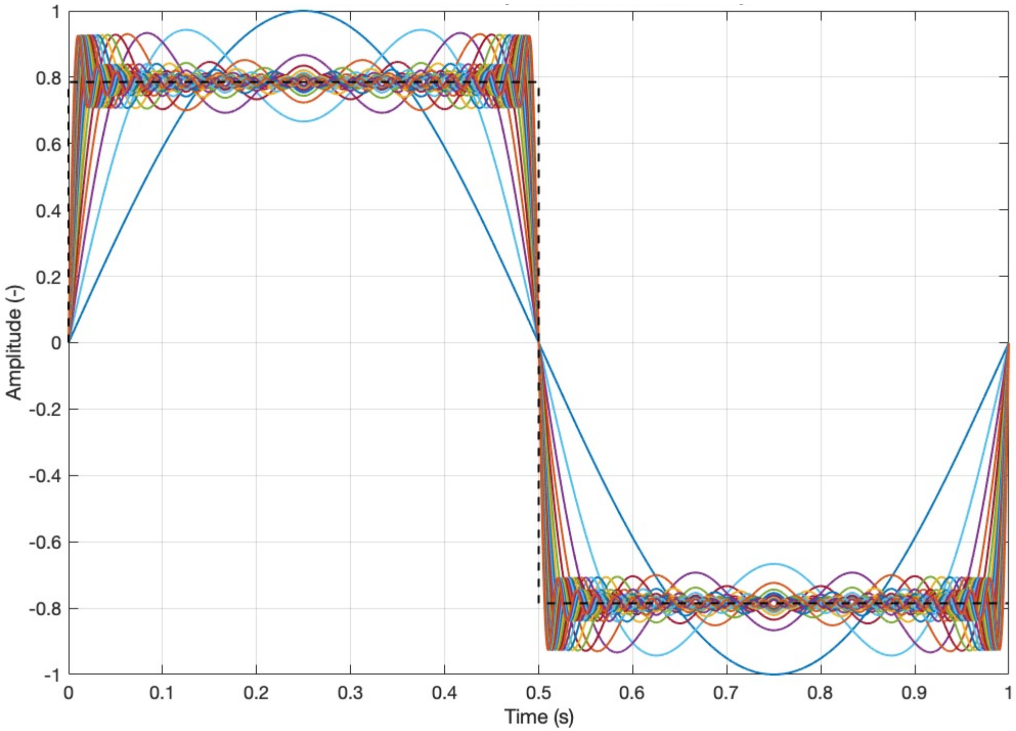
Tutorial: Introduction to Digital Signal Procesing (DSP)

7LS8M0: Architectural Acoustics

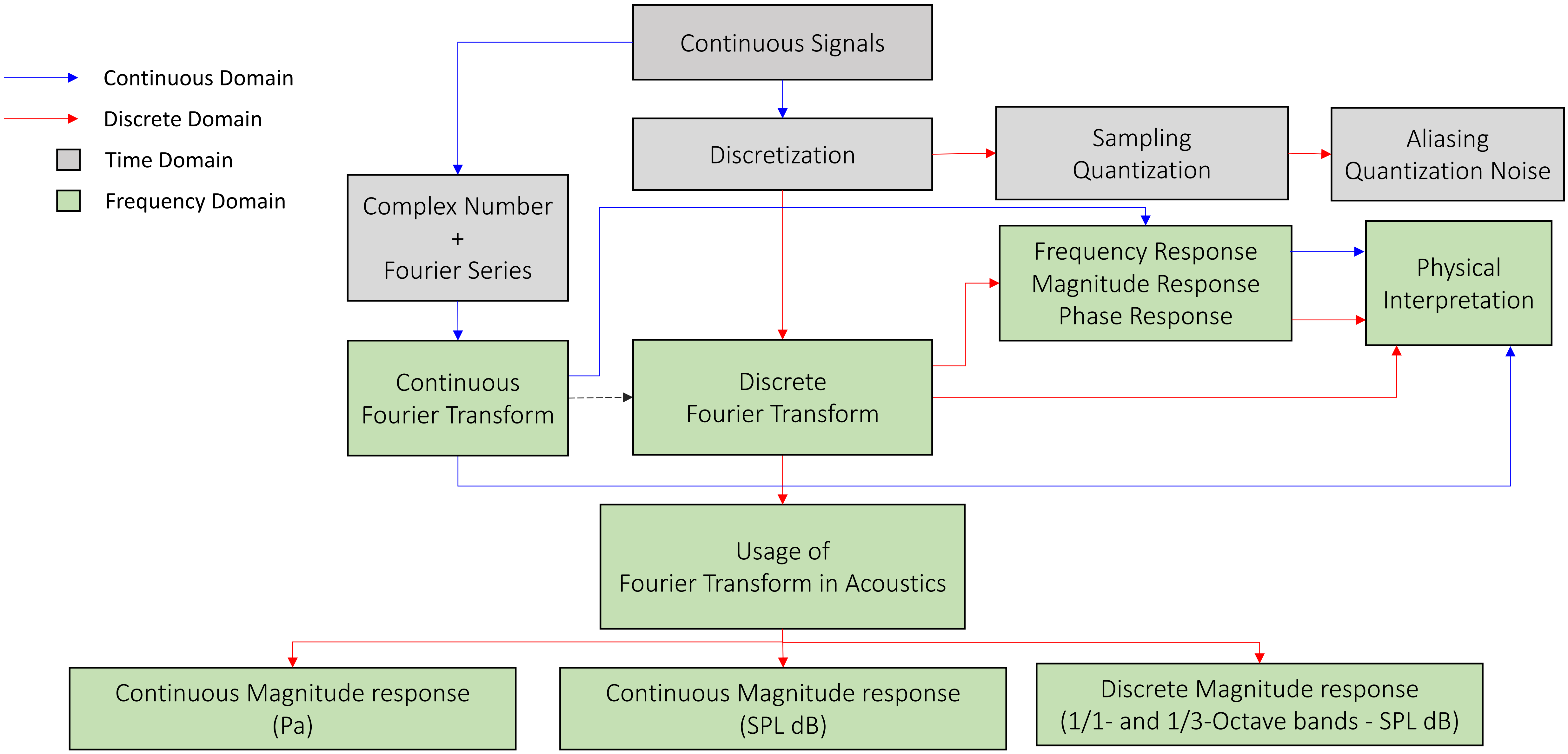
M.E. (Michalis) Terzakis

Building Acoustic Group



Department of the Built Environment

Block-Diagram of DSP Tutorial



What are we going to see in this tutorial?

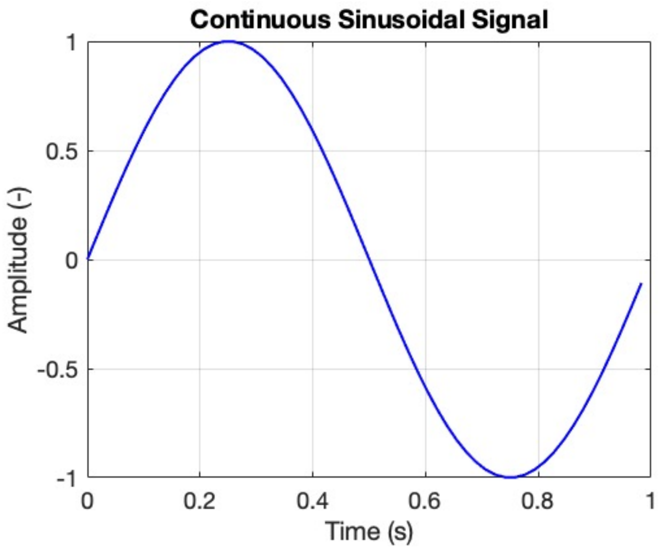
- What is a continuous and a discrete signal?
- What is sampling?
- What is aliasing?
- What is quantization?

Time-Domain

- What is the (Continuous) Fourier Transform?
- What is the Discrete Fourier Transform?
- What factors influence Fourier Transform?
- How is Fourier Transform used?

