VLASOV-POISSON SYSTEM

IKHAN CHOI

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1. Vlasov-Poisson system

Consider a Cauchy problem of the *Valsov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, \qquad (t,x,v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t,x) = -\nabla_x \Phi, \\ \Phi(t,x) = (-\Delta_x)^{-1} \rho, \\ \rho(t,x) = \int f\, dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $\gamma = \pm 1$. For example, we have *repulsive problem* $\gamma = +1$ for electrons in plasma and *attractive problems* $\gamma = -1$ for galaxies. (ρ denotes mass density.)

1.1. Representation of electric field.

For the boundaryless problem in which the potential function vanishes at infinity, we have

$$\Phi = \tfrac{1}{4\pi|x|} * \rho,$$

$$E = -\nabla_x \Phi = -\nabla_x(\tfrac{1}{4\pi|x|} * \rho) = \frac{x}{4\pi|x|^3} * \rho.$$

$$E(t,x) = \frac{1}{4\pi}\int \frac{(x-y)f(t,y,v)}{|x-y|^3}\,dy.$$

1.2. **Volume preservation.**

1.3. **Conservative laws.** The following results cannot be used in any local existence proof because they assume the existence of solutions, but they help understand the fundamental nature of solutions of the Vlasov-Poisson system and are used in the proof of global existence.

Lemma 1.1. *Let* $f \in C_c^1([0,T] \times \mathbb{R}^6)$.

(1) *(Continuity equation)*

$$\rho_t + \nabla_x \cdot j = 0, \qquad \textit{where} \quad j = \int vf\,dv.$$

(2) *(Energy conservation)*

$$\iint |v|^2 f\,dv\,dx + \gamma \int |E|^2\,dx = \mathrm{const}\,.$$

Proof.

(1) Integrate with respect to v to get

$$\begin{aligned} 0 &= \int f_t\,dv + \int v \cdot \nabla_x f\,dv \\ &= \rho_t + \nabla_x \cdot \int vf\,dv \\ &= \rho_t + \nabla_x \cdot j. \end{aligned}$$

(2) Multiply $|v|^2$ and integrate with respect to v and x to get

$$\begin{aligned} \frac{d}{dt}\iint |v|^2 f\,dv\,dx &= \iint |v|^2 f_t\,dv\,dx = -\iint |v|^2 \gamma E \cdot \nabla_v f\,dv\,dx \\ &= \iint 2v \cdot \gamma E f\,dv\,dx = -2\gamma \int \nabla_x \Phi \cdot j\,dx \\ &= 2\gamma \int \Phi \nabla_x \cdot j\,dx = 2\gamma \int \Phi \Delta_x \Phi_t\,dx \\ &= -\frac{d}{dt}\gamma \int |E|^2\,dx. \end{aligned}$$

Thus

$$\iint |v|^2 f\,dv\,dx + \gamma \int |E|^2\,dx = \mathrm{const}\,. \qquad \square$$

1.4. **Moment propagation.** We have a bound of kinetic energy even for $\gamma = -1$.

Lemma 1.2 ($L^{5/3}$ estimate of ρ)**.**

(1) $\|\rho(t)\|_{L_x^{5/3}} \lesssim \iint |v|^2 f\, dv\, dx$.
(2) $\iint |v|^2 f\, dv\, dx \lesssim 1$.

Proof.

(1) Note

$$\rho(t,x) = \int f(t,x,v)\, dv \leq \int_{|v|<R} f\, dv + \frac{1}{R^2}\int_{|v|\geq R} |v|^2 f\, dv$$
$$\lesssim R^3 + R^{-2}\int |v|^2 f\, dv.$$

Set $R^3 = R^{-2}\int |v|^2 f\, dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f\, dv.$$

(2) By the Hardy-Littlewood-Sobolev inequality,

$$\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$$

for $p = 2$, $d = 3$, and $\alpha = 1$ implies $q = 6/5$, hence the bound of $\|E(t)\|_2$

$$\|E(t)\|_2 \simeq \|\frac{1}{|x|^{d-\alpha}} *_x \rho(t,x)\|_{L_x^2} \lesssim \|\rho(t)\|_{6/5}.$$

