Dispersion for the Schrödinger equation

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1. Introduction

In this article, the purpose is on proving a dispersive inequality for solutions of an initial value problem of the Schödinger equation for no potential

$$\begin{cases} i\partial_t u(t,x) + \Delta u(t,x) = 0, \qquad t \in \mathbb{R},\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

The statemaent of our dispersive inequality is given as follows.

Theorem 1.1 (Dispersive estimate)**.** *Let u be a solution of (1). Then,*

$$\|u(t,x)\|_{L^\infty(\mathbb{R}^d)} \le C_d t^{-\frac{d}{2}} \|u(0,x)\|_{L^1(\mathbb{R}^d)}.$$

We can observe the inequality implies that the solution decays as time flows.

2. What is dispersion?

2.1. **Optics.** In optics, the *dispersion* is the phenomenon that the index of refracion depends on the wavelength.

A dispersion relation of light in transparent material is given by the Cauchy formula, an approximate empirical equation,

$$n(\lambda) = B + \frac{C}{\lambda^2}.$$

For exmaples,

2.2. **Quantization.** We can make a PDE form a dispersion relation by quantization. Mathematically, we can simply define a "wave" as a superposition of complex exponential functions on timespace $\mathbb{R}^{1+d}$ that have the form

$$\psi(t,x) = e^{i(k\cdot x - \omega t)},$$

where the parameters $\omega \in \mathbb{R}$ and $k \in \mathbb{R}^d$ are related by dispersion relation. On the space of wave functions, the multiplication operators with respect to ω and k are same with partial differential operators on wave functions:

$$i\partial_t \psi = \omega\psi, \quad -i\partial_{x_i}\psi = k_i\psi \quad (1 \le i \le d).$$

Consider the Schrödinger equation as an example. The energy conservation

$$E = \frac{p^2}{2m} + V$$

and the de Broglie relation

$$E = \hbar\omega, \quad p = \hbar k$$

give the dispersion relation

$$\hbar\omega = \frac{|\hbar k|^2}{2m} + V,$$

which is in fact the Schrödinger equation

$$i\hbar\partial_t = -\frac{\hbar^2}{2m}\Delta + V.$$

2.3. **Dispersive equation.**

Definition 2.1. If the dispersive relation is not of the form $\omega(k) \propto |k|$, the wave is called dispersive.

3. Method I: representation formula

3.1. **Basics on Fourier transform.**

3.2. **Multiplier.**

Definition 3.1. The *time evolution operator* is the multiplier operator associated with e^{-iHt}, and is denoted by $e^{-i\Delta t}$:

$$\widehat{e^{-i\Delta t}}u(x) := e^{-iHt}\widehat{u}(k)$$

Definition 3.2. The *propagator* is inverse transform of e^{-iHt}, and is denoted by $K(x;t)$.

$$\widehat{e^{-i\Delta t}u} = e^{-iHt}\widehat{u} = \widehat{K * u}.$$

3.3. **Fundamental solution.** First,

$$P(D)K(t,x) = 0, \quad t > 0$$

implies

$$\begin{aligned}
i\partial_t K + \Delta_x K &= 0 \\
i\partial_t \widehat{K} - |k|^2\widehat{K} &= 0 \\
\partial_t(\log\widehat{K}) &= -i|k|^2 \\
\widehat{K}(t,k) &= C(k)e^{-i|k|^2 t}.
\end{aligned}$$

And then,

$$K(0,x) = \delta(x), \quad t = 0$$

implies

$$\widehat{K}(t,k) = e^{-i|k|^2 t}.$$

Differentiaing,

$$\begin{aligned}
\nabla_k\widehat{K} &= -i2kt\widehat{K} \\
xK &= -i2t\nabla_x K \\
\nabla_x(\log K) &= i\frac{x}{2t} \\
K(t,x) &= C(t)e^{i\frac{|x|^2}{4t}}.
\end{aligned}$$