Frequency-Domain

Time-based (discrete) signals



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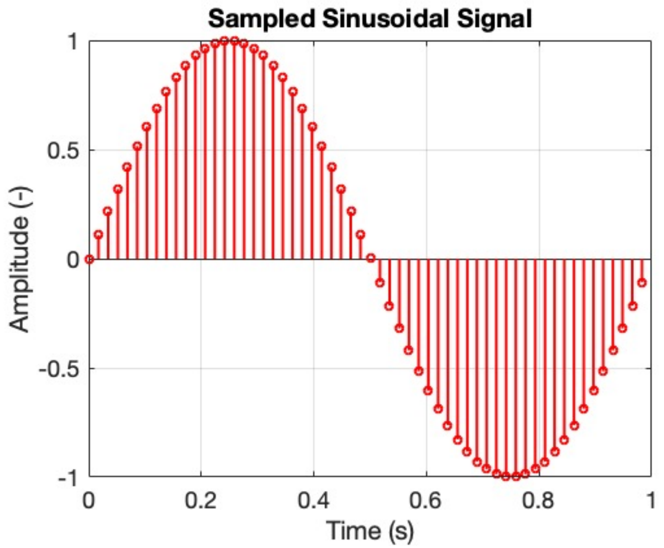


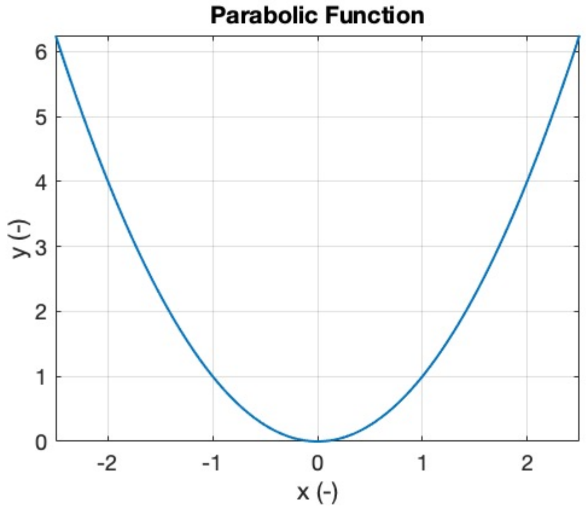
What is a signal?

Definition:

Any physical quantity varies with time, space, or any other independent variable(s).

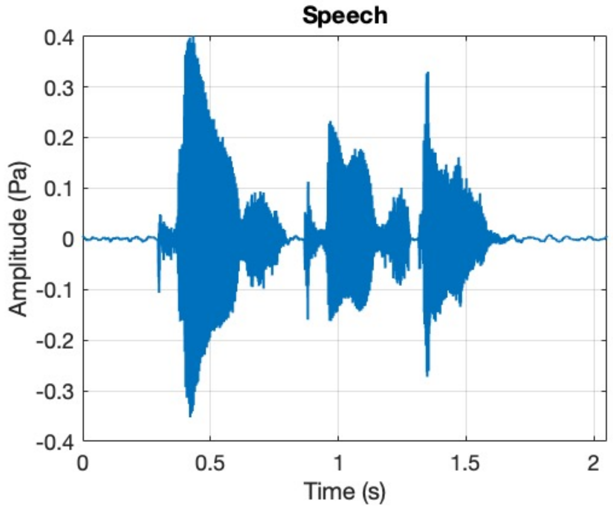
Types of signals:

Functionally Described



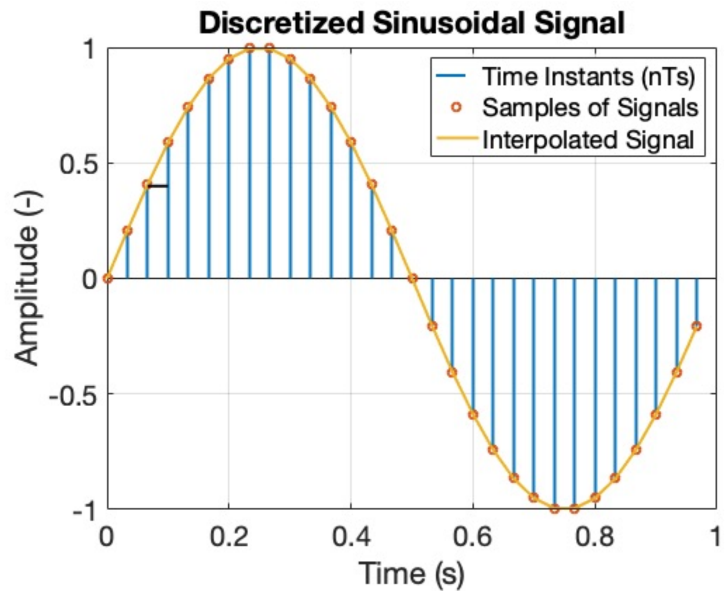
$$f(x) = x^2$$

Non-Functionally Described



$$\text{segment} \sim \sum A_i(t)\sin\left(2\pi f_i(t)t + \theta_i(t)\right)$$

Continuous-Time vs Discrete-Time Signals



Continuous Signals:

$$s(t) \triangleq s(t), \quad \forall t \in \mathbb{R}$$

Discrete Signals:

$$s[t] \triangleq s(nT_s), \quad \forall t, n \in \mathbb{R} \text{ and } T_s \in \mathbb{R}^{\{+\}}$$

indicating periodic signals with sampling period $T_s = \frac{1}{f_s}$ (i.e., f_s: sampling frequency in Hz).

Physical Meaning:

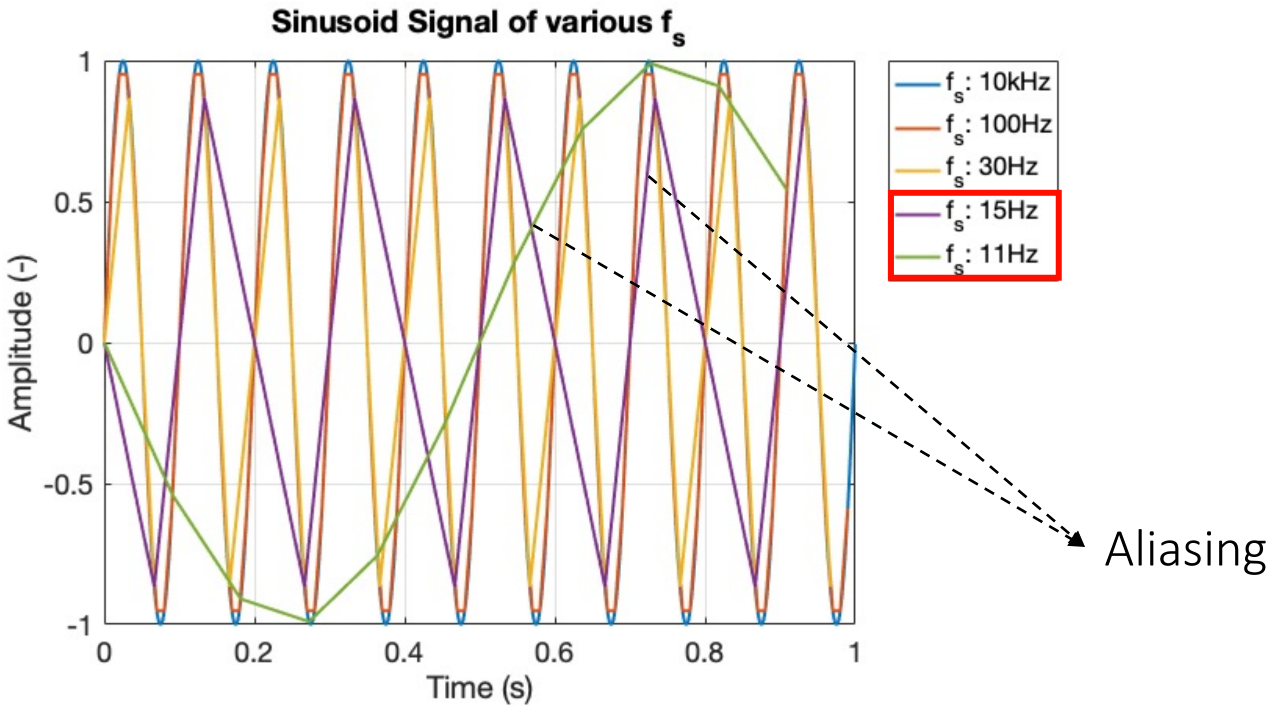
"A discrete-time signal is a sequence of values in particular instants of time (i.e., *sampling period*) and its values correpond to the values in sampling time (i.e., *samples of signals*)."

Time vector: $t = [0: T_s: (N-1)T_s]$, where N is the length of the discrete signal.

Sampling a Sinusoidal Signal

Consider the signal $s(t) = sin(2\pi 10t)$, where $f = 10Hz$.

What should be the sampling frequency (f_s) in order to capture the signal $s(t)$?



What is the proper sampling frequency?