So, interpolation with Hölder's inequality gives

$$\|E(t)\|_2 \lesssim \|\rho\|_1^{7/12}\|\rho\|_{5/3}^{5/12} \simeq \|\rho\|_{5/3}^{5/12}.$$

Thus (1) gives

$$\iint |v|^2 f\, dv\, dx = C + \|E(t)\|_2^2 \lesssim C + (\iint |v|^2 f\, dv\, dx)^{1/2},$$

so the kinetic energy $\iint f\, dv\, dx$ is bounded. As a corollary, $\|\rho\|_{5/3}$ is also bounded. □

Lemma 1.3 (L^∞-estimate of E)**.** *For* $0 \leq \frac{1}{p} < \frac{\alpha}{d} < \frac{1}{q} \leq 1$,

$$\|\frac{1}{|x|^{d-\alpha}} * \rho(t,x)\|_{L_x^\infty}^{\frac{1}{q}-\frac{1}{p}} \lesssim \|\rho(t)\|_p^{\frac{1}{q}-\frac{\alpha}{d}}\|\rho(t)\|_q^{\frac{\alpha}{d}-\frac{1}{p}}.$$

Proof. Fix time t. For $(d-\alpha)p < d < (d-\alpha)q$,

$$\begin{aligned}
|\frac{1}{|x|^{d-\alpha}} * \rho(t,x)| &\le \int_{|x-y|<R} \frac{\rho(t,y)}{|x-y|^{d-\alpha}}\,dy + \int_{|x-y|\ge R} \frac{\rho(t,y)}{|x-y|^{d-\alpha}}\,dy \\
&\le \|\rho\|_{p'} (\int_{|y|<R} \frac{dy}{|y|^{(d-\alpha)p}})^{1/p} + \|\rho\|_{q'} (\int_{|y|\ge R} \frac{dy}{|y|^{(d-\alpha)q}})^{1/q} \\
&\simeq \|\rho\|_{p'} (\int_0^R r^{d-1-(d-\alpha)p}\,dr)^{1/p} + \|\rho\|_{q'} (\int_R^\infty r^{d-1-(d-\alpha)q}\,dr)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} + \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}.
\end{aligned}$$

By choosing R such that $\|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} = \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}$, we get

$$\|E(t)\|_\infty \lesssim \|\rho(t,x)\|_{p'}^{\frac{1-\frac{\alpha}{d}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho(t,x)\|_{q'}^{\frac{\frac{1}{p}-1+\frac{\alpha}{d}}{\frac{1}{p}-\frac{1}{q}}},$$

hence the inequality by interchaning p and q with their conjugates. □

2. Local existence

2.1. **Approximate solution.**

Suppose $f_0 \in C_c^1$. With initial term f_0, inductively define f_{n+1} by letting the electric field E be known:

$$\begin{cases}
\partial_t f_{n+1} + v\cdot\nabla_x f_{n+1} + \gamma E_n \cdot \nabla_v f_{n+1} = 0, \\
E_n(t,x) = -\nabla_x \Phi_n, \\
\Phi_n(t,x) = (-\Delta_x)^{-1}\rho_n, \\
\rho_n(t,x) = \int f_n\,dv, \\
f_{n+1}(0,x,v) = f_0(x,v).
\end{cases}$$

Characteristics.

Then, the equation becomes a linear PDE for f_{n+1}.

Definition 2.1.

$$Q_n(t) = \sup\{\, |v| : f(s,x,v) \ne 0,\ s\in[0,t]\,\}$$

2.2. **A priori estimates of ρ_n and E_n.**

Lemma 2.1.

(1) *For all $t\in[0,\infty]$, $\|\rho_n(t)\|_\infty \lesssim Q_n^3(t)$ and $\|E_n(t)\|_\infty \lesssim Q_n^2(t)$*
(2) *For sufficiently small $t < c(f_0)$, $Q_n(t) \lesssim 1$ indendent on n.*

Proof.

