Computer Problem Set 3

Simulation of the Brownian motion

The present problem set is attached to Chapters 5 and 6 of the lectures notes. All implementations should be run with the value $T = 1$. For a positive integer n, we denote $\Delta T := 2^{-n}T$, $t_i^n := i\ \Delta T$, $i = 0, \ldots, 2^n$. We consider a Brownian motion W, and we denote $\Delta W_{t_i^n} := W_{t_i^n} - W_{t_{i-1}^n}$, $i = 1, \ldots, 2^n$.

1. In view of the approximation of $\int_0^T W_s dW_s$, we consider the three following quantities:

$$I_n := \sum_{i=1}^{2^n} W_{t_{i-1}^n} \Delta W_{t_i^n},\ J_n := \sum_{i=1}^{2^n} W_{t_i^n} \Delta W_{t_i^n},\ K_n := \sum_{i=1}^{2^n} \frac{W_{t_i^n} + W_{t_{i-1}^n}}{2} \Delta W_{t_i^n}.$$

   (a) Simulate a sample of $N = 1000$ copies of the randoms variables $\frac{1}{2}W_T^2 - I_n$, $\frac{1}{2}W_T^2 - J_n$, and $\frac{1}{2}W_T^2 - K_n$.

   (b) Compute the corresponding sample means, and comment on the results.

   (c) Vary the value of n from 10 to 20, and provide a graph of the resulting sample means, together with the corresponding confidence intervals.

2. Address the previous questions with the random variables

$$A_n := \sum_{i=1}^{2^n} e^{t_{i-1}^n} \Delta W_{t_i^n},\ B_n := \sum_{i=1}^{2^n} e^{t_i^n} \Delta W_{t_i^n}, \quad \text{and} \quad C_n := \sum_{i=1}^{2^n} e^{\frac{t_i^n + t_{i-1}^n}{2}} \Delta W_{t_i^n}.$$

3. We now consider the random variables

$$A_n \quad := \quad \sin(W_T) - 2^{-n-1} \sum_{i=1}^{2^n} \sin(W_{t_{i-1}^n}).$$

   (a) Simulate a sample of $N = 1000$ copies of A_n, and plot the corresponding sample mean, with the appropriate confidence interval, as a function of $n \in \{10, \ldots, 20\}$.

   (b) Comment the graph with appropriate justification.