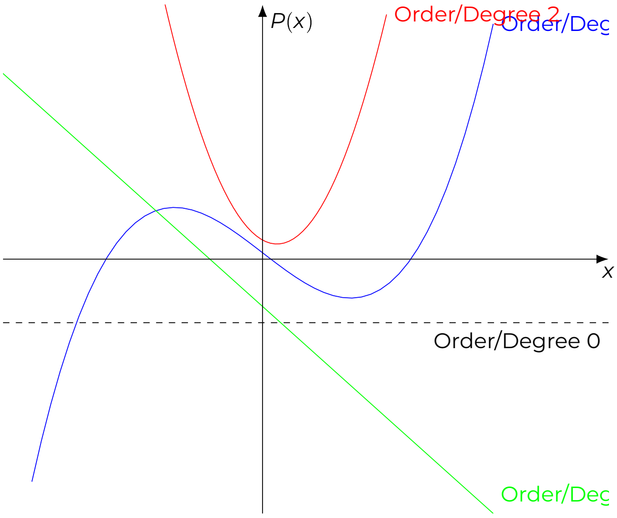

Figure A.4.2: Typical polynomials

The degree/order is also how many *roots* a polynomial has — this is values of the input variable where the output is zero. Looking at the typical graphs in Figure A.4.2 the roots are where the line crosses the horizontal axis (this is the values of input variable where output is 0). If the curve does not cross the horizontal axis then the roots are complex numbers rather than purely real (see Chapter A.10).

The MATLAB® software has a large number of useful facilities for us as engineers. In particular it provides a method of easily finding the roots of a polynomial via the `roots` function. This function takes an argument of a vector of length $n + 1$ where n is the polynomial degree. The first element in the vector is the coefficient of x^n and the last would be constant. So for example in equation A.4.5, this vector would be C=[a_n a_{n-1} a_{n-2} ... a_2 a_1 a_0].

Listing A.4.1 shows a simple example of using MATLAB® to find the roots of $f(x) = x^3 - 7x + 1$. Note that there is no x^2 term in this expression so the second value in the vector C is 0:

Listing A.4.1: MATLAB® Example

```
>> C=[1 0 −7 1];
>> r = roots(C);
>> r =
        −2.7145
        2.5712
        0.1433
```

Questions

1. State the order/degree of the following polynomial expressions:

    a) $z^3 - 3z^2 + 7z - 4$ b) $2t^2 - t^4 + 2 - 6t^3$
    c) $2w - 4w^2 + 3w^5$ d) $3x - x^2$
    e) $2(3t^2 + 4t + 1)$ f) $2z(z + 4)(2z - 5)$

2. Use MATLAB® to find the toots of the following equations:

    a) $3x^3 - x^2 + 2x + 1 = 0$ $-2 \leqslant x \leqslant 2$
    b) $x^4 + \frac{x^3}{3} - \frac{5x^2}{2} + x - 1 = 0$ $-3 \leqslant x \leqslant 2$
    c) $x^5 - x^2 + 2 = 0$ $-2 \leqslant x \leqslant 2$

4.4 Rational functions

A rational function is an algebraic fraction made up of two polynomials $P(x)$ and $Q(x)$

A rational function has form:

$$R(x) = \frac{P(x)}{Q(x)} \tag{A.4.6}$$

This is the form that many system functions take and are extensively used in control and system analysis (including digital signal processing). As both the numerator and denominator are polynomials, both have roots. The roots of the numerator are where $R(x)$ becomes 0, and are known as the *zeros* of the function. More important in many ways are the roots of the denominator, which are known as the *poles* of the function. The poles are particularly important to engineers as they enable us to determine the stability of a system amongst other things.

It can be useful to sketch the graph of a rational function $y = f(x)$ via drawing up a table of x and y values[*]. The plot can be useful to answer questions such as:

- As x becomes very large (either positive or negative), how does the function behave?
- What is value of function when x = 0?
- What values of x is the denominator 0 at?

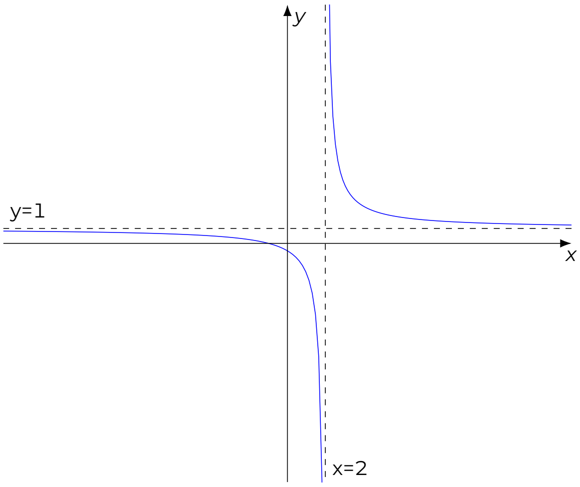

Figure A.4.3: Plot of $y = \dfrac{x+1}{x-2}$

Figure A.4.3 shows a plot of the function $y = \dfrac{x+1}{x-2}$. As x tends to infinity, the value of y approaches 1 which is written as:

$$y \to 1 \qquad \text{as} \qquad x \to \pm\infty$$

Similarly y tends to infinity as the value of x approaches 2 or:

$$y \to \pm\infty \qquad \text{as} \qquad x \to 2$$

The straight lines at $y = 1$ and $x = 3$ are known as *asymptotes* of the graph. The vertical asymptote indicates that there is a pole at $x = 2$ — this is another way of finding the poles of a rational function as well as using the MATLAB® `roots` function on the denominator polynomial.

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[*] This can also be done in MATLAB® or Microsoft® Excel® as discussed in Section 6.3

Engineering application A.4.2: Equivalent resistance

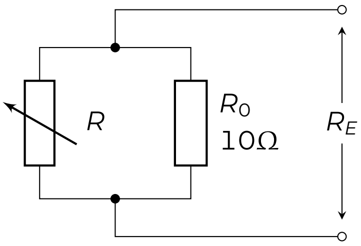


Figure A.4.4: Two resistors in parallel

An example of where this type of function can occur is shown in Figure A.4.4 where you have two resistors in parallel — one with a fixed value of 10Ω and one with a variable value R. Circuit theory tells us that the reciprocal of the equivalent resistance R_E is equal to the sum of the reciprocals of the 2 resistors:

$$\frac{1}{R_E} = \frac{1}{R} + \frac{1}{R_0} = \frac{1}{R} + \frac{1}{10} = \frac{10+R}{10R}$$

Hence:

$$R_E = \frac{10R}{R+10}$$

So the equivalent resistance is given by a rational function of R with the domain $R \geqslant 0$. If we plot this function we get the plot shown in Figure A.4.5. You can see that as R gets large, R_E tends to 10Ω.

$$R_E \to 10\Omega \qquad \text{as} \qquad R \to \infty$$

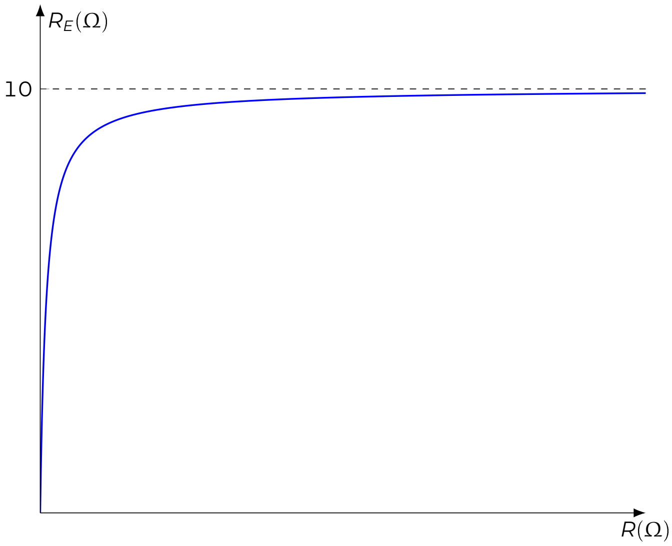


Figure A.4.5: Plot of equivalent resistance

Questions

1. State the poles of the following rational functions:

   a) $y(x) = \dfrac{x+5}{x-1}$ b) $y(x) = \dfrac{x-2}{x+5}$

   c) $y(t) = \dfrac{2t-1}{2t-3}$ d) $y(t) = \dfrac{t-1}{t^2-9}$

   e) $H(s) = \dfrac{4}{s^2+4s+4}$ f) $G(s) = \dfrac{2s+7}{s^2+3s-18}$

   g) $f(x) = \dfrac{9}{x^3-4x}$ h) $p(t) = \dfrac{3t-6}{t^2-6t+t^3}$

2. Describe the horizontal asymptotes of the following functions:

   a) $y(x) = 4 + \dfrac{1}{x}$ b) $h(t) = \dfrac{2}{t} - 3$ c) $f(x) = 6 - \dfrac{2}{5x}$

   d) $G(s) = \dfrac{3+s}{s}$ e) $H(s) = \dfrac{2s-3}{4s}$

3. Describe the vertical asymptotes of the following functions:

   a) $y(x) = \dfrac{x+5}{x-3}$ b) $y(f) = \dfrac{f-2}{2f+5}$

   c) $p(t) = \dfrac{4}{(t-3)(t+4)}$ d) $h(t) = \dfrac{1}{t^2-9}$

   e) $H(s) = \dfrac{s^2}{s^2-4}$ f) $G(s) = \dfrac{2s+7}{s^2+6s+9}$

   g) $f(x) = \dfrac{9}{x^3-x}$ h) $q(t) = \dfrac{3t-6}{t^2-t-12}$

4.5 Exponential Functions

An *exponent* is another name for a power or index as discussed in Section 1.3.

An exponential function has the form:

$$f(x) = a^x \tag{A.4.7}$$

where the *base* a is a positive constant.

It is worth noting that exponential expressions can be manipulated and simplified using the laws of indices expressed in Equation A.1.5.

The most widely used exponential function in engineering is **the exponential function**:

$$f(x) = \mathrm{e}^x \tag{A.4.8}$$

where e is an irrational constant ($e = 2.718281828...$) that is usually known as the *exponential constant*.

This function is so widely used that when an engineer refers to the exponential function they inevitably mean this one.

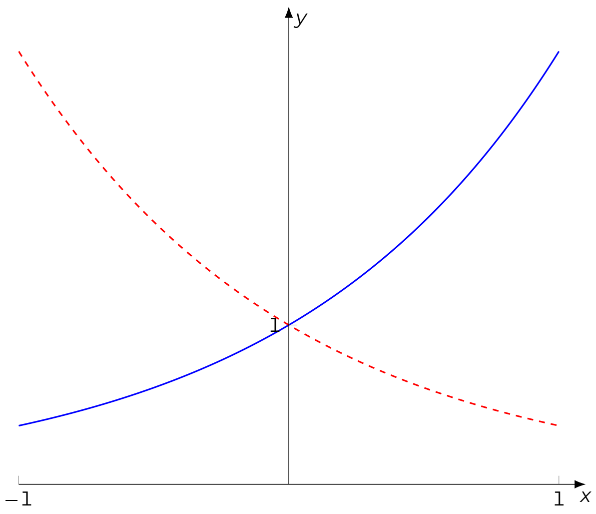


Figure A.4.6: Graph of $y = \mathrm{e}^x$ (solid) & $y = \mathrm{e}^{-x}$ (dashed) showing exponential growth and decay

As x increases positively, e^x increases rapidly — hence term *exponential growth*. The term *exponential decay* comes from the function $y = \mathrm{e}^{-x}$ where as x increases positively y decreases rapidly, The plots in Figure A.4.6 show these functions for values of $-1 \leqslant x \leqslant 1$.

But why is the exponential function important in engineering. Well put simply it can be used to describe the behaviour of various components — application A.4.3 shows how the discharge of a charged capacitor can be modelled using the negative exponential function.

Engineering application A.4.3: Discharge of a capacitor

The *capacitor* is considered to be one of the three fundamental electronic components (the others being the *resistor* and the *inductor*). In its simplest form, a capacitor consists of two parallel conducting plates separated by a *dielectric* which is an insulating material. Electrical energy is stored in the capacitor by charging the capacitor to a known potential which builds up positive charge on one plate and negative charge on the other plate thereby creating an electric field across the dielectric. For a more detailed explanation on capacitors see Blom 2013.

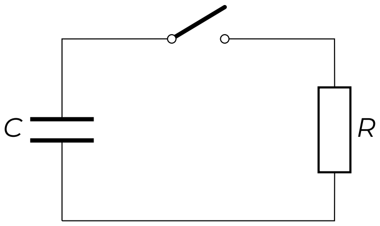


Figure A.4.7: Circuit for capacitor discharge

The capacitor C in Figure A.4.7 has voltage V across it before the switch is closed at time $t = 0$. Once the switch is closed a current flows, and the voltage v across the capacitor decays with time. The model of $v(t)$ is:

$$v(t) = \begin{cases} V & t < 0 \\ Ve^{-t/\tau} & t \geqslant 0 \end{cases} \tag{A.4.9}$$

where the time constant of the circuit is given by $\tau = RC$.
As the value of τ increases — so as value of resistance R increases assuming C is constant — then the time taken for v to decay is longer. This is shown in Figure A.4.8

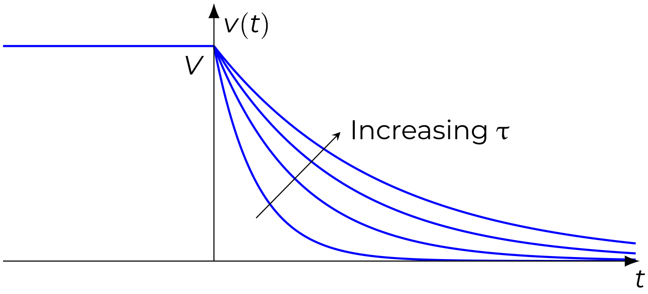


Figure A.4.8: Capacitor takes longer to discharge as time constant τ increases

There are other instances of electrical or electronic components where some factor can be modelled by the exponential function including for diodes and other semiconductor devices.

Engineering application A.4.4: The diode equation

A diode is an electric component that only allows current to flow easily in one direction whereas in other direction it is difficult. It is created from a *pn-junction* which is basically two types of semiconductor material (*p-type* and *n-type*). You will look at these devices in greater detail in Analogue Electronics at Level 5.
A diode can be modelled by the equation

$$I = I_s\left(e^{\frac{qV}{nkT}} - 1\right)$$

where:

V = applied voltage (V)
I = diode current (A)
I_s = reverse saturation current (A)

$k = 1.38 \times 10^{-23}$J/K — Boltzmann constant

$q = 1.60 \times 10^{-19}$C — charge of an electron

T = temperature (K)

n = ideality factor

This equation relates the current through a diode to the voltage across it. The ideality factor n typically is a value between 1 and 2 and relates to how the diode is manufactured and from what material it is made. When $n = 1$ then the diode is ideal and at room temperature (assumed to be 25∘ °C) $q/kT \approx 40$ so equation becomes

$$I = I_s\left(e^{40V} - 1\right)$$

. For negative voltages as $e^{40V} \approx 0$ the equation becomes

$$I \approx -I_s$$

Figure A.4.9 shows a graph of I against V although importantly note that the magnitude of I_s has been exaggerated in this graph as usually it is very small (region of 10^{-12} A).

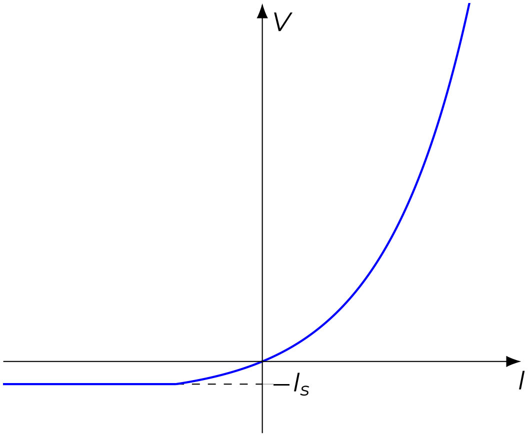


Figure A.4.9: Ideal diode characteristic

Questions

1. Simplify the following expressions using indices laws in equation A.1.5:

   a) $\dfrac{e^{2x}}{2e^{x}}$
   b) $\dfrac{e^{2t-1}}{2e^{2}}$
   c) $\dfrac{e^{x}(e^{x} - e^{2x})}{e^{2x}}$
   d) $\dfrac{e^{-2}e^{-6}}{e^{4}e^{-2}}$

2. Consider the RC circuit in Figure A.4.10. Given that the internal capacitor voltage is initially 10V plot the variation of capacitor voltage with time for the following pairs of values given the switch is closed at time $t = 0$. In each case also calculate the time constant, τ:

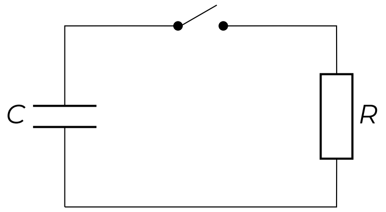


Figure A.4.10: Circuit for capacitor discharge

   a) $R = 100\,\text{k}\Omega, C = 1\,\text{nF}$
   b) $R = 1\,\text{M}\Omega, C = 1\,\text{nF}$
   c) $R = 330\,\Omega, C = 1\,\mu\text{F}$
   d) $R = 56\,\text{k}\Omega, C = 0.1\,\mu\text{F}$

4.6 Logarithm functions

What is a logarithm?

Effectively a logarithm is the inverse of a power. Taking a simple example we know that $2^4 = 16$. So we can say in logarithm form this identity becomes;

$$\log_2 16 = 4$$

— which we say 'log to the base 2 of 16 is 4'. So the log to the base a of a number is simply the power of a that equals the number. In general

$$\text{if } y = x^m, \text{ then } m = \log_x y \tag{A.4.10}$$

Practically in electronic engineering, we tend to mainly logarithms with base 10 or base e (*natural logarithms*) with some use of base 2 (in binary/digital modelling). Conventionally we abbreviate $\log_{10} x$ to $\log x$ and $\log_e x$ to $\ln x$ — both of which tend to be found on most scientific calculators (see Chapter C.1). Looking at base 10 logs we can say that:

$$\begin{aligned} \text{if } y &= 10^x, \text{ then } x = \log y \\ \text{and if } y &= \log x, \text{ then } x = 10^y \end{aligned} \tag{A.4.11}$$

Similarly for natural logs (base e):

$$\begin{aligned} \text{if } y &= e^x, \text{ then } x = \ln y \\ \text{and if } y &= \ln x, \text{ then } x = e^y \end{aligned} \tag{A.4.12}$$

Looking at this it can be seen that there will be a relationship between logarithm and general exponential functions in that a logarithm is the inverse of an exponential. In particular, the inverse of the exponential function is the natural logarithm

$$\text{If } f(x) = e^x, \text{ then } f^{-1}(x) = \ln x$$

Similarly in general:

$$\text{If } f(x) = a^x \text{, then } f^{-1}(x) = \log_a x$$

There are a number of laws of logarithms that can be used to manipulate expressions containing logarithms, It is important to note that all the laws require the logarithms to all be in the same base.

$$\begin{aligned} \log_x X + \log_x B &= \log_x(XB) \\ \log_x X - \log_x B &= \log_x\left(\frac{X}{B}\right) \\ n\log_x X &= \log_x(X^n) \\ log_x x &= 1 \end{aligned} \tag{A.4.13}$$

Use of logarithms — decibels

A very common use of logarithms in engineering is as a way of expressing the ratio between two signal levels. For instance, considering an amplifier (a device that increases signal amplitudes), the output signal is usually considerably larger in real terms than the input signal so we express this *gain* of the output signal as a logarithmic ratio with respect to the input signal in *decibels* (dB):

$$\text{power gain (dB)} = 10\log\left(\frac{P_{out}}{P_{in}}\right) \tag{A.4.14}$$

The advantage of using this ratio as a measure of gain is that in multi part systems we can simply add power gains in dBs of each part together to get the overall gain.

As engineers, we are often interested in the voltage gain of a system as opposed to power gain. We know that $P \propto V^2$ so we define the voltage gain of a system using equation A.4.14:

$$\text{voltage gain (dB)} = 10\log\left(\frac{V_{out}^2}{V_{in}^2}\right) = 20\log\left(\frac{V_{out}}{V_{in}}\right) \tag{A.4.15}$$

When we fix the value of P_{in} in equation A.4.14, we can gain a slightly altered version that uses a reference level and makes the decibel an absolute value as opposed to a relative value. We find output power of components in many communication systems expressed in either dBW or dBm. These are the actual power (P_o) referenced either to 1 W (dBW) or 1 mW (dBm). Assuming P_o is measured in watts then:

$$\text{power gain (dBW)} = 10\log P_o \tag{A.4.16}$$

$$\text{power gain (dBm)} = 10\log\left(\frac{P_o}{10^{-3}}\right) \tag{A.4.17}$$

Example A.4.1

If we have a device with an expressed output power of 20 dBm then the actual power is calculated as follows:

$$20 = 10\log\left(\frac{P_o}{10^{-3}}\right)$$

Dividing both sides by 10:

$$2 = \log\left(\frac{P_o}{10^{-3}}\right)$$

So performing the inverse of a log:

$$10^2 = \frac{P_o}{10^{-3}}$$

Therefore the actual power is:

$$P_o = 100 \times 10^{-3} = 0.1W = 100mW$$

Another example of the use of decibels is for general reference levels. Above we saw that the suffix 'm' in dBm to indicate the provision of a specific reference level (in this case 1 mW). We can generalise this concept to other quantities than power - so for instance sound pressure P in air. In this case the normal reference level is 20 µPa r.m.s. which corresponds approximately to the human hearing level for a 1 kHz sinusoidal signal. So commonly we quote sound pressure with reference to this level which is written as dB re 20 µPa r.m.s. or in shorthand dB SPL (sound pressure level (SPL)). Mathematically we have:

$$\text{sound pressure level (dB SPL)} = 20\log\left(\frac{P}{20 \times 10^{-6}}\right)$$

Negative values of dB SPL indicate sound that is too quiet to be heard by the average person; 0 dB SPL indicates a sound that can just be heard and positive values of dB SPL indicate sounds that are fully audible.

Logarithm functions

A logarithm functions is defined as functions involving logarithms:

$$f(x) = b\log_a(cx) \qquad x > 0 \tag{A.4.18}$$

where a is the base, and b & c are simply constants

Thinking about the simple functions $f(x) = \log x$ and $f(x) = \ln x$ it is key to note that their ranges are both $(-\infty, \infty)$ with domains of $(0, \infty)$. Looking at the graphs in Figure A.4.11, it is easy to see why functions of $\log_a(x)$ are defined only for positive x values as the functions tends to zero as x tends to $-\infty$.

Indeed for the functions $f(x) = \log x$ and $f(x) = \ln x$, the following properties are true:

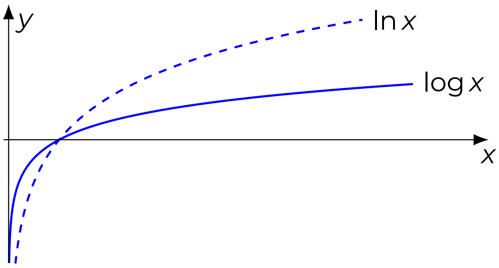


Figure A.4.11: Graphs of $\log x$ and $\ln x$

$$\left.\begin{aligned}\log x &\to \infty\\ \ln x &\to \infty\end{aligned}\right\}\text{ as } x \to \infty \qquad \left.\begin{aligned}\log x &\to -\infty\\ \ln x &\to -\infty\end{aligned}\right\}\text{ as } x \to 0$$

$$\log 1 = \ln 1 = 0 \qquad \log 10 = 1 \quad \log \mathrm{e} = 1$$

Questions

1. Evaluate e to 3 d.p.:

   a) $\log_2(16)$
   b) $\log_2(7)$
   c) $\log_{16}(48)$
   d) $\log_{16}(70)$

2. Simplify each of the following to a single log term:

   a) $\log 5 + \log y$
   b) $\log x + \log y + \log z$
   c) $\ln z - \ln 5$
   d) $2\log x + \log y$
   e) $\ln(yt) + \ln(t^2)$
   f) $\log(2x^2) + \log(5x)$
   g) $\frac{1}{3}\log x^6 - 2\log x$
   h) $\log t^3 - \log 3t + 2\log t$
   i) $\dfrac{\log(6x)}{2} - \log\left(\dfrac{2}{3x}\right)$

3. Solve the following equations:

   a) $10^x = 40$
   b) $10^x = 0.125$
   c) $10^{-x} = 0.0625$
   d) $6(10^x) = 30$
   e) $10^{3x} = 50$
   f) $10^{2x-1} = 30$
   g) $2(10^{x+2}) = 20$
   h) $10^{-x} = 0.5$
   i) $2(10^{-3x}) = 60$
   j) $\dfrac{9}{10^x} = 6$
   k) $(10^{-x})^3 = 27$
   l) $\sqrt{10^{4x}} = 4$
   m) $40(10^{2x}) = 10^{4x}$
   n) $10^{2x} - 5(10^x) + 6 = 0$

4. Solve the following equations:

a) $e^x = 20$	b) $e^x = \frac{2}{3}$
c) $e^{-x} = 1$	d) $5(e^x) = 25$
e) $e^{2x} = 50$	f) $e^{3x-1} = 40$
g) $3(e^{x+3}) = 30$	h) $3(e^{-2x}) = 60$
i) $\frac{4}{e^x} = 2$	j) $(e^{-x})^3 = 8$
k) $\sqrt{e^{2x}} = 2$	l) $30(e^{3x}) = e^{6x}$
m) $e^{2x} - 6(e^x) + 8 = 0$	

5. Solve:

a) $\log x = 2.5$	b) $\log 2x = 2.5$
c) $\log(4 + x) = 2.5$	d) $\log(3x - 2) = 3.4$
e) $3\log(x^2) = 3.4$	f) $\log\left(\frac{x+1}{2}\right) = 0.8$
g) $\ln x = 1.5$	h) $\ln 2x = 1.5$
i) $\ln(4 + x) = 1.5$	j) $\ln(3x - 2) = 2.4$
k) $3\ln(x^2) = 2.4$	l) $\ln\left(\frac{x+1}{2}\right) = 0.8$

6. Calculate in decibels the voltage gain of the following amplifiers:
   - a) input signal, V_{in} =0.5 V, output signal, V_{out} =5 V
   - b) input signal, V_{in} =10 mV, output signal, V_{out} =1 V
   - c) input signal, V_{in} =2 mV, output signal, V_{out} =5 V
   - d) input signal, V_{in} =30 mV, output signal, V_{out} =1 V

7. An audio amplifier consists of two stages: a pre-amplifier and a main amplifier. Given the following data for these stages, calculate, in decibels, the individual gain for each stage and the overall gain of the full amplifier.

preamplifier:	input signal $V_{pre(in)}$ =5 mV
	output signal $V_{pre(out)}$ =100 mV
main amplifier:	input signal $V_{main(in)}$ =200 mV
	output signal $V_{main(out)}$ =2 V

8. A Bluetooth radio system operating in class 2 has a maximum output power of 4dBm. Calculate the maximum output power in watts (W).

9. A electric heater has an output power of 2 kW. Calculate the output power in dBm.

10. Express 0 dB SPL as a sound pressure in μPa

11. An audio speaker has an output sound pressure level of 65 dB SPL at a distance of 50 cm. What is the sound pressure at that point in pascals (Pa).

12. The source sound pressure level of a sonar system is 100 kPa at 1 m. Express this value in dB re 1 μPa.

4.7 Modulus function

This is really a mathematical function but is one used a lot in electrical engineering. The *modulus* of a positive number is simply the number whereas the *modulus* of a negative number is the positive number with the same magnitude. So, for example the modulus of 5 is 5 and the modulus of −5 is also 5. Mathematically the modulus function is defined as:

The modulus function is defined as:

$$f(x) = |x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \tag{A.4.19}$$

We use the modulus function when we are simply interested in the *magnitude* of a quantity not its sign. It always gives a positive value which is useful if we want to know the absolute distance between two points or the amplitude of a signal. Consider 2 points a and b on the horizontal axis as shown in Figure A.4.12. Then

$$|a-b| = |b-a| = \text{distance from } a \text{ to } b$$

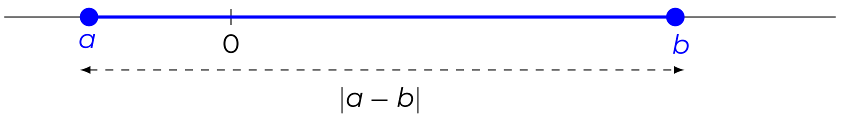


Figure A.4.12: Distance from a to b

Example A.4.2

Find the distance from:

a) $x = 4$ to $x = 7$ b) $x = -4$ to $x = 7$ c) $x = -4$ to $x = -7$

Solution:

a) Distance $= |4-7| = |-3| = 3$

b) Distance $= |-4-7| = |-11| = 11$

c) Distance $= |-4-(-7)| = |3| = 3$

From equation A.4.19 the following statements can be deduced:

If $|x| = a$ then either $x = a$ or $x = -a$
If $|x| < a$ then $-a < x < a$
If $|x| > a$ then either $x > a$ or $x < -a$

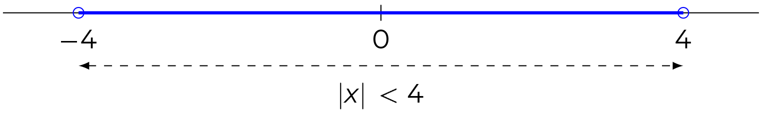
We can use these facts to describe intervals (in one dimension) or regions (in two dimensions with inequalities in x and y). The following example shows examples of this:

Example A.4.3

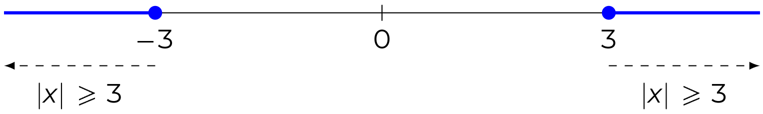
1. Sketch the following intervals:
   a) $|x| < 4$
   b) $|x| \geqslant 3$
   c) $|x - 1| \leqslant 2$
2. Sketch the following regions:
   a) $|x| < 4, |y| < 2$
   b) $|x^2 + y^2| \leqslant 4$
   c) $|x - 1| \leqslant 2, |y + 1| \leqslant 3$

Solution:

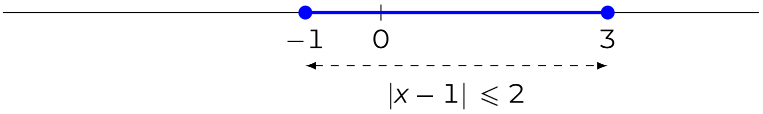
1. a) Describes interval $-4 < x < 4$



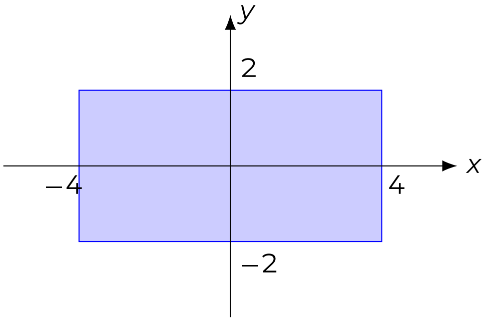
   b) Describes $x \leqslant -3$ and $x \geqslant 3$



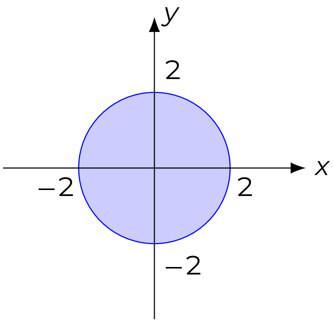
   c) As from modulus function, $x - 1 \leqslant 2 \implies x \leqslant 3$, and $x - 1 \geqslant -2 \implies x \geqslant -1$, this describes interval $-1 \leqslant x \leqslant 3$



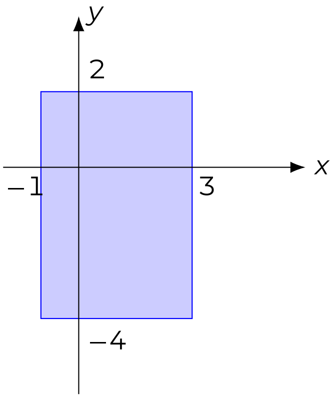
2. a) This indicates that $-4 < x < 4$ and $-2 < y < 2$ which describes a rectangular region as shown below without the boundary as using $<$ not $\leqslant$ in question



b) Describes region where $x^2 + y^2 \leqslant 4$ as square is always positive. Using Pythagoras, the distance from the origin r of any point $P(x, y)$ that satisfies this equation is given by $r^2 = x^2 + y^2$ so this inequality describes a circle of radius 2 ($\sqrt{4}$).



c) This indicates that $-1 \leqslant x \leqslant 3$ and $-4 \leqslant y \leqslant 2$ which describes a rectangular region shown below with the boundary included as both inequalities use $\leqslant$



Questions

1. State (without modulus sign) and sketch the intervals defined by:

a) $\lvert x\rvert > 2$	b) $\lvert y\rvert \leqslant 3$
c) $\lvert x+1\rvert \leqslant 2$	d) $\lvert 2t-3\rvert < 7$

2. Sketch the regions defined by:

   a) $|x| > 2, |y| < 1$ b) $|x| \leqslant 3, |y| \leqslant 3$
   c) $|x+1| \leqslant 2, |y+1| \leqslant 3$ d) $|x^2+y^2| \leqslant 9$

3. Express the following intervals using the modulus function:

   a) $-4 \leqslant x \leqslant 4$ b) $0 < y < 4$
   c) $x < -2$ and $x > 4$ d) $y \leqslant 2$ and $y \geqslant 4$

4.8 Unit step function, $u(t)$

Often called the Heaviside function, this represents a function which is zero for all negative arguments and 1 for all positive arguments as defined in equation A.4.20. In electrical engineering terms it models the behaviour of a switch (off/on) or indeed any sudden change in voltage or current within a circuit.

Heaviside or unit step function is defined as:

$$u(t) = \begin{cases} 1 & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{A.4.20}$$

More generally a unit step function can be defined at any point in time a as defined in equation A.4.21:

General unit step function is defined as:

$$u(t-a) = \begin{cases} 1 & \text{if } t \geqslant a \\ 0 & \text{if } t < a \end{cases} \tag{A.4.21}$$

;

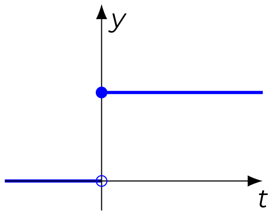


Figure A.4.13: $u(t)$

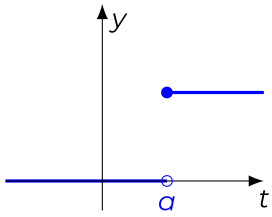


Figure A.4.14: $u(t-a)$

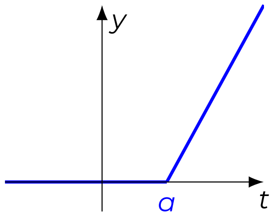


Figure A.4.15: $R(t-a)$

Figure A.4.13 shows a plot of the standard unit step function as described in equation A.4.20 while Figure A.4.14 shows the plot of the more general step function as described in equation A.4.21.

The following example shows how unit step functions cna be used to create defined width pulses and also used to create limited functions.

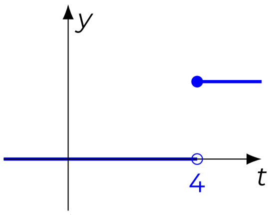
Example A.4.4

Sketch the following functions:

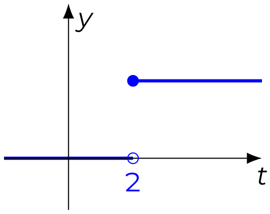
a) $f(t) = u(t-4)$
b) $f(t) = u(t-2)$
c) $f(t) = u(t-2) - u(t-4)$
d) $f(t) = u(t-4) - u(t-2)$
e) $f(t) = e^{2t}u(t)$
f) $f(t) = e^{-t}u(t)$

Solutions:

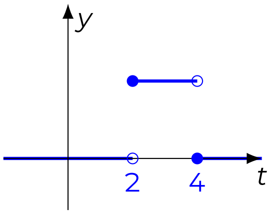
a)



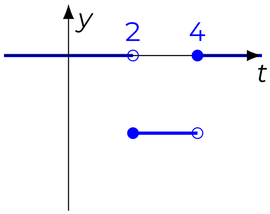
b)



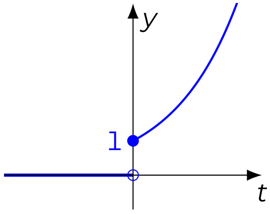
c)



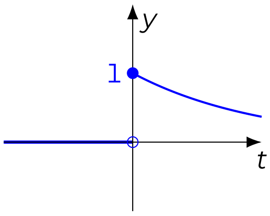
d)



e)



f)



Ramp function

The ramp function is shown in Figure A.4.15 and is defined by:

$$R(t-a) = \begin{cases} c(t-a) & \text{if } t \geqslant a \quad \text{where } c \text{ is a constant} \\ 0 & \text{if } t < a \end{cases} \tag{A.4.22}$$

In actual fact the ramp function is directly related to the unit step function as $R(t-a) = (t-a)u(t-a)$. It is used with periodic functions to create triangle and sawtooth waveforms as well as in motor drives and also in control theory.

Questions

1. Sketch the following functions:

   a) $f(t) = u(t+2)$
   b) $f(t) = u(t-1)$
   c) $f(t) = 2u(t-1)$
   d) $f(t) = 3u(t+2)$
   e) $f(t) = u(t+2) - u(t-1)$
   f) $f(t) = u(t-1) - u(t+2)$
   g) $f(t) = u(t+2) - 2u(t-1)$
   h) $f(t) = 3u(t+2) - 2u(t-1)$

2. A ramp function, $f(t)$, is defined by:

$$f(t) = \begin{cases} 3t & \text{if } t \geqslant 0 \\ 0 & \text{if } t < 0 \end{cases}$$

   Sketch the following expressions for $-2 \leqslant t \leqslant 2$:

   a) $f(t)$
   b) $u(t)f(t)$
   c) $u(t-1)f(t)$
   d) $u(t+1)f(t) - u(t-1)f(t)$

4.9 Unit impulse function or Dirac delta function, $\delta(t)$

This is an important function in the theoretical modelling of engineering systems. Effectively, an ideal impulse function is zero everywhere except at the origin ($t = 0$) as defined in equation A.4.23 where it is infinitely high — in fact it has zero width but an area of one.

The delta or unit impulse function is defined as:

$$\delta(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases} \tag{A.4.23}$$

This is of course impossible to create — you can think of as the limit of a rectangular pulse centred on ($t = 0$) that starts with a width of 1 and a height of 1 as it doubles in height and halves in width (thereby keeping area as 1). The start of sequence is shown in Figure A.4.16 and you can see how this gets close to the ideal impulse.

The two important properties of the impulse function are:

1. The generalised impulse function at $t = a$

$$\delta(t-a) = 0 \qquad \text{for } t \neq a \tag{A.4.24}$$

2. Property that expresses mathematically that the area enclosed by an impulse function is always 1

$$\int_{-\infty}^{\infty} \delta(t)dt = 1 \tag{A.4.25}$$

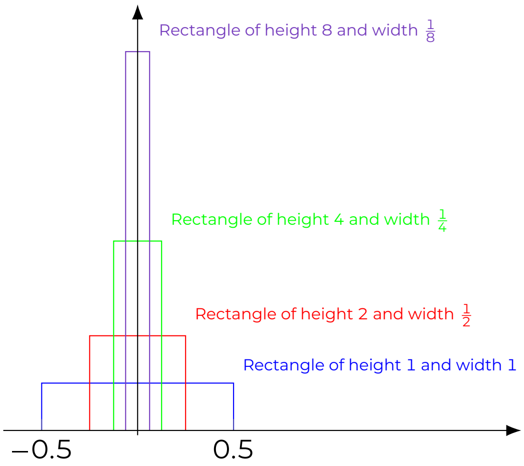


Figure A.4.16: Sequence of rectangular pulses of area 1

To plot impulse functions we put the area enclosed on the vertical axis and use a vertical area to the appropriate height. So if the function is $2\delta(t)$ then the arrow would go to 2. The generic representation of an impulse is $k\delta(t-d)$ which depicts an impulse of strength k at time $t = d$.

Example A.4.5

We have a train of impulse responses given by

$$f(t) = 2\delta(t) + 3\delta(t-1) + 0.5\delta(t-2)$$

. This leads to a graphical depiction as shown in Figure A.4.17

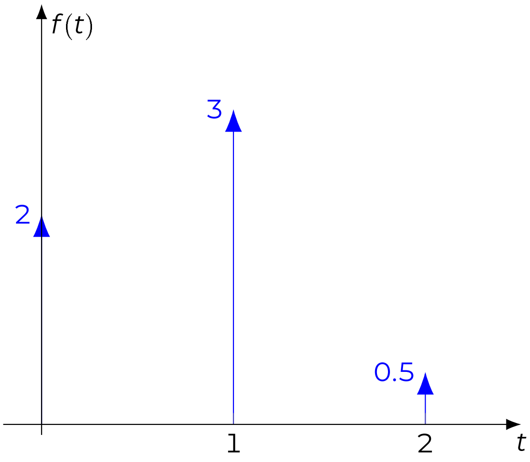


Figure A.4.17: Impulse train $f(t) = 2\delta(t) + 3\delta(t-1) + 0.5\delta(t-2)$

In electrical & electronic engineering the impulse function has many uses including describing mathematically how we sample an analogue signal (multiply signal by a chain of impulses at $t = 0, t = T, t = 2T, t = 3T$... where T is the sampling period). It is also used extensively to test systems.

Engineering application A.4.5: Impulse response of a system

The impulse response of a system is used to mathematically describe the output of a system when a unit impulse is applied as the input. This is very useful in analysing Linear Time Invariant (LTI) systems in both control theory and signal processing theory. Figure A.4.18 shows a LTI system with an impulse response of $h(t)$

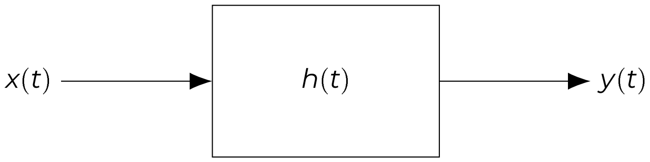


Figure A.4.18: Linear time invariant system

The impulse response of a system allows us to predict the time domain response of a LTI system to any input — mathematically in the time domain the output $y(t)$ is the convolution of the input $x(t)$ with the impulse response $y(t)$.

Continuous time systems (analogue if you like):

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \tag{A.4.26}$$

Discrete time systems — use sample number n instead of time t as variable

$$y[n] = \sum_{k=0}^{\infty} x[k]h[n-k] \tag{A.4.27}$$

Questions

1. Sketch the following impulse trains:
   a) $f(t) = \delta(t) + 2\delta(t-1) + 3\delta(t-3)$
   b) $f(t) = 2\delta(t-1) - \delta(t-2) + 2\delta(t-3)$
   c) $f(t) = 2(\delta(t) + 2\delta(t-1) - \delta(t-2) - 2\delta(t-3))$
   d) $f(t) = 0.5(4\delta(t) - 3\delta(t-1) + 2\delta(t-2) - \delta(t-3))$
   e) $f(t) = 2\delta(t) + \delta(t+1) + \delta(t-1) - \delta(t+3) - \delta(t-3)$
   f) $f(t) = \delta(t) - \delta(t+1) - \delta(t-1) + 0.5\delta(t+2) + 0.5\delta(t-2)$

4.10 Summary

This chapter has introduced a number of general and specific functions that are commonly used in electronic engineering. You will meet many of these throughout your course and see how they are used to model systems.

Trigonometry

Required Background: §A.3, Algebra (Page 45); §A.4, Functions (Page 65); §C.1, Calculator Skills (Page 333)

5.1 Introduction

As mentioned in the Introduction to Chapter A.4, in engineering we use the concept of mathematical modelling a lot. This chapter also covered the basics of many common engineering functions. However the trigonometric functions and trigonometry in general deserves a separate treatment given how important it is in electronic and electrical engineering. In particular in alternating current (AC) circuit analysis trigonometric functions are sued to describe alternating currents and voltages. They are also used extensively in the modelling of wireless communication systems where the radio frequency carrier wave is usually a trigonometric function.

In this chapter, we first of all explore the basics of trigonometry in terms of the fundamental ratios and functions. Then we take a look at the identities that can be used to simplify functions before taking a look at some of the uses of trigonometry in electrical and electronic engineering.

5.2 Basics

Degrees and radians

Before we get into trigonometry it is worth talking about the two units that can be used to measure angles — that is degrees (symbol °) and radians (conventionally no symbol but unit abbreviated to rad). A complete revolution is defined as 360° or 2π radians so it is easy to convert between the two units.

$$\begin{aligned} 360^\circ &= 2\pi \text{ radians} \\ 1^\circ &= \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians} \\ 1 \text{ radian} &= \frac{180}{\pi} \approx 57.3^\circ \end{aligned}$$

Note the following angles in degrees and radians (these angles crop up a lot)

$$\begin{aligned} 30^\circ &= \frac{\pi}{6} \text{ radians} & 45^\circ &= \frac{\pi}{4} \text{ radians} \\ 60^\circ &= \frac{\pi}{3} \text{ radians} & 90^\circ &= \frac{\pi}{2} \text{ radians} \\ 180^\circ &= \pi \text{ radians} & 270^\circ &= \frac{3\pi}{2} \text{ radians} \end{aligned} \tag{A.5.1}$$

For the remainder of this section we will mainly work in radians but it is worth recalling the corresponding angles in degrees as shown above in A.5.1.

Trigonometric Ratios

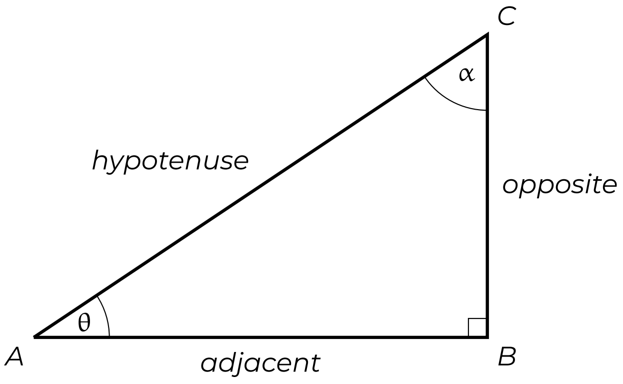


Figure A.5.1: Right-angled triangle ABC

In Figure A.5.1, considering the angle θ, we can define the trigonometric ratios *sine*, *cosine* and *tangent* as in Equation A.5.2:

$$\begin{aligned} \sin\theta &= \frac{\text{side opposite to angle}}{\text{hypotenuse}} = \frac{BC}{AC} \\ \cos\theta &= \frac{\text{side adjacent to angle}}{\text{hypotenuse}} = \frac{AB}{AC} \\ \tan\theta &= \frac{\text{side opposite to angle}}{\text{side adjacent to angle}} = \frac{BC}{AB} \end{aligned} \tag{A.5.2}$$

Note that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \tag{A.5.3}$$

$$\text{As } \tan\theta = \frac{BC}{AB} = \frac{BC}{AC} \times \frac{AC}{AB}$$

Also looking at other angle in Figure A.5.1 (α)

$$\sin\alpha = \frac{AB}{AC} = \cos\theta \qquad \cos\alpha = \frac{BC}{AC} = \sin\theta$$

Recall that for a triangle the total sum of angles in 180°, so as one angle is 90°then:

$$\theta + \alpha = 90^\circ = \frac{\pi}{2}$$
$$\implies \alpha = (\frac{\pi}{2} - \theta)$$

This means that:

$$\sin\theta = \cos(\frac{\pi}{2} - \theta) \qquad \cos\theta = \sin(\frac{\pi}{2} - \theta) \tag{A.5.4}$$

The ratios for common angles are defined in Table A.5.1.

θ in degrees θ in radians	0° 0	30° $\frac{\pi}{6}$	45° $\frac{\pi}{4}$	60° $\frac{\pi}{3}$	90° $\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	

Table A.5.1: Trigonometric ratios for common angles

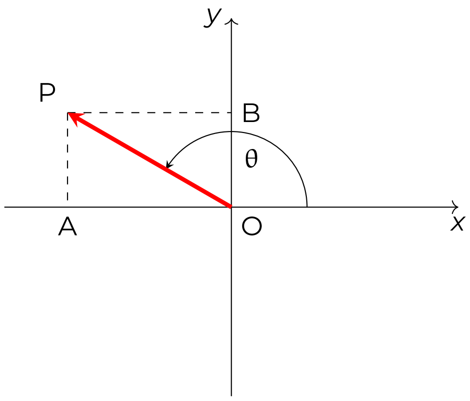


Figure A.5.2: Arm *OP* with projections *OA* and *OB*

To extend the trigonometric ratios to angles greater than $\pi/2$ we have to introduce a definition that is applicable to any size of angle. This is done by using polar plot as shown in Figure A.5.2 with an arm *OP* that is fixed at the origin *O* and rotates anticlockwise with the angle θ in radians being the angle between the positive x axis and the arm. The arm can be projected onto the x-axis *OA* and the y-axis *OB* with these projections being used to define the ratios. Then we can redefine the trigonometric ratios in A.5.2 to those in A.5.5:

$$\begin{aligned}\sin\theta &= \frac{\text{projection of } OP \text{ onto y axis}}{OP} = \frac{OB}{OP}\\ \cos\theta &= \frac{\text{projection of } OP \text{ onto x axis}}{OP} = \frac{OA}{OP}\\ \tan\theta &= \frac{\text{projection of } OP \text{ onto y axis}}{\text{projection of } OP \text{ onto x axis}} = \frac{OB}{OA}\end{aligned} \tag{A.5.5}$$

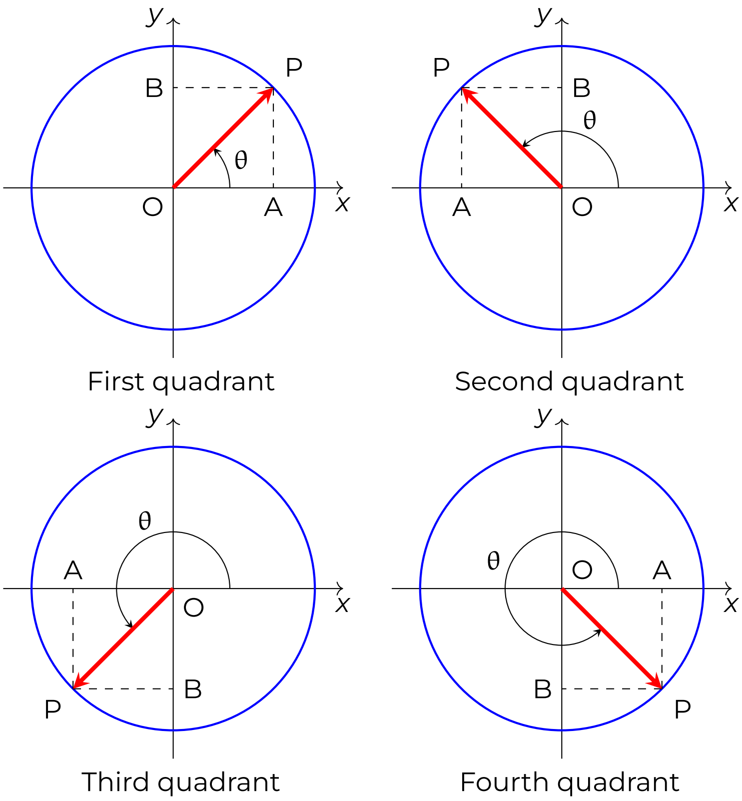


Figure A.5.3: Evaluating ratios in each quadrant

The projections OA and OB are positive or negative (hence determining the signs of $\sin\theta$, $\cos\theta$ and $\tan\theta$) depending on which quadrant the arm is in as shown in Figure A.5.3. So in the first quadrant where $0 \leqslant \theta < \pi/2$, both projections are positive so the values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ are all positive. In the second quadrant where $\pi/2 \leqslant \theta < \pi$, OB is positive and OA is negative so $\sin\theta$ is positive whereas $\cos\theta$ and $\tan\theta$ are both negative. In the third quadrant where $\pi \leqslant \theta < 3\pi/2$, both projections are negative so $\tan\theta$ is positive whereas $\cos\theta$ and $\sin\theta$ are both negative. In the final quadrant where $3\pi/2 \leqslant \theta < 2\pi$, OA is positive and OB is negative so $\cos\theta$ is positive whereas $\sin\theta$ and $\tan\theta$ are both negative. It is important to know which quadrant an angle lies in as the inverse trigonometric functions can give a number of different results due to the one-to-many relationship of the functions.

For angles greater than 2π, the arm simply rotates more than one revolution so the ratios can be calculated as above with θ being given by $\theta = x - n \cdot 2\pi$ where x is given angle in radians and n is number of complete revolutions the arm makes.

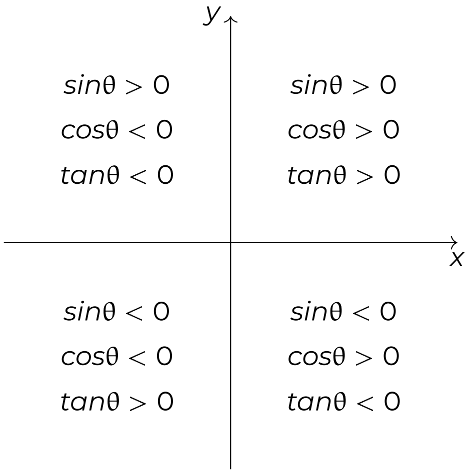


Figure A.5.4: Sign of trigonometric ratios in each quadrant

For negative angles the arm simply moves clockwise so for any angle $-\theta$:

$$\begin{aligned} \sin(-\theta) &= \sin(2\pi - \theta) \\ \cos(-\theta) &= \cos(2\pi - \theta) \\ \tan(-\theta) &= \tan(2\pi - \theta) \end{aligned} \tag{A.5.6}$$

The final set of trigonometric ratios are the reciprocal ratios which are donated as *cosecant* (cosec or csc) , *secant* (sec) and *cotangent* (cot):

$$\begin{aligned} \operatorname{cosec}\theta &= \frac{1}{\sin\theta} \\ \sec\theta &= \frac{1}{\cos\theta} \\ \cot\theta &= \frac{1}{\tan\theta} \end{aligned} \tag{A.5.7}$$

The inverse of all the trigonometric ratios — i.e. function to get the angle from a given ratio value are donated by the use of $^{-1}$. So inverse of $\sin\theta$ is $\sin^{-1}(\theta)$ and so on.

Trigonometric functions

The sine, cosine and tangent functions are basically the ratios as defined above — so $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$. Graphs of the functions can be constructed by using a table and a scientific calculator to find the values of $f(x)$ at various values of x as shown in Figures A.5.5, A.5.6 and A.5.7.

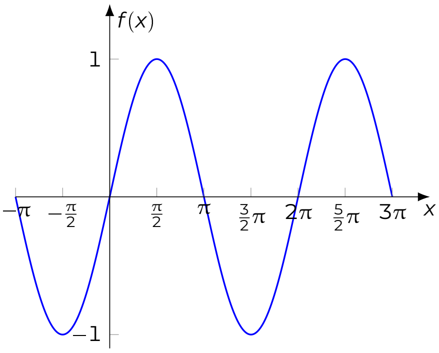


Figure A.5.5: Plot of $f(x) = \sin x$

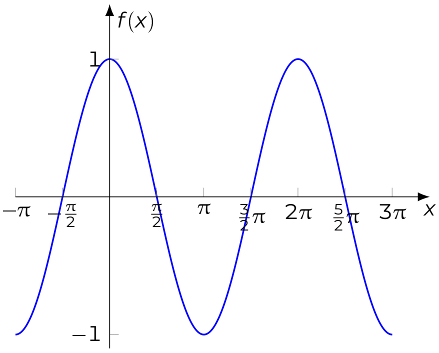


Figure A.5.6: Plot of $f(x) = \cos x$

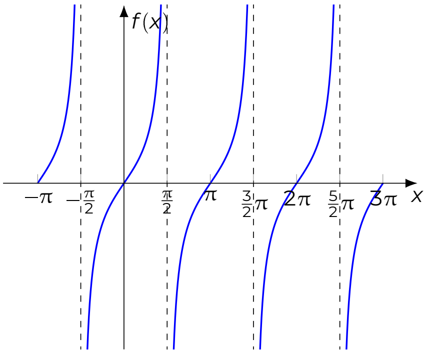


Figure A.5.7: Plot of $f(x) = \tan x$

It can be seen from the graphs of sine (Figure A.5.5) and cosine (Figure A.5.6) that if you shift the cosine graph right by $\pi/2$ then you obtain the sine function and similarly if you shift the sine function left by $\pi/2$ you get the cosine function. this interchangeability leads to these functions being commonly referred to as the *sinusoidal* functions. It is also possible to see two important properties of these functions:

$$\begin{aligned} \sin x &= -\sin(-x) \\ \cos x &= \cos(-x) \end{aligned} \tag{A.5.8}$$

The sinc x function

The *cardinal sine function* (aka sinc x) is a function that appears frequently in electronic engineering in applications from power electronics to communication. It is defined mathematically in Equation A.5.9 with a plot in Figure A.5.8:

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \tag{A.5.9}$$

It is worth noting that we define the function at $x = 0$ to avoid the issues that arise when dividing by 0. As it happens as $x \to 0$ we can say that $\operatorname{sinc} x \to 1$ as for instance $^{\sin 0.01}/_{0.01} = 0.9999833$ which is very close to 1.

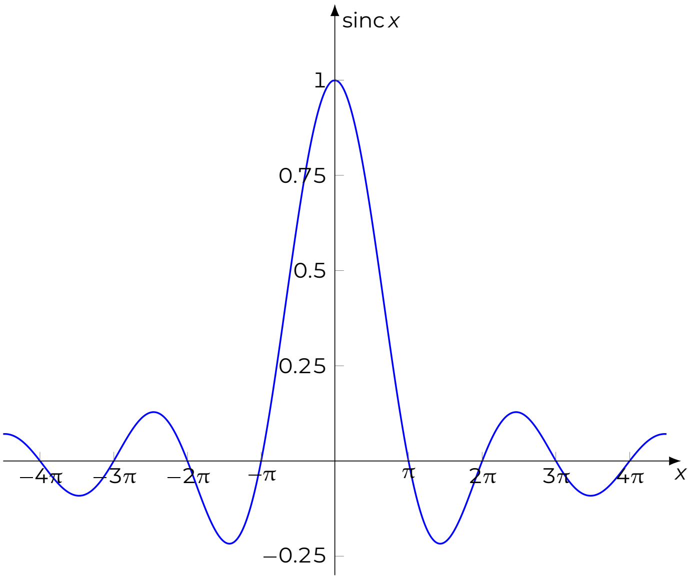


Figure A.5.8: Standard sinc function

Questions

1. Solve the following equations for $0 \leqslant \theta \leqslant 2\pi$

   a) $\sin(\theta) = 0.8426$

   b) $\sin(\theta) = 0.5681$

   c) $\sin(\theta) = -0.4316$

   d) $\sin(\theta) = -0.9042$

   e) $\cos(\theta) = 0.4243$

   f) $\cos(\theta) = 0.3500$

   g) $\cos(\theta) = -0.5618$

   h) $\cos(\theta) = -0.7433$

   i) $\tan(\theta) = 2.0612$

   j) $\tan(\theta) = 0.8436$

   k) $\tan(\theta) = -1.2501$

   l) $\tan(\theta) = -1.5731$

2. Solve the following equations for $0 \leqslant \theta \leqslant 2\pi$

   a) $\sin(2\theta) = 0.2516$

   b) $\sin(3\theta) = -0.6347$

   c) $\sin\left(\dfrac{\theta}{3}\right) = 0.4316$

   d) $\sin(2\theta - 1) = -06230$

   e) $\cos(3\theta) = 0.4243$

   f) $\cos\left(\dfrac{\theta}{2}\right) = 0.3500$

   g) $\cos(3\theta + 2) = -0.5618$

   h) $\cos\left(\dfrac{2\theta - 1}{3}\right) = -0.7433$

   i) $\tan(2\theta) = 2.0612$

   j) $\tan\left(\dfrac{\theta}{2}\right) = 0.8436$

   k) $\tan(2\theta + 1) = -1.2501$

   l) $\tan\left(\dfrac{3\theta - 1}{2}\right) = -1.5731$

3. a) An angle θ is such that $\cos\theta > 0$ and $\sin\theta > 0$. In which quadrant does θ lie?
   b) An angle θ is such that $\cos\theta < 0$ and $\sin\theta > 0$. In which quadrant does θ lie?
   c) An angle θ is such that $\cos\theta > 0$ and $\tan\theta < 0$. In which quadrant does θ lie?
   d) An angle θ is such that $\sin\theta < 0$ and $\tan\theta > 0$. In which quadrant does θ lie?

5.3 Identities

An identity is an equation where the left-hand side is always equal to the right hand side for all values of the variables. We have already seen trigonometric identities in equations A.5.3, A.5.4 and the reciprocal identities in equation A.5.6. The following identities are taken under a Creative Commons Attribution 4.0 International License from `http://evgenii.com/blog/basic-trigonometric-identities`

Even/odd

$$\begin{aligned}\sin(-x) &= -\sin x\\ \cos(-x) &= \cos x\\ \tan(-x) &= -\tan x\end{aligned}$$

Pythagorean

$$\begin{aligned}\sin^2 x + \cos^2 x &= 1\\ 1 + \tan^2 x &= \sec^2 x\\ 1 + \cot^2 x &= \operatorname{cosec}^2 x\end{aligned}$$

Co-function

$$\begin{aligned}\sin(\frac{\pi}{2} - x) &= \cos x\\ \cos(\frac{\pi}{2} - x) &= \sin x\\ \tan(\frac{\pi}{2} - x) &= \cot x\\ \cot(\frac{\pi}{2} - x) &= \tan x\\ \sec(\frac{\pi}{2} - x) &= \operatorname{cosec} x\\ \csc(\frac{\pi}{2} - x) &= \sec x\end{aligned}$$

Sum and difference of angles

$$\begin{aligned}\sin(x \pm y) &= \sin x\cos y \pm \cos x\sin y\\ \cos(x \pm y) &= \cos x\cos y \mp \sin x\sin y\\ \tan(x \pm y) &= \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}\end{aligned}$$

Double angles

$$\begin{aligned}\sin(2x) &= 2\sin x\cos x\\ \cos(2x) &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\\ \tan(2x) &= \frac{2\tan x}{1 - \tan^2 x}\end{aligned}$$

Half angles

$$\begin{aligned}\sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}}\\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}}\\ \tan\frac{x}{2} &= \frac{1-\cos x}{\sin x}\\ &= \frac{\sin x}{1+\cos x}\end{aligned}$$

Power reducing formulas

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2}\\ \cos^2 x &= \frac{1+\cos 2x}{2}\\ \tan^2 x &= \frac{1-\cos 2x}{1+\cos 2x}\end{aligned}$$

Product to sum

$$\sin x \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\tan x \tan y = \frac{\tan x + \tan y}{\cot x + \cot y}$$
$$\tan x \cot y = \frac{\tan x + \cot y}{\cot x + \tan y}$$

Sum to product

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x \cos y}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x \cos y}$$

Questions

1. Use the identities $\sin(x \pm y)$, $\cos(x \pm y)$ and $\tan(x \pm y)$ to simplify the following:
   - a) $\sin\left(\theta - \frac{\pi}{2}\right)$
   - b) $\cos\left(\theta - \frac{\pi}{2}\right)$
   - c) $\tan(\theta - \pi)$
   - d) $\tan(\theta + 2\pi)$
   - e) $\sin(\theta - 2\pi)$
   - f) $\sin(\theta + \pi)$
   - g) $\cos(\theta - \pi)$
   - h) $\cos(\theta + 3\pi)$
   - i) $\sin\left(2\theta - \frac{3\pi}{2}\right)$
   - j) $\cos\left(3\theta + \frac{\pi}{2}\right)$
2. Simplify the following expressions:
   - a) $\cos\theta\tan\theta$
   - b) $\sin\theta\cot\theta$
   - c) $\sin^2(\theta) + 2\cos^2(\theta)$
   - d) $2\cos^2(\theta) - 1$
   - e) $\frac{\sin 2\theta}{\cos 2\theta}$
   - f) $\frac{\sin\theta}{\sin 2\theta}$
   - g) $2\sin^2(\theta) + \cos 2\theta$

5.4 Uses

Basics

One of the major uses of trigonometry in engineering is in modelling waves — looking at the graphs in Figures A.5.5 and A.5.6 you can see that they have a shape similar to periodic waves. In fact any periodic wave can in general be modelled using a combination of sine and cosine functions [*]. The main waves we use in electronic & electrical engineering are ones that vary with time so the main variable is *t* for time.

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[*] See Chapter 23 in (Croft et al. 2017)

A periodic waveform has the key properties of *period* (and/or *frequency*), *amplitude* and *phase*. The *amplitude* of a wave is the maximum displacement of a wave from its mean position. So looking at the sine and cosine functions in Figures A.5.5 and A.5.6 we can say the the amplitude(A) of $\sin t$ and $\cos t$ is 1. The *period* of a periodic wave is defined as the time taken to complete one cycle — so in case of $\sin t$ and $\cos t$ the period (T) is 2π seconds. The *frequency* (f) of a periodic wave in Hertz(Hz) is defined as the number of cycles it completes in one second.

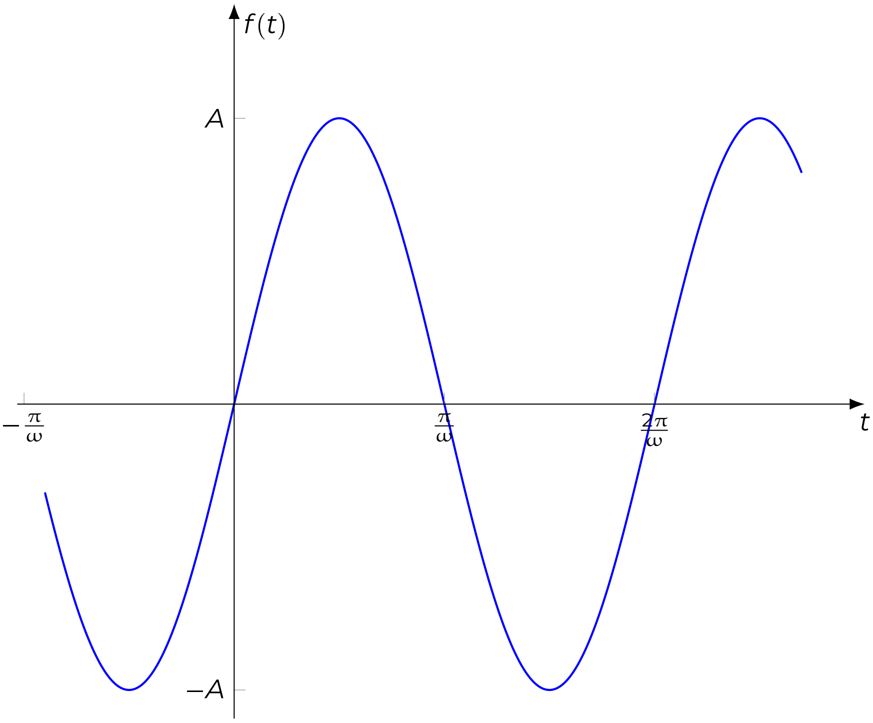


Figure A.5.9: Plot of $f(t) = A \sin \omega t$

A more general wave is defined as $f(t) = A \sin \omega t$ as shown in Figure A.5.9 or $f(t) = A \cos \omega t$ where ω represents the *angular frequency* of the wave — which is measured in radians per second (rad s^-1 or rad/s). The amplitude of the signal is A and the period is $T = \dfrac{2\pi}{\omega}$ as this is the time t when $\omega t = 2\pi$ as we know the number of radians for one cycle is 2π.

Given we known the frequency of a signal is the number of cycles in one second and the period is the time in seconds it takes to complete one cycle. So the relationship between f and T is:

$$f = \frac{1}{T} \tag{A.5.10}$$

So given for the general signal $T = \dfrac{2\pi}{\omega}$ we can therefore say:

$$\begin{aligned} f &= \frac{\omega}{2\pi} \\ \therefore \omega &= 2\pi f \end{aligned} \tag{A.5.11}$$

Using equation A.5.11 we can say that the wave $f(t) = A\sin\omega t$ can be rewritten as $f(t) = A\sin 2\pi f t$.

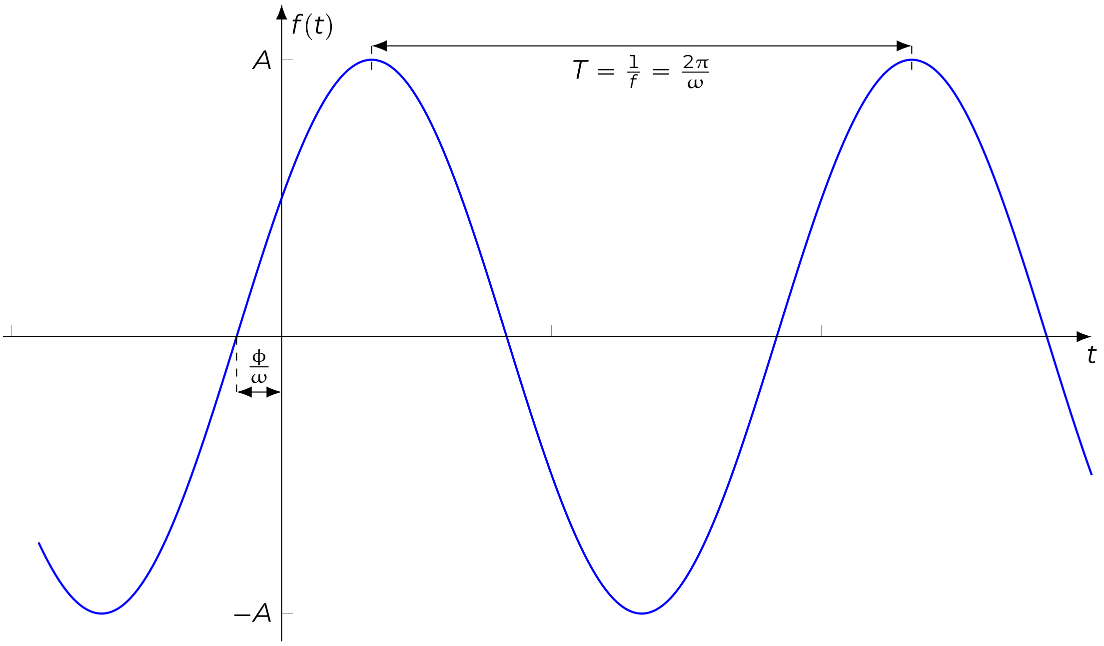


Figure A.5.10: Plot of $f(t) = A\sin(\omega t + \phi)$

The final parameter of a general wave is the *phase*, *phi* (or phase angle). This shifts the wave along the time axis and also allows either a sine or cosine function to represent the same wave as we know from equation A.5.4 that there is a $\frac{\pi}{2}$ phase shift between sine and cosine. In general we represent a sinusoid waveform as:

$$f(t) = A\sin(\omega t + \phi) = A\sin\omega\left(t + \frac{\phi}{\omega}\right) \tag{A.5.12}$$

The actual movement of the wave along the time axis as shown in Figure A.5.10 is given by the quantity $\frac{\phi}{\omega}$ in equation A.5.12 which is known as the *time displacement*.

Engineering application A.5.1: Electricity Supply

Alternating current (A.C.) waveforms are sinusoidal. Indeed the electricity supplied to our houses in the UK and Europe has a frequency of 50Hz and an nominal root mean square amplitude of 230V (V_{rms}). In fact, the voltage amplitude is approximately $v_{peak} = 230 \times \sqrt{2} \approx 325$V with $\omega = 50 * 2 * pi \approx 314.2$rad/s.

$$\begin{aligned} v(t) &= V_{peak}\sin(\omega t + \phi) \\ &= \sqrt{2}V_{rms}\sin(2\pi f t + \phi) \\ &\approx 325\sin(314.2t + \phi) \end{aligned}$$

The amplitude of the current waveform depends on the system but for a home power supply the frequency is 50Hz. A.C. is used to transmit power as it is far better for transmitting over long distances as the energy losses are less

Combining waves

You can combine sinusoidal waves together using the trigonometric identities in section 5.3. An example is given for two sinusoidal voltage signals in Example A.5.1

Example A.5.1: Combining sinusoidal voltages

Two voltage signals, $v_1(t)$ and $v_2(t)$ have the following mathematical expressions:

$$\begin{aligned} v_1(t) &= 2\cos 2t \\ v_2(t) &= 3\sin 2t \end{aligned}$$

a) State amplitude and angular frequency of the two signals.

b) Obtain an expression for the signal $v_3(t)$ given by

$$v_3(t) = 2v_1(t) + v_2(t)$$

c) Reduce expression for $v_3(t)$ to a single sinusoid and hence find the amplitude and phase of $v_3(t)$.

