# Functional Analysis I : Topological Vector Space

Lecture by Ikhan Choi
Notes by Ikhan Choi

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CHAPTER 1

# Category of topological vector spaces

## 1. Elementary properties

definition - how to use the continuity of vector space operations effectively homeomorphism by translation and dialation: local base at 0 uniformity pseudometrics, basic classification translation invariant metric completely regular (up to 3.5) boundedness and continuity

## 2. Classification

$$
\begin{tikzcd}
 & \text{inner product} \arrow[d] & \\
 & \begin{array}{c}\text{finite}\\ \text{seminorms}\\ \text{(= normable)}\end{array} \arrow[d] & \\
 & \begin{array}{c}\text{countable}\\ \text{seminorms}\end{array} \arrow[dl] \arrow[dr] & \\
\begin{array}{c}\text{countable}\\ \text{pseudometrics}\\ \text{(= metrizable)}\end{array} \arrow[dr] & & \begin{array}{c}\text{uncountable}\\ \text{seminorms}\\ \text{(= locally convex)}\end{array} \arrow[dl] \\
 & \begin{array}{c}\text{uncountable}\\ \text{pseudometrics}\end{array} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Hilbert space} \arrow[d] & \\
 & \text{Banach space} \arrow[d] & \\
 & \text{Fréchet space} \arrow[dl] \arrow[dr] & \\
\text{F-space} \arrow[dr] & & \text{(no name)} \arrow[dl] \\
 & \text{(no name)} &
\end{tikzcd}
$$

Proposition 2.1. *Let $\rho$ be a pseudometric. Then,*

$$B(0,1) \subset \frac{B(0,1)+B(0,1)}{2} \subset \frac{1}{2}B(0,2).$$

*If $\rho$ is a seminorm, then the equalities hold.*

I say this as $\frac{1}{2}B(0,2)$ is "fatter" than $B(0,1)$.

CHAPTER 2

# Barreled spaces and bornological spaces

### 0.1. The Baire category theorem.

### 0.2. Uniform boundedness principle.

THEOREM 0.2 (Uniform boundedness principle). *Let $X$ be a barreled space and $Y$ be a topological vector space. Let $\mathcal{F} \subset B(X,Y)$. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is equicontinuous.*

### 0.3. Open mapping theorem.

THEOREM 0.3 (Open mapping theorem). *Let $X$ be a topological vector space and $Y$ be a metrizable barreled space. Let $T\colon X \to Y$ be linear. If $T$ is surjective and continuous, then $T$ is open.*

PROOF. If we let $U$ be an open neighborhood in $X$, then we want to show $TU$ is a neighborhood. Because $T$ is surjective so that $\overline{TU}$ is absorbent, $\overline{TU}$ is a neighborhood. Note that an open set intersects $\overline{TU}$ also intersects $TU$.

If there exist two sequences of balanced open neighborhoods $U_n \subset X$ and $V_n \subset Y$ with

(1) $U_1 + \cdots + U_n \subset U$,
(2) $V_n \subset \overline{TU_n}$,
(3) $\bigcap_{n\in\mathbb{N}} V_n = \{0\}$,

then we can show $V_1 \subset TU$. Here is the proof: Suppose $y \in V_1$. Then,

$$\begin{tikzcd}
y \cap V_1 \neq \varnothing \arrow[r] & y \cap \overline{TU_1} \neq \varnothing \arrow[r] & (y+V_2)\cap TU_1 \neq \varnothing \arrow[dll] \\
(y+TU_1)\cap V_2 \neq \varnothing \arrow[r] & (y+TU_1)\cap \overline{TU_2} \neq \varnothing \arrow[r] & ((y+TU_1)+V_3)\cap TU_2 \neq \varnothing \arrow[dll] \\
(y+TU_1+TU_2)\cap V_3 \neq \varnothing \arrow[r] & \cdots .
\end{tikzcd}$$

From the first columns, and by the conditions (1) and (3), we obtain

$$(y+TU) \cap \bigcap_{n\in\mathbb{N}} V_n \neq \varnothing.$$

Therefore, the set $y + TU$ contains 0, hence $y \in TU$.

Let us show the existence of such sequences. At first, take $U_n = 2^{-n}U$ for (1). Then we can take $\{V_n\}_n$ with (2) as we mentioned above. Simultaneously we can have it satisfy (3) because $Y$ is metrizable. $\square$

Corollary 0.4. *Let $X$ be metrizable and $Y$ be barreled. Then, the open mapping theorem holds.*

Proof. The quotient of metrizable space is also metrizable, so $Y$ is a metrizble barreled space. □

Corollary 0.5 (The Banach Isomorphy). *A continuous linear bijection onto a metrizable barreled space is a homeomorphism, i.e. topological isomorphism.*

Corollary 0.6 (The first isomorphism theorem). *Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Then, the induced map $X/\ker T \to \operatorname{im} T$ is a topological isomorphism.*

CHAPTER 3

# Locally convex spaces

## 1. Seminorms

minkowski functional locally boundedness polar

## 2. The Hahn-Banach theorem

## 3. Weak topology

CHAPTER 4

# Operators on Banach space

DO NOT contain topics tht can be generalized within Banach algebras or any other operator algebras(e.g. polar decomposition, Gelfand theory, functional calculus, spectral resolution)

THEOREM 0.1. *Let $X$ be complete and $Y$ be complete metrizable. The range of a continuous operator $T : X \to Y$ is closed if and only if the induced linear isomorphism*

$$\frac{X}{\ker T} \to \operatorname{im} T$$

*has a continuous inverse so that it becomes a topological isomorphism.*

PROOF. One direction is easy.

For the other direction, suppose $\operatorname{im} T$ is closed in $Y$. Note that the metrizability condition of $Y$ is set in order to apply the open mapping theorem. $\square$

COROLLARY 0.2. *Let $T : X \to Y$ be a bounded operator between Banach spaces. Then, $T$ is bounded below if and only if $\operatorname{im} T$ is closed and $T$ is injective.*

## 1. Spectral theory

When a Banach algebra is realized as a concrete operator space, then the spectral theory on it changes drastically. For example we can categorize three cases for a linear operator between Banach spaces to fail the invertibility:

(1) it is not injective; (point spectrum)
(2) it is injective, its range is not dense; (residual spectrum)
(3) it is injective, its range is dense, but not closed; (continuous spectrum)

## 2. Compact operators

## 3. Unbounded operators

## 4. Nuclear operators

## 5. Fredholm theory

## 6. Perturbation theory