Since

$$C(t) = K(t,0) = \int e^{-i|k|^2 t}\, dk = (\pi i t)^{-\frac{d}{2}},$$

we get

$$K(t,x) = (\pi i t)^{-\frac{d}{2}} e^{i\frac{|x|^2}{4t}}.$$

4. Method II: oscillatory integral

4.1. **Reducing problem.** By Fourier transform, we get an ODE

$$i\partial_t \widehat{u}(t,k) + |k|^2 \widehat{u}(t,k) = 0,$$

solved by

$$\widehat{u}(t,k) = \widehat{u}_0(k) e^{-i|k|^2 t}.$$

Taking inverse Fourier transform, the solution of the original equation is given by

$$u(t,x) = \int \widehat{u}_0(k) e^{i(k\cdot x - |k|^2 t)}\, dk.$$

Define phase

$$\Phi(t,x,k) := k\cdot x - |k|^2 t$$

and an oscillatory integral

$$I(t,x) := \int \widehat{u}_0(k) e^{i\Phi}\, dk,$$

which is exactly the same with the general solution.

The statemaent of our dispersive inequality is given as follows.

Theorem 4.1 (Dispersive estimate)**.** *We have an asymptotic inequality*

$$\sup_x |I(t,x)| \lesssim t^{-\frac{d}{2}}$$

for $t \gtrsim 1$.

We can observe the inequality implies that the solution decays as time flows.

To prove this, we are going to make pointwise estimates for I.

To begin with, fix x. Note that the equation (1) has a rotational symmetry on physical space. Thus we assume $x_2 = \cdots = x_d = 0$. Also, suppose the support of $a(\xi)$ is restricted to an annulus $\frac{1}{2} < |\xi| < 2$. This assumption is valid because a simple dyadic decomposition guarantees the generality.

4.2. **Stationary and nonstationary phases.**

Definition 4.1. Let Φ be the phase defined as above. A *stationary* point $\xi^o(t,x)$ is a point at which $\nabla_\xi \Phi$ vanishes.

The word "stationary" is not with respect to time flows, but change of ξ. I think it would be fantastic if here is a 3d image to explain how the principles of stationary and nonstationary phase work.

The idea is to divide the oscillatory integral.

$$I = I_{stat} + I_{nonstat}.$$

We have *height* $\times$ *base* $\sim$ *base* for I_{stat}, and cancellation for $I_{nonstat}$. If one has a nice estimate, then the other must be bigger. Therefore, a natural question comes up with: how can we choose the suitable boundary to get an optimal estimate? Here, what we are going to use is a heuristic technique called "Linearizing phase". In fact, it is not a linearization but a *Taylorization*, but never mind. By the technique, we can expect to read out a suitable estimate: this is exactly $-\frac{d}{2}$, which is mentioned at the statement of our theorem. Proof will begin after getting $-\frac{d}{2}$.

Let us give a toy example to catch the idea.

Example 4.2 (Fresnel type integral)**.** Let

$$I(\lambda) = \int a(\xi)e^{i\lambda\xi^n}\,d\xi$$

and $\Phi(\lambda,\xi) = \lambda\xi^n$. In this problem, λ plays a similar role with t.

4.3. **Heuristic method by linearizing phase.** At first, the stationary point ξ^o is computed as

$$\xi_1^o = \left(\frac{|x_1|}{\alpha t}\right)^{\frac{1}{\alpha-1}}, \qquad \xi_2^o = \cdots = \xi_d^o = 0.$$

Note $|\xi^o| = \xi_1^o$. Although ξ^o depends on t, since the amplitude function $a(\xi)$ is supported on $\frac{1}{2} < |\xi|$, we can let $|\xi^o|$ as a constant asymptotically. Let $\xi' = \xi - \xi^o$ be an auxiliary variable for localization at ξ^o.