Sampling Theorem (Nyquist-Shannon)

If the highest frequency contained in an analog signal $s(t)$ is f_{max} and the signal is sampled at a rate of

$$f_s > 2f_{max}$$

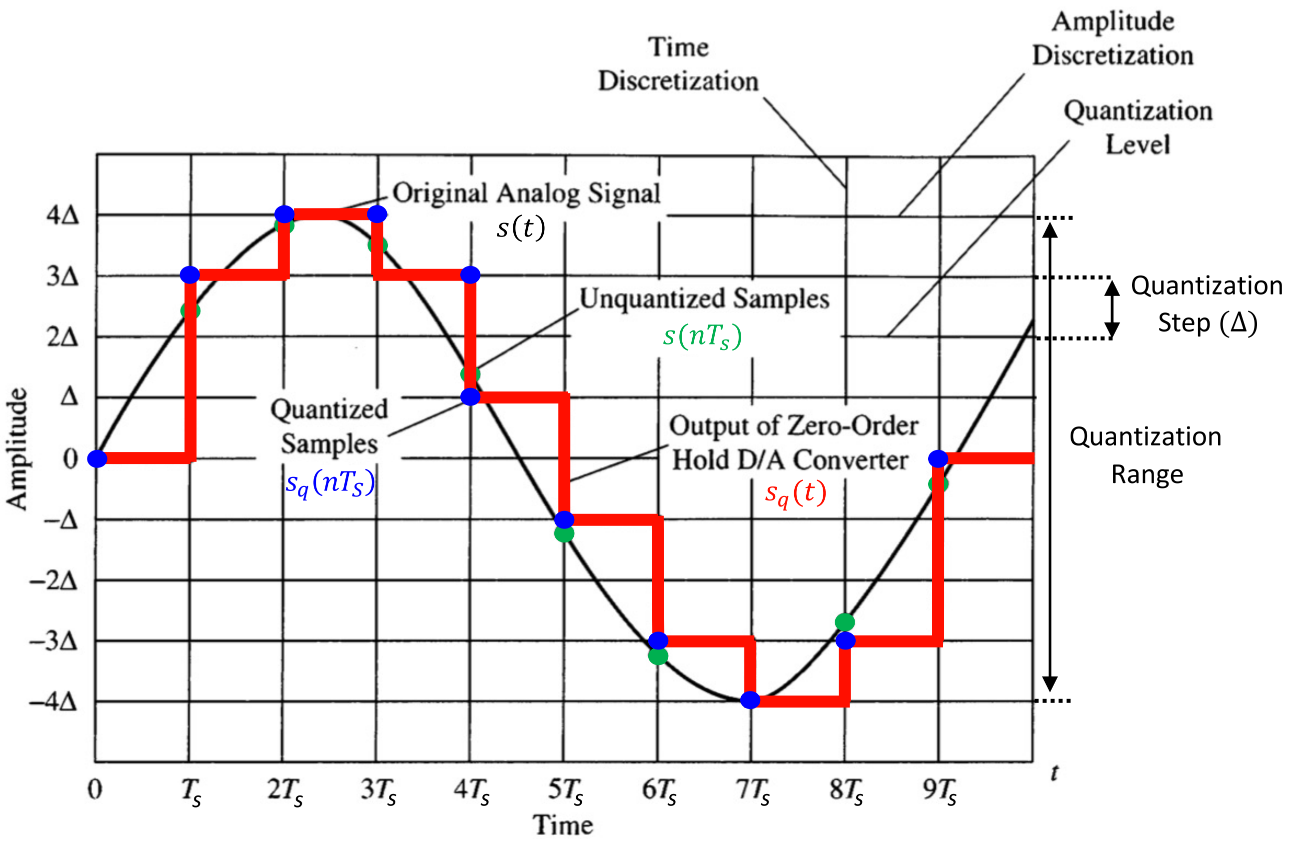
then, the signal $s(t)$ can be exactly reconstructed from its sample values $s[t]$.

The highest (maximum) represented frequency in a sampled signal $s[t]$ corresponds to the Nyquist frequency (f_N) and it is defined as the half of the sampling frequency f_s,

$$f_N = \frac{f_s}{2}\,.$$

Sampling in the Values of Time-Signals (Quantization)

Sampling is also referred to the amplitude of the continuous time-signals $s(t)$.



Amplitude: 2^N values, where N is bits.
Quantization step:

Amplitude of $s(t)$

$$\Delta = \frac{2A}{2^N}$$

Dynamic range is computed, such,

$$D = 20\log_{10}(2^N).$$

2-Bits 4-Bits 8-Bits 16-Bits

Proakis, J. G. and Manolakis, D. G. *Digital Signal Processing: PearsonNew International Edition*, Pearson, Fourth Edition, 2014.

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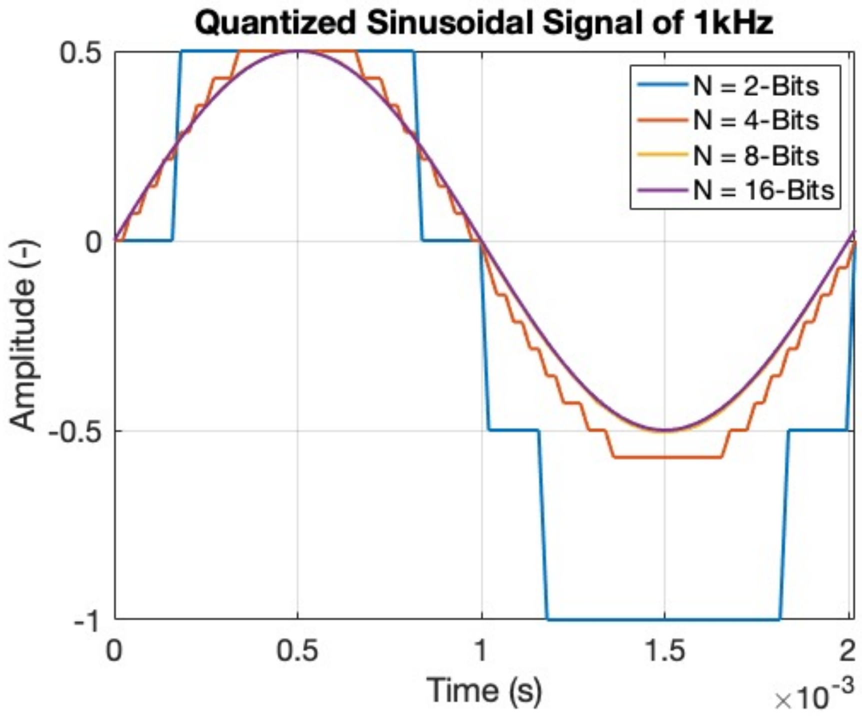

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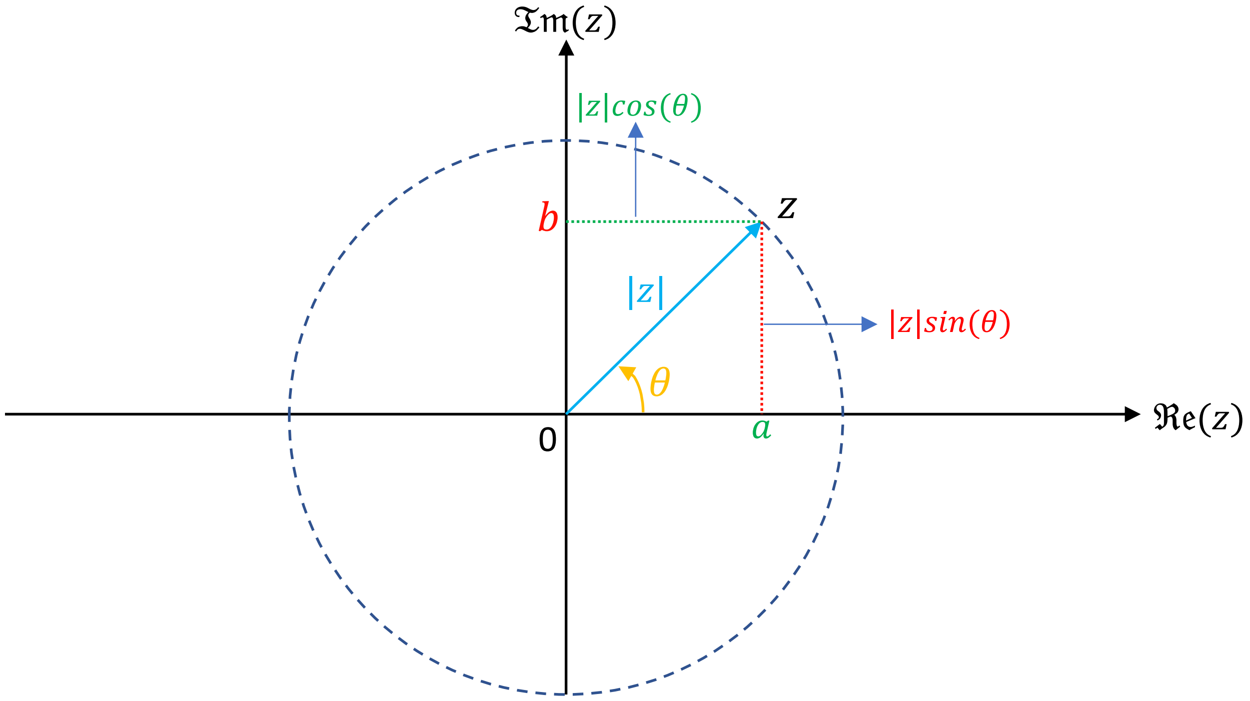
2-Bits 4-Bits 8-Bits 16-Bits

