Differential Geometry

-Problem set-

Ikhan Choi

September 13, 2018

Contents

1 Manifolds **3**
Lie derivatives 3
Curves 3

2 Lie groups **4**

3 Connections **5**
Affine connection 5
Ehresmann connection 7

4 Riemannian geometry **9**
Riemannian metric 9
Geodesics 9
Hodge theory 9

1 Manifolds

Lie derivatives

1.1. *Cartan's magic formula.*

Cartan's magic formula is

$$L_X\alpha = \iota_X d\alpha + d\iota_X\alpha.$$

(a) Show that
$$d\alpha(X,Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]).$$

Curves

Newton's dot notation will be used for only curves and scalar-valued functions.

2 Lie groups

3 Connections

Affine connection

3.1. *Covariant derivative.*

Let E be a vector bundle over a manifold M. A connection on E is an $\mathbb{R}$-linear map $\nabla\colon \Gamma(E) \to \Gamma(T^*M \otimes E)$ such that the Leibniz rule $\nabla(fs) = f\nabla s + df \otimes s$ for $s \in \Gamma(E)$ and $f \in C^\infty(M)$.

3.2. *Affine property.*

tensor

(a) Show that $\nabla_1 - \nabla_2$ is a tensor.

3.3. *Christoffel symbol.*

Let ∇ be a affine connection on a manifold M. Let $\{e_i\}_i$ be a local frame on M. The Christoffel symbol is a set of functions characterizing a connection on the tangent bundle, an affine connection, defined by

$$\nabla_i e_j = \Gamma_{ij}^k e_k.$$

Let X, Y be vector fields on U.

(a) Show that
$$\nabla_X Y = X^i(\partial_i Y^k + \Gamma_{ij}^k Y^j)e_k.$$

(b) Use a partition of unity to show that every manifold admits an affine connection.

Solution.

(a) This is easy.

(b) This is not easy.

3.4. *Dependnecy.*

Let M be a manifold. Let p be a point on M. Let X_1, X_2, Y_1, Y_2 be vector fields on a neighborhood of p.

(a) Show that if X_1 and X_2 are same at p, then $\nabla_{X_1} Y = \nabla_{X_2} Y$ at p for every vector field Y.

(b) Show that if Y_1 and Y_2 are same on a neighborhood of p, then $\nabla_X Y_1 = \nabla_X Y_2$ at p for every vector field X.

3.5. *Parallel transport of vector.*

Let ∇ be an affine connection on a manifold M. A vector field X is called *parallel* along with γ if $\nabla_{\dot\gamma} X = 0$.

(a) Deduce X satisfies a first order ODE

$$\dot{X}^k(t) + \Gamma^k_{ij}\dot{\gamma}^i(t)X^j(t) = 0.$$

(b) Prove that given a tangent vector $v_0 \in T_{\gamma(t_0)}M$, there exists a unique parallel vector field X along γ such that $X(t_0) = v$.

Parallel transport is defined by the previous subproblem

(c) Show that the parallel trasport is

(d) parallel frame

(e) Show that

$$\nabla_{\dot{\gamma}} X(t_0) = \lim_{t \to t_0} \frac{P^{-1}_{t_0 \to t} X(t) - X(t_0)}{t - t_0}.$$

3.6. *Second covariant dervative.*

The second covariant dervative is defined by

(a) Note that

$$\nabla^2_{X,Y} f = [\nabla(\nabla f)](X,Y) = \langle\langle \nabla(df), X\rangle, Y\rangle = \langle \nabla_X(df), Y\rangle = \nabla_X \langle df, Y\rangle - \langle df, \nabla_X Y\rangle = XYf - (\nabla_X Y$$

(b) Prove

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}.$$

3.7. *Torsion tensor.*

Let ∇ be an affine connection on a manifold M. The torsion tensor is a tensor field of (1,2)-type defined by

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y].$$

(a) If torsion-free, then second covariant derivative is symmetric(Hessian is symmetric)

3.8. *Riemannian curvature tensor.*

This is an empty problem. A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers.

(a) Show that the equation $f(x) = 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.

(b) Find the formula of solutions.

3.9. *Connection on natural constructions.*

This is an empty problem. A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers.

(a) Show that the equation $f(x) = 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.

(b) Find the formula of solutions.

Ehresmann connection

3.10. *Horizontal subbundle.*

Let $\pi\colon E \to M$ be a fiber bundle. A distribution of E defined by $\ker \pi$ is called vertical subbundle and denoted by VE. An Ehresmann connection is a choice of a subbundle HE of TE called horizontal subbundle that satisfies $TE = VE \oplus HE$.

(a) definitions by (1) choice of horizontal subbundle (2) projection to vericals (3) right action is adjoint + fundamental vector fields.

