

## Computer Problem Set 4

### The Black-Scholes hedging strategy

The present problem set is attached to Chapter 6 of the lectures notes. All implementations should be run with the value  $T = 1$ . For a positive integer  $n$ , we denote  $\Delta T := 2^{-n}T$ ,  $t_i^n := i \Delta T$ ,  $i = 0, \dots, 2^n$ . We consider a Brownian motion  $W$ , and we introduce the process

$$S_t := S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \quad t \geq 0,$$

where  $\mu \in \mathbb{R}$  and  $\sigma > 0$  denote the drift and the volatility parameters.

We recall that, in the context of the Black-Scholes model, the no-arbitrage price of a European call option on an underlying asset with price process  $\{S_t, t \geq 0\}$  is given by

$$BS(S_0, K, T) := S_0 \mathbf{N}(\mathbf{d}_+(S_0, K e^{-rT}, \sigma^2 T)) - K e^{-rT} \mathbf{N}(\mathbf{d}_-(S_0, K e^{-rT}, \sigma^2 T)),$$

where  $r$  is the instantaneous interest rate,  $K, T$  denote the strike and the maturity of the option, respectively, and

$$\mathbf{d}_{\pm}(s, k, v) := \frac{\ln(s/k)}{\sqrt{v}} \pm \frac{1}{2}\sqrt{v}.$$

The corresponding optimal hedging strategy consists in holding  $\Delta_t$  shares of the underlying asset at each time  $t$ , with

$$\Delta_t(K) = \text{Delta}(S_t, K, T - t) := \mathbf{N}(\mathbf{d}_+(S_t, K e^{-r(T-t)}, \sigma^2(T-t))).$$

1. Build a program which produces a sample of  $N = 1000$  copies of the discrete path  $\{S_{t_i}, i = 0, \dots, 2^n\}$ . Take  $S_0 = 100$ ,  $\sigma = 0.3$ ,  $r = 0.05$ , use three values of  $\mu$ : 0.05, 0.02 and 0.45. Compute the corresponding sample mean and variance. Comment the results.
2. Denote

$$e^{-rT} X_T^n(K) := BS(S_0, K, T) + \sum_{i=1}^{2^n} \Delta_{t_{i-1}^n}(K) (e^{-rt_i^n} S_{t_i^n} - e^{-rt_{i-1}^n} S_{t_{i-1}^n})$$

- (a) Simulate a sample of  $N = 1000$  copies of  $X_T^n$  for each value of  $\mu$ . Use the values of  $K \in \{100 \pm i, i = 0, \dots, 20\}$ .
- (b) Compute the corresponding *Profit and Loss*

$$\text{PL}_T^n(K) := X_T^n(K) - (S_T - K)^+.$$

- (c) For each value of  $\mu$  and  $K$ , compute the sample mean and variance of  $\text{PL}_T^n(K)$ , and provide the corresponding plots in terms of the number of steps  $n$  and the strike  $K$ .