$+1\text{Bit} \approx +6dB$ Signal to Quantization Noise Ratio (SQNR)

Proakis, J. G. and Manolakis, D. G. *Digital Signal Processing: PearsonNew International Edition*, Pearson, Fourth Edition, 2014.

before moving to the analysis of signals in the frequency domain… a review in complex numbers…

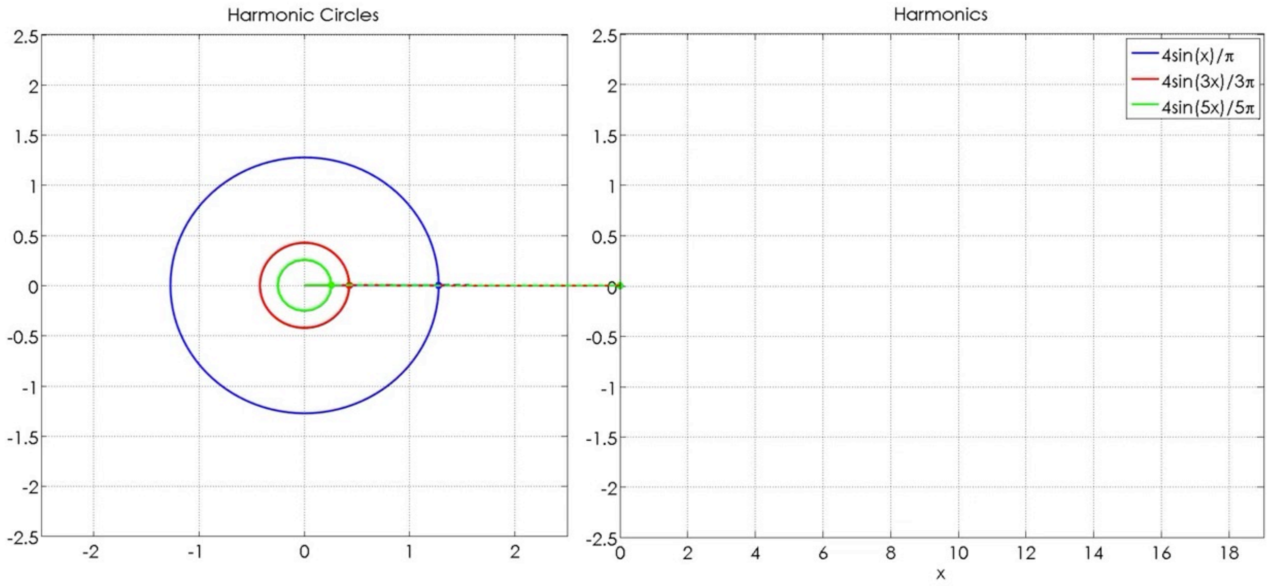
$$z = a + bi = |z|\cos(\theta) + i|z|\sin(\theta) = |z|e^{i\theta}, \forall \alpha, b \in \mathbb{R}\,, z \in \mathbb{C}, i \triangleq \sqrt{-1}$$



$$|z| = \sqrt{\mathfrak{Re}(z)^2 + \mathfrak{Im}(z)^2}, |z| \in \mathbb{R}^+$$

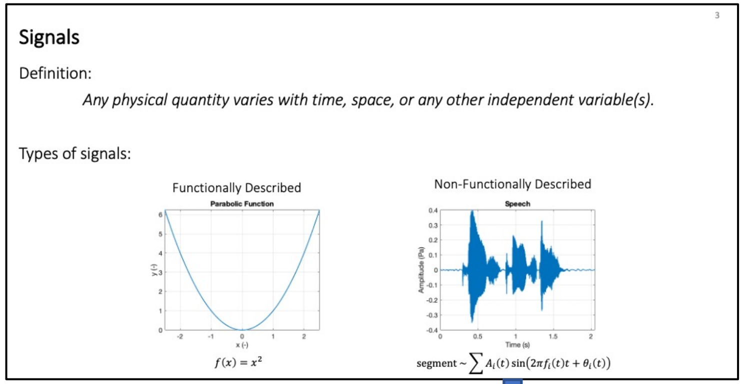
$$\theta = \tan^{-1}\left(\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)}\right), \theta \in (0,2\pi]$$

before moving to the analysis of signals in the frequency domain… Simple Harmonic Motions…

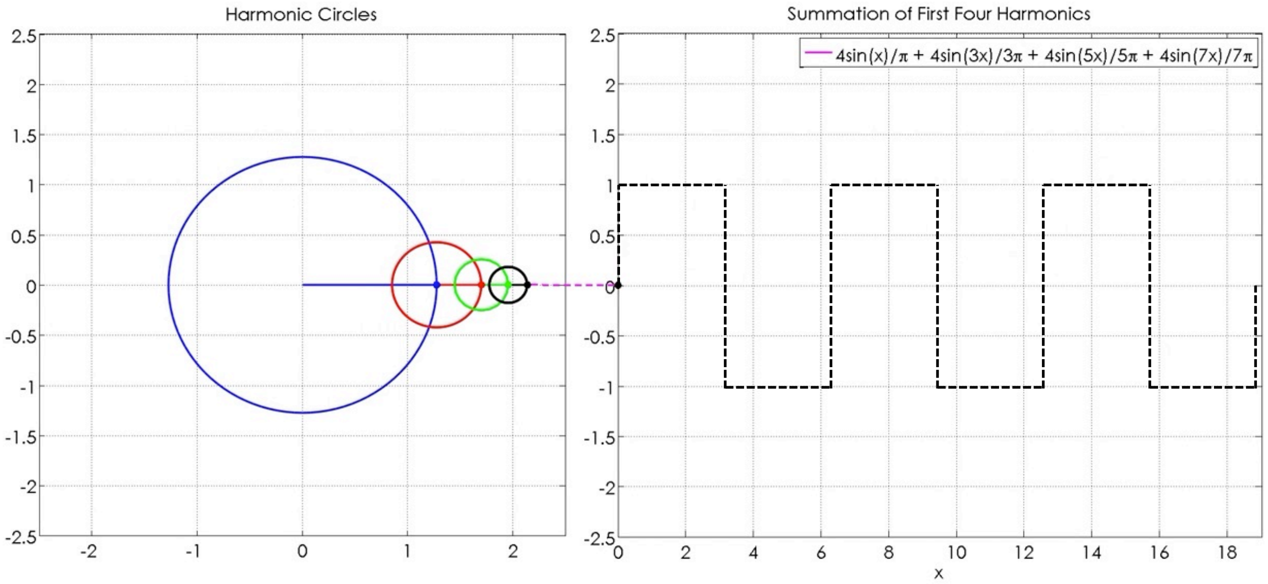


https://www.youtube.com/watch?v=LznjC4Lo7lE&ab_channel=meyavuz

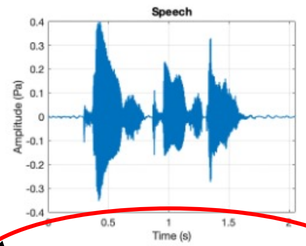
before moving (again) to the analysis of signals in the frequency domain…



introduction to Fourier Series…



Non-Functionally Described



Assuming Periodicity

segment ~ $\sum A_i(t)\sin\left(2\pi f_i(t)t + \theta_i(t)\right)$

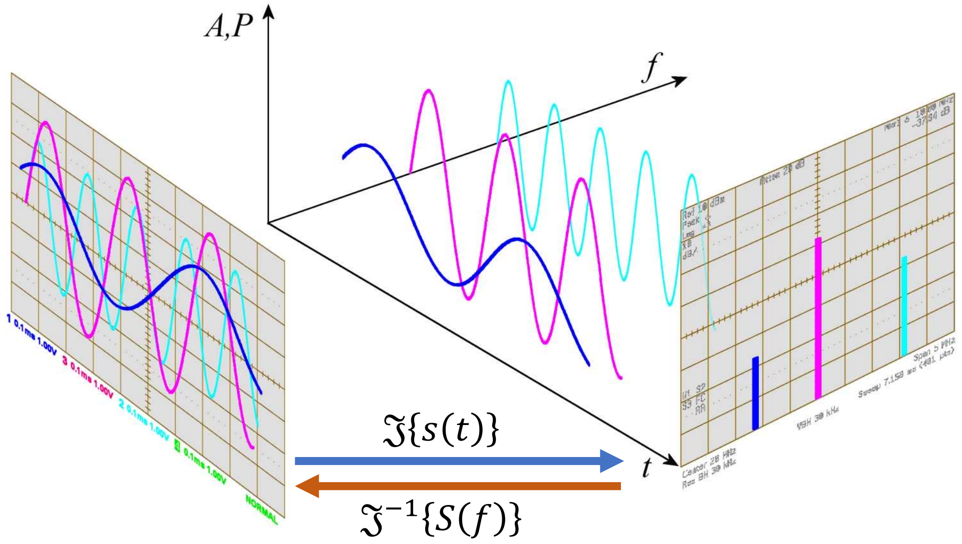
Any periodic function can be rewritten as a weighted sum of sines and cosines of different frequencies.

