Classical differential geometry

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1. Introduction

1.1. Two ways to represent curves or surfaces.

1.2. Coordinates and parametrizations.

Definition 1.1. An *m-dimensional parametrization* is a smooth map $U \to \mathbb{R}^n$ such that

(1) $U \subset \mathbb{R}^m$ is open and connected,
(2) α is one-to-one (optional),
(3) $d\alpha$ is nondegenerate; $\{\partial_i \alpha\}_{i=1}^m$ is linearly independent.

The third condition is important; in language of manifolds, the third condition defines what we call *immersed submanifolds*. We will see that the second condition is not important at all.

Definition 1.2. A *regular curve* is a subset of $\mathbb{R}^n$ that is the image of a one-dimensional parametrization.

Definition 1.3. A *regular surface* is a subset of $\mathbb{R}^n$ that is the image of a two-dimensional parametrization.

2. Curves in a space

2.1. Arc-length parameterization.

Theorem 2.1. *For every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. Suppose we have a parametrization $\beta : I_t \to \mathbb{R}^d$. Define $\tau : I_t \to I_s$ such that

$$\tau(t_0) := \int_0^{t_0} \|\beta'(t)\| \, dt.$$

Then, s is a diffeomorphism. Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta' \left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}.$$ □

Definition 2.1 (Frenet-Serret frame)**.** Let α be a nondegenerate curve. We define *tangent unit vector*, *normal unit vector*, *binormal unit vector* by:

$$\mathbf{T} := \frac{\alpha'}{\|\alpha'\|}, \qquad \mathbf{N} := \frac{\mathbf{T}'}{\|\mathbf{T}'\|}, \qquad \mathbf{B} := \mathbf{T} \times \mathbf{N},$$

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and *curvature* and *torsion* by:

$$\kappa := \langle \mathbf{T}', \mathbf{N} \rangle, \quad \tau := -\langle \mathbf{B}', \mathbf{N} \rangle.$$

Note that κ cannot vanish by definition.

Theorem 2.2 (Frenet-Serret formula)**.** *Let α be a unit speed curve.*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Proof. Step 1: $\mathbf{T}', \mathbf{B}', \mathbf{N}$ are parallel. Two vectors $\mathbf{T}'$ and $\mathbf{N}$ are parallel by definition. Since $\langle \mathbf{T}, \mathbf{B} \rangle = 0$ and $\langle \mathbf{B}, \mathbf{B} \rangle = 1$ are constant, we have

$$\langle \mathbf{B}', \mathbf{T} \rangle = \langle \mathbf{B}, \mathbf{T} \rangle' - \langle \mathbf{B}, \mathbf{T}' \rangle = 0, \qquad \langle \mathbf{B}', \mathbf{B} \rangle = \tfrac{1}{2} \langle \mathbf{B}, \mathbf{B} \rangle' = 0,$$

which show $\mathbf{B}'$ and $\mathbf{N}$ are parallel. □

Theorem 2.3. *Let α be a unit speed curve.*

$$\begin{aligned} \alpha' &= \mathbf{T}, \\ \alpha'' &= \kappa \mathbf{N}, \\ \alpha''' &= -\kappa^2 \mathbf{T} + \kappa' \mathbf{N} + \kappa\tau \mathbf{B}. \end{aligned}$$

Skew-symmetricity is due to the fact the differential of an orthogonal matrix forms a skew symmetric matrix.

- Aim for finding the coefficients of a special vector and its derivatives with respect to the Frenet-Serret frame.
- In particular, differentiate equations of the form

  $$\langle v^{(k)}, \mathbf{T} \text{ or } \mathbf{N} \text{ or } \mathbf{B} \rangle = \text{const}$$

  to get useful information.
- Heuristically find represent a vector and show what you want directly.

Example 2.1. Let α be a curve in $\mathbb{R}^3$. If the normal line of α always passes through a fixed point, then α is contained in a circle.

Proof. Step 1: Formulate conditions. Reparametrize α to become a unit speed curve. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and $\mathbf{N}$ are parallel so that we have

$$\langle \alpha - p, \mathbf{T} \rangle = 0, \qquad \langle \alpha - p, \mathbf{B} \rangle = 0.$$

Our goal is to show that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v \rangle = 0$.

Step 2: Collect information. Differentiate $\langle \alpha - p, \mathbf{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathbf{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathbf{B} \rangle = 0$ to get

$$\tau = 0.$$

Step 3: Complete proof. We can deduce that $\|\alpha - p\|$ is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p\rangle' = 2\langle \alpha - p, \mathbf{T}\rangle = 0.$$

Also, if we let $v = \mathbf{B}$, then $\mathbf{B}$ is constant since

$$v = -\tau\mathbf{N} = 0,$$

and $\langle \alpha - p, v\rangle = 0$ □

Example 2.2 (Plane curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α lies on a plane,
(2) $\tau = 0$,
(3) the osculating plane constains a fixed point.

Example 2.3 (Helices)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α is a helix,
(2) $\tau/\kappa = \text{const}$,
(3) normal lines are parallel to a plane.

Example 2.4 (Sphere curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α lies on a sphere,
(2) $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$,
(3) $\tau/\kappa = (\kappa'/\tau\kappa^2)'$,
(4) normal planes contain a fixed point.

* A sphere curve of constant curvature lies in a circle.

Example 2.5 (Bertrand mates)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α has a Bertrand mate,
(2) there are two constants $\lambda \neq 0, \mu$ such that $1/\lambda = \kappa + \mu\tau$.

* A curve is a circular helix iff it has more than one Bertrand mates.

3. Surfaces in a space

$$\nu_x = S(\alpha_x) = \kappa_1 \alpha_x$$

4. Curves on a surface