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FOLDING AND CUTTING «CLOSED RECTIFIABLE CURVE» UNDER CONICAL ORIGAMI

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Abstract.

1. Introduction

2. Definitions of Origami and Cut

In this section, we propose the definitions of words that is necessary to present the problem, that there is a proper planar straight cut drawing the given graph. As a first, we define *piecewise* C^1 for Lipschitz-continuous function, and a kind of rigid map *origami* that is modelling paper folding imbedded in $\mathbb{R}^3$. See [1]. Next, in special case, we define *conical origami* as an origami the image is (general) cone, and present expansion of *cutting* paper into *non-flat origami model*. To refer to usual geometrical approach to origami, see [123]

Definition. *(Piecewise C^1).* Let $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz continuous map that is differntiable for almost everywhere $x \in \Omega$. We call *singular set* of the map f the set of points $\Sigma_f \subset \Omega$ where f is not differentiable. We say that f is *piecewise* C^1 if the following conditions hold:

- Σ_f is closed in Ω;
- f is C^1 on every connected component of $\Omega \backslash \Sigma_f$;
- for every compact set $K \subset \Omega$ the number of connected components of $\Omega \backslash \Sigma_f$ which intersect K is finite.

We define *origami* as a piecewise C^1 map with orthogonal gradient and exclusion to intersect itself. To make more practical physical model of origami, we allow precise overlappings which can be approximated by injective maps, that means it is possible to be tangent to itself but not to transverse. For example, the map $u(x, y) = (|x|, y, 0)$ is not injective but can be obtained as $k \to \infty$ of the injective maps $u_k(x, y) = (|x| \cos \frac{1}{k}, y, x \sin \frac{1}{k})$ which represent actual folding process along time (see [2]).

Definition. *(Conical Origami).* Let $\Omega \subset \mathbb{R}^2$ be a connected set. A Lipschitz continuous map $u : \Omega \to \mathbb{R}^3$ is an *conical origami* if u is piecewise C^1 map, such that the gradient Du is orthogonal 3×2 matrix for all $x \in \Omega \backslash \Sigma_u$, and there exists a sequence of maps $u_k : \Omega \to \mathbb{R}^3$ that are Lipschitz continuous and injective such that $u_k \to u$ in the uniform convergence, and the image of u is a cone, not a plane.

If we exclude the condition which describes the image of u is a cone, then the map u is called just *origami*. See [1].

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In comparison with flat origami model, singular set Σ_u and conical origami are corresponding to *crease pattern* and *single-vertex flat folding* of flat origami respectively. The conical origami is defined in order to treat single-vertex folding which is easier than global flat folding (there are two or more vertices).

Fold-and-cut theorem states that we can find a flat folding of paper, so that one complete straight cut on folding creates any desired plane graph of cuts made up with straight sides. Similarly, but in 3-dimensional space, we ask that there is an origami map such that a certain planar straight cut on folding creates the given curve, especially «*closed curve*», on plane. If there is such an origami, we call the curve *cut by the origami.* For more detail contents apropos of fold-and-cut theorem, see [3].

Definition. Let us define that a «curve» $\gamma : I \to \mathbb{R}^2$ is *cut by origami* $u : \Omega = \mathbb{R}^2 \to \mathbb{R}^3$ if there exists a plane $S \subset \mathbb{R}^3$ such that $S \cap \operatorname{im} u = u(\operatorname{im} \gamma)$.

From now on, we will investigate the properties of «closed curve» that is cut by conical origami, and suggest a concrete illustration of conical origami which have the given curve satisfying condition (1) in Theorem 4.1 be cut.

3. Properties of «Closed Curve» that is Cut by Conical Origami

Let O be a point on plane such that the image of O under a conical origami is the vertex of the cone. We suggest the way to deal with the problem in polar coordinate which has O be the pole, by representing the given closed curve by a polar equation. The following two theorems show the *necessary condition* for the closed curve to be cut by conical origami, related on representing the «closed curve» as a simple polar equation.