Intuitively, if $|\Phi(\xi) - \Phi(\xi^o)|$ is greater than 2π, then ξ belongs to the region of nonstationary. Thus, our plan is to find the region $\{|\xi'| : |\Phi(\xi) - \Phi(\xi^o)| \lesssim 1\}$ since 2π is same with 1. Apply the Taylor expansion. Since

$$\partial_{\xi_i}\partial_{\xi_j}\Phi(\xi) = \alpha|\xi|^{\alpha-2}t(\delta_{ij} + (\alpha-2)\frac{\xi_i}{|\xi|}\frac{\xi_j}{|\xi|},$$

we have

$$\mathrm{Hess}_{\xi^o}[\Phi] = \begin{pmatrix} \alpha(\alpha-1)|\xi^o|^{\alpha-2}t & 0 \\ 0 & \alpha|\xi^o|^{\alpha-2}t\cdot \mathrm{id}_{d-1}\end{pmatrix}.$$

Therefore, by the Taylor expansion,

$$\begin{aligned} 1 &\gtrsim |\Phi(\xi) - \Phi(\xi^o)| \\ &\sim |\operatorname{Hess}_{\xi^o}[\Phi](\xi',\xi')| \\ &\sim \alpha|\xi^o|^{\alpha-2}t[(\alpha-1)|\xi_1'|^2 + |\xi_2'|^2 + \cdots + |\xi_d'|^2] \\ &\sim t|\xi'|^2. \end{aligned}$$

This lets us know the desired boundary $|\xi'| \lesssim t^{-\frac{1}{2}}$: say, $I_{stat} \sim \int \chi_{|\xi'|<t^{-1/2}}(\xi')a(\xi)e^{i\Phi}\,d\xi$.

4.4. **Dyadic decomposition.** This section begins the real proof. The stationary part is a piece of cake: since the area of base is aymptotically d-power of the radius $t^{-\frac{1}{2}}$,

$$|I_{stat} \lesssim 1 \times t^{-\frac{d}{2}} = t^{-\frac{d}{2}}.$$

Now, what we have to do is to show $|I_{nonstat}| \lesssim t^{-\frac{d}{2}}$.

Both the height and the volume of the base are roughly 1 on the nonstationary region $|\xi'|$, we need to do something genius. The height cannot be reduced under 1, so we are going to decompose the supports of the amplitude a: take a partition of unity

$$\chi_{<1}(\xi') = \chi_{<t^{-1/2}}(\xi') + \sum_{\substack{t^{-1/2} < \mu t^{-1/2} \leq 1 \\ \log_2 \mu \in \mathbb{Z}}} \chi_{\mu t^{-1/2}}(\xi'),$$

where $\chi_{<k}(\xi')$ and $\chi_k(\xi')$ are smooth functions supported on $|\xi'| < 2k$ and $\frac{1}{2}k < |\xi'| < 2k$ respectively. If we define

$$I_\mu := \int \chi_{\mu t^{-1/2}}(\xi') a(\xi) e^{i\Phi}\, d\xi,$$

then we have

$$\begin{aligned} I &= \int a(\xi) e^{i\Phi}\, d\xi \\ &= \int \chi_{<1}(\xi') a(\xi) e^{i\Phi}\, d\xi \\ &= \int \chi_{<t^{-1/2}}(\xi') a(\xi) e^{i\Phi}\, d\xi + \sum_{\substack{t^{-1/2} < \mu t^{-1/2} \leq 1 \\ \log_2 \mu \in \mathbb{Z}}} I_\mu \\ &= I_{stat} + I_{nonstat}. \end{aligned}$$

Notice that $|\xi'| \sim \mu t^{-1/2} \Leftrightarrow t^{-1}|\xi'|^{-2} \sim \mu^{-2}$ and the base gets reduced to $(\mu t^{-1/2})^d \sim t^{-\frac{d}{2}}$. This is a reason why the dyadic decomposition is useful. For some reasons that will be seen, dyadic decomposition becomes a powerful tool to estimate an oscillatory integral for polynomial phase. However, even though the sum in $I_{nonstat}$ is finite, the number of μ's that are summed is dependent on t, so we want to get rid of such dependency by summing up with a constant bound, which is impossible as of now since $\mu = 2, 4, 8, 16, \cdots$ and $d > 0$. In this situation, we can compress the size of μ^d by the *repeated integral by parts*. The power of magic number $t^{-\frac{1}{2}}$ arises in this procedure.