https://www.youtube.com/watch?v=LznjC4Lo7lE&ab_channel=meyavuz

(Continuous) Fourier-Transform

Transformation of signals from the time domain to the frequency domain.

$$\mathfrak{I}\{s(t)\} \triangleq S(f) = \int_{-\infty}^{+\infty} s(t) e^{-i2\pi f t} dt$$



Transformation of signals from frequency domain (back) to the time domain.

$$\mathfrak{I}^{-1}\{S(f)\} \triangleq s(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(f) e^{i2\pi f t} dt$$

https://www.radartutorial.eu/22.messpraxis/mp06.en.html

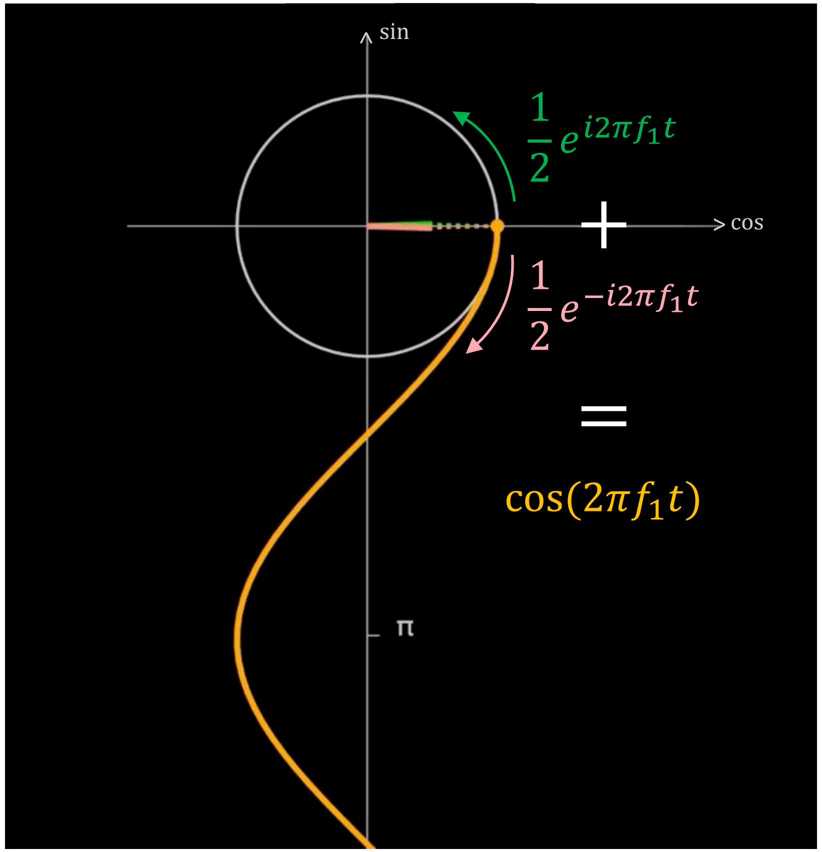
What information is extracted from Fourier Transform?

Suppose that $s(t) = \cos(2\pi f_1 t)$. Compute the $\mathfrak{F}\{s(t)\}$.

$$\mathfrak{F}\{s(t)\} = \int_{-\infty}^{+\infty} \cos(2\pi f_1 t) e^{-i2\pi f t} dt$$

Euler's Formula

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}}{2} \right) e^{-i2\pi f t} dt$$



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Euler's Formula

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}}{2}\right) e^{-i2\pi f t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} \left(e^{i2\pi f_1 t} + e^{-i2\pi f_1 t}\right) e^{-i2\pi f t} dt$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{i2\pi f_1 t} \cdot e^{-i2\pi f t} \, dt + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-i2\pi f_1 t} \cdot e^{-i2\pi f t} dt$$

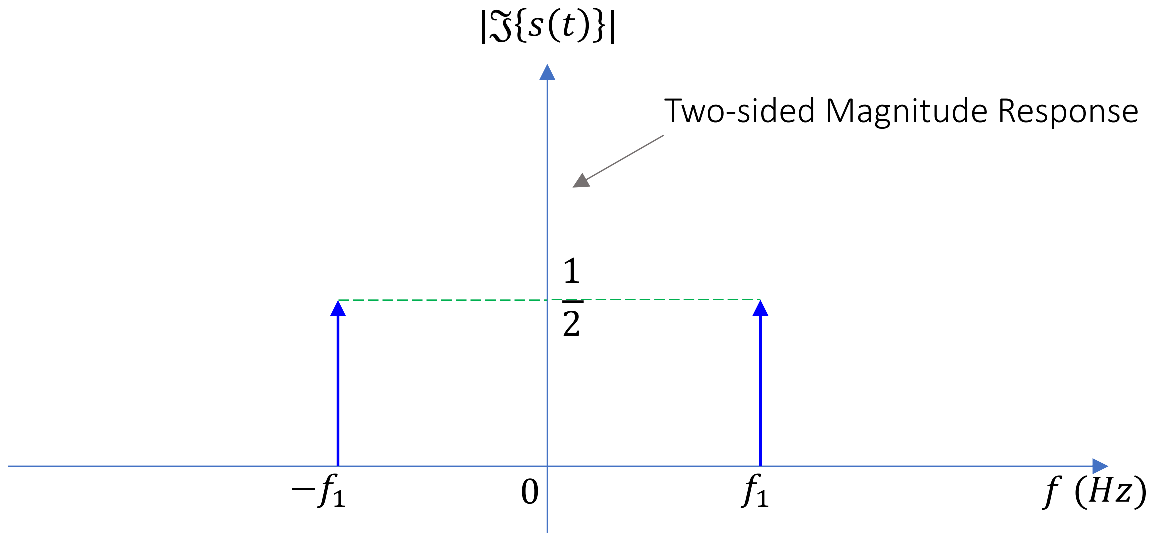
$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{i2\pi (f_1 - f) t} \, dt + \frac{1}{2} \int_{-\infty}^{+\infty} e^{i2\pi (-f_1 - f) t} dt$$

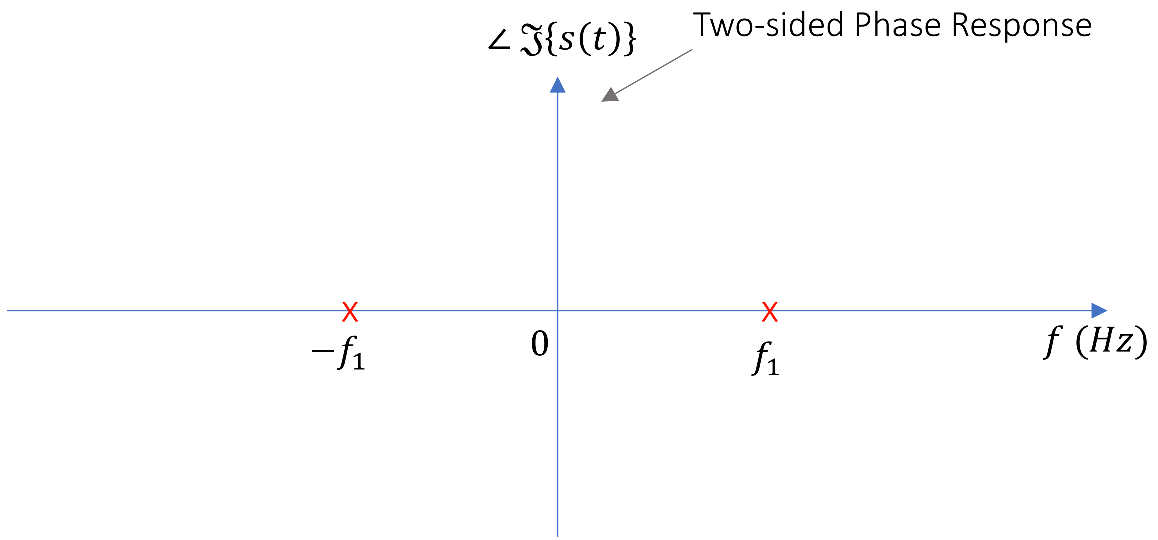
$|z| e^{\theta i}$

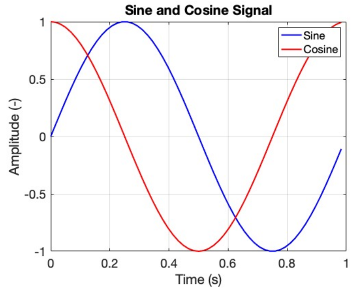
$$= \frac{1}{2} \delta(f_1 - f) + \frac{1}{2} \delta(-f_1 - f)$$