Theorem 3.1. *Let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be a conical origami and O be a point on $\mathbb{R}^2$ such that $u(O)$ is vertex of the cone. If a closed curve $\gamma : S^1 \to \mathbb{R}^2$ is cut by u, then arbitrary half line starting at O meets γ at one point that is not O.*

Proof. □

Theorem 3.2. *Let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be a conical origami and O be a point on $\mathbb{R}^2$ such that $u(O)$ is vertex of the cone. If a closed curve $\gamma : S^1 \to \mathbb{R}^2$ is cut by u and there exists a point γ is differentiable, then arbitrary half line starting at O is not tangent to γ.*

Proof. □

As the «curve» gratifying two conditions right above, we give a definition of *cut graph* that is the curve that we have to investigate whether it can be cut by conical origami. The words was borrowed from [3].

Definition. A «Lipschitz continuous simple closed curve» $\gamma : S^1 \to \mathbb{R}^2$ and a point O in the interior of γ are called *cut graph* and *skeleton vertex of* γ respectively if arbitrary half line starting at O meets γ at one point but is not tangent to γ.

Now, we can deal with the closed curve by putting in polar coordinate which has O be the pole. Following corollary defines this polar coordinate system.

Corollary 3.3. *If a «Lipschitz continuous simple closed curve» $\gamma : S^1 \to \mathbb{R}^2$ is cut graph and a point O in the interior of γ is skeleton vertex of γ, then there exists a positive real valued function $r : [0, 2\pi] \to (0, \infty)$ such that polar eqation $\rho = r(\psi)$*

represents the curve γ, in polar coordinate system (ρ, ψ) such that O is the pole and a certain half line starting at O is polar axis.

The following theorem is our third necessary condition for the closed curve to be cut by conical origami.

Theorem 3.4. *Let r be a piecewise C^1 function defined as same manner with Corollary 3.3 for a cut graph γ. If a cut graph γ is cut by conical origami, then the function r is piecewise C^1.*

Proof. □

4

We have inquired some necessary conditions in Section 3. For the curve γ satisfying the conditions, we directly propose conical origami by which γ be cut in Theorem 4.1.

Definition. Let r be a piecewise C^1 function defined as same manner with Corollary 3.3 for a cut graph γ, and L_r defined as an open interval such that:

$$L_r := \left(0 \, , \inf_{\psi \in [0,2\pi] \setminus \Sigma_r} \frac{r(\psi)^2}{\sqrt{r(\psi)^2 + r'(\psi)^2}} \right).$$

For the function r, a function $A_z : [0, 2\pi] \to \mathbb{R}$ is defined such that:

$$A_z(\psi) := \int_{[0,\psi] \setminus \Sigma_r} \left(1 + \frac{z^2}{r(\theta)^2 - z^2} \left(1 - \frac{r'(\theta)^2}{r(\theta)^2 - z^2} \right) \right)^{\frac{1}{2}} d\theta$$

for $z \in L_r$, where r' is derivative of the function r and Σ_r is singular set.

Theorem 4.1. *Consider the polar coordinate system (ρ, ψ) and let r be a piecewise C^1 function defined as same manner with Corollary 3.3. for a cut graph γ satisfying:*

$$\sup_z A_z(2\pi) \geq 2\pi. \tag{1}$$

If we define a map $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ such that:

$$\varphi(\rho, \psi)_{polar} = \left(\rho \sqrt{1 - \frac{z^2}{r(\psi)^2}} \, , \int_0^\psi (1 - 2\chi_\kappa(\theta)) \frac{\partial A_z(\theta)}{\partial \theta} \, d\theta \, , \, z \left(1 - \frac{\rho}{r(\psi)} \right) \right)_{cylindrical}$$

for fixed $z \in L_r$ and interval $\kappa \subset [0, 2\pi]$, then there exists a pair (κ, z) such that φ is origami.

