TD5: Topological Persistence
Report

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1 Question 3: Complexity of the Reduction Algorithm

```
void reduction() {
    // Reminder:
    // lowIndices stores the row index of the lowest non-zero entry for each column.
    // lowColumn[k] stores the index of the column which has its lowest entry at row k.

    // We loop through all columns
    for (int j = 0; j < this.n; j++) {
        int low = getLow(j);
        // While the column is not empty and there exists a previous column with the same pivot:
        while (low != -1 && lowColumn[low] != -1) {
            // We update the j-th column
            int i = lowColumn[low];
            reductColumns(i, j);
            // We compute the new pivot
            low = getLow(j);
        }
        // Final pivot
        lowIndices[j] = low;
        // If the column is non-empty, we complet lowColumn
        if (low != -1) {
            lowColumn[low] = j;
        }
    }
}
```

We have 3 loops. The **`for`** loop runs through all the columns, so m times, where m is the number of simplices. The **`while`** loop cycles through all previous columns, so $\mathcal{O}(m)$ iterations. Finally, the method `reductColumns` does a symmetric difference on the i-th and j-th columns, and thus requires at most m operations. Therefore, the total complexity of the reduction algorithm is $\mathcal{O}(m^3)$.

2 Questions 5-6: Classical Spaces

The filtrations for the spheres in dimension d are generated by the following Python script:

```python
from itertools import combinations

if __name__ == "__main__":
    for d in range(11):
        f = open("sphere_"+str(d)+".txt", 'w')
        n = d + 2
        for i in range(1, n):
            simplices = set(combinations(range(n), i))
            for x in simplices:
                f.write(str(i-1) + " " + str(i-1) + " ")
                for j in range(len(x)):
                    f.write(str(x[j]) + " ")
                f.write("\n")
```

The results corresponds to the ones obtained by hand in the exercise session.

3 Question 7: Timings

filtration	number of simplices	creation time (s)	reduction time (s)	time / m^3	time / $m\sqrt{m}$
A	428,643	2.365	24.654	3.13E-13	8.78E-05
B	108,161	6.93	1.319	1.04E-12	3.71E-05
C	180,347	2.797	2.821	4.80E-13	3.68E-05
D	2,716,431	162.546	263.103	1.31E-14	5.87E-05

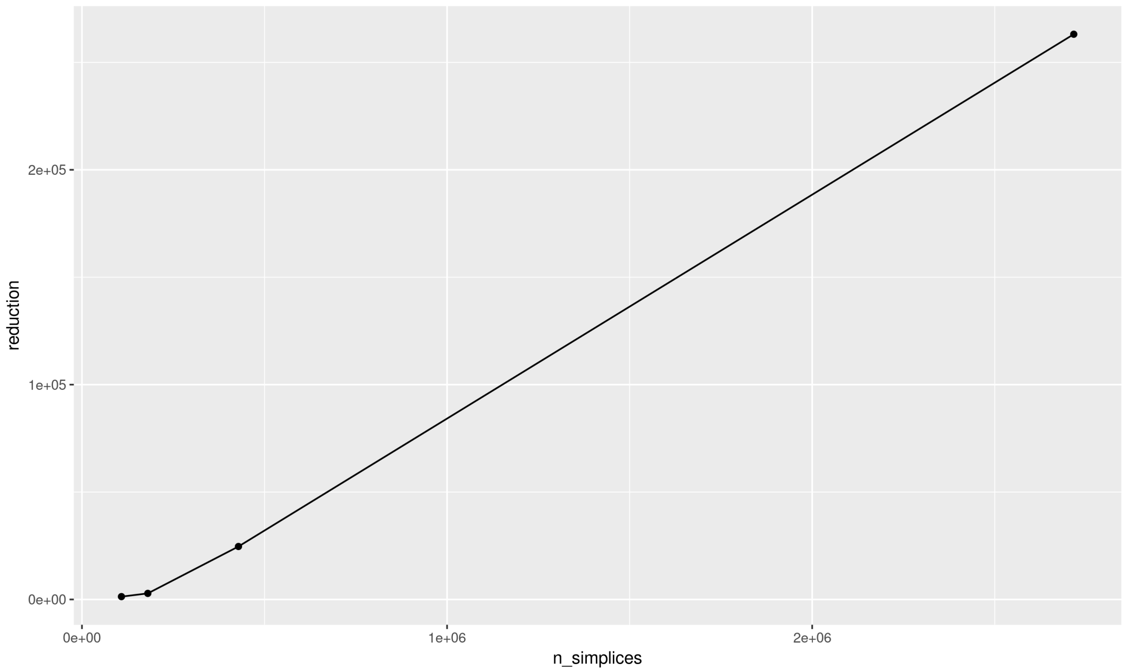

Figure 1: Reduction time vs. number of simplices in the filtration

In the table, we show the time that the program uses to create the (sparse) matrix and to reduce it. We then try to test if it is indeed cubic in the number of simplices. Experimentally, we have a slightly better complexity (around $\mathcal{O}(m^{1.5})$), probably due to the fact that most columns are empty or that there are very few columns with the same pivot.

4 Question 8: Topological Structure

4.1 Filtration A

This barcode corresponds to a sphere. If we visualize points on the surface of a sphere, and balls slowly growing around them, we can infer the good number of holes and voids to match the barcode.

First, there are many connected components, so in H_0 many segments. After a while, all these components merge together, thus reducing gradually the number of holes (i.e. the number of segments in H_1).

When the last hole has disappeared, a single void appears (in H_2). After some times, all the little ball have grown and this void disappears, leaving no hole or void, and only one connectefd component.

4.2 Filtration B

We apply the same reasoning. This time, the underlying topological space is a set of 8 spheres, position at the vertices of a cube, and tangent to each other. There is only one connected component, and initially 5 holes (because there are 6 openings on the sides of the cube formed by the sphere, but only 5 independant cycles). There are also 8 voids (the interiors of the spheres).

When every hole has disappeared, a void appears in the inside of the cube.

4.3 Filtrations C and D

These filtrations corresponds to a torus. Once the many connected components have merged together, there are many holes on the surface. The disappear progressively, leaving only the two independant cycles on the surface of the torus.

Afterwards, the torus fills itself, making one of the cycles to disappear, and simultaneously, the void inside disappears. (We have a little bit of noise because the void does not fill itself immediately; a lot of smaller voids appear and disappear very quickly.) Then the central hole in the torus disappears also, and the torus has become a single ball.