Diachrony of Spectra

Ikhan Choi

Postech - Unist - Kaist Joint Seminar

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Introduction

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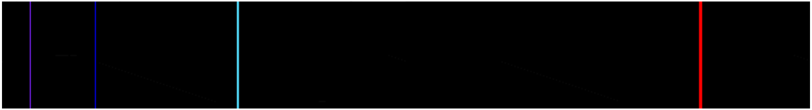
Question

Why is it defined like this?

Contents

Hydrogen spectral series



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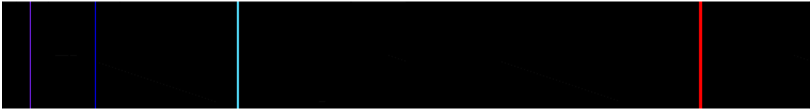

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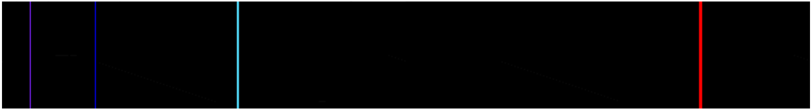

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Question

How can we explain and compute this phenomenon?

Rydberg's formula : Bohr model

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The constant h is called the Planck constant and $\hbar := \frac{h}{2\pi}$.

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$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

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Proposition (Rydberg formula)

The wavelengths λ of absorbed or emitted photons from a hydrogen atom is estimated by the following formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \textit{for} \quad n_1, n_2 \in \mathbb{N},$$

where $R := \frac{k^2e^4m}{4\pi\hbar^3c}$ is the Rydberg constant.

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Let's solve.

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- All functions that look like

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- Since $\phi_E(t) \propto e^{-iEt}$ is easily solved, the main difficulty is ψ_E.

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Don't be so pedantic in doing physics.

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- Eigenvalues embody the possible energies of an electron, so we can give the Rydberg formula a reasonable explanation.
- This result explains not only the discretized energy spectrum but also the number of each orbitals!

Conclusion of Section 1

$$\begin{array}{c}\text{Partial Differential Equations with Time Evolution}\\ \Big\downarrow \text{Separation of variables}\\ \text{Simultaneous Eigenvalue Problems}\\ \Big\downarrow\\ \text{Study of Eigenvalues} = \text{Study of Hydrogen Spectrum}\end{array}$$

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Spectral theory?

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In this section,

- we review the spectral theory on finite dimensional vector spaces
- and introduce Hilbert spaces — a typical example of infinite dimensional vector spaces — to state some results which extend the spectral theory to infinite dimensional spaces.

Spectral theorem of normal matrices

Hilbert space

Spectral theorem of compact operators

Spectral theorem of elliptic operators (Skipped)

Contents

Banach algebras and C*-algebras

Example 1 : Bounded operators

Example 2 : Continuous functions

Spectra, multiplicative homomorphisms, maximal ideals

Gelfand-Naimark theorem

Algebraic variety

Coordinate ring

Maximal ideal is a point

Problem of unified codomains

Functoriality