Classical differential geometry

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1. Coordinates and parametrizations

Definition 1.1. An *m-dimensional parametrization* is a smooth map $U \to \mathbb{R}^n$ such that

(1) $U \subset \mathbb{R}^m$ is open and connected,
(2) α is one-to-one (optional),
(3) $d\alpha$ is nondegenerate; $\{\partial_i \alpha\}_{i=1}^m$ is linearly independent.

The third condition is important; in language of manifolds, the third condition defines what we call *immersed submanifolds*. We will see that the second condition is not important at all.

Definition 1.2. A *regular curve* is a subset of $\mathbb{R}^n$ that is the image of a one-dimensional parametrization.

Definition 1.3. A *regular surface* is a subset of $\mathbb{R}^n$ that is the image of a two-dimensional parametrization.

2. Curves in a space

Theorem 2.1. *For every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. Suppose we have a parametrization $\beta : I_t \to \mathbb{R}^d$. Define $\tau : I_t \to I_s$ such that

$$\tau(t_0) := \int_0^{t_0} \|\beta'(t)\| \, dt.$$

Then, s is a diffeomorphism. Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta' \left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}.$$ □

2.1. Frenet-Serret theory.

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2.1.1. *Theory.*

Definition 2.1. We say a curve parametrized as $\alpha : I \to \mathbb{R}^3$ is *degenerate* if the normalized tangent vector $\alpha'/\|\alpha'\|$ is never locally constant everywhere. In other words, α is nowhere straight.

Definition 2.2 (Frenet-Serret frame)**.** Let α be a nondegenerate curve. We define *tangent unit vector*, *normal unit vector*, *binormal unit vector* by:

$$\mathbf{T}(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad \mathbf{N}(t) := \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}, \qquad \mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t),$$

and *curvature* and *torsion* by:

$$\kappa(t) := \langle \mathbf{T}'(t), \mathbf{N}(t) \rangle, \quad \tau(t) := -\langle \mathbf{B}'(t), \mathbf{N}(t) \rangle.$$

Note that κ cannot vanish by definition.

Theorem 2.2 (Frenet-Serret formula)**.** *Let α be a unit speed nondegenerate curve.*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Proof. Note that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an orthonormal basis.

Step 1: Show that $\mathbf{T}', \mathbf{B}', \mathbf{N}$ are parallel. Two vectors $\mathbf{T}'$ and $\mathbf{N}$ are parallel by definition. Since $\langle \mathbf{T}, \mathbf{B} \rangle = 0$ and $\langle \mathbf{B}, \mathbf{B} \rangle = 1$ are constant, we have

$$\langle \mathbf{B}', \mathbf{T} \rangle = \langle \mathbf{B}, \mathbf{T} \rangle' - \langle \mathbf{B}, \mathbf{T}' \rangle = 0, \qquad \langle \mathbf{B}', \mathbf{B} \rangle = \tfrac{1}{2} \langle \mathbf{B}, \mathbf{B} \rangle' = 0,$$

which show $\mathbf{B}'$ and $\mathbf{N}$ are parallel. By the definition of κ and τ, we have

$$\mathbf{T}' = \kappa \mathbf{N}, \qquad \mathbf{B}' = -\tau \mathbf{B}.$$

Step 2: Describe $\mathbf{N}'$. Since

$$\begin{aligned} \langle \mathbf{N}', \mathbf{T} \rangle &= -\langle \mathbf{N}, \mathbf{T}' \rangle = -\kappa, \\ \langle \mathbf{N}', \mathbf{N} \rangle &= \tfrac{1}{2} \langle \mathbf{N}, \mathbf{N} \rangle' = 0, \\ \langle \mathbf{N}', \mathbf{B} \rangle &= -\langle \mathbf{N}, \mathbf{B}' \rangle = \tau, \end{aligned}$$

we have

$$\mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B}.$$ □

Remark. Skew-symmetricity in the Frenet-Serret formula is not by chance. Let $\mathbf{X}(t)$ be the curve of orthogonal matrices $(\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t))^T$. Then, the Frenet-Serret formula reads

$$\mathbf{X}'(t) = A(t)\mathbf{X}(t)$$

for a matrix curve $A(t)$. Since $\mathbf{X}(t+h) = R_t(h)\mathbf{X}(t)$ for a family of orthogonal matrices $\{R_t(h)\}_h$ with $R_t(0) = I$, we can describe $A(t)$ as

$$A(t) = \left. \frac{dR_t}{dh} \right|_{h=0}.$$

By differentiating the relation $R_t^T(h)R_t(h) = I$ with respect to h, we get to know that $A(t)$ is skew-symmetric for all t. In other words, the tangent space $T_I\mathrm{SO}(3)$ forms a skew symmetric matrix.