$$\delta(f - x) = \begin{cases} 1, f = x \\ 0, f \neq x \end{cases}$$

$$= \frac{1}{2} e^{0i} \delta(f - f_1) + \frac{1}{2} e^{0i} \delta(f + f_1)$$







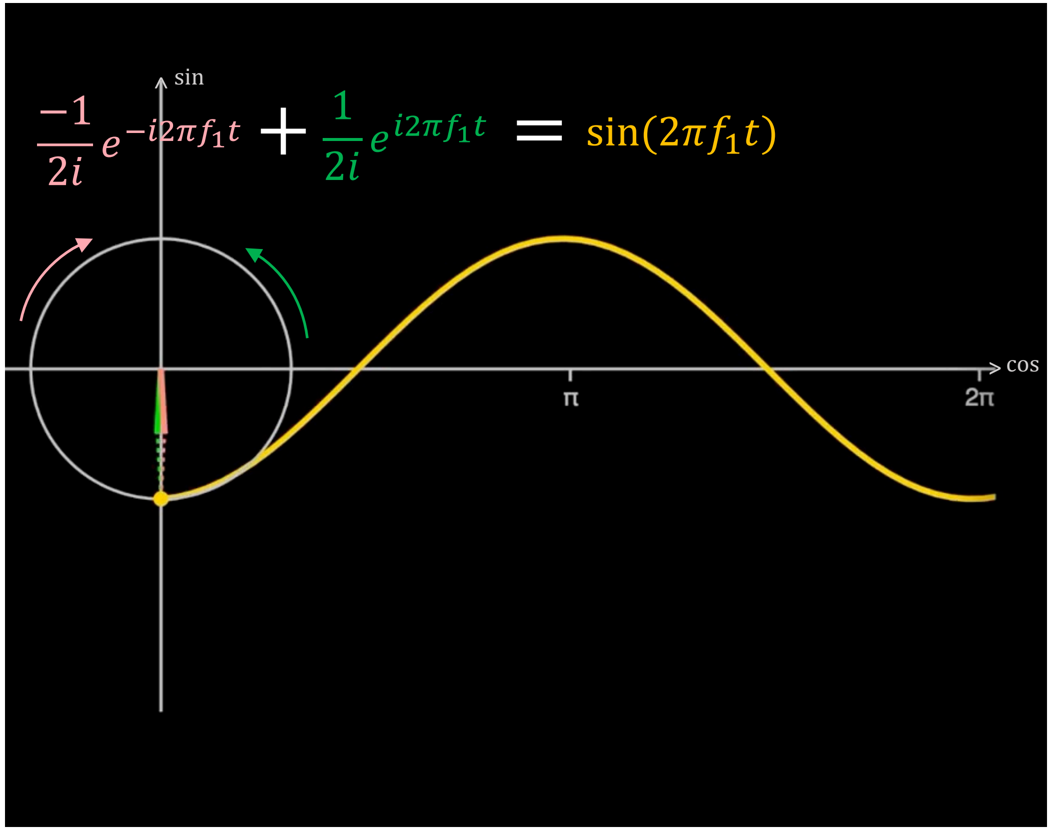
What information is extracted from Fourier Transform?

Suppose that $s(t) = sin(2\pi f_1 t)$. Compute the $\mathfrak{F}\{s(t)\}$.

$$\mathfrak{F}\{s(t)\} = \int_{-\infty}^{+\infty} sin(2\pi f_1 t) e^{-i2\pi f t} dt$$

Euler's Formula

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{i2\pi f_1 t} - e^{-i2\pi f_1 t}}{2i} \right) e^{-i2\pi f t} dt$$



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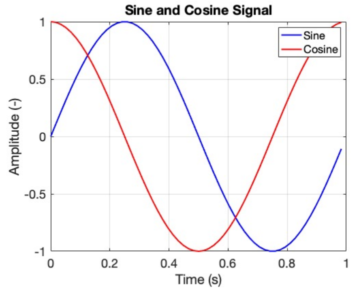

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Euler's Formula

$$= \int_{-\infty}^{+\infty} \left(\frac{e^{i2\pi f_1 t} - e^{-i2\pi f_1 t}}{2i} \right) e^{-i2\pi f t} dt$$

$$= \frac{1}{2i} \int_{-\infty}^{+\infty} \left(e^{i2\pi f_1 t} - e^{-i2\pi f_1 t} \right) e^{-i2\pi f t} dt$$

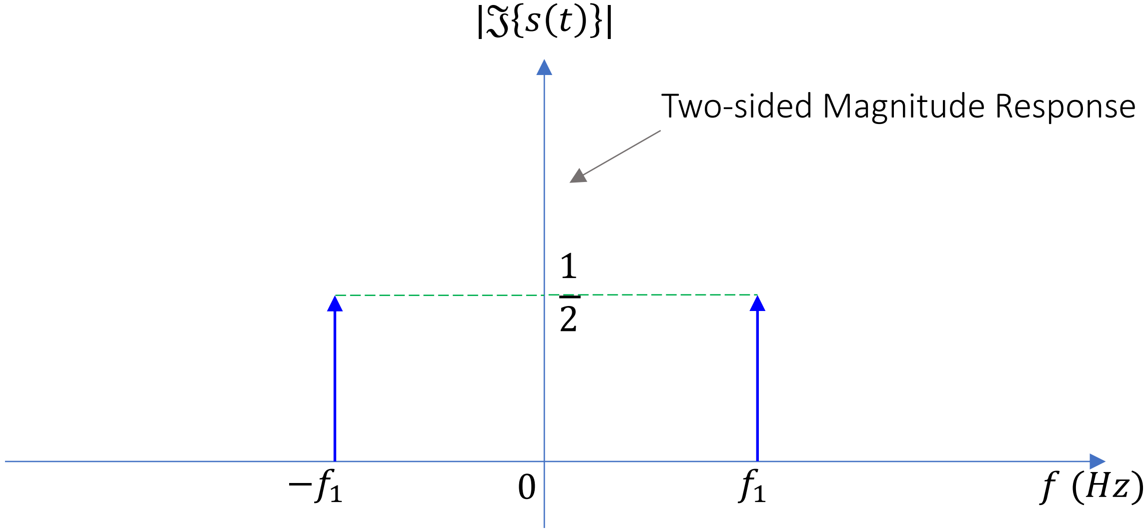
$$= \frac{1}{2i} \int_{-\infty}^{+\infty} e^{i2\pi f_1 t} \cdot e^{-i2\pi f t}\, dt - \frac{1}{2i} \int_{-\infty}^{+\infty} e^{-i2\pi f_1 t} \cdot e^{-i2\pi f t} dt$$