Solution

a) $v_1(t)$ has an amplitude of 2 volts and an angular frequency of 2 rad/s. $v_2(t)$ has an amplitude of 3 volts and an angular frequency of 2 rad/s.

b)

$$\begin{aligned} v_3(t) &= 2v_1(t) + v_2(t) \\ &= 2(2\cos 2t) + 3\sin 2t \\ &= 4\cos 2t + 3\sin 2t \end{aligned}$$

c) We want to write $v_3(t) = A\sin(2t + \phi)$ where A is amplitude of single sinusoid and ϕ is phase.
Using trigonometric identity $\sin(x+y) = \sin x\cos y + \cos x\sin y$ and letting $x = 2t$ and $y = \phi$

$$\begin{aligned} 4\cos 2t + 3\sin 2t &= A\sin(2t + \phi) \\ &= A(\sin 2t\cos\phi + \cos 2t\sin\phi) \\ &= (A\cos\phi)\sin 2t + (A\sin\phi)\cos 2t \end{aligned}$$

hence we can say that:

$$\begin{aligned} 3 &= A\cos\phi \\ 4 &= A\sin\phi \end{aligned}$$

In order to solve these equations we need to make use of the Pythagorean identity $\sin^2 x + \cos^2 x = 1$. So we square both equations and add the results together:

$$9 = A^2\cos^2\phi$$

$$\begin{aligned} 16 &= A^2 \sin^2 \phi \\ A^2(\sin^2 + cos^2 \phi) &= 9 + 16 \\ \therefore A^2 &= 25 \\ \implies A &= \sqrt{25} = 5 \end{aligned}$$

Now we use the expression for tan in equation A.5.3 and divide the second equation by the first equation:

$$\begin{aligned} \frac{A \sin \phi}{A \cos \phi} &= \frac{4}{3} \\ \tan \phi &= \frac{4}{3} \\ \therefore \phi &= tan^{-1}\left(\frac{4}{3}\right) \end{aligned}$$

Looking at the equations both $\sin \phi$ and $\cos \phi$ are positive so result is in first quadrant (see Figure A.5.4). Therefore using a calculator for above expression we can say that $\phi = 0.927$rad. So the final expression for $v_3(t)$ is

$$v_3(t) = 5 \sin(2t + 0.927)$$

It has an amplitude of 5 volts and a phase of 0.927 radians.

In general if we want to express the expression $a \cos \omega t + b \sin \omega t$ as a single cosine wave $R \cos(\omega t - \theta)$ then $R = \sqrt{(a^2 + b^2)}$ and $\tan \theta = \frac{b}{a}$. We can prove this as shown in the next example.

Example A.5.2: Combining general sinusoids

Prove that to express $a \cos \omega t + b \sin \omega t$ as a single cosine wave $R \cos(\omega t - \theta)$ then $R = \sqrt{(a^2 + b^2)}$ and $\tan \theta = \frac{b}{a}$

Solution:
Let

$$a \cos \omega t + b \sin \omega t = R \cos(\omega t - \theta)$$

From the identity for $\cos(x - y) = \cos x \cos y + \sin x \sin y$ we can say that:

$$\begin{aligned} a \cos \omega t + b \sin \omega t &= R \cos(\omega t - \theta) \\ &= R(\cos \omega t \cos \theta + \sin \omega t \sin \theta) \\ &= (R \cos \theta) \cos \omega t + (R \sin \theta) \sin \omega t \end{aligned}$$

So we can say by equating the coefficients:

$$\begin{aligned} a &= R \cos \theta \\ b &= R \sin \theta \end{aligned}$$

If we square and add these equations as $\sin^2 x + \cos^2 x = 1$ then

$$a^2 + b^2 = R^2(cos[2](\theta) + \sin[2](\theta))$$

So we can say that

$$R = \sqrt{(a^2 + b^2)} \tag{A.5.13}$$

Division of the equations for the two coefficients recalling that $\tan\theta = \dfrac{\sin\theta}{\cos theta}$ leads to:

$$\tan\theta = \frac{b}{a} \tag{A.5.14}$$

Wavelength, wave number and horizontal shift

Up to this point we have considered time varying waves but we could have waves where the independent variable is distance x say. In this case the general waveform equation from equation A.5.12 becomes:

$$f(t) = A\sin(kx + \phi) \tag{A.5.15}$$

In this case A is still the amplitude of the wave, k is the *wave number* and the phase ϕ shifts the graph horizontally. The *wavelength* λ is the length of one cycle which is related to k as $\lambda = \dfrac{2\pi}{k}$.

Questions

1. State the amplitude, angular frequency, frequency, phase and time displacement of the following waves:
    a) $3\sin 3t$
    b) $\dfrac{1}{2}\sin 4t$
    c) $\sin(t + \pi)$
    d) $2\sin(0.4t)$
    e) $\sin(2t + \frac{\pi}{2})$
    f) $5\sin(2\pi t - 10)$
2. A sinusoidal function has an amplitude of $\frac{1}{4}$ and a period of 0.5s. State one possible form of the function.
3. Write each of the following in the form $A\sin(\omega t + \phi)$:
    a) $2\sin t + 3\cos t$
    b) $4\sin 2t + \cos 2t$
    c) $-2\sin 5t + \cos 5t$
    d) $-3\sin 3t + 3\cos 3t$

5.5 Summary

This chapter has taken a look at trigonometry and in particular its uses in engineering. You should hopefully now have a grasp of what a sinusoidal wave looks like and what are the key elements of such a wave.

Graphs

6.1 Introduction

In Chapter A.4 the use of functions to model engineering systems is explored. In this chapter the various ways to draw a plot of a function are examined - using different coordinate systems. In effect, a coordinate system is a way of representing the position of a point - be it in terms of Cartesian coordinates x and $y = f(x)$ or polar coordinates r and θ. A function can be plotted on a graph using ordered pairs of numbers (in two dimensions) — the plotting of such a graph can be very useful in understanding what the model of a system is showing.

6.2 Plotting Graphs

2D Cartesian coordinates

When it comes to functions it is often useful to plot a graph of what the function looks like. We also plot results to enable us to see how a system behaves. In order to plot a function in two dimensions we need *ordered* pairs of numbers. So for instance we want to plot the function $y = 2x - 1$ for values of $-4 \leqslant x \leqslant +4$. To do this we calculate the value of y at each integer value of x:

x	-4	-3	-2	-1	0	1	2	3	4
y	-9	-7	-5	-3	-1	1	3	5	7

Table A.6.1: Order pairs for $y = 2x - 1$ where $-4 \leqslant x \leqslant +4$

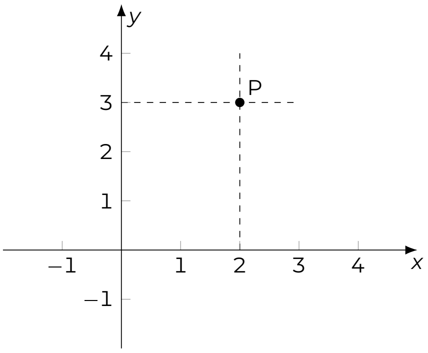


Figure A.6.1: Cartesian system in 2D

Now we can plot ordered pairs using *Cartesian* system. For 2D this is where you have two axes

— a horizontal one (conventionally x) and a vertical one (conventionally y) which intersect at the point (0, 0) which is known as the *origin*. By convention the positive part of the x-axis lies to the right of the origin and the positive part of the y-axis lies above the origin. Now consider a point $P(x_P, y_P)$ with an x-coordinate, x_P, that is the horizontal distance of the point from the y-axis and a y-coordinate, y_P, which is the vertical distance from the x-axis. So if we want to plot the point (2, 3) we need to place the point at the intersection of lines representing $x = 2$ and $y = 3$ as shown in Figure A.6.1.

Now we can plot our data from Table A.6.1 as shown in Figure A.6.2

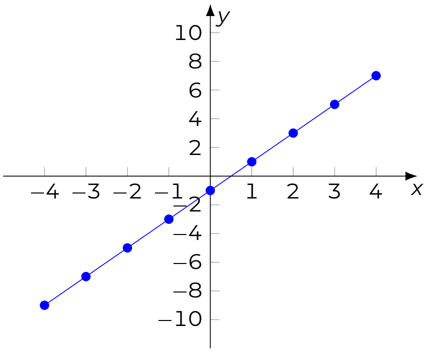


Figure A.6.2: Plot for $y = 2x - 1$ where $-4 \leqslant x \leqslant +4$

How about plotting the graph of $y = x^2 - 1$ for domain $|x| \leqslant 5$? Well again we start with the order pairs in Table A.6.2 and then plot the points as shown in Figure A.6.3

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
x^2	25	16	9	4	1	0	1	4	9	16	25
y	24	15	8	3	0	−1	0	3	8	15	24

Table A.6.2: Order pairs for $y = x^2 - 1$ where $|x| \leqslant 5$

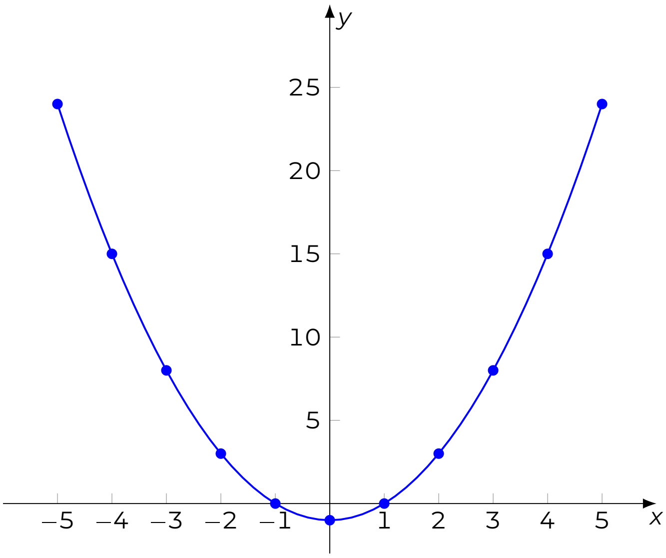


Figure A.6.3: Plot for $y = x^2 - 1$ where $|x| \leqslant 5$

We use the same technique for more complex functions as shown in Example A.6.1

Example A.6.1: More complex functions

Plot graphs for the following functions:

a) $y = x^3 - x^2 - 3x + 2$ for range $-2 \leqslant x \leqslant +3$

b) $y = \dfrac{1}{x-1}$ for range $0 \leqslant x \leqslant +1$

Solutions

a) Calculate each part for each value of x at intervals of 0.5 and then add together to get y

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
x^3	-8	-3.375	-1	-0.125	0	0.125	1	3.375	8	15.625	27
$-x^2$	-4	-2.25	-1	-0.25	0	-0.125	-1	-2.25	-4	-6.25	-9
$-3x$	6	4.5	3	1.5	0	-1.5	-3	-4.5	-6	-7.5	-9
y	-5	-0.125	2	4.125	1	-0.5	-2	-2.375	-1	2.875	10

Table A.6.3: Order pairs for $x^3 - x^2 - 3x + 1$

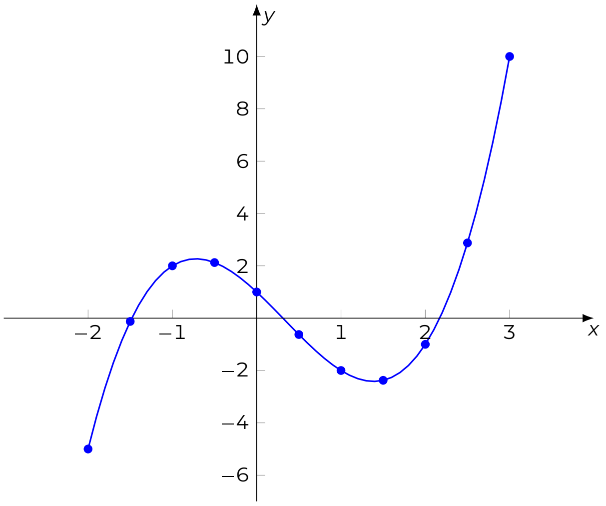


Figure A.6.4: Plot for $x^3 - x^2 - 3x + 1$

b) Calculate for intervals of x every 0.2. However not for $x = 1$ as this would cause denominator to be 0 hence $y = \pm\infty$.

x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
$x-1$	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y	-1	-1.25	$-1.\dot{6}$	-2.5	-5	$\pm\infty$	5	2.5	$1.\dot{6}$	1.25	1

Table A.6.4: Order pairs for $y = \dfrac{1}{x-1}$

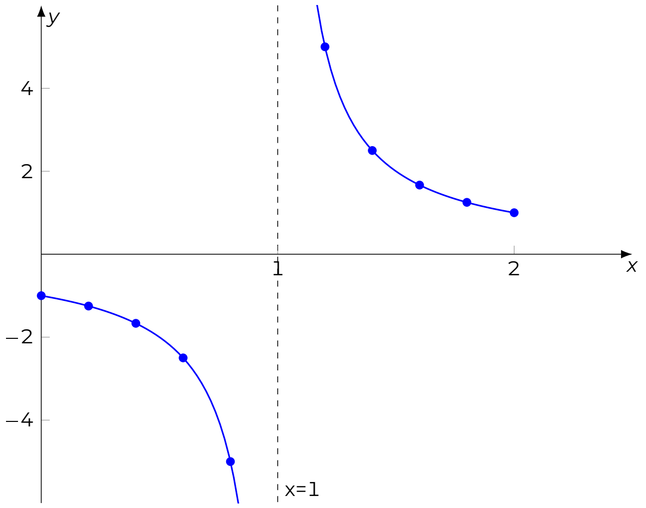


Figure A.6.5: Plot for $y = \dfrac{1}{x-1}$

Polar coordinates

We have in fact met this type of plot in Chapter A.5 when talking about extending the trigonometric ratios beyond a right-angled triangle. Basically it is an alternative way of describing position of point P as shown in Figure A.6.6.

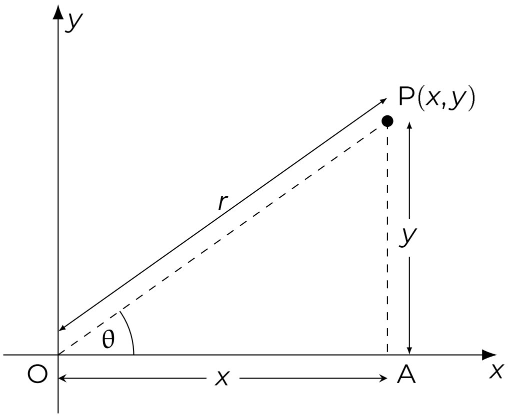


Figure A.6.6: Polar coordinates for P($r\angle\theta$)

Considering this diagram P has the Cartesian coordinates (x,y) so

$$\text{OA} = x \qquad \text{AP} = y$$

. The length OP is the distance of P from the origin which for polar coordinates we donate as r (note $r \geqslant 0$ is always true). The angle between OP and the positive x-axis is donated as θ and given we know a full circle is 2π rads or 360°, the limits for θ are $0 \leqslant \theta < 2\pi$ or $0° \leqslant \theta < 360°$. So the full polar coordinates for P are written conventionally as $r\angle\theta$.

But where would we use polar coordinates — well you may be surprised. One area that you may come across them is robotics when defining movement about a rotating point. Polar curves are used to describe antenna field patterns in two dimensions.

Converting from polar coordinates to Cartesian coordinates is simple when you think of trigonometry with reference to the △OPA in Figure A.6.6.

$$\begin{aligned} \cos\theta &= \frac{x}{r} & \therefore x &= r\cos\theta \\ \sin\theta &= \frac{y}{r} & \therefore y &= r\sin\theta \end{aligned} \tag{A.6.1}$$

Similarly, converting to polar coordinates from Cartesian coordinates is looks at the △OPA in Figure A.6.6. We have a right-angled triangle so can apply Pythagoras theorem which states that

Pythagoras

For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides

So in this case

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \therefore r &= +\sqrt{x^2 + y^2} \end{aligned} \tag{A.6.2}$$

Positive square root taken as we know $r \geqslant 0$. For the angle we use the tan definition as:

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ \therefore \theta &= \tan^{-1}\frac{y}{x} \end{aligned} \tag{A.6.3}$$

For the angle it is a good idea to sketch a plot of the Cartesian system as we use the tangent to find θ but as you should recall this ia a many-to-one function - so several values of θ have the same tangent value so the plot we enable you to work out what the actual value is.

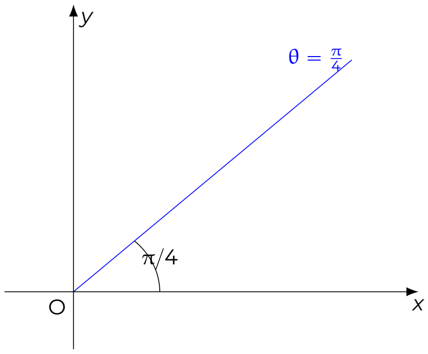


Figure A.6.7: Straight line when θ fixed and r varies

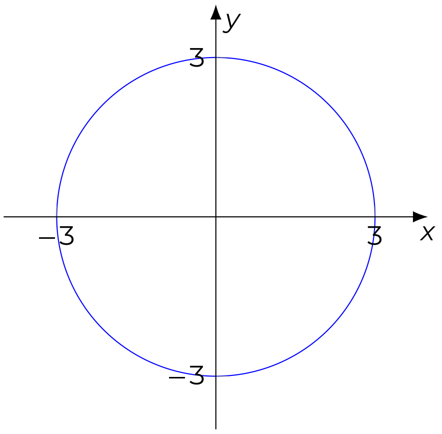


Figure A.6.8: Circle when θ varies and r fixed

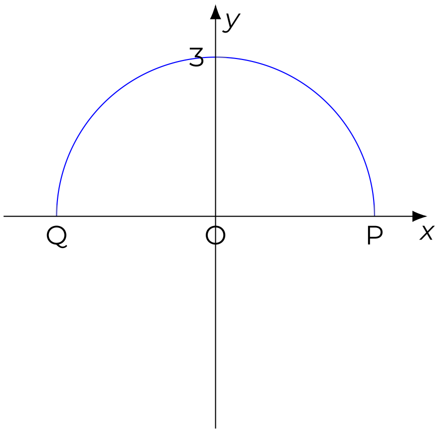


Figure A.6.9: Semicircle when θ varies between 0 & π radians (*r* fixed)

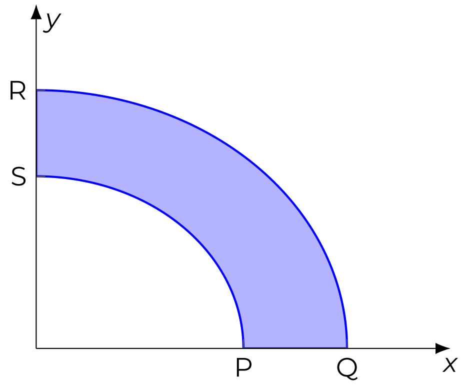


Figure A.6.10: Surface defined by $2 \leqslant r \leqslant 3$ & $0 \leqslant \theta \leqslant \frac{\pi}{2}$

For sketching polar curves, we tend to be given a range of values for one or both of r and θ. For instance looking at Figures A.6.7 — A.6.10 we can see that a straight line has a fixed θ value and r varies, whereas a circle or arc (part of a circle) has a fixed r and a varying θ. We can describe surfaces using polar coordinates by varying both variables as shown in Figure A.6.10. In general a polar curve is given by the equation $r = f(\theta)$. Let us consider a more complex function — polar co-ordinates are commonly used to describe the electric field strength at a fixed distance from an antenna as shown below in application A.6.1. This also shows a three dimensional plot of the radiation pattern (radius and 2 angles where the pattern is symmetrical in one angle)

Engineering application A.6.1: Antenna radiation pattern

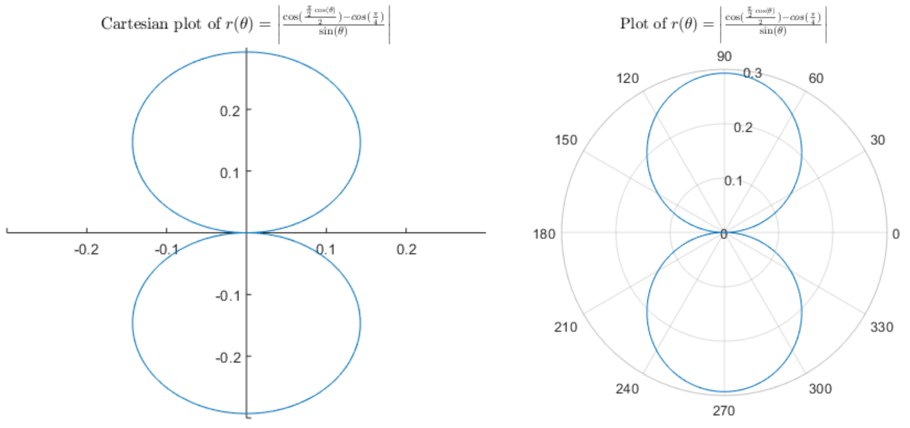
The field pattern of a quarter-wave dipole at a given distance is proportional to the polar function in equation A.6.4.

$$r(\theta) = \left| \frac{\cos(\frac{\frac{\pi}{2}\cos(\theta)}{2}) - \cos(\frac{\pi}{4})}{\sin(\theta)} \right| \tag{A.6.4}$$

Note that $|x|$ stands for the modulus function of x which is given by

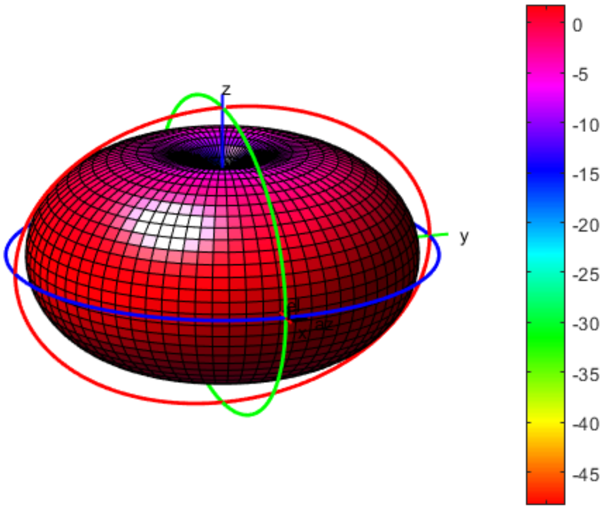
$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0 \end{cases} \tag{A.6.5}$$

We can use a polar plot to show this field pattern at a fixed distance as shown in Figure A.6.11. All these plots were both drawn using MATLAB® which we explore a little later in this section. The first two figures are slices through the antenna parallel to the z-axis shown in c).



(a) Cartesian

(b) Polar



(c) Spherical radiation pattern

Figure A.6.11: The field strength for a quarter-wave dipole

Questions

1. Plot the points A, B and C whose Cartesian coordinates are:

   a) $A(3, 4)$ b) $B(-2, -1)$ c) $C(3, -1)$

2. State the coordinates of points A, B and C as shown in Figure A.6.12

3. a) If a point A lies on the x-axis what is its y coordinate?

   b) If a point B lies on the y-axis what is its x coordinate?

4. Sketch the plots of the following functions

   a) $y = x - 1$ for range $-5 \leqslant x \leqslant +5$

   b) $y = 3 - 2x$ for range $-5 \leqslant x \leqslant +5$

   c) $y = x^2 + 2x + 1$ for range $-5 \leqslant x \leqslant +5$

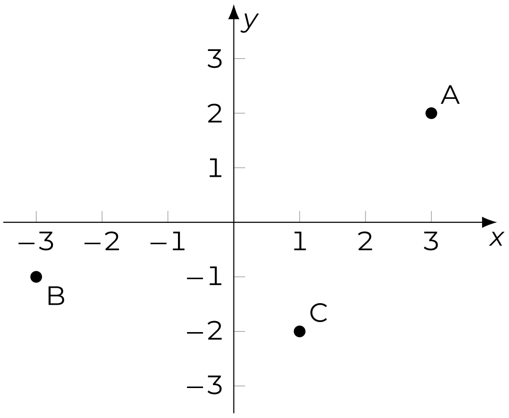


Figure A.6.12: Cartesian plot

d) $y = \frac{1}{x}$ for range $-1 \leqslant x \leqslant +1$

5. Plot the points P, Q and R whose polar coordinates are:

   a) $P(2, \frac{\pi}{3})$ b) $Q(1, \frac{5\pi}{6})$ c) $R(3, \frac{5\pi}{3})$

6. What are the Cartesian coordinates for the points P, Q and R in question 5?

7. The Cartesian coordinates of P are $(4, 3)$ and those of Q are $(-3, 2)$. Calculate the polar coordinates of P and Q.

8. Sketch the plots of the following functions

   a) The curve defined by $0 \leqslant r \leqslant 4, \theta = \frac{\pi}{4}$

   b) The curve defined by $r = 3, 0 \leqslant \theta \leqslant \frac{\pi}{2}$.

   c) The curve defined by $0 \leqslant r \leqslant 3, \frac{\pi}{6} \leqslant \theta \leqslant \frac{\pi}{3}$.

   d) The curve defined by $2 \leqslant r \leqslant 3, \frac{\pi}{2} \leqslant \theta \leqslant \pi$.

6.3 Using software to create graphs

Introduction

In order to draw graphs for use in word processed documents we tend to use specific software. Simple charts can be drawn directly in the common word processing software Microsoft® Word®, but for more complex plots we are better using MATLAB® or Microsoft® Excel® . The latter can actually be called from within Microsoft® Word® . You can also use the pgfplots within LaTeX which is how many of the plots in this book have been drawn but this pretty advanced — and indeed it is often easier to use MATLAB® to draw the plots before exporting them as *.png files and including them in a LaTeX document (as seen in Figure A.6.11). The following explores the basics of using both software packages to produce plots.

MATLAB®

As we see in application A.6.1 we can use MATLAB® to help us plot functions. This section is going to only give an introduction to this topic — it is worth you using the extensive help files and investigate further uses.

Cartesian Plots

The basic function of 2D Cartesian plots in MATLAB® is `plot`[*]. For a plot of $y = f(x)$ then two arguments are required with the first being the independent variable x and the second being the dependent values y. You will mainly be using equal length vectors for these variables.

As an example let us look at how we can use MATLAB® to plot the graphs shown in Figures A.6.3 & A.6.4[†].

Listing A.6.1: MATLAB® code for plotting $y = x^2 - 1$

```
 1  x=linspace(-5,5,11);
 2  for i=1:11
 3      y(i)=x(i)^2-1;
 4  end
 5  % Create figure with white background
 6  figure('Color','white')
 7  %plot figure with solid line and markers
 8  plot(x,y,'-o')
 9  %Add title
10  title('Plot of y=x^2 - 1')
11  % Move axes to centre
12  ax = gca;
13  ax.XAxisLocation = 'origin';
14  ax.YAxisLocation = 'origin';
```

Listing A.6.1 shows the code to create Figure A.6.13. Going through the lines let us explore how this script works. Note that a comment in MATLAB® is indicated by '`%`' symbol at start of comment.

Line 1: Creates a vector for x using `linspace(x1, x2, n)` function which takes in this case three arguments — minimum value ($x1$), maximum value ($x2$) and number of points (n). This creates a vector with n linearly spaced points starting at $x1$ and ending at $x2$. If n is not specified then 100 points are generated.

Line 2: Start `for` loop with `i` being variable. Goes up in integer steps of 1 from minimum value (1) to maximum value (11). The end of loop is indicated by `end` in line 4.

Line 3: This calculates the ith value of y (`y(i)`) using function $y = x^2 - 1$ with ith value of x.

Line 6: This creates a new figure with white background (default is light grey).

---

[*] See https://uk.mathworks.com/help/matlab/ref/plot.html

[†] For all MATLAB® within this book the code will be mainly provided in a script file format.

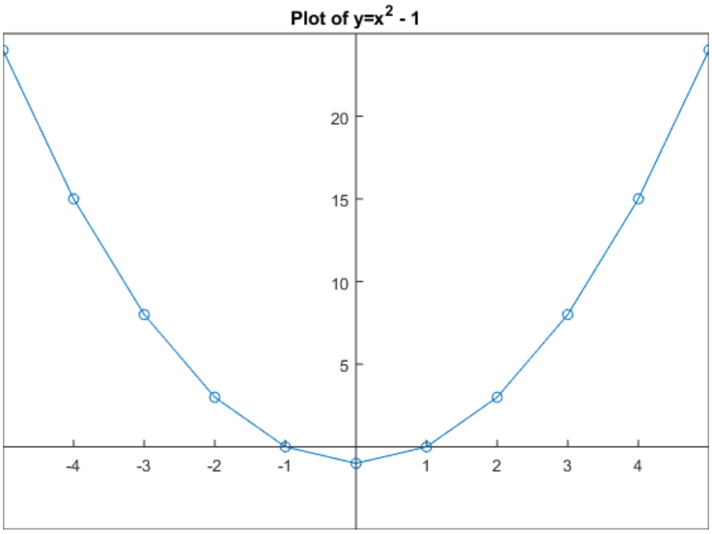


Figure A.6.13: Matlab plot of $y = x^2 - 1$

Line 8: Creates 2d plot of function with a solid line and circle markers - indicated by `'-o'` as option to `plot`.

Line 10: Adds title to current plot. Note by default Matlab uses TEX interpreter so can use subset of TEX commands for Maths formatting[*].

Lines 12 — 14: These move the axes from the left & bottom of figure frame (where they are by default) t cross at origin.

Listing A.6.2: MATLAB® code for plotting $y = x^3 - x^2 - 3x + 1$

```
x=-2:0.5:3;
for i=1:11
    y(i)=x(i)^3-x(i)^2-3*x(i)+1;
end
% Create figure with white background
figure('Color','white')
%plot figure with solid line and markers
plot(x,y,'-o')
xlim([-3 4])
ylim([-7 12])
%Add title
title('Plot of y=x^3-x^2-3x+1')
% Move axes to centre
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```

---

[*] See here for details of which commands can be used in text labels for MATLAB® .

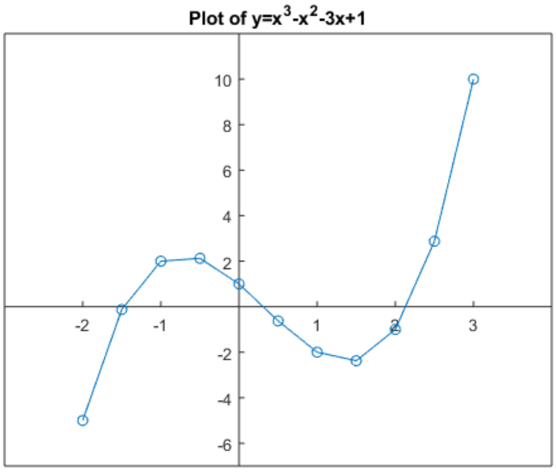


Figure A.6.14: Matlab plot of $y = x^3 - x^2 - 3x + 1$

Listing A.6.2 shows the code to create Figure A.6.14. Many of the code lines are similar to those for Listing A.6.1 but there are a few differences:

Line 1: This line uses colon notation to generate a vector with points between -2 and 3 with 0.5 interval. Notation is `x1:int:x2` with x1 and x2 being minimum and maximum values and int being interval spacing. If there are only two numbers (`x1:x2`) then default interval of 1 is used.

Lines 9 & 10: These two lines set the limits for the and y axes - so in this case $-3 \rightarrow 4$ for x-axis and $-7 \rightarrow 12$ for y-axis.

You can see looking at both Figures A.6.13 & A.6.14 that the curves are not as smooth as those shown for Figures A.6.3 & A.6.4. This is due to those in the earlier figures being plotted with more points (so in fact 51 samples in both figures using LaTeX pgfplots package). We can make the curves smoother by increasing the number of points - so taking $y = x^2 - 1$ example we can change code to that shown in Listing A.6.3 which results in Figure A.6.15. In this example we add a second set of values (xs & ys) which we create as 51 point vectors. The calculation of ys is done on Line 6 using elementwise .^ function to say that we create ys by squaring each element in xs and subtracting 1 from the result. The other change is to plot function on Line 10 where we just use red markers (`'ro'`)for 11 point vectors and a red line (`'r-'`) for 51 point vectors.

Listing A.6.3: MATLAB® code for plotting $y = x^2 - 1$

```
1  x=linspace(-5,5,11);
2  for i=1:11
3      y(i)=x(i)^2-1;
4  end
5  xs=linspace(-5,5,51);
6  ys=xs.^2-1;
```

```
% Create figure with white background
figure('Color','white')
%plot figure with solid line and markers
plot(x,y,'ro',xs,ys,'r-')
%Add title
title('Plot of y=x^2 - 1')
% Move axes to centre
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```

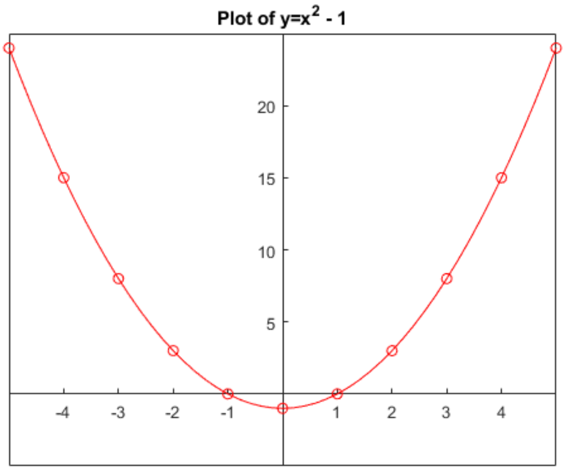
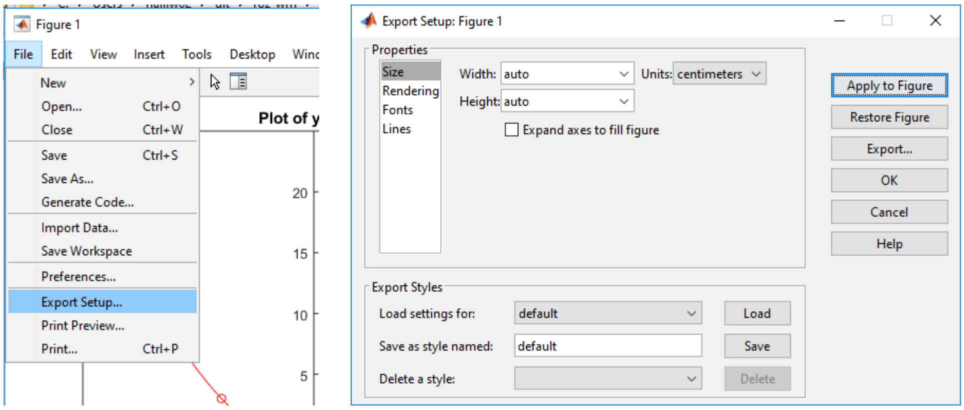


Figure A.6.15: Matlab plot of $y = x^2 - 1$

This section has literally touched on the basics of plotting Cartesian plots using Matlab. There is a large amount more things that can be done but the best way is to experiment yourselves.



Width: auto
Units: centimeters
Height: auto
☐ Expand axes to fill figure
Load settings for: default
Save as style named: default

(a) (b)

Figure A.6.16: Exporting MATLAB® Figures

However the last concept it is useful to know is how to export the figures into a format that can be used in Microsoft® Word® or other word processing packages. This is done from the

Figure window. Click on the File menu and select Export Setup... as shown in Figure A.6.16(a). This will open the dialogue box shown in Figure A.6.16(b). The default options are generally fine so click on Export... button, choose an appropriate format from drop-down list (usually Portable Network Graphics (*.png) is good) and save the figure somewhere appropriate.

Polar plots

As with Cartesian plots MATLAB® is powerful in enabling the plotting of polar functions. We saw some outputs in Figure A.6.11. The code used to plot (a) and (b) in this figure is given in Listing A.6.4. This code shows an example of using the LaTeX text interpreter in the `title` functions on Lines 13 & 23 to create the overall function. It also shows how the equations in A.6.1 can be used to create a Cartesian plot from a given set of polar coordinates (see Lines 4 & 5). Note that for the `polarplot(theta, r)` function, the angle vector (theta) needs to be in radians although the resulting plot is in degrees.

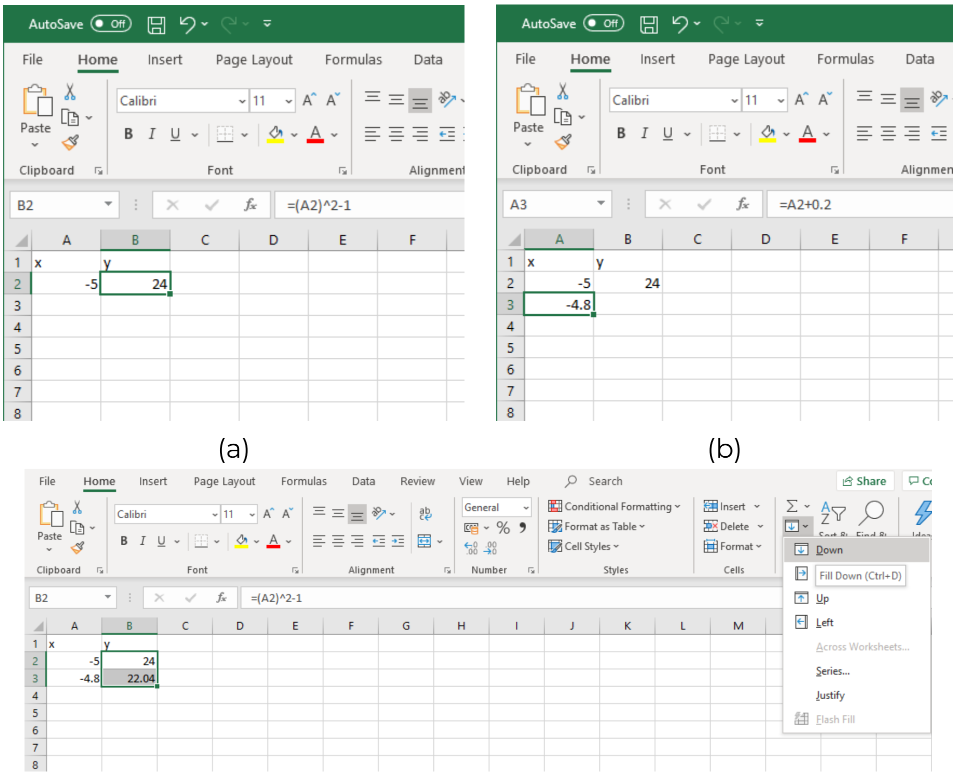
Listing A.6.4: MATLAB® code for plotting $r(\theta) = \left|\frac{\cos(\frac{\frac{\pi}{2}\cos(\theta)}{2}) - \cos(\frac{\pi}{4})}{\sin(\theta)}\right|$

```
1  theta=linspace(0, 2*pi, 721);
2  for i=1:721
3      r(i)=abs((cos(pi/2*cos(theta(i))/2)-cos(pi/4))/sin(theta(i)));
4      x(i)=r(i)*cos(theta(i));
5      y(i)=r(i)*sin(theta(i));
6  end
7
8  %Cartesian plot
9  figure('Color','white')
10 plot(x,y)
11 xlim([-0.3 0.3])
12 ylim([-0.3 0.3])
13 title('Cartesian plot of $r(\theta) = \biggl|\frac{\cos(\frac{\frac{\pi
       }{2}\cos(\theta)}{2})-cos(\frac{\pi}{4})}{\sin(\theta)}\biggr|$','
       Interpreter','latex')
14 % Move axes to centre
15 ax = gca;
16 ax.XAxisLocation = 'origin';
17 ax.YAxisLocation = 'origin';
18 ax.Box='off';
19
20 % Polar plot
21 figure('Color','white')
22 polarplot(theta,r)
23 title('Plot of $r(\theta) = \biggl|\frac{\cos(\frac{\frac{\pi}{2}\cos(\
       theta)}{2})-cos(\frac{\pi}{4})}{\sin(\theta)}\biggr|$','Interpreter',
       'latex')
```

As the result of the `polarplot(theta, r)` function is a MATLAB® figure it can be exported as described earlier to file suitable for inserting into a word processing software.

Microsoft® Excel®

Microsoft® Excel® is a spreadsheet package that can be sued to manipulate and plot data. It is particularly suited in engineering to plotting the results of experiments where you have a number of measurements of a result at different values of a variable as you can use scatter plots to plot the variable against the result to see what the resulting curve looks like. This can also be done in MATLAB® but involves putting the data into vectors manually or reading it in from a formatted file. Microsoft® Excel® is less suited than MATLAB® to plotting complex functions as its ability to calculate the value of the function at given input values is not as powerful as in MATLAB® which is after all a Mathematical modelling software at its core level (MATLAB® originally came from for MATrix LABoratory).



B2 =(A2)^2-1

	A	B
1	x	y
2	-5	24

(a)

A3 =A2+0.2

	A	B
1	x	y
2	-5	24
3	-4.8	

(b)

B2 =(A2)^2-1

	A	B
1	x	y
2	-5	24
3	-4.8	22.04

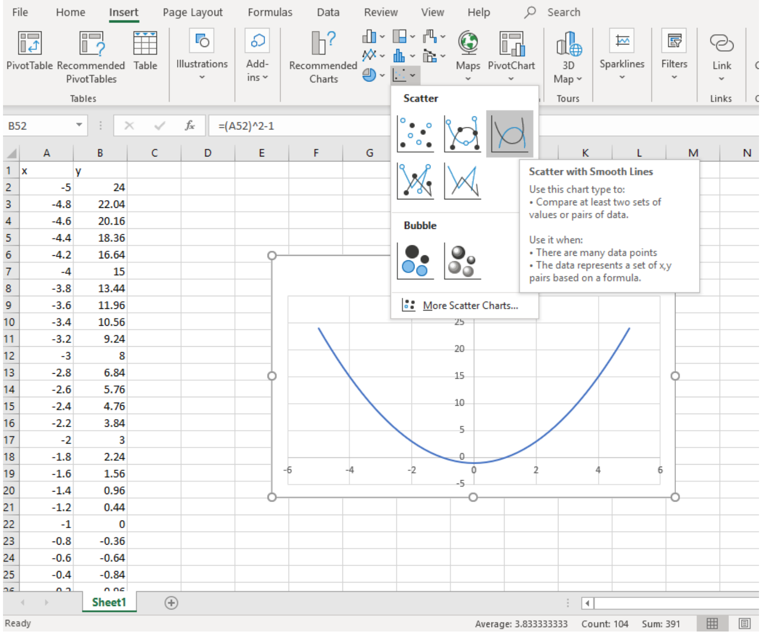
(c)

Figure A.6.17: Creating Microsoft® Excel® data for $y = x^2 - 1$

Taking the same example of $y = x^2 - 1$ with $-5 \leqslant x \leqslant 5$ let us see how we can plot it in Microsoft® Excel® . We are going to plot 51 points as in Figure A.6.15. To do this we need to create the data in a spreadsheet. The start is shown in Figure A.6.17(a) where we label column A as x and B as 1 in cells A1 & B1 respectively. Then in A2 we put the minimum value of x (-5) and in B2 we put the formula `=(A2)^2-1` which is $x^2 - 1$ which for -5 becomes 24 (compare to Table A.6.2). Then we need to create the next value of x in cell A3. Given we want 50 more values the interval is $int = \frac{5-(-5)}{50} = 0.2$) so we put the formula `=A2+0.2` into A3 as shown in Figure A.6.17(b). Then we can either put formula `=(A3)^2-1` in B3 or use the fill-down ability - by which you select A2 and A3 and either use `Ctrl-D` 9or the Fill Down command as shown in Figure A.6.17(c). The easy way to create the remaining 49 pairs of data is to select from A3 to B52 and use fill down again. Check that your bottom row (in A52

& B52) read 5 & 24 respectively.

Having created the data we now need to plot it using an x-y scatter graph. To do this select all the data (from A1 to B52) and on insert tab of toolbar select Scatter smooth curve as shown in Figure A.6.18. This creates the chart - which can be moved to its own sheet by right-clicking on it and selecting Move Chart ... and then selecting New Sheet.



B52 =(A52)^2-1

	A	B
1	x	y
2	-5	24
3	-4.8	22.04
4	-4.6	20.16
5	-4.4	18.36
6	-4.2	16.64
7	-4	15
8	-3.8	13.44
9	-3.6	11.96
10	-3.4	10.56
11	-3.2	9.24
12	-3	8
13	-2.8	6.84
14	-2.6	5.76
15	-2.4	4.76
16	-2.2	3.84
17	-2	3
18	-1.8	2.24
19	-1.6	1.56
20	-1.4	0.96
21	-1.2	0.44
22	-1	0
23	-0.8	-0.36
24	-0.6	-0.64
25	-0.4	-0.84

Scatter with Smooth Lines

Use this chart type to:
- Compare at least two sets of values or pairs of data.

Use it when:
- There are many data points
- The data represents a set of x,y pairs based on a formula.

Average: 3.833333333 Count: 104 Sum: 391

Figure A.6.18: Creating Microsoft® Excel® chart for $y = x^2 - 1$

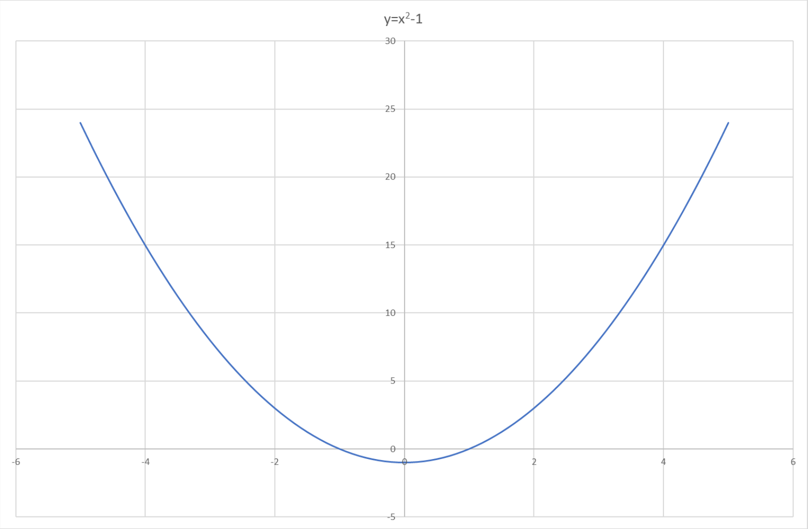


Figure A.6.19: Microsoft® Excel® chart for $y = x^2 - 1$

To export the chart there is not an option as in MATLAB® — instead you have to select the chart and then copy and paste it to your word processing document or an image software such as Paint to save it as an image file. The latter is how Figure A.6.19 was created.

It is worth noting that you can call Microsoft® Excel® directly from Word® on the Insert tab using Insert Table and selecting Excel spreadsheet form the list. You can then use the steps above to create a chart which is directly linked to the data in spreadsheet.

Microsoft® Excel® is really only useful for Cartesian plots in terms of plotting function although there are a large variety of charts available that can be used to display histograms, pie charts etc. So it is worth exploring in further depth so you can draw appropriate diagrams if needed for reports.

Summary

The software programmes MATLAB® and Microsoft® Excel® make it easy for engineers to draw up nice plots for reports but it is important to use the right tool for the job, and there will be occasions when just sketching a plot of paper will be more appropriate. As stated above if you are trying to plot a set of experimental data then Microsoft® Excel® is more likely to be your best option whereas MATLAB® is much better for plotting complex functions including in three dimensions and using non-Cartesian coordinates. Later in this chapter we will explore the use of logarithmic axes as opposed to the standard linear axes.

Questions

1. Using appropriate techniques use MATLAB® and Microsoft® Excel® to plot the functions defined in Question 6.2.4
2. Using appropriate techniques use MATLAB® to plot the functions defined in Question 6.2.8

6.4 Interpretations

Looking at the Cartesian plots earlier in this chapter they are all linear scaled — that is both axes use linear scales which ahs been fine for all the plots thus far. However let us consider a situation in which linear scales are not so useful. Suppose we wish to plot

$$y(x) = x^5 \qquad 1 \leqslant x \leqslant 10$$

This seems like a trivial problem and we would start by creating a table of ordered pairs as shown in Table A.6.5.

x	1	2	3	4	5	6	7	8	9	10
y	1	32	243	1024	3124	7776	16807	32768	59049	100000

Table A.6.5: Ordered pairs for $y(x) = x^5$ where $1 \leqslant x \leqslant 10$

However now look at the y values which start at 1 and go up to 100000 meaning if we plotted this on standard linear Cartesian axes several of the points would not be seen. This can be overcome by using a *log* scale for y — in other words we plot $\log y$ against x instead. Note that:

$$\log y = \log x^5 = 5 \log x$$

This means that that as x varies from 1 to 10, $\log y$ varies from 0 ($\log 1 = 0$) to 5 ($\log 10 = 1$). This type of plot as shown in Figure A.6.20 where one axis is linear and the other is log is known as a *log-linear* or *semilog* graph.

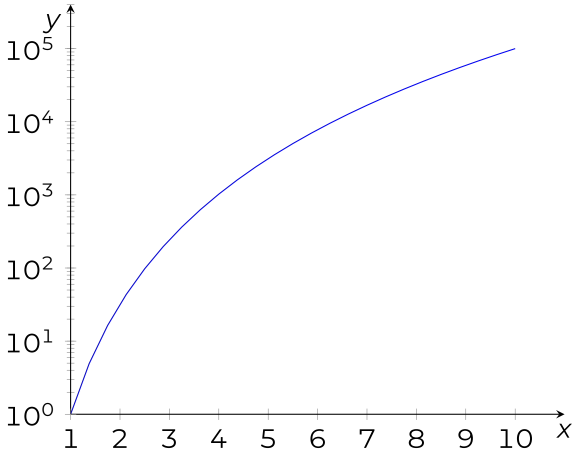


Figure A.6.20: Log-linear (semilogy) plot of $y(x) = x^5$

As with linear plots MATLAB® can be used to produce semilog graphs. The code in Listing A.6.5 with the resulting figure shown in Figure A.6.21

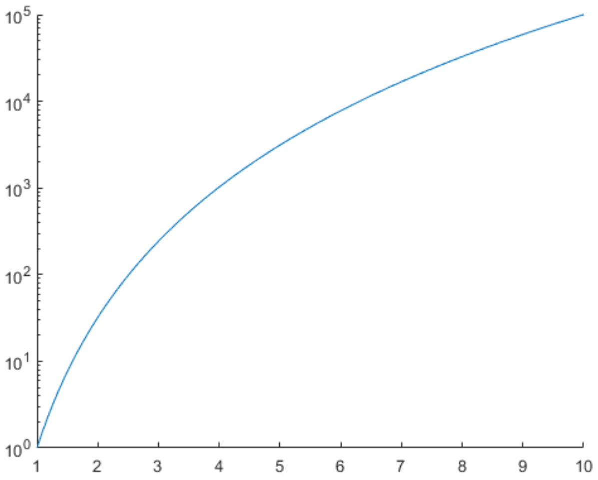


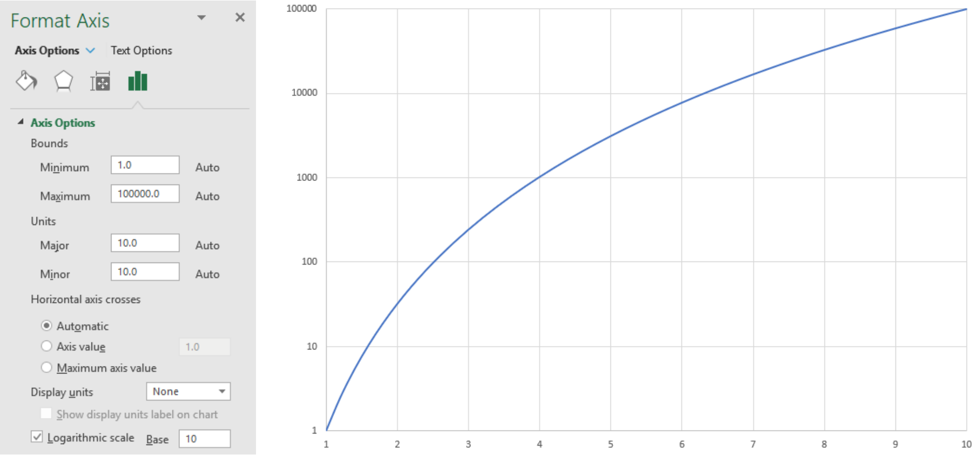
Figure A.6.21: MATLAB® plot of $y(x) = x^5$

Listing A.6.5: MATLAB® code for plotting $y(x) = x^5$

```
x=linspace(1,10);
y=x.^5;
% Create figure with white background
figure('Color','white')
semilogy(x,y)
% Move axes to centre
```

```
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
ax.Box = 'off';
```

Microsoft® Excel® can also be used to produce log-linear scatter graphs by changing the relevant axis to a logarithmic scale rather than a linear one within the Format Axis menu. There is a Logarithmic scale option that can be selected as shown in Figure A.6.22(a) with the plot of $y(x) = x^5$ using Microsoft® Excel® is shown in Figure A.6.22(b).


Format Axis
Axis Options | Text Options
Axis Options
Bounds
Minimum 1.0 Auto
Maximum 100000.0 Auto
Units
Major 10.0 Auto
Minor 10.0 Auto
Horizontal axis crosses
● Automatic
○ Axis value 1.0
○ Maximum axis value
Display units None
☐ Show display units label on chart
☑ Logarithmic scale Base 10

(a) (b)

Figure A.6.22: Creating Microsoft® Excel® plot of $y(x) = x^5$

As well as a logarithmic y-axis there are situations where a logarithmic x-axis (semilogx in MATLAB®) is required. For instance in plotting the frequency response of a system between 0.1Hz and 1MHz it makes sense to use a logarithmic scale for frequency as this is between 10^{-1}Hz and 10^6Hz. We also can get log-log graphs where both axes use a logarithmic scale.

Application A.6.2 shows how we can use MATLAB® to look at the response of an electrical system to a sinusoidal input signal. This will be discussed more in Instrumentation & Control for analysis of control systems.

Engineering application A.6.2: Bode plot

It is often very informative to look at the response of a electric circuit to a sinusoidal input signal. In fact we will often look at the response of a system to a range of fixed-amplitude sinusoids with different frequencies. If the system is a linear system, the output response will be a sinusoidal signal at same frequency but with different amplitude and phase (this is part of the definition of a linear system). A *Bode plot* plots the variance in amplitude and phase over the frequency range. In fact it consists of:

1. Plot of the ratio of output and input signal amplitudes versus frequency — usually amplitude ratio is plotted in dB using voltage gain definition in equation A.4.15. The actual values can have a wide range but using dB effectively reduce the scale to a linear scale.

2. Plot of the phase shift (difference in phase between input & output signals)

Let us consider the simple lumped element lowpass filter (attenuates high frequency components) shown in Figure A.6.23.

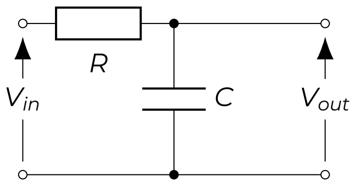


Figure A.6.23: Simple passive lowpass filter

This circuit has a transfer function of

$$H(s) = \frac{1}{1 + sRC}$$

The frequency response of $H(s)$ is given by setting $s = \mathrm{j}\omega$ (ω is angular frequency & j is complex notation — see Chapter A.10), so function becomes

$$H(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega RC}$$

In terms of the Bode plot, the magnitude of the frequency response, $|H(\mathrm{j}\omega)|$, gives the amplitude ratio and the phase of the frequency response, $\angle H(\mathrm{j}\omega)$ gives the phase shift. These are both non trivial to do by hand but we can use MATLAB® to produce the Bode plots. We need to convert ω to frequency in HZ but we know that $\omega = 2\pi f$. We can also state from circuit theory that the angular cut-off frequency ω_c is given by $\omega_c = 1/RC$. So for a cut-off frequency of 1kHz ($f_c = \omega_c/2\pi$), we say that

$$RC = \frac{1}{2\pi} \times 10^{-3}$$

then the frequency response becomes

$$H(f) = \frac{1}{1 + (jf \times 10^{-3})}$$

as the factors of 2π cancel each other out. So now we can use MATLAB® to produce the plots. Listing A.6.6 gives the code used to produce plots in Figure A.6.24.

Looking at the plots in Figure A.6.24 it is fairly easy to see that for a magnitude of −3 dB the frequency is 10^3Hz which is 1kHz. The definition of cut-off frequency is where the ratio of the output to input signal amplitudes is 3dB which is where output power is half the input power (or output voltage is $\frac{1}{\sqrt{2}}\times$input voltage).

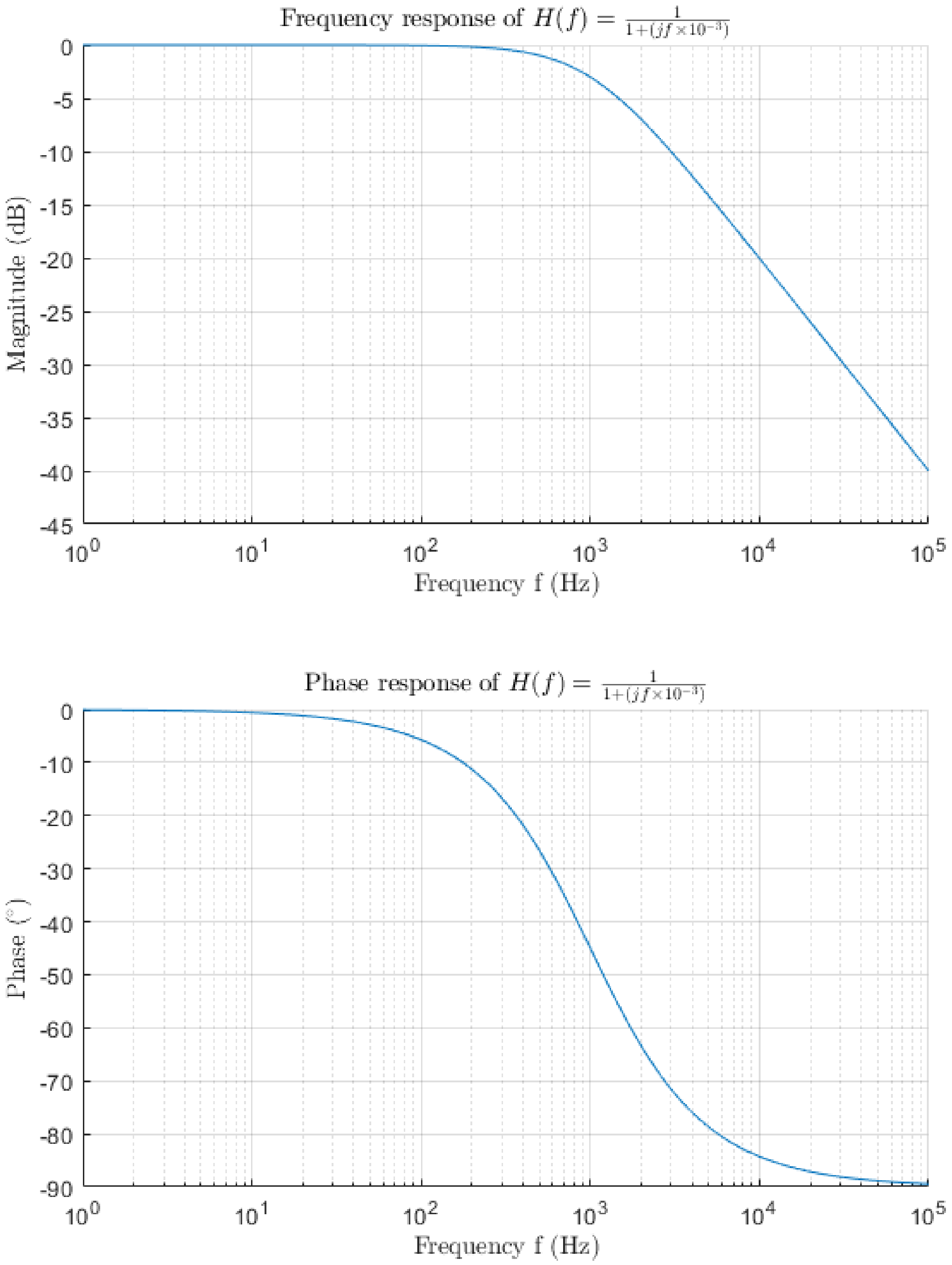


Figure A.6.24: MATLAB® produced Bode plots for lowpass filter

Listing A.6.6: MATLAB® code for plotting Bode plots of lowpass filter

```
f=1:1:100000;
fr=20*log10(abs(1./(1+j*f*1e-3)));
pr=180.*angle(1./(1+j*f*1e-3))./pi;
% Create figure with white background
figure('Color','white')
% Create sub plot with frequency response
ax1 = subplot(2,1,1); % top subplot
semilogx(f,fr)
grid
title(ax1, 'Frequency response of $H(f)=\frac{1}{1+(jf \times 10^{-3})}$', 'Interpreter','latex')
xlabel(ax1,'Frequency f (Hz)', 'Interpreter','latex')
ylabel(ax1,'Magnitude (dB)', 'Interpreter','latex')
ax1.Box = 'off';
% Create sub plot with phase response
ax2 = subplot(2,1,2); % top subplot
semilogx(f,pr)
grid
title(ax2, 'Phase response of $H(f)=\frac{1}{1+(jf \times 10^{-3})}$', 'Interpreter','latex')
xlabel(ax2,'Frequency f (Hz)', 'Interpreter','latex')
ylabel(ax2,'Phase ($^\circ$)', 'Interpreter','latex')
ax2.Box = 'off';
```

Questions

1. Create (using MATLAB®) the Bode plot of the lowpass filter shown in Figure A.6.25 for the following values of R & C.

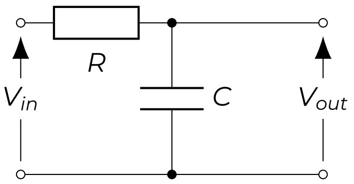


Figure A.6.25: Simple passive lowpass filter

   a) R=1 kΩ and C=1 nF
   b) R=2 kΩ and C=100 nF
   c) R=10 MΩ and C=10 pF
   d) R=2 MΩ and C=100 nF

2. Create (using MATLAB®) the Bode plot of the highpass filter shown in Figure A.6.26 for the following values of R & C. Note that the expression for the transfer function is $H(\mathrm{j}\omega) = \dfrac{\mathrm{j}\omega RC}{1+\mathrm{j}\omega RC}$ and the cut-off frequency is $\omega_c = \dfrac{1}{RC}$

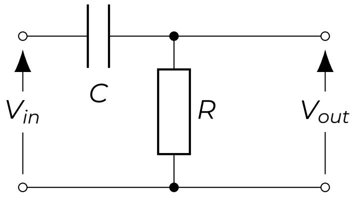


Figure A.6.26: Simple passive highpass filter

   a) R=1 kΩ and C=1 nF
   b) R=2 kΩ and C=100 nF
   c) R=10 MΩ and C=10 pF
   d) R=2 MΩ and C=100 nF

6.5 Summary

This chapter has introduced the concepts of graphs for both Cartesian and polar coordinates and how we can plot functions. It has also looked at how we can use computer software to produce plots of functions and the concept of logarithmic plots which are used in engineering when there is a wide range of values in one or both axes.

Discrete Mathematics

7.1 Introduction

The term *discrete* refers to objects that can only assume distinct, separated values whereas *continuous* refers to objects that can vary smoothly as we deal with in much of maths. Discrete mathematics as a branch is used a lot in electrical & electronic engineering due to the fact that we are often dealing with systems where the quantities are discrete (digital signals are discrete in time and amplitude). This chapter gives a brief introduction to *set theory* which provides a precise mathematical language often using in formal software verification and then looks mainly at *logic* and *Boolean algebra* both of which are mathematical tools that can be used to describe basic digital electronic circuits - that is circuits that confine themselves to two effective voltage levels rather than the range of levels used by analogue circuits. In the Level 5 mathematics module on we will look at digital systems in terms of z transforms and difference equations (Wyatt-Millington & Love 2022).

7.2 Set theory

A set is any collection of objects, things or states.

Literally a set can consist of anything under discussion including numbers,m letters or days of the week. To write down the members of a set we can write down the whole collection of *elements* (members) enclosed in a set of curly braces ({ }).

$A = \{1, 0\}$ the set of binary digits one and zero.

$B = \{\text{on}, \text{off}\}$ the set of possible states of a two-state system.

$C = \{\text{high}, \text{low}\}$ the set of effective voltage levels in a digital electronic circuit.

$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ the set of decimal digits

Note that we by convention use a capital letter to represent a set and we use the symbol $\in$ (means ‘is a member of’) to state that a particular object belongs to a particular set. So using above sets we can write:

$1 \in A \qquad \text{low} \in C \qquad 6 \in D$

There is another symbol $\notin$ which means ‘is not a member of’ that is used when an object is not a member of the set:

$5 \notin A \qquad \text{high} \notin B \qquad \text{off} \notin C$

For sets with a small number of elements it is fine to list them as above, but with larger sets this is just not feasible. Clearly for instance we could not write down the entire set of positive integer numbers as there are an infinite number of them. Fortunately specific symbols have be introduced to represent some commonly used sets in numbers:

$\mathbb{N}_0$	the set of non-negative integer numbers $0, 1, 2, 3\ldots$
$\mathbb{N}_{>0}$	the set of positive integer numbers $1, 2, 3\ldots$
$\mathbb{Z}$	the set of integer numbers, positive, negative & zero $\cdots -3, -2, -1, 0, 1, 2, 3\ldots$
$\mathbb{R}$	the set of all real numbers — any number in interval $(-\infty, \infty)$
$\mathbb{R}_{>0}$	the set of positive real numbers
$\mathbb{R}_{<0}$	the set of negative real numbers
$\mathbb{Q}$	the set of rational numbers

A way of defining a large set of numbers is to give a rule by which all members can be found. For instance:

$$A = \{x : x \in \mathbb{R} \text{ and } x < 2\}$$

This reads as 'A is the set of values of x such that x is a member of the set of real numbers and x is less than 2'. So the interval $(-\infty, 2)$ corresponds to A. See Section 4.2 for more on interval notations - but in simple terms a normal bracket '(' or ')' indicates not including that value, whereas a square bracket '[' or ']' indicates you do include this number. We can therefore define the sets of positive and negative real numbers as:

$$\mathbb{R}^+ = \{x : x \in \mathbb{R} \text{ and } x > 0\}$$
$$\mathbb{R}^- = \{x : x \in \mathbb{R} \text{ and } x < 0\}$$

Example A.7.1

Use set notation to describe the following intervals on the x axis:

a) $[0, 4]$ b) $[0, 4)$ c) $(-6, 6)$

Solutions

a) $\{x : x \in \mathbb{R} \text{ and } 0 \leqslant x \leqslant 4\}$ as square brackets indicate that the interval does include the numbers.

b) $\{x : x \in \mathbb{R} \text{ and } 0 \leqslant x < 4\}$ as normal brackets indicate that the interval does not include the number.

c) $\{x : x \in \mathbb{R} \text{ and } -6 < x < 6\}$

At times we use an English description of a set as opposed to mathematical expressions. So for instance:

M is the set of inductors made by machine M

N is the set of inductors made by machine N

Q is the set of faulty inductors

We can say that two sets are equal if they contain exactly the same elements. So for instance, the sets $\{2, 4, 7\}$ and $\{7, 4, 2\}$ are identical — the order in which we write the elements is immaterial. If elements are repeated in a set then this can be ignored so the sets $\{2, 4, 7, 4\}$ and $\{7, 4, 2\}$ are equal.

Venn diagrams

These are a graphical way of picturing sets aiding understanding — the sets are drawn as regions, usually circles, from which various properties can be observed. One important property is the *universal set*, $\mathbb{E}$, which is defined as:

The universal set, $\mathbb{E}$ is the set containing all elements of interest.

Figure A.7.1 shows an example digram where the *universal set*, $\mathbb{E} = \{x : x \in \mathbb{N}_{>}0 \text{ and } x \leqslant 10\}$ which is all positive integers between 1 and 10 and shown by the rectangle. Then we have two sets $A = \{1, 3, 5, 6\}$ and $B = \{2, 3, 6, 7, 9\}$ shown by the two intersecting circles.

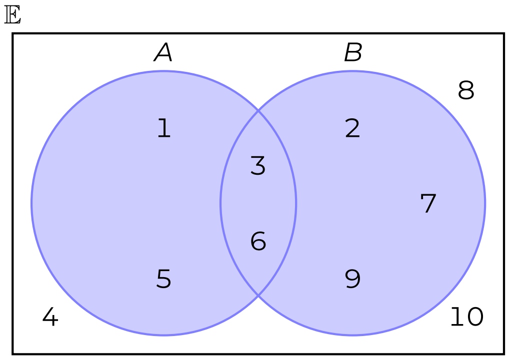


Figure A.7.1: Example Venn Diagram

Set algebra

From the Venn diagram we can see which elements are in common between the sets — this is the *intersection* of A and B donated as

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

In Figure A.7.1 we can see that $A \cap B = \{3, 6\}$. If the set $a \cap B$ has no elements then we say that the sets A and B are *disjoint sets* and write $A \cap B = \varnothing$ where $\varnothing$ donates the *empty set*.

refs	Group	Rule or law
1	Commutative laws	$A \cup B = B \cup A$
2		$A \cap B = B \cap A$
3	Associative laws	$A \cup (B \cup C) = (A \cup B) \cup C$
4		$A \cap (B \cap C) = (A \cap B) \cap C$
5	Distributive laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
6		$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7	Identity laws	$A \cap \mathbb{E} = A$
8		$A \cup \varnothing = A$
9	Complement laws	$A \cup \overline{A} = \mathbb{E}$
10		$A \cap \overline{A} = \varnothing$
11		$\overline{\overline{A}} = A$
12	Absorption rules	$A \cup (A \cap B) = A$
13		$A \cap (A \cup B) = A$
14		$A \cup (\overline{A} \cap B) = A \cup B$
15	Minimization rules	$(A \cap B) \cup (A \cap \overline{B}) = A$
16		$(A \cup B) \cap (A \cup \overline{B}) = A$
17	De Morgan's rules	$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$
18	De Morgan's rules	$\overline{(A \cap B)} = \overline{A} \cup \overline{B}$

Table A.7.1: Laws of Set Algebra

The empty set, $\varnothing$, is a set with no elements.
If $A \cap B = \varnothing$ then sets A and B are *disjoint sets*

The Venn diagram also shows the *union* of the two sets — that is all the elements in A and B:

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}$$

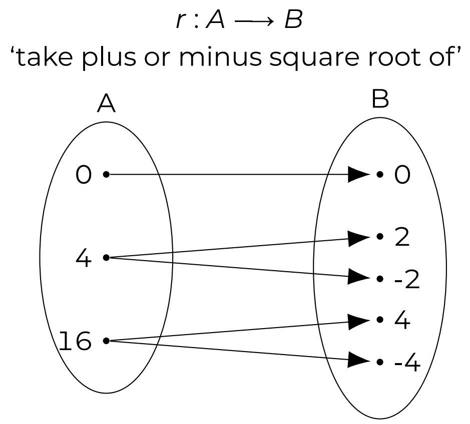
In Figure A.7.1 we can see that $A \cup B = \{1, 2, 3, 5, 6, 7, 9\}$.

If all the elements in set A are also members of set B then we can say that A is a *subset* of B, written as $A \subset B$. We have already met some subsets in that $\mathbb{N}_0 \subset \mathbb{Z}$ and $\mathbb{Z} \subset \mathbb{R}$.

If $A \subset \mathbb{E}$, then the members of $\mathbb{E}$ not in A are called the *complement* of A, written as $\overline{A}$. Clearly $A \cup \overline{A} = \mathbb{E}$ and $A \cap \overline{A} = \varnothing$.

There are a number of laws of set algebra for any sets A, B, C with a universal set $\mathbb{E}$ as defined in Table A.7.1

It is worth noting the rules from 12 to 18 can be derived from the laws in 1 to 11.



$r : A \longrightarrow B$

'take plus or minus square root of'

Figure A.7.2: A relation between sets A and B

Sets and functions

With two sets A and B it can be a useful exercise to examine the relationship in terms of the rules between the elements in the two sets. Take for instance the two sets $A = \{0, 4, 16\}$ and $B = \{-4, -2, 0, 2, 4\}$ where the relationship is that each element of B is either the plus or minus square root of an element in A as shown in Figure A.7.2. The rule that when applied to an element in A gives an element in B is called a *relation*, r where 'the relation r maps elements of set A to elements of set B':

$$r : A \longrightarrow B$$

In this example the relation is generally $r : x \longrightarrow \pm\sqrt{x}$. The set from which we take our inputs is known as the *domain*; conversely the set to which we map is known as the *co-domain*; the subset of the co-domain which we actually use is known as the *range* — this is not necessarily the whole of the co-domain.

A relation r maps element of a set D (the domain), to one of more elements in set C (the co-domain):

$$r : D \longrightarrow C$$

Example A.7.2

If $E = \{0, 1, 2, 3, 4, 5\}$ and $F = \{1, 3, 5, 7, 9, 11, 13, 15\}$ and the relation, s is defined by $s : E \longrightarrow F, s : m \longrightarrow 2m + 1$, identify the domain and co-domain of s. Draw a mapping diagram to illustrate the relation and what is the range of s?