Proposition 2.3. *Let α be a nondegenerate space curve.*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}.$$

Proof. If we let $s = \|\alpha'\|$, then

$$\begin{aligned} \alpha' &= s\mathbf{T}, \\ \alpha'' &= s'\mathbf{T} + s^2\kappa\mathbf{N}, \\ \alpha''' &= (s'' - s^3\kappa^2)\mathbf{T} + (3ss'\kappa + s^2\kappa')\mathbf{N} + (s^3\kappa\tau)\mathbf{B}. \end{aligned}$$

Now the formulas are easily derived. $\square$

2.1.2. *Problems.* Let α be a nondegenerate unit speed space curve, and let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet-Serret frame for α. Consider a diagram as follows:

$$\begin{array}{ccccc} \langle \alpha, \mathbf{T} \rangle = ? & \longleftrightarrow & \langle \alpha, \mathbf{N} \rangle = ? & \longleftrightarrow & \langle \alpha, \mathbf{B} \rangle = ? \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ \langle \alpha', \mathbf{T} \rangle = 1 & & \langle \alpha', \mathbf{N} \rangle = 0 & & \langle \alpha', \mathbf{B} \rangle = 0. \end{array}$$

Here the arrows indicate which information we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t), \mathbf{T}(t) \rangle = f(t),$$

then we can obtain by differentiating it

$$\langle \alpha(t), \mathbf{N}(t) \rangle = \frac{f'(t) - 1}{\kappa(t)}$$

since we have known $\langle \alpha', \mathbf{T} \rangle$ but not $\langle \alpha, \mathbf{N} \rangle$, and further

$$\langle \alpha(t), \mathbf{B}(t) \rangle = \frac{\left(\frac{f'(t)-1}{\kappa(t)} \right)' + \kappa(t) f(t)}{\tau(t)}$$

since we have known $\langle \alpha, \mathbf{T} \rangle$ and $\langle \alpha', \mathbf{N} \rangle$ but not $\langle \alpha, \mathbf{B} \rangle$. Thus, $\langle \alpha, \mathbf{T} \rangle = f$ implies

$$\alpha(t) = f(t) \cdot \mathbf{T} + \frac{f'(t) - 1}{\kappa(t)} \cdot \mathbf{N} + \frac{\left(\frac{f'(t)-1}{\kappa(t)} \right)' + \kappa(t) f(t)}{\tau(t)} \cdot \mathbf{B}.$$

Suggested a strategy for space curve problems:

- Formulate the assumptions of the problem as the form
  $$\langle \text{ (interesting vector)},\ \text{(Frenet-Serret basis) } \rangle \ = \ \text{(some function)}.$$
- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of κ and τ by comparing with assumptions.
- Heuristically find a constant vector and show what you want directly.

Example 2.1 (Plane curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α lies on a plane,
(2) $\tau = 0$,
(3) the osculating plane constains a fixed point.

Example 2.2 (Helices)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α is a helix,
(2) $\tau/\kappa = \text{const}$,
(3) normal lines are parallel to a plane.

Example 2.3 (Sphere curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α lies on a sphere,
(2) $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$,
(3) $\tau/\kappa = (\kappa'/\tau\kappa^2)'$,
(4) normal planes contain a fixed point.

Example 2.4 (Bertrand mates)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α has a Bertrand mate,
(2) there are two constants $\lambda \neq 0, \mu$ such that $1/\lambda = \kappa + \mu\tau$.

2.2. **Example problems.** Here we give an example solution of several problems.

Example 2.5. A space curve whose normal lines always pass through a fixed point lies in a circle.

Proof. Step 1: Formulate conditions. Reparametrize α to become a unit speed curve. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and $\mathbf{N}$ are parallel so that we have

$$\langle \alpha - p, \mathbf{T} \rangle = 0, \qquad \langle \alpha - p, \mathbf{B} \rangle = 0.$$

Our goal is to show that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v \rangle = 0$.

Step 2: Collect information. Differentiate $\langle \alpha - p, \mathbf{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathbf{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathbf{B} \rangle = 0$ to get

$$\tau = 0.$$

Step 3: Complete proof. We can deduce that $\|\alpha - p\|$ is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p \rangle' = 2\langle \alpha - p, \mathbf{T} \rangle = 0.$$

Also, if we heuristically define a vector $v := \mathbf{B}$, then v is constant since

$$v' = -\tau\mathbf{N} = 0,$$

and clearly $\langle \alpha - p, v \rangle = 0$ □

Example 2.6. A sphere curve of constant curvature lies in a circle.

Example 2.7. A curve is a circular helix iff it has more than one Bertrand mates.

3. Surfaces in a space

$$\nu_x = S(\alpha_x) = \kappa_1 \alpha_x$$

4. Curves on a surface