Short Review in Complex Numbers

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Contents

1 Introduction

The need of expressing all the roots of polynomials led mathematicians to the introduction of complex numbers during the 16^{th} century. In the beginning, the complex number was defined such $\sqrt{-1}$. Later, this number was introduced, using the notation of i and $-i$, for the categorization of the two square roots of -1. In addition, the introduction of complex numbers led to their interpretation in different directions, with the most common being their geometrical interpretation and their sound mathematical footing by Hamilton in the early 19^{th} century. In physics and engineering the complex numbers are also symbolized with the letter j. However, here the i symbol is used.

2 Complex Number Definitions

A complex number is defined in the complex domain, symbolized by $\mathbb{C}$, in which, $\mathbb{R} \subseteq \mathbb{C}$. A complex number can be written in the following Cartesian form, such

$$z = a + bi \tag{1}$$

where,

a: The real part of the complex number z, expressed such,

$$\mathfrak{Re}(z) = a, \;\; a \in \mathbb{R}, \tag{2}$$

b: The imaginary part of the complex number z, expressed such,

$$\mathfrak{Im}(z) = b, \;\; b \in \mathbb{R}, \tag{3}$$

i: The imaginary unit, defined such,

$$i = \sqrt{-1} \Longleftrightarrow i^2 = -1. \tag{4}$$

The Cartesian form indicates that the complex number z can be expressed by a point with respect to a pair of the real numbers a and b in the complex Cartesian two-dimensional axle system, where the x and y axes represent the real and imaginary part of the complex number z, respectively (Figure 1).

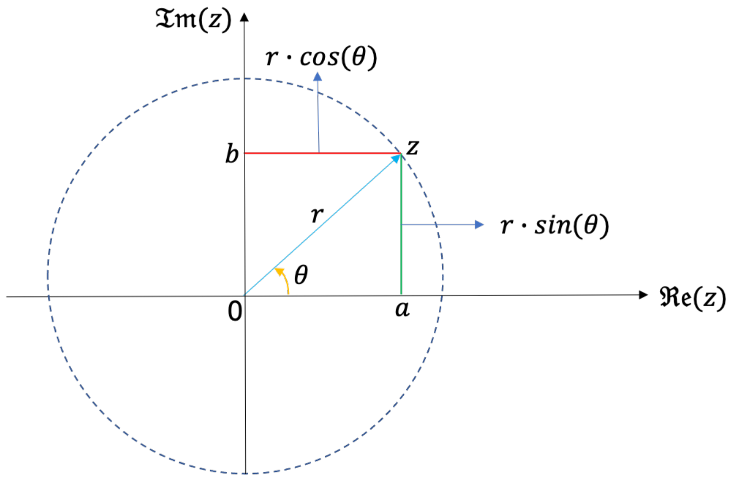

Figure 1: Graphical representation of a complex number z in the Cartesian axle system. The r represents the modulus or the amplitude of the complex number z and the θ represents its angle.

This fact indicates that a complex number z is a vector, with modulus or amplitude $r \in \mathbb{R}^{\{0,+\}}$ and phase $\theta \in [0, 2\pi)$ or $(-\pi, \pi]$, expressed such,

$$|z| \equiv r = \sqrt{\mathfrak{Re}(z)^2 + \mathfrak{Im}(z)^2} = \sqrt{a^2 + b^2} \tag{5}$$

$$\angle z \equiv \theta = \tan^{-1}\left(\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)}\right) = \tan^{-1}\left(\frac{b}{a}\right). \tag{6}$$

The representation of the complex numbers in the Cartesian domain indicates that complex numbers can also be expressed in polar coordinates, corresponding to the trigonometric form of complex numbers, which is written such,

$$z = r\cos(\theta) + ir\sin(\theta). \tag{7}$$

By equating the expressions (1) and (7), it can be seen that

$$\mathfrak{Re}(z) = a = r\cos(\theta) \tag{8}$$

$$\mathfrak{Im}(z) = b = r\sin(\theta). \tag{9}$$

A complex exponential form or a polar form of complex numbers can be derived by using the Taylor expansion of cosine and sine functions, concluded to the simplified form,

$$z = re^{i\theta}, \forall z \neq 0. \tag{10}$$

3 Complex Conjugate Number

The transformation of a complex number to its symmetrical real axis corresponds to the complex conjugate number. The complex conjugate of a complex number z is notated by either the asterisk symbol $(\cdot)^*$, or the macron symbol $\bar{(\cdot)}$. Here, the macron symbol is considered. The complex conjugate of z can be expressed such,

Cartesian:

$$\bar{z} = a - bi \tag{11}$$

Trigonometric:

$$\bar{z} = r\cos\theta - ir\sin(\theta) \tag{12}$$

Polar:

$$\bar{z} = re^{-i\theta} \tag{13}$$

4 Complex Numbers-based Operations

By considering the following complex numbers z_1 and z_2, with $a, b, c, d \in \mathbb{R}$, then

$$z_1 = a + bi \tag{14}$$

$$z_2 = c + di \tag{15}$$

Their modulus and phases can be calculated, such

$$|z_1| \equiv r_1 = \sqrt{a^2 + b^2} \text{ and } \angle z_1 \equiv \theta_1 = \tan^{-1}\left(\frac{b}{a}\right) \tag{16}$$

$$|z_2| \equiv r_2 = \sqrt{c^2 + d^2} \text{ and } \angle z_2 \equiv \theta_2 = \tan^{-1}\left(\frac{d}{c}\right) \tag{17}$$

The four basic operations (i.e., addition, subtraction, multiplication and division) can be conducted with respect to the form of complex number.