(1) Since

$$\|\rho_n(t)\|_\infty \le Q^3(t)\|f_0\|_\infty \lesssim Q_n^3(t),$$

a rough estimate for $\|E\|_\infty$ gives

$$\|E_n(t)\|_\infty \le \|\rho_n(t)\|_\infty^{2/3} \|\rho_n(t)\|_1^{1/3} \lesssim Q_n^2(t)$$

(2) Let $c = c(f_0)$ be a constant such that $\|E_n(t)\| \le cQ^2(t)$. We claim that

$$Q_n(t) \le \frac{Q_0}{1 - cQ_0 t}$$

for all n. Easily checked for $n = 0$; $Q_0(t) = Q_0 \le \frac{Q_0}{1-cQ_0 t}$.

Assume $Q_n(t) \le \frac{Q_0}{1-cQ_0 t}$. Then,

$$\begin{aligned} |V_{n+1}(s;0,x,v)| &\le |v| + \int_0^t |E_n(s;0,x,v)|\,ds \\ &\le Q_0 + c\int_0^t Q_n^2(s)\,ds \end{aligned}$$

implies

$$\begin{aligned} Q_{n+1}(t) &\le Q_0 + c\int_0^t Q_n^2(s)\,ds \\ &\le Q_0 + c\int_0^t \left(\frac{Q_0}{1-cQ_0 s}\right)^2 ds = \frac{Q_0}{1-cQ_0 t}. \end{aligned}$$

By induction, $Q_n(t) \lesssim 1$ for all n and $t \in [0,T]$, where $T < (cQ_0)^{-1}$. □

2.3. A priori estimates of $\nabla_x \rho_n$ and $\nabla_x E_n$.

2.4. Convergence of f_n.

(1)

$$\|f_{n+1}(t) - f_n(t)\|_\infty \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds.$$

(2)

$$\|f_{n+1}(t) - f_n(t)\|_\infty \lesssim \int_0^t \|f_n(s) - f_{n-1}(s)\|_\infty^{1/3}\,ds.$$

Therefore, f_n converges uniformly.

Proof.

(1) Take T such that

$$T < \min\{\|\nabla_x E_n(t)\|_\infty^{-1}, 1\} \quad \text{and} \quad Q_n(t) \lesssim 1$$

for all n and $t < T$. This can be done thanks for the local a priori bound for $\nabla_x E_n$.

The C^1 regularity of f_0 lets

$$\begin{aligned} &|f_{n+1}(t,x,v) - f_n(t,x,v)| \\ &\quad = |f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)) - f_0(X_n(0;t,x,v), V_n(0;t,x,v))| \\ &\quad \lesssim |X_{n+1}(0;t,x,v) - X_n(0;t,x,v)| + |V_{n+1}(0;t,x,v) - V_n(0;t,x,v)|. \end{aligned}$$

Because

$$X_n(s;t,x,v) = x - \int_s^t V_n(s';t,x,v)\,ds',$$
$$V_n(s;t,x,v) = v - \int_s^t E_{n-1}(s', X_n(s;t,x,v))\,ds',$$

we have

$$\begin{aligned}
&|V_{n+1}(s;t,x,v) - V_n(s;t,x,v)| \\
&\quad\le \int_0^t |E_n(s, X_{n+1}(s;t,x,v)) - E_{n-1}(s, X_n(s;t,x,v))|\,ds \\
&\quad\le \int_0^t |E_n(s, X_{n+1}) - E_n(s, X_n)| + |E_n(s, X_n) - E_{n-1}(s, X_n)|\,ds \\
&\quad\le T \sup_{s\in[0,t]} \|\nabla_x E_n(s)\|_\infty |X_{n+1}(s) - X_n(s)| + \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds
\end{aligned}$$

and

$$\begin{aligned}
|X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| &\le \int_0^t |V_{n+1}(s;t,x,v) - V_n(s;t,x,v)|\,ds \\
&\le T \sup_{s\in[0,t]} |V_{n+1}(s) - V_n(s)|.
\end{aligned}$$

