Classical differential geometry

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1. Introduction

1.1. **Two ways to represent curves or surfaces.**

1.2. **Coordinates and parametrizations.**

Definition 1.1. A *parametrization* is a smooth map $U \to \mathbb{R}^d$ such that

(1) $U \subset \mathbb{R}^c$ is open and connected,
(2) α is one-to-one,
(3) $d\alpha$ is nondegenerate; $\{\partial_i \alpha\}_{i=1}^c$ is linearly independent.

Definition 1.2. A *regular curve* is a subset of $\mathbb{R}^d$ that is the image of some parametrization $\alpha : I \subset \mathbb{R} \to \mathbb{R}^d$.

Definition 1.3. A *regular surface* is a subset of $\mathbb{R}^d$ that is the image of some parametrization $\alpha : U \subset \mathbb{R}^2 \to \mathbb{R}^d$.

2. Curves in a space

2.1. **Arc-length parameterization.**

Theorem 2.1. *For every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. Suppose we have a parametrization $\beta : I_t \to \mathbb{R}^d$. Define $\tau : I_t \to I_s$ such that

$$\tau(t_0) := \int_0^{t_0} \|\beta'(t)\| \, dt.$$

Then, s is a diffeomorphism. Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta' \left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}. \qquad \square$$

Definition 2.1 (Frenet-Serret frame)**.** Let α be a curve such that $\kappa \neq 0$. Define *tangent unit vector*, *normal unit vector*, *binormal unit vector* by:

$$\mathbf{T} := \frac{\alpha'}{\|\alpha'\|}, \qquad \mathbf{N} := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}, \qquad \mathbf{B} := \mathbf{T} \times \mathbf{N}.$$

Definition 2.2.

$$\kappa := \mathbf{T}' \cdot \mathbf{N}, \quad \tau := -\mathbf{B}' \cdot \mathbf{N}.$$

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Theorem 2.2 (Frenet-Serret formula)**.** *Let α be a unit speed curve.*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Proof. The vectors $\mathbf{T}', \mathbf{B}', \mathbf{N}$ are collinear. □

Theorem 2.3. *Let α be a unit speed curve.*

$$\begin{aligned} \alpha' &= \mathbf{T} \\ \alpha'' &= \kappa \mathbf{N} \\ \alpha''' &= -\kappa^2 \mathbf{T} + \kappa' \mathbf{N} + \kappa\tau \mathbf{B} \end{aligned}$$

Skew-symmetricity is due to the fact the differential of an orthogonal matrix forms a skew symmetric matrix.

Example 2.1. Let α be a curve in $\mathbb{R}^3$. If the normal line always passes through a point, then α is contained in a circle.

Proof. Let α be a unit speed curve. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and $\mathbf{N}$ are parallel so that we have

$$\langle \alpha - p, \mathbf{T} \rangle = 0, \qquad \langle \alpha - p, \mathbf{B} \rangle = 0.$$

Our goal is that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v \rangle = 0$.

$$0 = \langle \alpha - p, \mathbf{T} \rangle' = \langle \alpha', \mathbf{T} \rangle + \langle \alpha - p, \kappa \mathbf{N} \rangle = 1 + \kappa \langle \alpha - p, \mathbf{N} \rangle.$$

$$0 = \langle \alpha - p, \mathbf{B} \rangle' = \langle \alpha - p, -\tau \mathbf{N} \rangle = -\tau \cdot (-\frac{1}{\kappa})$$

$$\begin{aligned} (\|\alpha - p\|^2)' &= \langle \alpha - p, \alpha - p \rangle' \\ &= 2\langle \alpha - p, \alpha' \rangle \\ &= 2\langle \alpha - p, \mathbf{T} \rangle \\ &= 0 \end{aligned}$$

$$\mathbf{B}' = -\tau \mathbf{N} = 0.$$

□

3. Surfaces in a space

$$\nu_x = S(\alpha_x) = \kappa_1 \alpha_x$$

4. Curves on a surface