Addition

Cartesian:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i \tag{18}$$

Trigonometric:

$$\begin{aligned} z_1 + z_2 &= (r_1 \cos(\theta_1) + ir_1 \sin(\theta_1)) + (r_2 \cos(\theta_2) + ir_2 \sin(\theta_2)) \\ &= (r_1 \cos(\theta_1) + r_2 \cos(\theta_2)) + (r_1 \sin(\theta_1) + r_2 \sin(\theta_2))i \\ &= (a + c) + (b + d)i \end{aligned} \tag{19}$$

Polar:

$$\begin{aligned} z_1 + z_2 &= r_1 e^{i\theta_1} + r_2 e^{i\theta_2} \\ &= (r_1 \cos(\theta_1) + ir_1 \sin(\theta_1)) + (r_2 \cos(\theta_2) + ir_2 \sin(\theta_2)) \\ &= (r_1 \cos(\theta_1) + r_2 \cos(\theta_2)) + (r_1 \sin(\theta_1) + r_2 \sin(\theta_2))i \\ &= (a + c) + (b + d)i \end{aligned} \tag{20}$$

Subtraction

Cartesian:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i \tag{21}$$

Trigonometric:

$$\begin{aligned} z_1 - z_2 &= (r_1 \cos(\theta_1) + ir_1 \sin(\theta_1)) - (r_2 \cos(\theta_2) + ir_2 \sin(\theta_2)) \\ &= (r_1 \cos(\theta_1) - r_2 \cos(\theta_2)) + (r_1 \sin(\theta_1) - r_2 \sin(\theta_2))i \\ &= (a - c) + (b - d)i \end{aligned} \tag{22}$$

Polar:

$$\begin{aligned} z_1 - z_2 &= r_1 e^{i\theta_1} - r_2 e^{i\theta_2} \\ &= (r_1 \cos(\theta_1) + ir_1 \sin(\theta_1)) - (r_2 \cos(\theta_2) + ir_2 \sin(\theta_2)) \\ &= (r_1 \cos(\theta_1) - r_2 \cos(\theta_2)) + (r_1 \sin(\theta_1) - r_2 \sin(\theta_2))i \\ &= (a - c) + (b - d)i \end{aligned} \tag{23}$$

Multiplication

Cartesian:

$$z_1 \cdot z_2 = (a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i \tag{24}$$

Trigonometric:

$$\begin{aligned} z_1 \cdot z_2 &= (r_1 \cos(\theta_1) + ir_1 \sin(\theta_1)) \cdot (r_2 \cos(\theta_2) + ir_2 \sin(\theta_2)) \\ &= r_1 \cos(\theta_1) r_2 \cos(\theta_2) + ir_1 \cos(\theta_1) r_2 \sin(\theta_2) + ir_1 \sin(\theta_1) r_2 \cos(\theta_2) - r_1 \sin(\theta_1) r_2 \sin(\theta_2) \\ &= r_1 r_2 (\cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2)) + r_1 r_2 (\cos(\theta_1) \sin(\theta_2) + \sin(\theta_1) \cos(\theta_2))i \\ &= (r_1 r_2 \cos(\theta_1 + \theta_2)) + (r_1 r_2 \sin(\theta_1 + \theta_2))i \end{aligned} \tag{25}$$

Polar:

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)} \tag{26}$$

Division

Cartesian:

$$\frac{z_1}{z_2} = \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{ac-adi+bci+bd}{c^2-cdi+cdi+d^2} \tag{27}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right) i$$

Trigonometric:

$$\frac{z_1}{z_2} = \frac{r_1\cos(\theta_1) + ir_1\sin(\theta_1)}{r_2\cos(\theta_2) + ir_2\sin(\theta_2)} \tag{28}$$
$$= \frac{r_1}{r_2}\frac{(\cos(\theta_1) + i\sin(\theta_1))(\cos(\theta_2) - i\sin(\theta_2))}{(\cos(\theta_2) + i\sin(\theta_2))(\cos(\theta_2) - i\sin(\theta_2))}$$
$$= \frac{r_1}{r_2}\frac{\cos(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)i + \sin(\theta_1)\cos(\theta_2)i + \sin(\theta_1)\sin(\theta_2)}{\cos^2(\theta_2) + \sin^2(\theta_2)}$$
$$= \frac{r_1}{r_2}\frac{\cos(\theta_1-\theta_2) + \sin(\theta_1-\theta_2)i}{\cos^2(\theta_2) + \sin^2(\theta_2)}$$
$$= \left(\frac{r_1}{r_2}\cos(\theta_1-\theta_2)\right) + \left(\frac{r_1}{r_2}\sin(\theta_1-\theta_2)\right) i$$

Polar:

$$\frac{z_1}{z_2} = \frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)} \tag{29}$$

As it can be seen by the latter basic operations, when addition and subtraction operations are considered, the Cartesian form provides the dominant form in comparison to other forms. In the case that multiplication and division operations are needed to be conducted, the polar form is considered as a first choice, implying exponent properties. Focusing on the division operation with respect to Cartesian and trigonometric form, the multiplication of both numerator and denominator with the complex conjugate of the complex number in the denominator is implied. On this way, the denominator becomes a purely real number.