Solution:
The domain of s is the set from which we take our inputs so in this case that is $E = \{0, 1, 2, 3, 4, 5\}$. The co-domain of s is the set to which we map so in this case $F = \{1, 3, 5, 7, 9, 11, 13, 15\}$. The rule $s : m \longrightarrow 2m + 1$ enables to do the mapping such as $s : 3 \longrightarrow 7$ as shown in Figure A.7.3. The range of s is the subset of F that is actually used so $\{1, 3, 5, 7, 9, 11\}$ — this is an example when not all the co-domain set is in the range of

the relation.

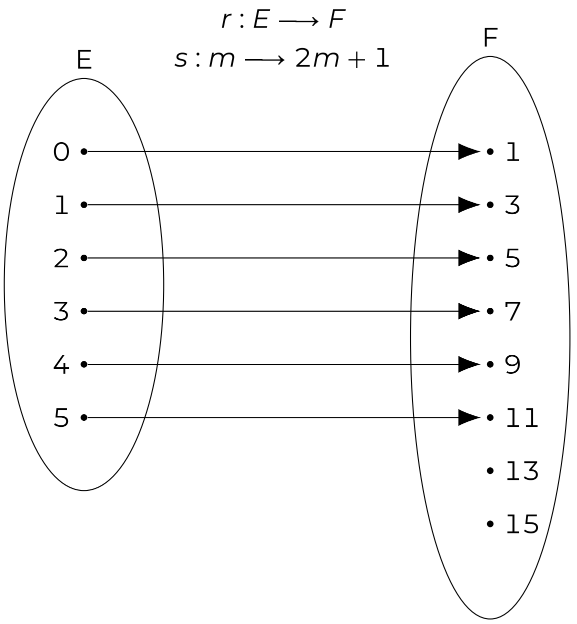


Figure A.7.3: A relation between sets E and F

If you think back to Section 4.2 then the notation used for relations is very similar to that for functions. In fact a function is a special form of a relation — recall that the definition fo a function is 'a rule which when given an input produces a single output'. If we study the two relationship r (in Figure A.7.2) and s (in Figure A.7.3) then we can see that r is not a function as one input can generate two outputs, but s is a function as each input only produces one output. So we can rigorously define a function as:

A function f is a relation which maps each element of a set D (the domain), to one element of a set C (the co-domain):

$$f : D \longrightarrow C$$

This means we can find the domain of a function from set theory and write it down. All the functions seen in chapter A.4 have domains which are subsets of the real numbers $\mathbb{R}$ and for continuous functions graphs can replace the mapping diagrams.

Example A.7.3

Find the domain, D, of the rational function $f : D \longrightarrow \mathbb{R}$ given by:

$$f : x \longrightarrow \frac{2x}{x-3}$$

Solution:
No domain is given so we choose the largest set possible. This is basically all real numbers

except for the point $x = 3$ where f is not defined (recall Figure A.4.3 where y tends to ∞ as x tends to 2). So this means we have result $D = \{x : x \in \mathbb{R}, x \neq 3\}$.

Questions

1. Use set notation to describe the following intervals on the x axis:

   a) $(-2, 3)$ b) $[0, 3]$ c) $[-3, 4)$
   d) $(-1, 5]$ e) $|x| < 4$

2. Using number lines sketch the following sets (recall Figure A.4.1):

   a) $\{x : x \in \mathbb{R} \text{ and } 1 \leqslant x \leqslant 3\}$
   b) $\{x : x \in \mathbb{R} \text{ and } -1 < x \leqslant 1\}$
   c) $\{x : x \in \mathbb{R} \text{ and } 0 \leqslant x < 2\}$
   d) $\{x : x \in \mathbb{R} \text{ and } -2 < x < 0\}$

3. Given $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 4, 6, 7, 8\}$ find:

   a) $A \cap B$ b) $A \cup B$

4. Given $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{3, 6, 9\}$ find:

   a) $A \cap B$ b) $B \cap C$ c) $A \cap C$
   d) $A \cap B \cap C$ e) $(A \cup C) \cap B$ f) $(B \cap C) \cup A$

5. Write out all members of the following sets:

   a) $A = \{x : x \in \mathbb{N}_0 \text{ and } x < 9\}$
   b) $B = \{x : x \in \mathbb{N}_0 \text{ and } 0 \leqslant x \leqslant 9 \text{ and } x \text{ is divisible by } 4\}$

6. Given $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6\}$, $C = \{3, 6, 9\}$ and $\mathbb{E} = \{1, 2, 3 \ldots 10\}$

   a) Draw a Venn diagram of this system
   b) State $A \cup B$
   c) State $B \cap C$
   d) State $A \cap B \cap C$
   e) State $\overline{A}$
   f) State $\overline{B} \cap \overline{C}$
   g) State $\overline{A \cup B}$

7. For the sets $A = \{0, 1, 2\}$ and $B = \{3, 4\}$ draw mapping diagrams to represent the following relations and determine which are functions. State your reason if you decide relation is not a function

   a) $r : A \longrightarrow B, r : 0 \longrightarrow 3, r : 1 \longrightarrow 4, r : 2 \longrightarrow 4$
   b) $s : A \longrightarrow B, s : 0 \longrightarrow 3, s : 0 \longrightarrow 4, s : 1 \longrightarrow 3, s : 2 \longrightarrow 4$
   c) $t : A \longrightarrow B, t : 0 \longrightarrow 3, t : 1 \longrightarrow 4$

7.3 Logic

This is the basis of digital electronics with Boolean algebra being the manipulation of logic statements which can then be used to analyse digital circuits. In terms of logic as engineers there are a number of basic logic gates that implement logic statements.

We define logic in terms of binary digits (base 2) - that is values of 0 (LOW voltage level) and 1 (HIGH voltage level). So a logic variable can only take two discrete values of 0 and 1.

The OR gate

This is a logic function that gives a HIGH output when any of its inputs are HIGH. It can be represented in a DC circuit as switches in parallel as shown in Figure A.7.4 for a two input gate. The bulb F will light if either or both switches are closed. Figure A.7.5 is the symbol for a two input OR gate, with Table A.7.2 giving the *truth table*. This defines the output F for all possible combinations of the inputs A and B where OR is represented by the symbol $+$ in discrete maths.

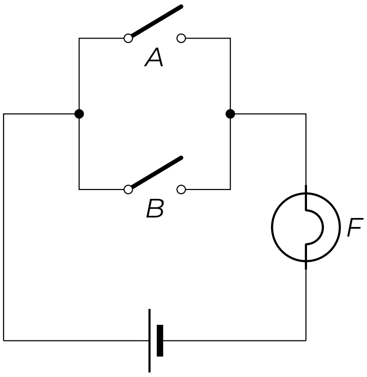


Figure A.7.4: OR - equivalent electrical circuit

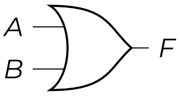


Figure A.7.5: OR - symbol

A	B	$F = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

Table A.7.2: OR - Truth Table

An OR gate can have any number of inputs with the same fact that the output is HIGH if any input is HIGH. So for instance, Figure A.7.6 shows the electric circuit representation of a 3 input OR gate with Figure A.7.7 showing the symbol and Table A.7.3 giving the truth table.

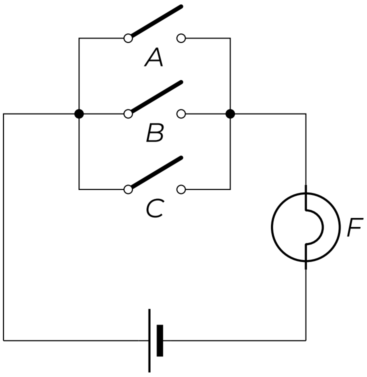


Figure A.7.6: 3 input OR - equivalent electrical circuit

A	B	C	$F = A + B + C$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Table A.7.3: 3 input OR - Truth Table

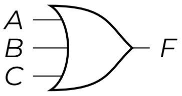


Figure A.7.7: 3 input OR - symbol

The AND gate

This is a logic function that gives a HIGH output when all of its inputs are HIGH. It can be represented in a DC circuit as switches in series as shown in Figure A.7.8 for a two input gate. The bulb F will light only if both switches are closed. Figure A.7.9 is the symbol for a two input AND gate, with Table A.7.4 giving the truth table. This shows the output F for all possible combinations of the inputs A and B where AND is represented by the symbol $\cdot$ (or a simple variable multiplication AB) in discrete maths.

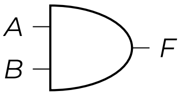


Figure A.7.9: AND - symbol

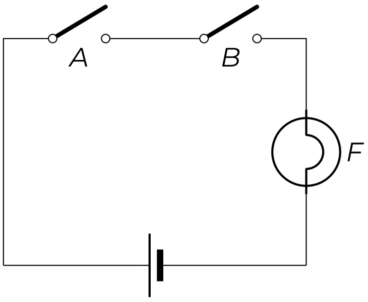


Figure A.7.8: AND - equivalent electrical circuit

A	B	$F = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

Table A.7.4: AND - Truth Table

The inverter or NOT gate

This is a logic function with one input where the output is the inverse of the input. Figure A.7.10 shows the symbol for the inverter and Table A.7.5 gives the truth table. In discrete maths a logic variable is shown to be inverted by a bar over the variable $\overline{A}$ - this is also known as the complement of A.

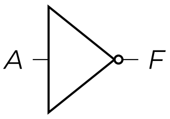


Figure A.7.10: Inverter - symbol

A	$F = \overline{A}$
0	1
1	0

Table A.7.5: Inverter - Truth Table

The NOR gate

This is a logic function that gives a HIGH output only when all its inputs are LOW which is logically equivalent to an OR gate followed by an inverter as shown in Figure A.7.11 for a two input gate. Figure A.7.12 is the symbol for a two input NOR gate, with Table A.7.6 giving the truth table. This defines the output F for all possible combinations of the inputs A and B.

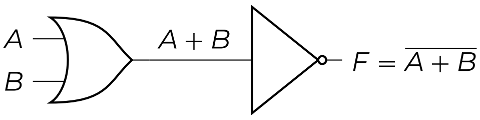


Figure A.7.11: OR & NOT in series

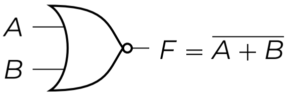


Figure A.7.12: NOR - symbol

A	B	$F = \overline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0

Table A.7.6: NOR - Truth Table

The NAND gate

This is a logic function that gives a LOW output only when all its inputs are HIGH which is logically equivalent to an OAND gate followed by an inverter as shown in Figure A.7.13 for a two input gate. Figure A.7.14 is the symbol for a two input NAND gate, with Table A.7.7 giving the truth table. This defines the output F for all possible combinations of the inputs A and B.

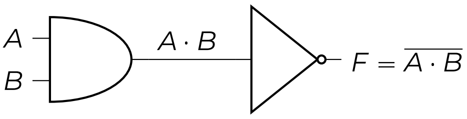


Figure A.7.13: AND & NOT in series

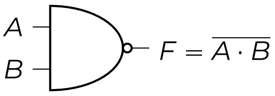


Figure A.7.14: NAND - symbol

A	B	$F = \overline{A \cdot B}$
0	0	1
0	1	1
1	0	1
1	1	0

Table A.7.7: NAND - Truth Table

Notes on logic gates

The five gates discussed above form the basis of combinatorial logic — two or more input states define one or more output states, where the resulting state or states are related by defined rules that are independent of previous states. As shown for the OR gate with a 3 input gate, all the gates except for inverter can have more the two inputs with the same rules applying:

- OR: HIGH output when any input is HIGH
- AND: HIGH output only when all inputs HIGH
- NOR: HIGH output only when all inputs are LOW
- NAND: HIGH output when any input is LOW

Logic gates form the basic building blocks for more complex digital circuits, which we can describe mathematically using Boolean algebra.

Example A.7.4: Logic to truth table exclusive OR

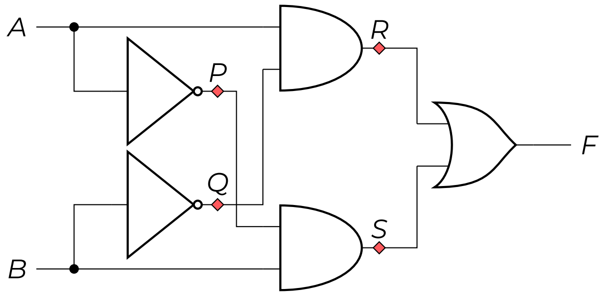


Figure A.7.15: Example 2

To construct the truth table we look at output at each intermediate state (P, Q, R and S) and formulate the output from them:

A	B	$P=\overline{A}$	$Q=\overline{B}$	$R=A\cdot\overline{B}$	$S=\overline{A}\cdot B$	F
0	0	1	1	0	0	0
0	1	1	0	0	1	1
1	0	0	1	1	0	1
1	1	0	0	0	0	0

This is in fact the output of a special gate known as the exclusive OR gate (XOR) — that is a gate that outputs a HIGH when only one and only one input is HIGH .

$$F=A\cdot\overline{B}+\overline{A}\cdot B=A\oplus B$$

Example A.7.5: Logic to truth table exclusive NOR

Construct the truth tables for the following circuit:

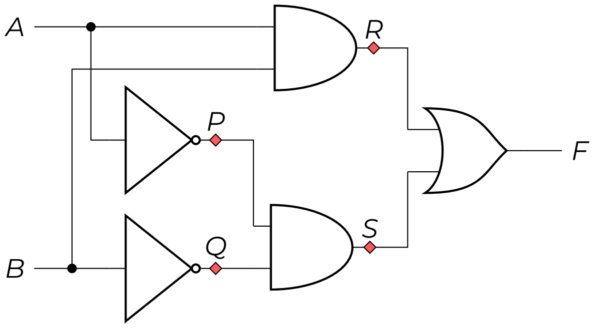


Figure A.7.16: Example 1

To construct the truth table we look at output at each intermediate state (P, Q, R and S) and formulate the output from them:

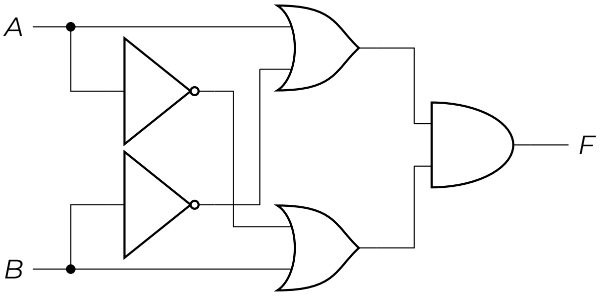
A	B	$P=\overline{A}$	$Q=\overline{B}$	$R=A\cdot B$	$S=\overline{A}\cdot\overline{B}$	F
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

This is in fact the output of a special gate known as the XNOR gate - that is a gate that outputs a HIGH when all inputs match.

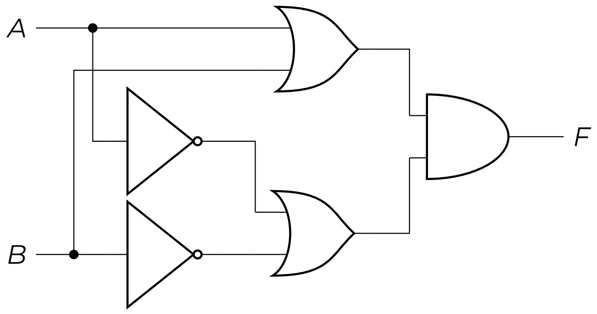
$$F=\overline{A}\cdot\overline{B}+A\cdot B=\overline{A\oplus B}$$

Questions

1. Find the truth table for the following circuit:



2. Find the truth table for the following circuit:



7.4 Boolean Algebra

Introduction

Boolean Algebra is the mathematical representation of logic circuits which developed from the concepts described by George Boole in his books Boole (1847) and Boole (1854) which effectively founded the algebraic logic discipline in mathematics. What we see as Boolean algebra today was developed further by later mathematicians. The principles are that variables can only take two values (true or false, 1 or 0) and that there are three prime operations: the conjunction or AND $(\cdot)$ which is multiplication in standard algebra), the disjunction or OR $(+)$ which is addition in standard algebra) and the negation/complement NOT (represented by bar over variable $\overline{A}$). Boolean algebra has been fundamental in the development of digital electronics, but it turns up in other areas of discrete maths such as set theory and statistics.

We have already met some basic Boolean algebra in the way we described the 5 basic logic gates above.

Laws of Boolean Algebra

There are a number of laws that apply to Boolean algebra some of which are similar to those we met in Section 3.1 for standard algebra. It is also worth noting that the AND $(\cdot)$ operation has precedence over the OE $(+)$ operation in Boolean algebra much as multiplication has higher precedence to addition. Table A.7.8 sets out the main rules for basic Boolean algebra.

To see how these laws apply it is easier to look at a number of problems and see how we can use the rules to simplify the expressions. This is done in Example A.7.6

Ref	Group	Rule or law
1	Commutative laws	$A + B = B + A$
2		$A \cdot B = B \cdot A$
3	Associative laws	$A + (B + C) = (A + B) + C$
4		$A \cdot (B \cdot C) = (A \cdot B) \cdot C$
5	Distributive laws	$A \cdot (B + C) = (A \cdot B) + (A \cdot C)$
6		$A + (B \cdot C) = (A + B) \cdot (A + C)$
7	Double negation rule	$\overline{\overline{A}} = A$
8	Sum rules	$A + 0 = A$
9		$A + 1 = 1$
10		$A + A = A$
11		$A + \overline{A} = 1$
12	Product rules	$A \cdot 0 = 0$
13		$A \cdot 1 = A$
14		$A \cdot A = A$
15		$A \cdot \overline{A} = 0$
16	Absorption rules	$A + (A \cdot B) = A$
17		$A \cdot (A + B) = A$
18		$A + (\overline{A} \cdot B) = A + B$

Table A.7.8: Laws of Boolean Algebra

Example A.7.6: Boolean problems

Simplify the expressions:

1. $\overline{A} \cdot \overline{B} + \overline{A} \cdot B + A \cdot \overline{B}$

2. $(A + \overline{A} \cdot B) \cdot (B + A \cdot \overline{B})$

3. $A \cdot B \cdot \overline{C} + A \cdot B \cdot C + \overline{A} \cdot B \cdot C$

4. $\overline{A} \cdot \overline{B} \cdot C + \overline{A} \cdot B \cdot C + A \cdot \overline{B} \cdot C + A \cdot B \cdot C$

5. $A \cdot \overline{C} + \overline{A} \cdot (B + C) + A \cdot B \cdot (C + \overline{B})$

Solutions

Applying rules from Table A.7.8

1. *Ref*

$\overline{A} \cdot \overline{B} + \overline{A} \cdot B + A \cdot \overline{B}$	
$= \overline{A} \cdot (\overline{B} + B) + A \cdot \overline{B}$	5
$= \overline{A} \cdot 1 + A \cdot \overline{B}$	11
$= \overline{A} + A \cdot \overline{B}$	13
$= \overline{A} + \overline{B}$	18

2.

	Ref
$(A+\overline{A}\cdot B)\cdot(B+A\cdot\overline{B})$	
$=A\cdot(B+A\cdot\overline{B})+\overline{A}\cdot B\cdot(B+A\cdot\overline{B})$	5
$=A\cdot B+A\cdot A\cdot\overline{B}+\overline{A}\cdot B\cdot B+\overline{A}\cdot B\cdot A\cdot\overline{B}$	5
$=A\cdot B+A\cdot\overline{B}+\overline{A}\cdot B+\overline{A}\cdot B\cdot A\cdot\overline{B}$	14
$=A\cdot B+A\cdot\overline{B}+\overline{A}\cdot B+0\cdot 0$	15
$=A\cdot(B+\overline{B})+\overline{A}\cdot B$	5
$=A\cdot 1+\overline{A}\cdot B$	11
$=A+\overline{A}\cdot B$	13
$=A+B$	18

3.

	Ref
$A\cdot B\cdot\overline{C}+A\cdot B\cdot C+\overline{A}\cdot B\cdot C$	
$=A\cdot B\cdot(\overline{C}+C)+\overline{A}\cdot B\cdot C$	5
$=A\cdot B\cdot 1+\overline{A}\cdot B\cdot C$	11
$=B\cdot(A+\overline{A}\cdot C)$	5 and 13
$=B\cdot(A+C)$	18

4.

	Ref
$\overline{A}\cdot\overline{B}\cdot C+\overline{A}\cdot B\cdot C+A\cdot\overline{B}\cdot C+A\cdot B\cdot C$	
$=\overline{B}\cdot C\cdot(\overline{A}+A)+B\cdot C\cdot(\overline{A}+A)$	5
$=\overline{B}\cdot C\cdot 1+B\cdot C\cdot 1$	11
$=C\cdot(\overline{B}+B)$	5
$=C$	11 and 13

5. *Ref*

$$\begin{aligned}
&A \cdot \overline{C} + \overline{A} \cdot (B + C) + A \cdot B \cdot (C + \overline{B}) && \\
&= A \cdot \overline{C} + \overline{A} \cdot B + \overline{A} \cdot C + A \cdot B \cdot C + A \cdot B \cdot \overline{B} && 5 \\
&= A \cdot \overline{C} + \overline{A} \cdot B + \overline{A} \cdot C + A \cdot B \cdot C + A \cdot 0 && 15 \\
&= A \cdot \overline{C} + \overline{A} \cdot B + \overline{A} \cdot C + A \cdot B \cdot C && 12 \\
&= A \cdot (\overline{C} + B \cdot C) + \overline{A} \cdot B + \overline{A} \cdot C + && 5 \\
&= A \cdot (\overline{C} + B) + \overline{A} \cdot B + \overline{A} \cdot C && 18 \\
&= A \cdot \overline{C} + A \cdot B + \overline{A} \cdot B + \overline{A} \cdot C && 5 \\
&= A \cdot \overline{C} + B \cdot (A + \overline{A}) + \overline{A} \cdot C && 5 \\
&= A \cdot \overline{C} + B + \overline{A} \cdot C && 11 \text{ and } 13
\end{aligned}$$

De Morgan’s Laws

These are laws that relate NAND and NOR operations to NOT AND and NOT OR. They state that

$$\overline{(A + B)} = \overline{A} \cdot \overline{B} \tag{A.7.1}$$

$$\overline{(A \cdot B)} = \overline{A} + \overline{B} \tag{A.7.2}$$

We can verify these through a truth table. So for instance for equation A.7.1, in Table A.7.9, columns 4 ($\overline{(A + B)}$) and column 7 ($\overline{A} \cdot \overline{B}$) are identical:

A	B	$A + B$	$\overline{A + B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cdot \overline{B}$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Table A.7.9: Truth Table for $\overline{(A + B)} = \overline{A} \cdot \overline{B}$

Now we can simplify a couple of problems using De Morgan’s laws and the rules of Boolean algebra in Table A.7.8

Example A.7.7: Boolean simplification

1. Simplify $\overline{(\overline{A}\cdot B)}+\overline{(\overline{A}+B)}$
2. Simplify $\overline{(A\cdot\overline{B}+C)}\cdot(\overline{A}+\overline{\overline{B}\cdot C})$

Solutions

1. $\overline{(\overline{A}\cdot B)}+\overline{(\overline{A}+B)}$
   Applying De Morgan's laws to first term:

   $$\overline{(\overline{A}\cdot B)}=\overline{\overline{A}}+\overline{B}=A+\overline{B} \qquad \text{from rule 7 in Table A.7.8}$$

   Applying De Morgan's laws to second term:

   $$\overline{(\overline{A}+B)}=\overline{\overline{A}}\cdot\overline{B}=A\cdot\overline{B} \qquad \text{from rule 7 in Table A.7.8}$$

   Thus full expression is:

   $$\overline{(\overline{A}\cdot B)}+\overline{(\overline{A}+B)}=(A+\overline{B})+A\cdot\overline{B}$$

   Removing bracket and re-ordering gives:

   $$A+A\cdot\overline{B}+\overline{B}$$

   Now by rule 16 in Table A.7.8 $A+A\cdot\overline{B}=A$ as doesn't matter what value of B is. Thus:

   $$\overline{(\overline{A}\cdot B)}+\overline{(\overline{A}+B)}=A+\overline{B}$$

2. $\overline{(A\cdot\overline{B}+C)}\cdot(\overline{A}+\overline{\overline{B}\cdot C})$
   Applying De Morgan's laws to first term:

   $$\begin{aligned}\overline{(A\cdot\overline{B}+C)}&=\overline{A\cdot\overline{B}}\cdot\overline{C}=(\overline{A}+\overline{\overline{B}})\cdot\overline{C}\\&=(\overline{A}+B)\cdot\overline{C}=\overline{A}\cdot\overline{C}+B\cdot\overline{C}\end{aligned}$$

   Applying De Morgan's laws to second term:

   $$\overline{A}+\overline{\overline{B}\cdot C}=\overline{A}+(\overline{\overline{B}}+\overline{C})=\overline{A}+(B+\overline{C})$$

   Thus full expression is:

   $$\begin{aligned}&\overline{(A\cdot\overline{B}+C)}\cdot(\overline{A}+\overline{\overline{B}\cdot C})\\&=(\overline{A}\cdot\overline{C}+B\cdot\overline{C})\cdot(\overline{A}+B+\overline{C})\\&\qquad=\overline{A}\cdot\overline{A}\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}+B\cdot\overline{A}\cdot\overline{C}+B\cdot B\cdot\overline{C}+\overline{C}\cdot\overline{A}\cdot\overline{C}+\overline{C}\cdot B\cdot\overline{C}\end{aligned}$$

   But rule 14 in table A.7.8 means that $\overline{A}\cdot\overline{A}=\overline{A}$, $B\cdot B=B$ and $\overline{C}\cdot\overline{C}=\overline{C}$

   Thus the Boolean expression becomes using rule 11:

   $$\begin{aligned}&\overline{A}\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}+B\cdot\overline{C}+\overline{A}\cdot\overline{C}+B\cdot\overline{C}\\&\quad=\overline{A}\cdot\overline{C}+\overline{A}\cdot B\cdot\overline{C}+B\cdot\overline{C}\\&\quad=\overline{C}\cdot(\overline{A}+\overline{A}\cdot B+B)=\overline{C}\cdot(\overline{A}+B)\end{aligned}$$

   Thus: $\overline{(A\cdot\overline{B}+C)}\cdot(\overline{A}+\overline{\overline{B}\cdot C})=\overline{C}\cdot(\overline{A}+B)$

Questions

Using the laws in Table A.7.8 and De Morgan's laws simplify the following Boolean expressions:

1. $\overline{A} \cdot \overline{B} + \overline{A} \cdot B$
2. $\overline{A} \cdot \overline{B} + \overline{A} \cdot B + A \cdot B$
3. $\overline{A} \cdot \overline{B} + A \cdot B + A \cdot \overline{B} + \overline{B} \cdot (A + \overline{A})$
4. $(A + A \cdot B) \cdot (B + A \cdot B)$
5. $\overline{A} \cdot \overline{B} \cdot C + \overline{A} \cdot B \cdot C + A \cdot \overline{B} \cdot C$
6. $\overline{A} \cdot \overline{B} \cdot \overline{C} + \overline{A} \cdot \overline{B} \cdot C + A \cdot \overline{B} \cdot \overline{C} + A \cdot \overline{B} \cdot C$
7. $\overline{C} \cdot (A \cdot B + A \cdot \overline{B}) + \overline{C} \cdot (\overline{A} \cdot \overline{B} + \overline{A} \cdot B)$
8. $(\overline{A} \cdot \overline{B}) \cdot (\overline{\overline{A} \cdot B})$
9. $(A + \overline{B \cdot C}) + (\overline{A \cdot B} + C)$
10. $(\overline{\overline{A} \cdot B + B \cdot \overline{C}}) \cdot \overline{A \cdot \overline{B}}$
11. $(\overline{A \cdot \overline{B}} + \overline{B \cdot \overline{C}}) + (\overline{\overline{A} \cdot B})$
12. $(\overline{A \cdot \overline{B}} + \overline{\overline{A} \cdot C}) \cdot (\overline{A} \cdot \overline{B \cdot C})$

7.5 Summary

This section has given an overview of set theory followed by a brief overview of basic logic gates along with an explore of Boolean algebra — especially the laws and rules that apply when you are doing discrete maths rather than the more usual continuous maths. You will find you do a lot more on Boolean and optimisation in Digital Electronics where you will cover Karnaugh maps and also sequential logic.

Number representation

8.1 Introduction

This chapter looks at how we can represent numbers in a format that can be understood by digital systems — that is in a binary form. We start off with looking at how positive and negative numbers can be represented using signed numbers and then we look at floating point representation for very small, very large and non-integer numbers.

8.2 Signed numbers

Digital systems such as computers need to be able to handle both positive and negative numbers — this is where signed binary numbers with both sign and magnitude information are used. There are three forms of representation for signed integer numbers in binary which are sign-magnitude (least used), 1's complement and 2's complement (most used).

Sign Bit

With signed binary numbers, the MSB (i.e. the left-hand most bit) is the *sign bit* which tells you whether the number is positive or negative.

> Positive numbers are indicated by a *0* in the sign bit.
> Negative numbers are indicated by a *1* in the sign bit.

Representation of signed numbers

As stated above there are three ways that signed numbers can be represented:

1. Sign-magnitude form where the MSB represents the sign and the remaining bits are the magnitude bits in true binary form. So the only difference between +25 and -25 is the sign bit.
2. 1's complement form where a positive number is represented as in sign-magnitude form and a negative number is represented by the 1's complement of the corresponding positive number.
3. 2's complement form where again a positive number is represented as in sign-magnitude form and a negative number is represented by the 2's complement of the corresponding positive number.

As an example let us look at representing +35 and -35 in the three forms using 8-bit signed number:

Example A.8.1

For all forms +35 is expressed as:

$$\underset{\text{Sign bit}}{0}\underbrace{0100011}_{\text{Magnitude bits}}$$

For sign-magnitude form -35 is expressed as:

$$10100011$$

For 1's complement form -35 is expressed as:

$$11011100$$

For 2's complement form -35 is expressed as:

$$11011101$$

So the rules for representing negative signed numbers are:

> *Sign-magnitude form*: The negative number is represented by the same magnitude bits as the corresponding positive number with a sign bit of 1.
> *1's complement form*: The negative number is represented by the 1's complement of the corresponding positive number.
> *2's complement form*: The negative number is represented by the 2's complement of the corresponding positive number

Engineering application A.8.1: Computers & signed numbers

The 2's complement form of signed numbers is used for arithmetic operations in computers due to the fact that subtracting a binary number is the same as adding the 2's complement of the binary number.

Signed number to decimal conversion

Sign-magnitude

Conversion of sign-magnitude form binary numbers to their decimal equivalent is simple — you convert the magnitude part from binary to the equivalent decimal value (see Section 2.3) and then use the sign bit to determine whether the number is positive of negative as shown in Example A.8.2.

Example A.8.2

Find the decimal value of the following signed binary numbers expressed in sign-magnitude format:

a) 01000111 b) 11100101

Solution

a) Seven magnitude bits in terms of their powers of 2 are:

$$\begin{array}{ccccccc} 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

Summing the values to get decimal magnitude value:

$$64 + 4 + 2 + 1 = 71$$

Sign bit is 0; therefore answer is **+71**

b) Seven magnitude bits in terms of their powers of 2 are:

$$\begin{array}{ccccccc} 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{array}$$

Summing the values to get decimal magnitude value:

$$64 + 32 + 4 + 1 = 101$$

Sign bit is 1; therefore answer is **-101**

1's complement

Conversion of 1's complement form of signed binary numbers is simply using the binary to decimal conversion discussed in Section 2.3 for positive numbers (sign bit 0). When the sign bit is 1 (negative number), then you need to make the sign bit weight negative, sum the powers of 2 and then add 1 to the answer to get correct decimal value. Example A.8.3 shows this in action:

Example A.8.3

Find the decimal value of the following signed binary numbers expressed in 1's complement format:

a) 01010101 b) 10101010

Solution

a) The bits in terms of their powers of 2:

$$\begin{array}{cccccccc} -2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

Summing the values to get decimal value:

$$64 + 16 + 4 + 1 = \mathbf{+75}$$

b) Seven magnitude bits in terms of their powers of 2 are:

-2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	1	0	1	0	1	0

Summing the values to get decimal value:

$$-128 + 32 + 8 + 2 = -76$$

Adding 1 to the answer to get negative value

$$-76 + 1 = \mathbf{-75}$$

2's complement

Conversion of 2's complement form of signed binary numbers is simply using the binary to decimal conversion discussed in Section 2.3 with the weight of the sign bit being a negative number. Example A.8.4 shows this in action:

Example A.8.4

Find the decimal value of the following signed binary numbers expressed in 2's complement format:

a) 01101010 b) 10010110

Solution

a) The bits in terms of their powers of 2:

-2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
0	1	1	0	1	0	1	0

Summing the values to get decimal value:

$$64 + 32 + 8 + 2 = \mathbf{+106}$$

b) Seven magnitude bits in terms of their powers of 2 are:

-2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
1	0	0	1	0	1	1	0

Summing the values to get decimal value:

$$-128 + 16 + 4 + 2 = \mathbf{-106}$$

You can see from the above examples one of the reasons why 2's complement is preferred form for signed numbers as whether the number is positive or negative the decimal value is simply found by summing together the appropriate powers of 2.

Range of signed integer numbers

The examples of signed numbers have all been 8 bits (so a byte) which means you can uniquely represent 256 (2^8) numbers. With 16 bits (2 bytes) you can uniquely represent 65536 (2^{16}) numbers and with 32 bits or 4 bytes you can uniquely represent 4294967296 (2^{32}) numbers. The number of unique numbers you can represent with n bits is given by (2^n).

For 2's complement signed integers using n bits there is 1 sign bit and $n - 1$ magnitude bits — this means that the range of integer numbers that can be represented is given in equation A.8.1

$$\text{Range} = -(2^{(n-1)})\text{to} + (2^{(n-1)} - 1) \qquad \text{(A.8.1)}$$

So for instance with 8 bits you can represent values from $-2^7 = -128$ (in binary 10000000) to $(2^7 - 1) = +127$ (in binary 01111111). There is one less positive number than negative number as zero is represented as a positive number with all zeros (00000000 for 1 byte). With 2 bytes the range is -32768 to $+32767$ and so on.

Arithmetic with signed numbers

As 2's complement is the most widely used form of signed integer numbers within processors and microprocessors, this is going to be solely about addition, subtraction, multiplication & division using this form of signed integers.

Addition

In an addition of two numbers there are four cases that can occur when using signed binary numbers:

1. Both numbers are positive (sign bits both 0) — leads to a positive number answer.
2. Positive number is larger than the negative number — leads to a positive number answer.
3. Negative number is larger than the positive number — leads to a negative number answer (so answer in 2's complement form).
4. Both numbers are negative (sign bits both 1) — leads to a negative number answer (so answer in 2's complement form).

Example A.8.5

Two +ve numbers:

$$\begin{array}{ccccccccccc}
 & 0&0&1&1&0&0&1&1 & \quad & 51 \\
+ & 0&0&1&1&0&1&0&1 & & +53 \\
 & &\scriptstyle 1&\scriptstyle 1& &\scriptstyle 1&\scriptstyle 1&\scriptstyle 1& & & \\
\cline{2-9} \cline{11-11}
 & 0&1&1&0&1&0&0&0 & & 104
\end{array}$$

The sum is +ve and therefore in true binary

Addition of two +ve numbers results in a +ve number

+ve number greater than -ve number:

$$\begin{array}{ccccccccccc}
 & 0&0&1&1&0&0&1&1 & \quad & 51 \\
+ & 1&1&1&0&1&1&0&0 & & +(-20) \\
\scriptstyle 1 &\scriptstyle 1&\scriptstyle 1& & & & & & & & \\
\cline{2-9} \cline{11-11}
\color{red}{1} & 0&0&0&1&1&1&1&1 & & 31
\end{array}$$

The sum is +ve and therefore in true binary

Addition of +ve number with smaller -ve number results in smaller +ve number

-ve number greater than +ve number:

$$\begin{array}{ccccccccccc}
 & 0&0&0&1&1&1&1&1 & \quad & 31 \\
+ & 1&1&0&0&1&1&0&1 & & +(-51) \\
 & & &\scriptstyle 1&\scriptstyle 1&\scriptstyle 1&\scriptstyle 1&\scriptstyle 1& & & \\
\cline{2-9} \cline{11-11}
 & 1&1&1&0&1&1&0&0 & & -20
\end{array}$$

The sum is -ve & therefore in 2's complement

Addition of +ve number with larger -ve number results in smaller -ve number

2 -ve numbers :

$$\begin{array}{ccccccccccc}
 & 1&1&0&0&1&1&0&1 & \quad & -51 \\
+ & 1&1&0&0&1&0&1&1 & & +(-53) \\
\scriptstyle 1 &\scriptstyle 1& & &\scriptstyle 1&\scriptstyle 1&\scriptstyle 1&\scriptstyle 1& & & \\
\cline{2-9} \cline{11-11}
\color{red}{1} & 1&0&0&1&1&0&0&0 & & -104
\end{array}$$

The sum is -ve & therefore in 2's complement

Addition of 2 -ve numbers results in a -ve number

Discard any final carry bits over 8 bits

As can be seen in example A.8.5 with negative numbers in 2's complement form we discard any final carry bits. So general rule for addition process is

Add two numbers and discard any final carry bit when negative numbers are involved.

What about when we add two positive 8 bit numbers together and result is greater than 127? In other words the number required to represent the sum exceeds the number of bits. This is known as an *overflow* and only occurs when we are adding two positive or two negative numbers together. The computer will check the sign bit of result and if it differs from the sign bit of original two numbers an overflow is indicated as what happens on addition is there is a carry into the sign bit.

If you are adding multiple numbers together the principle is to add the first pair together, then add the next number to the result of the first addition and so on. This is how computers add together strings of numbers.

Subtraction

This is in fact a special case of addition as subtracting 5 from 10 is the same as adding −5 to 10 - the result of a subtraction is known as the *difference*. So in order to subtract one signed number from another a computer will take the 2's complement of the number being subtracted (subtrahend) and add it to the number being subtracted from (minuend). The general rule is

> *Take the 2's complement of the subtrahend and add — discarding any final carry bit.*

Example A.8.6

Find the results of each of the following signed number subtractions:

a) 00001010 – 00000101

b) 00001010 – 11111011

c) 11110110 – 00000101

d) 11110110 – 11111011

Solution

As with addition the decimal subtractions are given for reference

a) This is $10 - 5 = 10 + (-5) = 5$

$$\begin{array}{r@{}cccccccc@{\qquad}l}
 & 0&0&0&0&1&0&1&0 & \text{minuend } (+10)\\
+ & 1&1&1&1&1&0&1&1 & \text{2's complement of subtrahend } (-5)\\
{\scriptstyle 1} & {\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}& &{\scriptstyle 1}& & & \\ \cline{2-9}
1 & 0&0&0&0&0&1&0&1 & \text{difference } (+5)
\end{array}$$

b) This is $10 - (-5) = 10 + (+5) = 15$

$$\begin{array}{r@{}cccccccc@{\qquad}l}
 & 0&0&0&0&1&0&1&0 & \text{minuend } (+10)\\
+ & 0&0&0&0&0&1&0&1 & \text{2's complement of subtrahend } (+5)\\ \cline{2-9}
 & 0&0&0&0&1&1&1&1 & \text{difference } (+15)
\end{array}$$

c) This is $-10 - 5 = -10 + (-5) = -15$

$$\begin{array}{r@{}cccccccc@{\qquad}l}
 & 1&1&1&1&0&1&1&0 & \text{minuend } (-10)\\
+ & 1&1&1&1&1&0&1&1 & \text{2's complement of subtrahend } (-5)\\
{\scriptstyle 1} & {\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}&{\scriptstyle 1}& & & \\ \cline{2-9}
1 & 1&1&1&1&0&0&0&1 & \text{difference } (-15)
\end{array}$$

d) This is $-10 - 5 = -10 + (-5) = -15$

$$\begin{array}{r@{}cccccccc@{\qquad}l}
 & 1&1&1&1&0&1&1&0 & \text{minuend } (-10)\\
+ & 0&0&0&0&0&1&0&1 & \text{2's complement of subtrahend } (+5)\\
 & & & & &{\scriptstyle 1}& & & & \\ \cline{2-9}
 & 1&1&1&1&1&0&1&1 & \text{difference } (-5)
\end{array}$$

Discard any final carry bit

Multiplication

As with subtraction there are names for the numbers in a multiplication — the number of top line is the *multiplicand*, which is multiplied by the *multiplier* with the result being the *product*. In computers there are two common methods used to implement the multiplication: *direct addition* and *partial products*.

Direct addition is simply adding the multiplicand to itself by the number of times equal to the multiplier. So if the multiplier is 00000011 which is 5 in decimal the multiplicand is added to itself 3 times — so if multiplicand is 00000100 then the product is given by 00000100 + 00000100 + 00000100 = 00001000 + 00000100 = 00001100. This gets complicated quickly as the multiplier increases in value — for instance for the sum 25 × 20 you would have to add 25 to itself 20 times.

Partial products is much more commonly used as it basically replicates long multiplication for denary numbers as discussed in Section 2.2 where each time the multiplicand is multiplied by a digit in the multiplier a partial product is result — to get product you simply add the partial products together.

With signed numbers the sign of a product depends on the signs of the multiplicand and multiplier according to the following rules:

- The product is positive if both signs are the same.
- The product is negative if the signs are different.

The steps in multiplication of binary signed numbers are:

Step 1: Determine the sign of the product using the rules defined above.
Step 2: Change any negative numbers to the true un-complemented binary form
Step 3: Using only magnitude bits, starting with the LSB in the multiplier generate the partial products — if multiplier bit is 1 then partial product is the multiplicand whereas for a multiplier bit of 0 the partial product is 0
Step 4: Add each partial product to the sum of the previous partial products to get the result
Step 5: If sign of product determined in step 1 is negative then take the 2's complement of the result to get the final result. If the sign was determined to be positive then leave the product in true binary form. Then attach the sign bit to the result.

Example A.8.7

Calculate the result of 01100100 × 11001110

Solution:

Step 1: The sign bits differ — multiplicand is 0 and multiplier is 1 — so the sign bit of the product will be 1 (negative).

Step 2: Take the 2's complement of the multiplier to make it true binary:

$$11001110 \longrightarrow 00110010$$

Steps 3 & 4: The multiplication proceeds as follows:

1100100	Multiplicand
×0110010	Multiplier
0000000	1st partial product
+1100100	2nd partial product
11001000 0	Sum of 1st & 2nd
+0000000	3rd partial product
011001000	Sum
+0000000	4th partial product
0011001000	Sum
+1100100	5th partial product
1 1	
11100001000	Sum
+1100100	6th partial product
1 1	
1001110001000	Sum
+0000000	7th partial product
1001110001000	Final product

Step 5: Since the sign bit was determined to be 1, take the 2's complement of the product and add the sign bit

1001110001000 ⟶ 0110001111000

Attach sign bit

1 0110001111000

Division

The numbers is a division are the number on top line (numerator) which is the *dividend*, the *divisor* which is number on bottom line (denominator) with the result being the *quotient*. In computers, division is done using subtraction - in fact by addition of 2's complement.

The *quotient* is the number of times the divisor will go into the dividend — or in other words the number of times the divisor can be subtracted from the dividend before result becomes 0 or negative. With signed numbers the sign of a quotient depends on the signs of the dividend and divisor according to the same rules as for multiplication:

- The quotient is positive if both signs are the same.
- The quotient is negative if the signs are different.

The steps in division of binary signed numbers are:

Step 1: Determine the sign of the quotient using the rules defined above. Initiate value of the quotient to zero.

Step 2: Subtract the divisor from the dividend using 2's complement addition (so 2's complement a positive divisor and use the negative divisor as is) to get first partial remainder and add 1 to the quotient. If this partial remainder is positive go to step 3 otherwise division is complete if remainder is zero or negative (sign bit 1) and you go to step 4.

Step 3: Subtract divisor from partial remainder and add 1 to the quotient — if new partial remainder is positive repeat this step for next partial remainder. Otherwise division is complete if remainder is zero or negative (sign bit 1) and you go to step 4.

Step 4: Count up the number of times you have subtracted the divisor and this is the final quotient (check value is correct). If sign of quotient determined in step 1 is negative then take the 2's complement of the result to get the final result. If the sign was determined to be positive then leave the quotient in true binary form. If you final partial remainder is negative you can 2's complement it to find out the remainder of the division if required.

It is worth noting that is the divisor is a negative number (sign bit of 1) then you can just use that value for the 2's complement addition in steps 2 & 3. If the divisor is positive (sign bit of 0) then you will need to 2's complement it before doing the 2's complement addition in steps 2 & 3. Similarly the dividend needs to be in true binary form — so if it is negative (sign bit of 1) you will need to 2's complement it before starting on step 2.

Example A.8.8

Calculate the result of 01100100 ÷ 11100111

Solution:

Step 1: The sign bits differ — dividend is 0 and divisor is 1 — so the sign bit of the quotient will be 1 (negative). Set value of quotient to be 0000000.

Step 2 Subtract true binary form of divisor from dividend using 2's complement addition (final carries are discarded)

01100100	Dividend
+11100111	Divisor (already negative so leave as is)
1 11 1	
01001011	Positive 1st partial remainder

Add 1 to quotient: 00000000 + 00000001 = 00000001

Step 3 Subtract true binary form of divisor from 1st partial remainder using 2's complement addition

01001011	1st partial remainder
+11100111	Divisor (already negative so leave as is)
1 1 1111	
00110010	Positive 2nd partial remainder

Add 1 to quotient: 00000001 + 00000001 = 00000010

Step 3r Subtract true binary form of divisor from 2nd partial remainder using 2's complement addition

00110010	2nd partial remainder
+11100111	Divisor (already negative so leave as is)
1 11 11	
00011001	Positive 3rd partial remainder

Add 1 to quotient: 00000010 + 00000001 = 00000011

Step 3r Subtract true binary form of divisor from 3rd partial remainder using 2's complement addition

00011001	3rd partial remainder
+11100111	Divisor (already negative so leave as is)
1 1111111	
00000000	Zero remainder

Add 1 to quotient: 00000011 + 00000001 = 00000100 — final quotient as process is complete.

Step 4: Since the sign bit was determined to be 1, take the 2's complement of the product to give final answer

00000100 ⟶ **11111100**

Questions

1. Convert the following 2's complement form signed binary numbers to decimal:

a) 00001011_2
b) 10010110_2
c) 10001011_2
d) 01100101_2
e) 10101001_2
f) 11001100_2
g) 01000111_2
h) 00100101_2

2. Convert the following decimal numbers to i) sign-magnitude, ii) 1's complement & iii) 2's complement form signed binary:

a) -13_{10}

b) 13_{10}

c) 24_{10}

d) -24_{10}

e) 56_{10}

f) -56_{10}

g) 117_{10}

h) -117_{10}

3. Do the following signed binary (2's complement) sums:

a) $1110_2 + 0100_2$

b) $1010_2 + 1100_2$

c) $00110110_2 + 11001100_2$

d) $10101010_2 + 01100110_2$

4. Do the following signed binary (2's complement) sums:

a) $1110_2 - 0100_2$

b) $1010_2 - 1100_2$

c) $00110110_2 - 11001100_2$

d) $10101010_2 - 01100110_2$

5. Do the following signed binary (2's complement) sums:

a) $0011_2 \times 0010_2$

b) $0101_2 \times 1110_2$

c) $00110110_2 \times 11001100_2$

d) $10101010_2 \times 01100110_2$

6. Do the following signed binary (2's complement) sums:

a) $0110_2 \div 0010_2$

b) $0110_2 \div 1101_2$

c) $01100110_2 \div 00110010_2$

d) $01100110_2 \div 11100111_2$

8.3 Floating point numbers

Basics

To represent very large *integer* (whole) numbers using standard binary many bits are required. Similarly representing numbers with integer and fractional parts such as 23.45678 is also a problem with the binary number representation we have met so far. This is where the floating point number system, which is based on scientific notation, is used — it is a way of representing large numbers or numbers with fractional components without an increase in number of bits.

A floating point number (or real number) is made up of two parts plus a sign. The part of the floating point number that represents the magnitude between 0 and 1 is call the *mantissa* or *significand*. The other part of the floating point number, called the *exponent*, represents the number of places that the decimal (or binary) point is moved. So considering 23.4678 — the significand is .234678 and the exponent is 2. Considering a larger number, 432567891 — the significand is .432567891 and the exponent is 9 and the number is represented as

$$0.432567891 \times 10^9$$

The IEEE Standard 754-2008 [*] defines the formats for binary floating point numbers in three basic forms: 32 bits (*single-precision*), 64 bits (*double-precision*) and 128 bits (*quad-precision*). The same standard also defines decimal format floating point numbers in 64 and 128 bits forms. The set of numbers representable within all formats is defined by the following integer parameters:

- b is the radix or base which is either 2 (binary) or 10 (decimal).
- p is the precision — the number of digits in significand
- $emax$ is the maximum exponent e
- $emin = 1 - emax$ is the minimum exponent

Table A.8.1 defines these parameters for each format. The general format for a floating point number is:

- Signed numbers of the form $(-1)^s \times b^e \times m$.
  - Sign bit s is 0 or 1.
  - Exponent e is any integer $emin \leqslant e \leqslant emax$
  - Mantissa/significand m is number of form $d_0.d_1.d_2...d_{p-1}$ where d_i is integer digit $0 \leqslant d_i < b$ which means $(0 \leqslant m < b)$.
- Two infinities $+\infty$ and $-\infty$
- Two not a number (NAN) values — the qNAN and sNAN

	Binary format $b = 2$			**Decimal format** $b = 10$	
parameter	**bin32**	**bin64**	**bin128**	**dec64**	**dec128**
p, digits	24	53	113	16	24
$emax$	+127	+1023	+16383	+384	+6144

Table A.8.1: Parameters for basic floating point formats

For any format, b^{emin} represents the smallest magnitude of a floating point number that can be represented by the full precision — numbers with magnitudes equal to or greater than b^{emin} are known as *normal* numbers. If the magnitude of a number is less than b^{emin} then it is known as a *subnormal* number. Usually with floating point numbers the leading zeros in a significand are removed so $0.0123 = 1.23 \times 10^{-2}$ but with *subnormal* numbers the leading zeros are left once the minimum exponent value has been reached.

Binary formats

In binary formats each floating point number has a unique value. For all formats the format is encoded using three fields:

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[*] found at *IEEE Standard for Floating-Point Arithmetic* 2008

1. Sign bit S - single bit that is 0 for positive numbers and 1 for negative numbers.
2. Exponent field $E = e + bias$ which is w bits in length where $w = length - p$. The *bias* is the maximum value of exponent (*emax*) which allows the minimum ($1 - emax$) to maximum exponent (in fact $emax + 1$) values to be represented with no need for a separate sign bit. The presence of *subnormal* numbers is always represented by an exponent of 0.
3. Trailing significand field $T = d_1d_2\dots d_(p - 1)$ as the leading bit d_0 is implicitly always 1 for *normal* numbers given the way floating point numbers are formed. This field is $p - 1$ bits in length

As an example a *single-precision* floating point binary number is 32 bits in total in length. Table A.8.1 says $p = 24$ so $w = 32 - 24 = 8$. The format of this type is shown in Figure A.8.1.

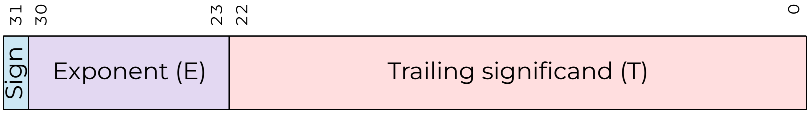


Figure A.8.1: Single-precision binary floating point number

As an example let us express 1010010111000011 as a floating point number. first off we need to express it as 1 plus a fractional binary number times by the appropriate power of 2. in this case this means moving the decimal point 15 places to the left such that:

$$1.010010111000011 \times 2^{15}$$

As this is a positive number the sign bit is 0. The exponent (e) is 15 which needs to be converted to the biased exponent (E) by adding 127 to it so $E = 15 + 127 = 142 = 10001110_2$. The trailing significand is created from the fractional part with zeros used to fill in remaining bits to create a 23 bit number ($T = 01001011100001100000000$ so the complete floating point number is shown in Figure A.8.2

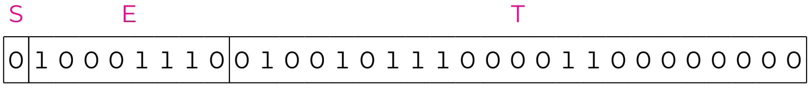


Figure A.8.2: Example binary floating point number

As a second example let us express -134.8515625 as a floating point number. First off we need to express the absolute value 134.8515625 as a binary number. The part before decimal point is easy as that is $134_{10} = 10000110_2$. To do bit after the decimal point need to use 2^{-n} and repeatedly subtract till we get to 0.

$2^{-1} = 0.5$	$0.8515625 > 0.5$	
	$\therefore 0.8515625 - 0.5 = 0.3515625$	.1
$2^{-2} = 0.25$	$0.3515625 > 0.25$	
	$\therefore 0.3515625 - 0.25 = 0.1015625$	.11
$2^{-3} = 0.125$	$0.1015625 < 0.125$	
	$\therefore 0$	.110
$2^{-4} = 0.0625$	$0.1015625 > 0.0625$	
	$\therefore 0.1015625 - 0.0625 = 0.0390625$	.1101
$2^{-5} = 0.03125$	$0.0390625 > 0.03125$	
	$\therefore 0.0390625 - 0.03125 = 0.0078125$	.11011
$2^{-6} = 0.015625$	$0.0078125 < 0.015625$	
	$\therefore 0$	.110110
$2^{-7} = 0.0078125$	$0.0078125 = 0.0078125$	
	$\therefore 0.0078125 - 0.0078125 = 0$	.1101101

So this means $134.8515625_{10} = 1000110.1101101_2$. As above we need to convert this to 1 plus a fractional binary number times by the appropriate power of 2 (base 2 scientific notation). In this case this means moving the decimal point 6 places to the left such that:

$$1.0001101101101 \times 2^{16}$$

As this is a negative number the sign bit is 1. The exponent (e) is 6 which needs to be converted to the biased exponent (E) by adding 127 to it so $E = 6 + 127 = 133 = 10000101_2$. The trailing significand is created from the fractional part with zeros used to fill in remaining bits to create a 23 bit number ($T = 00011011011010000000000$ so the complete floating point number is shown in Figure A.8.3

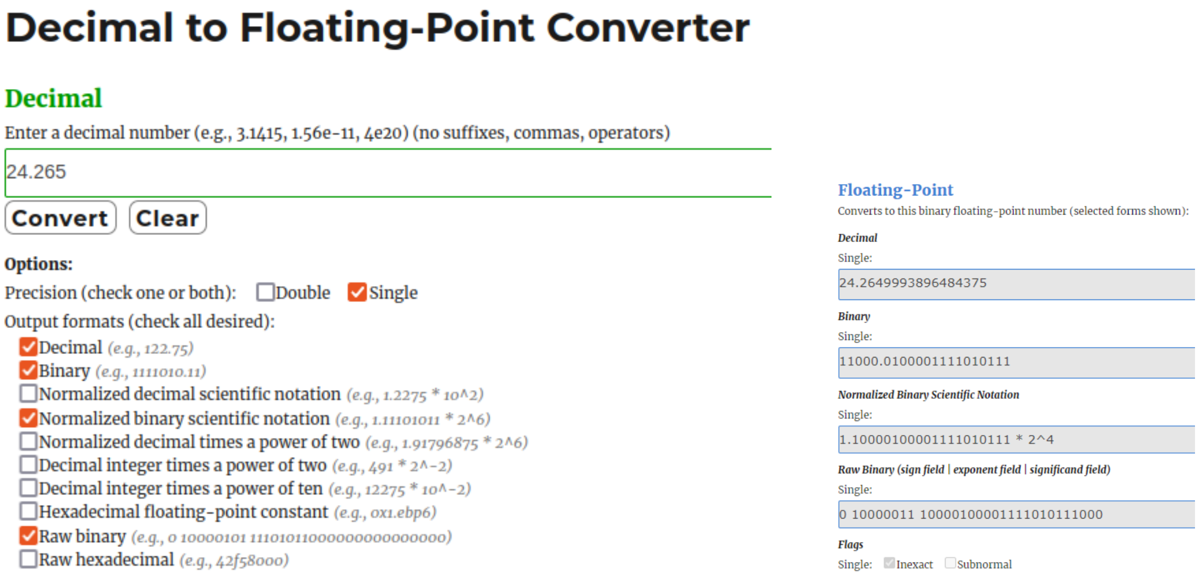
Figure A.8.3: Example binary floating point number for -134.8515625

In order to work out the binary floating point representation of floating point numbers which do not exactly convert to binary — so for example 24.265 where 0.265 is not an exact binary value we can keep going as we did above till we run out of digits, but in fact there are converters online that will do this for us.

Two such tools are:

- `https://www.exploringbinary.com/floating-point-converter/`
- `https://www.h-schmidt.net/FloatConverter/IEEE754.html`

The first tool allows you to convert a decimal number to various representations — the example shown in Figure A.8.4 shows the single precision (32 bit) conversion of 24.265 to decimal, binary, normalised binary and finally 'raw binary' which is the value stored by computer in a `float` variable. The second tool simply converts a decimal number to its single precision binary floating point representation and tells you the error as shown in Figure A.8.5.



Decimal to Floating-Point Converter

Decimal

Enter a decimal number (e.g., 3.1415, 1.56e-11, 4e20) (no suffixes, commas, operators)

24.265

Options:

- Precision (check one or both): ☐Double ☑Single
- Output formats (check all desired):
  - ☑Decimal *(e.g., 122.75)*
  - ☑Binary *(e.g., 1111010.11)*
  - ☐Normalized decimal scientific notation *(e.g., 1.2275 * 10^2)*
  - ☑Normalized binary scientific notation *(e.g., 1.11101011 * 2^6)*
  - ☐Normalized decimal times a power of two *(e.g., 1.91796875 * 2^6)*
  - ☐Decimal integer times a power of two *(e.g., 491 * 2^-2)*
  - ☐Decimal integer times a power of ten *(e.g., 12275 * 10^-2)*
  - ☐Hexadecimal floating-point constant *(e.g., 0x1.ebp6)*
  - ☑Raw binary *(e.g., 0 10000101 11101011000000000000000)*
  - ☐Raw hexadecimal *(e.g., 42f58000)*

Floating-Point

Converts to this binary floating-point number (selected forms shown):

Decimal

Single: 24.2649993896484375

Binary

Single: 11000.0100001111010111

Normalized Binary Scientific Notation

Single: 1.10000100001111010111 * 2^4

Raw Binary (sign field | exponent field | significand field)

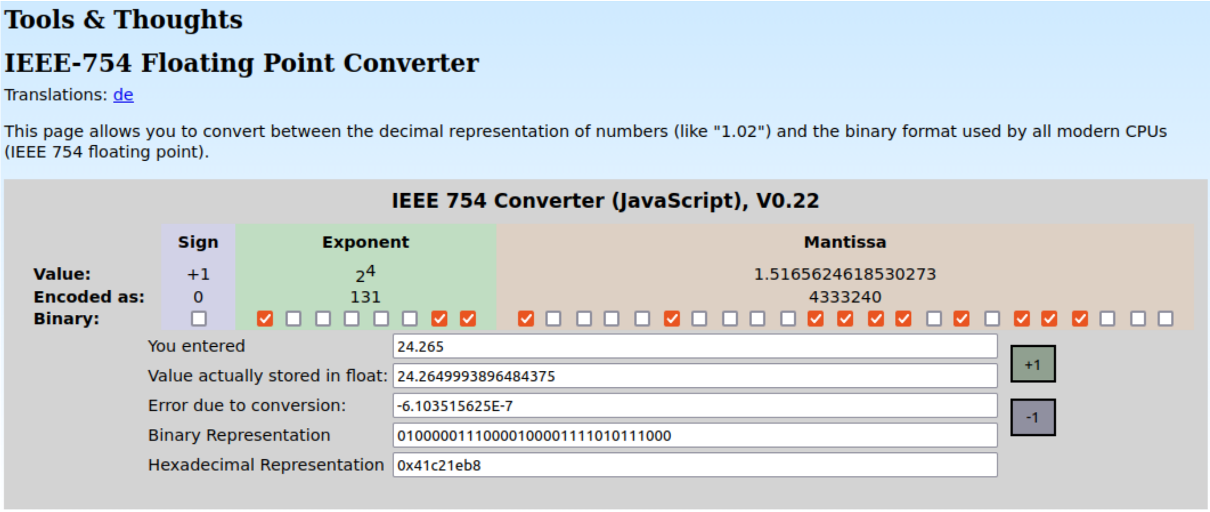
Single: 0 10000011 10000100001111010111000

Flags

Single: ☑Inexact ☐Subnormal

a) Input & output selection **b)** Output of tool

Figure A.8.4: Input and Output of "Exploring Binary" Floating Point Converter



Tools & Thoughts

IEEE-754 Floating Point Converter

Translations: de

This page allows you to convert between the decimal representation of numbers (like "1.02") and the binary format used by all modern CPUs (IEEE 754 floating point).

IEEE 754 Converter (JavaScript), V0.22

	Sign	Exponent	Mantissa
Value:	+1	2^4	1.5165624618530273
Encoded as:	0	131	4333240
Binary:	☐	☑☐☐☐☐☐☑☑	☑☐☐☐☐☑☐☐☐☐☑☑☑☑☐☑☐☑☑☑☐☐☐

You entered: 24.265
Value actually stored in float: 24.2649993896484375
Error due to conversion: -6.103515625E-7
Binary Representation: 01000001110000100001111010111000
Hexadecimal Representation: 0x41c21eb8

Figure A.8.5: Output of the "H-Schmidt" Float converter

To convert a binary floating point number back to a binary number the general formula is:

$$Number = (-1)^S(1+T)(2^{E-emax}) \tag{A.8.2}$$

Example A.8.9

Convert the following binary single precision number to it's binary equivalent:

S	E	T
1	1 0 0 0 1 0 0 1	1 0 0 0 1 1 0 0 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0

The biased exponent is $10001001_2 = 137$ so applying formula:

$$\begin{aligned} Number &= (-1)^1(1.10001100111)(2^{137-127}) \\ &= (-1)(1.10001100111)(2^{10}) = -11000110011.1 \end{aligned}$$

This is equivalent to -1587.5 in decimal.

As the biased exponent can be any number from 1 to 255 for single precision numbers this means the the exponent can be any value from -126 to $+127$ thus allowing very small and very large numbers to be expressed. Using floating point numbers a binary integer number with 128 bits can be expressed as a 32 bit number and as the Example A.8.9 shows floating point numbers can be used to express numbers with both integer and fractional parts.

For binary floating point numbers, the number 0.0 is represented by all zeros and $\pm\infty$ are represented by the relevant sign bit, all 1's in the exponent (E) field and all 0's in the trailing significand (T) field. The two NAN are represented by by the relevant sign bit, all 1's in the E field and a non-zero T field. If the E field is 0 but the T field is not zero then the number is a subnormal number as discussed above.

Questions

1. Convert the following numbers to their binary single precision floating point number:
    a) -124.75
    b) 112345.6875
    c) 0.2265625
    d) -5432.96875

2. Convert the following binary single precision floating point number to decimal:
    a) 11000000101001001000000000000000
    b) 01000110010000011111010010000000
    c) 10111110110110000000000000000000
    d) 01000000111001110000000000000000

8.4 Summary

This chapter has looked at how we represent signed numbers (so positive and negative) for storage in computers and also looked at floating point number representation. This is all around how we store numbers in computer systems which effectively just work in binary numbers.

Sequences and series

9.1 Introduction

This chapter looks at some fundamental concepts that you will comes across time and again in engineering. Of especial significance to us as electronic & electrical engineers is the concept of sampling a continuous waveform or signal to obtain a sequence of measure values. This is 'sampling' and will come into greater importance as you study digital signals. Equally we get a sequence when we obtain an approximate set of equations that model physical phenomena. These concepts are extremely important when we want to obtain a solution using a digital computer which is often the only practical solution to solving a problem. Signal processing which is the manipulation/modification of signals to improve them is an important part of electronic & electrical engineering — when this involves digital signals this means manipulating a sequence or series using digital signal processing.

9.2 Sequences

A ***sequence*** is a set of numbers or algebraic terms, not necessarily distinct, written down in a definite order.

We write sequences down by writing each term and putting a comma between them. If the sequence continues then we use '...' to indicate this. So the sequence of integer numbers between 1 and 10 can be written as $1, 2, 3, \ldots, 10$. This is a finite sequence with a definite end point, but we can also have sequences with an infinite number of terms where we just end with the ... notation. The following show examples of finite and infinite sequences:

$$1, 3, 5, \ldots, 15 \qquad\qquad 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{64}$$

$$2, 4, 6, 8, \ldots \qquad\qquad 1, -1, 1, -1, \ldots$$

In general we write a sequence as

$$x[1], x[2], x[3], \ldots \qquad \text{or} \qquad x_1, x_2, X_3, \ldots$$

Or more compactly as:

$$x[k] \quad k = 1, 2, 3, \ldots \qquad \text{or} \qquad x_k \quad k = 1, 2, 3, \ldots$$

The first syle is often used in signal processing where the terms in the sequence represent the values of the signal and the second style is often found in the numerical solutions of equations. $x[1]$ often will be the first term in the sequence but not necessarily as we can have sequences such as:

$$\begin{aligned} &\ldots, x[-3], x[-2], x[-1], x[0], x[1], x[2], x[3], \ldots \\ \text{or } x[k] \quad & k = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \end{aligned}$$

A sequence can be regarded as a function whose domain is a subset of integers. For example the function defined by

$$x : \mathbb{N} \longrightarrow \mathbb{R} \qquad x : k \longrightarrow \frac{3k}{2}$$

is the sequence

$$x[0] = 0 \quad x[1] = \frac{3}{2} \quad x[2] = 3 \quad x[3] = \frac{9}{2}$$

The terms of the sequence are the values in the range of the function with an independent variable of k. This function differs from those seen in Chapter A.4 as the independent variable is not selected from a continuous interval but rather is a ***discrete*** value. It is possible however to graph discrete functions as shown in examples A.9.1 - A.9.3 (adapted from Croft et al. (2017)).

Example A.9.1

Graph the sequences given by

a) $x[k] = \begin{cases} 0 & k < 0 \\ 1 & k \geqslant 0 \end{cases} \qquad k = \ldots, -2, -1, 0, 1, 2, \ldots, \text{ that is } k \in \mathbb{Z}$

b) $x[k] = \begin{cases} 1 & k \text{ even} \\ -1 & k \text{ odd} \end{cases} \qquad k \in \mathbb{Z}$

Solution:

a) This is the ***unit step sequence*** — denoted as $u[k]$ as shown in Figure A.9.1

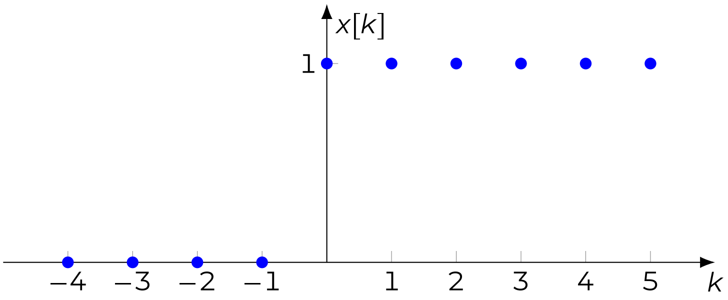


Figure A.9.1: The unit step sequence

b) The sequence is shown below:

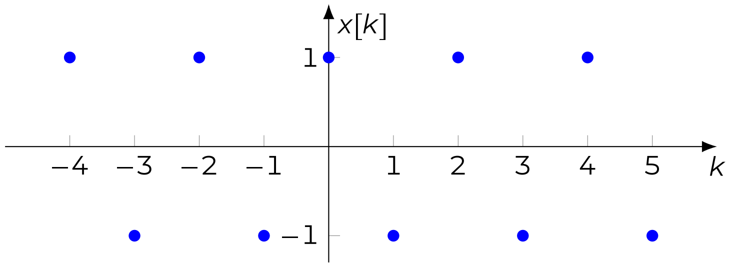


Figure A.9.2: Sequence in question b)

Example A.9.2

Graph the sequence defined as:

$$x[k] = \begin{cases} 1 & k = 0 \\ -1 & k \neq 0 \end{cases} \qquad k \in \mathbb{Z}$$

Solution:
This is the ***Kronecker delta sequence*** - the discrete version of the Dirac delta function $\delta(t)$

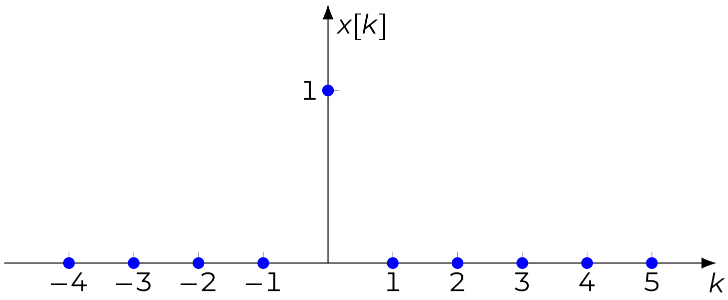


Figure A.9.3: The Kronecker delta sequence

Example A.9.3

The sequence $x[k]$ is obtained from ***sampling*** the function $f(t) = \cos t, t \in \mathbb{R}$ at $t = -2\pi, -3\pi/2, -\pi, -\pi/2, 0, \pi/2, \pi, 3\pi/2, 2\pi$. Sampling is measuring the amplitude of continuous function at defined values of t
Write down the values of $x[k]$ and graph the result.

Solution:
Figure A.9.4 shows the function $f(t) = \cos t$ for the range $-2\pi \leqslant t \leqslant 2\pi$ along with the samples taken at the requisite points. The resulting sequence is

$$x[k] = 1, 0, -1, 0, 1, 0, -1, 0, 1 \qquad \text{at } k = -4, -3, \ldots, 3, 4$$

This is shown graphically in Figure A.9.5

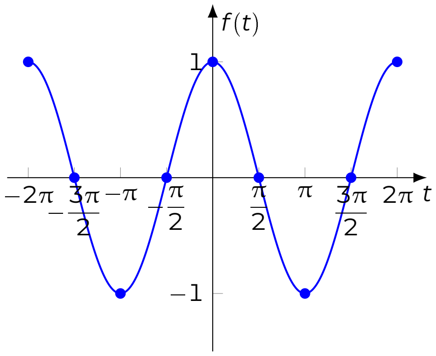


Figure A.9.4: Function $f(t) = \cos t$

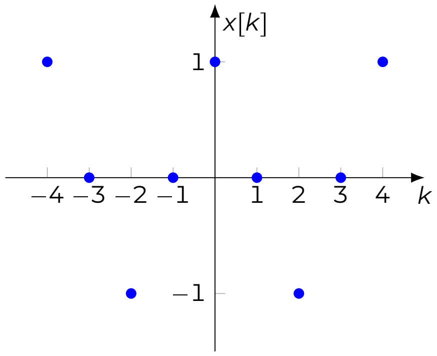


Figure A.9.5: Sequence formed by sampling $\cos t$

For some sequences we can use a rule to describe the kth term so for example $x[k] = 2^k, k = 0, 1, 2, 3, \ldots$ describes the sequence $1, 2, 4, 8, \ldots$. At times this rule may give $x[k]$ in terms of earlier values of the sequence. This is then a sequence defined ***recursively*** and the formula is known as a ***recursive relation*** or a ***difference equation***. These are important in digital signal processing so you will meet them in later levels of the course.

Example A.9.4

Given the following rule for $x[k]$ write down the terms for $k = 0, 1, 2, \ldots 6$ given that $x[0] = 1$ and $x[1] = 1$

$$x[k] = x[k-2] + x[k-1]$$

Solution:
We have the values of $x[0]$ and $x[1]$ so form the rule we can find the remaining terms:

$$\begin{aligned}
x[2] &= x[0] + x[1] = 1 + 1 = 2 \\
x[3] &= x[1] + x[2] = 1 + 2 = 3 \\
x[4] &= x[2] + x[3] = 2 + 3 = 5 \\
x[5] &= x[3] + x[4] = 3 + 5 = 8 \\
x[6] &= x[4] + x[5] = 5 + 8 = 13
\end{aligned}$$

This is in fact the first seven terms of the Fibonacci sequence

Arithmetic & geometric progressions

An ***arithmetic progression*** is a sequence where each term is found by adding to the previous term a fixed quantity, called the ***common difference***. So for instance the sequence $1, 4, 7, 10, 13, \ldots$ is an arithmetic progression with the first term being 1 and the common difference is 3.

The general arithmetic progression with a first term a and common difference term of d is written as:

$$a, a + d, a + 2d, a + 3d, \ldots$$

You should be able to see that the kth term is written as

$$a + (k - 1)d$$

and the difference equation is $x[k] = x[k - 1] + d$

Arithmetic progression: $a, a + d, a + 2d, a + 3d, \ldots$
$a =$ first term, $d =$common difference, kth term $= a + (k - 1)d$

A *geometric progression* is a sequence where each term is found by multiplying the previous term by a fixed quantity, called the ***common ratio***. So for instance the sequence $1, 2, 4, 8, \ldots$ is a geometric progression with a first term of 1 and a common ratio of 2.

