

# Classical differential geometry

IKHAN CHOI

## 1. INTRODUCTION

### 1.1. Two ways to represent curves or surfaces.

### 1.2. Coordinates and parametrizations.

**Definition 1.1.** A *parametrization* is a smooth map  $U \rightarrow \mathbb{R}^d$  such that

- (1)  $U \subset \mathbb{R}^c$  is open and connected,
- (2)  $\alpha$  is one-to-one,
- (3)  $d\alpha$  is nondegenerate;  $\{\partial_i \alpha\}_{i=1}^c$  is linearly independent.

**Definition 1.2.** A *regular curve* is a subset of  $\mathbb{R}^d$  that is the image of some parametrization  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{R}^d$ .

**Definition 1.3.** A *regular surface* is a subset of  $\mathbb{R}^d$  that is the image of some parametrization  $\alpha : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^d$ .

## 2. CURVES IN A SPACE

### 2.1. Arc-length parameterization.

**Theorem 2.1.** For every regular curve, there is a parametrization  $\alpha$  such that  $\|\alpha'\| = 1$ .

*Proof.* Suppose we have a parametrization  $\beta : I_t \rightarrow \mathbb{R}^d$ . Define  $\tau : I_t \rightarrow I_s$  such that

$$\tau(t_0) := \int_0^{t_0} \|\beta'(t)\| dt.$$

Then,  $s$  is a diffeomorphism. Define  $\alpha : I_s \rightarrow \mathbb{R}^d$  by  $\alpha := \beta \circ \tau^{-1}$ . Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt} \frac{d\tau^{-1}}{ds} = \beta' \left( \frac{d\tau}{dt} \right)^{-1} = \frac{\beta'}{\|\beta'\|}. \quad \square$$

**Definition 2.1** (Frenet-Serret frame). Let  $\alpha$  be a curve such that  $\kappa \neq 0$ . Define *tangent unit vector*, *normal unit vector*, *binormal unit vector* by:

$$\mathbf{T} := \frac{\alpha'}{\|\alpha'\|}, \quad \mathbf{N} := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}, \quad \mathbf{B} := \mathbf{T} \times \mathbf{N}.$$

**Definition 2.2.**

$$\kappa := \mathbf{T}' \cdot \mathbf{N}, \quad \tau := -\mathbf{B}' \cdot \mathbf{N}.$$

**Theorem 2.2** (Frenet-Serret formula). *Let  $\alpha$  be a unit speed curve.*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

*Proof.* The vectors  $\mathbf{T}', \mathbf{B}', \mathbf{N}$  are collinear. □

**Theorem 2.3.** *Let  $\alpha$  be a unit speed curve.*

$$\begin{aligned} \alpha' &= \mathbf{T} \\ \alpha'' &= \kappa \mathbf{N} \\ \alpha''' &= -\kappa^2 \mathbf{T} + \kappa' \mathbf{N} + \kappa \tau \mathbf{B} \end{aligned}$$

Skew-symmetry is due to the fact the differential of an orthogonal matrix forms a skew symmetric matrix.

**Example 2.1.** Let  $\alpha$  be a curve in  $\mathbb{R}^3$ . If the normal line always passes through a point, then  $\alpha$  is contained in a circle.

*Proof.* Let  $\alpha$  be a unit speed curve. By the assumption, there is a constant point  $p \in \mathbb{R}^3$  such that the vectors  $\alpha - p$  and  $\mathbf{N}$  are parallel so that we have

$$\langle \alpha - p, \mathbf{T} \rangle = 0, \quad \langle \alpha - p, \mathbf{B} \rangle = 0.$$

Our goal is that  $\|\alpha - p\|$  is constant and there is a constant vector  $v$  such that  $\langle \alpha - p, v \rangle = 0$ .

$$0 = \langle \alpha - p, \mathbf{T} \rangle' = \langle \alpha', \mathbf{T} \rangle + \langle \alpha - p, \kappa \mathbf{N} \rangle = 1 + \kappa \langle \alpha - p, \mathbf{N} \rangle.$$

$$0 = \langle \alpha - p, \mathbf{B} \rangle' = \langle \alpha - p, -\tau \mathbf{N} \rangle = -\tau \cdot \left(-\frac{1}{\kappa}\right)$$

$$\begin{aligned} (\|\alpha - p\|^2)' &= \langle \alpha - p, \alpha - p \rangle' \\ &= 2\langle \alpha - p, \alpha' \rangle \\ &= 2\langle \alpha - p, \mathbf{T} \rangle \\ &= 0 \end{aligned}$$

$$\mathbf{B}' = -\tau \mathbf{N} = 0.$$

□

### 3. SURFACES IN A SPACE

### 4. CURVES ON A SURFACE