5 Complex Number Properties

The most important properties related to both complex and complex conjugate numbers have been summarized below, assuming $z, z_1, z_2, z_3 \neq 0$.

Additional Properties

$$z_1 + z_2 = z_2 + z_1 \tag{30}$$

$$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \tag{31}$$

Multiplication Properties

$$z_1z_2 = z_2z_1 \tag{32}$$

$$z_1(z_2z_3) = (z_1z_2)z_3 \tag{33}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \tag{34}$$

Conjugate Properties

$$\overline{(\bar{z})} = z \tag{35}$$

$$z + \bar{z} = 2\mathfrak{Re}(z) \tag{36}$$

$$z - \bar{z} = 2\mathfrak{Im}(z) \tag{37}$$

$$z\bar{z} = \mathfrak{Re}^2(z) + \mathfrak{Im}^2(z) \tag{38}$$

$$\overline{(z_1 + z_2)} = \overline{z_1} + \overline{z_2} \tag{39}$$

$$\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2} \tag{40}$$

$$\overline{(z_1 \cdot z_2)} = \overline{z_1} \cdot \overline{z_2} \tag{41}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\overline{z_1}}{\overline{z_2}}\right) \tag{42}$$

$$z_1\overline{z_2} \pm \overline{z_1}z_2 = 2\mathfrak{Re}(\overline{z_1}z_2) = 2\mathfrak{Im}(z_1\overline{z_2}) \tag{43}$$

$$\mathfrak{Re}(z) = \frac{z + \bar{z}}{2} \tag{44}$$

$$\mathfrak{Im}(z) = \frac{z - \bar{z}}{2i} \tag{45}$$

Modulus Properties

$$|z| = |-z| = |\bar{z}| = |-\bar{z}| \tag{46}$$

$$|z|^2 = z\bar{z} \tag{47}$$

$$|z^n| = |z|^n, \forall n \in \mathbb{Z} \tag{48}$$

$$|\sqrt[n]{z}| = \sqrt[n]{|z|}, n \geq 2 \text{ and } n \in \mathbb{N} \tag{49}$$

$$|z_1 z_2| = |z_1||z_2| \tag{50}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, z_2 \neq 0 \tag{51}$$

$$|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2\mathfrak{Re}(\overline{z_1}z_2) \tag{52}$$

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2) \tag{53}$$

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{54}$$

$$|z_1 - z_2| \geq |z_1| - |z_2| \tag{55}$$

6 Exercises

1. Consider the complex numbers $z_1 = 5 + 3i$ and $z_2 = 2 - 3i$. Calculate the following operations as well as their modulus and their phases. Then, represent them in the complex Cartesian domain.

$$z_1 + z_2, z_1 - z_2, z_1 z_2, \frac{z_1}{z_2}, z_1 \overline{z_2}, \overline{z_1 z_2} \tag{56}$$

2. Consider the general complex number of the form $z = a + bi$, where $a, b \in \mathbb{R}$. Show that $z^2 \neq |z|^2$.

3. Calculate the value of the complex number i^{2021}.

4. Calculate the value of the complex number i^{2i}. Is the result a real or complex number?

5. Prove the following Euler equations.

$$\frac{e^{i\theta} + e^{-i\theta}}{2} = \cos(\theta) \tag{57}$$

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta) \tag{58}$$

6. Consider the following cosine and sine waves with amplitude equal to 1,

$$x = \cos(2\pi f t)\ ,\ x = \sin(2\pi f t) \tag{59}$$

where, f is the frequency in Hz, and t the propagation time in s.

(a) Express both waves in Euler's form.

(b) What are you observing?

(c) How would you interpret the results?

7*. Consider the complex numbers z_1, z_2, z_3 with $|z_1| = |z_2| = |z_3| = 3$.

(a) Show that $\overline{z_1} = \frac{9}{z_1}$.

(b) Show that $\frac{z_1}{z_2} + \frac{z_2}{z_1}$ is a real number.

(c) Show that $|z_1 + z_2 + z_3| = \frac{1}{3}|z_1 z_2 + z_2 z_3 + z_3 z_1|$

8. Consider the complex number $z = \frac{2+ai}{a+2i}$, where $a \in \mathbb{R}$.

(a) Show that the polar coordinates of the complex number z are projected to a circle with centre $O(0, 0)$ and a modulus of $r = 1$.

(b) Suppose that z_1 and z_2 are the complex numbers arise from the complex number z for $a = 0$ and $a = 2$, respectively.

(i) What are the modules of z_1 and z_2?

(ii) What is the distance between z_1 and z_2?