The general geometric progression with a first term a and common ratio term of d is written as:

$$a, ar, ar^2, ar^3, \ldots$$

You should be able to see that the kth term is written as

$$ar^{(k-1)}$$

and the difference equation is $x[k] = rx[k - 1]$

Geometric progression: $a, ar, ar^2, ar^3, \ldots$
$a =$ first term, $r =$common ratio, kth term $= ar^{(k-1)}$

More general sequences

We have met infinite sequences which have no defined end. For example:

$$1, 3, 5, 7, 9, \ldots \tag{1}$$

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \tag{2}$$

It is easy to see that the terms of sequence (1) go on increasing without any bound — so this sequence is ***unbounded***. In sequence (2) the successive terms clearly get smaller and smaller so as $k \to \infty, x[k] \to 0$ — this is an illustration of the concept of a ***limit***. we can say that for sequence (2) 'the limit of $x[k]$ as k tends to infinity is 0' which is written as:

$$\lim_{k \to \infty} x[k] = 0$$

The sequence (2) converges to 0 and it is a ***bounded*** sequence as it terms do not increase without bound.

Formally, a sequence $x[k]$ ***converges*** to a limit l if as we proceed along the sequence far enough all subsequent terms can be said to lie closer & closer to l. If a sequence is not convergent as in sequence (1) then it is said to be a ***divergent*** sequence.

A bounded sequence is not necessarily a convergent sequence. Take the example of

$$x[k] = 1, -1, 1, -1, 1, \ldots$$

This clearly is bounded within a range of values ± 1 but equally clearly as $k \to \infty$ it fails to have a limit so is not convergent — in fact this sequence is said to ***oscillate***.

There are a number of rules that apply to limits of sequences that we will just state as the proof is beyond the level of this book.

If $x[k]$ and $y[k]$ are two sequences where $\lim_{k\to\infty} x[k] = l_1$ and $\lim_{k\to\infty} y[k] = l_2$ where l_1 and l_2 are finite then:

(1) The sequence given by $x[k] \pm y[k]$ has limit $l_1 \pm l_2$

(2) The sequence given by $cx[k]$ where c is a constant has limit cl_1

(3) The sequence $x[k]y[k]$ has limit $l_1 l_2$

(4) The sequence $\dfrac{x[k]}{y[k]}$ has limit $\dfrac{l_1}{l_2}$ provided $l_2 \neq 0$

Plus we can always assume the following:

$$\lim_{k\to\infty} \frac{1}{k^m} = 0 \qquad \text{for any constant } m > 0$$

Example A.9.5

Find, if possible the limit of each of the following sequences

a) $x[k] = \dfrac{1}{k}$ $\quad k \in \mathbb{N}_{>0}$

b) $x[k] = 2$ $\quad k \in \mathbb{N}_{>0}$

c) $x[k] = 4 + \dfrac{1}{k}$ $\quad k \in \mathbb{N}_{>0}$

d) $x[k] = \dfrac{1}{k+2}$ $\quad k \in \mathbb{N}_{>0}$

e) $x[k] = k^3$ $\quad k \in \mathbb{N}_{>0}$

Solution:

a) This sequence is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Successive terms are getting smaller and smaller so we can say as $k \to \infty, x[k] \to 0$, so

$$\lim_{k\to\infty} x[k] = \lim_{k\to\infty} \frac{1}{k} = 0$$

b) This sequence is $2, 2, 2, \ldots$ so the limit is clearly 2.

c) This is a combination of two sequences $4, 4, 4, \dots$ with a limit of 4 and $\frac{1}{k}$ which has a limit of 0. So using rule (1) we can say that:

$$\lim_{k\to\infty}\left(4+\frac{1}{k}\right) = 4+0 = 4$$

d) This sequence gives

$$\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$$

which has a limit of 0 (as terms are getting smaller and smaller)

e) The sequence is $1, 8, 27, 64, \dots$ which is an unbounded sequence are the terms increase with no bound. So there is no limit for this sequence.

Example A.9.6

Find the limit as $k \to \infty$ of the sequence with a general term of $x[k] = \frac{4k-1}{2k+1}$.

Solution:
In this case we cannot just simply say $k = \infty$ to obtain the limit as $\frac{\infty - 1}{\infty + 1}$ makes no sense. Instead what we do is rewrite the sequence in a form where we can let $k \to \infty$. If we dive both the numerator and denominator by k then we get:

$$\frac{4k-1}{2k+1} = \frac{4-(1/k)}{2+(1/k)}$$

We know that as $k \to \infty$, $1/k \to 0$ so using rules (1) and (4):

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{4-(1/k)}{2+(1/k)}\right) &= \frac{\lim_{k\to\infty}(4-(1/k))}{\lim_{k\to\infty}(2+(1/k))}\\ &= \frac{4-0}{2+0}\\ &= 2\end{aligned}$$

Example A.9.7

Find the limit as $k \to \infty$ of the sequence with a general term of $x[k] = \frac{2k^2-3k+4}{4k^2+3k-4}$.

Solution:
In this case we divide the numerator and denominator by highest power of k in denominator (to avoid dividing by 0) to give all terms that tend to 0 as $k \to \infty$. So in this case that is k^2 :

$$\frac{2k^2-3k+4}{4k^2+3k-4} = \frac{2-(3/k)+(4/k^2)}{4+(3/k)-(4/k^2)}$$

We know that as $k \to \infty, 1/k \to 0$ so using rules (1) and (4):

$$\begin{aligned}\lim_{k\to\infty}\left(\frac{2-(3/k)+(4/k^2)}{4+(3/k)-(4/k^2)}\right) &= \frac{\lim\limits_{k\to\infty}(2-(3/k)+(4/k^2))}{\lim\limits_{k\to\infty}(4+(3/k)-(4/k^2))} \\ &= \frac{2-0+0}{4+0-0} \\ \lim_{k\to\infty} x[k] &= 0.5\end{aligned}$$

Example A.9.8

Investigate the behaviour of $\dfrac{k^3}{2k^2+3}$ as $k \to \infty$

Solution:

Divide numerator and denominator by k^2 as this is the highest power of k in denominator:

$$\frac{k^3}{2k^2+3} = \frac{k}{2+(3/k^2)}$$

As $k \to \infty$, the denominator $(2+(3/k^2)) \to 2$, but the numerator $k \to \infty$. So this sequence diverges to infinity.

Questions

1. Graph the sequences for range $0 \leqslant k \leqslant 4$ given by:
   a) $x[k] = k \qquad k \in \mathbb{N}$
   b) $x[k] = \begin{cases} 2 & k = 2 \\ 0 & \text{otherwise} \end{cases} \qquad k \in \mathbb{N}$
   c) $x[k] = 2e^{-k} \qquad k \in \mathbb{N}$
2. The sequence $x[k]$ is obtained from sampling $f(t) = \sin\left(2t{-}\pi\right), t \in \mathbb{R}$ at $t = 0, \pi/4, \pi/2, 3\pi/4, \pi, 5\pi/4, \ldots$. Write down the first 6 terms of the sequence $x[k]$.
3. Write down and plot the first 5 ($n = 1, 2, 3, 4, 5$) terms of the sequences defined recursively as:
   a) $x[n] = \dfrac{x[n-1]}{4}, \qquad x[0] = 0$
   b) $x[n] = 3x[n-1] - 2x[n-2], \qquad x[0] = 1, x[1] = -1$
   c) $x[n+1] = x[n] + 3, \qquad x[0] = 1$
   d) $x[n+1] = x[n] + \dfrac{n^2}{2}, \qquad x[0] = 1$
   e) $x[n+2] - x[n+1] = 2x[n], \qquad x[0] = 1, x[1] = 2$
4. For the following arithmetic progressions write down the 6th, 10th and kth terms:
   a) $7, 12, 13, \ldots$

   b) $9, 6, 3, \ldots$
   c) $c, \frac{3c}{4}, \frac{c}{2}, \frac{c}{4}, 0, \ldots$

5. Write down the 6th and 10th terms of the geometric progression $16, 8, 4, \ldots$

6. What are the 5th and 15th terms of the geometric progression with first term of 3 and a common ratio of 2.

7. Find the limit if it exists of the following expressions as $k \to \infty$:

   a) $x[k] = \dfrac{2k+3}{k}$
   b) $x[k] = \dfrac{2k+3}{k^2+4}$
   c) $x[k] = k^4$
   d) $x[k] = \dfrac{k+3}{k-4}$
   e) $x[k] = \dfrac{k^2-3k+3}{k^2-2k+5}$
   f) $x[k] = \left(\dfrac{3k+4}{2k-8}\right)^2$

8. Find the limit if it exists of the following expressions as $k \to \infty$:

   a) $x[k] = (-2)^k$
   b) $x[k] = 3 - \dfrac{k}{5}$
   c) $x[k] = \left(\dfrac{1}{2}\right)^k$
   d) $x[k] = \dfrac{3k^3+4k-2}{2k^3+k^2-5}$
   e) $x[k] = \left(\dfrac{1}{3}\right)^2 k$

9.3 Series

A ***series*** is simply the result of adding the terms of a sequence together. For example taking the simple sequence $1, 2, 3, 4$ we obtain the series S by adding the terms together:

$$S = 1 + 2 + 3 + 4$$

This is a ***finite series*** as it ends after 4 terms — if the series does not end then it is called an ***infinite series***.

Given an arbitrary sequence $x[k]$ we can use the sigma notation introduced in Section 3.4 to donate the series

$$S_n = \sum_{k=0}^{n} x[k]$$

which means the sum $x[0] + x[1] + x[2] + \cdots + x[n]$. If the series is infinite then we would write

$$S = \sum_{k=0}^{\infty} x[k]$$

Sum of a finite arithmetic series

An arithmetic series is simply formed from summing the terms of an arithmetic progression. It is simple to do this or a small number but who do we handle a large number of terms. Will let us consider the series formed from the sequence $1, 2, 3, 4, 5$. Going back to basics addition is a commutative process so the order in which we write the numbers does not matter so we can create the series in two ways:

$$\begin{aligned} S &= 1 + 2 + 3 + 4 + 5 \\ S &= 5 + 4 + 3 + 2 + 1 \end{aligned}$$

If we add the two equations together we get

$$2S = 6 + 6 + 6 + 6 + 6$$

There are five terms so we can write:

$$\begin{aligned} 2S &= 5 \times 6 = 30 \\ \therefore S &= 15 \end{aligned}$$

More generally for an arithmetic progression with $k$terms can be written in the following two ways (starting from first term a or last term $(a + (k-1)d)$)

$$\begin{aligned} S_k &= a + (a+d) + (a+2d) + \cdots + (a + (k-1)d) \\ S_k &= (a + (k-1)d) + (a + (k-2)d) + (a + (k-3)d) + \cdots + a \end{aligned}$$

Adding together the first terms gives $2a + (k-1)d$ and similarly adding together the second two terms gives $2a + (k-1)d$. In fact this is the case for each pair of terms - with k pairs of terms. So we end up with

$$2S_k = k \times (2a + (k-1)d)$$

Therefore we can say that the sum to k terms of an arithmetic series with a first term of a and a common difference of d is given by:

Sum of an arithmetic series:

$$S_k = \frac{k}{2}(2a + (k-1)d) \tag{A.9.1}$$

Example A.9.9

Find the sum of the arithmetic series containing 30 terms with a first term of 1 and a common difference of 3.

Solution:
For this we use the formula for the sum of an arithmetic series $S_k = \frac{k}{2}(2a + (k-1)d)$ with $k = 30, a = 1$ and $d = 3$

$$\begin{aligned} S_{30} &= \frac{30}{2}(2(1) + (30-1)3) \\ &= 15(2 + (29 \times 3)) = 1335 \end{aligned}$$

Example A.9.10

Find the sum of the arithmetic series containing 30 terms with a first term of 0, a common difference of 4 and a last term of 100.

Solution:
For this we first need to find the number of terms for which we use the formula for the kth term of an arithmetic progression as in $a + (k-1)d$

$$\begin{aligned} 100 &= 0 + 4(k-1) \\ \implies 4(k-1) &= 100 \\ k - 1 &= 25 \\ k &= = 26 \end{aligned}$$

We can now use the formula for the sum of an arithmetic series $S_k = \frac{k}{2}(2a + (k-1)d)$ with $k = 26, a = 0$ and $d = 4$

$$\begin{aligned} S_{26} &= \frac{26}{2}(2(0) + (26-1)4) \\ &= 13(25 \times 4) = 1300 \end{aligned}$$

Sum of finite geometric series

A geometric series is simply the sum of the terms of a geometric progression. If we take the geometric progression $1, 2, 4, 8, 16$ then it is simple task to sum the terms but as the number of terms increases it becomes more difficult so again we consider a method for doing this as we did with arithmetic series. For geometric series this means taking the difference between the series and the prdouct of the common ratio and series series multiplied:

$$S = 1 + 2 + 4 + 8 + 16 \tag{A.9.2}$$

$$2S = 2 + 4 + 8 + 16 + 32 \tag{A.9.3}$$

Subtracting equation A.9.3 from equation A.9.3 we find that:

$$S - 2S = 1 - 32$$

since most terms cancel out. Hence $-S = 3 - 1$ so $S = 31$.

As we found with arithmetic progressions, we can apply this procedure more generally to a geometric progression with a first term of a and a common ratio of r. The sum to k terms is:

$$S_k = a + ar + ar^2 + \cdots + ar^{k-1}$$

Multiplying this by r gives

$$rS_k = ar + ar^2 + \cdots + ar^{k-1} + ar^k$$

Subtraction gives us

$$S_k - rS_k = a - ar^k$$

Therefore we can say that the sum to k terms of a geometric series with a first term of a and a common ratio of 4 is given by:

Sum of a finite geometric series:

$$S_k = \frac{a(1-r^k)}{1-r} \qquad r \neq 1 \qquad \text{(A.9.4)}$$

Sum of an infinite series

The situation with infinite series is more complex — in general an arithmetic series will not converge to a given value as the value of each term increases (if $d > 0$) or decreases (if $d < 0$) by the common difference so effectively $lim_{k\to\infty} S_k \to \infty$ if $d > 0$ and $k \to \infty S_k \to -\infty$ if $d < 0$.

For geometric series we can consider the series defined by geometric progression with k th term being $\frac{1}{2^k}$. We can plot each successive addition (***partial sum*** of S_1, S_2, S_3 etc) on a line as shown in Figure A.9.6.

$$S_1 = \sum_{k=1}^{1} x[k] = x[1] = 1$$

$$S_2 = \sum_{k=1}^{2} x[k] = x[1] + x[2] = 1 + \frac{1}{2} = 1\frac{1}{2}$$

$$S_3 = \sum_{k=1}^{3} x[k] = 1 + \frac{1}{2} + \frac{1}{4} = 1\frac{3}{4}$$

$$S_4 = \sum_{k=1}^{4} x[k] = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1\frac{7}{8}$$

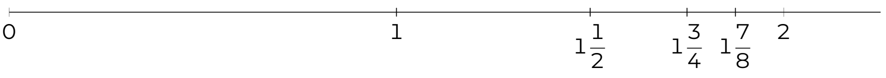


Figure A.9.6: Graphical interpretation of geometric series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

As we can see at each stage the value of the sum moves half the remaining distance to 2 and therefore the total distance can never exceed 2 so we say the series converges to 2. On the other hand the series created from $1, 2, 4, 8, 16, \ldots$ never converges to a value so the series is said to be divergent.

Thinking about the formula for the sum of a geometric series

$$S_k = \frac{a(1-r^k)}{1-r} \qquad r \neq 1$$

in order to think about the sum of a infinite geometric series we need to consider what happens as $k \to \infty$. If $-1 < r < 1$ then $r^k \to 0$ so $S_k \to \dfrac{a}{1-r}$.

This means we can say that:

Sum of an infinite geometric series:

$$S_\infty = \frac{a}{1-r} \qquad -1 < r < 1 \tag{A.9.5}$$

Example A.9.11

Find the sum to k terms of the following geometric series and deduce their sums to infinity

a) $1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots$ b) $20 + 10 + 5 + 2\frac{1}{2} + 1\frac{1}{4} + \ldots$

Solution:

a) This is the sum of a geometric progression with first term $a = 1$ and common ratio $r = 1/4$ so using formula:

$$S_k = \frac{a(1-r^k)}{1-r} = \frac{1(1-(1/4)^k)}{1-(1/4)} = \frac{4}{3}\left(1 - \left(\frac{1}{4}\right)^k\right)$$

As $k \to \infty$, $\left(\frac{1}{4}\right)^k \to 0$ so that $S_\infty = \frac{4}{3}$

b) This is the sum of a geometric progression with first term $a = 20$ and common ratio $r = 1/2$ so using formula:

$$S_k = \frac{a(1-r^k)}{1-r} = \frac{20(1-(1/2)^k)}{1-(1/2)} = 40\left(1 - \left(\frac{1}{2}\right)^k\right)$$

As $k \to \infty$, $\left(\frac{1}{2}\right)^k \to 0$ so that $S_\infty = 40$

Questions

1. An arithmetic series has a first term of 4 and its 30th term is 120. Find its sum to 30 terms.
2. Find the sum to 25 terms of the following arithmetic series:

a) $a = 4, d = 3$
b) $a = 4, d = -3$
c) $a = 2, d = 4$
d) $a = 2, d = -4$

3. If the sum to 10 terms of an arithmetic series is -140 with $d = -4$, what is the value of its first term, a?

4. The sum to 10 terms is equal to the sum to 11 terms of an arithmetic series with a $d = -2$. What is the first term (a) of the series?

5. Find the sum to five terms of the geometric series with $a = 1$ and $r = 1/4$. What is the sum to infinity of this series?

6. Find the sum to 15 terms of the geometric series with $a = 3$ and $r = 1.5$

7. The sum to infinity of a geometric series is equal to five times its first term. What is its common ratio?

8. The sum to infinity of a geometric series is equal to four times the sum of the first two terms. What are the possible values of r?

9. Write down the following series in sigma notation

$$1 - 2 + 3 - 4 + \cdots$$

10. Write down the first six terms of the series $\sum_{k-0}^{\infty}(-1)^{k+1}z^{-k}$

9.4 Binomial Theorem

We can easily expand the expression $(a+b)^2$ to $a^2 + 2ab + b^2$ and although it is slightly more complicated we can still expand the expression $(a+b)^3$ to $a^3 + 3a^2b + 3ab^2 + b^3$. However it is much more work to expand expressions such as $(a+b)^5$ and $(a+b)^8$. A simple technique for expanding expressions of format $(a+b)^n$ where n is a positive integer is using Pascal's triangle.

Pascal's triangle is shown in Figure A.9.7 — it is basically a triangle of numbers where every entry is obtained from adding the two entries either side on the preceding row, always starting and ending row with a '1'. You can see that the third row gives the coefficients of the expansion of $(a+b)^2$ and the fourth row gives the coefficients of the expansion of $(a+b)^3$. So to expand $(a+b)^4$ we take the coefficients from the row starting '1 4' with the appropriate powers of a and b. Therefore $(a+b)^4 = 1a^4 + 4a^3b + 10a^2b^2 + 4ab^3 + 1b^4$.

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & + & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Figure A.9.7: Pascal's triangle

Example A.9.12

Use Pascal's triangle to expand $(a+b)^6$

Solution:
The row starting '1 6' is 1 6 15 20 15 6 1 so we can say that:

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^5$$

Example A.9.13

Use Pascal's triangle to expand $(1+x)^8$

Solution:
Forming the row starting '1 8'

1 8 28 56 70 56 28 8 1

In this case $a = 1$ and $b = x$ so:

$$(1+x)^9 = 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$$

Pascal's triangle is useful for relatively small values of n but as n increases it gets somewhat unwieldy. This is where the binomial theorem comes in which enable us to work out the coefficients for any positive n

The binomial theorem states that when n is a positive integer

$$\begin{aligned}(a+b)^n =& a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 \\ &+ \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n\end{aligned} \tag{A.9.6}$$

Equation A.9.6 is often re-written with $a = 1$ and $b = x$, so that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \tag{A.9.7}$$

Example A.9.14

Expand $(1+x)^{12}$ up to the term in x^3

Solution:

We could use Pascal's triangle to answer this question and look to the row starting '1 12' but a lot of calculations will be required so it easier to use the binomial theorem in equation A.9.7. Setting $n = 12$ we get:

$$\begin{aligned}(1+x)^{12} &= 1 + 12x + \frac{12(11)}{2!}x^2 + \frac{12(11)(10)}{3!} + \cdots \\ &= 1 + 12x + \frac{132}{2}x^2 + \frac{1320}{6}x^3 + \cdots \\ &= 1 + 12x + 66x^2 + 220x^3 + \cdots\end{aligned}$$

Therefore, the expansion of $(1+x)^{12}$ up to the term in x^3 is

$$1 + 12x + 66x^2 + 220x^3$$

Equation A.9.7 will eventually terminate provided n is a positive integer. What happens however when n is not a positive integer? In this case, provided that $|x| < 1$, we end up with an infinite series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \tag{A.9.8}$$

Example A.9.15

Expand $\dfrac{1}{1+x}$ for powers up to x^3

Solution:

$\dfrac{1}{1+x} = (1+x)^{-1}$ so using equation A.9.8

$$\begin{aligned}(1+x)^{-1} &= 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \frac{(-1)(-2)(-3)}{3!} + \cdots \\ &= 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1\end{aligned}$$

So if we meet the series $1 - x + x^2 - x^3 + \cdots$ we can write it in form $(1+x)^{-1}$ which is easier to handle than an infinite series.

We can rewrite functions which involve terms such as $\frac{1}{x} = x^{-1}$ as shown in the next example to obtain the expansion in terms of ascending powers of x rather than descending powers or vice versa which can be very useful sometimes.

Example A.9.16

A function, $f(x)$ is given by

$$f(x) = \left(1 + \frac{1}{x}\right)^{1/2}$$

a) Obtain the first four terms in the expansion of $f(x)$ in descending powers of x in expansion of $f(x)$ and state the range of values of x for which expansion is valid.

b) Show that $f(x)$ can be rewritten as $f(x) = x^{-1/2}(1+x)^{1/2}$. Obtain the first four terms of the expansion of this function in ascending powers of x and state the range of values of x for which expansion is valid.

Solution:

a) Use the binomial theorem in equation A.9.8 with $x = x^{-1}$ and $n = \dfrac{1}{2}$

$$\begin{aligned}\left(1 + \frac{1}{x}\right)^{1/2} =& 1 + \left(\frac{1}{2}\right)(x^{-1}) + \left(\frac{(1/2)(-1/2)}{2!}\right)(x^{-1})^2 \\ &+ \left(\frac{(1/2)(-1/2)(-3/2)}{3!}\right)(x^{-1})^3 \\ &+ \left(\frac{(1/2)(-1/2)(-3/2)(-5/2)}{4!}\right)(x^{-1})^4 \\ =& 1 + \frac{1}{2}x^{-1} - \frac{1/4}{2}x^{-2} + \frac{3/8}{6}x^{-3} - \frac{15/16}{24}x^{-4} \\ =& 1 + \frac{1}{2x} - \frac{1}{8x^2} + \frac{1}{16x^3} - \frac{1}{16x^4}\end{aligned}$$

To work out validity we know that $|x^{-1}| < 1 \implies \boxed{|x| > 1}$

b) Rewrite the function as follows:

$$\left(1 + \frac{1}{x}\right)^{1/2} = \left(\frac{x+1}{x}\right)^{1/2} = \left(\frac{1}{x}\right)^{1/2}(x+1)^{1/2} = x^{-1/2}(x+1)^{1/2}$$

Then expand it using equation A.9.8 with $n = \dfrac{1}{2}$

$$\begin{aligned}x^{-1/2}(1+x)^{1/2} =& x^{-1/2}\left(1 + \left(\frac{1}{2}\right)x + \left(\frac{(1/2)(-1/2)}{2!}\right)x^2\right. \\ &+ \left(\frac{(1/2)(-1/2)(-3/2)}{3!}\right)x^3 \\ &\left.+ \left(\frac{(1/2)(-1/2)(-3/2)(-5/2)}{4!}\right)x^4\right) \\ =& x^{-1/2}\left(1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{x^4}{16}\right)\end{aligned}$$

Expansion is valid for $|x| < 1$

Questions

1. Use the binomial theorem to expand

   a) $(1+x)^3$ b) $(1+x)^5$ c) $\left(1+\frac{x}{2}\right)^5$

   d) $\left(1-\frac{x}{2}\right)^4$ e) $\left(2+\frac{x}{4}\right)^4$ f) $\left(4-\frac{x}{2}\right)^4$

2. Use Pascal's triangle to expand

   a) $(a+b)^9$ b) $(3x+2y)^3$ c) $(a-2b)^4$

3. Use the binomial theorem to find the first four terms of the following expansions

   a) $(2-3x)^5$ b) $(3+4x)^6$ c) $\left(1+\frac{x}{2}\right)^{10}$

4. Use the binomial theorem to find the first five terms of the following expansions. State the range of values of x for which the expansion is valid in each case.

   a) $(1+x)^{1/2}$ b) $(1-2x)^{-3}$ c) $\left(1+\frac{x}{2}\right)^{-4}$

   d) $\left(2+\frac{x}{4}\right)^{-2}$ e) $\left(1+\frac{x}{3}\right)^{-1/2}$ f) $\left(3+\frac{x}{3}\right)^{-3}$

5. The function $f(x)$ is defined as

   $$f(x) = (1-\frac{1}{x})^{1/2}$$

   a) Obtain the first four terms in the expansion of $f(x)$ in descending powers of x in expansion of $f(x)$ and state the range of values of x for which expansion is valid.

   b) By re-writing $f(x)$ in a suitable form obtain the first four terms of the expansion of this function in ascending powers of x and state the range of values of x for which expansion is valid.

6. The function $f(x)$ is defined as

   $$f(x) = (1+\frac{1}{x^2})^{1/2}$$

   a) Obtain the first four terms in the expansion of $f(x)$ in descending powers of x in expansion of $f(x)$ and state the range of values of x for which expansion is valid.

   b) Show that $f(x)$ can be rewritten as $f(x) = x^{-1/2}(x-1)^{1/2}$. Obtain the first four terms of the expansion of this function in ascending powers of x and state the range of values of x for which expansion is valid.

9.5 Power series

Power series are an important class of series which are infinite series involving integer powers of x (or any other variable name). So for instance both the following series are examples of power series:

$$1+x+x^2+x^3+x^4+\cdots$$

$$1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \cdots$$

As you can see a power series can be regarded as an infinite polynomial. We can express many common functions as a power series. For example:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \text{ in radians}$$

This converges for any value of x. Taking $x = 1$:

$$\cos(1) = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} + \cdots$$

If we just take the first four terms and compare it to the real value of cos 1

$$\cos(1) \approx 1 - \frac{1}{2} + \frac{1}{24} = 1 - 0.5 + 0.041\dot{6} - 0.0013\dot{8} = 0.54027\dot{8}$$

$$\cos(1) = 0.54030231$$

We can already see that the power series is giving a close approximation to the actual value.

In general a power series is only useful if the series converges for the value of x that is used. We can define an important quantity R which is the maximum value that if x is chosen in region $-r < x < R$ then the series will converge. R is known as the ***radius of convergence*** and the interval $(-R, R)$ is known as the ***interval of convergence***.

Three common power series expansions are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad x \text{ in radians} \tag{A.9.9}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad x \text{ in radians} \tag{A.9.10}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{A.9.11}$$

All of these converge for any value of x

For small values of x all the above series converge quickly which gives us using the sin and cos series the ***small angle approximations***

If x is small and measured in radians:

$$\sin x \approx x \qquad \text{and} \qquad \cos x \approx 1 - \frac{x^2}{2}$$

Questions

1. Using the first four terms of the power series expansion for e^x when $x = 0, 0.1, 0.5$ and 1 compare and comment on the results with the value of e^x obtained from your calculator.

2. Using the power series expansion for $\sin x$:
    a) Write down the power series expansion for $\sin 2x$
    b) Write down the power series expansion for $\sin 0.5x$
    c) By considering power series expansion for $\sin -x$ show that $\sin(-x) = -\sin(x)$
3. What is the power series expansion for e^{-x}
4. Obtain a quadratic approximation to $e^x \cos x$

9.6 Summary

Sequences are a significant tool in digital signal processing as they represent the sampling a continuous signal. Series are simply the summation of the terms of a sequence and are again a useful tool in our mathematical knowledge. You will come across sequences and series again — both in semester 2 and more especially with z-transforms at Level 5.

Complex Numbers

10.1 Introduction

Complex numbers are a more general system of numbers than the real number system that you are used to. Basically complex number theory introduces the concept of numbers that when squared result in negative numbers — the real number system only leads to positive numbers when a number is squared. On first introduction complex numbers often appear strange as we grapple with the concept of *imaginary* numbers. However they are a very powerful tool for solving engineering problems. In particular, the analysis of AC circuits is one of the main areas that complex number theory is used in electrical and electronic engineering.

Another major area where complex number theory is applied, is in the analysis and processing of signals where we can crate mathematical models using complex numbers. This is due to the fact that we tend to represent signals using sinusoidal models and these can be manipulated more easily using complex numbers. In communications, complex number theory is very important in the design of filters and other signal processing systems.

The third area that you are likely to come across complex numbers is in control engineering where many engineers tend to use a 'complex plane' representation of systems as opposed to a 'time domain' representation. This is because again working in complex domain makes the mathematical model easier to manipulate.

10.2 Complex numbers & Arithmetic

What is a complex number?

We met the concept of quadratic expressions back in Chapter A.3 so we know that both the expressions in equation A.10.1 represents a quadratic expression.

$$\begin{aligned} x^2 - x - 12 &= 0 \\ x^2 - 4x + 13 &= 0 \end{aligned} \tag{A.10.1}$$

From Chapter A.3, equation A.3.9 gives us the general formula for finding the roots of the expression $ax^2 + bx + c = 0$. For ease this is repeated in equation A.10.2

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{A.10.2}$$

So let us apply this for the expressions in equation A.10.1 starting with $x^2 - x - 12 = 0$

$$\begin{aligned} x &= \frac{+1 \pm \sqrt{(-1)^2 - 4(1)(-12)}}{2(1)} \\ &= \frac{1 \pm \sqrt{49}}{2} \\ &= \frac{1 \pm 7}{2} \end{aligned} \tag{A.10.3}$$

$\therefore$ roots are $x = 4$ and $x = -3$. Now let us apply A.10.2 to $x^2 - 4x + 13 = 0$

$$\begin{aligned} x &= \frac{+4 \pm \sqrt{(-4)^2 - 4(1)(13)}}{2(1)} \\ &= \frac{4 \pm \sqrt{-36}}{2} \end{aligned} \tag{A.10.4}$$

Now we have a problem as we need to find the square root of a negative number which we know with real numbers does not exist as the result of both 6^2 and $(-6)^2$ is 36. So we need to introduce a technique to deal with the situation that arise with using equation A.10.2 when $b^2 - 4ac < 0$, which leads to the theory of complex numbers.

So let us introduce the concept of a number, which we will call j [*] with the property that

$$\mathrm{j}^2 = -1$$

This cannot be a real number as the square of a real number is always positive so we say the number j is *imaginary*. We can now find the solution to the problem in equation A.10.4.

$$\begin{aligned} \sqrt{-36} &= \sqrt{36 \times (-1)} \\ &= \sqrt{36 \times \mathrm{j}^2} \\ &= 6\mathrm{j} \end{aligned}$$

$$\begin{aligned} \therefore x &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6\mathrm{j}}{2} \\ &= 2 \pm 3\mathrm{j} \end{aligned}$$

$\therefore$ the roots are $x = 2 + 3\mathrm{j}$ and $x = 2 - 3\mathrm{j}$. These numbers are called *complex numbers* and they consist of two parts — a *real part* which is 2 in both roots and an *imaginary* part which is $\pm 3\mathrm{j}$. Generally we refer to complex numbers using format $z = a + b\mathrm{j}$ where $a = \mathrm{Re}(z)$ and $b = \mathrm{Im}(z)$. The set of all complex numbers is donated by $\mathbb{C}$ so we can say that:

$z = a + b\mathrm{j}$, z is a member of the set of complex numbers — $z \in \mathbb{C}$
a and b are both real numbers — $a, b \in \mathbb{R}$

---

[*] In other areas of engineering and mathematics in general, the symbol i is used, but as *i* or i can represent alternating current in electrical engineering j is preferred

Complex conjugate

An important concept in complex numbers is the *complex conjugate* which is the number with the same real part but with an imaginary part that is equal in magnitude but opposite in sign. So the complex conjugate of $z = a + b\mathrm{j}$ is $\bar{z} = a - b\mathrm{j}$[*].

Recall from above that the roots of the quadratic expression $x^2 - 4x + 13 = 0$ were found to be $x = 2 + 3\mathrm{j}$ and $x = 2 - 3\mathrm{j}$. These are a complex conjugate pair that arise because of the $\pm\sqrt{b^2 - 4ac}$ term in the quadratic roots equation A.10.2. In general we can say that:

If all the coefficients of a polynomial equation $P(x) = 0$ are real numbers, then any complex roots will always occur in conjugate pairs.

There are a few rules about complex conjugate numbers that are worth noting — in following z and w are complex numbers unless otherwise noted:

$$\begin{aligned}
\overline{z + w} &= \bar{z} + \bar{w} \\
\overline{z - w} &= \bar{z} - \bar{w} \\
\overline{zw} &= \bar{z} \cdot \bar{w} \\
\overline{\left(\frac{z}{w}\right)} &= \frac{\bar{z}}{\bar{w}}, \text{provided that } w \neq 0 \\
\bar{z} = z &\iff z \in \mathbb{R}
\end{aligned}$$

Addition and subtraction

Addition of two complex numbers is relatively straight forward as the real part of the answer simply comes from adding the real parts together and similarly the imaginary part of the answer comes from adding the imaginary parts together. Think about algebra when we gathered like terms together — in the case of complex numbers we are doing the same in gathering the real and imaginary parts.

$$\begin{aligned}
(5 + 4\mathrm{j}) + (2 + 3\mathrm{j}) &= 5 + 4\mathrm{j} + 2 + 3\mathrm{j} \\
&= (5 + 2) + \mathrm{j}(4 + 3) \\
&= 7 + 7\mathrm{j}
\end{aligned}$$

Similarly with subtraction of two complex numbers we do the subtraction on the real parts and the subtraction on the imaginary parts.

$$\begin{aligned}
(5 + 4\mathrm{j}) - (2 + 3\mathrm{j}) &= (5 - 2) + \mathrm{j}(4 - 3) \\
&= 3 + \mathrm{j}
\end{aligned}$$

So the general rules for complex addition & subtraction of two numbers are:

---

[*] the complex conjugate of z is indicated symbolically either as $\bar{z}$ or z^*

$$(a+b\mathrm{j})+(c+d\mathrm{j})=(a+c)+\mathrm{j}(b+d)$$
$$(a+b\mathrm{j})-(c+d\mathrm{j})=(a-c)+\mathrm{j}(b-d) \qquad (A.10.5)$$

Multiplication

In many ways this is similar to the way we do algebraic multiplication as effectively we are simply removing brackets. It is important to remember the fact that $\mathrm{j}^2 = -1$.

$$\begin{aligned}(5+4\mathrm{j})(2+3\mathrm{j}) &= 5(2+3\mathrm{j})+4\mathrm{j}(2+3\mathrm{j})\\ &= 10+15\mathrm{j}+8\mathrm{j}+12(\mathrm{j}^2)\\ &= (10-12)+\mathrm{j}(15+8)\\ &= -2+23\mathrm{j}\end{aligned}$$

So generalising complex multiplication is given by:

$$(a+b\mathrm{j})(c+d\mathrm{j})=(ac-bd)+\mathrm{j}(ad+bc) \qquad (A.10.6)$$

An interesting case arises when we calculate $z \cdot \bar{z}$ as given $z = a + b\mathrm{j}$:

$$\begin{aligned}z\cdot\bar{z} &= (a+b\mathrm{j})(a-b\mathrm{j})\\ &= a^2+ba\mathrm{j}-ba\mathrm{j}-b^2\mathrm{j}^2\\ &= a^2+b^2\end{aligned}$$

So again in general the sum of two squares is derived from multiplying a complex number by its conjugate —- compare to equation A.3.7 which is the difference of two squares:

$$z\cdot\bar{z}=a^2+b^2 \qquad (A.10.7)$$

Division

To divide one complex number by another we multiply both the numerator and denominator by the complex conjugate of the denominator and simplify the result from there. This makes the denominator wholly real using equation A.10.7 which makes the simplification much easier.

$$\begin{aligned}\frac{5+4\mathrm{j}}{2+3\mathrm{j}} &= \frac{(5+4\mathrm{j})}{(2+3\mathrm{j})}\cdot\frac{(2-3\mathrm{j})}{(2-3\mathrm{j})}\\ &= \frac{10-15\mathrm{j}+8\mathrm{j}-12\mathrm{j}^2}{4+9}\\ &= \frac{10-7\mathrm{j}+12}{13}\\ &= \frac{22-7\mathrm{j}}{13}\\ &= \frac{22}{13}-\frac{7}{13}\mathrm{j}\end{aligned}$$

So again generalising complex division is given by equation A.10.8

$$\frac{(a+bj)}{(c+dj)} = \frac{(ac+bd)+j(bc-ad)}{c^2+d^2} \qquad (A.10.8)$$

Questions

1. Solve the following equations:

   a) x^2+4 b) x^2+16 c) $3x^2+7$
   d) x^2+2x+2 e) $-x^2+2x-4$ f) x^2+5x+3

2. Write down the complex conjugates of

   a) $5+j3$ b) $3-j2$

3. If $z=2+j$ and $w=3-j$ evaluate:

   a) $z+w$ b) $w-z$ c) $3z-2w$
   d) $2z+5w$ e) $j(w-2z)$ f) $2jw-3jz$

In the following questions evaluate the answers in $a+jb$ form given $Z_1=1+j2, Z_2=4-j3$, $Z_3=-2+j3$ and $Z_4=-5-j$

4. a) $Z_1+Z_2-Z_3$ b) $Z_2-Z_1+Z_4$
5. a) Z_1Z_2 b) Z_3Z_4
6. a) $Z_1Z_3+Z_4$ b) $Z_1Z_2Z_3$
7. a) $\dfrac{Z_1}{Z_2}$ b) $\dfrac{Z_1+Z_3}{Z_2-Z_4}$
8. a) $\dfrac{Z_1Z_3}{Z_1+Z_3}$ b) $Z_2\dfrac{Z_1}{Z_4}+Z_3$
9. a) $\dfrac{1+j}{1-j}$ b) $\dfrac{1}{1-j}$
10. Show that $\dfrac{-25}{2}\left(\dfrac{1+j2}{3+j4}-\dfrac{2-j5}{-j}\right)=57+j24$

10.3 Representation

Graphical— Argand diagrams

We can plot the complex number $z=a+bj$ on Cartesian axes by plotting the real part a on horizontal axis, and the imaginary part b on the vertical axis and joining the resulting point to the origin by a straight line as shown in Figure A.10.1. The x-axis is the *real axis* and the y-axis is the *imaginary axis* and the x-y plane is often called the *complex plane*.. This type of diagram is called an *Argand diagram* after the Swiss mathematician Jean-Robert Argand (1768-1822) who first devised this way of representing complex numbers in a privately

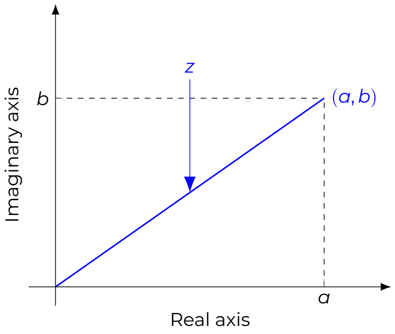


Figure A.10.1: Argand diagram of $z = a + b\text{j}$

published essay in 1806*. In another seminal paper, Argand (1814) delivered a proof of the fundamental theorem of algebra which states that "every non-constant single-variable polynomial with complex coefficients has at least one complex root.".

Example A.10.1

Plot the following complex numbers on an Argand diagram:

a) $z_1 = 5 + 7\text{j}$ b) $z_2 = -2 + 4\text{j}$ c) $z_3 = -3 - 4\text{j}$ d) $z_4 = -3\text{j}$

Solution

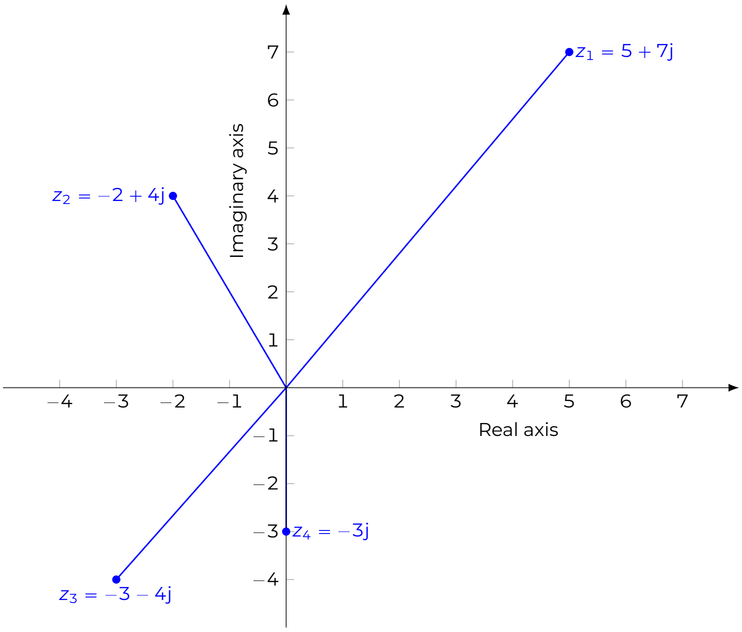


Figure A.10.2: Argand diagram for various complex numbers

Polar form

In many situations it is often useful to swap the Cartesian coordinate form of complex numbers, (a, b), for polar coordinates, (r, θ) as shown in Figure A.10.3. The conversion between the forms is as defined in Chapter A.6 — repeated below for ease.

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[*] This essay was republished in Argand (1813)

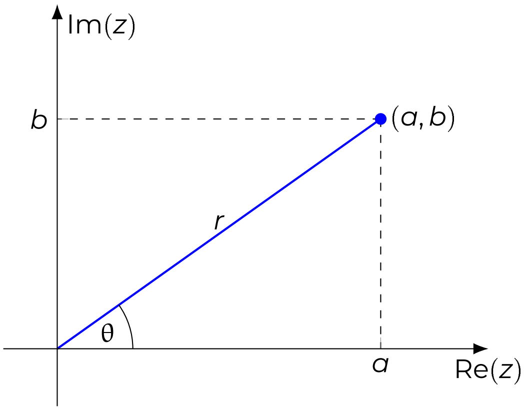


Figure A.10.3: Polar and Cartesian forms of a a complex number

From Figure A.10.3 and knowledge of trigonometric ratios we can say that:

$$\begin{aligned} \cos\theta &= \frac{a}{r} \qquad &\therefore a = r\cos\theta \\ \sin\theta &= \frac{b}{r} \qquad &\therefore b = r\sin\theta \end{aligned} \tag{A.10.9}$$

To find θ we use the tangent ratio so:

$$\tan\theta = \frac{b}{a} \tag{A.10.10}$$

For the length r we use Pythagoras theorem so $r = +\sqrt{a^2 + b^2}$. We can then rewrite the complex number $z = a + b\mathrm{j}$ in the polar form as

$$z = r\cos\theta + \mathrm{j}r\sin\theta = r(\cos\theta + \mathrm{j}\sin\theta)$$

this often in fact is written as $z = r\angle\theta$. Clearly r is the distance from the origin to point (a, b) and is called the *modulus* of the complex number z — this is donated mathematically as $|z|$. Note that $|z| \geqslant 0$ is always true. The angle θ is known as the *argument* of z, arg(z), and is conventionally measured as shown from the x axis — with positive angles being measure din anticlockwise direction and negative angles being those measured in a clockwise direction. Usually we say that $-\pi < \theta \leqslant \pi$ as we know that this measures all angles given that a full circle is 2π.

All of this leads to the complex notation being:

Cartesian: $z = a + b\mathrm{j}$

Polar: $z = r(\cos\theta + \mathrm{j}\sin\theta) = r\angle\theta$

$|z| = r = +\sqrt{a^2 + b^2}$

$a = r\cos\theta \qquad b = r\sin\theta \qquad \tan\theta = \dfrac{b}{a}$

What about the complex conjugate in polar form? Well let us look at number $z = r\angle(-\theta)$:

$$\begin{aligned} r\angle(-\theta) &= r(\cos(-\theta) + \mathrm{j}\sin(-\theta)) \\ &= r(\cos(\theta) - \mathrm{j}\sin(\theta)) \\ &= a - b\mathrm{j} \\ &= \overline{z} \end{aligned}$$

So for complex conjugates the rule is:

$$\boxed{\text{If } z = a + \mathrm{j} = r\angle\theta \text{ then } \overline{z} = a - b\mathrm{j} = r\angle(-\theta)}$$

Multiplication and division using polar form

The polar form of complex numbers is really useful when we are talking about multiplication and division as we can use the trigonometric identities to get a useful result. Suppose we want to multiply two complex numbers $z_1 = r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2)$.

$$\begin{aligned} z_1 z_2 =& (r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1))(r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2)) \\ =& r_1 r_2((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ & + \mathrm{j}(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)) \end{aligned} \tag{A.10.11}$$

Using the following trigonometric identities we can simplify equation A.10.11

$$\begin{aligned} \sin(-x) &= -sin(x) \\ \sin x \sin y &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right] \\ \cos x \cos y &= \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \\ \sin x \cos y &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right] \end{aligned}$$

$$\begin{aligned} z_1 z_2 =& \frac{r_1 r_2}{2}(((\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)) \\ & - (\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2))) \\ & + \mathrm{j}((\sin(\theta_1 + \theta_2) + \sin(\theta_1 - \theta_2)) \\ & + (\sin(\theta_2 + \theta_1) + \sin(\theta_2 - \theta_1)))) \\ =& \frac{r_1 r_2}{2}(2(\cos(\theta_1 + \theta_2)) + \mathrm{j}(2(\sin(\theta_1 + \theta_2)))) \\ =& r_1 r_2(\cos(\theta_1 + \theta_2) + \mathrm{j}(\sin(\theta_1 + \theta_2))) \end{aligned} \tag{A.10.12}$$

Comparing the resulting expression in equation A.10.12 with the general polar form of a complex number $z = r(\cos\theta + \mathrm{j}\sin\theta)$ we can see that this represents a number with a modulus of $r_1 r_2$ and an argument of $(\theta_1 + \theta_2)$. So to multiply two complex numbers together we multiply their moduli and add their arguments.

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2) \tag{A.10.13}$$

For division we can do a similar set of manipulation remembering to divide we multiply the numerator and denominator by the complex conjugate of the denominator (which is $r_2\angle(-\theta_2)$. we can use equations A.10.12 and A.10.8:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{(r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1))}{(r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2))} \\ &= \frac{(r_1(\cos\theta_1 + \mathrm{j}\sin\theta_1))}{(r_2(\cos\theta_2 + \mathrm{j}\sin\theta_2))}\frac{(r_2(\cos(-\theta_2) + \mathrm{j}\sin(-\theta_2)))}{(r_2(\cos(-\theta_2) + \mathrm{j}\sin(-\theta_2)))} \\ &= \frac{r_1 r_2(\cos(\theta_1 - \theta_2) + \mathrm{j}(\sin(\theta_1 - \theta_2)))}{r_2^2(\cos^2\theta_2 + \sin^2\theta_2)} \\ &= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \mathrm{j}(\sin(\theta_1 - \theta_2)) \end{aligned} \tag{A.10.14}$$

Again comparing the result in equation A.10.14 to a general complex number we can see that to divide one complex number by the other is to divide the moduli and subtract the angles.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2) \tag{A.10.15}$$

Exponential form

The theory of this form is somewhat advanced but it is useful. Later in the book we will talk about power series and how many functions possess a power series expansion — in other words they can be expressed as the sum of a sequence of terms involving integer powers of the variable. For this form, we need to know the following power series expansions (z is a complex number and x is a real number):

$$\mathrm{e}^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots = \sum_{n=0}^{\infty}\frac{z^n}{n!} \tag{A.10.16}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots. = \sum_{n=0}^{\infty}(-1)^n\frac{x^{(2n+1)}}{(2n+1)!} \tag{A.10.17}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots. = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} \tag{A.10.18}$$

So for the generic polar form of

$$z = r(\cos\theta + \mathrm{j}\sin\theta)$$

using equations A.10.17 and A.10.18 we can say that:

$$\begin{aligned} z &= r\left(\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + \mathrm{j}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\right) \\ &= r\left(1 + \mathrm{j}\theta - \frac{\theta^2}{2!} - \mathrm{j}\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \mathrm{j}\frac{\theta^5}{5!} - \ldots\right) \end{aligned}$$

From equation A.10.16 we can say that expansion of $e^(j\theta)$ is given by:

$$\begin{aligned} e^{j\theta} &= 1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + ... \\ &= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + ... \end{aligned}$$

So given that the previous two equations end in same result we can say that:

$$z = r(\cos\theta + j\sin\theta) = e^{j\theta}$$

This is the *exponential form* of a complex number. It is worth noting that:

$$e^{j\theta} = \cos\theta + j\sin\theta \quad (A.10.19)$$

$$e^{-j\theta} = \cos\theta - j\sin\theta \quad (A.10.20)$$

$$\therefore \bar{z} = r(\cos\theta - j\sin\theta) = r\,e^{-j\theta} \quad (A.10.21)$$

From these expressions we can also say that:

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Questions

1. Represent the following on an Argand diagram:
   a) $z_1 = 3 + 2j$ b) $z_2 = -1 + 3j$ c) $z_3 = -3 - j$ d) $z_4 = -2j$
2. Evaluate the following sums and show the results on an Argand diagram:
   a) $(1 + j2) + (2 - j)$ b) $(-2 + j3) - (1 - 6j)$ c) $(1 + j)^2$

In the next two questions express the complex numbers in polar form, leaving answers in surd form (i.e. with any irrational numbers as $\sqrt{x}$ etc.) Calculate θ for values $-180° < \theta \leqslant 180°$

3. a) $3 + 2j$ b) $-1 + 3j$ c) $-3 - j$ d) $-2j$
4. a) $(1 + j2) + (2 - j)$ b) $(-2 + j3) - (1 - 6j)$ c) $(1 + j)^2$

In the next two questions express the results in Cartesian form $a + jb$ with answers correct to 4 significant figures

5. a) $2\angle 30°$ b) $3\angle 60°$ c) $2\angle 45°$ d) $6\angle 125°$
6. a) $3\angle\pi$ b) $2.5\angle\frac{-\pi}{6}$ c) $3.4\angle\frac{2\pi}{3}$ d) $4.2\angle\frac{-3\pi}{4}$

Evaluate the next two questions in polar form

7. a) $2\angle 30^\circ \times 3\angle 60^\circ$

   b) $3.4\angle 15^\circ \times 4.4\angle -25^\circ$

8. a) $2.5\angle\frac{-\pi}{6} \div 0.5\angle\frac{\pi}{3}$

   b) $3\angle\frac{5\pi}{6} \times 2\angle\frac{\pi}{3} \div 1\angle\frac{-\pi}{4}$

9. Find the real and imaginary parts and the modulus and argument of the following complex numbers

   a) $2\,\mathrm{e}^{\mathrm{j}\pi/3}$ b) $4\,\mathrm{e}^{-\mathrm{j}\pi/6}$ c) $5\,\mathrm{e}^{\mathrm{j}\pi/4}$ d) $3\,\mathrm{e}^{-\mathrm{j}5\pi/6}$

10. Express $z = 3(\cos 60^\circ + \mathrm{j}\sin 60^\circ)$in exponential form and plot it on an Argand diagram to find the real and imaginary parts of z

11. If $\sigma, \omega, T \in \mathbb{R}$ find the real and imaginary parts of $\mathrm{e}^{(\sigma+\mathrm{j}\omega)T}$

12. Express the following complex numbers in form $a + b\mathrm{j}$:

    a) $\mathrm{e}^{(1-\mathrm{j}\pi/2)}$ b) $\mathrm{e}^{(2+\mathrm{j}\pi/3)}$

13. Express the following complex numbers in exponential form:

    a) $z_1 = 1 - \mathrm{j}$ b) $z_3 = \frac{1}{2} + \frac{1}{3}\mathrm{j}$ c) $z_2 = 4 + 7\mathrm{j}$ d) $z_4 = \frac{1-\mathrm{j}}{\sqrt{2}+\mathrm{j}}$

10.4 Uses

Having done all the above what is the actual implication of complex numbers. Well remember from Chapter A.5 that we use sinusoids a lot to represent analogue periodic signals — that is signals whose amplitude varies with time with a defined pattern that repeats. In general we can represent say a voltage as

$$v(t) = V\cos(\omega t + \phi) = v\cos(2\pi f t + \phi)$$

where V is maximum or peak amplitude, ω is the angular frequency, f is the frequency and ϕ is the phase relative to some reference waveform. This is known as the *time domain representation*. In an analogue circuit we may have various components which can vary the phase and amplitude of the voltage — and to analyse the circuit in the time domain actually becomes quite tedious.

However how about we introduce another representation of the waveform known as a *phasor* which is a an entity consisting of a magnitude (V) and an angle (ϕ). This sounds familiar from the polar representation of complex numbers doesn't it? The time dependency of the signal is represented by rotating the phasor about the origin in the anticlockwise direction at a rate of ω. The phasor form of the general time varying voltage is shown in Figure A.10.4.

If we consider at a sinusoidal waveform $A(t) = \sin(t)$ then at time $t = 0$ the instantaneous amplitude of the wave will be $\sin(0) = 0$ with the phasor pointing in horizontal direction and at time $t = \frac{\pi}{3}$ the amplitude will be $\frac{\sqrt{3}}{2}$. This can be seen in Figure A.10.5 which shows a complete period of a sine wave in phasor and time domains.

Phasors really come into their own when you are combining AC quantities which are not necessarily in phase with each other. Let us take the example of two currents where $i_2(t)$

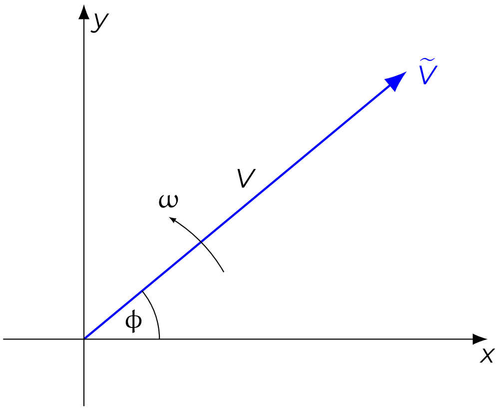


Figure A.10.4: Phasor form of $v(t) = V\cos(\omega t + \phi)$

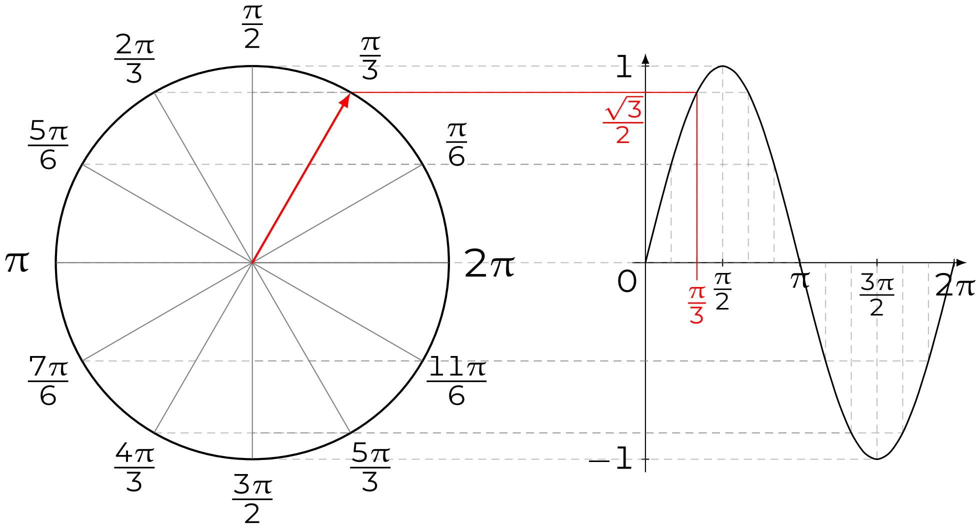


Figure A.10.5: Phasor diagram for $\sin t$

leads $i_1(t)$ by an angle of ϕ with the phasor of $i_1(t)$ being on horizontal axis at $t = 0$. We can model these currents mathematically as given in equation A.10.22 and represent them by the phasor and time domain diagrams in Figure A.10.6. In this figure, the phase difference is $\phi = \frac{\pi}{4}$:

$$\begin{aligned} i_1(t) &= I_1 \sin(\omega t) \\ i_2(t) &= I_2 \sin(\omega t + \phi) \end{aligned} \tag{A.10.22}$$

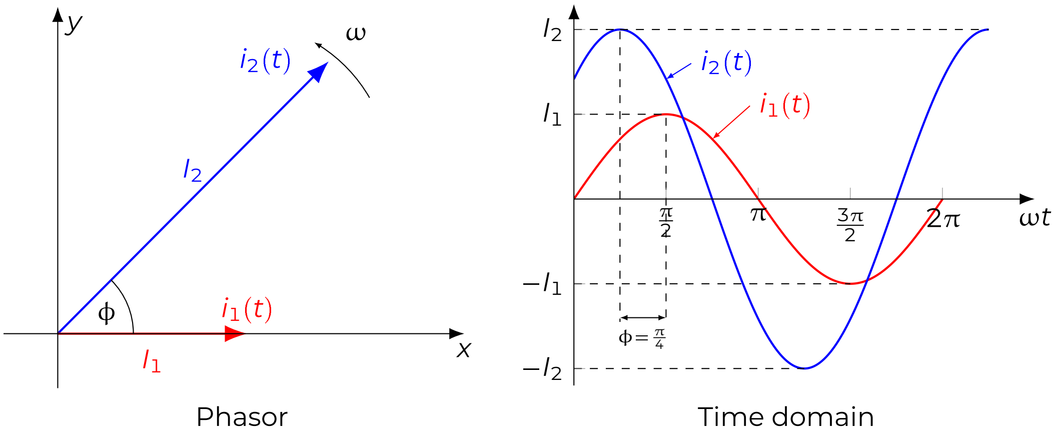


Figure A.10.6: Phase difference of sinusoidal waveforms

The term *lead* in phase "phasor *a* leads phasor *b*" implies that on a phasor diagram *a* is ahead of *b* in the direction of rotation. Similarly the term *lag* in phase "phasor *a* lags phasor *b*" implies that on a phasor diagram *b* is ahead of *a* in the direction of rotation. So looking at Figure A.10.6 we can say the $i_2(t)$ leads $i_1(t)$ or $i_1(t)$ lags $i_2(t)$. Which term is used depends on which phasor is chosen as reference phasor. In terms of the phase difference is a phasor leads then ϕ is positive whereas if it lags then ϕ is negative.

But why do we use phasors at all in AC analysis? Well when you look at adding the two currents in Figure A.10.6 together it is non trivial as there is a phase difference. If both current were in phase with each other then the calculation would simply be

$$i_1(t) + i_2(t) = I_1 \sin(\omega t) + I_2 \sin(\omega t) = (I_1 + I_2) \sin(\omega t)$$

But there is a phase difference of ϕ. We can do this graphically by drawing out the phasors accurately on graph paper and forming the parallelogram. The resultant current is the diagonal line from the origin to opposing point on parallelogram as shown in Figure A.10.7.

However these relies on accurate drawing to scale. So ideally an analytical method is preferred and this is where complex numbers come in (you may have been wandering up to this point why we are discussing phasors). Mathematically, we add two phasors together by finding their "horizontal" and "vertical" components and then adding the "horizontal" components to find the "horizontal" component of the resultant phasor and the same for the "vertical" component. Going back to the polar form of a complex number as shown in equation A.10.9 we can see that the horizontal component is given by $r \cos \theta$ and the vertical component by $r \sin \theta$. Remembering that a phasor is simply a snapshot of a signal at a single point in time we can say that

$$I_m = I_m(\cos(\phi) + \mathrm{j}\sin(\phi))$$

expresses a complex sinusoid.

So the addition of the two phasors can be simplified to the addition of two complex numbers.

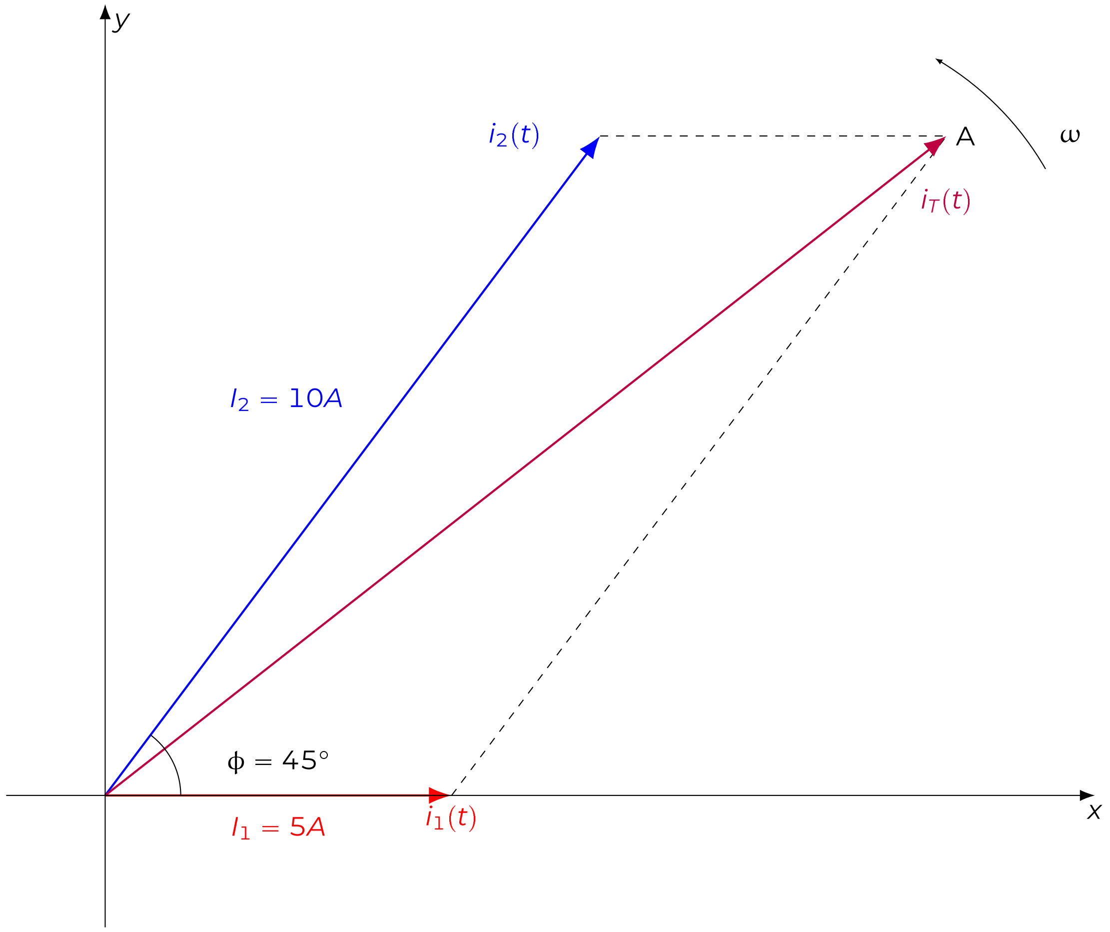


Figure A.10.7: Adding two phasors graphically

Given the facts that $I_1 = 5A$, $I_2 = 10A$ and $\phi = \frac{\pi}{4}$ we ca convert $i_1(t)$ and $i_2(t)$ to complex form:

$$\begin{aligned}\text{Horizontal of } i_1(t) &= 5\cos(0) = 5\\ \text{Vertical of } i_1(t) &= 5\sin(0) = 0\\ i_1(t) &= 5 + \text{j}0 = 5\end{aligned}$$

$$\begin{aligned}\text{Horizontal of } i_2(t) &= 10\cos(\frac{\pi}{4}) = 10 \cdot 0.7071 = 7.071\\ \text{Vertical of } i_2(t) &= 10\sin(\frac{\pi}{4}) = 10 \cdot 0.7071 = 7.071\\ i_2(t) &= 7.071 + \text{j}7.071\end{aligned}$$

So adding the two complex phasors together gives us $i_T(t)$

$$\begin{aligned}i_T(t) &= i_1(t) + i_2(t)\\ &= 5 + 7.071 + \text{j}7.071\\ i_T(t) &= 12.071 + \text{j}7.071\end{aligned} \tag{A.10.23}$$

So overall we can take the complex value for $i_T(t)$ and using equations to convert from

Cartesian to polar form we get that:

$$\begin{aligned} I_t &= \sqrt{12.071^2 + 7.071^2} = 13.99A \\ \phi &= \tan^{-1}\frac{7.071}{12.071} = 0.53 \text{ rad} = 30.36° \\ \therefore i_T(t) &= 13.99 \sin(\omega t + 0.53) \end{aligned} \tag{A.10.24}$$

Similarly phasors are very powerful when you need to subtract one AC quantity from another. In this case the graphical representation is the other diagonal of the parallelgram - starting from end of phasor that is being subtracted.

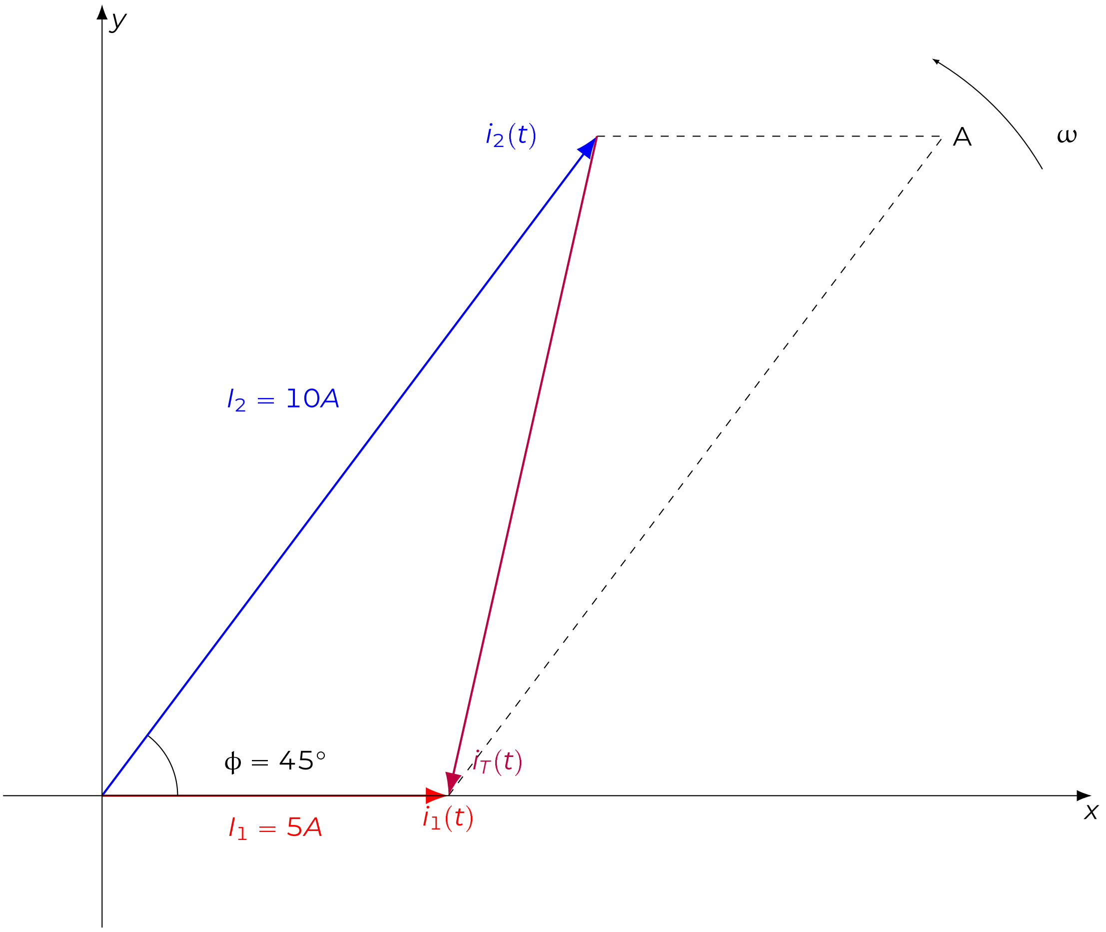


Figure A.10.8: Subtracting two phasors graphically

Analytically we convert the original phasors to complex notation and then subtract one from the other to give $i_T(t)$

$$\begin{aligned} i_T(t) &= i_1(t) - i_2(t) \\ &= 5 - (7.071 + j7.071) \\ i_T(t) &= -2.071 - j7.071 \end{aligned} \tag{A.10.25}$$

So overall we can take the complex value for $i_T(t)$ and using equations to convert from

Cartesian to polar form we get (according to calculator):

$$\begin{aligned} I_t &= \sqrt{(-2.071)^2 + (-7.071)^2} = 7.37A \\ \phi &= \tan^{-1}\frac{-7.071}{-2.071} = 1.286 \text{ rad} = 73.68^\circ \end{aligned} \tag{A.10.26}$$

However we know that $\tan^{-1}$ is a *one to many* function given tan is (see Figure A.5.7) so we need to look at which quadrant the result will be in. In this case as both real and imaginary parts of $i_T(t)$ are negative, it will be in third quadrant so using convention $-\pi < \phi \leqslant \pi$ we say that $\phi = -\pi + 1.286 = -1.856 = -106.32^\circ$. So now final answer is

$$i_T(t) = 7.37\sin(\omega t - 1.856)$$

. The above is all about using phasors for AC analysis where complex numbers are very powerful tools. The effect of multiplying a phasor by j rotates it 90°in a positive direction whereas multiplying it by –j rotates phasor in negative direction by 90°— in both cases the length remains the same. Lets now look at some simple serial resistor-inductor/capacitor circuits which lead to a complex impedance. With an inductor, the voltage across the inductor (V_L) leads the AC current by 90°so the voltage is written mathematically as $\mathrm{j}V_L$ whereas the voltage across a capacitor lags the current by 90°so the voltage is written mathematically as $-\mathrm{j}V_C$. This means that in the R-L circuit shown in Figure A.10.9a $V_R + \mathrm{j}V_L = V$. By Ohm's Law we know that $V_R = IR$ and if the reactance of the inductor is $X_L = 2\pi fL$, then $V_L = IX_L$. So generally the impedance of the circuit is $Z = R + \mathrm{j}X_L$

a) RL circuit

b) RC circuit

c) RL phasors

d) RC phasors

Figure A.10.9: Serial Impedance Circuits

Similarly for the circuit in Figure A.10.9b, we can say that $V = V + R - \mathrm{j}V_C$, so the impedance is $Z = R - X_C$ where the reactance of the capacitor is $X_C = \dfrac{1}{2\pi fC}$.

Example A.10.2

Determine the resistance and series inductance or capacitance for each of the following complex impedances, assuming a frequency of 50 Hz (mains):

a) $(3 + 4\mathrm{j})\Omega$ b) $-15\mathrm{j}\Omega$ c) $10\angle - 60°\Omega$

Solutions:

a) Impedance $Z = (3 + 4\mathrm{j})\Omega$ so resistance (real part of impedance), $R = 3\Omega$.

Since imaginary part of impedance is positive the reactance is for an inductor, $X_L = 2\pi f L = 4\Omega$.

$$L = \frac{X_L}{2\pi f} = \frac{4}{2\pi(50)} = 0.0127\,\text{H or } 12.7\,\text{mH}$$

b) Impedance $Z = -15\mathrm{j}\Omega$ so is wholly negative imaginary which represents a capacitative reactance. So $R = 0\Omega$

Reactance $X_C = 15\Omega = \dfrac{1}{2\pi f C}$

Therefore:

$$\begin{aligned}\text{capacitance, } C &= \frac{1}{2\pi f X_C} = \frac{1}{2\pi(50)(15)}\text{text}\\ &= \frac{10^6}{2\pi(50)(15)}\,\mu\text{F} = 212.2\mu\text{F}\end{aligned}$$

c) Impedance

$$\begin{aligned}Z &= 10\angle - 60°\Omega\\ &= 10[\cos(-60) + \mathrm{j}\sin(-60)]\\ &= (5 - \mathrm{j}8.66)\Omega\end{aligned}$$

Hence, resistance $R = 5\Omega$ and capacitive reactance $X_C = 8.66\Omega$ Therefore:

$$\begin{aligned}\text{capacitance, } C &= \frac{1}{2\pi f X_C} = \frac{10^6}{2\pi(50)(8.66)}\mu\text{F}\\ &= 367.6\mu\text{F}\end{aligned}$$

Complex numbers can be used to analyse parallel AC circuits as shown in application A.10.1.

Engineering application A.10.1: Parallel circuit analysis

For circuit in Figure A.10.10, from the generalised Ohm's Law $V = IZ$ we can saw that $I = \dfrac{V}{Z}$. So we need to calculate Z or the admittance $Y = \dfrac{1}{Z}$ so that $I = VY$.