3.11. *Connection form.*

Let $\pi\colon P \to M$ be a principal G-bundle and VP be the vertical subbundle. Let $v\colon TP \to VP$ be a projection vector bundle homomorphism so that $\ker v =: HP$ defines an Ehresmann connection on P.

(a) Provide a vector bundle isomorphism $VP \to P \times \mathfrak{g}$.

(b) Show that v determines a $\mathfrak{g}$-valued one-form on P. This is called connection form and will be denoted by ω. blablbalblablbala

(c) fundamental vector field $\sigma_p\colon \mathfrak{g} \to T_pP$ is defined by the differential of the orbit map $g \mapsto pg$:

$$\sigma_p(X) = \frac{d}{dt}(p \exp(tX))|_{t=0}.$$

3.12. *Parallel transport.*

This is an empty problem. A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers.

(a) Show that the equation $f(x) = 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.

(b) Find the formula of solutions.

3.13. *Principal connection.*

This is an empty problem. A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers.

(a) Show that the equation $f(x) = 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.

(b) Find the formula of solutions.

3.14. *Exterior covariant derivative.*

de Rham

(a) Note that exterior derivative can be realized as the unique connection on the trivial bundle $M \times \mathbb{R} \to M$.(I guess it is true, but I should check)

3.15. *Curvature form.*

This is an empty problem. A quadratic function is defined by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers.

(a) Show that the equation $f(x) = 0$ has a real solution if and only if $b^2 - 4ac \geq 0$.

(b) Find the formula of solutions.

3.16. *Torsion form.*

(a) Show that $d\,\mathrm{id} \in \Omega^1(M; TM)$ and $d^\nabla d\,\mathrm{id} = T$.

3.17. *Lie algebra-valued differential form.*

wedge product and cartan structural equation

4 Riemannian geometry

Riemannian metric

4.1. *Musical isomorphism.*

sharp flat

(a) sharp flat

4.2. *Levi-civita connection.*

Let M be a Riemannian manifold. A Levi-civita connection is a connection on M such that

(1) $\nabla_X Y$

4.3. *Riemannian curvature tensor.*

Let ∇ be the Levi-civita connection on a Riemannian manifold M. The Riemannian curvature tensor is a tensor field of type (1,3) defined by

$$R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Geodesics

4.4. *Geodesic equation.*

Let ∇ be an affine connection on a manifold M. A smooth curve $\gamma\colon I \to M$ is called geodesic if

$$\nabla_{\dot\gamma} \dot\gamma(t) = 0$$

at every t.

(a) Deduce the geodesic equation

$$\ddot\gamma^k(t) + \Gamma^k_{ij} \dot\gamma^i(t) \dot\gamma^j(t) = 0.$$

(b) Prove that for $p \in M$, $v \in T_pM$, and $t_0 \in \mathbb{R}$, there exists a locally defined geodesic curve $\gamma\colon I \to M$ with $\gamma(t_0) = p$ and $\dot\gamma(t_0) = v$.

Hodge theory

4.5. *Normal bundle.*

Let $\tilde M$ be an oriented Riemannian manifold. Let M be an embedded manifold in $\tilde M$. The normal bundle is defined as the quotient...

(a) Show that M is orientable if and only if there is a globally nonvanishing normal vector.

(b) Show show show

4.6. *Volume form.*

Let M be an oriented Riemannian manifold with boundary.

(a) Show that if $\partial M = \varnothing$, then $d\operatorname{vol}_M$ is not exact.

4.7. *Divergence.*

The divergence is defined by

$$\operatorname{div} X \, d\operatorname{vol} := d(\iota_X d\operatorname{vol}) = L_X d\operatorname{vol}.$$

(a) Show that

$$\iota_F d\operatorname{vol} = F^1\, dy \wedge dz + F^2\, dz \wedge dx + F^3\, dx \wedge dy$$

for a vector field $F = F^1 \partial_x + F^2 \partial_y + F^3 \partial_z$ on $\mathbb{R}^3$.

(b) Deduce that

$$\operatorname{div} F = \frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}.$$

4.8. *Divergence theorem.*

Let M be an oriented Riemannian manifold with boundary. Let X be a vector field on M and ν be a global normal vector field on ∂M. Let $d\operatorname{vol}_M$ and $d\operatorname{vol}_{\partial M}$ be invariant volume forms.

(a) Show that

$$\langle X, \nu \rangle \; d\operatorname{vol}_{\partial M} = \iota_X \, d\operatorname{vol}_M$$

on ∂M

(b) Deduce the divergence theorem:

$$\int_{\partial M} \langle X, \nu \rangle \; d\operatorname{vol}_{\partial M} = \int_M \operatorname{div} X \, d\operatorname{vol}_M .$$