Moreover, if φ is origami, then φ is conical origami and γ is cut by φ.

In the Theorem 4.1, the function χ_κ is *indicator function* such that:

$$\chi_\kappa(\psi) = \begin{cases} 1 & , \psi \in \kappa \\ 0 & , \text{otherwise.} \end{cases}$$

If we put any point on γ in φ, then the z-coordinate in cylindrical coordinate becomes 0, on the other hand, the z-coordinate is 0 implies that $\rho = r(\psi)$. So we get $S \cap \operatorname{im} \varphi = \varphi(\operatorname{im} \gamma)$ where we let S be plane $z = 0$, it is trivial that γ is cut by φ if φ is origami.

Proof of Theorem 4.1 is obtained by theorems from 4.2 to 4.5 that the gradient $D\varphi$ is orthogonal by Theorem 4.2, there exists a sequence of maps φ_k that are injective such that $\varphi_k \to \varphi$ in uniform convergence by Theorem 4.3, and φ_k is Lipscitz continuous and piecewise C^1 by Theorem 4.4, at last, we can prove that φ is conical origami through Theorem 4.5.

Precisely, we show the metric tensor is preserved by φ in Theorem 4.2, and present the way how we set an interval κ which can makes a sequence of injective maps φ_k that uniformly converges to φ in Theorem 4.3. And then using the consequence of Theorem 4.3, we show it is possible to take z and κ simultaneously with keeping Lipschitz continuity of φ. Recall that since the function r is piecewise C^1, there exists an interval in which r is increasing or decreasing monotonically. It says there is no problem to take κ in Theorem 4.3 to prove Theorem 4.1.

Theorem 4.2. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 4.1. The map φ is local isometric immersion, i.e. the gradient $D\varphi$ is orthognal.*

Proof. □

Theorem 4.3. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 4.1, and $[a, b] \subset [0, 2\pi]$ be an interval such that the function r is increasing or decreasing monotonically for $\psi \in [a, b]$. Take $\kappa = [\alpha, \beta]$ where*

$$A_z(\alpha) = \frac{5}{6}A_z(a) + \frac{1}{6}A_z(b)\,,\ A_z(\beta) = \frac{2}{3}A_z(a) + \frac{1}{3}A_z(b)$$

for arbitrary $z \in L_r$. Then there exists a sequence of maps $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^3$ that are injective such that $\varphi_k \to \varphi$ in uniform convergence.

Proof. □

Theorem 4.4. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 4.1, and $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by replacing r to r_k defined such that:*

$$r_k(\psi) = r(\psi) - \frac{1}{k}\chi_{[a,b]}(\psi)\left(2\chi_{[0,\infty)}(r(b) - r(a)) - 1\right)(\psi - a)(\psi - b)$$

for a positive integer k and real numbers $a, b \in [0, 2\pi]$ satisfying $a < b$. For all k, there exists a pair (κ, z) such that the map φ_k is Lipschitz continuous and r is increasing or decreasing monotonically for $\psi \in [a, b]$ where $\kappa = [\alpha, \beta]$ and

$$A_z(a) = 2A_z(\alpha) - A_z(\beta)\,,\ A_z(b) = 5A_z(\beta) - 4A_z(\alpha).$$

Moreover, if φ_k is Lipschitz continuous, then it is also piecewise C^1.

Proof. □

Theorem 4.5. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 4.1. For a certain pair (κ, z), if φ is origami , then φ is also conical origami such that $\varphi(O)$ is vertex of the cone where O is skeleton vertex.*

Proof. □

Proof of Theorem 4.1. □

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Theorem. If X is an n-dimensional non-singular projective algebraic variety with ample tangent vector bundle defined over an algebraically closed field k, then X is (algebraically) isomorphic to $\mathbf{P}^n$ over k.

References

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