For the three branch parallel circuit, we can say that:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

where $Z_1 = 2 + \mathrm{j}$, $Z_2 = 5$ and $Z_3 = 3 - \mathrm{j}$.

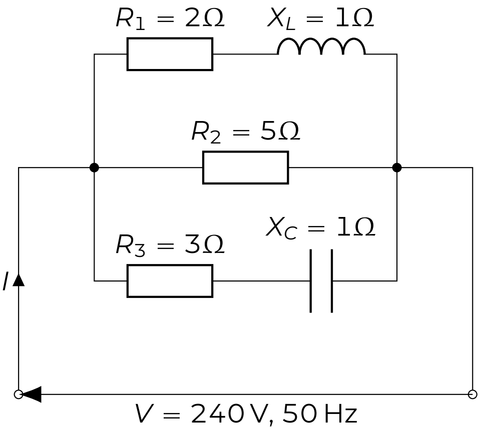


Figure A.10.10: Parallel RLC circuit

$$\begin{aligned}
\text{Admittances: } Y_1 &= \frac{1}{Z_1} = \frac{1}{2+1\text{j}} \\
&= \frac{1}{2+\text{j}} \times \frac{2-\text{j}}{2-\text{j}} = \frac{2-\text{j}}{2^2+1^2} \\
&= 0.4 - 0.2\text{jS (siemens — unit of admittance)} \\
Y_2 &= \frac{1}{Z_2} = \frac{1}{5} = 0.2\text{S} \\
Y_3 &= \frac{1}{Z_3} = \frac{1}{3-\text{j}} \\
&= \frac{1}{3-\text{j}} \times \frac{3+\text{j}}{3+\text{j}} = \frac{3+\text{j}}{3^2+1^2} \\
&= 0.3 + 0.1\text{jS}
\end{aligned}$$

$$\begin{aligned}
\text{Total admittance: } Y &= Y_1 + Y_2 + Y_3 \\
&= 0.4 - 0.2\text{j} + 0.2 + 0.3 + 0.1\text{j} \\
&= 0.9 - 0.1\text{j} \\
&= 0.906\angle - 6.34^\circ\text{S} \\
\text{Current } I &= YZ \\
&= (240\angle 0^\circ)(0.906\angle - 6.34^\circ) \\
&= 217.44\angle - 6.34^\circ A
\end{aligned}$$

Other than AC analysis, where else are we likely to come across complex numbers in electrical & electronic engineering. Well one of the major areas is communications where we use phasor type diagrams to describe the digital modulation of data onto radio waves where we talk about in-phase (real) and quadrature (imaginary) with the quadrature component leading the in-phase component by one-quarter cycle ($\pi/2$ radians). Plus with many transfer functions we model them using rational functions and complex numbers allow us find the poles and zeros (where system response becomes zero).

Complex numbers also crop up a lot in control for similar reasons as we are often dealing with periodic functions - and complex numbers (in their polar form) encapsulate the idea of rotations in algebra. To engineers, complex numbers are simply a representational tool for modelling systems — as indeed is most of mathematics.

Questions

1. What is the resulting voltage $v_T(t) = v_1(t) + v_2(t)$ for following values:
   a) $v_1(t) = 5\sin(2t + \frac{\pi}{2})$ and $v_2(t) = 10\sin\left(2t + \frac{\pi}{6}\right)$
   b) $v_1(t) = 2\sin\left(4t - \frac{\pi}{6}\right)$ and $v_2(t) = 3\sin\left(4t + \frac{\pi}{6}\right)$
2. What is the resulting current $i_T(t) = i_1(t) + i_2(t)$ for following values:
   a) $i_1(t) = 4\sin\left(2t - \frac{\pi}{4}\right)$ and $i_2(t) = 3\sin\left(2t + \frac{\pi}{4}\right)$
   b) $i_1(t) = 3\sin\left(t + \frac{\pi}{6}\right)$ and $i_2(t) = 2\sin\left(t - \frac{\pi}{3}\right)$
3. Determine the series resistance R and series inductance L or capacitance C for each of the following complex impedances assuming that the frequency is 50Hz.
   a) $(5 + 8j)\Omega$ b) $(3 - 7j)\Omega$ c) $20\angle - 30°\Omega$ d) $50\angle 60°\Omega$
4. Two impedances, $Z_1 = (2 + j4)\Omega$ and $Z_2 = (5 - j3)\Omega$ are connected in series to a supply voltage of 80V. Determine the magnitude of the current and its phase angle relative to the voltage.
5. If the two impedances in question 4 are connected in parallel determine the magnitude of the current and its phase angle relative to the voltage.

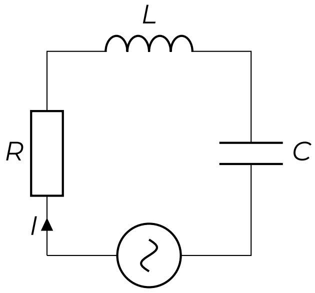


Figure A.10.11: Series RLC circuit

6. For the series circuit shown in Figure A.10.11, calculate the current flowing and its phase relative to the 240V, 50Hz supply voltage for the following values of R, L & C.
   a) $R = 1.2$ kΩ, $L = 10$ mH and $C = 470$ nF.
   b) $R = 8.2$ MΩ, $L = 0.5$ mH and $C = 10$nF.
7. For the circuit in Figure A.10.12, determine the current flowing and its phase relative to the supply voltage.

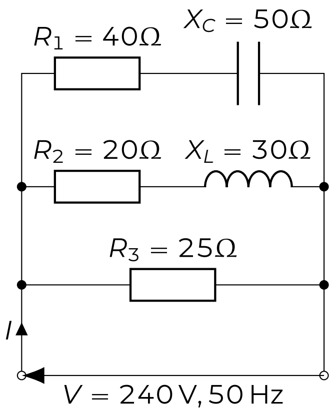


Figure A.10.12: Parallel RLC circuit

10.5 Summary

This chapter has introduced the concepts of complex numbers in terms of what they mean mathematically, how we can do arithmetic with them and the different representations of complex numbers. It has also given an introduction into how we use them in electrical & electronic engineering to model/represent systems and to analyse AC systems. This is covered in greater detail in the relevant technical modules.

PART B

CALCULUS & STATISTICS

Differentiation

1.1 What is differentiation?

Differentiation is a mathematical technique that enables us to analyse how a function changes — specifically how rapidly it changes at any specific point. The *derivative* of a function $f(x)$ with respect to the variable x is simply a measure of how sensitive the output of the function is with respect to changes in the input value. As a simple example, the rate of change of the position of an object with respect to time is the velocity ('speed') of the object.

As engineers we often want to know the rate at which some variable is changing. So for instance the rate at which the voltage across a capacitor is changing with time or the rate at which the temperature changes in a chemical reaction. Rapid rates of change can indicate an issue of concern. These rates of change can be negative (so output variable decreasing), positive (output variable increasing) or zero (output variable remaining constant).

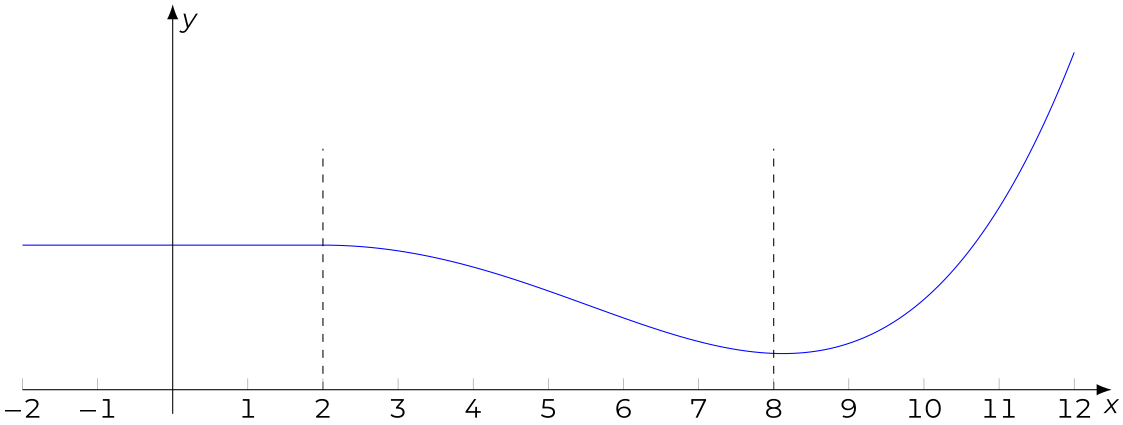


Figure B.1.1: Function $y = f(x)$ changes at different rates

Looking at graph in Figure B.1.1, up to ($x = 2$) the function does not change value at all so the rate of change is zero. In area $2 < x < 8$ the function decreases in value so the rate of change is negative. Above $x = 8$ the rate of change is positive as y is increasing. Initially rate of change is small as rate of increase is slight, but it gradually increases. The aim of *differential calculus* is to specify the exact rate of change of a function — as it is not sufficient to specify it is a small positive rate of change for instance. But how do we find the rate of change? Let's start with a simple straight line

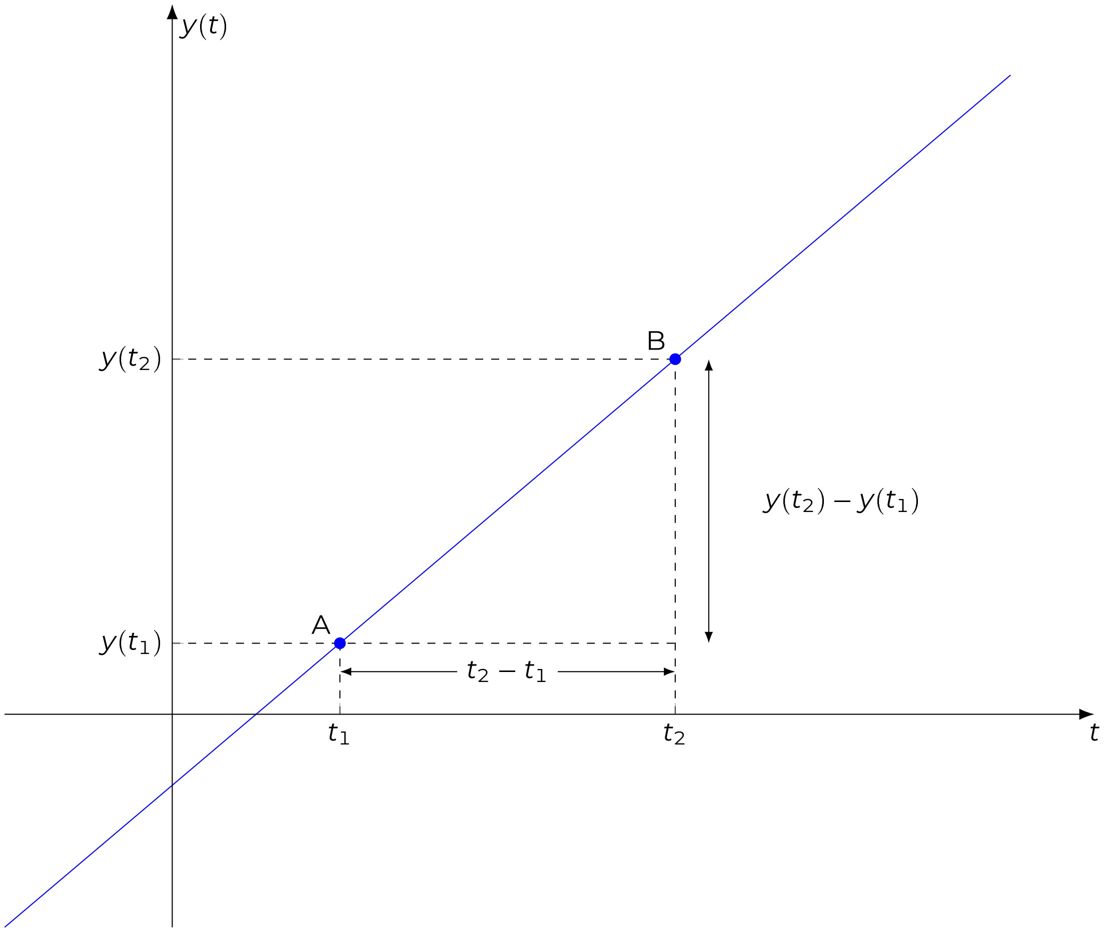


Figure B.1.2: Gradient of a line

Gradient of a line

For a straight line graph finding the derivative is easy as the rate of change is the gradient. Looking at Figure B.1.2, the gradient can be calculated between points A & B as:

$$\frac{\text{change in } y}{\text{change in } x} = \frac{y(t_2) - y(t_1)}{t_2 - t_1} \tag{B.1.1}$$

This means that the rate of change is given by B.1.1 so if I say the equation of the line is $y(t) = 2t - 1$ and $t_1 = 1$ & $t_2 = 3$ then we can say that

$$\text{rate of change} = \frac{(2 \cdot 3 - 1) - (2 \cdot 1 - 1)}{3 - 1} = \frac{5 - 1}{2} = 2$$

This is a constant value for a straight line and indeed we can say the derivative of an arbitrary straight line $y(t) = ax + c$ is $\frac{\mathrm{dy}}{\mathrm{dx}} = a$.

Rate of change at given point

But what about other functions? Well in general we say that the rate of change of a function $y(x)$ at a particular value of x is given by the gradient of a tangent to the curve at that value of x. Let us consider the simple second order equation $y(x) = x^2 + 1$ and take tangents at $x = \pm 1$ and $x = 0$. These are shown in Figure B.1.3.

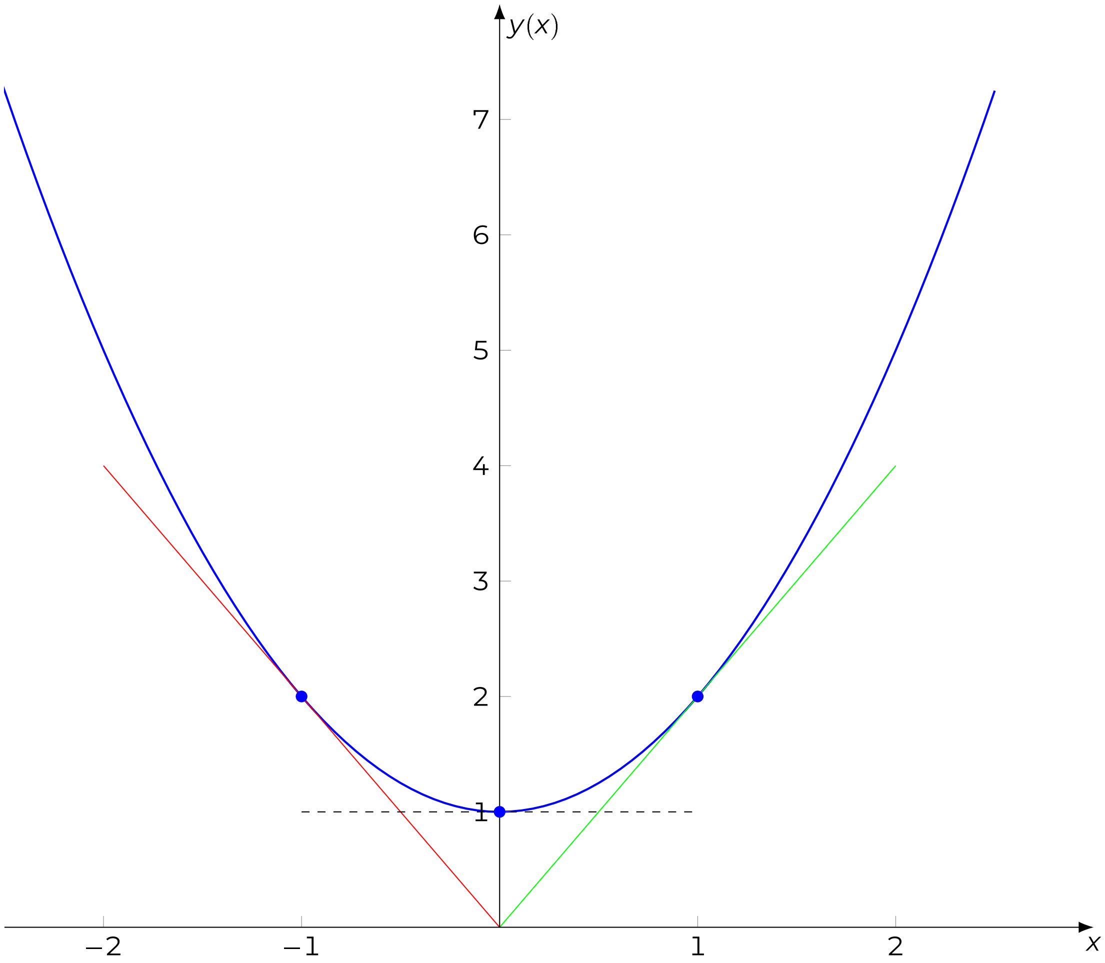


Figure B.1.3: Tangents to $y(x) = x^2 + 1$

But how do we find the gradient of the tangents. Well what we do is basically take the value of x and add a very small amount to it, donated by δx, and then find the limit of the expression as we let δx tend to 0. An example is shown in Example B.1.1.

Example B.1.1

Find the rate of change of $y = x^2 + 1$ at $x = 1$ by considering interval $[1, 1 + \delta x]$ and let δx tend to 0.

Solution:

When $x = 1, y(1) = 2$. When $x = 1 + \delta x$ then:

$$\begin{aligned} y(1 + \delta x) &= (1 + \delta x)^2 + 1 \\ &= 1 + 2\delta x + (\delta x)^2 + 1 \qquad &= (\delta x)^2 + 2\delta x + 2 \end{aligned}$$

So,

$$\begin{aligned}\text{average rate of change of } y \text{ across } [1, 1+\delta x] &= \frac{\text{change in } y}{\text{change in } x}\\ &= \frac{((\delta x)^2 + 2\delta x + 2) - 2}{\delta x}\\ &= \frac{(\delta x)^2 + 2\delta x}{\delta x}\\ &= \delta x + 2\end{aligned}$$

We now let δx tend to 0, so the result is the limit of the expression as this occurs.

$$\text{average rate of change of } y \text{ when } x \text{ is } 1 = \lim_{\delta x \to 0} (\delta x + 2) = 2$$

Rate of change at a general point

All of the above is at a given value — what about a general point x for a function $y(x)$? Effectively the same process happens — so we find the average rate of change in y over an interval $[x, x + \delta x]$ and allow the interval to shrink to a point x by letting δx tend to 0 and evaluating the limit of the expression.

$$\text{rate of change of } y = \lim_{\delta x \to 0} (\frac{y(x + \delta x) - y(x)}{\delta x}) = \lim_{\delta x \to 0} (\frac{y \delta y}{\delta x}) \tag{B.1.2}$$

Let us consider this for the expression $y = x^2 + 2x$ as shown in Example B.1.2.

Example B.1.2

Find the rate of change of $y = x^2 + 2x$.

Solution:

Given $y(x) = x^2 + 2x$:

$$\begin{aligned}y(x + \delta x) &= (x + \delta x)^2 + 2(x + \delta x)\\ &= x^2 + 2x\delta x + (\delta x)^2 + 2x + 2\delta x\end{aligned}$$

hence

$$y(x + \delta x) - y(x) = (\delta x)^2 + 2x\delta x + 2\delta x$$

So,

$$\begin{aligned}\text{rate of change of } y &= \lim_{\delta x \to 0} \left(\frac{y(x + \delta x) - y(x)}{\delta x}\right)\\ &= \lim_{\delta x \to 0} \left(\frac{(\delta x)^2 + 2x\delta x + 2\delta x}{\delta x}\right)\\ &= \lim_{\delta x \to 0} (\delta x + 2x + 2) = 2x + 2\end{aligned}$$

The rate of change of y is the called the *first derivative* of y and is donated by y' or $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ (known as 'dee y by dee x').

To stress that y is a dependent variable of the independent variable x we will often talk about the 'rate of change of y with respect to x' or the 'derivative of y with respect to x'. The process of finding y' is differentiation which so far we have just seen from first principles. We can use y' to find the gradient of a tangent to the curve $y(x)$ at a value of x by evaluating the derivative value at the point. So for a general point $x = x_0$ we can say that the value of the gradient is donated by one of:

$$\frac{\mathrm{d}y}{\mathrm{d}x} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} \qquad y'(x_0) \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x_0}$$

Engineering application B.1.1

One example of where we would find a derivative in electrical engineering is in the relationship of the voltage, v, across an inductor of inductance L with the current i through the inductor. In this case both v and i are functions of time t as they vary. The current is usually an AC so we can use an extension of the DC Ohm's Law:

$$v(t) = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Figure B.1.4 shows the relationship between the current flowing through an inductor and the magnetic flux lines passing through the coils.

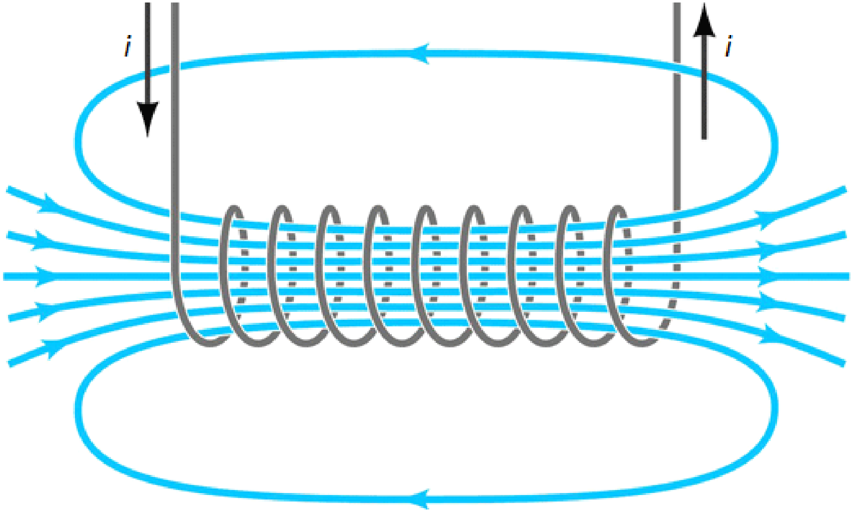


Figure B.1.4: Inductor: Magnetic flux lines & current (Components 101 2018)

This relationship actually quantifies Faraday's Law which states that the voltage induced in a coil is proportional to the rate of change of the magnetic flux within it. When the current in a coil inductor varies then this changes the magnetic flux. It is worth noting if the variation in current with respect to time is very high then v is very high — hence the reason we have to be careful when switching the current supplied to an inductor off as it causes high voltages to be induced.

Engineering application B.1.2

Similarly to inductors capacitors also have a derivative relationship between the voltage v across the capacitor with capacitance C and the current through the capacitor i.

$$i(t) = C\frac{\mathrm{d}v}{\mathrm{d}t}$$

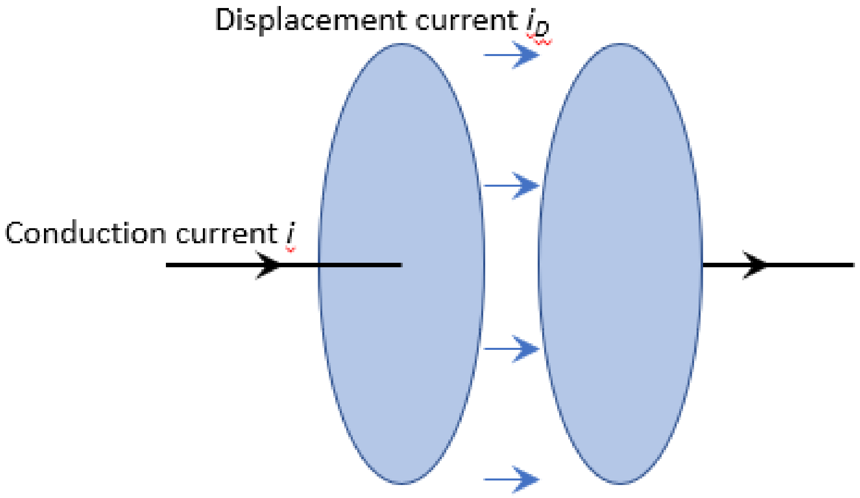


Figure B.1.5: Capacitor: conduction and dispalcement currents

The actual mechanism by which current apparently flows through a capacitor is complicated as there is no flow of charge through a capacitor. Instead charge builds-up on the plates which leads to a voltage (potential difference) across the capacitor. If the current flow is large then the charge builds up quicker which in turn means the rate of change of the this voltage is large. The current flowing through the capacitor wires is termed a *conduction current* while the apparent flow between the plates is termed as a *displacement current*. The latter is basically a virtual current that flows between the plates due to the build-up of positive charge on one plate and negative charge on the other plate. There is no real current flowing as the plates are separated by an insulating material.

Common derivatives

The first principles method described above by using a shrinking interval to limit when δx tends to 0. In practice we tend to find most derivatives using a table of common derivatives. A table of the ones you are most likely to require are given in Table B.1.1 — note that a, b and n are all constants, and that for all the trigonometric functions that $(ax + b)$ quantity which is an angle must be in radians for the derivatives to be valid.

Function $y(x)$	Derivative y'	Function $y(x)$	Derivative y'
constant	0	$\cos(ax+b)$	$-a\sin(ax+b)$
ax^n	anx^{n-1}	$\tan(ax+b)$	$a\sec^2(ax+b)$
e^x	e^x	$\operatorname{cosec}(ax+b)$	$-a\operatorname{cosec}(ax+b)\cot(ax+b)$
e^{-x}	$-e^{-x}$	$\sec(ax+b)$	$a\sec(ax+b)\tan(ax+b)$
e^{ax}	$a\,e^{ax}$	$\cot(ax+b)$	$-a\operatorname{cosec}^2(ax+b)$
$\ln x$	$\frac{1}{x}$	$\sin^{-1}(ax+b)$	$\frac{a}{\sqrt{1-(ax+b)^2}}$
$\sin x$	$\cos x$	$\cos^{-1}(ax+b)$	$\frac{-a}{\sqrt{1-(ax+b)^2}}$
$\cos x$	$-\sin x$	$\tan^{-1}(ax+b)$	$\frac{a}{1+(ax+b)^2}$
$\sin(ax+b)$	$a\cos(ax+b)$		

Table B.1.1: Derivatives of common functions

Example B.1.3

Use Table B.1.1 to find y' for the following functions:

a) $y = e^{-4x}$ b) $y = 2x^4$ c) $y = \sin(2x+6)$
d) $y = \tan(\omega t + \phi)$ e) $y = \frac{1}{\sqrt{x}}$ f) $y = \frac{1}{x^4}$

Solution

a) From table, if

$$y = e^{ax} \quad \text{then} \quad y' = a\,e^{ax}$$

In this case, $a = -4$ so

$$y' = -4\,e^{-4x}$$

b) From table:

$$y = ax^n \quad \text{then} \quad y' = anx^{n-1}$$

In this case, $a = 2$ and $n = 4$ so

$$y' = 8x^3$$

c) From table:

$$y = \sin(ax+b) \quad \text{then} \quad y' = a\cos(ax+b)$$

In this case, $a = 2$ and $b = 6$ so

$$y' = 2\cos(2x+6)$$

d) From table:

$$y = \tan(ax+b) \quad \text{then} \quad y' = a\sec^2(ax+b)$$

In this case, $a = \omega$ and $b = \phi$, and the independent variable is t rather than x so

$$y' = \frac{dy}{dt} = \omega\cos(\omega t + \phi)$$

e) First of we need to remember that $\frac{1}{\sqrt{x}} = x^{-1/2}$.
From table:

$$y = ax^n \quad \text{then} \quad y' = anx^{n-1}$$

In this case, $a = 1$ and $n = -\frac{1}{2}$ so

$$y' = -\frac{1}{2}x^{-3/2}$$

f) Again we need to remember that $\frac{1}{x^4} = x^{-4}$.
From table:

$$y = ax^n \quad \text{then} \quad y' = anx^{n-1}$$

In this case, $a = 1$ and $n = -4$ so

$$y' = -4x^{-5}$$

Questions

1. Calculate the gradient of the following functions at the specified point:
   a) $y = 3x^2$ at $(1, 3)$
   b) $y = 3x - x^2$ at $(0, 0)$
   c) $y = 1 + 2x + x^2$ at $(2, 9)$
   d) $y = x^2 + 2$ at $(2, 6)$
2. A function, y has a derivative $\frac{dy}{dx}$ that is always constant. What can you say about y?
3. Differentiate $y = 3x^2 + 6$. What is the rate of change of y when $x = 3, -2, 1, 0$?
4. Differentiate the following functions with respect to x:
   a) $y = e^{2x}$
   b) $y = 4\ln(x)$
   c) $y = \sin(x - 5)$
   d) $y = \cos(5x + 2)$
   e) $y = \frac{1}{\sqrt[3]{x}}$
   f) $y = \frac{2}{x^2}$
   g) $y = \tan^{-1}(2x - 4)$
   h) $y = \sec(3x + \pi)$
   i) $y = \frac{1}{e^{3x}}$
5. Given the expression for the voltage, v across an inductor with an inductance L is related to the current through the inductor i by $v = L\frac{di}{dt}$:

a) What is expression for the voltage when $i = 3\cos(\omega t)$ where ω is the angular frequency $\omega = 2\pi f$

b) What is expression for voltage when the current takes the form of a sine wave with an amplitude of 3A and a period of 0.02 sec.

1.2 Rules of differentiation

There are a number of rules that apply to differentiation when a function is actually the combination of two functions.

Linearity

Differentiation is a linear operator which means the derivative of two functions added together $(f(x) + g(x))$ is equal to the sum of the individual derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x) + g(x)) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} \tag{B.1.3}$$

The other property related to linearity is the derivative of a function multiplied by a constant value is simply the derivative of the function multiplied by the same constant value.

$$\frac{\mathrm{d}}{\mathrm{d}x}(kf(x)) = k\frac{\mathrm{d}f}{\mathrm{d}x} \tag{B.1.4}$$

Example B.1.4: Linearity

Differentiate $5x^2 - 3x + 6$

Solution

Let $y = 5x^2 - 3x + 6$, then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2 - 3x + 6\right)\\ &= 5\frac{\mathrm{d}}{\mathrm{d}x}(x^2) - 3\frac{\mathrm{d}}{\mathrm{d}x}(x) + \frac{\mathrm{d}}{\mathrm{d}x}(6) \text{ using linearity}\\ &= 5(2x) - 3 \text{ as derivative of a constant is always 0}\\ &= 10x - 3\end{aligned}$$

Product rule

This is a simple rule that allows us to calculate the derivative of a function that is formed by two functions multiplied together:

Product rule states that when

$$y(x) = u(x)v(x)$$

then

$$\frac{dy}{dx} = \frac{du}{dx}v + \frac{dv}{dx}u = u'v + v'u \qquad \text{(B.1.5)}$$

Engineering application B.1.3: Product rule

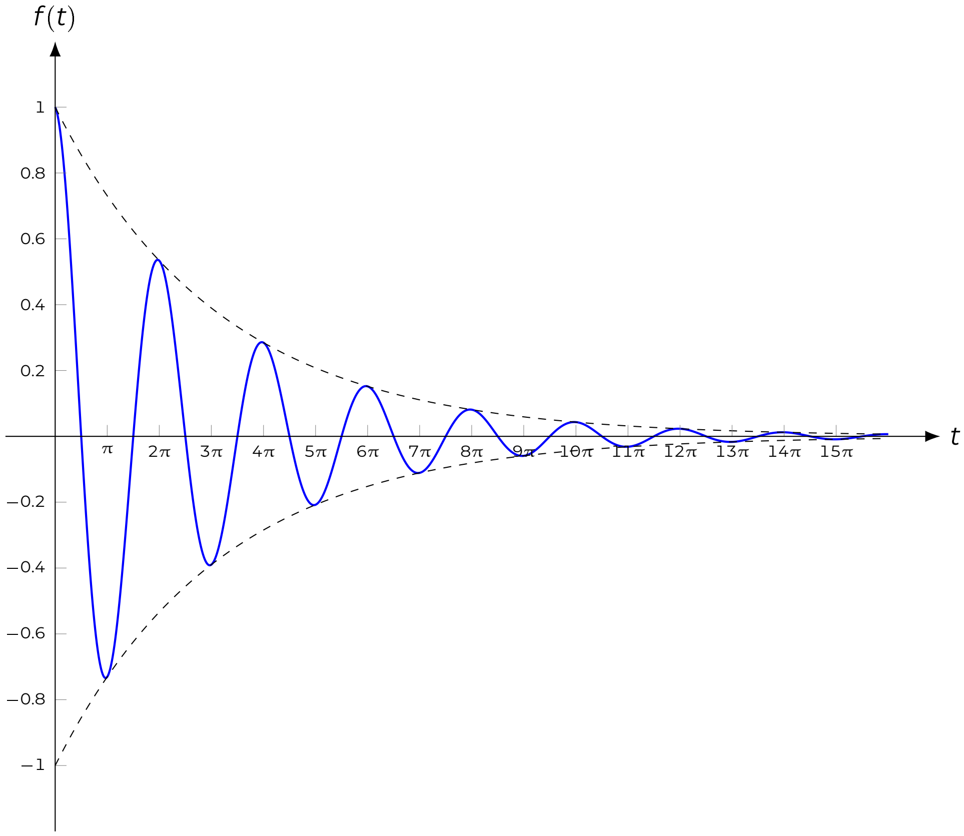


Figure B.1.6: Damped sinusoidal wave

The product rule crops up in electronic engineering is various places. A common one is looking at the *damped sinusoidal function* which simply represents a sinusoid whose amplitude decays exponentially with time:

$$f(t) = e^{-kt} \cos at$$

An example plot of this function with $a = 1$ and $k = 0.1$ is shown in Figure B.1.6. This type of function approximates the way the body of a car reacts when it drives over a large bump — initially there is a large amplitude but the shock absorbers swiftly dampen the oscillations.

With this type of response we often want to know how rapidly the system reacts and at what rate of change — so we need to find the derivative of $f(t)$. We can see that $f(t)$ is the product of two functions $u(t) = e^{-kt} = e^{-0.1t}$ and $v(t) = \cos at = \cos t$. So we can use the product rule to find the derivative.

$$\frac{du}{dt} = -0.1\, e^{-0.1t}$$

$$\frac{dv}{dt} = -a\sin(at) = -\sin t$$

So this means that:

$$\begin{aligned}\frac{df}{dx} &= \frac{du}{dx}v + \frac{dv}{dx}u \\ &= -0.1\,e^{-0.1t}\cos t + e^{-0.1t}(-\sin t) \\ \therefore \frac{df}{dx} &= -\,e^{-0.1t}(0.1\cos t + \sin t)\end{aligned}$$

Quotient Rule

This rule relates to differentiating of a function formed by one function divided by another such as $\frac{x^2}{\cos x}$ and $\frac{t^3 - t - 1}{t^2 + 1}$.

Quotient rule states that when

$$y(x) = \frac{u(x)}{v(x)}$$

then

$$\frac{dy}{dx} = \frac{\frac{du}{dx}v - \frac{dv}{dx}u}{v^2} = \frac{u'v - v'u}{v^2} \qquad \text{(B.1.6)}$$

Example B.1.5: Quotient rule

Find $\frac{dy}{dx}$ for $y = \frac{\sin x}{x}$.

Solution

Let $u = \sin x$ and $v = x$, then $y = \frac{u}{v}$ so we can use the quotient rule:

$$\begin{aligned}u' &= \frac{du}{dx} = \cos x \\ v' &= \frac{dv}{dx} = 1 \\ \implies y' &= \frac{u'v - v'u}{v^2} \\ &= \frac{x\cos x - \sin x}{x^2}\end{aligned}$$

Chain rule

This is a rule used to help differentiate complex equations where we can use a substitution to simplify the equation. For instance suppose $y(x) = ((\cos 2x)^2 - (2\cos 2x))$, then if we declare $z = \cos 2x$, then we can simplify the first equation to $y = z^2 - 2z$. But how do we find the derivative of y with respect to x — that is $\frac{dy}{dx}$.

Chain rule states that

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} \qquad \text{(B.1.7)}$$

Example B.1.6: Chain rule

Find $\frac{dy}{dx}$ for $y = \ln(x^2 + 3x + 2)$.

Solution

Let $z = x^3 + 3x + 2$ then $y = \ln z$ Now

$$\frac{dy}{dz} = \frac{1}{z}$$

$$\frac{dz}{dx} = 3x^2 + 3$$

So using chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx}\\ &= \frac{1}{z} \times (3x^2 + 3) \qquad \text{replace } z \text{ by } x^3 + 3x + 2\\ &= \frac{3x^2 + 3}{x^3 + 3x + 2}\end{aligned}$$

In fact example B.1.6 shows an important known result for functions where it is the natural logarithm of another function:

$$\text{When } y = \ln f(x) \quad \text{then} \quad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$$

Questions

1. Differentiate the following functions:

   a) $y = 2x^3 - x$

   b) $y = 2x^4 - 3x^2 + 2$

   c) $y = \sin 3t - 2\cos t + 2t$

   d) $y = \tan 5z$

   e) $y = 3\,e^{2x} - 2\sin 2x - 10$

   f) $y = \dfrac{2}{z^3} + \dfrac{\cos 3z}{3}$

   g) $y = \dfrac{1}{2}x^2 - \dfrac{1}{3}x^3$

   h) $y = 2\ln 3t + \cos 2t$

2. Differentiate the following functions using the product rule:

   a) $y = 2x\cos x$

   b) $y = \sin 2x \tan 2x$

   c) $y = \ln t \sin t$

   d) $y = 3\,e^{0.2t}\cos(2t + \pi)$

   e) $y = 3\,e^{2x}(x^2 - 1)$

   f) $y = e^{2x}\sqrt{2x}$

   g) $y = \left(1 + \cos\dfrac{t}{2}\right)\sin\dfrac{t}{2}$

   h) $y = 4\sin(t + 1)\cos(t - 1)$

3. Differentiate the following functions using the quotient rule:

   a) $y = \dfrac{\cos x}{x}$ b) $y = \dfrac{\sin x}{\cos x}$

   c) $y = \dfrac{\ln t}{\tan t}$ d) $y = \dfrac{x^2 + 4x + 4}{x - 1}$

   e) $y = \dfrac{1 + \mathrm{e}^{2t}}{t^2 - 1}$ f) $y = \dfrac{2 - \mathrm{e}^{-2t}}{2 + \mathrm{e}^{-t}}$

4. Differentiate the following functions using the chain rule:

   a) $y = (x^4 + 1)^5$ b) $y = \ln(x^2 - 4)$

   c) $y = 2\sqrt{4t - 3}$ d) $y = \sqrt{\sin(2t - 1)}$

   e) $y = 4\,\mathrm{e}^{-x^2}$ f) $y = \dfrac{1}{t^2 - 1}$

1.3 Higher order differentiation

Up to now we have just used first order differentiation but you can differentiate a first order differential $y'(x)$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ using same rules to get a second order differential $y''(x)$ or $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Similarly the third order differential is the differential of the second order and so on.

But why do we want higher order differentials? Well consider our simple example of velocity of an object being the rate of change of distance of object with respect to time — so first order differential of distance with respect to time. We could be interested in the acceleration of the object which is the rate of change of velocity — the second order differential of distance with respect to time. This is one place that second order differentials are used — another is when finding the maxima and minima of functions and points of inflexion as discussed below.

Maximum & minimum points

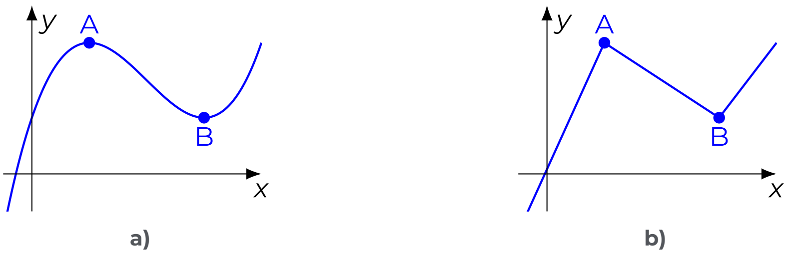


Figure B.1.7: Local maximum (A) and minimum (B) points

The maximum and minimum values of a functions are often important in engineering — for instance if we want to find the maximum power transferred from a voltage source to a load resistor. we can locate these points by using the differential of the function rather than the function itself.

First lets consider what we mean by maximum and minimum points — they are not the largest/smallest values a function can take as you might expect. Instead consider the functions in Figure B.1.7 where point A is a *local maximum* and point B is a *local minimum*. The use of 'local' stresses that A is only a maximum in its locality and B is only a minimum in its locality — not the lowest point of the graph.

How can we use differentiation to find these local maxima and minima (which are useful in certain situations). Well note as we move away from A, both the left and right, the value of y decreases, and similarly as we move away from B the value of y increases in both directions. If we plot tangents at points A & B to plot in Figure B.1.7a as shown in Figure B.1.8 then we can see the tangents are parallel to the x-axis. That is $\frac{\mathrm{dy}}{\mathrm{dx}} = 0$ at these points.

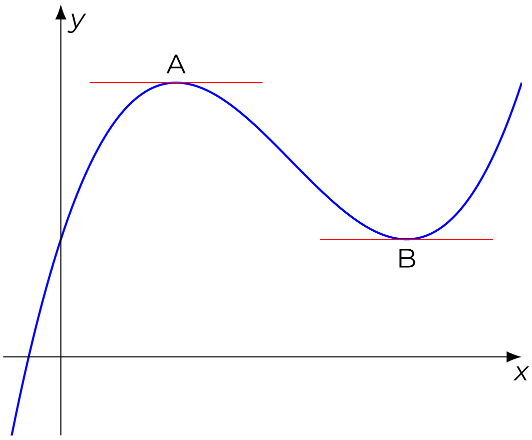


Figure B.1.8: Tangents to A & B

Considering plot in Figure B.1.7b we cannot draw tangents at points A & B: that is they do not exist. Therefore at these points $\frac{\mathrm{dy}}{\mathrm{dx}}$ does not exist.

So we can state that:

At maximum and minimum points of a curve, $\frac{\mathrm{dy}}{\mathrm{dx}} = 0$ or $\frac{\mathrm{dy}}{\mathrm{dx}}$ does not exist.

So we can find maxima and minima by finding points where this rule is true.

First derivative test

We can find points that will be maxima or minima by finding points when the above rule is true. But when we find these points how do we know whether it is a maximum or a minimum point? There are two main methods.

The first method is known as the *first derivative test*. Having found the points we then check the value of the derivative on either side of the point, as a maximum to the left of the point $y' > 0$ as y is increasing, but to the right of the point $y' < 0$ as y is decreasing whereas for a minimum $y' < 0$ to left of point but $y' > 0$ to right of point.

The first derivative test is for deciding whether a point is a maximum or a minimum:

- To the left of a maximum point $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive; to the right $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative.
- To the left of a minimum point $\frac{\mathrm{d}y}{\mathrm{d}x}$ is negative; to the right $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive

Example B.1.7

Determine the position of any maximum and minimum points for the functions:
a) $y = x^2 - 2x$ b) $y = -x^2 + x + 1$ c) $y = \frac{t^3}{3} - \frac{t^2}{2} - 3t + 3$

Solution

a) Differentiation of $y = x^2 - 2x$ gives us $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - 2$ which exists for all values of x. So we now need to solve $y' = 0$.

$$\begin{aligned} y' &= 0 \\ 2x - 2 &= 0 \\ 2x &= 2 \\ \implies x &= 1 \end{aligned}$$

So now only know that a maximum or minimum point can be found where $x = 1$. To find out what type it is we apply the first-derivative test. To the left of $x = 1$, at $x = 0, y' = 2(0) - 2 = -2$ and at $x = -1, y' = 2(-1) - 2 = -4$ so the first derivative is negative. To the right of $x = 1$ x is positive and therefore $y' > 0$. So since y' is going from negative to positive there must be a minimum point at $x = 1$ where $y = -1$.

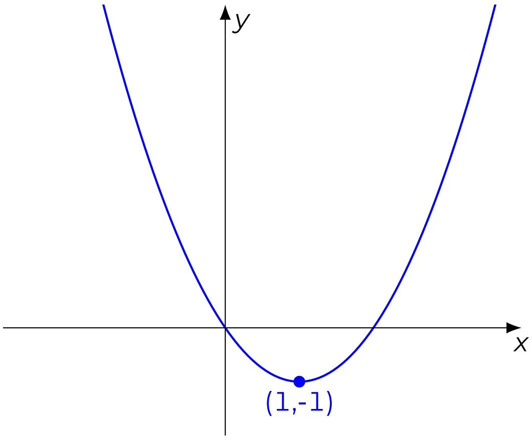


Figure B.1.9: Minimum of $y = x^2 - 2x$

b) Differentiation of $y = -x^2 + x + 1$ gives us $\frac{\mathrm{d}y}{\mathrm{d}x} = -2x + 1$ which exists for all values of x. So we now need to solve $y' = 0$.

$$y' = 0$$

$$\begin{aligned} -2x+1 &= 0 \\ 2x &= 1 \\ \implies x &= \frac{1}{2} \end{aligned}$$

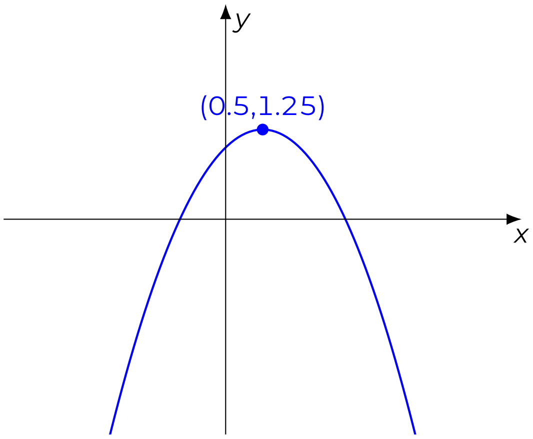


Figure B.1.10: Maximum of $y = -x^2 + x + 1$

So now only know that a maximum or minimum point can be found where $x = 0.5$. To find out what type it is we apply the first-derivative test. To the left of $x = 0.5$, at $x = 0, y' = -2(0) + 1 = 1$ so the first derivative is positive. To the right of $x = 0.5$, at $x = 1, y' = -2(1) + 1 = -1$ so the first derivative is negative. So since y' is going from positive to negative there must be a maximum point at $x = 0.5$ where $y = 1.25$.

c) Differentiation of $y = \frac{t^3}{3} - t^2 - 3t + 3$ gives us $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - 2t - 3$ which exists for all values of t. So we now need to solve $y' = 0$.

$$\begin{aligned} y' &= 0 \\ t^2 - 2t - 3 &= 0 \\ (t-3)(t+1) &= 0 \\ \implies t &= 3, -1 \end{aligned}$$

So now only know that there are two turning points (maximum or minimum) — one at $t = -1$ and one at $t = 3$. To find out what type they are we apply the first-derivative test. Taking the first point, to the left of $t = -1$, at $t = -2, y' = (-2)^2 - 2(-2) - 3 = 5$ so the first derivative is positive. To the right of $t = -1$, at $t = 0, y' = (0)^2 - 2(0) - 3 = -3$ so the first derivative is negative. So since y' is going from positive to negative there must be a maximum point at $t = -1$ where $y = 5\frac{5}{6}$. For the second point to the left of $t = 3$, at $t = 1, y' = (1)^2 - (1) - 3 = -$ so the first derivative is negative. To the right of $t = 3$, at $t = 4, y' = (4)^2 - (4) - 3 = 9$ so the first derivative is positive. So since y' is going from negative to positive there must be a minimum point at $t = 3$ where $y = -1\frac{1}{2}$.

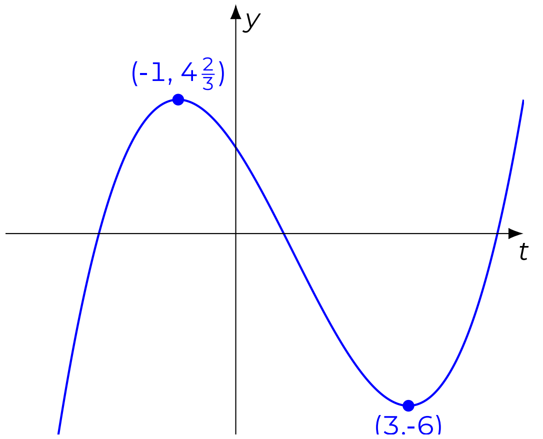


Figure B.1.11: Turning points of $y = \frac{t^3}{3} - t^2 - 3t + 3$

Second derivative test

The other method of working out whether a turning point is a maximum or a minimum is the *second derivative test* . Thinking about the first derivative test we know that a maximum point y' changes from positive to 0 to negative, hence y' is decreasing. This means that y'' will be negative. Similarly at a minimum point y' goes from negative to 0 to positive so is increasing so we know that y'' will be positive. Therefore we can say:

The second derivative test is for deciding whether a point is a maximum or a minimum:

- If $y' = \frac{dy}{dx} = 0$ and $y'' = \frac{d^2y}{dx^2} < 0$ at a point, then the point is a maximum turning point.
- If $y' = \frac{dy}{dx} = 0$ and $y'' = \frac{d^2y}{dx^2} > 0$ at a point, then the point is a minimum turning point.
- If $y' = \frac{dy}{dx} = 0$ and $y'' = \frac{d^2y}{dx^2} = 0$ at a point, then the second derivative test fails and you must use the first derivative test.

Example B.1.8

Determine the position of any maximum and minimum points for the functions in example B.1.7

Solution

a) Given $y = x^2 - 2x$ then $y' = 2x - 2$ and $y'' = 2$. To locate the turning point we now need to solve $y' = 0$ which gives us a turning point at $x = 1$. To find out what type it is we apply the second -derivative test. $y'' = 2$ at all points which is positive so the turning point is a minimum at (1,-1)

b) Given $y = -x^2 + x + 1$ then $y' = -2x + 1$ and $y'' = -2$. To locate the turning point we now need to solve $y' = 0$ which gives a turning point at $x = \frac{1}{2}$. Applying the second derivative test, $y'' < 0$ so this point is a minimum at (0.5,1.25)

c) Given $y = \frac{t^3}{3} - t^2 - 3t + 3$ then $y' = t^2 - 2t - 3$ and $y'' = 2t - 2$ To locate the turning point we now need to solve $y' = 0$ which gives us turning points at $t = 3$ and $t = -1$. Now apply the second derivative test at both these points: $y''_{t=-1} = 2(-1) - 2 = -4$ and $y''_{t=3} = 2(3) - 2 = 4$. So we have a maximum turning point at (-1, $4\frac{2}{3}$) and a minimum turning point at (3,-6).

Engineering application B.1.4

Maximum power transfer

Consider the system in Figure B.1.12 where a non-ideal voltage source (with an internal resistance of R_S) is connected to a variable load resistance (R_L). We want to calculate the value of R_L which results in the maximum power being transferred from the source to the load resistor. This is often used in power systems design when it can be important to consider how we can transfer the maximum power from the source to where it is being consumed.

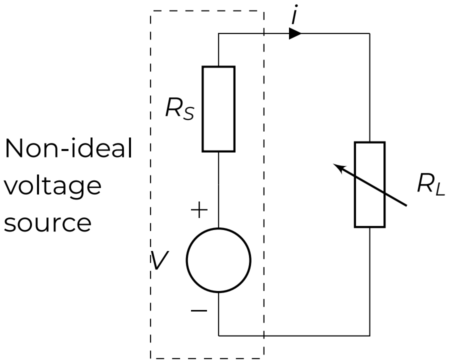


Figure B.1.12: Maximum power transfer

Solution

If we let i be current flowing in the circuit and V be the source voltage, then by Ohms Law and Kirchoff's voltage law:

$$V = i(R_S + R_L)$$

If we let P be the power developed in the load resistor then:

$$P = i^2 R_L = \frac{V^2 R_L}{(R_s + R_L)^2}$$

Clearly P depends on value of R_L so if we differentiate above equation with respect to R_L using the quotient rule in equation B.1.7:

$$\frac{dP}{dR_L} = V^2 \frac{(R_s + R_L)^2 - R_L 2(R_S + R_L)}{(R_S + R_L)^4}$$

$$= V^2 \frac{(R_s + R_L)^{\cancelto{1}{2}} - 2R_L\cancel{(R_S + R_L)}}{(R_S + R_L)^{\cancelto{3}{4}}}$$

$$= V^2 \frac{(R_s + R_L) - 3R_L}{(R_S + R_L)^3}$$
$$= V^2 \frac{R_s - R_L}{(R_S + R_L)^3}$$

To find turning point need to solve $\frac{\mathrm{d}P}{\mathrm{d}R_L} = 0$

$$V^2 \frac{R_s - R_L}{(R_S + R_L)^3} = 0$$

which is only true when $R_L = R_S$.
So we have a turning point when the load resistance is equal to the source resistance but is this a maximum point (as we need for maximum power transfer). Applying second-order differentiation rule:

$$\frac{\mathrm{d}P}{\mathrm{d}R_L} = V^2 \frac{R_S - R_L}{(R_S + R_L)^3}$$
$$\frac{\mathrm{d}^2P}{\mathrm{d}R_L^2} = V^2 \frac{-(R_S + R_L)^3 - (R_S - R_L)3(R_s + R_L)^2}{(R_S + R_L)^6}$$
$$= V^2 \frac{-(R_S + R_L)^{\cancelto{1}{3}} - 3(R_S - R_L)\cancel{(R_s + R_L)^2}}{(R_S + R_L)^{\cancelto{4}{6}}}$$
$$= V^2 \frac{-(R_S + R_L) - 3(R_S - R_L)}{(R_S + R_L)^4}$$
$$= V^2 \frac{2R_L - 4R_S}{(R_S + R_L)^4}$$
$$= 2V^2 \frac{(R_L - 2R_S)}{(R_S + R_L)^4}$$

When $R_L = R_S$ then the second derivative evaluates to:

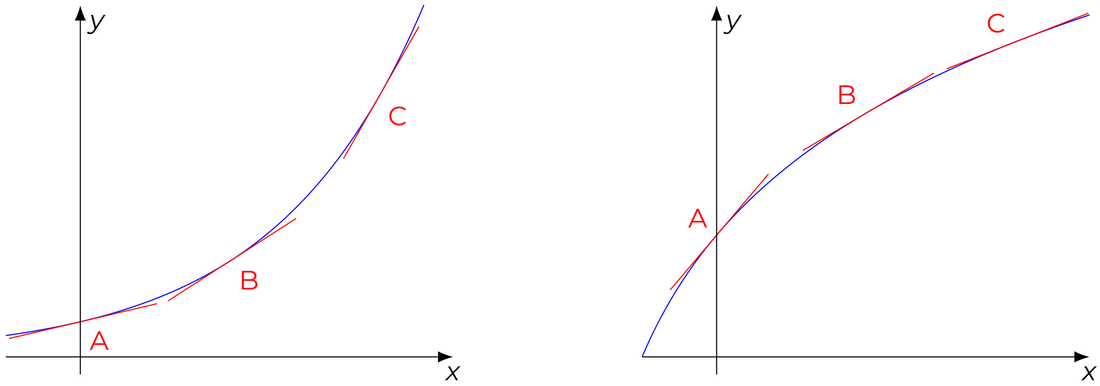
$$\frac{\mathrm{d}^2P}{\mathrm{d}R_L^2} = 2V^2 \frac{(R_S - 2R_S)}{(R_S + R_S)^4} = \frac{-V^2}{8R_S^3}$$

This is a negative value so the turning point is a maximum - therefore the maximum power transfer occurs when the load resistance equals the source resistance.

Points of inflexion

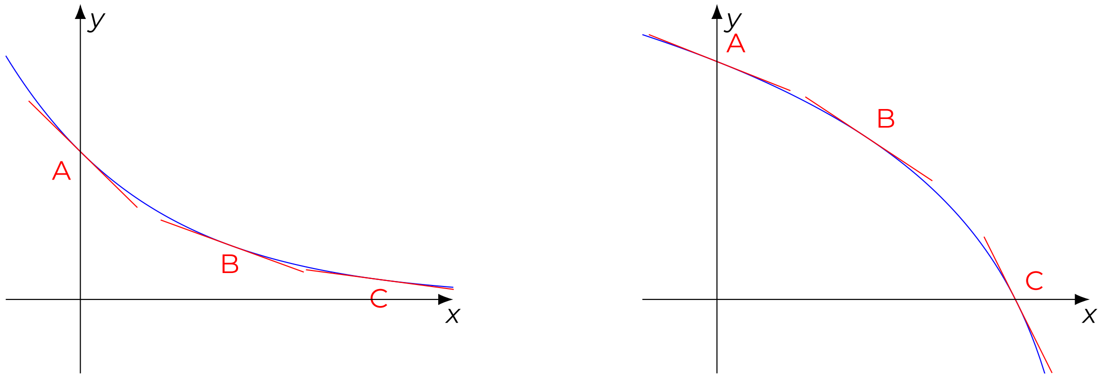
The first and second differentials of a function can be used to determine the nature of functions — that is the function increasing or decreasing. For instance, considering $y = x^2$, y is increasing to the right of the y-axis (as x increases y increases), and decreasing to the left of the y-axis (as x increases y decreases). Looking at Figures B.1.13a & B.1.13b we can see that y increases as x increases and the tangents at A, B & C are all positive (that is $y' > 0$). Whereas for at Figures B.1.13c & B.1.13d as x increases y decreases and the tangent gradients are all negative (that is $y' < 0$). However there is clearly a difference in the way y increases in Figures B.1.13a & B.1.13b.

Considering the tangents in Figure B.1.13a, we can see that as x increases the gradients (y') increase which implies that $y'' > 0$ whereas in Figure B.1.13b the gradients of the tangents decrease as x increases so is $y'' < 0$. We define a function as *concave up* when y' increases as x increases and *concave down* when y' decreases with increasing x.



a) y is concave up

b) y is concave down



c) y is concave up

d) y is concave down

Figure B.1.13: Increasing & decreasing functions

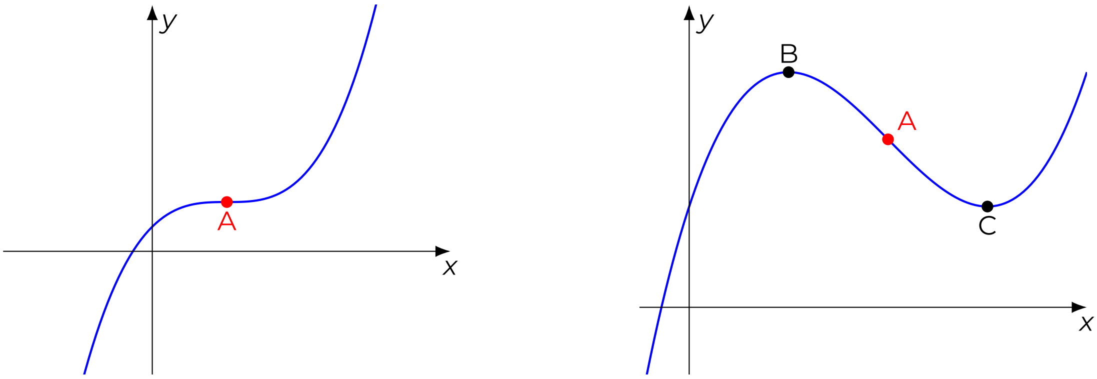
Considering the curve in Figure B.1.13c we can so that as x increases the gradient (y') is increasing (becoming less negative) which implies that $y'' > 0$ therefore the curve can be said to be concave up. Similarly for Figure B.1.13d the gradient y' is decreasing (becoming more negative) as x increases so $y'' < 0$ and thus the curve in concave down.

A point of inflexion is the point at which the concavity of a curve changes, that is the curve goes from concave up to concave down or vice versa without being a local maximum or minimum. This is the point where the second differential is zero ($y'' = 0$) or exceptionally has no value.

We can state that:

When y' is increasing then $y'' > 0$, the function is concave up.
When y' is decreasing then $y'' < 0$, the function is concave down
A point of inflexion is when $y'' = 0$ or has no value, but it is necessary to consider the concavity of the function either side of this point to ensure it is a point of inflexion (i.e. is concavity changing)

Looking at curves in Figure B.1.14, you can see that in both figures there is a point of inflexion at A - but Figure B.1.14b also has a local maximum at B and a local minimum at C.



a) $x^3 - 3x^2 + 3x + 1$ **b)** $x^3 - 6x^2 + 9x + 3$

Figure B.1.14: Points of inflexion

Example B.1.9

Looking at functions in Figure B.1.14, let us work out where all turning points & points of inflexion are.

For $y = x^3 - 3x^2 + 3x + 1$:

$$y' = 3x^2 - 6x + 3$$
$$y'' = 6x - 6$$

Turning points when $y' = 0$ with type given by sign of y''

$$3x^2 - 6x + 3 = 0$$
$$3(x^2 - 2x + 1) = 0$$
$$3(x - 1)^2 = 0x \qquad = 1$$
$$\text{At } x = 1:$$
$$y'' = 6(1) - 6 = 0$$

No turning point as $y'' = 0$ and inspection of the graph (and knowledge that $y' = 3(x - 1)^2$ so always positive) shows that $y' > 0$ for both sides at $x = 1$ but we can infer that there is a point of inflexion at (1, 2).

For $x^3 - 6x^2 + 9x + 3$:

$$\begin{aligned} y' &= 3x^2 - 12x + 9 \\ y'' &= 6x - 12 \end{aligned}$$

Turning points when $y' = 0$ with type given by sign of y''

$$\begin{aligned} 3x^2 - 12x + 9 &= 0 \\ 3(x^2 - 4x + 3) &= 0 \\ 3(x - 1)(x - 3) &= 0 \\ x &= 1, 3 \end{aligned}$$

$$\begin{aligned} &\text{At } x = 1: \\ &\qquad y'' = 6(1) - 12 = -6 \implies \text{ a local maximum at } (1, 7) \\ &\text{At } x = 3: \\ &\qquad y'' = 6(3) - 12 = 6 \implies \text{ a local minimum at } (3, 3) \end{aligned}$$

Point of inflexion when $y'' = 0$:

$$\begin{aligned} 6x - 12 &= 0 \\ 6x &= 12 \\ x &= 2 \end{aligned}$$

Therefore there is a point of inflexion at (2, 5) with a maximum turning point at (1, 7) and a minimum turning point at (3, 3).

Not that it is not always the case that if $y' = y'' = 0$ then there is a point of inflexion — this is only true if y' has same side both sides of the point. As an example let us consider $y = x^4$:

Example B.1.10

Find all turning points and points of inflexion of $y = x^4$.

Differentials: $y' = 4x^3$ and $y'' = 12x^2$. Both of these are zero at $x = 0$ but to the left of this value $y' < 0$ as cube of a negative number is a negative number but to the right of this point $y' > 0$ as cube of a positive number is a positive number so we can say that this is a minimum turning point (by the first derivative test)

Questions

Locate the maximum points, minimum points and points of inflexion for the following functions:

1. $y = 3x^2 + 6x - 1$
2. $y = 4 - t - t^2$
3. $y = x^3 + 3x^2 - 9x + 7$
4. $y = t^6$
5. $y = x^4 - 2x^2 + 1$
6. $y = t + \frac{1}{t}$
7. $3x^5 - 5x^3$
8. $y = 3t^{1/3}$

1.4 Summary

This chapter has given a brief revision/introduction to differentiation and explored how it is used in electronic & electrical engineering. You will look at this in more depth as an when it is required in your other modules, but in effect differentiation is a way of representing th rate of change of a variable with respect to another varying quantity. In AC circuit analysis we are often looking that rate of change of current or voltage with respect to time. However much of the time we are interested in the opposite to differentiation — that is integration as explored in the next chapter.

Integration

2.1 Introduction

When we have a function $f(x)$ we know we can differentiate it to obtain the derivative $\frac{df}{dx}$, but what about the reverse process. That is to obtain $f(x)$ when we know its derivative — this reverse process is called *integration*. So integration is the reverse of differentiation.

Integration is also related to the problem of finding the area between a curve and the x-axis — this is not immediately obvious but we explore it in the next section. However why would we need to know this value — well consider a plot of current flow into a capacitor against time. The area under the curve in this case represents the total charge stored by the capacitor as current is simply a measure of rate of charge moving over time.

Integration is used a lot in analogue circuit analysis as well as in classic control theory where often we have an input that represents the rate of change of a variable with respect to time and we are interested in this variable. As a simple example consider the distance travelled by a car that is travelling at a known velocity for a known time. In order to get the distance we have to integrate the velocity over the time as velocity is imply the rate of change of distance with time.

2.2 Basic Integration

Let us consider the simple expression $\frac{dy}{dx} = 2x$ — what is value of function y? Well we know that the differentiation of function $y = x^2 + c$ results in $\frac{dy}{dx} = 2x$ for any value of c. This implies that the solution to our problem is $y = x^2 + c$ where c is the *constant of integration* and can take any value. We found this solution by knowledge of what the derivative of y is — we can say that $2x$ has been *integrated* to give $x^2 + c$. Mathematically we represent integration by $\int$ so we represent the above problem as:

$$\frac{dy}{dx} = 2x \qquad \Longrightarrow y = \int 2x \, dx = x^2 + c$$

In general, if: $\frac{dy}{dx} = f(x)$

then $y = \int f(x) \, dx$

Engineering application B.2.1: Distance travelled by an object

Constant speed

Let us consider a simple example of a car travelling for 1 minute at 36 kph (kilometres per hour) where we want to know how far the car has travelled. Well we know velocity is the derivative of distance with respect to time. First of all we need to work out what the speed (velocity) is in m/s

$$\begin{aligned}36\,\text{kph} &= 36 \times 1000\text{m/h}\\ &= \frac{36000}{60 \times 60}\text{m/s}\\ &= 10\,\text{m/s}\end{aligned}$$

In graphical terms, a plot of velocity against time for a constant velocity is a straight line as shown by blue line in Figure B.2.1.

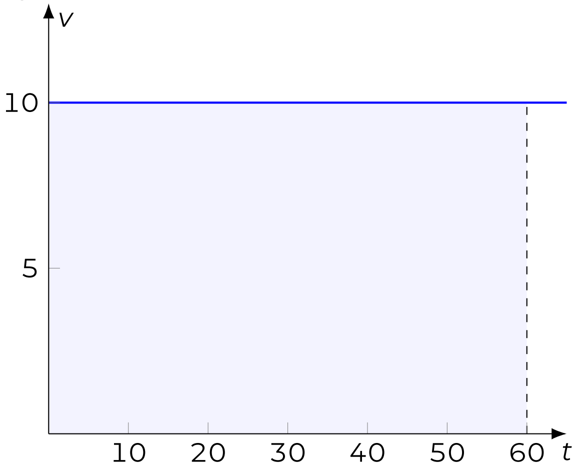


Figure B.2.1: Plot of $v = 10$ m/s

The total distance travelled in 60 sec is equivalent to the area under the line as shown by shaded area in Figure B.2.1 which is equal to 600 m. This could also have been calculated by integrating the velocity with respect to time over 0 to 60 s to give distance s.

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= 10\\ s &= \int_0^{60} 10\,\mathrm{d}t\\ &= [10t + c]_0^{60}\\ &= 10(60) + c - (10(0) + c) \qquad \text{the two constants cancel out}\\ &= 600\,\text{m}\end{aligned}$$

This shows that integration of a function is equal to the area under a curve for a definite integral (one with limits).

Constant acceleration

Let us consider a slightly more complex scenario when the object is travelling at a constant acceleration of 1 m/s^2 from rest at ($t = 0$ s (so velocity = 0 m/s). How far as the car travelled after 60 sec?

Let us first remember that acceleration is the second differential of distance with respect to time. So a constant acceleration is the gradient of the velocity against time plot - i.e. $v = at + c$, but we know in this case at $t = 0$, $v = 0$ so $c = 0$ and $v = t$ as $a = 1$. We can also get this equation by integrating a with respect to t.

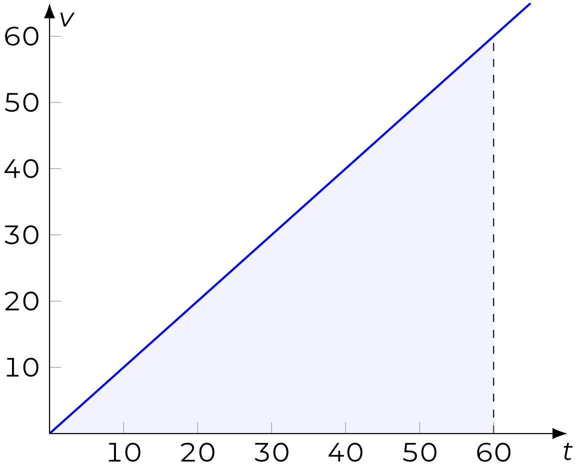


Figure B.2.2: Plot of $v = 5t$

Again the distance travelled in 1 minute is the shaded area in Figure B.2.2 which we can calculate from the formula for the area of a right angled triangle ($a = \frac{1}{2}$base × height).

$$\begin{aligned} \text{distance} &= \frac{1}{2} \times 60 \times 60 \\ &= \frac{1}{2} \times 3600 \\ &= 1800\,\text{m} = 1.8\,\text{km} \end{aligned}$$

As with the first example we can integrate the velocity equation with respect to time over the limits 0 to 60 s to give distance s.

$$\begin{aligned} \frac{\text{ds}}{\text{dt}} &= t \\ s &= \int_0^{60} t\,\text{d}t \\ &= \left[\frac{1}{2}(t^2 + c)\right]_0^{60} \\ &= \frac{1}{2}(60^2 + c - (0^2 + c)) = \frac{1}{2}(3600) \\ &= 1800\,\text{m} = 1.8\,\text{km} \end{aligned}$$

Common Integrals

As with differentiation we tend to use a table of common integrals to find the result of integration as opposed to first principles or deducing the result from knowledge of differentiation. Table B.2.1 lists the common functions that you are most likely to require — note that although x is the variable used throughout, it can be replaced by any other variable such as t. Also note that a, b and n are all constants, and that for all the trigonometric functions that all

Function $f(x)$	Integral $\int f(x)\,\mathrm{d}x$	Function $f(x)$	Integral $\int f(x)\,\mathrm{d}x$
k, *constant*	$kx + c$	$\cos(ax+b)$	$\dfrac{\sin(ax+b)}{a} + c$
x^n	$\dfrac{x^{n+1}}{n+1} + c \quad n \neq -1$	$\tan x$	$\ln\lvert\sec x\rvert\ + c$
$x^{-1} = \dfrac{1}{x}$	$\ln\lvert x\rvert\ + c$	$\tan ax$	$\dfrac{\ln\lvert\sec ax\rvert}{a} + c$
e^x	$\mathrm{e}^x + c$	$\tan(ax+b)$	$\dfrac{\ln\lvert\sec(ax+b)\rvert}{a} + c$
e^{-x}	$-\mathrm{e}^{-x} + c$	$\operatorname{cosec}(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\operatorname{cosec}(ax+b)$
e^{ax}	$\dfrac{\mathrm{e}^{ax}}{a} + c$		$-\cot(ax+b)\rvert\} + c$
$\sin x$	$-\cos x + c$	$\sec(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sec(ax+b)$
$\sin ax$	$\dfrac{-\cos ax}{a} + c$		$+\tan(ax+b)\rvert\} + c$
$\sin(ax+b)$	$\dfrac{-\cos(ax+b)}{a} + c$	$\cot(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sin(ax+b)\rvert\} + c$
$\cos x$	$\sin x + c$	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a} + c$
$\cos ax$	$\dfrac{\sin x}{a} + c$	$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

Table B.2.1: Integrals of common functions

angles must be in radians for the integrals to be valid.