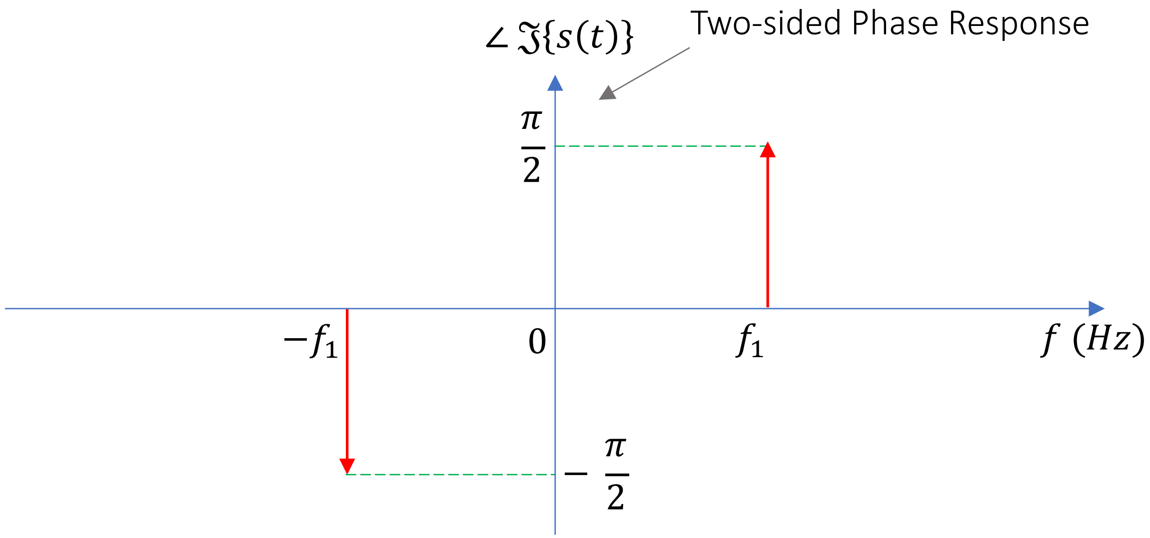
$$= \frac{1}{2i} \int_{-\infty}^{+\infty} e^{i2\pi (f_1 - f) t}\, dt - \frac{1}{2i} \int_{-\infty}^{+\infty} e^{i2\pi (-f_1 - f) t} dt$$

$|z| e^{\theta i}$

$$= \frac{1}{2i} \delta(f_1 - f) - \frac{1}{2i} \delta(-f_1 - f)$$

$$= \frac{1}{2} e^{\frac{\pi}{2} i} \delta(f - f_1) + \frac{1}{2} e^{-\frac{\pi}{2} i} \delta(f + f_1)$$





[Summary] What information is extracted from (Continuous) Fourier Transform?

Frequency response:

$$s(t) \in \mathbb{R} : \xrightarrow{\mathfrak{F}\{s(t)\}} S(f) \in \mathbb{C}.$$

Magnitude response:

$$|S(f)| \in \mathbb{R}^{\{+\}}$$

Phase response:

$$\angle S(f) \in [0,2\pi) \text{ or } [-\pi, \pi)$$

[Remarks] (Continuous) Fourier Transform

- Orthogonality-based transformation: Projection of signal’s frequency components onto orthonormal base (i.e., frequencies).

- Symmetrical spreding of the frequency components with respect to the center of axes.

$$\{\cos(2\pi f t), \sin(2\pi f t)\} = \frac{e^{i2\pi f t} \pm e^{-i2\pi f t}}{2i}.$$

- Real-valued signal in time-domain becomes complex-valued in the frequency domain.

- Time-shift (time-domain) is associated to phase-shift (frequency domain).

- The energy in time-domain is equal to the energy in frequency-domain (Parseval-Theorem).

- The energy is distributed symmetrically in magnitude response between $(-\infty, 0)$ and $(0, +\infty)$.

➢ !!Signals are not infinite!!, leading to the definition of the discrete Fourier transform (DFT).

MATLAB fft() calculates the Fourier Transform of the discrete time signal based on this principle.

Discrete Fourier Transform (DFT)

The DFT of a sampled and periodic signal is defined such,

$$S[k] = \sum_{n=0}^{N-1} s[n] e^{-i\frac{2\pi k}{N}n}, k = 0, 1, \ldots, N-1$$

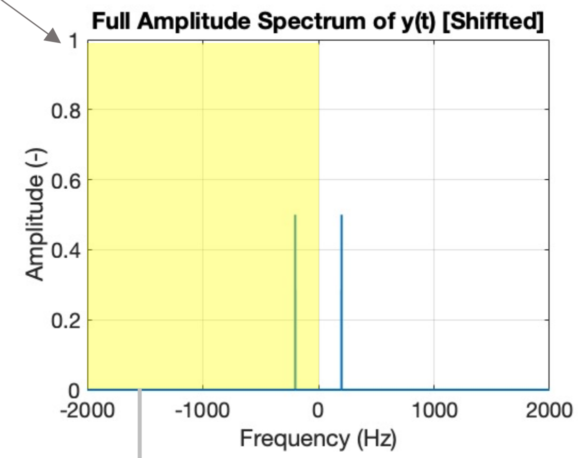
and the inverse discrete Fourier transform (IDFT) is defined such,

$$s[n] = \frac{1}{N} \sum_{k=0}^{N-1} S[k] e^{i\frac{2\pi k}{N}n}, n = 0, 1, \ldots, N-1$$

$\left[-\frac{f_s}{2}, \frac{f_2}{2}\right]$

$[0, f_s]$

"Continuous"



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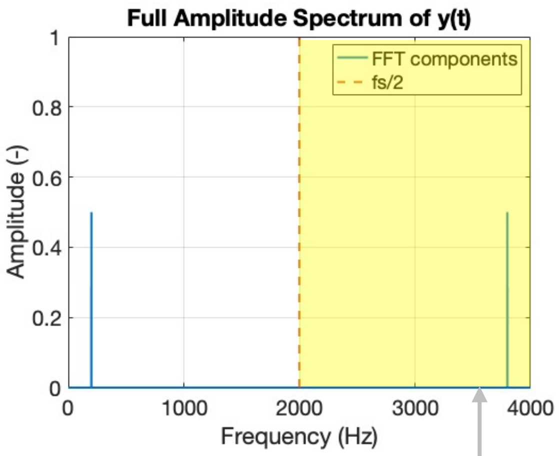
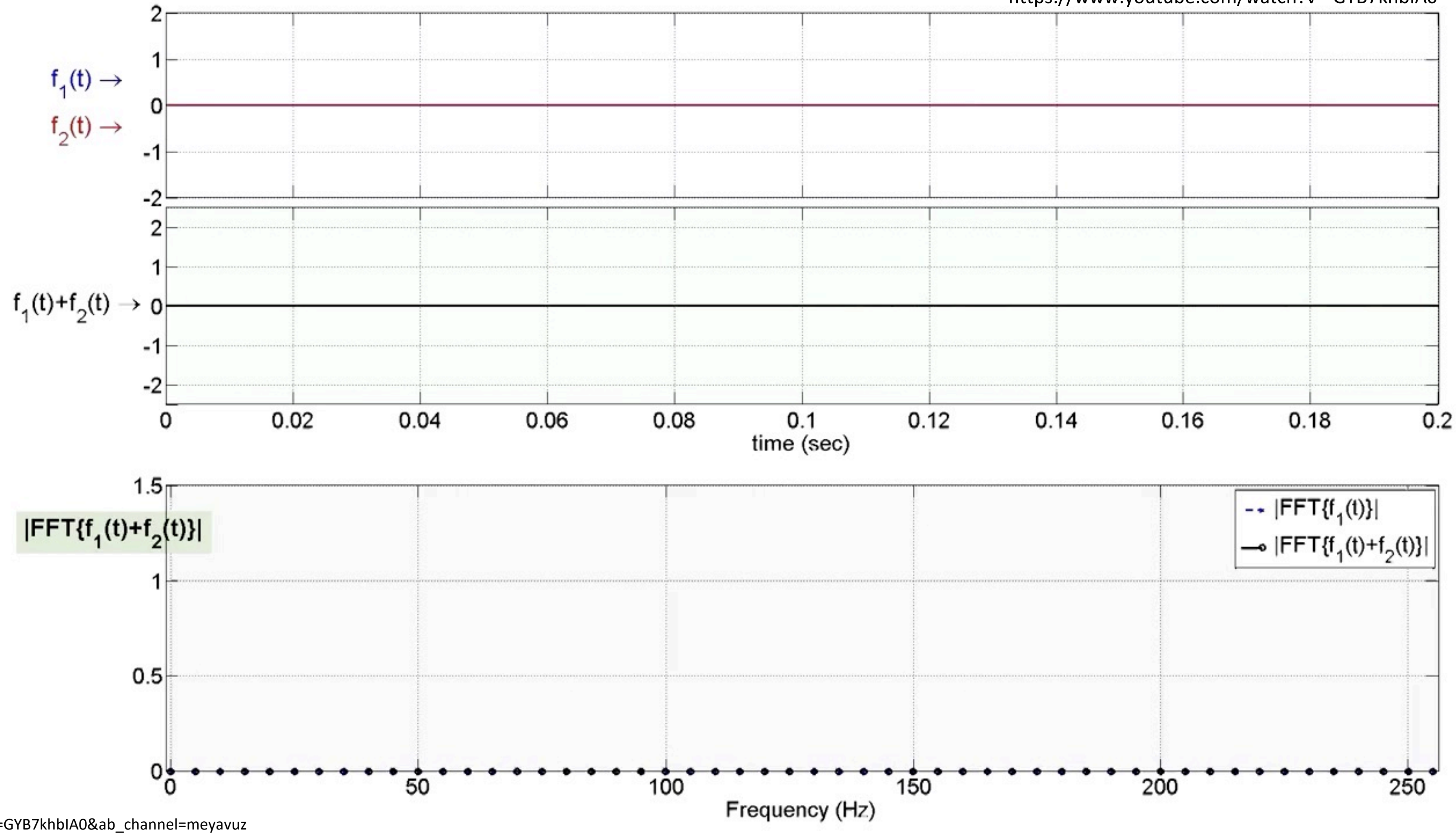


Discrete

The Fourier Transform of Sionusoidal Discrete Signals

What do you observe?

https://www.youtube.com/watch?v=-GYB7khbIA0



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DFT-based Remarks

- Only the positive frequency components (i.e., one-sided magnitude response) are considered.

$$\left|S\left[f\left(0:\frac{f_s}{2}\right)\right]\right|$$

- Multiplying the positive components by a factor of two, maintaining the energy of two-sided magnitude response to one-sided magnitude response.

