The de Rham theorem

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Let $\omega \in \Omega^k(M)$.

First representation:

$$\omega = \frac{1}{k!} \sum_{|\alpha|=k} \omega_\alpha \, dx^\alpha$$

where α runs through all multi-indices. Then, note that

$$\omega_\alpha = \operatorname{sgn}(\sigma)\omega_{\sigma(\alpha)}$$

implies

$$\omega_\alpha \, dx^\alpha = \omega_{\sigma(\alpha)} \, dx^{\sigma(\alpha)}.$$

Second representation:

$$\omega = \sum_{|I|=k} \omega_I \, dx^I$$

where I runs through all increasing multi-indices.

When $\alpha = I$, then $\omega_\alpha = \omega_I$.

Theorem 0.1 (The Poincare lemma; original version)**.** *Let M be a star-shaped submanifold with boundary embedded in $\mathbb{R}^d$. Then every closed form on M is exact.*

Proof. Let $\omega \in \Omega^k(M)$. Represent as

$$\omega = \frac{1}{k!} \sum_{|\alpha|=k} \omega_\alpha \, dx^\alpha$$

using multi-indices α.

Last Update: April 23, 2019.

Define
$$\eta_i := \frac{1}{(k-1)!} \sum_{\substack{i\in\alpha \\ |\alpha|=k}} \omega_\alpha \, dx^\alpha$$
and
$$\eta := \sum_{i=1}^n x^i \eta_i.$$
Then,
$$d\eta = \sum_{i=1}^n dx^i \wedge \eta_i + \sum_{i=1}^n x^i \, d\eta_i.$$
The first term is
$$\begin{aligned}\sum_{i=1}^n dx^i \wedge \eta_i &= \sum_{i=1}^n dx^i \wedge \left[\frac{1}{(k-1)!} \sum_{\substack{i\in\alpha \\ |\alpha|=k}} \omega_\alpha \, dx^\alpha\right] \\ &= \end{aligned}$$
HaHa

The first term is
$$\begin{aligned}\sum_{i=1}^n dx^i \wedge \eta_i &= \sum_{i=1}^n dx^i \wedge \left[\frac{1}{(k-1)!} \sum_{|\alpha|=k-1} \omega_{(i,\alpha)} \, dx^\alpha\right] \\ &= \frac{1}{(k-1)!} \sum_{i=1}^n \sum_{|\alpha|=k-1} \omega_{(i,\alpha)} \, dx^i \wedge dx^\alpha \\ &= \frac{1}{(k-1)!} \sum_{|\alpha|=k} \omega_\alpha \, dx^\alpha \\ &= k\omega.\end{aligned}$$
The second term is
$$\begin{aligned}d\eta_i &= \frac{1}{(k-1)!} \sum_{|\alpha|=k-1} d\omega_{(i,\alpha)} \wedge dx^\alpha \\ &= \frac{1}{k!} \sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x^i} \, dx^\alpha\end{aligned}$$
$$d\omega_{(i,\alpha)} = \sum_{j=1}^n \frac{\partial \omega_{(i,\alpha)}}{\partial x^j} \, dx^j$$
$$\begin{aligned}d\eta_i &= \frac{1}{(k-1)!} \sum_{j=1}^n \sum_{|\alpha|=k-1} \frac{\partial \omega_{(i,\alpha)}}{\partial x^j} \, dx^j \wedge dx^\alpha \\ &= \frac{1}{(k-1)!} \sum_{j,j'=1}^n \sum_{|\alpha|=k-2} \frac{\partial \omega_{(i,j',\alpha)}}{\partial x^j} \, dx^j \wedge dx^{j'} \wedge dx^\alpha \\ &= \end{aligned}$$

Note that we have

$$
\begin{aligned}
d\omega &= \frac{1}{k!}\sum_{|\alpha|=k} d\omega_\alpha \wedge dx^\alpha \\
&= \frac{1}{k!}\sum_{|\alpha|=k}\sum_{i=1}^{n} \frac{\partial \omega_\alpha}{\partial x^i}\, dx^i \wedge dx^\alpha \\
&= \sum_{i=1}^{n} dx^i \wedge \left[\frac{1}{k!}\sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x^i}\, dx^\alpha\right]
\end{aligned}
$$

and

$$
d\omega = \sum_{i=1}^{n} dx^i \wedge \left[\frac{1}{k} d\eta_i\right].
$$

Consider a map $\sum_{i=1}^{n} dx^i \wedge -$.

$$
\begin{aligned}
&= \frac{1}{k!}\sum_{i=1}^{n}\sum_{|\alpha|=k-1} d\omega_{(i,\alpha)} \wedge dx^i \wedge dx^\alpha \\
&= \frac{1}{k!}\sum_{i=1}^{n}\sum_{|\alpha|=k-1}\sum_{j=1}^{n} \frac{\partial \omega_{(i,\alpha)}}{\partial x^j}\, dx^j \wedge dx^i \wedge dx^\alpha \\
&= \frac{1}{k!}\sum_{j=1}^{n} dx^j \wedge \left[\sum_{i=1}^{n}\sum_{|\alpha|=k-1} \frac{\partial \omega_{(i,\alpha)}}{\partial x^j}\, dx^i \wedge dx^\alpha\right] \\
&= \frac{1}{k!}\sum_{j=1}^{n} dx^j \wedge \left[\sum_{|\alpha|=k} \frac{\partial \omega_\alpha}{\partial x^j}\, dx^\alpha\right] \\
&= \frac{1}{k!}\sum_{|\alpha|=k+1}
\end{aligned}
$$

$$
d\omega_{(i,\alpha)} = \sum_{j=1}^{n} \frac{\partial \omega_{(i,\alpha)}}{\partial x^j}\, dx^j
$$

□