Example B.2.1

Use Table B.2.1 to integrate following functions where k & n are constants:

a) x^3 b) $\sin kx$ c) $\cos(2x+5)$ d) 2.4
e) $\tan(3t-2)$ f) e^{-2z} g) $\dfrac{1}{x^3}$ h) $\sin(20\pi nt)$

Solution

a) From table, we know $\displaystyle\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ provided that $n \neq -1$. In this case, $n = 3$ so

$$\int x^3\,\mathrm{d}x = \frac{x^4}{4} + c$$

b) From table: $\displaystyle\int \sin ax\,\mathrm{d}x = \frac{-\cos ax}{a} + c$. In this case, $a = k$ so

$$\int \sin kx\,\mathrm{d}x = \frac{-\cos kx}{k} + c$$

c) From table: $\displaystyle\int \cos(ax+b)\,\mathrm{d}x = \frac{\sin(ax+b)}{a} + c$. In this case, $a = 2$ and $b = 5$ so

$$\int \cos(2x+5)\,\mathrm{d}x = \frac{\sin(2x+5)}{2} + c$$

d) From table: $\int k \, \mathrm{d}x = kx + c$. In this case $k = 2.4$ so

$$\int 2.4 \, \mathrm{d}x = 2.4x + c$$

e) From table: $\int \tan(ax+b) \, \mathrm{d}x = \dfrac{\ln|\sec(ax+b)|}{a} + c$. In this case, $a = 3$ and $b = -2$ with a variable of t not x so

$$\int \tan(3t-2) \, \mathrm{d}t = \frac{\ln|\sec(3t-2)|}{3} + c$$

f) From table: $\int \mathrm{e}^{ax} \, \mathrm{d}x = \dfrac{\mathrm{e}^{ax}}{a} + c$. In this case $a = -2$ and the variable is z so

$$\int \mathrm{e}^{-2z} \, \mathrm{d}z = -\frac{\mathrm{e}^{-2z}}{2} + c$$

g) First of we need to remember that $\dfrac{1}{x^3} = x^{-3}$.
From table: $\int x^n \, \mathrm{d}x = \dfrac{x^{n+1}}{n+1} + c$ provided that $n \neq -1$. In this case, $n = -3$ so

$$\int \frac{1}{x^3} \, \mathrm{d}x = \int x^{-3} \, \mathrm{d}x = \frac{x^{-2}}{-2} + c = c - \frac{1}{2x^2}$$

h) When integrating $\sin(20\pi nt)$ we can note that $20\pi n$ is a constant so this simply is the integration of $\sin at$ with respect to t. Using table and setting $a = 20\pi n$

$$\int \sin 20\pi nt \, \mathrm{d}t = \frac{-\cos 20\pi nt}{20\pi n} + c$$

Linearity

As with differentiation, we can say that integration is a linear operator so uf f and g are functions of x then the integral of the sum of functions $f(x) + g(x)$ is the sum of the integrals of the individual functions:

$$\int (f(x) + g(x)) \, \mathrm{d}x = \int f(x) \, \mathrm{d}x + \int g(x) \, \mathrm{d}x$$

The other property related to linearity is the integral of a function multiplied by a constant value is simply the integral of the function multiplied by the same constant value.

$$\int Af(x) \, \mathrm{d}x = A \int f(x) \, \mathrm{d}x$$

In general these two properties are simply special cases of the general linearity property given in equation B.2.1:

$$\int (Af(x) + Bg(x))\,dx = A\int f(x)\,dx + B\int g(x)\,dx \qquad \text{(B.2.1)}$$

Engineering application B.2.2: Voltage across a capacitor

Recall from Engineering application B.1.2 that the current, $i(t)$, through a capacitor depends on the time t

$$i(t) = C\frac{dv}{dt}$$

where $v(t)$ is voltage across the capacitor and C is the capacitance. So we can derive an expression for $v(t)$ using integration:

$$\begin{aligned}\frac{d}{dv}(t) &= \frac{i}{C}\\ \therefore v &= \int \frac{i}{C}\,dt = \frac{1}{C}\int i\,dt \quad \text{using linearity}\end{aligned}$$

So if we know the expression for $i(t)$ through a capacitor we can find the voltage across it.

Example B.2.2: Linearity

Use Table B.2.1 and the linearity property in equation B.2.1 to integrate following functions:

a) $3x^3 - \sqrt{x}$

b) $4\sin(3t + 4)$

c) $(x + 1)^2$

d) $3t - e^{-t}$

e) $\frac{1}{x}$

f) $3\cos\left(\frac{\pi}{3}\right)$

Solution

a)

$$\begin{aligned}\int (3x^3 - \sqrt{x})\,dx &= 3\int x^3\,dx - \int x^{1/2}\,dx \quad \text{using linearity}\\ &= 3\left(\frac{x^4}{4}\right) - \left(\frac{x^{3/2}}{3/2}\right) + c \quad \text{from Table B.2.1}\\ &= \left(\frac{3x^4}{4}\right) - \left(\frac{2x^{3/2}}{3}\right) + c\end{aligned}$$

b)

$$\begin{aligned}\int 4\sin(3t + 4)\,dt &= 4\int \sin(3t + 4)\,dt \quad \text{using linearity}\\ &= \frac{-4\cos(3t + 4)}{3} + c\end{aligned}$$

c)

$$\begin{aligned}\int (x + 1)^2\,dx &= \int x^2 + 2x + 1\,dx\\ &= \int x^2\,dx + 2\int x\,dx + \int 1\,dx \quad \text{using linearity}\end{aligned}$$

$$= \frac{x^3}{3} + x^2 + x + c$$

d)

$$\int 3t - \mathrm{e}^{-t}\,\mathrm{d}t = \int 3t\,\mathrm{d}t - \int \mathrm{e}^{-t}\,\mathrm{d}t \quad \text{using linearity}$$

$$= \frac{3t^2}{2} + \mathrm{e}^{-t} + c$$

e)

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + \quad \text{from Table B.2.1}$$

However if we write $c = \ln|A|$ then answer becomes $\ln|x| + \ln|A|$ and we can use the law of logarithms to rewrite the answer as

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|Ax|$$

f) Looking at $3\cos\left(\frac{\pi}{3}\right)$ we can say that $\cos\left(\frac{\pi}{3}\right)$ is a constant as it has no variable so this becomes:

$$\int 3\cos\left(\frac{\pi}{3}\right)\mathrm{d}x = 3\cos\left(\frac{\pi}{3}\right)x + c$$

Definite & Indefinite Integrals

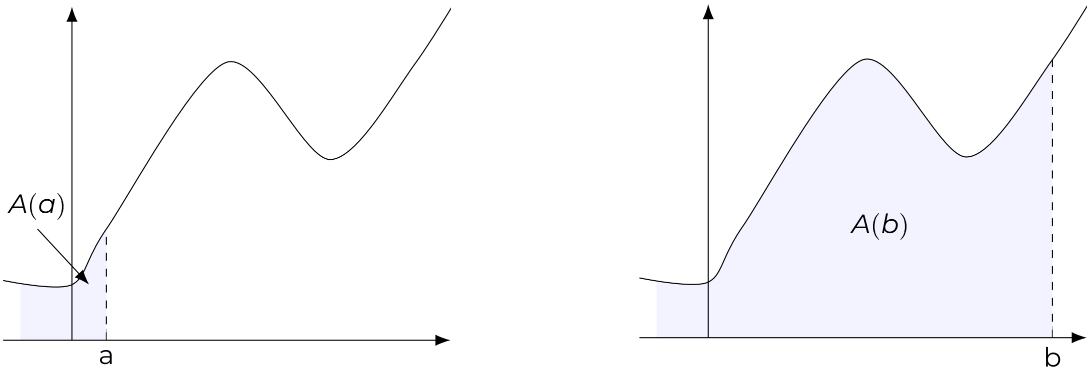
Most of the integrals met so far have been *indefinite integrals* — that is ones that contain a constant of integration. However in Engineering application B.2.1 we saw examples of *definite integrals* — that is integration over a limited area. In general:

The area under the curve, $y(x)$, between $x = a$ and $x = b$ is denoted as

$$\int_{x=a}^{x=b} y(x)\,\mathrm{d}x \qquad \text{or more compactly} \qquad \int_a^b y(x)\,\mathrm{d}x$$

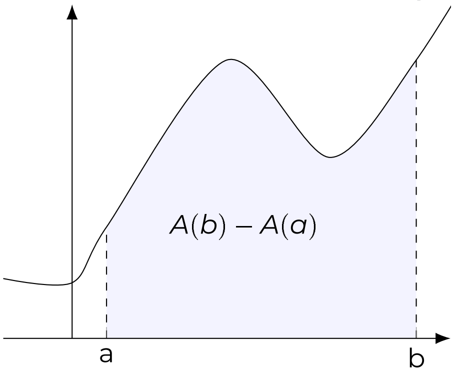
The *limits* of the integral are represented by the constants a for the *lower limit* and b for the *upper limit*.

But what does this mean in practice? Well let's consider Figure B.2.3 that illustrates a general function that we integrate between limits a and b — leading to a definite area hence the term *definite integral*. The area under the curve up to the vertical line defined by $x = a$ is $A(a)$ as shown in Figure B.2.3a. Similarly, as shown in Figure B.2.3b, the area under the curve up to the vertical line defined by $x = b$ is given by $A(b)$. So the area between $x = a$ and $x = b$ is $A(b) - A(a)$ as shown in Figure B.2.3c.



a) Area bounded by limit a

b) Area bounded by limit b



c) Area between limits $x = a$ and $x = b$

Figure B.2.3: Definite integral: area under curve

The area between $x = a$ and $x = b$ is given by

$$\text{Area} = \int_a^b y(x)\,\mathrm{d}x = A(b) - A(a)$$

So to calculate the results of a definite integral, the integral is evaluated at the upper limit b and the lower limit a. The result is then the difference between these two values.

Often, we will find the expression $A(b) - A(a)$ written as $[A(x)]_a^b$. In similar vein, $[x^2 - 1]_2^4$ is evaluated as the value of $x^2 - 1$ at $x = 4$ minus the value of $x^2 - 1$ at $x = 2$:

$$[x^2 - 1]_2^4 = (4^2 - 1) - (2^2 - 1) = (16 - 1) - (4 - 1) = 15 - 3 = 12$$

In general

$$[f(x)]_a^b = f(b) - f(a)$$

Note that as

$$\int_a^b y(x)\,\mathrm{d}x = A(b) - A(a)$$

then if we interchange the upper and lower limits

$$\int_b^a y(x)\,\mathrm{d}x = A(a) - A(b) = -\{A(b) - A(a)\}$$

Therefore in general we can say that:

$$\int_a^b y(x)\,\mathrm{d}x = -\int_b^a y(x)\,\mathrm{d}x$$

Interchanging the limits of an integral changes the sign of the integral

In addition to the rule above interchanging limits with definite integrals there are three other points to note:

1. The integrated function is evaluated at the upper and lower limits, and the difference is found.
2. There is no need for a constant of integration (unlike with indefinite integrals).
3. All angles are measured in radians — this applies to limits as well as in functions themselves.

Example B.2.3

Evaluate the following definite integrals:

a) $\int_1^3 (x-1)\,\mathrm{d}x$ b) $\int_3^1 (x-1)\,\mathrm{d}x$ c) $\int_0^{\pi/2} \cos x\,\mathrm{d}x$

Solutions

a) First we do the integration:

$$\int_1^3 (x-1)\,\mathrm{d}x = \left[\frac{x^2}{2} - x\right]_1^3$$

We now evaluate the integral at the upper and lower limits and find the difference:

$$\left(\frac{3^2}{2} - 3\right) - \left(\frac{1^2}{2} - 1\right) = \frac{9}{2} - 3 - \left(\frac{1}{2} - 1\right) = \frac{3}{2} - \left(\frac{-1}{2}\right) = \frac{4}{2} = 2$$

b) This is the same integral as above with the limits interchanged so we can apply the rule that this changes the sign of the result

$$\int_3^1 (x-1)\,\mathrm{d}x = -\int_1^3 (x-1)\,\mathrm{d}x = -2$$

c) $$\int_0^{\pi/2} \cos x\,\mathrm{d}x = [\sin x]_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin 0 = 1 - 0 = 1$$

Integration by parts

This is a technique to integrate a product (i.e. where the integrand is two functions multiplied together) derived from the product rule for differentiation (equation B.1.5). If u and v are functions of x, the the product rule of differentiation states that

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + \frac{dv}{dx}u$$

Rearranging this leads to

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating this equation gives us:

$$\int u\frac{dv}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int v\frac{du}{dx}\,dx$$

Recognising that integration is simply the inverse of differentiation means:

$$\int \frac{d}{dx}(uv)\,dx = uv$$

which leads to formula for integration by parts

For indefinite internals:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \qquad \text{(B.2.2)}$$

For definite integrals:

$$\int_a^b u\frac{dv}{dx}\,dx = [uv]_a^b - \int_a^b v\frac{du}{dx}\,dx \qquad \text{(B.2.3)}$$

Example B.2.4

Evaluate the following integrals:

a) $\int x\cos x\,dx$

b) $\int_0^3 x\,e^x\,dx$

c) $\int_0^3 x^2\,e^x\,dx$

d) $\int e^t\cos t\,dt$

Solution:

a) Recognise that integrand is product of x and $\cos x$ so let:

$$u = x \quad \text{and} \quad \frac{dv}{dx} = \cos x$$

Then:

$$\frac{du}{dx} = 1 \quad \text{and} \quad v = \sin x$$

So applying the product rule:

$$\begin{aligned}\int x\cos x\,dx &= x(\sin x) - \int (\sin x)1\,dx \\ &= x\sin x - \cos x + c\end{aligned}$$

b) Let:

$$u = x \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

Then:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 1 \quad \text{and} \quad v = \mathrm{e}^x$$

So applying the product rule:

$$\begin{aligned}\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x\,\mathrm{d}x &= [x(\mathrm{e}^x)]_0^3 - \int_0^3 (\mathrm{e}^x)1\,\mathrm{d}x\\ &= 3\,\mathrm{e}^3 - [\mathrm{e}^x]_0^3\\ &= 3\,\mathrm{e}^3 - [\mathrm{e}^3 - 1]\\ &= 2\,\mathrm{e}^3 + 1\end{aligned}$$

c) This is an example where we have to use product rule twice.
Let:

$$u = x^2 \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x$$

Then:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \text{and} \quad v = \mathrm{e}^x$$

So applying the product rule:

$$\begin{aligned}\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x\,\mathrm{d}x &= [x^2(\mathrm{e}^x)]_0^3 - \int_0^3 2x(\mathrm{e}^x)\,\mathrm{d}x\\ &= 9\,\mathrm{e}^3 - 2\int_0^3 x(\mathrm{e}^x)\,\mathrm{d}x\end{aligned}$$

Now we calculated $\int_0^3 x(\mathrm{e}^x)\,\mathrm{d}x$ in b) above so we can say that:

$$\begin{aligned}\int_0^3 x\,\mathrm{e}^x\,\mathrm{d}x\,\mathrm{d}x &= 9\,\mathrm{e}^3 - 2(2\,\mathrm{e}^3 + 1)\\ &= 5\,\mathrm{e}^3 - 2\end{aligned}$$

d) This is an example in which the integral reappears after repeated application of integration by parts.
let:

$$u = \mathrm{e}^t \quad \text{and} \quad \frac{\mathrm{d}v}{\mathrm{d}t} = \cos t$$

Then:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \mathrm{e}^t \quad \text{and} \quad v = \sin t$$

So applying the product rule:

$$\int e^t \cos t \, dt = e^t \sin t - \int e^t \sin t \, dt + c \tag{B.2.4}$$

Now we apply product rule to $\int e^t \sin t \, dt$. We let:

$$u = e^t \quad \text{and} \quad \frac{dv}{dt} = \sin t$$

Then:

$$\frac{du}{dt} = e^t \quad \text{and} \quad v = -\cos t$$

so:

$$\int e^t \sin t \, dt = -e^t \cos t + \int e^t \cos t \, dt \tag{B.2.5}$$

Substituting equation B.2.5 into equation B.2.4 leads to:

$$\begin{aligned}\int e^t \cos t \, dt &= e^t \sin t) - (-e^t \cos t + \int e^t \cos t \, dt) + c \\ &= e^t \sin t + e^t \cos t - \int e^t \cos t \, dt + c\end{aligned}$$

Rearranging gives:

$$2 \int e^t \cos t \, dt = e^t \sin t + e^t \cos t + c$$

From which we can see that:

$$\int e^t \cos t \, dt = \frac{e^t \sin t + e^t \cos t + c}{2}$$

Questions

1. Integrate the following expressions:

a) x^9	b) 4	c) $x^{3/2}$	d) $\sqrt[3]{t}$
e) $\frac{1}{t}$	f) $\frac{1}{z^4}$	g) $z^{-1/2}$	h) $(z^3)^2$

2. Integrate the following expressions:

a) e^{2x}	b) e^{7x}	c) e^{-3x}	d) $\frac{1}{e^{2x}}$
e) $e^{-0.5x}$	f) $e^{x/3}$	g) e^{-2t}	h) $e^{1.5t}$

3. Integrate the following expressions:

    a) $\sin 2x$ b) $\sin\left(\frac{t}{2}\right)$ c) $\cos\left(\frac{3t}{2}\right)$

    d) $\cos(-4x)$ e) $\tan 3t$ f) $\sin(3x+1)$

    g) $\cos(2-\frac{t}{2})$ h) $\tan(4t-3)$ i) $\operatorname{cosec}(2x+3)$

    j) $\sec(4-3x)$ k) $\cot(4t+5)$ l) $\operatorname{cosec}(\pi-t)$

4. Integrate the following expressions:

    a) $\frac{1}{4+x^2}$ b) $\frac{1}{\sqrt{9-x^2}}$ c) $\frac{1}{\sqrt{0.25-x^2}}$

    d) $\frac{1}{0.01+t^2}$ e) $\frac{1}{\sqrt{2-x^2}}$ f) $\frac{1}{z^2+\frac{1}{16}}$

5. Integrate the following expressions (where k is a constant):

    a) $3-2x+\frac{1}{x}$ b) $e^{3x}-e^{-3x}$

    c) $3\cos 2t-\sin 2t$ d) $\tan(2t+\pi)+\operatorname{cosec}\left(\frac{t}{2}-\pi\right)$

    e) $\sec\left(\frac{x}{2}\right)+\cot(3x-\pi)$ f) $\cos x+\frac{x}{2}+\frac{1}{e^x}$

    g) $\left(z+\frac{1}{z}\right)^2$ h) $\frac{1}{2\,e^{3t}}$

    i) $1+3\cot 2z$ j) $4(1-x^3)+e^{-2x}$

    k) $(x-k)^2$ l) $k\sin t-\cos kt$

    m) $\frac{1}{z^2+36}$ n) $\frac{1}{\sqrt{36-x^2}}$

    o) $\frac{4}{1+x^2}+\frac{1+x^2}{4}$

6. An object has an acceleration, $a(t)=1+\frac{t}{2}$. Find expressions for the speed of the object, $v(t)$ and distance travelled by the object $s(t)$.

7. Evaluate the following integrals:

    a) $\int_1^4 x^3\,dx$ b) $\int_1^5 \frac{1}{x}\,dx$

    c) $\int_0^2 3\,dx$ d) $\int_{-1}^1 e^x\,dx$

    e) $\int_{-\pi/3}^0 \sin t\,dt$ f) $\int_0^{\pi/2} \cos t\,dt$

    g) $\int_0^\pi \cos(t+2)\,dt$ h) $\int_0^2 \sin \pi t\,dt$

i) $\int_0^{\pi/4} \tan t \, \mathrm{d}t$

8. Evaluate the following integrals:

a) $\int_0^1 x^2 + 1 \, \mathrm{d}x$

b) $\int_0^2 \frac{1}{x^2+1} \, \mathrm{d}x$

c) $\int_1^2 3\,\mathrm{e}^{2x} + 2\,\mathrm{e}^{3x} \, \mathrm{d}x$

d) $\int_{-1}^1 (x+1)(x+2) \, \mathrm{d}x$

e) $\int_0^{\pi} 2 \sin 3t \, \mathrm{d}t$

f) $\int_0^{\pi} 2\cos\left(\frac{t}{3}\right) \mathrm{d}t$

g) $\int_0^2 3\sin(4t - \pi) + 5\cos\left(3t + \frac{\pi}{2}\right) \mathrm{d}t$

h) $\int_0^1 \frac{2}{x^2+1} - \frac{1}{\sqrt{9-x^2}} \, \mathrm{d}x$

i) $\int_0^{\pi} 2\tan(t+2) \, \mathrm{d}t$

9. Evaluate the following integrals:

a) $\int_1^2 x^9 \, \mathrm{d}x$

b) $\int_{-1}^1 4 \, \mathrm{d}x$

c) $\int_1^3 x^{3/2} \, \mathrm{d}x$

d) $\int_1^2 \frac{1}{t} \, \mathrm{d}t$

e) $\int_1^3 z^{-1/2} \, \mathrm{d}z$

f) $\int_{-1}^1 \mathrm{e}^{2x} \, \mathrm{d}x$

g) $\int_0^2 \mathrm{e}^{-3x} \, \mathrm{d}x$

h) $\int_{-2}^2 \frac{1}{\mathrm{e}^{2x}} \, \mathrm{d}x$

i) $\int_{-1}^1 \mathrm{e}^{1.5t} \, \mathrm{d}t$

j) $\int_{-\pi/2}^{\pi/2} \sin\left(\frac{x}{2}\right) \mathrm{d}x$

k) $\int_0^{\pi} \cos\left(\frac{3t}{2}\right) \mathrm{d}t$

l) $\int_0^{\pi} \tan(4t-3) \, \mathrm{d}t$

m) $\int_0^{\pi} \operatorname{cosec}(2x+3) \, \mathrm{d}x$

n) $\int_{-\pi/2}^{\pi/2} \sec(4-3x) \, \mathrm{d}x$

o) $\int_{-\pi/2}^{\pi/2} \cot(4t+5) \, \mathrm{d}t$

10. Evaluate the following integrals where a, b are constants:

a) $\int x \sin(2x) \, \mathrm{d}x$

b) $\int t\,\mathrm{e}^{4t} \, \mathrm{d}t$

c) $\int x \ln x \, \mathrm{d}x$

d) $\int 2t \cos\left(\frac{t}{2}\right) \mathrm{d}t$

e) $\int x \cos(ax+b) \, \mathrm{d}x$

f) $\int x\,\mathrm{e}^{(ax+b)} \, \mathrm{d}x$

11. Evaluate the following integrals:

a) $\int_0^{\pi/2} x \sin(2x) \, \mathrm{d}x$

b) $\int_{-2}^2 t\,\mathrm{e}^{4t} \, \mathrm{d}t$

c) $\displaystyle\int_1^3 x^2 \ln x \, dx$

d) $\displaystyle\int_0^2 t^2 e^{2t} \, dt$

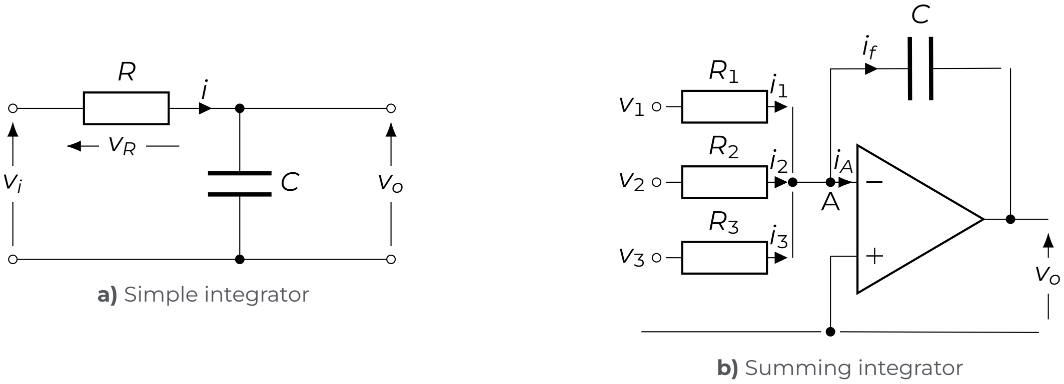
e) $\displaystyle\int_0^{\pi} t^2 \cos\left(\frac{t}{2}\right) dt$

f) $\displaystyle\int_0^{\pi/2} e^{2x} \sin x \, dx$

2.3 Application of integration

Electronic integrators

At times in electronic engineering there is a need to integrate electrical signals - particularly when looking at AC circuit analysis. There are a number of circuits that can integrate signals — one of the simplest is the resistor-capacitor circuit shown in Figure B.2.4a.



a) Simple integrator

b) Summing integrator

Figure B.2.4: Electronic integrators

Looking at this circuit and applying Kirchoff's Voltage Law where v_i is input voltage, v_o is output voltage and v_R is voltage drop across the resistor then:

$$v_i = v_R + v_o \tag{B.2.6}$$

Using Ohm's Law we can say that for the resistor:

$$v_R = iR \tag{B.2.7}$$

For the capacitor using the fact stated earlier we can say the that current i we can say that the current through the capacitor:

$$i = C\frac{dv_o}{dt} \tag{B.2.8}$$

Combining equations B.2.6 to B.2.8 gives us:

$$v_i = RC\frac{dv_o}{dt} + v_o \tag{B.2.9}$$

If v_i is a time-varying sinusoidal signal of frequency f Hz then we can specify the reactance of the capacitor X_c given by:

$$X_C = \frac{1}{2\pi f C}$$

It can be seen that as f increases then X_C decreases. We can therefore say that for frequencies where X_C is much smaller than R most of the voltage drop takes place across the resistor — that is when $X_C << R$ then $v_o << V_R$. So we can rewrite equation B.2.9 as:

$$v_i = RC\frac{\mathrm{d}v_o}{\mathrm{d}t} \qquad \text{for frequencies where } X_C << R \tag{B.2.10}$$

Rearranging this equation leads to:

$$\begin{aligned} \frac{\mathrm{d}v_o}{\mathrm{d}t} &= \frac{v_i}{RC} \\ \implies v_o &= \frac{1}{RC}\int v_i \,\mathrm{d}t \end{aligned}$$

So the output voltage of the circuit in Figure B.2.4a is an integrated version of the input voltage scaled by a factor of $\frac{1}{RC}$.

Figure B.2.4b shows a better electronic integrator using an operational amplifier (opamp). This circuit has the advantage of a high input impedance but low output impedance which makes it very useful in analogue control and signal processing. An opamp amplifies the difference between its inverting (−) and non-inverting (+) inputs — with a high gain a relatively small difference will give a high output voltage. In this circuit the capacitor provides *negative feedback* — that is a proportion of the output voltage is fed back to the input which reduces the output. The consequence of this is that the gain is limited and the opamp works at an equilibirum state where the voltage at point A (*virtual earth*) is the same as the voltage at the non-inverting input which is connected to earth (0 V). Using Ohm's Law with assumption voltage at A is 0 V:

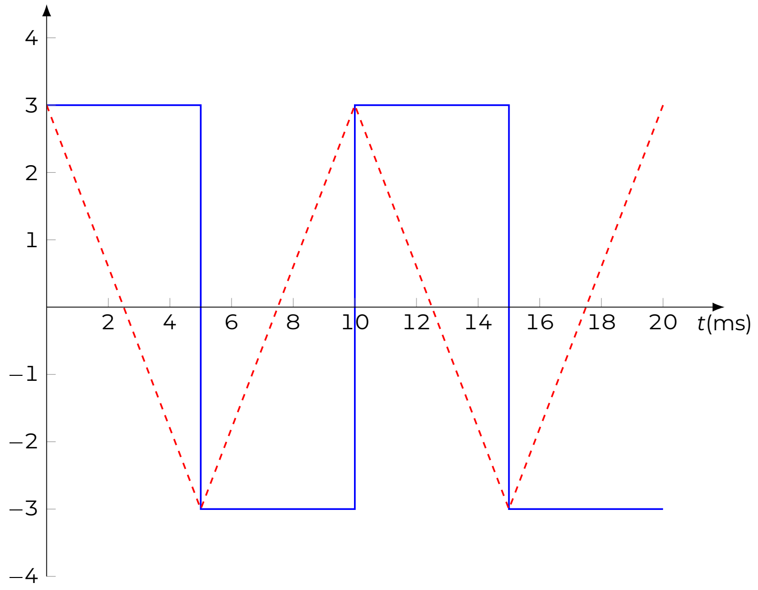


Figure B.2.5: Output from Figure B.2.4b with v_1 being a square wave and $v_2 = v_3 = 0$

$$i_1 = \frac{v_1}{R_1}, \quad i_2 = \frac{v_2}{R_2}, \quad i_3 = \frac{v_3}{R_3} \tag{B.2.11}$$

We can assume that the current flowing into the opamp i_A is negligible ($\approxeq$)0 A, so Kirchoffs Current law means that:

$$i_f = i_1 + i_2 + i_3 \tag{B.2.12}$$

We also know that for the capacitor (current flow direction makes equation negative):

$$i_f = -C\frac{\mathrm{d}v_o}{\mathrm{d}t} \tag{B.2.13}$$

Combing equations B.2.11 to B.2.13 leads to:

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} = -C\frac{\mathrm{d}v_o}{\mathrm{d}t}$$

By rearrangement we can prove this is an integrator circuit — the negative sign indicates that it is an *inverting circuit*.

$$\begin{aligned} v_o &= -\int \frac{v_1}{R_1C} + \frac{v_2}{R_2C} + \frac{v_3}{R_3C}\,\mathrm{d}t \\ v_o &= -\frac{1}{RC}\int (v_1 + v_2 + v_3)\,\mathrm{d}t \qquad \text{if } R = R_1 = R_2 = R_3 \end{aligned}$$

If we make v_1 a square wave input of amplitude 3V and frequency 100 Hz and $v_2 = v_3 = 0$ then the output of this circuit is given by the dashed red line in Figure B.2.5 (assuming a scaling factor $\dfrac{1}{RC} = 1$)

Use of definite integrals

The need to evaluate and area under a curve is common in engineering — often we will know the rate of change of a variable with respect to time and we need to calculate the value of the variable which is the area under the curve between two points. Let's look at a couple of examples* where this is the case.

Engineering application B.2.3: Energy used by an electric motor

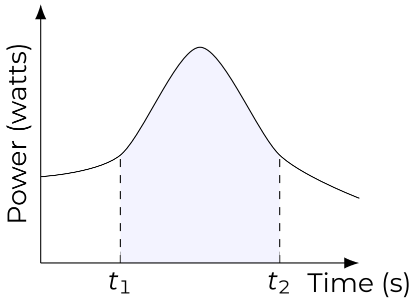


Figure B.2.6: Power supplied by motor

Consider a small DC electric motor being used to drive an electric device such as an electric screwdriver — in this case the power supplied to the motor by the battery depends on the load on the device. The power supplied is therefore a function of time with Figure B.2.6

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[*] Taken from Croft et al. (2017)

showing a typical curve.
Now we can say that:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t}$$

where P = power (W) and E = energy (J). So to calculate the energy used by the motor between times t_1 and t_2 we would use the definite integral:

$$E = \int_{t_1}^{t_2} P\,\mathrm{d}t$$

This is the same as the shaded area shown in Figure B.2.6.

Engineering application B.2.4: Capacitance of a coaxial cable

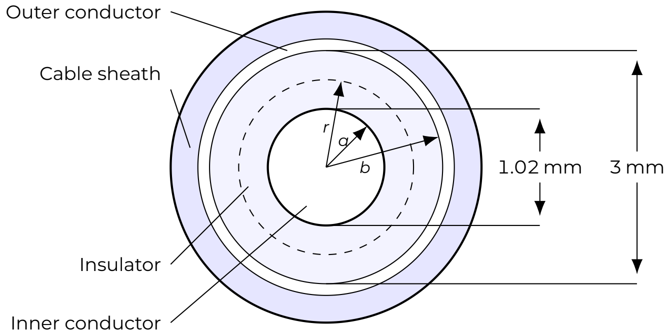


Figure B.2.7: Cross section of a coaxial cable

A coaxial cable consists of two conductors separated by a insulating layer (with a relativity permittivity of 1.55) all surrounded by a protective (& insulating) cable sheath as shown in Figure B.2.7.
Before we solve this problem let us first derive the general formula for the capacitance of a coaxial cable. If we say the inner conductor has a charge of $+Q$ per metre and the outer conductor has a charge of $-Q$ per metre, and assume the cable is long and we are analysing a central section which allows the end effects to be ignored.

Let us consider a cylindrical surface of radius r and length l within the insulating layer (or dielectric) — Gauss's theorem states that the electric flux out of any closed surface is equal to the charge enclosed by the surface. For a cylinder enclosing charge the electric flux does not radiate out the ends — just radially outwards due to symmetry and the enclosed charge (inner conductor) is Ql. For a cylinder the radiating surface (curved surface) is given by $2\pi rl$ so by Gauss's theorem (with D representing electric flux density)

$$D \times 2\pi rl = Ql$$

For a dielectric or insulator the electric flux density is given by $D = \epsilon_r\epsilon_0 E$ where E is electric field strength, ϵ_r is relative permittivity and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m is the permittivity of free space. Therefore:

$$\epsilon_r\epsilon_0 E \times 2\pi rl = Ql$$

$$\implies E = \frac{Q}{2\pi\epsilon_r\epsilon_0 r} \tag{B.2.14}$$

We now have the equation for the electric field within the insulator/dielectric which is effectively the gap between the two 'plates' of the capacitor (the two conductors). To calculate the capacitance of the cable we need to know the voltage difference between the two conductors — we donate the voltage of the inner conductor to be V_a and the voltage of the outer conductor to be V_b.

An electric filed is in effect the measure of the rate of change of the voltage with position — so if the voltage is changing rapidly with position this indicates a high magnitude of the electric field. A positive electric field E corresponds to a decreasing voltage V so the simplified relationship in general (provided r is in same direction as electric field) is:

$$E = -\frac{\mathrm{d}V}{\mathrm{d}r} \tag{B.2.15}$$

If r is not in same direction as electric field then we have to use a vector form of equation B.2.15. For a coaxial cable, the direction of r and E are the same so we can use this equation to calculate the voltage difference between the two conductors. We can rearrange equation B.2.15 to find the voltage at an arbitrary point as:

$$V = -\int E\,\mathrm{d}r$$

This means that the difference in voltage between points $r = a$ and $r = b$ in Figure B.2.7 is given by:

$$\begin{aligned} V_b - V_a &= -\int_a^b E\,\mathrm{d}r \\ &= -\frac{Q}{2\pi\epsilon_r\epsilon_0}\int_a^b \frac{1}{r}\,\mathrm{d}r \quad \text{using equation B.2.14} \\ &= -\frac{Q}{2\pi\epsilon_r\epsilon_0}\Big[\ln r\Big]_a^b \\ &= -\frac{Q}{2\pi\epsilon_r\epsilon_0}\ln\left(\frac{b}{a}\right) \end{aligned}$$

This is voltage of the outer conductor relative to the inner conductor so we can say that the voltage of the inner conductor relative to the outer conductor is given by:

$$V_a - V_b = \frac{Q}{2\pi\epsilon_r\epsilon_0}\ln\left(\frac{b}{a}\right)$$

However we know that the capacitance of a pair of conductors is given by $C = Q/V$ where V is the voltage difference. Therefore we can rearrange the above equation to give us the capacitance per unit length of a coaxial cable with an inner conductor of radius a and an outer conductor of inner radius b to be:

$$C = \frac{Q}{V_a - V_b} = \frac{2\pi\epsilon_r\epsilon_0}{\ln\left(\frac{b}{a}\right)} \tag{B.2.16}$$

For this specific example we know that $\epsilon_r = 1.55$, $\epsilon_0 = 8.85 \times 10^{-12}$ F/m, $a = 1.02 \times 10^{-3}/2 = 5.1 \times 10^{-4}$m and $b = 3 \times 10^{-3}/2 = 1.5 \times 10^{-3}$m so:

$$C = \frac{2\pi \times 1.55 \times 8.85 \times 10^{-12}}{\ln\left(\dfrac{1.5 \times 10^{-3}}{5.1 \times 10^{-4}}\right)} = 7.99 \times 10^{-11} \approx 80 \text{ pF/m} \tag{B.2.17}$$

Average value of a function

In AC circuits currents and voltages often vary with time and we may wish to know the average value of such a current or voltage over a given time interval. Mathematically we can define such a value in terms of integrals.

Let us take a function $f(t)$ across an interval $a \leqslant t \leqslant b$. The areas A under $f(t)$ is given by:

$$A = \int_a^b f(t)\,\mathrm{d}t$$

We know that a rectangle of height h with a base spanning interval $[a, b]$ has and area of $h(b-a)$. Now let us chose the value of h such that the area of the rectangle and A are equal to each other. This means that:

$$\begin{aligned} h(b-a) &= \int_a^b f(t)\,\mathrm{d}t \\ h &= \frac{\int_a^b f(t)\,\mathrm{d}t}{b-a} \end{aligned}$$

This value of h is known as the *average value* of the function across the interval $[a, b]$ as illustrated in Figure B.2.8.

$$\text{average value} = \frac{\displaystyle\int_a^b f(t)\,\mathrm{d}t}{b-a}$$

It is worth noting that the average value can change depending on the interval considered. For instance, consider $f(t) = -t^2$ over the intervals $[-3, -1]$ and $[2, 5]$:

$$\begin{aligned} \text{average value} &= \frac{-\int_{-3}^{-1} t^2\,\mathrm{d}t}{-1-(-3)} = -\frac{1}{2}\left[\frac{t^3}{3}\right]_{-3}^{-1} = -\frac{13}{3} \\ \text{average value} &= \frac{-\int_2^5 t^2\,\mathrm{d}t}{5-2} = -\frac{1}{3}\left[\frac{t^3}{3}\right]_2^5 = -13 \end{aligned}$$

As can be seen the average value changes between the two intervals by a factor of 3.

Engineering application B.2.5: Sawtooth waveform

One periodic waveform that engineers often make use of is the ‘sawtooth’ waveform — and it can be useful to know the average value of such a waveform over a single period.

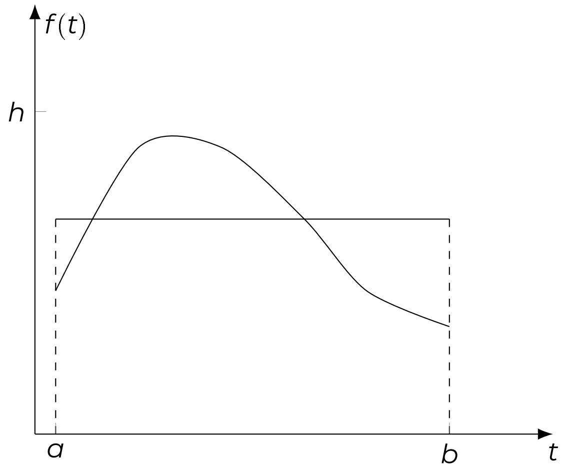


Figure B.2.8: Area under curve and area of rectangle are equal

Consider the waveform in Figure B.2.9 — what is the average value over a complete period?

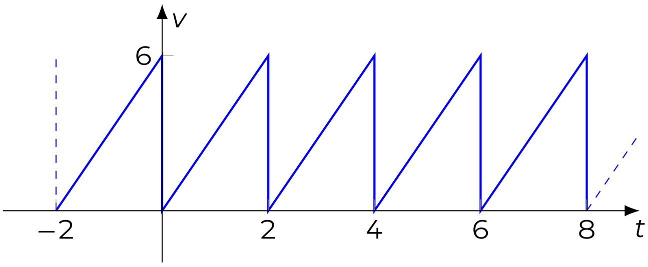


Figure B.2.9: Sawtooth waveform

Solution

First we need to obtain the equation for the waveform. Looking at the interval $0 \leqslant t < 2$ we can see that in this interval (which is a complete period) the waveform takes the form of a straight line which in general has form $v(t) = mt + c$.

When $t = 0$, $v = 0$

$$\begin{aligned} 0 &= 0 + c \\ \implies c &= 0 \end{aligned}$$

When $t = 2$, $v = 6$

$$\begin{aligned} 6 &= m(2) \\ \implies m &= 3 \end{aligned}$$

Hence the waveform in interval $0 \leqslant t < 2$ is given by:

$$v = 3t$$

So the average value over interval $0 \leqslant t < 2$ is given by:

$$v_{av} = \frac{1}{2}\int_0^2 3t\,\mathrm{d}t = \frac{1}{2}\left[\frac{3t^2}{2}\right]_0^2$$

$$v_{av} = \frac{1}{2}\left(\frac{3 \times 4}{2} - 0\right) = \frac{12}{4} = 3\,\mathrm{V}$$

Engineering application B.2.6: Average value of full-wave rectifier

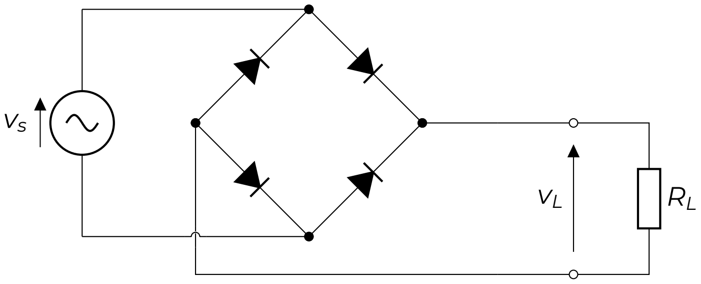


Figure B.2.10: A full-wave rectifier

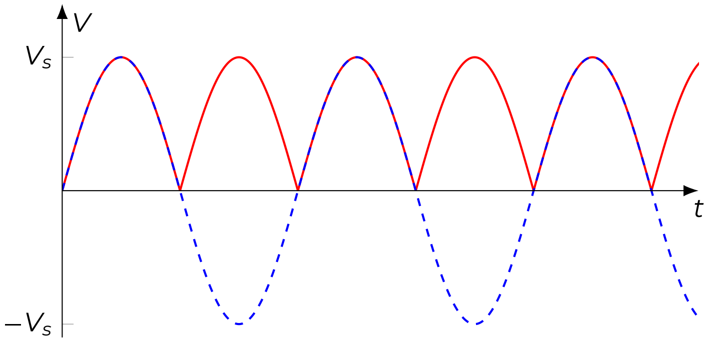


Figure B.2.11: Input (dashed blue) & output (solid red) of full-wave recitifier

Figure B.2.10 shows the circuit for a full wave rectifier. A rectifier is a device that basically takes an input AC source where the current flow periodically change directions (and hence voltage changes sign) as shown by dashed line in Figure B.2.11 to one where the direction of current flow remains the same - i.e. a DC source). A full-wave rectifier converts the whole the input waveform to one of constant polarity (positive or negative) as shown by red line in Figure B.2.11 — this is a pulsating DC voltage with the apparent DC voltage given by the average voltage of the function. The instantaneous value of the source voltage is given by

$$v_S(t) = V_S \sin(\omega t) = V_S \sin(2\pi f t) = V_S \sin\left(\frac{2\pi t}{T}\right)$$

So the average value of v_L is given by taking the average of value of this function over interval $[0, T/2]$

$$\begin{aligned}\frac{1}{T/2 - 0}\int_0^{T/2} v_S \,\mathrm{d}t &= \frac{2}{T}\int_0^{T/2} V_S \sin\left(\frac{2\pi t}{T}\right)\mathrm{d}t \\ &= \frac{2}{T}\frac{T}{2\pi}\left[-\cos\left(\frac{2\pi t}{T}\right)\right]_0^{T/2} \\ &= \frac{V_s}{\pi}\left(-\cos\left(\frac{2\pi T}{T2}\right) - (-\cos(0))\right) \\ &= \frac{V_s}{\pi}(-\cos\pi + \cos 0) = \frac{2V_s}{\pi}\end{aligned}$$

Root mean square value of a function

As with the average value, the Root Mean Square (RMS) value of a function is used in AC circuit analysis. It effectively is a measure of the average magnitude delivered by a time-varying signal. Practically, for a given resistive load, the RMS value of an AC voltage is equivalent to the DC voltage that would be needed to provide the same average power dissipation across the load. Mathematically we can define such a value in terms of integrals.

Let us take a function $f(t)$ across an interval $a \leqslant t \leqslant b$. The RMS value of the function is defined as:

$$\text{RMS} = \sqrt{\frac{\displaystyle\int_a^b (f(t))^2 \, \mathrm{d}t}{b-a}}$$

Example B.2.5

Find the RMS values of the following functions:

a) $f(t) = A \sin t$ across interval $[0, 2\pi]$

b) $f(t) = A \sin(\omega t + \phi)$ across interval $[0, 2\pi/\omega]$

Solution:

a) Recall that from the trigonometric identities in Section 5.3 $\sin^2 x = \dfrac{1-\cos 2x}{2}$

$$\begin{aligned}
\text{rms} &= \sqrt{\frac{\int_0^{2\pi} A^2 \sin^2(t)\,\mathrm{d}t}{2\pi}} \\
&= \sqrt{\frac{A^2 \int_0^{2\pi} (1-\cos 2t)/2 \,\mathrm{d}t}{2\pi}} \\
&= \sqrt{\frac{A^2}{4\pi}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}} \\
&= \sqrt{\frac{A^2 2\pi}{4\pi}} \quad \text{as } \sin 2\pi = \sin 0 = 0 \\
&= \frac{A}{\sqrt{2}} = 0.707A
\end{aligned}$$

So the RMS value of a sine wave is 0.707× the amplitude.

b) Using same identity as above:

$$\begin{aligned}
\text{rms} &= \sqrt{\frac{\int_0^{2\pi/\omega} A^2 \sin^2(\omega t + \phi)\,\mathrm{d}t}{2\pi/\omega}} \\
&= \sqrt{\frac{A^2\omega}{4\pi}\int_0^{2\pi/\omega} 1 - \cos 2(\omega t + \phi)\,\mathrm{d}t}
\end{aligned}$$

$$= \sqrt{\frac{A^2\omega}{4\pi}\left[t - \frac{sin2(\omega t+\phi)}{2\omega}\right]_0^{2\pi/\omega}}$$

$$= \sqrt{\frac{A^2\omega}{4\pi}\left(\frac{2\pi}{\omega} - \frac{sin2(2\pi+\phi)}{2\omega} + \frac{sin2\phi}{2\omega}\right)}$$

Now $\sin 2(2\pi+\phi) = \sin(4\pi+2\phi)$ and since a generic sine wave has a period of 2π we can say that $\sin(4\pi+2\phi) = \sin 2\phi$. So:

$$\text{rms} = \sqrt{\frac{A^2\omega}{4\pi}\left(\frac{2\pi}{\omega} - \frac{sin2\phi}{2\omega} + \frac{sin2\phi}{2\omega}\right)}$$

$$= \sqrt{\frac{A^2\omega}{4\pi}\frac{2\pi}{\omega}} = \sqrt{\frac{A^2}{2}} = \frac{A}{\sqrt{2}} = 0.707A$$

The second example in example B.2.5 is for a general sine wave $f(t) = A\sin(\omega t + \phi)$ which has a period of $2\pi/\omega$. Both the above examples illustrate a general result:

The root mean square value of any sinusoidal waveform across an interval that is of one period in length is:

$0.707 \times$ the amplitude of the waveform

The root mean square value is an effective way of measuring the energy transfer capability of a time-varying electric current. Let us consider a couple of specific engineering examples:

Engineering application B.2.7: Average power developed across a resistor

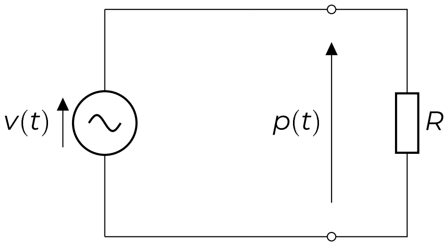


Figure B.2.12: Power dissipated $p(t)$ due to voltage $v(t)$ across a resistor

Consider the circuit in Figure B.2.12 which shows a voltage $v(t)$ that develops a power $p(t)$ in a resistor R. let us consider the time interval $[t_1, t_2]$ where the average power dissipated by the resistor is $P_{av}(t)$. We want the total energy transfer E within this period to be the same - i.e. the power dissipated by the time-varying current to be P_{av}.

$$E = P_{av}(t_2 - t_1) = \int_{t_1}^{t_2} p(t)\,\mathrm{d}t$$

Now we know that the relationship between voltage, power, and resistance is given by:

$$p(t) = \frac{(v(t))^2}{R}$$

Therefore:

$$P_{av}(t_2 - t_1) = \int_{t_1}^{t_2} \frac{(v(t))^2}{R}\,\mathrm{d}t$$

If we consider the value of P_{av} dissipated by the resistor to be the result of an effective direct voltage V_{eff} with $P_{av}(t_2 - t_1) = \frac{V_{eff}^2}{R}$ then:

$$\begin{aligned}
\frac{V_{eff}^2}{R}(t_2 - t_1) &= \int_{t_1}^{t_2} \frac{(v(t))^2}{R}\,\mathrm{d}t \\
V_{eff}^2(t_2 - t_1) &= \int_{t_1}^{t_2} (v(t))^2\,\mathrm{d}t \\
V_{eff}^2 &= \frac{\int_{t_1}^{t_2} (v(t))^2\,\mathrm{d}t}{(t_2 - t_1)} \\
V_{eff} &= \sqrt{\frac{\int_{t_1}^{t_2} (v(t))^2\,\mathrm{d}t}{(t_2 - t_1)}}
\end{aligned}$$

This shows us that the equivalent direct voltage is the RMS value of the time-varying voltage. A similar proof can be done replacing voltage by current as $P = I^2 R$

Engineering application B.2.8: root mean square voltage of full-wave rectifier

Let us go back to our full-wave rectifier from example B.2.6 and calculate the RMS of $v_L(t)$. We know that the instantaneous value of the source voltage is given by

$$v_s(t) = V_S \sin\left(\frac{2\pi t}{T}\right)$$

So the RMS value of v_L is calculated by taking this function over the interval $[0, T/2]$

$$\begin{aligned}
\text{rms} &= \sqrt{\frac{\int_0^{T/2} V_s^2 \sin^2\left(\frac{2\pi t}{T}\right)\mathrm{d}t}{T/2}} \\
&= \sqrt{\frac{2V_s^2 \int_0^{T/2} \left(1 - \cos 2\left(\frac{2\pi t}{T}\right)\right)/2\,\mathrm{d}t}{T}} \\
&= \sqrt{\frac{2V_s^2}{2T}\left[t - \frac{\sin\left(\frac{4\pi t}{T}\right)}{4\pi/T}\right]_0^{T/2}} \\
&= \sqrt{\frac{V_s^2}{T}\left(\frac{T}{2} - \frac{\sin\left(\frac{4\pi T/2}{T}\right)}{4\pi/T} - 0 + \frac{\sin\left(\frac{4\pi 0}{T}\right)}{4\pi/T}\right)} \\
&= \sqrt{\frac{V_s^2}{T}\left(\frac{T}{2} - \frac{\sin 2\pi}{4\pi/T} - \frac{\sin 0}{4\pi/T}\right)} \\
&= \sqrt{\frac{V_s^2}{T}\frac{T}{2}} \quad \text{as } \sin 2\pi = \sin 0 = 0
\end{aligned}$$

$$= \sqrt{\frac{V_s^2}{2}} = \frac{V_s}{\sqrt{2}}$$

Which is the value we would expect over a single period of full-wave rectifier given it is effectively half a sine wave.

Questions

1. The velocity, $v(t)$ of an object is given by

   $$v(t) = 2 + e^{-t}$$

   a) Obtain an expression for the distance travelled by the object.
   b) Calculate the distance travelled between t=0 s to 4 s.
   c) Obtain an expression for the acceleration (rate of change of velocity with time) experienced by the object.
   d) What is the acceleration experienced by the object at t=4 s

2. A capacitor of capacitance 10^{-6} F has a current $i(t)$ through it given by:

   $$i(t) = 1 - e^{-t}$$

   a) Find an expression for the voltage across the capacitor $v(t)$
   b) Given that at t =0 s, $v(t)$ =0 V what is the voltage across the capacitor at 2 s

3. Calculate the average values of the following functions across the specified interval:

   a) $f(t) = 1 + t$ across $[0, 3]$
   b) $f(x) = 3x - 2$ across $[-1, 1]$
   c) $f(t) = t^2$ across $[-1, 1]$
   d) $f(t) = t^2$ across $[0, 3]$
   e) $f(x) = x^2 + x$ across $[2, 3]$
   f) $f(x) = \sqrt{x}$ across $[0, 4]$
   g) $f(x) = \frac{1}{x}$ across $[-2, -1]$
   h) $f(t) = \frac{1}{t^2}$ across $[1, 2]$
   i) $f(t) = t^3 - 1$ across $[-1, 1]$

4. Calculate the average values of the following functions across the specified interval:

   a) $f(t) = \sin t$ across $\left[0, \frac{\pi}{2}\right]$
   b) $f(t) = \sin \omega t$ across $[0, \pi]$
   c) $f(t) = \sin 2\omega t$ across $\left[0, \frac{\pi}{\omega}\right]$

   d) $f(t) = \cos t$ across $\left[0, \frac{\pi}{2}\right]$
   e) $f(t) = \cos \omega t$ across $[0, \pi]$
   f) $f(t) = \cos \omega t$ across $\left[0, \frac{\pi}{\omega}\right]$
   g) $f(t) = \sin \omega t + \cos \omega t$ across $[0, \pi]$
   h) $f(t) = \sin \omega t + \cos \omega t$ across $\left[0, \frac{\pi}{\omega}\right]$

5. Calculate the RMS values for the functions in question 3 above (across same interval).
6. Calculate the RMS values for the functions in question 4 above (across same interval).
7. Given a current $i(t)$ as shown in Figure B.2.13 which is defined mathematically as

$$i(t) = 10\,\mathrm{e}^{-t} \qquad 0 \leqslant t < 10 \qquad \text{period } T = 10$$

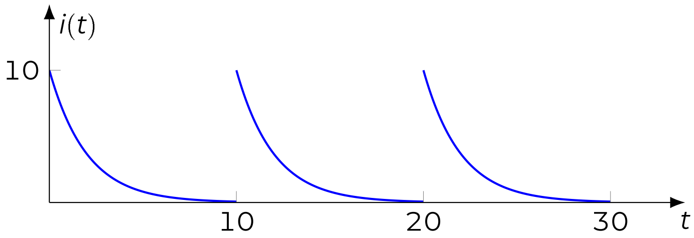


Figure B.2.13: Waveform for question 7

   a) Calculate the average value of the current over a complete period
   b) Calculate the RMS value of the current over a complete period

8. Given a periodic voltage $v(t)$ as shown in Figure B.2.14 with the waveform defined as

$$v(t) = \sin 4\pi t + \cos 2\pi t$$

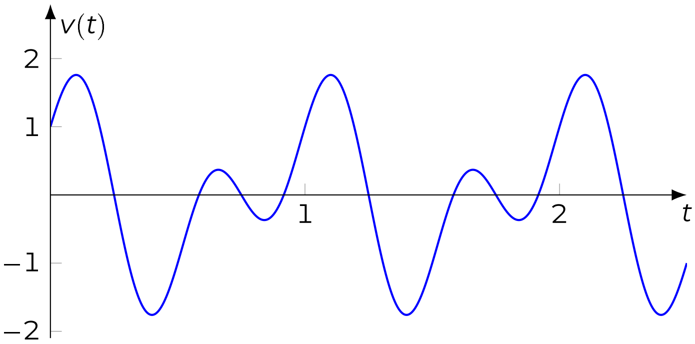


Figure B.2.14: Waveform for question 8

   a) Calculate the average value of the voltage over a complete period
   b) Calculate the RMS value of the voltage over a complete period

2.4 Further topics

This section looks at a number of other areas of interest that use integration that you may come across during your studies. The first one is solving integration using substitution methods then we look at orthogonal functions which is a key property used extensively in areas such as telecommunications. Then there are *improper* integrals which are integrals where one or both limits are infinity or where the integrand (function being integrated) becomes infinity at particular points (think of the graph of $\tan x$ in Figure A.5.7). This type of integral is used extensively in Laplace & Fourier transforms as discussed in the Level 5 course (Wyatt-Millington & Love 2022). The impulse or Dirac delta function, $\delta(t)$ was introduced in Section 4.9 but there are specific properties for its integral which are explored. Finally the integration of piecewise continuous functions is examined.

Algebraic substitutions

This is a method sued to solve integral problems which initially look like they cannot be solved. What we effectively do is we let u be equal to $f(x)$ so that the integral becomes $f(u)du$. It can be used with integrals of the forms:

$$k\int [f(x)]^n f'(x)\,\mathrm{d}x \quad \text{and} \quad k\int \frac{f'(x)}{f(x)}\,\mathrm{d}x$$

The best way of seeing how this works is to do some examples:

Example B.2.6

Evaluate the following definite integrals:

a) $\int (2x-1)^5\,\mathrm{d}x$

b) $\int 3\,\mathrm{e}^{2x-1}\,\mathrm{d}x$

c) $\int 3x(3x^2+2)^3\,\mathrm{d}x$

d) $\int_0^{\frac{\pi}{6}} 10\sin^4\theta\cos\theta\,\mathrm{d}\theta$

e) $\int_0^2 \frac{2x}{3x^2+2}\,\mathrm{d}x$

f) $\int_0^4 \frac{10}{5x+1}\,\mathrm{d}x$

Solution:

a) Let $u = 2x-1$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \implies \mathrm{d}x = \frac{\mathrm{d}u}{2}$
Hence:

$$\int u^5\frac{\mathrm{d}u}{2} = \frac{1}{2}\int u^5\,\mathrm{d}u = \frac{1}{2}\frac{u^6}{6} + c$$
$$= \boxed{\frac{1}{12}(2x-1)^6 + c}$$

b) Let $u = 2x-1$ then $\frac{\mathrm{d}u}{\mathrm{d}x} = 2 \implies \mathrm{d}x = \frac{\mathrm{d}u}{2}$
Hence:

$$\int 3\,\mathrm{e}^u\frac{\mathrm{d}u}{2} = \frac{3}{2}\int \mathrm{e}^u\,\mathrm{d}u = \frac{3}{2}\mathrm{e}^u + c$$
$$= \boxed{\frac{3}{2}\mathrm{e}^{(2x-1)} + c}$$

c) Let $u = 3x^2 + 2$ then $\dfrac{du}{dx} = 6x \implies dx = \dfrac{du}{6x}$

Hence:

$$\int 3x(u)^3 \frac{du}{6x} = \frac{1}{2}\int u^3\,du = \frac{1}{2}\frac{u^4}{4} + c$$

$$= \boxed{\frac{1}{8}(3x^2 + 2)^4 + c}$$

d) Let $u = \sin\theta$ then $\dfrac{du}{d\theta} = \cos\theta \implies d\theta = \dfrac{du}{\cos\theta}$

Hence:

$$\int 10u^4 \cos\theta \frac{du}{\cos\theta} = 10\int u^4\,du$$

$$= 10\frac{u^5}{5} + c = 2u^5 + c = 2\sin^5\theta + c$$

$$\therefore \int_0^{\frac{\pi}{6}} 10\sin^4\theta\cos\theta\,d\theta = \Big[2\sin[5]\theta\Big]_0^{\frac{\pi}{6}} = 2\sin^5\left(\frac{\pi}{6}\right) - 2\sin^0$$

$$= 2\left(\frac{1}{2}\right)^5 = \boxed{\frac{1}{16} \text{ or } 0.0625}$$

e) Let $u = 3x^2 + 2$ then $\dfrac{du}{dx} = 6x \implies dx = \dfrac{du}{6x}$

Hence:

$$\int \frac{2x}{u}\frac{du}{6x} = \frac{1}{3}\frac{1}{u}du$$

$$= \frac{1}{3}\ln u + c = \frac{1}{3}\ln(3x^2 + 1) + c$$

$$\therefore \int_0^2 \frac{2x}{3x^2 + 2}dx = \frac{1}{3}\Big[\ln(3x^2 + 2)\Big]_0^2$$

$$= \frac{1}{3}((\ln(3\cdot 2^2 + 2) - \ln(3\cdot 0^2 + 2))$$

$$= \frac{1}{3}(\ln 14 - \ln 2) = \frac{1}{3}(\ln 7) = \boxed{0.649}$$

f) Let $u = 5x + 1$ then $\dfrac{du}{dx} = 5 \implies dx = \dfrac{du}{5}$

Hence:

$$\int \frac{10}{u}\frac{du}{5} = 2\frac{1}{u}du$$

$$= 2\ln u + c = 2\ln(5x + 1) + c$$

$$\therefore \int_0^4 \frac{10}{5x + 1}dx = 2\Big[\ln(5x + 1)\Big]_0^4$$

$$= 2((\ln(5\cdot 4 + 1) - \ln(5\cdot 0 + 1))$$

$$= 1(\ln 11 - \ln 1) = 2(\ln 11) = \boxed{4.796}$$

Orthogonal functions

Two functions $f(x)$ and $g(x)$ are said to be orthogonal over the interval $[a, b]$ if

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0$$

In other words, two functions are orthogonal to each other over a specified interval if the integral of their product over that interval is zero. This is explored in the following examples (taken from Croft et al. (2017))

Example B.2.7

Show that $f(x) = x$ and $g(x) = (x - 1)$ are orthogonal over $\left[0, \frac{3}{2}\right]$.

Solution:

$$\begin{aligned}\int_0^{3/2} x(x-1)\,\mathrm{d}x &= \int_0^{3/2} x^2 - x\,\mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_0^{3/2} \\ &= \frac{(\frac{3}{2})^3}{3} - \frac{(\frac{3}{2})^2}{2} - \left(\frac{0}{3} - \frac{0}{2}\right) \\ &= \frac{27}{8 \times 3} - \frac{9}{4 \times 2} = \frac{9}{8} - \frac{9}{8} = 0\end{aligned}$$

Hence we have proved that f and g are orthogonal over the specified interval.

However if we looked at the same two functions in Example B.2.7 over the interval $[0, 1]$ then it is obvious that

$$\int_0^1 x(x-1)\,\mathrm{d}x \neq 0$$

So these two functions are not orthogonal over this interval. It is important to remember that functions may be orthogonal over one interval but not over other intervals.

We can show that more than two functions are mutually orthogonal by proving each possible pair of functions is orthogonal over the specified interval as shown in Example B.2.8.

Example B.2.8

Show that $f(t) = 1$, $g(t) = \sin t$ and $h(t) = \cos t$ are mutually orthogonal over $[-\pi, \pi]$

Solution:

We need to show that each pair of functions is orthogonal over the interval $[-\pi, \pi]$. Starting with $f(t)$ and $g(t)$ and $f(t)$ and $h(t)$

$$\begin{aligned}\int_{-\pi}^{\pi} 1 \sin t\,\mathrm{d}t = [-\cos t]_{-\pi}^{\pi} &= -\cos\pi - (-\cos(-\pi)) \\ &= -(-1) + (-1) = 0\end{aligned}$$

$$\int_{-\pi}^{\pi} 1\cos t\,\mathrm{d}t = [\sin t]_{-\pi}^{\pi} = \sin\pi - \sin(-\pi)$$
$$= 0 - 0 = 0$$

Now we can use the trigonometric identity $\sin(2x) = 2\sin x\cos x$ to calculate the orthogonality of $g(t)$ and $h(t)$

$$\begin{aligned}\int_{-\pi}^{\pi} \sin t\cos t\,\mathrm{d}t &= \int_{-\pi}^{\pi}\frac{1}{2}\sin(2t)\,\mathrm{d}t\\ &= -\left[\frac{\cos(2t)}{4}\right]_{-\pi}^{\pi}\\ &= -\frac{\cos(2\pi)-\cos(-2\pi)}{4}\\ &= -\frac{1-1}{4} = 0\end{aligned}$$

Hence we can say that the functions 1, $\sin t$ and $\cos t$ form an orthogonal set over $[-\pi,\pi]$

It is possible to extend the set in Example B.2.8 to

$$\{1, \sin t, \cos t, \sin(2t), \cos(2t), \sin(3t), \cos(3t), ..., \sin(nt), \cos(nt)\} \quad n \in \mathbb{N}_0$$

We can verify this by showing all combinations of $1, sinnt, sinmt, cosnt$ and $\cos mt$ are orthogonal ($n, m \in \mathbb{N}_0$) in interval $[-\pi, \pi]$:

$$\begin{aligned}\int_{-\pi}^{\pi} 1\sin nt\,\mathrm{d}t &= \left[\frac{-\cos nt}{n}\right]_{-\pi}^{\pi} = \frac{-\cos(n\pi)+\cos(n\pi)}{n}\\ &= \frac{-(-1)+(-1)}{n} = 0\end{aligned} \tag{B.2.18}$$

Similarly easy to show:

$$\begin{aligned}\int_{-\pi}^{\pi} 1\cos nt\,\mathrm{d}t &= \left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi} = \frac{\sin(n\pi)-\sin(n\pi)}{n}\\ &= \frac{0-0}{n} = 0\end{aligned} \tag{B.2.19}$$

Using the trigonometric identity $\sin x\cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$

$$\int_{-\pi}^{\pi} \sin mt\cos nt\,\mathrm{d}t = \frac{1}{2}\int_{-\pi}^{\pi} \sin(n+m)t + \sin(n-m)t\,\mathrm{d}t$$

We have already proved that $\int_{-\pi}^{\pi}\sin nt\,\mathrm{d}t = 0$ for any $n \in \mathbb{N}$ — plus we can note that both $(n+m) \in \mathbb{N}$ and $(n-m) \in \mathbb{N}$. So we can say that

$$\int_{-\pi}^{\pi} \sin(n+m)t + \sin(n-m)t\,\mathrm{d}t = 0$$

In a similar fashion we can see that

$$\int_{-\pi}^{\pi} \sin mt\sin nt\,\mathrm{d}t = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)t - \cos(n+m)t\,\mathrm{d}t = 0 \qquad n \neq m$$

$$\int_{-\pi}^{\pi} \cos mt \cos nt \, \mathrm{d}t = \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)t + \cos(n-m)t \, \mathrm{d}t = 0 \qquad n \neq m$$

Thus the functions form an orthogonal set across the interval $[-\pi, \pi]$

We can extend this further to say that:

$\{1, \sin t, \cos t, \sin 2t, \cos 2t, \ldots\}$ is an orthogonal set across any interval of length 2π

Indeed taking it more generally for a period of T we can say that

$$\left\{1, \sin\left(\frac{2\pi t}{T}\right), \cos\left(\frac{2\pi t}{T}\right), \sin\left(\frac{4\pi t}{T}\right), \cos\left(\frac{4\pi t}{T}\right), \cdots\right\}$$

is an orthogonal set over any interval of length T.

This is used in Fourier analysis which we examine at a basic level in Chapter B.3 and in further detail in the level 5 Advanced Maths course where we meet the Fourier Transform (Wyatt-Millington & Love 2022)

Improper integrals

Two cases of evaluation of integrals need special care:

(1) one or both limits of the integral are infinite

(2) the integrand tends to infinity at one of more points over the interval of integration.

If either of these are true then the integral is called an *improper integral*. The following examples show how we consider improper integrals in practice: Examples B.2.9 to B.2.11 consider integrals where the limits are infinite and Examples B.2.12 to B.2.14 consider integrals where the integrand tends to infinity at a specific point within the interval of integration.

Example B.2.9

Evaluate $\int_{2}^{\infty} \frac{2}{t^2} \, \mathrm{d}t$

Solution:

$$\int_{2}^{\infty} \frac{2}{t^2} \, \mathrm{d}t = \left[-\frac{2}{t}\right]_{2}^{\infty}$$

To evaluate $-\frac{2}{t}$ at the upper limit we consider $\lim_{t\to\infty} -\frac{2}{t}$. Clearly the limit is 0. Hence:

$$\int_{2}^{\infty} \frac{2}{t^2} \, \mathrm{d}t = 0 - \left(-\frac{2}{2}\right) = 1$$

Example B.2.10

Evaluate $\int_{-\infty}^{1} e^{3x}\,dx$

Solution:

$$\int_{-\infty}^{1} e^{3x}\,dx = \left[\frac{e^{3x}}{3}\right]_{-\infty}^{1}$$

To evaluate $\frac{e^{3x}}{3}$ at the lower limit we consider $\lim_{x\to-\infty} \frac{e^{3x}}{3}$. Clearly the limit is 0. Hence:

$$\int_{-\infty}^{1} e^{3x}\,dx = \left[\frac{e^{3x}}{3}\right]_{-\infty}^{1} = \left(\frac{e^{3}}{3} - 0\right) = 6.695$$

Example B.2.11

Evaluate $\int_{1}^{\infty} \cos t\,dt$

Solution:

$$\int_{1}^{\infty} \cos t\,dt = \left[\sin t\right]_{1}^{\infty}$$

Now $\lim_{t\to\infty} \sin t$ does not exist as it cannot be evaluated as the function $\sin t$ does not approach a limit as $t \to \infty$ so we cannot evaluate the integral — thus we say the integral *diverges* and does not exist.
Note: If the integral can be evaluated as in examples B.2.9 & B.2.10, then we say the integral *converges* and does exist.

Engineering application B.2.9: Series Capacitors

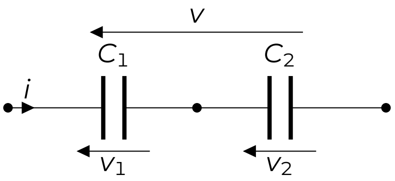


Figure B.2.15: Series capacitors

As we often need to simplify a circuit to make it easiest to analyse. One common arrangement of components is two in series. If we take two capacitors as shown in Figure B.2.15 then it can be useful to replace these by a single capacitor with an equivalent capacitance C - but what is the value of C.
Recall that in Engineering application B.2.2 we derived an expression for the voltage across a capacitor

$$v = \frac{1}{C}\int i\,dt$$

This can be written as a definite integral to give the voltage at a point in time t

$$v = \frac{1}{C}\int_{-\infty}^{t} i\,\mathrm{d}t$$

So for the two capacitors in Figure B.2.15 the voltages across them are

$$v_1 = \frac{1}{C_1}\int_{-\infty}^{t} i\,\mathrm{d}t \qquad v_2 = \frac{1}{C_2}\int_{-\infty}^{t} i\,\mathrm{d}t$$

By Kirchoff's voltage law[a]

$$\begin{aligned} v &= v_1 + v_2 \\ &= \frac{1}{C_1}\int_{-\infty}^{t} i\,\mathrm{d}t + \frac{1}{C_2}\int_{-\infty}^{t} i\,\mathrm{d}t \\ &= \left(\frac{1}{C_1} + \frac{1}{C_2}\right)\int_{-\infty}^{t} i\,\mathrm{d}t \end{aligned}$$

So the equivalent capacitance of the single capacitor obeys rule

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

Rearranging:

$$C = \frac{C_1 C_2}{C_1 + C_2}$$

---

[a] discussed in EEP1

Example B.2.12

Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

Solution:
The integrand $\dfrac{1}{\sqrt{x}}$ becomes infinite when $x = 0$, which is within the interval of integration (i.e. within the integration limits). To evaluate the integral we 'remove' the point from the interval and consider $\displaystyle\int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ where b is slightly greater than 0, & then let $b \to 0^+$

$$\int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = [2\sqrt{x}]t_b^1 = 2 - 2\sqrt{b}$$

Then:

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{b\to 0+}\int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{b\to 0+}(2 - 2\sqrt{b}) = 2$$

So in this case, the improper integral exists and has a value of 2.

Example B.2.13

Determine if $\int_0^1 \frac{1}{x}\,dx$ exists.

Solution:

As in Example B.2.12 the integrand is not defined at $x = 0$, so we consider $\int_b^1 \frac{1}{x}\,dx$ and let $b \to 0^+$

$$\int_b^1 \frac{1}{x}\,dx = [\ln|x|\,]t_b^1 = \ln 2 - \ln b$$

Then:

$$\lim_{b\to 0^+}\left(\int_b^1 \frac{1}{x}\,dx\right) = \lim_{b\to 0^+}(\ln 2 - \ln b)$$

Since $\lim_{b\to 0^+} b$ tends to infinity so does not exist the integral diverges and those does not exist.

Example B.2.14

Determine if $\int_{-2}^1 \frac{1}{x}\,dx$ exists.

Solution:

We 'remove' the point $x = 0$, and consider the integrals $\int_{-2}^b \frac{1}{x}\,dx$ as $b \to 0$ and $\int_c^1 \frac{1}{x}\,dx$ as $c \to 0$ If these integrals exist as $b \to 0$ and $c \to 0$ then $\int_{-2}^1 \frac{1}{x}\,dx$ exists. As we know from Example B.2.13, $\int_c^1 \frac{1}{x}\,dx$ as $c \to 0$ does not exist so in this case we can say $\int_{-2}^1 \frac{1}{x}\,dx$ diverges.