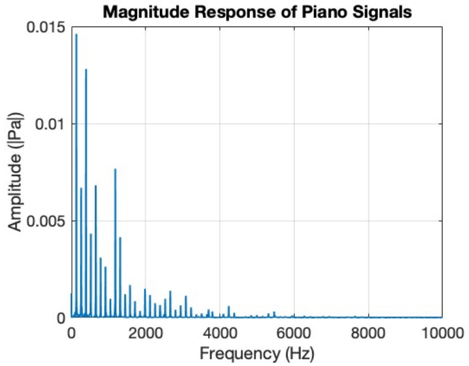
$$2\cdot\left|S\left[f\left(0:\frac{f_s}{2}\right)\right]\right|$$

- Discontinuities are responsible for the presence of “ghost” frequency components, called leakage.
- The Fourier transform of a $\mathbf{step}$ function is a $\mathbf{sinc}$ function.
- As the duration of the signal increases, the components are “closer” to delta pulse.
- The one-sided magnitude response is used *!!ONLY!!* for representation reasons (physical meaning).

How the magnitude response is used in Acoustics?

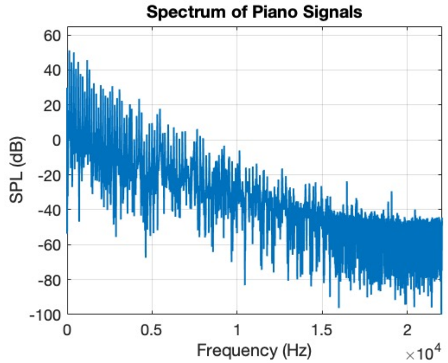
- Continuous magnitude response:

$$|S(f)|_{Pa} = 2\times|S(f)|\ ,\ f \in \left[0, \frac{f_s}{2}\right]$$



- Continuous, sound pressure level (SPL), magnitude response:

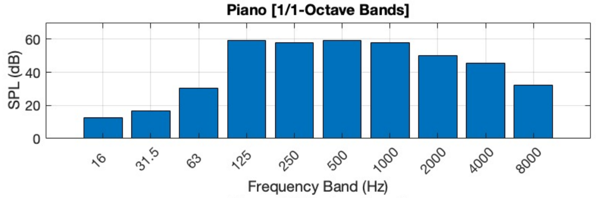
$$|S(f)|_{dB} = 10\log_{10}\left(\frac{2\times\left|\tilde{S}(f)\right|^2}{p_{ref}^2}\right), f \in \left[0, \frac{f_s}{2}\right]$$

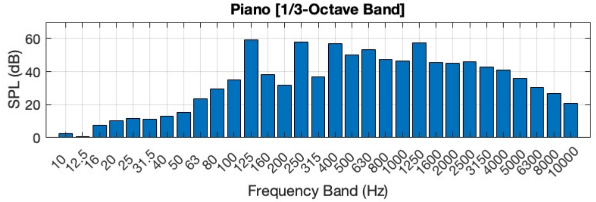


- Discrete, sound pressure level (SPL), magnitude response:

$$|S(f)|_{n^{th}-oct,dB} = 10\log_{10}\sum_{f_l}^{f_u}\left(\frac{2\times\left|\tilde{S}(f)\right|^2}{p_{ref}^2}\right), f \in \left[0, \frac{f_s}{2}\right]$$

$$\tilde{S}(f) = \frac{S(f)}{\sqrt{2}}$$

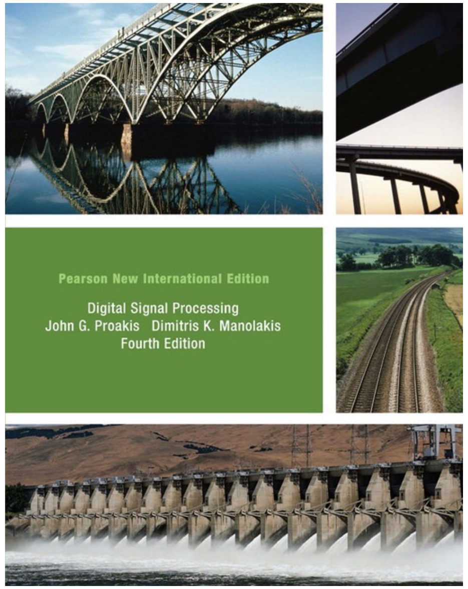




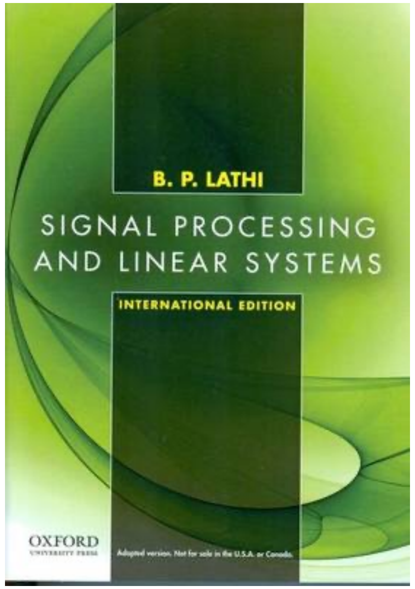
Summary – Main Points

- Sampling theorem ensures the proper discretization of the signals.
- Discretization of signals with sampling frequency $f_s < 2f_{max}$ leads to aliasing.
- The maximum representative frequency is related to the Nyquist frequency.
- Sampling is used also in the values of signals (Quantization).
- Fourier Transform is used to transfer signals from time domain to the frequency domain.
- Frequency response is a complex signal, including information related to magnitude and phase of signal.
- Discrete Fourier transform is used to finite size signals.
- Discontinuities in time signals associated to leakage to magnitude response.
- Magnitude and/or phase response is used to be one-sided (i.e., illustration reasons and physical meaning).
- One-sided magnitude response is multiplied by a factor two, retaining the two-sided energy.
- In acoustics different forms of magnitude responses are used.

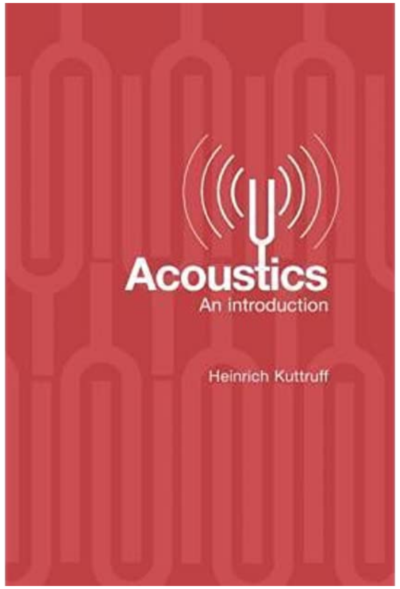
Corresponding and Interesting Literature



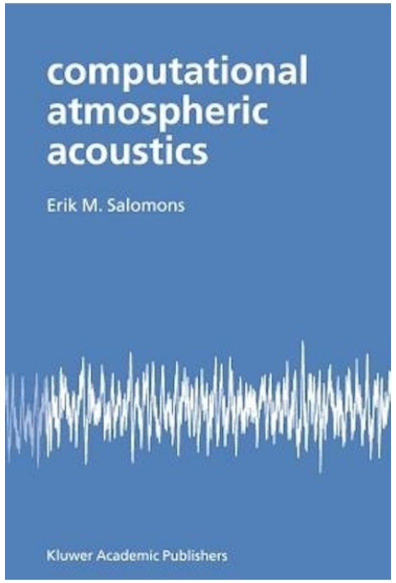
Chapter 1 and 7



Chapter 1 and 6



Chapter 2.9



Chapter B.4

Thank you!

One Tutorial (more) remains …