Since T is taken such that

$$T < \min\{\|\nabla E_n(t)\|_\infty^{-1}, 1\},$$

we get by combining the above two inequality,

$$\begin{aligned}
&\sup_{s\in[0,t]} |X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| + |V_{n+1}(s;t,x,v) - V_n(s;t,x,v)| \\
&\qquad\lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds.
\end{aligned}$$

(2) Notice that

$$\|E_n(t) - E_{n-1}(t)\|_\infty \lesssim \|\rho_n(t) - \rho_{n-1}(t)\|_1^{1/3} \|\rho_n(t) - \rho_{n-1}(t)\|_\infty^{2/3}.$$

For L^∞-norm,

$$\|\rho_n(t) - \rho_{n-1}(t)\| \lesssim Q_n^3(t) + Q_{n-1}^3(t) \lesssim 1.$$

For L^1-norm, since the support of f_n, f_{n-1} is bounded in both directions x, v,

$$\|\rho_n(t) - \rho_{n-1}(t)\|_1 \le \|f_n(t) - f_{n-1}(t)\|_1 \lesssim \|f_n(t) - f_{n-1}(t)\|_\infty.$$

□

2.5. **Uniqueness.**

3. Global existence

3.1. Prolongation criterion.

Theorem (Schaeffer, 1991)**.** *Let $f_0 \in C^1_{c,x,v}$ and $f_0 \geq 0$. Then, the Cauchy problem for the VP system has a unique C^1 global solution.*

Definition 3.1. For a local solution f,

$$Q(t) := 1 + \sup\{|v| : f(s,x,v) \neq 0 \text{ for some } s \in [0,t],\, x \in \mathbb{R}^3_x\}.$$

Decompose $[t-\Delta, t] \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ as

$$\begin{aligned} U &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq r \right\}, \\ B &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |v| \geq P \right\} \setminus U, \\ G &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| < P \quad \text{or} \quad |v| < P \right\}. \end{aligned}$$

(We can let $U \mapsto U \cap \{|v| \geq P\}$ to make the decomposition disjoint.) Later we choose

$$P = Q^{4/11}, \quad r = R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}, \quad R = Q^{16/33} \log^{1/2} Q.$$

Also, later we choose $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

3.2. Some observations.

Our goal is to obtain a priori estimate like

$$\|E(t)\|_\infty \lesssim Q(t)^a \qquad \text{for some } a < 1.$$

Since the force field E measures the maximal rate of changes in velocity, the estimate can be read very roughly as

$$Q'(t) \lesssim Q(t)^a,$$

which lead its polynomial growth.

So we need to bound the Riesz potential E. The following observation suggests a lower bound of relative velocity.

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$|y - \widehat{X}(s)| \geq \frac{1}{4}|v - \widehat{V}(t)||s - s_0|$$

for some $s_0 \in [t-\Delta, t]$, where $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

Proof. Since $\Delta\|E(s)\|_\infty < \frac{P}{4}$, we have

$$|v - w| < \frac{P}{4} \quad \text{and} \quad |\widehat{V}(t) - \widehat{V}(s)| < \frac{P}{4}.$$

The condition $|v - \widehat{V}(t)| \geq P$ implies

$$\frac{1}{2}|v - \widehat{V}(t)| \leq |v - \widehat{V}(t)| - \frac{P}{4} - \frac{P}{4} < |w - \widehat{V}(s)|.$$

Let $Z(s) := y - \widehat{X}(s)$. Then,

$$\begin{aligned} Z'(s) &= w - \widehat{V}(s), \\ Z''(s) &= \gamma[E(s,y,w) - E(s,\widehat{X}(s),\widehat{V}(s))]. \end{aligned}$$