Engineering application B.2.10: Energy stored in a capacitor

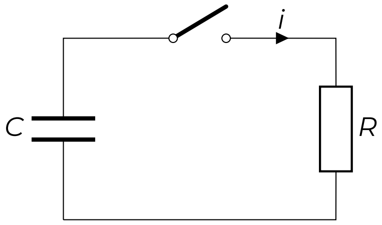


Figure B.2.16: Circuit for capacitor discharge

A charged capacitor effectively stores energy until it is discharged. Let us consider the circuit in Figure B.2.16 where a charged capacitor with voltage V across it at time $t = 0$ is connected via a switch to a series load resistor. When the switch is closed the electrical energy stored in the capacitor is converted to heat energy due to the current flowing through the resistor. Considering the circuit in Figure Figure B.2.16, we want to calculate the energy stored in the capacitor. The switch is closed at time $t = 0$ and a current i flows in the circuit. We know from Engineering application A.4.3 that the time-varying voltage v across the capacitor decays exponentially according to

$$v = V\,\mathrm{e}^{-t/RC}$$

So using Ohm's law we can find an expression for the current i:

$$i = \frac{v}{R} = \frac{V\,\mathrm{e}^{-t/RC}}{R}$$

The impact of closing the switch is to dissipate the energy stored in the capacitor in the resistor. So if we calculate the total energy dissipated in the resistor we will know the energy stored in the capacitor. However the energy dissipation rate, which is the power dissipated, is not a constant rate, but varies based on the current flowing through the resistor. We can use integration to calculate the total energy dissipated, E from the power dissipated in the resistor $P(t)$ at time t (as discussed above in Engineering application B.2.3)

$$E = \int_0^\infty P(t)\,\mathrm{d}t$$

We also know that the power law gives us $P = i^2R$. So we can say that:

$$\begin{aligned} P(t) &= \left(\frac{V\,\mathrm{e}^{-t/RC}}{R}\right)^2 R = \frac{V^2\,\mathrm{e}^{-2t/RC}\,R}{R^2} = \frac{V^2\,\mathrm{e}^{-2t/RC}}{R} \\ \therefore E &= \int_0^\infty \frac{V^2\,\mathrm{e}^{-2t/RC}}{R}\,\mathrm{d}t = \frac{V^2}{R}\int_0^\infty \mathrm{e}^{-2t/RC}\,\mathrm{d}t \\ &= \frac{-V^2RC}{2R}\left[\mathrm{e}^{-2t/RC}\right]_0^\infty \end{aligned}$$

Now

$$\lim_{t\to\infty}(\mathrm{e}^{-2t/RC}) = 0$$

So the energy stored in a capacitor is given by:

$$E = \frac{CV^2}{R}$$

Integral properties of the delta (impulse) function

The Dirac delta (impulse), $\delta(t-d)$, was introduced in Section 4.9. At times, we need to integrate the delta function — in particular we need to consider the improper integral:

$$\int_{-\infty}^{\infty} \delta(t-d)\,\mathrm{d}t$$

This integral gives the area under the curve and as the definition of the delta function is a rectangle whose area is 1 in the limit as the base length tends to 0 and as the height tends to infinity. So the result of this improper integral is 1

$$\int_{-\infty}^{\infty} \delta(t-d)\,\mathrm{d}t = 1$$

When we consider Laplace transforms in the level 5 Advanced Mathematics course (Wyatt-Millington & Love 2022) we need to consider the improper integral

$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t$$

where $f(t)$ is a known function of time. The definition of $\delta(t-d)$ is zero everywhere except at time $t = d$ where $f(t)$ has a value of $f(d)$ (which is a constant), so we can say that:

$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t = \int_{-\infty}^{\infty} f(d)\delta(t-d)\,\mathrm{d}t = f(d)\int_{-\infty}^{\infty} \delta(t-d)\,\mathrm{d}t$$

But we know $\displaystyle\int_{-\infty}^{\infty} \delta(t-d)\,\mathrm{d}t = 1$ and hence $\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t = f(d)$

$$\int_{-\infty}^{\infty} \delta(t-d)\,\mathrm{d}t = 1$$
$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t = f(d)$$

The result $\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t = f(d)$ is known as the *sifting property* of the delta function as by multiplying a function, $f(t)$ by the delta function $\delta(t-d)$ and integrating from $-\infty$ to ∞ we sift from the function the value $f(d)$

Example B.2.15

Evaluate the following integrals:

a) $\displaystyle\int_{-\infty}^{\infty} t^2\delta(t-3)\,\mathrm{d}t$

b) $\displaystyle\int_{0}^{\infty} \mathrm{e}^{-t}\,\delta(t-1)\,\mathrm{d}t$

Solution:

a) We use the sifting property

$$\int_{-\infty}^{\infty} f(t)\delta(t-d)\,\mathrm{d}t = f(d)$$

with $f(t) = t^2$ and $d = 3$ so:

$$\int_{-\infty}^{\infty} t^2\delta(t-3)\,\mathrm{d}t = f(3) = 3^2 = 9$$

b) We note that $\delta(t-1) = 0$ everywhere except at $t = 1$ so we can use the sifting property, as the limits include $t = 1$, with $f(t) = \mathrm{e}^{-t}$ and $d = 1$ so:

$$\int_{0}^{\infty} \mathrm{e}^{-t}\,\delta(t-1)\,\mathrm{d}t = f(1 = \mathrm{e}^{-1} = \frac{1}{\mathrm{e}}$$

Integration of piecewise continuous functions

When functions have a discontinuity that occurs within the limits of integration then the interval is divided into sub-intervals so that the integrand is continuous on each sub-interval. This is best illustrated by an example:

Example B.2.16

Given

$$f(x) = \begin{cases} 5 & 0 \leqslant x < 2 \\ t^2 & 2 < x \leqslant 4 \end{cases}$$

evaluate $\int_0^4 f(x)\,\mathrm{d}x$

Solution:

The function $f(x)$ is an example of a piecewise continuous function with a discontinuity at $t = 2$ which is within the limits of integration. Figure B.2.17 illustrates the function clearly showing the discontinuity. We can split the integration at the discontinuity:

$$\int_0^4 f(x)\,\mathrm{d}x = \int_0^2 f(x)\,\mathrm{d}x + \int_2^4 f(x)\,\mathrm{d}x$$

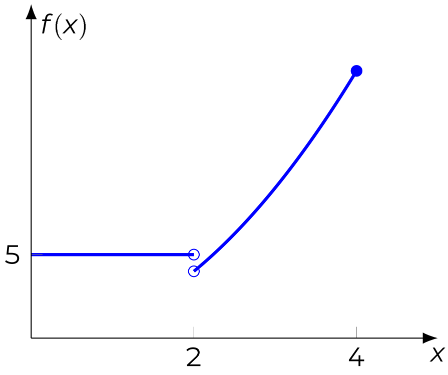


Figure B.2.17: Plot of $f(x)$

$f(x)$ is continuous in intervals $(0, 2)$ and $(2, 4)$ hence

$$\begin{aligned}\int_0^4 f(x)\,\mathrm{d}x &= \int_0^2 5\,\mathrm{d}x + \int_2^4 x^2\,\mathrm{d}x \\ &= \Big[5x\Big]_0^2 + \left[\frac{x^3}{3}\right]_2^4 \\ &= 5 + \left(\frac{64}{3} - \frac{8}{3}\right) = \frac{71}{3} = 23\frac{2}{3}\end{aligned}$$

Questions

1. Show the following pairs of functions are orthogonal over the specified intervals:
   a) $f(x) = x^2$ and $g(x) = 1 - x$ over $\left[0, \frac{4}{3}\right]$
   b) $f(x) = x^2$ and $g(x) = \frac{1}{x}$ over $[-k, k]$
   c) $f(x) = 1 + t$ and $g(x) = 1 - t$ over $[0, \sqrt{3}]$

d) $f(t) = e^t$ and $g(t) = 1 - e^{-2t}$ over $[-1, 1]$

2. Evaluate the following integrals if possible:

a) $\int_0^\infty e^{-2t}\,dt$

b) $\int_1^\infty \frac{1}{x}\,dx$

c) $\int_1^\infty \frac{2}{x^2}\,dx$

d) $\int_0^3 \frac{2}{x+2}\,dx$

e) $\int_0^\infty \sin 4t\,dt$

f) $\int_0^2 \frac{1}{x^2+4}\,dx$

g) $\int_{-\infty}^3 t\,e^t\,dt$

h) $\int_0^2 e^{-st}\cos t\,dt \quad s > 0$

i) $\int_0^2 e^{(2x+2)}\,dx$

j) $\int_0^2 \frac{x}{x^2+2}\,dx$

3. Evaluate the following integrals:

a) $\int_{-\infty}^\infty e^{2t}\,\delta(t)\,dt$

b) $\int_{-\infty}^\infty e^t\,\delta(t+3)\,dt$

c) $\int_{-\infty}^\infty t^3\delta(t-1)\,dt$

d) $\int_{-\infty}^\infty e^{-kt}\,\delta(t-a)\,dt$

e) $\int_{-\infty}^\infty \cos t\delta(t)\,dt$

f) $\int_{-\infty}^\infty \sin\left(\frac{t}{2}\right)\delta(t-\pi)\,dt$

g) $\int_{-\infty}^0 e^{-t}\,\delta(t+2)\,dt$

h) $\int_0^5 x^2\delta(x-3)\,dx$

i) $\int_{-1}^1 x^3\delta(x+3)\,dx$

j) $\int_{-\infty}^\infty t\cos 2t\delta(t-\pi)\,dt$

k) $\int_0^\infty \delta(t-d)\,dt$

l) $\int_{-a}^a \delta(t-d)\,dt$

4. Given

$$f(x) = \begin{cases} 3 & -3 \leqslant x < 0 \\ x^2 & 0 \leqslant x \leqslant 2 \\ 2x & 2 < x \leqslant 3 \end{cases}$$

Evaluate the following integrals:

a) $\int_{-1}^1 f(x)\,dx$

b) $\int_{-0.5}^{2.5} f(x)\,dx$

c) $\int_0^{2.5} f(x)\,dx$

d) $\int_{-3}^3 f(x)\,dx$

5. Given

$$g(t) = \begin{cases} 2t & 0 \leqslant x < 2 \\ 10 - 2t & 2 \leqslant x < 4 \\ 5 & 4 \leqslant x \leqslant 6 \end{cases}$$

Evaluate the following integrals:

a) $\int_0^2 g(t)\,\mathrm{d}t$
b) $\int_2^4 g(t)\,\mathrm{d}t$
c) $\int_1^3 g(t)\,\mathrm{d}t$
d) $\int_{1.5}^{3.5} g(t)\,\mathrm{d}t$
e) $\int_0^6 g(t)\,\mathrm{d}t$

6. Given $u(t)$ in the unit step function (see Section 4.8) evaluate the following integrals:
   a) $\int_0^2 u(t)\,\mathrm{d}t$
   b) $\int_{-4}^4 u(t)\,\mathrm{d}t$
   c) $\int_1^3 u(t-2)\,\mathrm{d}t$
   d) $\int_{-2}^4 2u(t+1)\,\mathrm{d}t$
   e) $\int_{-1}^1 tu(t)\,\mathrm{d}t$
   f) $\int_0^4 e^{kt}u(t-2)\,\mathrm{d}t$ k constant

2.5 Summary

This chapter has given an introduction to integration and explored its uses in electronic & electrical engineering. You will meet integration more in AC circuit analysis at level 5 and also in the Advanced Mathematics module at Level 5 where we look at various transforms (methods to move signals between domains). The next two chapters explore how we can use differentiation and integration to model systems.

Fourier Series

3.1 Introduction

In electronic & electrical engineering we often have to analyse waveforms of various types — Fourier analysis is a set of mathematical tools which enables us to break signals down into their constituent frequency components. The behaviour of the waveform can then be predicted from knowledge of the behaviour of the individual frequency components. Often it is useful to think of a signal in terms of its frequency components rather than its time domain representation — this is the frequency domain which is particularly useful in analysis of the effect of particular systems. The impact of filters (that is systems that remove unwanted frequencies from a signal) is often assessed using the frequency domain. In this chapter we will begin by looking at the essential properties of waveforms before describing how we can break the waveform down into its frequency components.

3.2 Properties of waveforms

Periodic waveforms

A periodic waveform (signal or function) is defined as one that repeats over time. Mathematically, for a given constant period T, at any time t, a periodic waveform $s(t)$ obeys the rule:

$$s(t+T) = s(t) \qquad -\infty < t < \infty \tag{B.3.1}$$

In particular we are going to be using the sine and cosine functions. Let us recall some simple facts about sinusoidal waves. A function $f(t) = A\sin(\omega t + \phi) = A\sin\omega\left(t + \frac{\phi}{\omega}\right)$ represents a sine wave with an amplitude A, an angular frequency ω, a frequency $f = \frac{\omega}{2\pi}$, a period $T = \frac{2\pi}{\omega}$ and a phase angle ϕ as shown in Figure B.3.1. The quantity $\frac{\phi}{\omega}$ is known as the time displacement which is a measure of how much the sine wave is horizontally shifted. Similar observations can be made about $f(t) = A\cos(\omega t + \phi)$.

Together these two functions form the class of functions called ***sinusoids*** or ***harmonics***. In the following sections it will be important to understand the integration of these functions. From Table B.2.1 we can see that:

$$\int \sin n\omega t \, \mathrm{d}t = -\frac{\cos n\omega t}{n\omega} + c \qquad \int \cos n\omega t \, \mathrm{d}t = \frac{\sin n\omega t}{n\omega} + c$$

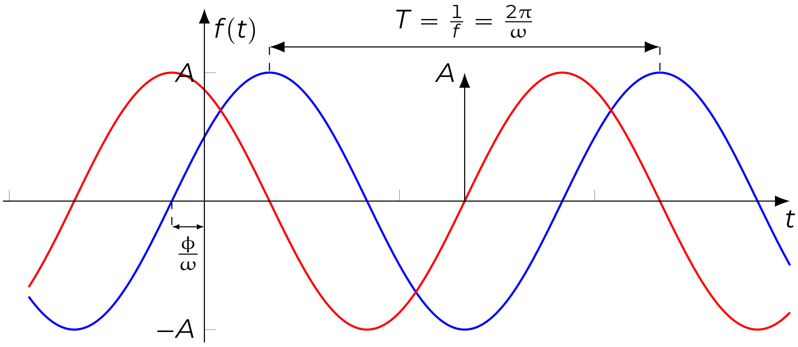


Figure B.3.1: Plot of $f(t) = A\sin(\omega t + \phi)$ and $f(t) = A\cos(\omega t + \phi)$

$$\text{for } n = \pm 1, \pm 2, \ldots$$

Sometimes a function will be created as the sum of a number of sinusoidal components. For example

$$f(t) = \sin \omega_1 t + 0.5 \sin 2\omega_1 t + 0.2 \sin 4\omega_1 t$$

This is a ***linear combination*** of sinusoids where of particular note is that all the angular frequencies are integer multiples of the ***fundamental angular frequency*** ω_1. The components are known as harmonics with $\sin \omega_1 t$ being the ***fundamental/first harmonic***, $0.5 \sin 2\omega_1 t$ being the second harmonic and $0.2 \sin 4\omega_1 t$ being the fourth harmonic — there is no third harmonic as there is no component that has $3\omega_1$ as the angular frequency. One consequence of this is that the function $f(t)$ is also a periodic function with the frequency being the fundamental frequency $f_1 = \dfrac{\omega_1}{2\pi}$.

Example B.3.1

Find the frequency and amplitude of the different harmonic components of

$$f(t) = \sin 20\pi t + 0.7 \sin 40\pi t = 0.4 \cos 120\pi t$$

Solution:
The fundamental angular frequency is 20π arising from the first term $\sin 20\pi t$. This means that the fundamental frequency is $f_1 = \frac{\omega_1}{2\pi} = 10$Hz with an amplitude of 1. The second harmonic term $0.7 \sin 40\pi t$ has a frequency of 20 Hz and an amplitude of 0.7. There are no third, fourth or fifth harmonics. The sixth harmonic has a frequency of 60 Hz and an amplitude of −0.2.

Example B.3.2

Express the function $\sin \omega_1 t + \sqrt{3} \cos \omega_1 t$ as a single sinusoid and hence determine its amplitude and phase.

Solution:
From Example A.5.2 we know that

$$R\cos(\omega t - \theta) = a \cos \omega t + b \sin \omega t$$

where $R = \sqrt{(a^2 + b^2)}$ and $\tan\theta = \dfrac{b}{a}$
In this case

$$R = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

and

$$\tan\theta = \frac{1}{\sqrt{3}} \qquad \Longrightarrow \theta = \frac{\pi}{6}\text{rad}$$

Therefore we can express $f(t)$ as:

$$f(t) = 2\cos\left(\omega_1 t - \frac{\pi}{6}\right)$$

Therefore this is a sinusoid with an amplitude of 2 and a phase angle of $-\dfrac{\pi}{6}\text{rad} = -30°$

Other periodic functions than the sinusoidal and harmonic waves arise in engineering applications — the key factor is they must obey the rule in equation B.3.1. In order to define a periodic function mathematically it is sufficient to give its equation over one period and state that period. For instance Figure B.3.2 shows the plot of the function defined as

$$f(t) = 1 - t \qquad 0 \leqslant t < 1 \qquad \text{period of } 1$$

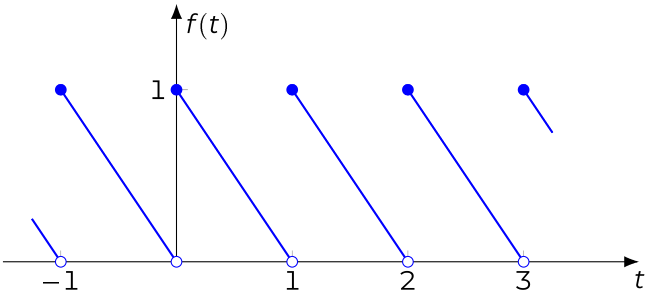


Figure B.3.2: Plot of periodic function $f(t) = 1 - t \quad 0 \leqslant t < 1$ period of 1

Example B.3.3

Find the mathematical expression for the periodic function shown in Figure B.3.3

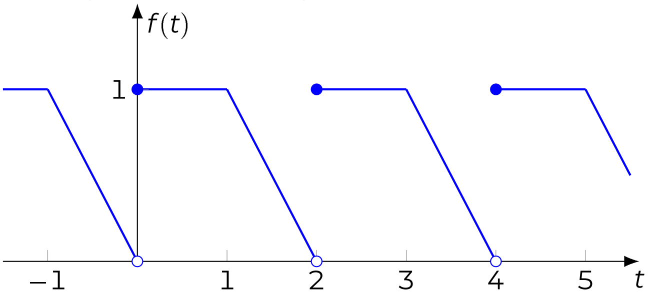


Figure B.3.3: Plot for example B.3.3

Solution:
The first thing to note is that the period of the function is 2 as the function repeats every 2 ticks of t. Therefore we only need to describe the function over the interval $0 \leqslant t < 2$. This can be split into two parts — firstly for the interval $0 \leqslant t < 1$ the graph remains at a constant value of 1 so $f(t) = 1$. For the interval $1 \leqslant t < 2$, the graph is a slope down from (1,1) to (2,0) so $f(t) = 2 - t$. Therefore the mathematical expression for the function in Figure B.3.3 is:

$$f(t) = \begin{cases} 1 & 0 \leqslant t < 1 \\ 2 - t & 1 \leqslant t < 2 \end{cases} \qquad \text{period of 2}$$

Odd and even waveforms

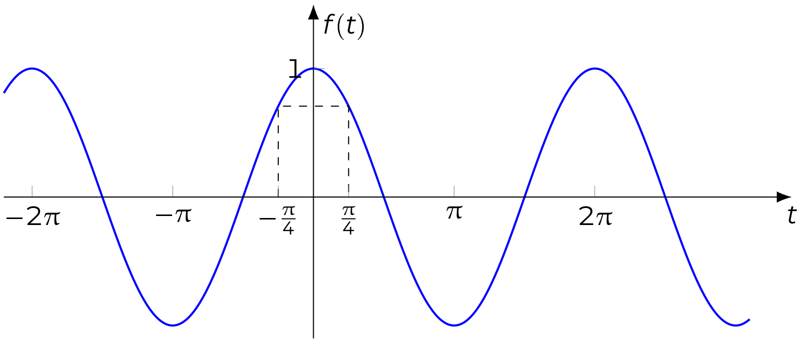


Figure B.3.4: Plot of $f(t) = \cos t$

Let us take a closer look at the sinusoidal functions $\sin t$ and $\cos t$ as they have properties that can be generalised to other functions. Looking at the plot of $\cos t$ in Figure B.3.4 we can see that the function value at $-t$ is the same as the value at $+t$ as the plot is symmetrical about the vertical axis. Look at the values for $\pm\frac{\pi}{4}$ as marked on the plot where this can be seen — we can therefore say $\cos(t) = \cos(-t)$ for any value of t. In general, functions where $f(-t) = f(t)$ are known as ***even*** functions.

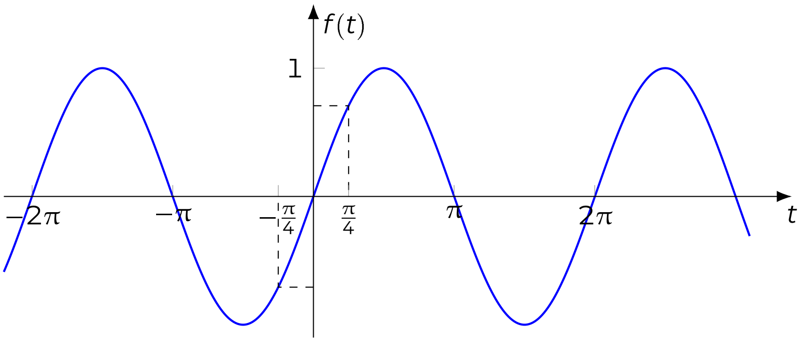


Figure B.3.5: Plot of $f(t) = \sin t$

Conversely if we look at the plot of $f(t) = \sin t$ in Figure B.3.5 we can see the value at $t = \frac{\pi}{4}$ is not the same as the value at $t = -\frac{\pi}{4}$ so the function is not symmetrical about the vertical axis.

In fact in this case $\sin\left(\frac{\pi}{4}\right) = -\sin\left(-\frac{\pi}{4}\right)$ so in general we can say $\sin(t) = -\sin(-t)$ for all values of t. More generally, we can say that where $f(t) = -f(-t)$ then the function is ***odd***

If $f(t) = f(-t)$ then the function is even.
If $f(t) = -f(-t)$ then the function is odd.

Example B.3.4

Find out if the following functions are even or odd:

a) $f(t) = 2$
b) $f(t) = t$
c) $f(t) = 2t^2$
d) $f(t) = 3t^3$

a) $f(t) = 2$ is a straight line at a value of 2 so $f(t) = f(-t)$ so the function is even.

b) In this case we find $f(t)$ and $f(-t)$

$$\begin{aligned} f(t) &= t \\ f(-t) &= -t \end{aligned}$$

So we can say that $f(t) = -f(-t)$ so the function is odd.

c) In this case we find $f(t)$ and $f(-t)$

$$\begin{aligned} f(t) &= 2t^2 \\ f(-t) &= 2(-t)^2 = 2t^2 \end{aligned}$$

So we can say that $f(t) = f(-t)$ so the function is even.

d) In this case we find $f(t)$ and $f(-t)$

$$\begin{aligned} f(t) &= 3t^3 \\ f(-t) &= 3(-t)^3 = -3t^3 \end{aligned}$$

So we can say that $f(t) = -f(-t)$ so the function is odd.

Sum & Product with even & odd functions

We can express any function as a sum of an odd component and an even component. We can say that:

$$\begin{aligned} f(t) &= f(t) + \frac{f(-t)}{2} - \frac{f(-t)}{2} \\ &= \frac{f(t)}{2} + \frac{f(t)}{2} + \frac{f(-t)}{2} - \frac{f(-t)}{2} \end{aligned}$$

Re-arranging this gives:

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2}$$

The first term is an even function as if we set it to be $g(t)$

$$\begin{aligned}g(t) &= \frac{f(t)+f(-t)}{2}\\ g(-t) &= \frac{f(-t)+f(-(-t))}{2} = \frac{f(-t)+f(t)}{2} = g(t)\end{aligned}$$

Similarly we can prove the second term is odd:

$$\begin{aligned}h(t) &= \frac{f(t)-f(-t)}{2}\\ h(-t) &= \frac{f(-t)-f(-(-t))}{2} = -\frac{f(t)-f(-t)}{2} = -h(t)\end{aligned}$$

So we have proved that any function can be written at the sum of even and odd components.

We can also state that the product of two even or two odd functions is an even function whereas the product of an even and odd function is an odd function.

If we have two functions $f(t)$ and $g(t)$ that are even, and we let the product be $P(t) = f(t)g(t)$ then:

$$\begin{aligned}P(-t) &= f(-t)g(-t)\\ &= f(t)g(t) \qquad \text{since } f \text{ and } g \text{ are even}\\ &= P(t)\end{aligned}$$

Therefore we have proved that $P(t) = P(-t)$ so the product of two even functions is an even function. If we make $f(t)$ and $g(t)$ odd then:

$$\begin{aligned}P(-t) &= f(-t)g(-t)\\ &= (-f(t))(-g(t)) \qquad \text{since } f \text{ and } g \text{ are odd}\\ &= f(t)g(t) \qquad\qquad = P(t)\end{aligned}$$

Therefore we have proved that $P(t) = P(-t)$ so the product of two odd functions is an even function. If we make $f(t)$ even and $g(t)$ odd then:

$$\begin{aligned}P(-t) &= f(-t)g(-t)\\ &= (f(t))(-g(t)) \qquad \text{since } f \text{ is even and } g \text{ is odd}\\ &= -f(t)g(t) \qquad\qquad = -P(t)\end{aligned}$$

So we have proved that $P(t-t) = -P(t)$ so the product is odd.

Summarizing the above rules:

$$\text{Any function } f(t) = (\text{even}) + (\text{odd}) \tag{B.3.2}$$

$$(\text{even}) \times (\text{even}) = (\text{even}) \tag{B.3.3}$$

$$(\text{odd}) \times (\text{odd}) = (\text{even}) \tag{B.3.4}$$

$$(\text{even}) \times (\text{odd}) = (\text{odd}) \tag{B.3.5}$$

Integral properties of even & odd functions

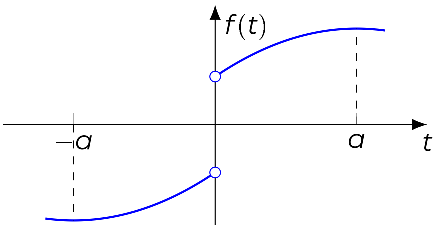


Figure B.3.6: A typical odd function $f(t)$

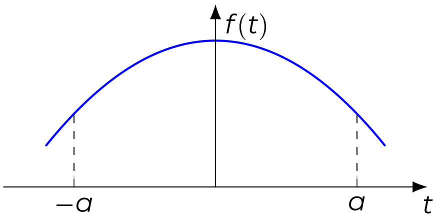


Figure B.3.7: A typical even function $f(t)$

Consider a typical odd function as shown in Figure B.3.6 where we want to evaluate $\int_{-a}^{a} f(t)\,\mathrm{d}t$ where the integration interval $[-a, a]$ is symmetrical about the vertical axis. This is the area bounded by the limits and the horizontal axis — and we can see that the are to left of the vertical axis is a negative value whereas the area to the right of the vertical axis is a positive value of the same magnitude. So these cancel each other out so for an odd function provided the integration limits are symmetrical about the vertical axis the definite integral $\int_{-a}^{a} f(t)\,\mathrm{d}t = 0$.

Similar analysis can be done on a typical even function as shown in Figure B.3.7. Looking at the result of evaluating $\int_{-a}^{a} f(t)\,\mathrm{d}t$ it can be seen that there is the same contribution from both sides of the vertical axis so provided the integration limits are symmetrical about the vertical axis, the integral can be written as

$$\int_{-a}^{a} f(t)\,\mathrm{d}t = 2\int_{0}^{a} f(t)\,\mathrm{d}t$$

Example B.3.5

Evaluate:

a) $\int_{-\pi}^{\pi} t\cos t\,\mathrm{d}t$

b) $\int_{-\pi}^{\pi} t\sin t\,\mathrm{d}t$

Solution:

a) This is a product of two functions where $f(t) = t$ is odd and $g(t) = \cos t$ is even — so the product is an odd function therefore the result of this integration is zero as the limits are symmetrical about the vertical axis.

b) This si the product of two odd functions so the product is even. Therefore using even function rule:

$$\int_{-\pi}^{\pi} t\sin t\,\mathrm{d}t = 2\int_{0}^{\pi} t\sin t\,\mathrm{d}t$$

We can now integrate by parts (equation B.2.3):

$$\begin{aligned} 2\int_{0}^{\pi} t\sin t\,\mathrm{d}t &= 2\left([-t\cos t]_0^{\pi} + \int_{0}^{\pi} \cos t\,\mathrm{d}t\right) \\ &= 2((-\pi\cos\pi) - (0) + [\sin t]_0^{\pi}) \\ &= 2\pi \end{aligned}$$

Orthogonality & other useful identities

We met the concept of orthogonality in Section 2.4 — two functions $f(t)$ and $g(t)$ are said to be orthogonal over the interval $[a, b]$ if

$$\int_a^b f(t)g(t)\,\mathrm{d}t = 0$$

Let us consider the two functions $\sin m\omega t$ and $\sin n\omega t$ where m & n are positive integers with $m \neq n$. Are these functions orthogonal over the interval $-\dfrac{\pi}{\omega} \leqslant t \leqslant \dfrac{\pi}{\omega}$? To answer this we need to evaluate

$$\int_{-\pi/\omega}^{\pi/\omega} \sin m\omega t \sin n\omega t\,\mathrm{d}t$$

We can use the trigonometric identity $\sin x \sin y = \dfrac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]$ to make the integral easier to evaluate:

$$\begin{aligned}
\int_{-\pi/\omega}^{\pi/\omega} \sin m\omega t \sin n\omega t\,\mathrm{d}t &= \frac{1}{2}\int_{-\pi/\omega}^{\pi/\omega} \big[\cos(m-n)\omega t - \cos(m+m)\omega t\big]\,\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{\sin(m-n)\omega t}{(m-n)\omega} - \frac{\sin(m+n)\omega t}{(m+n)\omega}\right]_{-\pi/\omega}^{\pi/\omega} \\
&= 0 \qquad \text{as } \sin(m \pm n)\pi = 0 \text{ for all integers } m, n
\end{aligned}$$

Provided $m \neq n$ (this makes $(m-n) \neq 0$) we can say that the two functions are orthogonal over the defined interval.

$$\begin{array}{ll}
\displaystyle\int_0^T \sin\frac{2n\pi t}{T}\,\mathrm{d}t = 0 & \text{for all integers } n \\[2ex]
\displaystyle\int_0^T \cos\frac{2n\pi t}{T}\,\mathrm{d}t = 0 & \text{for } n = 1, 2, 3, \ldots \\[2ex]
\displaystyle\int_0^T \cos\frac{2n\pi t}{T}\,\mathrm{d}t = T & \text{for } n = 0 \\[2ex]
\displaystyle\int_0^T \cos\frac{2m\pi t}{T}\cos\frac{2n\pi t}{T}\,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} & \\[2ex]
\displaystyle\int_0^T \sin\frac{2m\pi t}{T}\sin\frac{2n\pi t}{T}\,\mathrm{d}t = \begin{cases} 0 & m \neq n \\ T/2 & m = n \end{cases} & \\[2ex]
\displaystyle\int_0^T \cos\frac{2m\pi t}{T}\sin\frac{2n\pi t}{T}\,\mathrm{d}t = 0 & \text{for all integers } m \text{ and } n
\end{array}$$

Table B.3.1: Some useful integral identities

Questions

1. Describe the frequency and amplitude characteristics of the harmonic components of the following:

   a) $f(t) = 3\cos 50\pi t - 4\cos 100\pi t + 0.7\cos 150\pi t$

b) $f(t) = \sin 15t - 0.5\cos 45t + 0.3\cos 90t$

Use a suitable software package (MATLAB® or Microsoft® Excel®) to create plots of these two functions.

2. Express each of the following functions as a single sinusoid and hence find their amplitudes and phases.

   a) $f(t) = 3\cos t - 2\sin t$
   b) $f(t) = 2.2\cos t + 0.5\sin t$
   c) $f(t) = -2\cos 2t$
   d) $f(t) = 3\cos 3t + 2\sin 3t$

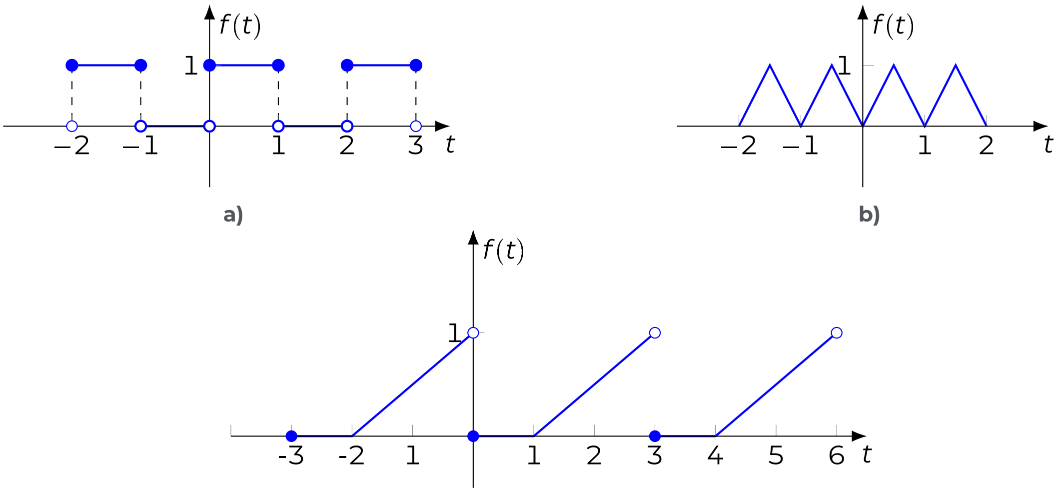
3. Sketch graphs of the following periodic functions:

   a) $f(t) = 0.5t^2, -2 \leqslant t \leqslant 2$, period 4

   b) $f(t) = \begin{cases} \cos t & 0 \leqslant t < \pi/2 \\ 0 & \pi/2 \leqslant t \leqslant \pi \end{cases}$, period π

   c) $f(t) = \begin{cases} -t & -1 \leqslant t < 0 \\ t & 0 \leqslant t \leqslant 2 \end{cases}$, period 3

4. Write down expressions to describe the functions shown in Figure B.3.8



a) b) c)

Figure B.3.8: Periodic functions for Question 4

5. Are the functions in Figure B.3.9 even, odd or neither?

6. Using the properties defined in equations B.3.3 - B.3.5 state whether the following functions are even, odd or neither:

   a) $f(t) = t^3 \cos t$
   b) $f(t) = t\sin t$
   c) $f(t) = t^2 \sin 2t$
   d) $f(t) = \sin t \sin 2t$
   e) $f(t) = \cos 2\omega t \sin \omega t$
   f) $f(t) = e^{2t} \cos t$

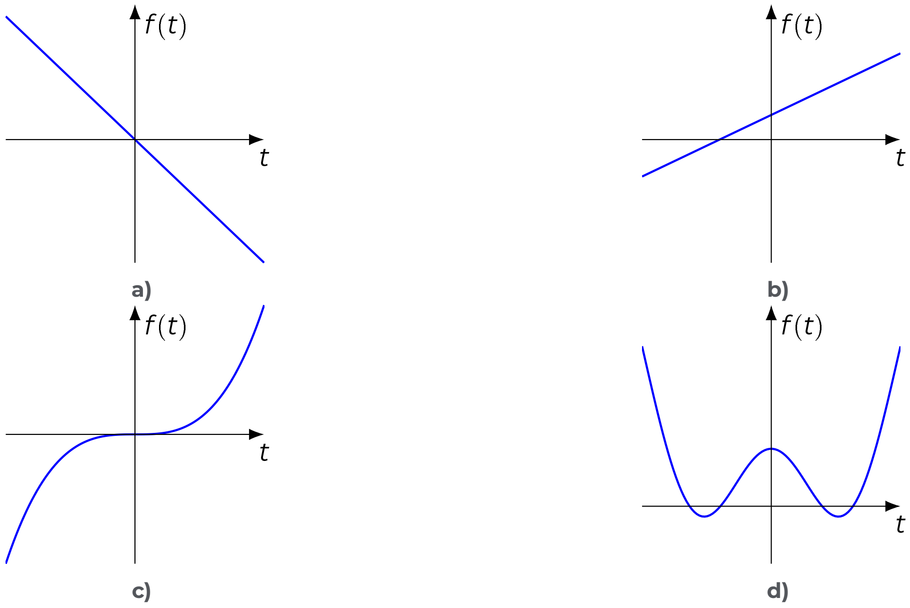


Figure B.3.9: Functions for Question 5

7. Evaluate the following integrals using the integral properties of odd and even functions where appropriate:

a) $\int_{-5}^{5} 2t^3 \, \mathrm{d}t$

b) $\int_{-5}^{5} t^3 \cos 2t \, \mathrm{d}t$

c) $\int_{-\pi}^{\pi} t^2 \cos t \, \mathrm{d}t$

d) $\int_{-\pi}^{\pi} t \sin t \, \mathrm{d}t$

e) $\int_{-1}^{1} 2|t| \, \mathrm{d}t$

f) $\int_{-1}^{1} 2t|t| \, \mathrm{d}t$

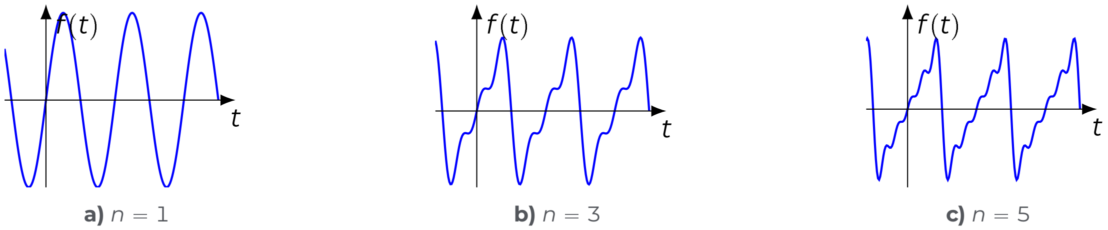
3.3 Fourier series

We know that the functions $\sin n\omega t$ and $\cos n\omega t$ where $n = 1, 2, 3, \ldots$ are periodic as are linear combinations of these functions. They are also convenient as they are easily differentiated and integrated etc. In addition they also have the property of ***completeness*** as almost any periodic function can be expressed as a linear combination of $\sin n\omega t$ and $\cos n\omega t$ without using any other function. In other words they are building blocks for periodic functions and we can construct periodic functions by adding particular multiples of them together. This is the foundation of Fourier series.

Let us take the example of the following infinite series and see what happens as we add more terms in as shown in Figure B.3.10:

$$\begin{aligned} f(t) &= 2\left(\sin t - \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t - \frac{1}{4}\sin 4t + \frac{1}{5}\sin 5t \cdots\right) \\ &= 2\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}\sin nt}{n} \end{aligned} \tag{B.3.6}$$

Looking at the figures in B.3.10, you can see that already for $n = 5$ a sawtooth wave is starting



a) $n = 1$ **b)** $n = 3$ **c)** $n = 5$

Figure B.3.10: Fourier synthesis for equation B.3.6

to appear which is in fact what equation B.3.6 represents. If you continued the sum to infinity then the waveform in Figure B.3.11 would be the result for any value of t except at the discontinuities (so at odd multiples of π or mathematically $\pm(2r-1)\pi$ for $r = 1, 2, 3 \dots$) where the sum will become 0.

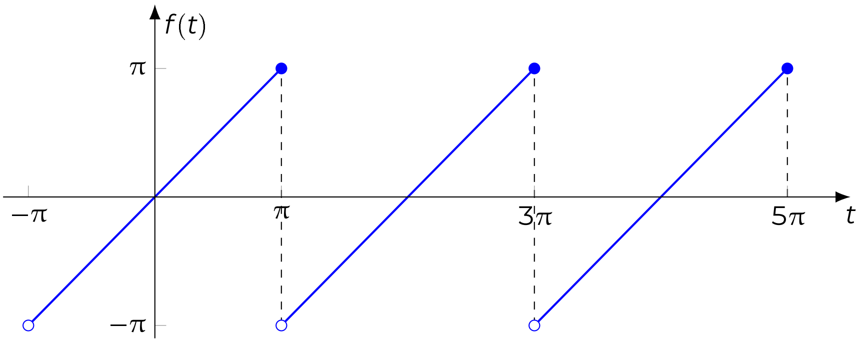


Figure B.3.11: Sawtooth waveform

This process of adding sinusoids together to make a new periodic waveform is called ***Fourier Synthesis*** — from equation B.3.6 we can see that the sawtooth waveform is formed from a series of harmonics of $\sin t$. This series is called the ***Fourier Series*** representation of $f(t)$ where we have broken the function down into its constituent harmonic components. More generally Fourier series require both sine and cosine waves — if we define the periodic function $f(t)$ with a period of T in interval $0 < t < T$ then under certain conditions the Fourier Series of $f(t)$ with ***Fourier coefficients*** a_n and b_n is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right) \tag{B.3.7}$$

Or equivalently as $\omega = 2\pi f = \dfrac{2\pi}{T}$:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} ((a_n \cos n\omega t + b_n \sin n\omega t) \tag{B.3.8}$$

The Fourier coefficients are calculated as:

$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t \tag{B.3.9}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2n\pi t}{T}\,\mathrm{d}t \qquad \text{for } n \text{ as a positive integer} \tag{B.3.10}$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t \qquad \text{for } n \text{ as a positive integer} \tag{B.3.11}$$

It is worth noting that the term $\frac{a_0}{2}$ represents the mean value of DC component of the waveform and also that the integrals in equations B.3.9 - B.3.11 can be evaluated over any complete period so $-\frac{T}{2}$ to $\frac{T}{2}$.

The conditions for the equations B.3.7 and B.3.8 to be valid are that the integral $\int |f(t)|\,\mathrm{d}t$ must be finite over a complete period and that there can only be a finite number of discontinuities in $f(t)$ in a finite interval. These conditions, known as the Dirichlet conditions, mean that the series converge to the value of $f(t)$. As in the sawtooth waveform it is worth noting that at a discontinuity the Fourier series converge to the average of the two values either side of the discontinuity.

We will now look at some examples of Fourier series starting with the simple square wave:

Example B.3.6

Find the Fourier series representation of the function with a period $T = 1$ given by:

$$f(t) = \begin{cases} 1 & 0 \leqslant t < 0.5 \\ 0 & 0.5 \leqslant t < 1 \end{cases}$$

Solution:
It is always useful to start with a sketch of the function as in Figure B.3.12.

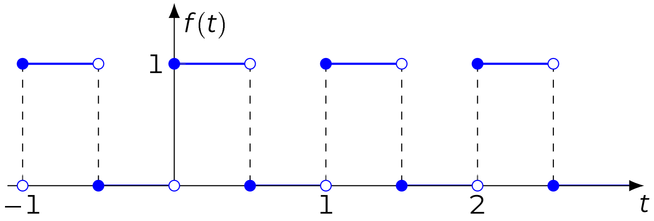


Figure B.3.12: Square wave of period $T = 1$

We know that the function is 0 between $t = 0.5$ and $t = 1$ so we only need to consider the interval $0 \leqslant t < 0.5$. Using equations B.3.9 - B.3.11 we get:

$$\begin{aligned} a_0 &= \frac{2}{1}\int_0^1 f(t)\,\mathrm{d}t = 2\int_0^{0.5} 1\,\mathrm{d}t + 2\int_{0.5}^1 0\,\mathrm{d}t \\ &= 2[t]_0^{0.5} = 1 \\ a_n &= 2\int_0^{0.5} \cos 2n\pi t\,\mathrm{d}t = 2\left[\frac{\sin 2n\pi t}{2n\pi t}\right]_0^{0.5} \\ &= 0 \qquad \text{as } \sin n\pi = 0 \end{aligned}$$

$$
\begin{aligned}
b_n &= 2\int_0^{0.5} \sin 2n\pi t\,dt = 2\left[-\frac{\cos 2n\pi t}{2n\pi t}\right]_0^{0.5} \\
&= -\frac{1}{n\pi}(\cos n\pi - cos 0)
\end{aligned}
$$

Now we know that $\cos n\pi = (-1)^n$ so:

$$
b_n = \frac{1}{n\pi}(1 - (-1)^n)
$$

If n is even then $(-1)^n = 1$ so $b_n = 0$ and if n is odd then $(-1)^n = -1$ so $b_n = \frac{2}{n\pi}$. Therefore the Fourier series representation of $f(t)$ is:

$$
\begin{aligned}
f(t) &= \frac{1}{2} + \frac{2}{\pi}\left(\sin 2\pi t + \frac{\sin 6\pi t}{3} + \frac{\sin 10\pi t}{5} + \cdots\right) \\
&= \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin 2(2n+1)\pi t}{(2n+1)}
\end{aligned}
$$

The average value of the waveform can be seen to be $\frac{1}{2}$ which is the zero frequency component. In this waveform only odd harmonics are present.

Example B.3.7

Find the Fourier series representation of the function $f(t) = 1 + t, -\pi \leqslant t < \pi$ with a period $T = 2\pi$

Solution:

Again we start with a sketch of the function as in Figure B.3.13.

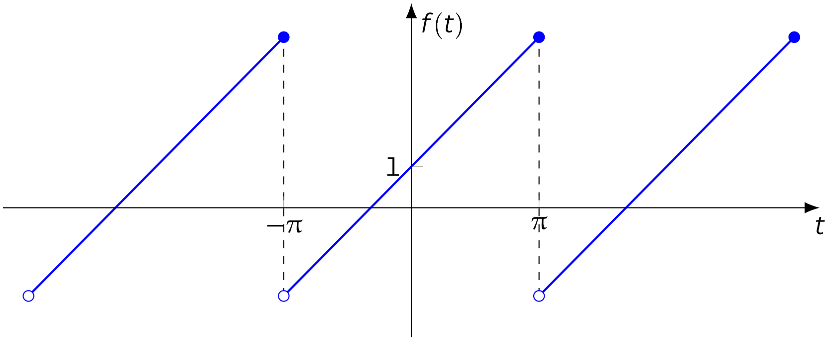


Figure B.3.13: Graph for Example B.3.7

Here we know that $T = 2\pi$ which means that $\omega = 1$. We shall use the integration interval of $[-\pi, \pi]$. So using the Fourier series coefficients equation B.3.9:

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(1+t)\,dt = \frac{1}{\pi}\left[t + \frac{t^2}{2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left(\left(\pi + \frac{\pi^2}{2}\right) - \left(-\pi + \frac{\pi^2}{2}\right)\right) \\
&= \frac{1}{\pi}(2\pi)
\end{aligned}
$$

$$= 2$$

Similarly using equation B.3.10:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+t)\cos nt\,\mathrm{d}t$$

Integrating by parts:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left(\left[(1+t)\frac{\sin nt}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin nt}{n}\,\mathrm{d}t\right)\\
&= \frac{1}{\pi}\left(0 + \left[\frac{\cos nt}{n^2}\right]_{-\pi}^{\pi}\right) \qquad \text{since } \sin \pm n\pi = 0\\
&= \frac{1}{n^2\pi}(\cos n\pi - \cos(-n\pi))\\
&= 0 \qquad \text{since } \cos n\pi = \cos(-n\pi)
\end{aligned}$$

For b_n we use equation B.3.11:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(1+t)\sin nt\,\mathrm{d}t$$

Integrating by parts:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\left(\left[-(1+t)\frac{\cos nt}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nt}{n}\,\mathrm{d}t\right)\\
&= \frac{1}{\pi}\left(-(1+\pi)\frac{\cos n\pi}{n} + (1-\pi)\frac{\cos(-n\pi)}{n} + \left[\frac{\sin nt}{n^2}\right]_{-\pi}^{\pi}\right)\\
&= \frac{1}{n\pi}(-2\pi\cos n\pi)\text{since } \sin \pm n\pi = 0\\
\implies b_n &= -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n
\end{aligned}$$

So the Fourier series representation of $f(t)$ (using equation B.3.8):

$$\begin{aligned}
f(t) &= 1 + 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \dots\\
&= 1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n \sin nt}{n}
\end{aligned}$$

Fourier series of odd and even functions

Let us consider the equations for a_n and b_n as shown in equation B.3.10 and B.3.11. First of let us think about a_n using interval $[-T/2, T/2]$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\frac{2n\pi t}{T}\,\mathrm{d}t$$

Now if $f(t)$ is odd then the product

$$f(t)\cos\frac{2n\pi t}{T}$$

is also odd from equation B.3.5 as $\cos t$ is an even function. We know that the integral of an odd function on an interval that is symmetrical about the vertical axis is zero. So we can say that if $f(t)$ is odd, then $a_n = 0$ for all n.

Similarly if $f(t)$ is an even function let us consider b_n using interval $[-T/2, T/2]$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t$$

In this case as $\sin t$ is an odd function the product

$$f(t)\sin\frac{2n\pi t}{T}$$

is odd so the integral resolves to a value of 0. So we can say that if $f(t)$ is even then $b_n = 0$ for all n.

These rules can vastly simplify Fourier series calculations as if the function you are trying to represent is odd you do not have to calculate a_n and if it is even you do not have to calculate b_n.

Parseval’s Theorem

This is a useful theorem for power calculations if you know the Fourier Series representation of a signal. For a periodic function$f(t)$ with period T and Fourier coefficients a_n and b_n, Parseval’s Theorem states that:

$$\frac{2}{T}\int_0^T (f(t))^2\,\mathrm{d}t = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \qquad \text{(B.3.12)}$$

Engineering application B.3.1: Average power of a signal

Find the average power developed by the current signal $i(t)$ through a

a) 1 Ω resistor

b) 2 Ω resistor

c) 0.5 Ω resistor

1 Ω resistor:

$$i(t) = \sin t - \frac{1}{3}\cos 2t + \frac{1}{4}\sin 4t \qquad \text{period } 2\pi$$

Solution:
We can see that $i(t)$ is expressed as a Fourier series (sum of harmonics) with $b_1 = 1, b_4 = \frac{1}{4}$ and $a_2 = -\frac{1}{3}$ and all other Fourier coefficients being 0. The instantaneous power is given by $(i(t))^2 R$ so the average power over one period is given by

$$P_aV = \frac{R}{2\pi}\int_0^{2\pi}(i(t))^2\,\mathrm{d}t$$

a) Applying Parseval's theorem from equation B.3.12 leads to:

$$P_a v = \frac{1}{2}\left(1^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right) = 0.68\text{W}$$

b) Applying Parseval's theorem from equation B.3.12 leads to:

$$P_a v = \left(1^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right) = 1.36\text{W}$$

c) Applying Parseval's theorem from equation B.3.12 leads to:

$$P_a v = \frac{1}{4}\left(1^2 + \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{4}\right)^2\right) = 0.34\text{W}$$

Questions

1. Find the Fourier series representation of the function:

$$f(t) = \begin{cases} 1 & -2 < t < 0 \\ 0 & 0 < t < 2 \end{cases} \qquad \text{period } 4$$

2. Find the Fourier series representation of the function:

$$f(t) = \begin{cases} 0 & -\pi < t < 0 \\ t & 0 < t < \pi \end{cases} \qquad \text{period } 2\pi$$

3. Find the Fourier series representation of the function:

$$f(t) = t^2 \qquad -\pi < t < \pi \qquad \text{period } 2\pi$$

4. Find the Fourier series representation of the function:

$$f(t) = \begin{cases} 0 & -\pi < t < -\frac{\pi}{2} \\ 2 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < t < \pi \end{cases} \qquad \text{period } 2\pi$$

5. Find the Fourier series representation of the function:

$$f(t) = \begin{cases} -2 & -\pi < t < 0 \\ 2 & 0 < t < \pi \end{cases} \qquad \text{period } 2\pi$$

6. Find the Fourier series representation of the function:

$$f(t) = \begin{cases} 1 & -\pi < t < -\frac{\pi}{2} \\ -1 & -\frac{\pi}{2} < t < \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < t < \pi \end{cases} \qquad \text{period } 2\pi$$

7. Consider a PWM signal with an amplitude A, a period of T and a 'on' time of m as shown in Figure B.3.14. The duty cycle (ratio of on time to period) of such a signal is given by $d = m/T$. What is the Fourier series representation of this signal?

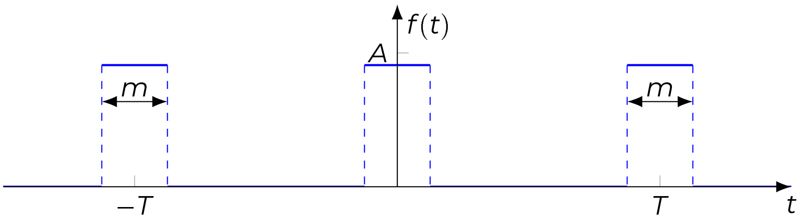


Figure B.3.14: A PWM signal of period T and duty cycle m/T

8. What is average power generated by the function $i(t) = 1 + 2\cos t - \sin t + 0.5\cos 3t$ over a period of 2π through a $2\,\Omega$ resistor?

9. What is average power generated by $v(t) = \cos t - 0.5\sin t + 0.2\cos 3t$ over a period of 2π across a $1\,\Omega$ resistor? Note that $P = \dfrac{V^2}{R}$

3.4 Frequency response

We know that *linear systems* have the useful property that the response of the system to several inputs being applied to the system simultaneously can be obtained by adding the responses of the system to the individual inputs. Another useful property of such systems is that when you apply a sinusoid to the input then the output will also be a sinusoid of the same frequency but with a modified amplitude and phase as shown in Figure B.3.15.

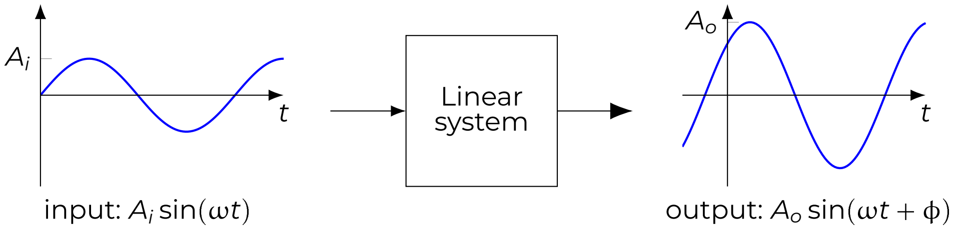


Figure B.3.15: Linear system

In Section 10.4, we saw how sinusoidal signals can be represented by complex numbers and how we can analyse AC circuits using complex numbers. This is true for linear systems in general where it is possible to define a ***complex frequency function*** $G(j\omega)$ with ω being frequency of the input and G being the transfer function that relates the output to the input of the system.

If a sine wave of amplitude A_i is applied to the system then the output amplitude A_o and phase shift ϕ are given by:

$$A_o = |G(j\omega)|A_i$$
$$\phi = \angle G(j\omega)$$

The notation $G(j\omega)$ looks a little odd, but arises, as we shall see in the Level 5 Advanced Mathematics course (Wyatt-Millington & Love 2022), from creating the complex frequency function from substituting $s = \mathrm{j}\omega$ into the Laplace transform transfer function, $G(s)$, of the system.

We can now analyse the effect of applying a more general periodic function to the input of a linear system. We do this by finding the Fourier components of the input waveform and then finding the amplitude and phase shift of the output components relating to these input Fourier components. Finally we find the overall output by adding the output components together using the linearity property. Let us look at an example of passing a square wave through an analogue low pass filter as originally discussed in Engineering application A.6.2

Engineering application B.3.2: Analogue lowpass filter

A simple circuit that acts as a low-pass filter is shown in Figure B.3.16. we want to derive the frequency response of this circuit and draw graphs of its amplitude and phase response before going on to consider its response to a square wave input.

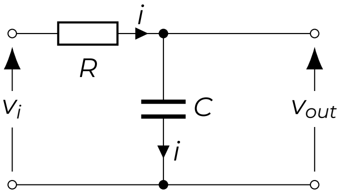


Figure B.3.16: Simple analogue lowpass filter

Using Ohm's law and the knowledge that the impedance of a capacitor os given by $Z = \dfrac{1}{\mathrm{j}\omega C}$we can say that:

$$\begin{aligned} v_i &= iR + v_o \\ v_o &= \frac{i}{\mathrm{j}\omega C} \\ \implies \quad i &= v_o \mathrm{j}\omega C \\ \therefore v_i &= v_o \mathrm{j}\omega CR + v_o = v_o(1 + \mathrm{j}\omega RC) \end{aligned}$$

Rearranging the last equation gives us:

$$\frac{v_o}{v_i} = G(\mathrm{j}\omega) = \frac{1}{1 + \mathrm{j}\omega RC} \tag{B.3.13}$$

This is the same as the frequency response given in Engineering application A.6.2 and relates the output of the system to the input of the system hence equalling $G(\mathrm{j}\omega)$. If we rewrite equation B.3.13 in polar form $r\angle\theta$ using the equations in Section 10.3 then:

$$\begin{aligned} G(\mathrm{j}\omega) &= \frac{1\angle 0}{\sqrt{1 + (\omega RC)^2}\angle \tan^{-1}(\omega RC)} \\ &= \frac{1}{\sqrt{1 + (\omega RC)^2}}\angle - \tan^{-1}(\omega RC) \end{aligned}$$

So we can say that the amplitude and phase of $G(\mathrm{j}\omega)$ are given by:

$$|G(\mathrm{j}\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{B.3.14}$$

$$\angle G(\mathrm{j}\omega) = -\tan^{-1}\omega RC \tag{B.3.15}$$

If we plot these responses we get the characteristic plots shown in Figure B.3.17

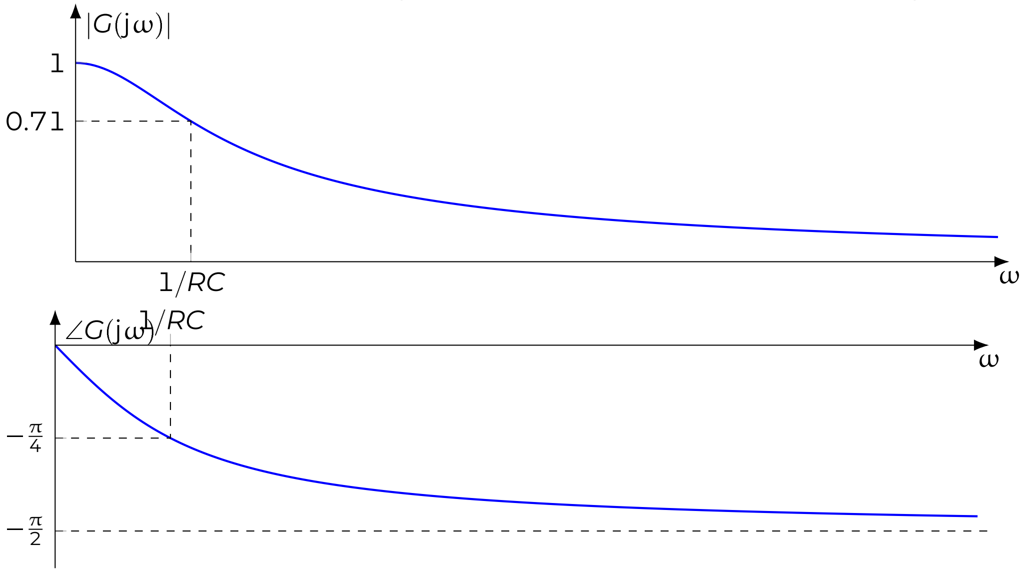


Figure B.3.17: Amplitude and phase characteristics of circuit in Figure B.3.16

The cut-off frequency — that is when significant frequency attenuation occurs — can be altered by varying the values of R and C to change the time constant RC. Let us consider a specific example where $RC = 0.4$ — in this case equations B.3.14 and B.3.15 become:

$$|G(\mathrm{j}\omega)| = \frac{1}{\sqrt{1+0.16\omega^2}} \tag{B.3.16}$$

$$\angle G(\mathrm{j}\omega) = -\tan^{-1} 0.4\omega \tag{B.3.17}$$

Let us now consider the response of the system to a square wave input of amplitude 1 and fundamental angular frequency of $\omega = 1 \implies T = 2\pi$ — this wave is shown in Figure B.3.18a. We need to first calculate the Fourier coefficients of this waveform — we note it has an average value of 0 so $a_0 = 0$ and also that it is odd so it will contain so cosine components thus $a_n = 0$. This means we just need to evaluate:

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\frac{2n\pi t}{T}\,\mathrm{d}t$$

As we know that $T = 2\pi$ and $f(t) = -1$ for $-\pi < t < 0$ and $f(t) = 1$ for $0 < t < \pi$ the coefficients calculation becomes:

$$\begin{aligned} b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin nt\,\mathrm{d}t + \int_{0}^{\pi}\sin nt\,\mathrm{d}t\right) \\ &= \frac{1}{\pi}\left(\left[\frac{\cos nt}{n}\right]_{-\pi}^{0} + \left[\frac{-\cos nt}{n}\right]_{0}^{\pi}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n\pi}(\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0) \\
&= \frac{1}{n\pi}(2 - 2\cos n\pi) \qquad \text{as } \cos(-n\pi) = \cos(n\pi) \\
&= \frac{2}{n\pi}(1 - \cos n\pi)
\end{aligned}
$$

The first few coefficients are:

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}(1 - \cos\pi) = \frac{4}{\pi} \\
b_2 &= \frac{2}{2\pi}(1 - \cos 2\pi) = 0 \\
b_3 &= \frac{2}{3\pi}(1 - \cos 3\pi) = \frac{4}{3\pi} \\
b_4 &= \frac{2}{4\pi}(1 - \cos 4\pi) = 0 \\
b_5 &= \frac{2}{5\pi}(1 - \cos 5\pi) = \frac{4}{5\pi} \\
b_n &= \frac{4}{n\pi} \qquad \text{for } n = 1, 3, 5, \ldots
\end{aligned}
$$

The next stage is to calculate the gain and phase changes on the Fourier coefficients from equations B.3.16 and B.3.17:

$n = 1$

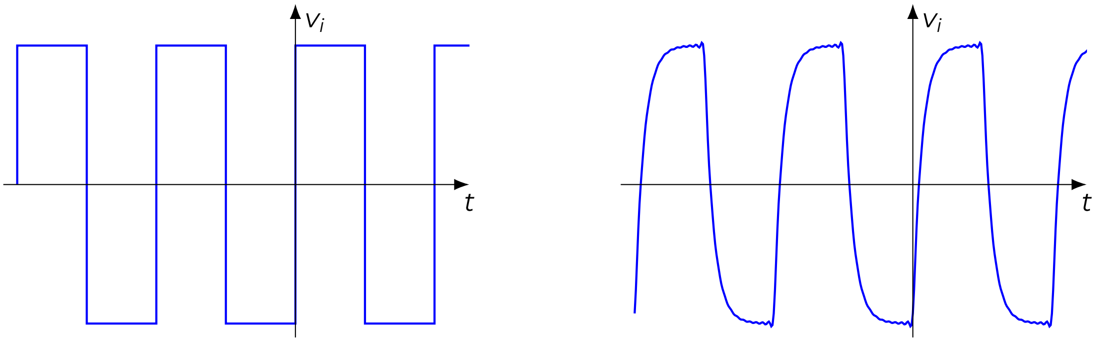
$$
\begin{aligned}
\omega_1 &= 1 \\
|G(j\omega_1)| &= \frac{1}{\sqrt{1 + 0.16}} = 0.928 \\
\angle G(j\omega_1) &= -\tan^{-1} 0.4 = \text{-}0.381\,\text{rad} = -21.8^\circ
\end{aligned}
$$

$n = 3$

$$
\begin{aligned}
\omega_3 &= 3 \\
|G(j\omega_3)| &= \frac{1}{\sqrt{1 + (0.16 \times 9)}} = 0.640 \\
\angle G(j\omega_3) &= -\tan^{-1}(0.4 \times 3) = \text{-}0.876\,\text{rad} = -50.2^\circ
\end{aligned}
$$

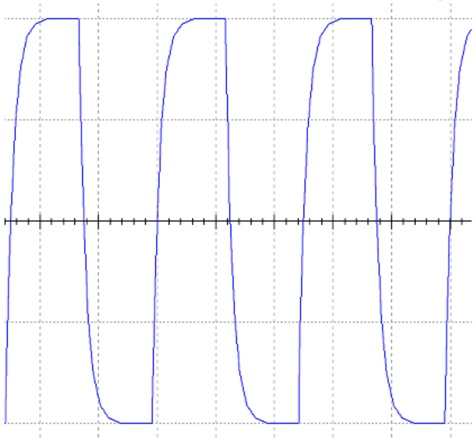
$n = 5$

$$
\begin{aligned}
\omega_5 &= 5 \\
|G(j\omega_5)| &= \frac{1}{\sqrt{1 + (0.16 \times 25)}} = 0.447 \\
\angle G(j\omega_5) &= -\tan^{-1}(0.4 \times 5) = \text{-}1.107\,\text{rad} = -63.4^\circ
\end{aligned}
$$



a) Square wave

b) Response for $n = 21$



c) Simulation trace of output

Figure B.3.18: Applying filter to square wave

As can be seen as n increases the higher frequency components are more attenuated (amplitude response decreases) and also have an increased phase shift. The effect of this is to round the rising and falling edges of the square wave as can be seen in Figure B.3.18b for a maximum value of $n = 21$. Figure B.3.18c shows the simulated output of the circuit to a square wave for comparison purposes. The general equation for the response using Fourier analysis is:

$$v_o(t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{\sin(nt - \tan^{-1}(0.4n))}{n\sqrt{1 + (0.4n)^2}}$$

3.5 Summary

This chapter has looked at various properties of functions (periodic, odd and even) and then has looked at the use of Fourier analysis to model general periodic functions. It has also taken a brief look at how you can use the Fourier series to obtain the response of a linear system to an input that is a periodic waveform. We will look more at Fourier and linear system responses in the level 5 Advanced Maths module.

Statistics

4.1 Introduction

First of what is statistics? According to Bird (2017, p. 619) “Statistics is the study of the collection, organisation, analysis and interpretation of data”. So statistics is involved in everything from how the data is collected (planning of surveys and/or experiments) to analysing the data to create mathematical models of dynamic systems from the measured information. The data can be analysed to discover the ***probability*** of a variable falling into a certain range — this can then be used in probabilistic design of products and systems.

The probability of something happening is the measure of the likelihood or chance that it will happen. It is important in reliability engineering which is to do with analysing the likelihood a system will fail but it is also used in communication engineering as noise is random in nature so is basically modelled using probability theory.

In this chapter we are first going to look at the basics of probability theory before going onto random variables and the difference between continuous & discrete variables and then a brief look at probability density functions. We are then going to look at the key basic measures of data (mean, median, mode & standard deviation) before finally looking at some commonly used statistical distributions (binomial, Poisson and normal)

4.2 Probability theory

Consider a machine making electronic components which must meet a minimum specification. The quality control department regularly samples the components and, on average, 93 in every 100 components meets the specification. Now let us randomly select a component and let A be the outcome that the component meets the requirements which has a likelihood of $\frac{93}{100}$ and B be the outcome it does not which has a chance of $\frac{7}{100}$. We can therefore say:

$$\begin{aligned} P(A) &= \text{probability of } A \text{ occurring} = 0.93 \\ P(B) &= \text{probability of } B \text{ occurring} = 0.07 \end{aligned}$$

The sum of all outcomes (events) is 1 which is the ***sample space***. We can represent this situation using a Venn diagram as shown in Figure B.4.1 (see Section 7.2). We can see that $B = \overline{A}$ — that is B is the ***complement*** of A as there are only two possible outcomes of each ***trial*** (selection of component)

.

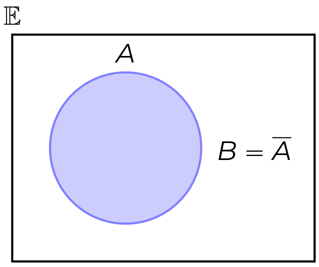


Figure B.4.1: Probability

More formally if event E occurs m times in n trials (where n is a large number), then the probability that E will occur is given by:

$$P(E) = \frac{m}{n} \qquad 0 \leqslant P(E) \leqslant 1 \tag{B.4.1}$$

The larger n is (i.e. the more trials that occur) then the more confident we can be of our estimate of $P(E)$. For instance tossing a coin 1000 times as opposed to 10 times is more likely to get an accurate estimate of the probability of a head. $P(E)$ has a maximum value of 1 (when $m = n$), which is a certain event (always occurs), and a minimum value of 0 (when $m = 0$), which is an impossible event (never occurs).

The sum of the probabilities of all possible outcomes must equal 1 — so for p outcomes, $E_1, E_2, E_3, \ldots, E_p$:

$$P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_p) = 1$$

Compound events

Suppose we define two events of outcomes from rolling a dice (6 possible outcomes), E_1 and E_2 as:

$$\begin{aligned} E_1 &= \{1,2,3,4\} && \text{that is a 1, 2 ,3 or 4 is obtained} \\ E_2 &= \{1,3,5\} && \text{that is an odd number is obtained} \\ \mathbb{E} &= \{1,2,3,4,5,6\} && \text{universal set} \end{aligned}$$

Now let us define another event E_3 as occurring if both E_1 and E_2 occur. This is a ***compound event*** which only occurs when both E_1 and E_2 occur at the same time. Using set notation:

$$\begin{aligned} E_3 &= E_1 \cap E_2 \\ &= \{1,2,3,4\} \cap \{1,3,5\} \\ &= \{1,3\} \end{aligned}$$

So E_3 occurs when a 1 or a 3 is rolled — so in two out of six equally likely outcomes. Therefore:

$$P(E_3) = P(E_1 \cap E_2) = \frac{2}{6} = \frac{1}{3}$$

This compound event represents the intersection of E_1 and E_2 as shown in Figure B.4.2.

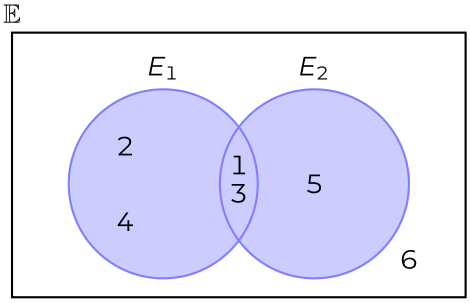


Figure B.4.2: Compound event — intersection of E_1 & E_2

We can also define another compound event E_4 which occurs when either E_1 occurs, or E_2 occurs, or both occur — which in set theory is the union of E_1 and E_2:

$$\begin{aligned} E_4 &= E_1 \cup E_2 \\ &= \{1, 2, 3, 4\} \cup \{1, 3, 5\} \\ &= \{1, 2, 3, 4, 5\} \end{aligned}$$

Mutually exclusive events: Addition law of probability

Consider a manufacturer who manufactures specific components that can fall into one of four categories:

top quality
standard
sub-standard
reject

After a lot of samples it is found that in every 100 samples, on average 20 will be top quality, 65 will be standard quality, 10 will be sub-standard quality and 5 will be rejected. None of these events can occur together, so if one occurs it excludes the occurrence of any of the others, so they are said to be ***mutually exclusive***. In general, using set notation (ϕ is the empty set – that is it contains no elements), we can say that if two events E_i and E_j are mutually exclusive then:

$$E_i \cap E_j = \phi$$

The compound event $E_i \cap E_j$ is an impossible event and will never occur. If we have p events $E_1, E_2, E_3, \ldots, E_p$ which are all mutually exclusive — that is in one trial only one event can occur which excludes all others then the addition law of probability applies (that is the probability of any of these events occurring is equal to the sum of the probabilities for each event):

$$\begin{aligned} P(E_1 \text{ or } E_2 \text{ or } \ldots \text{ or } E_p) &= P(E_1 \cup E_2 \cup \cdots \cup E_p) \\ &= P(E_1) + P(E_2) + \cdots + P(E_p) \end{aligned} \tag{B.4.2}$$

Engineering application B.4.1: Electrical component reliability

Electrical components fail after a certain length of time in use — this ***lifespan*** is not a fixed value but can vary. It is important to be able to quantify the reliability of these components. Let us consider the situation where the lifespans (L) of 5000 components have been measured and recorded as shown in Table B.4.1 below. What is the probability of a randomly selected component lasting

a) more than 4 years

b) between 4 and 6 years

c) less than 5 years

Lifespan of component (year)	*Number*
$L \leqslant 4$	300
$4 < L \leqslant 5$	1950
$5 < L \leqslant 6$	2250
$L > 6$	500

Table B.4.1: Lifespans of 5000 components

In probability terms we define four events $A, B, C\&D$:

A: component lasts 4 years or less
B: component lasts between 4 and 5 years
C: component lasts between 5 and 5 years
A: component lasts more than 6 years

From Table B.4.1, we can calculate the probability of each event occurring:

$$P(A) = \frac{300}{5000} = 0.06 \qquad P(B) = \frac{1950}{5000} = 0.39$$

$$P(C) = \frac{2250}{5000} = 0.45 \qquad P(D) = \frac{500}{5000} = 0.1$$

These events are mutually exclusive so the addition law can be applied.

a)
$$\begin{aligned} P(\text{components lasts more than 4 years}) &= P(B \cup C \cup D) \\ &= P(B) + P(C) + P(D) \\ &= 0.39 + 0.45 + 0.1 \\ &= 0.94 \end{aligned}$$

So there is a 94% chance the component will last more than 4 years.

b)
$$\begin{aligned} P(\text{components between 4 and 6 years}) &= P(B \cup C) \\ &= P(B) + P(C) \\ &= 0.39 + 0.45 \\ &= 0.84 \end{aligned}$$

84% chance component will last between 4 & 6 years.

c) $$\begin{aligned} P(\text{components lasts less than 5 years}) &= P(A \cup B) \\ &= P(A) + P(B) \\ &= 0.06 + 0.39 \\ &= 0.45 \end{aligned}$$ So there is a 45% chance the component will last less than 5 years.

Complementary events

Figure B.4.1 illustrates the complementary event scenario where there are only two possible events which are mutually exclusive. The rules for two events, A & B to be complementary are:

$$\begin{aligned} P(A) + P(B) &= 1 \\ P(A \cap B) &= 0 \end{aligned}$$

What this means if that if we know the probability of one event we can calculate the probability of the other event as $P(\overline{A}) = 1 - P(A)$. We have a good example of this in application B.4.1 above where in question a)we are asked to calculate the probability of a component lasting more than 4 years. This is the complementary event to event A which is lasts 4 years or less. So instead of calculating the answer from the addition law we could have used the complementary event knowledge

$$\begin{aligned} P(\text{components lasts more than 4 years}) &= P(\overline{A}) \\ &= 1 - P(A) \\ &= 1 - 0.06 \\ &= 0.94 \end{aligned}$$

Conditional probability: the multiplication law

Suppose two machines, N and M, both manufacture the same component, but 94% of the components manufactured by M are acceptable whereas only 85% of the components manufactured by N are acceptable. Consider now the event E: component is of an acceptable standard. $P(E) = 0.94$ if all components are manufactured by machine M and $P(E) = 0.85$ if all components are manufactured by machine N. However if half are manufactured by machine M and half by machine N what is $P(E)$?

Let us consider manufacturing 1000 components — 500 with machine M and 500 with machine N. Given the probability of an acceptable component from each machine we can say that the total number of acceptable components is given by $(500 * 0.94) + (500 * 0.85) = 470 + 425 = 895$. So the probability of an acceptable component is $P(E) = \dfrac{895}{1000} = 0.895$. Clearly if the number of components manufactured by each machine changes the value of $P(E)$ will change — this is the concept of ***conditional probability***. If we define two events:

A: component is manufactured by machine M
B: component is manufactured by machine N

Then the notation for the probability that a component is acceptable given it is manufactured by machine M is $P(E|A)$ which means this is the probability of event E given event A has already happened. Similarly the probability that a component is acceptable given it is manufactured by machine N is written as $P(E|B)$. In this example we say that

$$P(E|A) = 0.94 \qquad P(E|B) = 0.85$$

In fact all probabilities are conditional as the conditions surrounding any event can change. However in many cases (such as a dice throw where we assume an unweighted die) we tacitly assume a definite set of conditions and calculate an ***unconditional probability***.

Multiplication law

In the case, where two events A&B are not mutually exclusive — that is $A \cap B \neq \phi$ we may want to know the probability of B occurring given that event A has happened — that is of course $P(B|A)$. As we know that A has occurred we can restrict our interest to set A and we know that event B will occur if any outcome is in $A \cap B$. Thus:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

The multiplication law of probability states that:

$$P(A \cap B) = P(A)P(B|A) \qquad \text{(B.4.3)}$$

Since from set theory $A \cap B = B \cap A$ we can also say:

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B)$$

Engineering application B.4.2: Reliability of components

Following an extensive study a manufacturer finds that they can say for a certain component that: 78% of components will be reliable at least 5 years; 94% of components will be reliable for at least 3 years. What is the probability that a component that has remained reliable for 3 years will still be reliable after 5 years?
We can define two events:

A: component remains reliable for at least three years
B: component remains reliable for at least five years

The two probabilities are unconditional with $P(A) = 0.94$ and $P(B) = 0.78$. We need the conditional probability $P(A|B)$. Now equation B.4.3 tells us:

$$P(A \cap B) = P(A)P(B|A)$$

$P(A \cap B)$ is the compound event that a component remains reliable for at least three years and that it remains reliable for at least five years. Clearly this is the same as event B as all of event B has top be within event A. Thus:

$$P(A \cap B) = P(B)$$

So now we can say:

$$P(B) = P(A)P(B|A)$$

$$\therefore P(B|A) = \frac{P(B)}{P(A)} = \frac{0.78}{0.94} = 0.83 \qquad \text{to 2 d.p.}$$

So 83% of components that remain reliable for three years will remain reliable for five years.

Independent events

Two events are said to be independent if the outcome of one event does not influence the probability of the other event occurring

In set notation terms:

If two events A and B are independent the following rules apply:

(1) $P(A|B) = P(A,$ $\qquad P(B|A) = P(B)$

(2) $P(A \cap B) = P(A)P(B)$

This concept can be extended to more than two events — in this case the definition is that three or more events are independent if every pair of events is independent. If we have $p-$independent events $E_1, E_2, \ldots, E_p$ then

$$\begin{aligned} P(E_i|E_j) &= P(E_i) && \text{for any } i \text{ and } j, i \neq j \\ P(E_i \cap E_j) &= P(E_i)P(E_j) && i \neq j \end{aligned} \tag{B.4.4}$$

In fact the multiplication law for multiple events can be extended to any number of events:

$$\begin{aligned} P(E_1 \cap E_2 \cap E_3) &= P(E_1)P(E_2)P(E_3) \\ P(E_1 \cap E_2 \cap E_3 \cap E_4) &= P(E_1)P(E_2)P(E_3)P(E_4) \\ &\text{and so on.} \end{aligned}$$

So in general

$$P(E_1 \cap E_2 \cap \cdots \cap E_p) = P(E_1)P(E_2)\cdots P(E_n) \tag{B.4.5}$$

Questions

1. A fair die is rolled. The events $E_1, \ldots, E_5$ are defined as follows:

   E_1 : an odd number is obtained
   E_2 : an even number is obtained
   E_3 : a score of less than 3 is obtained
   E_4 : a 4 is obtained
   E_5 : a score of more than 5 is obtained

   Find:

   a) $P(E_1), P(E_2), P(E_3), P(E_4), P(E_5)$

   b) $P(E_2 \cap E_3)$
   c) $P(E_1 \cap E_5)$
   d) $P(E_1 \cap E_3)$
   e) $P(E_2 \cap E_4)$

2. A trial has four outcomes E_1, E_2, E_3 and E_4. E_1, E_2 and E_3 are all equally liekly to occur. E_4 is two times more likely to occur than E_1. Find $P(E_1), P(E_2), P(E_3), P(E_4)$

3. Chips are manufactured by two machines A and B where machine A makes 60% of the chips and machine B makes 40% of the chips. The probability of machine A manufacturing a faulty chip is 0.03 whereas the probability of machine B manufacturing a faulty chip is 0.05. A chip is selected at random - calculate the probability that it is
   a) faulty and manufactured by machine A
   b) faulty and manufactured by machine B
   c) faulty or manufactured by machine A
   d) faulty or manufactured by machine B
   e) faulty

4. The measured lifespans of 2000 components are recorded in Table B.4.2

Lifespan of component (hours)	*Number*
$L < 4$	25
$250 \leqslant L < 500$	93
$500 \leqslant L < 750$	132
$750 \leqslant L < 1000$	713
$1000 \leqslant L < 1250$	620
$1250 \leqslant L < 1500$	187
$1500 \leqslant L < 1750$	135
$L \geqslant 1750$	95

Table B.4.2: Lifespans of 2000 components

   a) Calculate the probability that the lifespan of a component is more than 1000 hours.
   b) Calculate the probability that the lifespan of a component is less than 750 hours.
   c) Calculate the probability that the lifespan of a component that is working after 500 hours will continue to work to at least 1500 hours.

5. Capacitors are manufactured by four machines A, B, C and D with variable probability of an acceptable component of 0.93, 0.95, 0.92 and 0.98 for machines A,B,C and D respectively.
   a) A capacitor is taken from each machine. What is the probability that all four capacitors are acceptable?
   b) Two capacitors are taken from machine C and two from machine D. What is the probability that all four capacitors are acceptable?

c) A capacitor is taken from each machine. What is the probability that at least three capacitors are acceptable?

d) A capacitor is taken from each machine. From these four capacitors one is selected at random:

  i. What is the probability that it is acceptable and made by machine A?
  ii. What is the probability that it is acceptable and made by machine C?

e) A capacitor is taken from each machine. From these four capacitors one is selected at random — what is the probability that it si acceptable?

4.3 Variables & Representation

Quantities that contain an element of chance are called ***random variables***. Some examples of these in engineering are:

(1) the diameter of a motor shaft of nominal size 0.3 m;

(2) the time a battery lasts;

(3) the nominal resistance value of resistors;

(4) the number of components manufactured by a machine in 1 minute.

The first two are both examples of ***continuous*** variables as the value can be anything in a stated range — so for instance the shaft length could vary between 0.297 m and 0.307 m. The last two however are ***discrete*** variables as the nominal resistance value can only be one of a stated set of values and the number of components has to be an integer value.

Probability distributions are used to represent sets of data in terms of the probability of a value occurring. So for instance we have a discrete random variable x which can take a value of 0, 1, 2, 3, 4, 5, or 6. We want to be able to answer questions such as 'Which value is most likely to occur?' and 'Is a 5 more likely to occur then a 4?' This means we need information on the probability of each value occurring. We can take a number of samples of x and record how many times each value occurs — thereby calculating the probability for each value as shown in Table B.4.3.

x	0	1	2	3	4	5	6
$P(x)$	0.05	0.1	0.15	0.3	0.25	0.1	0.05

Table B.4.3: Probability of a discrete value occurring

This is the ***probability distribution*** (probability mass or density function) for the random discrete variable x and can also be presented graphically as shown in Figure B.4.3. Note that the probabilities add up to 1 — the distribution shows how the total probability is distributed amongst the possible values.

What about when x is a continuous random variable that can take any value i interval $[0, 1]$? In this case it is impossible to list all possible values as the variable is continuous. However it can be useful to ask 'What is the probability of x falling in a sub-interval $[a, b]$?'. We can divide

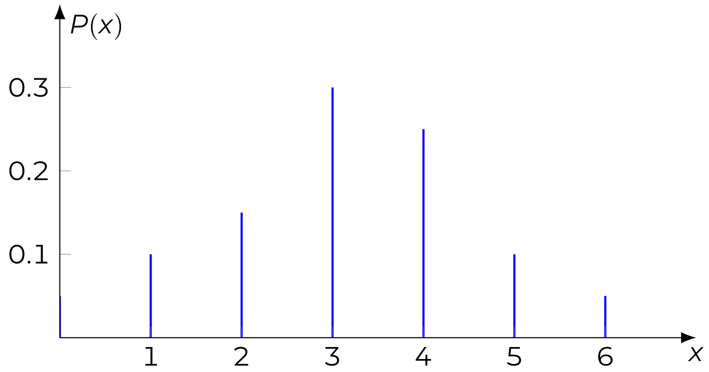


Figure B.4.3: Plot of data in Table B.4.3

the total interval into sub-intervals and find the probability that the variable lies in these sub-intervals as shown in Table B.4.4 and Figure B.4.4. We can make the intervals smaller and refine the distribution as shown in Table B.4.5 and Figure B.4.5

x	$[0, 0.2)$	$[0.2, 0.4)$	$[0.4, 0.6)$	$[0.6, 0.8)$	$[0.8, 1.0]$
$P(x)$	0.1	0.2	0.35	0.25	0.1

Table B.4.4: Probability that x lies in a given sub-interval

x	$[0, 0.1)$	$[0.1, 0.2)$	$[0.2, 0.3)$	$[0.3, 0.4)$	$[0.4, 0.5)$
$P(x)$	0.04	0.06	0.08	0.12	0.15
x	$[0.5, 0.6)$	$[0.6, 0.7)$	$[0.7, 0.8)$	$[0.8, 0.9)$	$[0.9, 1.0]$
$P(x)$	0.2	0.125	0.125	0.07	0.03

Table B.4.5: Refining sub-intervals of Table B.4.4

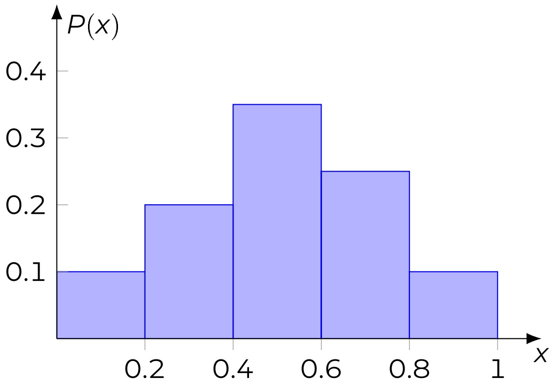


Figure B.4.4: Plot of data in Table B.4.4

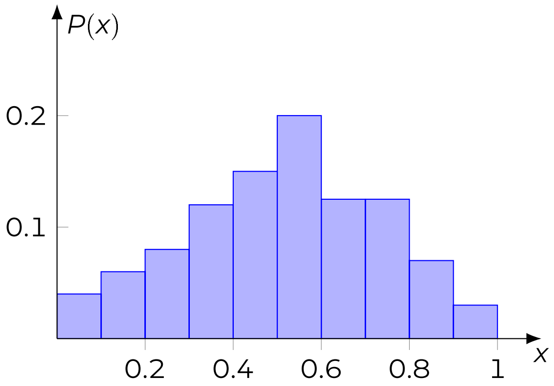


Figure B.4.5: Plot of data in Table B.4.5

Eventually what we want is a continuous function, $f(x)$, that represents the **pdf!** (**PDF!**). We can then answer the question 'What is probability of x being in interval $[a, b]$?'— that is $P(a \leqslant x \leqslant b)$ as the area under the curve between the two limits. This is shown in Figure B.4.6. Mathematically:

$$P(a \leqslant x \leqslant b) = \int_a^b f(x)\,\mathrm{d}x)$$

Note that the total area under a **PDF!** is always 1.

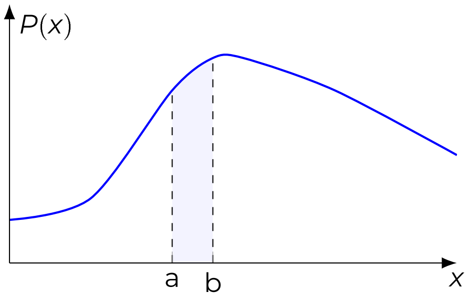


Figure B.4.6: Shaded area represents $P(a \leqslant x \leqslant b)$

Example B.4.1

Suppose x is a continuous random variable taking any value in interval $[1, 4]$. Its **PDF!** is given by

$$f(x) = \frac{1}{2\sqrt{x}} \qquad 1 \leqslant x \leqslant 4$$

a) Is this a suitable function for a **PDF!**.

b) What is probability that:

i) $1.5 \leqslant x \leqslant 2$ ii) $x \leqslant 2$ iii) $x > 3$

Solution:

a) To check if the function is suitable for use as a **PDF!** we need to see if the results of the integral over the specified interval is 1 as the total area under a **PDF!** is always 1.

$$\text{Does } \int_1^4 f(x)\,\mathrm{d}x = 1$$

$$\implies \int_1^4 \frac{1}{2\sqrt{x}}\,\mathrm{d}x = [\sqrt{x}]_1^4 = \sqrt{4} - \sqrt{1} = 1$$

So yes $f(x)$ is a suitable function

b) i) $P(1.5 \leqslant x \leqslant 2) = \int_{1.5}^2 \frac{1}{2\sqrt{x}}\,\mathrm{d}x = [\sqrt{x}]_{1.5}^2 = 0.189$

ii) $P(x \leqslant 2) = \int_1^2 \frac{1}{2\sqrt{x}}\,\mathrm{d}x = [\sqrt{x}]_1^2 = 0.414$

iii) $P(x > 3) = \int_3^4 \frac{1}{2\sqrt{x}}\,\mathrm{d}x = [\sqrt{x}]_3^4 = 0.268$

We can also represent probability distributions using the cumulative distribution function (CDF). For a real random variable x evaluated at z, the CDF is the probability that x will take a value less than or equal to z Mathematically this is calculated from the **PDF!** of x, $f(x)$ as

$$F_x(z) = P(x < z) = \int_{-\infty}^{z} f(x)\,\mathrm{d}x \tag{B.4.6}$$

Questions

1. Is the length of time a machine works a continuous or discrete variable?
2. State whether the following variables are continuous or discrete:
   a) the length of a tunnel
   b) the number of TV sockets in a house
   c) the number of capacitors in a circuit
   d) the size of a computer memory chip in bytes
   e) the output current of a system
3. The probability distribution for the random variable x is:

x	-2	-1	0	1	2	3
$P(x)$	0.07	0.1	0.21	0.36	0.19	0.07

   a) State $P(x = 2)$
   b) Calculate $P(x \geqslant 0)$
   c) Calculate $P(x < 1)$
   d) Calculate $P(x < -1)$
   e) Calculate $P(x > 0.5)$
   f) If x is sampled 20000 times how many times would you expect the value of x to be 1?
4. $f(x) = kx^2$ for $0 \leqslant x \leqslant 2$ is a **PDF!**.
   a) What is the value of the constant k?
   b) Calculate $P(x > 1)$
   c) If $P(x > c)$ =0.5 what is the value of c?
5. Consider the function $g(x) = \frac{1}{2}(1 - x)$ for interval $[-1, 1]$
   a) Prove the function is a suitable **PDF!**
   b) Calculate $P(0 \leqslant x \leqslant 0.5)$
   c) Calculate $P(-0.3 \leqslant x \leqslant 0.7)$
   d) Calculate $P(|x| < 0.5)$
   e) Calculate $P(x > 0.5)$
   f) Calculate $p(x \leqslant 0.7)$

4.4 Basic Measures

The ***mean value*** of a set of discrete numbers $\{x_1, x_2, \ldots, x_n\}$ denoted by $\overline{x}$ is given by:

$$\overline{x} = \frac{x_1 + x_2 + x_3 + \cdots + x_n}{n} = \frac{\sum x_i}{n}$$

Technically this is the ***arithmetic mean*** often referred to as the 'average' value.

Example B.4.2

What is the mean of the set of the set of data $x_i = \{-3.1, -1.7, 0, 0.5, 2.1\}$

$$\overline{x} = \frac{-3.1 + -1.7 + 0 + 0.5 + 2.1}{5} = \frac{-2.2}{5} = -0.44$$

If we have a set of values in groups (so a number of instances for a value – which is the frequency a value occurs) then the arithmetic mean is calculated as

If values $x_1, x_2, x_3, \ldots, x_p$ occur with frequencies $f_1, f_2, f_3, \ldots, f_p$ where total number of samples, $n = \sum f_i$ then

$$\overline{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{\sum x_i f_i}{n}$$

The following examples (adapted from Croft et al. (2017, p. 939) Bird (2017, p. 633)) show how the calculation of the mean of a grouped set of data works in practice.

Example B.4.3

A variable x can have values 4, 5, 6, 7 and 8. Many measurements of x are made (denoted as x_i) and the frequency that each possible value occurs is recorded f_i. The results are:

Value (x_i)	4	5	6	7	8
Frequency (f_i)	6	9	3	7	4

The mean of these values is given by:

$$\begin{aligned}\overline{x} &= \frac{\sum x_i f_i}{\sum f_i} \\ &= \frac{(4 \times 6) + (5 \times 9) + (6 \times 3) + (7 \times 7) + (8 \times 4)}{6 + 9 + 3 + 7 + 4} \\ &= \frac{24 + 45 + 18 + 49 + 32}{29} = \frac{168}{29} = 5.79\end{aligned}$$

Example B.4.4

The frequency distribution for the resistance of a nominal 220 Ωresistor for 48 resistors is given in table below with the values being grouped together in 5 Ω ranges. Determine the mean value of the resistance:

range	205-209	210-214	215-219
frequency (f_i)	3	10	11
range	220-224	224-229	230-234
frequency (f_i)	13	9	2

To determine the mean of these values we need to find the midpoint of each group to be used as x_i. The midpoints and frequencies are:

x_i	207	212	217	222	227	232
(f_i)	3	10	11	13	9	2

$$\begin{aligned}\bar{x} &= \frac{\sum x_i f_i}{\sum f_i}\\ &= \frac{(207 \times 3) + (212 \times 10) + (217 \times 11)}{48}\\ &\quad + \frac{(222 \times 13) + (227 \times 9) + (232 \times 2)}{48}\\ &= \frac{621 + 2120 + 2387 + 2886 + 2043 + 464}{48} = \frac{10521}{48}\\ &= 219.19\end{aligned}$$

Another measure is the ***median*** which is the middle point of a set — this can be useful when a set contains extreme values. It is obtained by:

(a) Ranking the members of the set in ascending order of magnitude

(b) If the number of members of the set is odd the median is the middle member. If the number of members is even then the median is the mean value of the two middle members.

For instance consider the set $\{7, 5, 64, 12\}$. The mean value of this set is 22, but the median value is found by ranking the members $\{5, 7, 12, 64\}$ and taking the mean value of 7, 12 which is 9.5 which has a lot more meaning in this case than the arithmetic mean as it tells us the 50% of the values are below 9.5 and 50% are above.

Another important measure is the ***standard deviation*** which is a measure of the spread of the data within a set. For instance, $\{-1, 0, 1\}$ and $\{-10, 0, 10\}$ both have a mean of zero but the second set is clearly more dispersed than the first set. If we have a set $x_1, x_2, x_3, \ldots, x - n$ with a mean $\bar{x}$ then the ***deviation*** of x_i is the quantity $(x_i - \bar{x})$ where some deviations will eb positive and some negative. The mean of the deviations is always 0, so we take the mean of the squared deviations $(x_i - \bar{x})^2$ which is the ***variance*** of the set. The standard deviation is simple the square root of the variance:

$$\text{variance } \sigma^2 = \frac{\sum (x_i - \overline{x})^2}{n} = \frac{\sum (x_i^2)}{n} - \overline{x}^2$$

$$\text{standard deviation } \sigma = \sqrt{variance} = \sqrt{\frac{\sum (x_i - \overline{x})^2}{n}} = \sqrt{\frac{\sum (x_i^2)}{n} - \overline{x}^2}$$

For grouped data the formula is:

If values $x_1, x_2, x_3, \ldots, x_p$ occur with frequencies $f_1, f_2, f_3, \ldots, f_p$ where $n = \sum f_i$ then

$$\text{variance } \sigma^2 = \frac{\sum f(x_i - \overline{x})^2}{n} = \frac{\sum f_i(x_i^2)}{n} - \overline{x}^2$$

$$\text{standard deviation } \sigma = \sqrt{\frac{\sum f(x_i - \overline{x})^2}{n}} = \sqrt{\frac{\sum f_i(x_i^2)}{n} - \overline{x}^2}$$

Example B.4.5

We want to determine the standard deviations for the data in Examples B.4.3 and B.4.4. Starting with Example B.4.3:

$$\begin{aligned}
\sigma &= \sqrt{\frac{\sum f_i(x_i^2)}{n} - \overline{x}^2} \\
&= \sqrt{\frac{(4^2 \times 6) + (5^2 \times 9) + (6^2 \times 3) + (7^2 \times 7) + (8^2 \times 4)}{29} - \left(\frac{168}{29}\right)^2} \\
&= \sqrt{\frac{96 + 225 + 108 + 343 + 256}{29} - \frac{28224}{841}} = \sqrt{\frac{1028}{29} - \frac{28224}{841}} \\
&= 1.374
\end{aligned}$$

For Example B.4.4:

$$\begin{aligned}
\sigma &= \sqrt{\frac{\sum f_i(x_i^2)}{n} - \overline{x}^2} \\
&= \left\{\left(\frac{(207^2 \times 3) + (212^2 \times 10) + (217^2 \times 11) + (222^2 \times 13)}{48}\right.\right. \\
&\qquad \left.\left. + \frac{+(227^2 \times 9) + (232^2 \times 2)}{48}\right) - \left(\frac{10521}{48}\right)^2\right\}^{1/2} \\
&= \sqrt{\frac{2308067}{48} - 48043.16} = 6.3447
\end{aligned}$$

For discrete variables where we have the probability distribution function (as in Table B.4.3 and Figure B.4.3) then the mean value is known as the ***expected value*** donated as μ and from this we can calculate the standard deviation σ.

If a discrete random variable, x can take values

$$x_1, x_2, x_3, \ldots, x_n$$

probabilities $P(x_1), P(x_2), P(x_3), \ldots, P(x_n)$ then

$$\text{expected value} = \mu = \sum_{i=1}^{i=n} x_i P(x_i)$$

The variance is given by

$$\text{variance} = \sigma^2 = \sum_{1}^{n} P(x_i)(x_i - \mu)^2$$

so the standard deviation is given by

$$\text{standard deviation} = \sigma = \sqrt{\sum P(x_i)(x_i - \mu)^2}$$

We can use the example of Table B.4.3 (repeated below for convenience) to show how this works in practice. For this example:

x	0	1	2	3	4	5	6
$P(x)$	0.05	0.1	0.15	0.3	0.25	0.1	0.05

$$\begin{aligned}\mu = \sum_{i=1}^{i=n} x_i P(x_i) &= (0 \times 0.05) + (1 \times 0.1) + (2 \times 0.15) + (3 \times 0.3) \\ &+ (4 \times 0.25) + (5 \times 0.1) + (6 \times 0.05) = 3.1\end{aligned}$$

$$\begin{aligned}\sigma &= \sqrt{\sum P(x_i)(x_i - \mu)^2} = \left\{\sum P(x_i)(x_i - \mu)^2\right\}^{1/2} \\ &= \Big\{((0 - 3.1)^2 \times 0.05) + ((1 - 3.1)^2 \times 0.1) + ((2 - 3.1)^2 \times 0.15) \\ &\quad + ((3 - 3.1)^2 \times 0.3) + ((4 - 3.1)^2 \times 0.25) \\ &\quad + ((5 - 3.1)^2 \times 0.1) + ((6 - 3.1)^2 \times 0.05)\Big\}^{1/2} \\ &= \sqrt{(0.4805 + 0.441 + 0.1815 + 0.003 + 0.2025 + 0.361 + 0.4205)} \\ &= \sqrt{2.09} = 1.446 \qquad \text{(to 3 d.p.)}\end{aligned}$$

For a continuous random variable, x, where we have the **PDF!** $f(x), a \leqslant x \leqslant b$ we can similarly find the expected value, μ and standard deviation σ using integration of $f(x)$ over the given interval.

If a continuous random variable, x has a **PDF!** $f(x)$ over the interval $a \leqslant x \leqslant b$ then

$$\mu = \int_a^b x f(x)\,\mathrm{d}x$$

The standard deviation is given by

$$\sigma = \sqrt{\int_a^b (x-\mu)^2 f(x)\,\mathrm{d}x}$$

Let us take an example **PDF!**, $f(x) = 2x, 0 \leqslant x \leqslant 1$ and calculate the expected value and standard deviation

$$\begin{aligned}
\mu &= \int_0^1 2x^2\,\mathrm{d}x \\
&= \left[\frac{2x^3}{3}\right]_0^1 = \frac{2}{3} \\
\sigma^2 &= \int_0^1 (x - \frac{2}{3})^2 2x\,\mathrm{d}x \\
&= \int_0^1 (x^2 - \frac{4x}{3} + \frac{4}{9}) 2x\,\mathrm{d}x = \int_0^1 2x^3 - \frac{8x^2}{3} + \frac{8x}{9}\,\mathrm{d}x \\
&= \left[\frac{x^4}{2} - \frac{8x^3}{9} + \frac{4x^2}{9}\right]_0^1 \\
&= \frac{1}{2} - \frac{8}{9} + \frac{4}{9} \\
&= \frac{1}{18} \\
\therefore \sigma &= \sqrt{\frac{1}{18}} = \frac{1}{3\sqrt{2}} = 0.236
\end{aligned}$$

Questions

1. The output, in volts from a system is measured 50 times with the following results recorded.

voltage (volts)	9	9.5	10	10.5	11	11.5	12
number of measurements	3	7	12	11	9	6	2

   a) What is the mean voltage?

   b) What is the standard deviation of the data?

2. The output, in amps, of a current through a resistor is measured 200 times with the following results recorded.

current (amps)	2	2.25	2.5	2.75	3	3.25	3.5
number of measurements	20	41	33	50	24	20	12

   a) What is the mean current?
   b) What is the standard deviation of the data?

3. Given the following discrete probability distribution calculate

   a) The expected value
   b) The standard deviation

x_i	1	2	3	4	5
$P(x_i)$	0.2	0.3	0.23	0.15	0.12

4. Given the following discrete probability distribution calculate

   a) The expected value
   b) The standard deviation

x_i	-1	-0.5	0	0.5	1
$P(x_i)$	0.2	0.25	0.3	0.2	0.05

5. Given the **PDF!** of a random variable x is given by $f(x)$ calculate

   a) The expected value
   b) The standard deviation

$$f(x) = \frac{3}{4}(1 - x^2) \qquad\qquad -1 \leqslant x \leqslant 1$$

6. Given the **PDF!** of a random variable z is given by $h(z)$ calculate

   a) The expected value
   b) The standard deviation

$$h(z) = \begin{cases} 1 + z & -1 \leqslant z < 0 \\ 1 - z & 0 \leqslant z \leqslant 1 \end{cases}$$

4.5 Permutations and Combinations

Consider a scenario where we have the three letters A, B and C — how many different ways can we order the three letters

$$ABC, ACB, BAC, BCA, CAB \text{ and } CBA$$

So the answer is there are six different ***permutations*** of three letters being taken 3 at a time $P(3,3) = {}^3P_3 = 6$. If we only want to take the same three letters two at a time then the permutations are

$$AB, AC, BA, BC, CA \text{ and } CB$$

So again $P(3,2) = {}^3P_2 = 6$ It is important to note that in permutations the order of the letters matter so $AB \neq BA$.

In general the number of permutations of n distinct objects take r at a times, $P(n,r)$ or ${}^nP_r$ is given by:

$$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!} \tag{B.4.7}$$

If the same selection of two from three letters is made without regard to the order of selection (i.e. $AB = BA$) then each distinct group is a ***combination***. For our three letter set A, B and C, then selecting two letters gives us three combinations

$$AB(= BA), AC(= CA) \text{ and } BC(= CB)$$

In general the number of combinations of n distinct objects taken r at a times, $\binom{n}{r}$ or ${}^nC_r$ is given by:

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!} \tag{B.4.8}$$

Example B.4.6

Applying Equation B.4.7 to selecting 3 letters from 3 letters gives us:

$$P(3,3) = {}^3P_3 = \frac{3!}{(3-3)!} = \frac{6}{0!} = \frac{6}{1} = 6$$

Applying Equation B.4.7 to selecting 2 letters from 3 letters gives us:

$$P(3,2) = {}^3P_2 = \frac{3!}{(3-2)!} = \frac{6}{1!} = \frac{6}{1} = 6$$

Applying Equation B.4.8 to selecting 2 letters from 3 letters gives us:

$$\binom{3}{2} = {}^3C_2 = \frac{3!}{2!(3-1)!} = \frac{6}{2 \times 1!} = \frac{6}{2} = 3$$

Questions

1. Evaluate:

   a) $P(8,5)$ b) $P(11,8)$ c) $P(10,6)$

   d) $\binom{8}{5}$ e) $\binom{11}{8}$ f) $\binom{10}{6}$

2. Write out explicitly:

   a) $\binom{m}{0}$ b) $\binom{m}{1}$ c) $\binom{m}{2}$

   d) $P(m,m)$ e) $P(m,(m-1))$ f) $P(m,(m-2))$

3. The expansion of $(a+b)^n$ (n is a positive integer) can be found with the help of the combination notation within the Binomial Theorem from equation A.9.6

   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

   Use this to expand the following:

   a) $(x+2)^4$ b) $(x-1)^3$ c) $(a+b)^6$

4. A nuclear power station is to be built on one of 15 possible sites. Engineers have been commissioned to examine the sites and rank the three most favourable sites. In how many ways can this be done?

5. A combination lock consists of two letters (A-Z) followed by two digits (0-9). How many possible ways are there of arranging the letters and digits? Is this more secure that a five digit lock? Is the term 'combination' mathematically correct?

4.6 Binomial distribution

In a single trial or experiment a particular result may be achieved or not — in an exam a student may pass or fail; or in testing a component it may work or it may not work. The two outcomes are complementary events and, if the probability of an outcome is fixed, then a trial of this nature is called a ***Bernoulli trial***. Let us now consider the situation where we have a Bernoulli trial with possible outcomes A ***success*** or B ***failure***. As $A\&B$ are complementary we know that $P(B) = 1-P(A)$ so if $P(A) = p$ then $P(B) = 1-p$. If n independent trials are observed, what is the probability that A occurs k times and hence B occurs $n-k$ times? The number of successful trials (result A) is a discrete random variable with a ***binomial distribution***.

We can define compound event C as the event A occurs k times and B occurs $n-k$ times so $P(C)$ is the answer to out earlier question. We know that the number of different ways k occurrences of A amongst n trials is given by the combination $\binom{n}{k}$. The probability of one distribution of k occurrences of A and $n-k$ occurrences of B happening is given by

$p^k(1-p)^{n-k}$ and since the number of distinct distributions is $\binom{n}{k}$ the binomial distribution for $P(C)$can be defined as:

$$P(C) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \text{for } k = 0, 1, 2, \ldots, n \tag{B.4.9}$$

This may not make a lot of sense so let us think about a simple example.

Engineering application B.4.3

A machine makes components with a 80% chance of the component being acceptable (outcome A) so the probability of successful outcome is $P(A) = 0.8$. So the probability of a component not being acceptable (outcome B) is $P(B) = 1 - P(A) = 0.2$. We want to know when sampling three components

a) What is the chance of the first two components being acceptable and the last one not being acceptable, i.e. we want to calculate $P(AAB)$?

b) What is the chance overall for two acceptable and one not acceptable component?

Solution:

a) As the three events are independent we know from equation B.4.5 that

$$P(AAB) = P(A)P(A)P(B) = 0.8 \times 0.8 \times 0.2 = 0.128$$

b) We are interested in the compound event where we have two successful outcomes and one unsuccessful outcome. This could occur as AAB, ABA or BAA. Given the fact that in all three cases the events are independent they all have the same probability as calculated above for $P(AAB)$.

If exactly two component are acceptable then either AAB or ABA or BAA occurs so the three possibilities are mutually exclusive. Thus we can calculate the overall probability from the addition law in equation B.4.2

$$\begin{aligned} P(\text{exactly two acceptable}) &= P(AAB) + P(ABA) + P(BAA) \\ &= 3 \times 0.128 = 0.384 \end{aligned}$$

Using the binomial distribution for three trials and two successes:

$$\begin{aligned} P(\text{exactly two acceptable}) &= \binom{3}{2} 0.8^2 (1-0.8)^{3-2} \\ &= \frac{3!}{2!(3-2)!}(0.74 \times 0.2) \\ &= \frac{6}{2 \times 1} \times 0.128 = 3 \times 0.128 \\ &= 0.384 \end{aligned}$$

The answers are the same — and you should see where the binomial distribution formula comes from in this simple example.

Example B.4.7

The probability that a component is acceptable is 0.95. The components are selected at random. What is the probability that:

a) At least 8 are acceptable

b) No more than 2 are acceptable

Solution:

a) In this case we need the probability that 8, 9 or 10 components are acceptable. Using binomial distribution:

$$\begin{aligned} P(\text{exactly 8 acceptable}) &= \binom{10}{8}(0.95)^8(0.05)^2 = 0.0746 \\ P(\text{exactly 8 acceptable}) &= \binom{10}{9}(0.95)^9(0.05) = 0.3151 \\ P(\text{exactly 8 acceptable}) &= \binom{10}{10}(0.95)^10 = 0.5987 \\ \implies P(\text{at least 8 acceptable}) &= 0.0746 + 0.3151 + 0.5987 \\ &= 0.9884 \end{aligned}$$

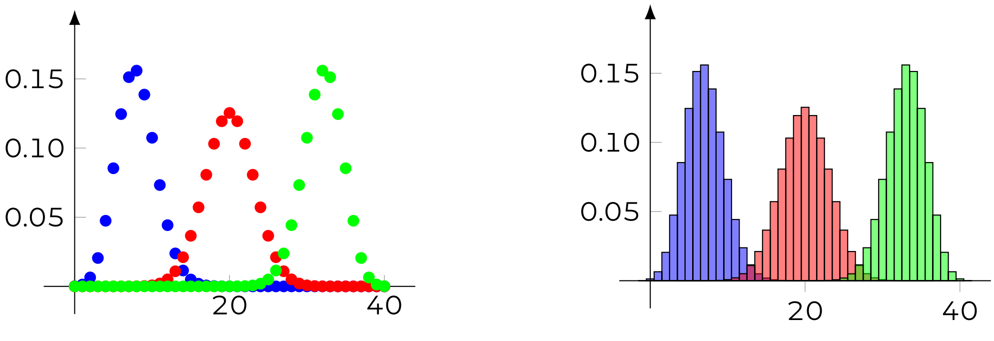
b) In this case we need the probability that 0, 1 or 2 components are acceptable. Using binomial distribution:

$$\begin{aligned} P(\text{exactly 0 acceptable}) &= \binom{10}{0}(0.05)^{10} \\ &= 9.766 \times 10^{-14} \\ P(\text{exactly 1 acceptable}) &= \binom{10}{1}(0.95)(0.05)^9 \\ &= 1.855 \times 10^{-11} \\ P(\text{exactly 2 acceptable}) &= \binom{10}{2}(0.95)^2(0.05)^8 \\ &= 1.586 \times 10^{-9} \\ P(\text{no more than 2 acceptable}) &= 9.766 \times 10^{-14} \\ &\quad + 1.855 \times 10^{-11} \\ &\quad + 1.586 \times 10^{-9} \\ &= 1.605 \times 10^{-9} \end{aligned}$$

In other words the chance of at most two components being acceptable is almost zero so it is virtually impossible.

We can represent the binomial distribution graphically for any values of n and p

Let's take the example of $n = 40$ with three different values of $p = 0.2, 0.5$ and 0.8. Figure B.4.7 shows the three distributions plotted as data points and as histograms.



a) Points

b) Histograms

Figure B.4.7: Binomial distributions for $n = 40$, $p = 0.2$, $p = 0.5$ and $p = 0.8$

Mean and standard deviation

Given a probability of success in a single trial being p with n trials, then the number of successes is a discrete random variable x with a binomial distribution. x can take any value from $\{0, 1, 2, 3 \ldots, n\}$ although clearly (as can be seen in Figure B.4.7) some values will be more likely than others depending on p. We can find the expected value and standard deviation of x given n and p. (Proof of these beyond this course)

For a binomial distribution with n trials and a success probability of p

$$\text{expected value} = \mu = np \tag{B.4.10}$$

$$\text{standard deviation} = \sigma = \sqrt{np(1-p)} \tag{B.4.11}$$

Most likely number of successes

Often rather than the mean we might want to know what is the most likely outcome. For example what is the most likely outcome in a sample of 5 components with a success probability of $p = 0.75$? To answer this we have to calculate the probability for each outcome x between 0 and 5 successes.

$$P(x=0) = \binom{5}{0}(0.75)^0(0.25)^5 = 9.766 \times 10^{-4}$$

$$P(x=1) = \binom{5}{1}(0.75)^1(0.25)^4 = 2.930 \times 10^{-3}$$

$$P(x=2) = \binom{5}{2}(0.75)^2(0.25)^3 = 0.0879$$

$$P(x=3) = \binom{5}{3}(0.75)^3(0.25)^2 = 0.2637$$

$$P(x=4) = \binom{5}{4}(0.75)^4(0.25)^1 = 0.3955$$

$$P(x=5) = \binom{5}{5}(0.75)^5(0.25)^0 = 0.2373$$

So the most likely outcome is 4 acceptable components

This is laborious to do when there are many trials so we can actually find i the most likely outcome from p and n as it can be proved it lies in a range:

$$p(n+1)-1 < i \leqslant p(n+1)$$

So for trial above

$$\begin{aligned}(0.75)(6)-1 < i &\leqslant (0.75)(6)\\ 3.5 < i &\leqslant 4.5\end{aligned}$$

Since i must be an integer $i = 4$ as found by the calculation above.

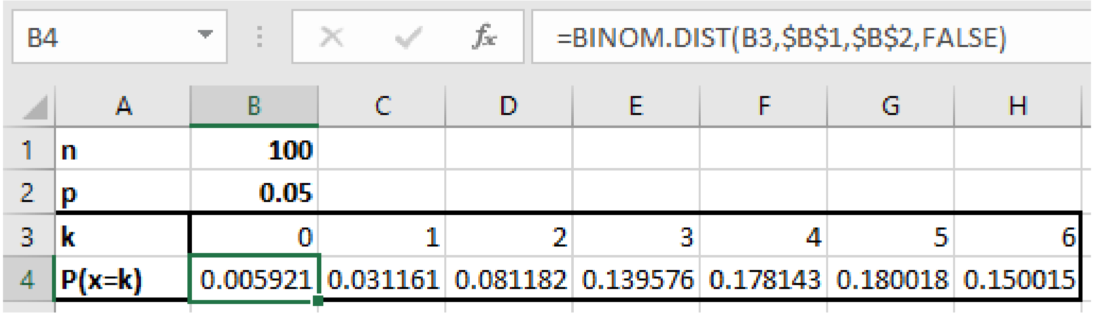
Using computer to calculate binomial distribution

In Microsoft® Excel® (2010 and later), there are two functions that can be used to calculate the binomial distribution probabilities:

- `BINOM.DIST(number_s,trials,probability_s,cumulative)` this function returns the individual binomial distribution probability of the number of successes defined by `number_s` occurring. The inputs are:
  - — `number_s` is the required number of successes which is k above.
  - — `trials` is the number of trials — i.e. n
  - — `probability_s` is the probability of a success — i.e. p
  - — `cumulative` is either TRUE or FALSE. If `cumulative` is TRUE, then the return value is the cumulative distribution function, which is the probability that there are at most `number_s` successes; if FALSE, it returns the probability mass function, which is the probability that there are `number_s` successes.
- `BINOM.DIST.RANGE(trials,probability_s,number_s,[number_s2])`. This is very similar to `BINOM.DIST` in that if the optional`number_s2` is not used then it returns the same value as `BINOM.DIST` when `cumulative`=FALSE. If a value is given for *number_s2* then it returns the binomial probability for the number of successes to be between `number_s` and `number_s2`. In this case `number_s2` must be greater than `number_s`

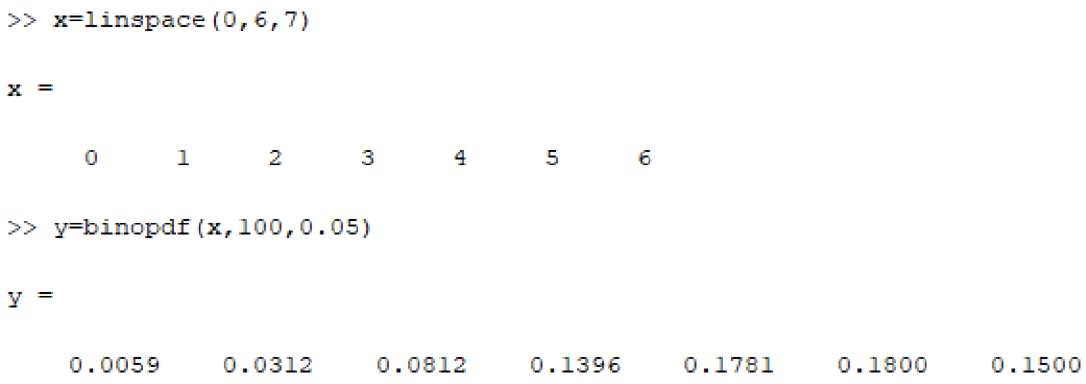
Using Microsoft® Excel® makes calculating the binomial probabilities much easier for larger distributions. For instance if we have 100 components so $n = 100$ and a probability that a component is not acceptable of 5% so$p = 0.05$ we can use Microsoft® Excel® to calculate the probabilities of between 0 and 6 components being not acceptable as shown in Figure B.4.8 — where formula for P(x=0) is shown. Note that n and p are both declared to be absolutely in these cells (using $ before the row & column values). This is because the remaining values were calculated by using the fill right from cell B4.

MATLAB® also has a number of functions relating to the binomial distribution in its Statistics and Machine Learning toolbox including one to calculate the CDF (mass) function . This is `binopdf(x,n,p)` where `x` is the value or a vector of values that we want the binomial distribution probabilities for, `n` is the number of trials and `p` is the probability of success. repeating the experiment in Figure B.4.8 we get what is shown in Figure B.4.9.



	A	B	C	D	E	F	G	H
1	n	100						
2	p	0.05						
3	k	0	1	2	3	4	5	6
4	P(x=k)	0.005921	0.031161	0.081182	0.139576	0.178143	0.180018	0.150015

Figure B.4.8: Using Microsoft® Excel® for binomial distribution calculation



```
>> x=linspace(0,6,7)

x =

     0     1     2     3     4     5     6

>> y=binopdf(x,100,0.05)

y =

    0.0059    0.0312    0.0812    0.1396    0.1781    0.1800    0.1500
```

Figure B.4.9: Using MATLAB® for binomial distribution calculation

Questions

1. The probability that a component is acceptable is 85%. Five components are randomly sampled. What is the probability that:
   a) Exactly one is acceptable
   b) All five are acceptable
   c) At least three are acceptable
2. Five coins are tossed simultaneously. Determine the probabilities of having 0, 1, 2, 3, 4 and 5 heads upwards. Draw a histogram of the results.
3. A machine has 6 microchips that all need to operate correctly in order for the acceptable operation of the machine. Each chip has a 99% chance of operating successfully.
   a) What is the probability that the machine is operating successfully?
   b) What is the probability that 5 out of 6 of the chips are operating successfully?
   c) The chips are replaced by three chips with a 98% chance of operating successfully. Is the new design more or less reliable?
4. The probability that a component is reliable for more that 1000 hours of operation is 0.85. Ten components are chosen at random. What is probability that
   a) eight components are reliable for more than 1000 hours?

   b) all the components are reliable for more than 1000 hours?
   c) at least seven components are reliable for more than 1000 hours?
   d) no more than two components are reliable for less than 1000 hours?

5. 92% of components manufactured by a machine are of an acceptable standard. Six components are selected randomly from a large batch.
   a) Calculate the probability that all the components are acceptable.
   b) Calculate the probability that none of the components are acceptable.
   c) Calculate the probability that fewer than five components are acceptable.
   d) Calculate the most likely number of acceptable components.
   e) Calculate the probability that less than three components are not acceptable.

4.7 Poisson Distribution

In order to model the number of occurrences of an even in a given interval we use the Poisson distribution — proposed by the French mathematician Siméon Denis Poisson in 1837 (Poisson 1837, pp. 205–207). It was actually proposed to model the number of wrongful convictions in a given country by focusing on certain random variables N that count, among other things, the number of discrete occurrences that take place during a time-interval of given length. In fact, the result was originally written about by Abraham de Moivre in his 1710 paper for the Royal Society (De Moivre 1710).

For instance, consider the number of emergency calls received by a plumber in one day. Experience shows that this is usually three or four calls, but some days it may be five or six or even more, and other days it could be one or two or even none. This is a good example of where a Poisson distribution would be very useful to model the number of calls received in one day.

If we let the number of occurrences of an event, E, in a given time period will be a discrete random variable denoted by X. What we want to know is the probability that $X = 0, X = 1, X = 2$ and so on. We assume that the occurrence of E in any time interval is not affected by its occurrence in the preceding time interval — think about measuring how many cars pass a point in 20 s. A car is not more or less likely to pass in the current 20 s if a car passed in the previous 20 s. We are saying the the occurrences are independent events.

Let the expected value of X be λ occurrences in the time interval (this is the average value of X when it is measured for many time periods). Under these conditions then X follows a Poisson distribution with the probability of X being a value r ($r = 0, 1, 2, 3, \ldots$) given by equation B.4.12:

$$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!} \tag{B.4.12}$$

The expected value (μ) and standard deviation (σ) of the distribution are:

$$\mu = \lambda \qquad \sigma = \lambda$$

Engineering application B.4.4: Number of emergency calls

Records show that on average four emergency calls are received a day. What is the probability that the following number of calls will be received on a particular day?

a) four b) three c) five

Solution:

The number of calls follows a Poisson distribution so:

a) $P(X = 4) = \frac{e^{-4}4^4}{4!} = 0.195$

b) $P(X = 3) = \frac{e^{-4}4^3}{3!} = 0.195$

c) $P(X = 5) = \frac{e^{-4}4^5}{5!} = 0.156$

So in a 200 day period, three calls will be received on approximately 39 days, four calls will be received on approximately 39 days and five call will be received on approximately 31 days

Poisson approximation to the binomial

	Binomial (n, p) $P(X = r); n = 20, p = 0.04$	*Poisson* (λ) $P(X = r), \lambda = 0.8$
$r = 0$	0.4420024	0.4493290
$r = 1$	0.3683354	0.3594632
$r = 2$	0.1457994	0.1437853
$r = 3$	0.0364499	0.0383427
$r = 4$	0.0064547	0.0076685
$r = 5$	0.0008606	0.0012270
$r = 6$	0.0000896	0.0001636

Table B.4.6: Probabilities for binomial and Poisson distributions

Consider the binomial distribution with n trials and the probability of success is p. As n increases it becomes more time-consuming the calculate the probabilities. If n is large and p is small such that the expected value $np \leqslant 5$ then we can actually use a Poisson distribution with an expected value of $\lambda = np$. Let us take the example of a binomial distribution with $n = 20$ and $p = 0.04$ so $\lambda = 0.8$. Table B.4.6 compares the values calculated for various numbers of successful outcomes with the remaining probabilities almost 0.

As n increases and p decreases (keeping np fixed) then the agreement between the distributions becomes closer.

Engineering application B.4.5: Workforce absentees

There are 500 people in a workforce with the probability that a person is absent on any one day is 0.01. What is the probability on any one day that the number of people absent is:

a) three b) seven

Solution:

This can be either be solved using a binomial distribution for a set of Bernoulli trials with $n = 500$ and $p = 0.01$ or as a Poisson distribution where $\lambda = 500 * 0.01 = 5$ as n is large.

Binomial solution:

$$\text{a)}\ \ P(X = 3) = \binom{500}{3}(0.01)^3(0.99)^497 = 0.140$$

$$\text{b)}\ \ P(X = 7) = \binom{500}{7}(0.01)^7(0.99)^493 = 0.105$$

Poisson solution:

$$\text{a)}\ \ P(X = 3) = \frac{e^{-5}(5)^3}{3!} = 0.140$$

$$\text{b)}\ \ P(X = 7)\frac{e^{-5}(5)^7}{7!} = 0.104$$

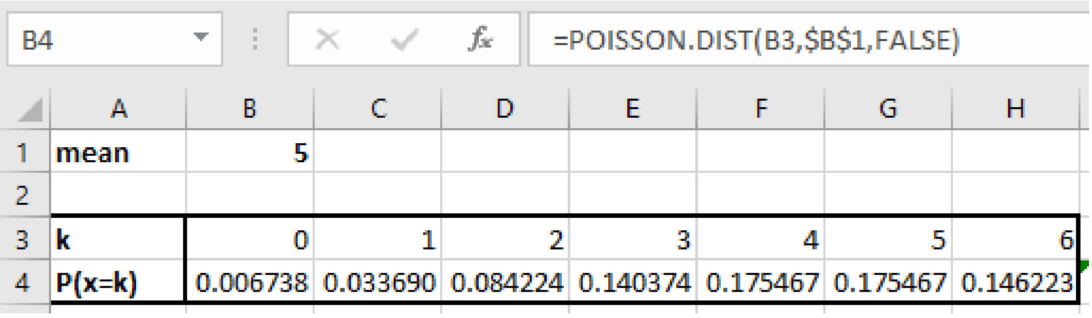
Using computer to calculate Poisson distribution

In Microsoft® Excel® (2010 and later), there is a specific function for calculating the Poisson distribution probability: `POISSON.DIST(x,mean,cumulative)` this function returns the individual Poisson distribution probability of the number of successes defined by x occurring. The inputs are:

- `x` is the required number of successes which is r above.
- `mean` is the mean value — i.e. $\lambda = np$
- `cumulative` is either TRUE or FALSE. If `cumulative` is TRUE, then the return value is the cumulative distribution function, which is the probability that there are at most `x` successes; if FALSE, it returns the probability mass function, which is the probability that there are `x` successes.

Using Microsoft® Excel® makes calculating the Poisson probabilities much easier. For instance for the binomial distribution in Figure B.4.8 we can use Microsoft® Excel® to calculate the Poisson probabilities of between 0 and 6 components with a mean of $\lambda = 5 = 100 \times 0.05$ being not acceptable as shown in Figure B.4.10 — where formula for P(x=0) is shown.

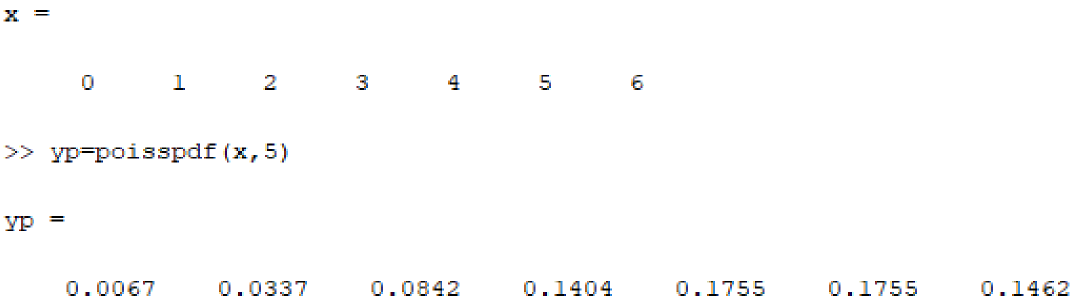
MATLAB® also has a number of functions relating to the Poisson distribution in its Statistics and Machine Learning toolbox including one to calculate the probability density (mass) function . This is `poisspdf(x,lambda)` where `x` is the value or a vector of values that we want the



	A	B	C	D	E	F	G	H
1	mean	5						
2								
3	k	0	1	2	3	4	5	6
4	P(x=k)	0.006738	0.033690	0.084224	0.140374	0.175467	0.175467	0.146223

Figure B.4.10: Using Microsoft® Excel® for Poisson distribution calculation

Poisson distribution probabilities for and `lambda` is the mean value. Repeating the experiment in Figure B.4.10 we get what is shown in Figure B.4.11.



```
x =

     0     1     2     3     4     5     6

>> yp=poisspdf(x,5)

yp =

    0.0067    0.0337    0.0842    0.1404    0.1755    0.1755    0.1462
```

Figure B.4.11: Using MATLAB® for Poisson distribution calculation

Exponential distribution

A related distribution to the Poisson distribution is the exponential distribution which models the time between the events in a Poisson distribution happening which is also a random variable but this time a continuous random variable often denoted as t with a **PDF!** of $f(t)$.

The probability of an event occurring in a time interval T is given by

$$P(T) = \int_0^T f(t)\,\mathrm{d}t$$

The **PDF!** $f(t)$ is given for any $\alpha > 0$:

$$f(t) = \begin{cases} \alpha\,\mathrm{e}^{-\alpha t}\,\mathrm{d}t & t \geqslant 0 \\ 0 & \text{otherwise} \end{cases} \tag{B.4.13}$$

The expected value of $f(t)$ is defined as

$$\mu = \int_{-\infty}^{\infty} tf(t)\,\mathrm{d}t = \int_0^{\infty} t\alpha\,\mathrm{e}^{-\alpha t}\,\mathrm{d}t = \frac{1}{\alpha} \tag{B.4.14}$$

Questions

1. A computer network contains several hundred computers. In any twelve hour period, the average number of computers not working is 8. Find the probability that during a twelve hour period the number of computers that do not work is:

   a) ten b) six c) eight

2. On average 4 people a day are absent from within a large workforce. What is the probability that the following number of people are absent on a typical day:

   a) four b) at least five c) less than six

3. The probability of a disk drive failing in any week is 0.7%. A computer service company maintains 800 disk drives. What is the probability of the following number of disk drive failures in a week:

   a) eight b) more than eight c) less than six

4. The probability that an employee is absent is 1.2%. A company has 750 employees – what is the probability that the following number of employees are absent on any one day?

   a) nine b) ten c) six

5. A machine manufactures electrical components at a rate of 700 per hour with a probability of 0.015 of a component being faulty. Use both the binomial distribution and the corresponding Poisson approximation find the probability that in a set of 250 components:

   a) none are faulty b) one is faulty
   c) two are faulty d) three are faulty
   e) more than three are faulty

6. The mean time between breakdowns for a machine is 500 hours. Calculate the probability that the time between breakdowns is

   a) less than 400 hours
   b) greater than 550 hours

4.8 Normal distribution

The normal probability density function, usually called the ***normal distribution***, is probably one of the most important distributions and certainly is widely used. It is used to calculate the probable values of continuous variables such as weight, length, height and error measurement and has been shown to reflect accurately those that would be found using actual data. If x is a continuous variable with a normal distribution $N(x)$, where expected value of $x = \mu$ and the standard deviation of $x = \sigma$ then equation B.4.15 defines $N(x)$

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2} \qquad -\infty < x < \infty \tag{B.4.15}$$

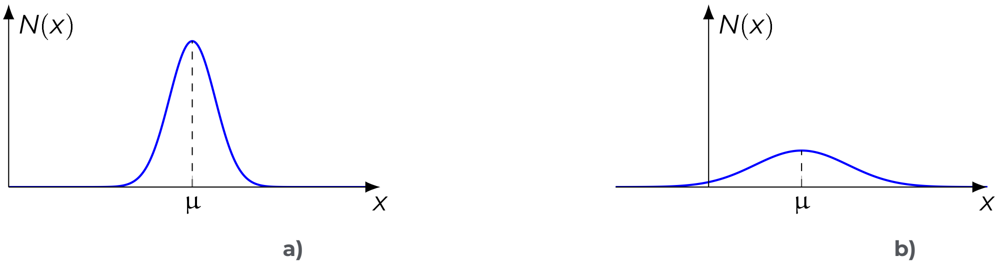


Figure B.4.12: Two typical normal distributions

All normal distribution curves are bell shaped and symmetrical about μ as can be seen in Figure B.4.12. In Figure B.4.12a the values of x are grouped closely to μ which indicates a small standard deviation, σ, whereas the values of x in Figure B.4.12b are more spread out indicating a higher standard deviation. Figure B.4.13 shows two normal curves with the same value of σ but different means μ. As for all distribution curves the area under a normal curve must be 1; that is

$$\int_{-\infty}^{\infty} N(x)\,\mathrm{d}x = 1$$

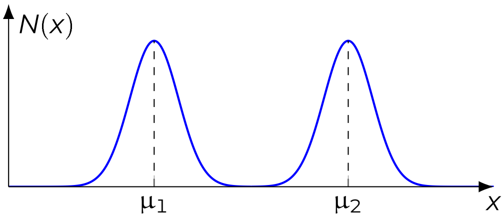


Figure B.4.13: Two normal distributions with same σ but different values of μ

The standard normal

As we have seen a normal distribution $N(x)$ is uniquely defined by its mean and standard deviation so we can say that the probability the x lies in interval $[a, b]$ is given by

$$P(a \leqslant x \leqslant b) = \int_a^b N(x)\,\mathrm{d}x$$

The issue we have is that the mathematical formula for $N(x)$ in equation B.4.15 makes analytic integration impossible so we would have to calculate all probabilities numerically which would change every time μ and/or σ changed. To combat this we introduce the concept of the ***standard normal*** which is a normal distribution with $\mu = 0$ and $\sigma = 1$.

Let us consider the probability that the random variable x has a value less than z — this is the CDF as defined in equation B.4.6 — which we can call $A(z)$ so:

$$A(z) = P(x < z) = \int_{-\infty}^{z} N(x)\,\mathrm{d}x$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7703	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.7	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.8	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999
3.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table B.4.7: Cumulative standard normal distribution to 4 d.p.

The values of the cumulative distribution function of $A(z)$ for values of $0 \leqslant z \leqslant 3.99$ are given in Table B.4.7[*]. As the normal function is symmetrical we can use this table to find the values for $z < 0$

The following examples show how we use this table in practice to calculate probabilities for variables that have a standard normal distribution

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[*] LaTeXcode for this table from Lee (2004)

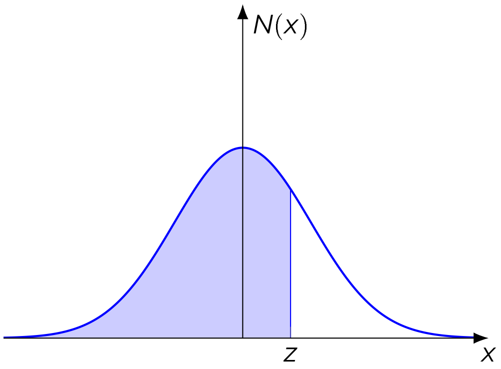

Figure B.4.14: Shaded area: $A(z) = P(x < z) = \int_{-\infty}^{z} N(x)\,\mathrm{d}x$

Example B.4.8

The variable x has a standard normal distribution. Calculate the probability that:

a) $x < 1.5$ b) $x > 1.5$ c) $x > -1.5$ d) $x < -1.5$

Solution:

a) From Table B.4.7

$$P(x < 1.5) = 0.9332$$

Shown in Figure B.4.15

b) $P(x > 1.5) = 1 - P(x < 1.5) = 1 - 0.9332 = 0.0668$
As depicted in Figure B.4.16

c) By symmetry $P(x > -1.5) = P(x < 1.5)$ (compare Figures B.4.15 & B.4.17). Thus

$$P(x > -1.5) = 0.9332$$

d) Similarly

$$P(x < -1.5) = P(x > 1.5) = 0.0668$$

Compare Figures B.4.16 & B.4.18

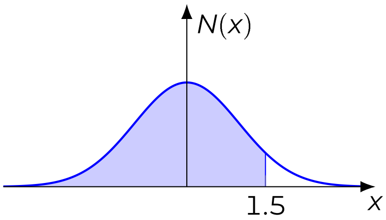

Figure B.4.15: $P(x < 1.5)$

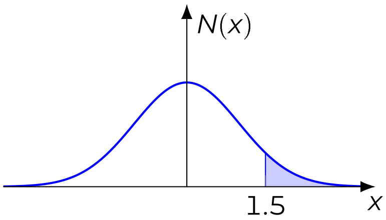

Figure B.4.16: $P(x > 1.5)$

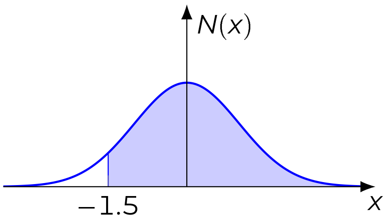

Figure B.4.17: $P(x > -1.5)$

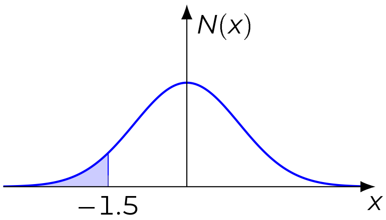

Figure B.4.18: $P(x < -1.5)$

Example B.4.9

The variable w has a standard normal distribution. Calculate the probability that:

a) $0 < w < 1$ b) $-1 < w < 1$ c) $-1 \leqslant w \leqslant 1.5$

Solution:

a) Figure B.4.19 shows the area required

$$P(w < 1) = 0.8413 \qquad \text{using Table B.4.7}$$
$$P(w > 0) = 0.5 \qquad \text{using symmetry}$$
$$P(0 < w < 1) = 0.8413 - 0.5 = 0.3413$$

b) Figure B.4.20 shows the area required

$$\begin{aligned} P(-1 < w < 1) &= 2 \times P(0 < w < 1) \qquad \text{using symmetry} \\ &= 2 \times 0.3413 = 0.6826 \end{aligned}$$

c) Figure B.4.21 shows the area required

$$\begin{aligned} P(w \leqslant 1.5) &= 0.9332 \\ P(w \leqslant -1) &= P(w \geqslant 1) = 1 - P(w < 1) \\ &= 1 - 0.8413 = 0.1587 \\ P(-1 \leqslant w \leqslant 1.5) &= 0.9332 - 0.1587 = 0.7745 \end{aligned}$$

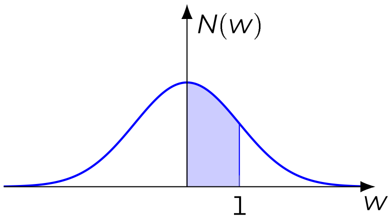

Figure B.4.19: $P(0 < w < 1)$

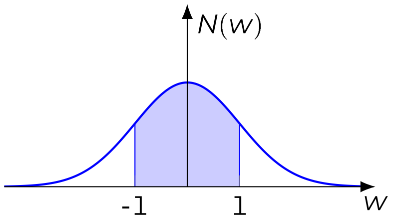

Figure B.4.20: $P(-1 < w < 1)$

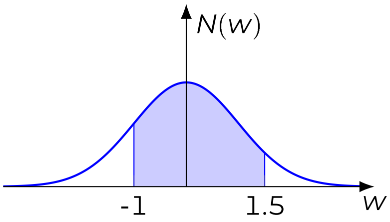

Figure B.4.21: $P(-1 \leqslant w \leqslant 1.5)$

Non-standard normal

We can use Table B.4.7 to calculate the probabilities for continuous random variables with non-standard normal distributions — that is normal distributions where $\mu \neq 0$ and/or $\sigma \neq 1$. Effectively we can transform a non-standard normal into a standard normal by applying the following rule:

$$\begin{aligned} \text{non-standard} &\longrightarrow \text{standard} \\ X &\longrightarrow \frac{X-\mu}{\sigma} \end{aligned} \tag{B.4.16}$$

This is best illustrated with an example

Example B.4.10

The random variable g has a normal distribution with a mean of 3 and a standard deviation of 2. Calculate the probability that:

a) $h > 5$ b) $h < 2$ c) $1 < h < 4$

Solution:

a) Applying the transformation in equation B.4.16 gives:

$$5 \longrightarrow \frac{5-3}{2} = 1$$

So we can say that $h > 5$ has the same probability as $x > 1$ in a standard normal distribution.

$$P(h > 5) = P(x > 1) = 1 - P(x < 1) = 1 - 0.8413 = 0.1587$$

b) Applying the transformation in equation B.4.16 gives:

$$2 \longrightarrow \frac{2-3}{2} = -0.5$$

So we can say that $h < 2$ has the same probability as $x < -0.5$ in a standard normal distribution.

$$P(h < 2) = P(x < -0.5)P(x > 0.5) = 1 - P(x < 0.5) = 1 - 0.6915 = 0.3085$$

c) he transformation in equation B.4.16 to both limits gives:

$$1 \longrightarrow \frac{1-3}{2} = -1 \qquad 4 \longrightarrow \frac{4-3}{2} = 0.5$$

This means that $P(1 < h < 4) = P(-1 < x < 0.5)$. So we need

$$P(x < 0.5) = 0.6915 \qquad P(x < -1) = 1 - 0.8413 = 0.1587$$

Therefore

$$P(1 < h < 4) = P(-1 < x < 0.5) = 0.6915 - 0.1587 = 0.5328$$

Questions

1. A random variable, x, has a standard normal distribution. Calculate the probability that x is in the intervals
   a) $(0.75, 1.25)$
   b) $(-0.4, 0.1)$
   c) $(-1.6, -0.1)$
   d) within 1.2 standard deviations of the mean
   e) More than 1.5 standard deviations from the mean
2. A random variable, x, has a normal distribution with a mean of 1 and a standard deviation of 2. Calculate the probability that
   a) $-1 \leqslant x \leqslant 2$
   b) $x > 0$
   c) $|x| < 0.9$
   d) $|x| > 2$
3. A random variable, t, has a normal distribution with a mean of 4 and a standard deviation of 0.5. Calculate the probability that
   a) $3 \leqslant t \leqslant 4.5$
   b) $2.5 < t < 3.8$
   c) $t > 4.6$
   d) $t < 4.2$
   e) t is within 0.4 of the the mean
4. The scores from an exam have a mean of 55% and a standard deviation of 10%. What mark should a student score to be described as being gin:
   a) top 25% of the marks (the top quartile)

b) top 10% of the marks

c) bottom 30% of the marks

5. The random variable x has a normal distribution as shown in Figure B.4.22. How many standard deviations above the mean must the point P be if the shaded are is to represent the following percentages of the total area:

   a) 20% b) 10% c) 5%

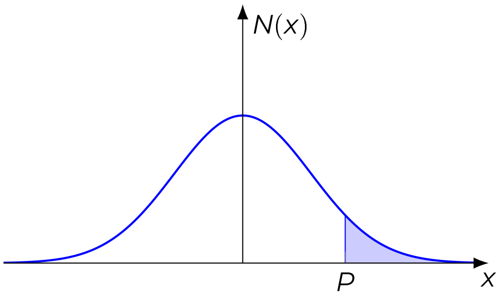

Figure B.4.22: Graph for Question 5

6. The random variable x has a normal distribution as shown in Figure B.4.23. How many standard deviations from the mean must the points A and B (they are equal distance from mean) be if the shaded area is to represent the following percentages of the total area:

   a) 50% b) 25% c) 10%

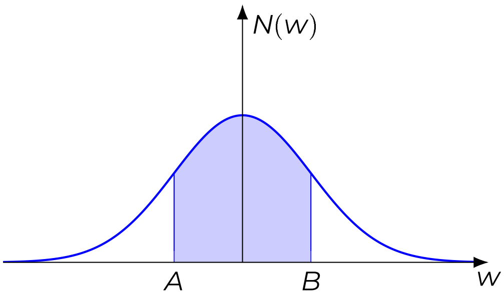

Figure B.4.23: Graph for Question 6

4.9 Summary

This chapter has given an introduction to some of the key areas of statistics that you may use in the course of your studies. Statistics can be very useful when we are trying to model systems with some uncertainty or where there is random chance. Wireless communications is an area where this becomes very true as the noise experienced by a signal is effectively a random variable. We also model the traffic within a system using statistical distributions including the Poisson and exponential distributions.

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PART C

APPENDICES

Calculator Skills

1.1 Powers and Indices

On a scientific calculator there tends to be an x^y or $x^{\square}$ function which allows you to calculate any power value by using the base number as x and putting the power in as other value.

1.2 Number conversion

Some scientific calculators can convert between denary, binary and hexadecimal numbers which can be useful. However not all can and it is important to check the required steps for your calculator.

1.3 Degrees and Radians

It is important when dealing with angles to ensure your calculator is in the correct mode for degrees or radians. This can generally be found in Mode or Setup menu. For instance on a Casio fx-83GT PLUS, I go into SETUP (SHIFT plus MODE key) and from there can select Deg (3) or Rad (4). Check your instruction manual or ask if you are not sure.

1.4 Logarithms

The vast majority of scientific calculators have both base 10 logs (log) and base e logs (ln). Some also have a generic logarithm for any base which is, for example, given by $\log_{\blacksquare}\square$ on a Casio fx-83GT PLUS.

However not all scientific calculators include a generic log function so in order to log to another base it becomes necessary to use the rule:

$$\log_a X = \frac{\log_b X}{\log_b a}$$

This is particularly used if you need to find log to the base 2 of a number — this occurs a lot in digital so we can say that

$$\log_2 X = \frac{\log X}{\log 2} = \frac{\log X}{0.30103}$$

Glossary

AC: alternating current

This describes a current where the flow of charge (that creates a current) changes direction periodically — the voltage level also reserve along with the current.

BCD: Binary Coded Decimal

An almost obsolete form of binary number representation which allows easy display of denary numbers using simple circuits from the underlying binary representation; at the expense of being harder to process for digital computers.

CDF: cumulative distribution function

This is a statistical function that gives the probability that a variable is equal to or less than the value it is being evaluated at.

DC: direct current

This describes a current where the flow of charge is always in the same direction. The level of the current (and hence voltage) can vary over time but the direction of flow does not change.

D.P.: Decimal Places

used for rounding decimal numbers where we count from after the decimal point. Once the requisite number of numerals has been reached the rest of the digits are discarded but final numeral is incremented by 1 if the first digit of the discarded digits is 5 or greater.

HCF: Highest Common Factor

Given two or more umbers, the number that is the largest (highest) number that is a factor of all given numbers (or the number that divides exactly into all given numbers). For example: HCF of 2 & 6 is 2 ($6 = 2 \times 3$) whereas HCF of 12 & 30 is 6 (as $12 = 2 \times 6$ and $30 = 5 \times 6$. For three numbers the HCF of 12, 21 & 30 is 3 ($12 = 4 \times 3, 21 = 7 \times 3$ and $30 = 10times3$)

LCM: Lowest Common Multiple

Given two or more umbers, the number that is the smallest (lowest) number where all given numbers are factors (or the number into which all given numbers will divide exactly). For example: LCM of 2 & 3 is 6 (2×6) whereas LCM of 2 & 4 is 4 as 2 is a factor of 4. For three numbers LCM of 2, 3 & 4 is 12 whereas LCM of 2, 4 & 8 is 8.

LSB: Least Significant Bit

This is the right hand most bit in any binary number - i.e the end bit.

LSD: Least Significant Digit
This is the right hand most digit in any number - i.e the end digit.

LTI: Linear Time Invariant
A type of system that has the following properties:

- ***linear***: if output of system to input $x_1(t)$ is $y_1(t)$ and output to input $x_2(t)$ is $y_2(t)$ then for all values of a and b:

$$H\{ax_1(t) + bx_2(t)\} = ay_1(t) + by_2(t)$$

- ***time invariant***: characteristics do not change with time — adding delay to input just adds delay to output. So for all values of τ, if inpout $x(t)$ maps to output $y(t)$ then:

$$H\{x(t-\tau)\} = y(t-\tau)$$

MSB: Most Significant Bit
This is the left hand most bit in any binary number i.e. the starting bit.

MSD: Most Significant Digit
This is the left hand most digit in any number i.e. the starting digit.

NAN: not a number
This is a numeric value representing an undefined or un-representable value.

PWM: Pulse Width Modulation
This is a signal that is a rectangular wave of a fixed time period T but a variable 'on' or ***mark time*** m. The ratio of on time to the period is known as the ***duty cycle*** $d = m/T$.

RAM: Random Access Memory
A type of computer memory where the values stored at any address can be accessed in (nominally) constant time.

RMS: Root Mean Square
This is a mathematical construct that represents the average magnitude of a time varying signal. The rms value of a time-varying quantity is the value where the varying signal delivers the same amount of power on average as a constant quantity of the same value (rms value).

SI: International System of Units
Abbreviated from the French ***Système international (d'unités)***; hence the standard acronym.

S.F.: Significant Figures
used for rounding decimal numbers where we count from the first non-zero digit from the left of the number. Once the requisite number of significant figures has been reached the rest of the digits are discarded but final significant numeral is incremented by 1 if the first digit of the discarded digits is 5 or greater.

SPL: sound pressure level
This is used to define the standard reference level for sound pressure which is 20 µPa r.m.s.