Let $s_0 \in [t-\Delta, t]$ minimize $s \mapsto |Z(s)|$ and expand Z as

$$Z(s) = Z(s_0) + Z'(s_0)(s-s_0) + \frac{Z''(\sigma)}{2}(s-s_0)^2$$

for some σ between s and s_0. Then,

$$|Z(s_0) + Z'(s_0)(s-s_0)| \geq |Z'(s_0)(s-s_0)| \geq \frac{1}{2}|v - \widehat{V}(t)||s-s_0|$$

and

$$\begin{aligned} |\frac{Z''(\sigma)}{2}(s-s_0)^2| \leq \|E(t)\|_\infty (s-s_0)^2 &\leq \|E(t)\|_\infty \Delta |s-s_0| \\ &\leq \frac{P}{4}|s-s_0| \leq \frac{1}{4}|v-\widehat{V}(t)||s-s_0| \end{aligned}$$

proves

$$|y - \widehat{X}(s)| = |Z(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|. \qquad \square$$

We introduce time averaging to use the above lower bound.

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$\int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim \frac{r^{-1}}{|v-\widehat{V}(t)|},$$

where $A = \{s : |y - \widehat{X}(s)| \geq r\}$.

Proof. Since $|y-\widehat{X}(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|$,

$$\begin{aligned} \int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds &\leq 16\int_{t-\Delta}^t \frac{1}{|v-\widehat{V}(t)|^2|s-s_0|^2}\chi_A(s)\,ds \\ &\leq 32\int_r^\infty \frac{1}{|v-\widehat{V}(t)|^3|s-s_0|^2}\,d(|v-\widehat{V}(t)||s-s_0|) \\ &= 32\,\frac{r^{-1}}{|v-\widehat{V}(t)|}. \qquad \square \end{aligned}$$

3.3. Divide and conquer.

3.3.1. *Ugly set estimate.* Therefore, if we let $r^{-1} \simeq \min\{|v|^3, |v - \widehat{V}(t)|^3\}$, then

$$\int_{t-\Delta}^{t} \frac{1}{|y - \widehat{X}(s)|^2} \chi_A(s)\, ds \lesssim |v|^2$$

so that we have

$$\iiint_U \frac{f(s,y,w)}{|y - \widehat{X}(s)|^2}\, dw\, dy\, ds \lesssim R^{-1} \int |v|^2 f(t,x,v)\, dv\, dx \lesssim R^{-1}$$

when

$$U \subset \{\, (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}\, \}.$$

3.3.2. *Bad set estimate.* Consider U^c. We need to control the union of two regions

$$|y - \widehat{X}(s)| < R|v|^{-3} \quad \text{and} \quad |y - \widehat{X}(s)| < R|v - \widehat{V}(t)|^{-3}.$$

Without any conditions, the integration of fundamental solution with respect to y gives

$$\int_{|y-\widehat{X}(s)|<r} \frac{1}{|y - \widehat{X}(s)|^2}\, dy \simeq r.$$

Claim. *If $|v| \geq P$ and $|v - \widehat{V}(t)| \geq P$, then*

$$\int_{U^c} \frac{1}{|y - \widehat{X}(s)|^2}\, dy \lesssim \max\{|w|^{-3}, |w - \widehat{V}(s)|^{-3}\}$$

for $s \in [t - \Delta, t]$.

Proof. It follows from

$$|w| \simeq |v|, \quad |w - \widehat{V}(s)| \simeq |v - \widehat{V}(t)|$$

for $|v| \geq P$ and $|v - \widehat{V}(t)| \geq P$. □

3.3.3. *Good set estimate.*

3.4. **Polynomial decay.**

Lemma 3.1. *Along the time of existence we have*

$$\|E(t)\|_{L^\infty_x} \lesssim Q(t)^{4/3}.$$

Proof. Note that we have

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

Since the velocity support of f is bounded by finite $Q(t)$,

$$\rho(t,x) = \int_{|v|<Q(t)} f(t,x,v)\, dv \lesssim Q(t)^3 \|f_0(x)\|_{L^\infty_v} \lesssim Q(t)^3,$$

so

$$\|E(t)\|_{L^\infty_x} \lesssim \|\rho(t)\|_{L^\infty_x}^{4/9} \lesssim Q(t)^{4/3}.$$ □