

Dispersion for the Schrödinger equation

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1. INTRODUCTION

In this article, the purpose is on proving a dispersive inequality for solutions of an initial value problem of the Schrödinger equation for no potential

$$(1) \quad \begin{cases} i\partial_t u + \Delta u = 0, & \text{in } \mathbb{R}_t \times \mathbb{R}_x^d, \\ u(0, x) = u_0(x). \end{cases}$$

The statement of our dispersive inequality is given as follows.

Theorem 1.1 (Dispersive estimate). *Let u be a solution of (1). Then,*

$$\|u(t, x)\|_{L^\infty(\mathbb{R}_x^d)} \lesssim_d t^{-\frac{d}{2}} \|u(0, x)\|_{L^1(\mathbb{R}_x^d)}.$$

We can observe the inequality implies that the solution decays as time flows.

2. WHAT IS DISPERSION?

2.1. Optics. In optics, the *dispersion* is the phenomenon that the index of refraction depends on the wavelength.

A dispersion relation of light in transparent material is given by the Cauchy formula, an approximate empirical equation,

$$n(\lambda) = B + \frac{C}{\lambda^2}.$$

For examples,

2.2. Quantization. We can make a PDE form a dispersion relation by quantization. Mathematically, we can simply define a “wave” as a superposition of complex exponential functions on timespace \mathbb{R}^{1+d} that have the form

$$\psi(t, x) = e^{i(k \cdot x - \omega t)},$$

where the parameters $\omega \in \mathbb{R}$ and $k \in \mathbb{R}^d$ are related by dispersion relation. On the space of wave functions, the multiplication operators with respect to ω and k are same with partial differential operators on wave functions:

$$i\partial_t \psi = \omega \psi, \quad -i\partial_{x_i} \psi = k_i \psi \quad (1 \leq i \leq d).$$

Consider the Schrödinger equation as an example. The energy conservation

$$E = \frac{p^2}{2m} + V$$

and the de Broglie relation

$$E = \hbar\omega, \quad p = \hbar k$$

give the dispersion relation

$$\hbar\omega = \frac{|\hbar k|^2}{2m} + V,$$

which is in fact the Schrödinger equation

$$i\hbar\partial_t = -\frac{\hbar^2}{2m}\Delta + V.$$

2.3. Dispersive equation.

Definition 2.1. If the dispersive relation is not of the form $\omega(k) \propto |k|$, the wave is called dispersive.

3. METHOD I: REPRESENTATION FORMULA

3.1. Basics on Fourier transform.

3.2. Multiplier.

Definition 3.1. The *time evolution operator* is the multiplier operator associated with e^{-iHt} , and is denoted by $e^{-i\Delta t}$:

$$\widehat{e^{-i\Delta t}u}(x) := e^{-iHt}\widehat{u}(k)$$

Definition 3.2. The *propagator* is inverse transform of e^{-iHt} , and is denoted by $K(x; t)$.

$$\widehat{e^{-i\Delta t}u} = e^{-iHt}\widehat{u} = \widehat{K * u}.$$

3.3. Fundamental solution. First,

$$P(D)K(t, x) = 0, \quad t > 0$$

implies

$$i\partial_t K + \Delta_x K = 0$$

$$i\partial_t \widehat{K} - |k|^2 \widehat{K} = 0$$

$$\partial_t(\log \widehat{K}) = -i|k|^2$$

$$\widehat{K}(t, k) = C(k)e^{-i|k|^2 t}.$$

And then,

$$K(0, x) = \delta(x), \quad t = 0$$

implies

$$\widehat{K}(t, k) = e^{-i|k|^2 t}.$$

Differentiating,

$$\nabla_k \widehat{K} = -i2kt\widehat{K}$$

$$xK = -i2t\nabla_x K$$

$$\nabla_x(\log K) = i\frac{x}{2t}$$

$$K(t, x) = C(t)e^{i\frac{|x|^2}{4t}}.$$

Since

$$C(t) = K(t, 0) = \int e^{-i|k|^2 t} dk = (\pi i t)^{-\frac{d}{2}},$$

we get

$$K(t, x) = (\pi i t)^{-\frac{d}{2}} e^{i \frac{|x|^2}{4t}}.$$

4. METHOD II: OSCILLATORY INTEGRAL

4.1. Reducing problem. By Fourier transform, we get an ODE

$$i \partial_t \hat{u}(t, k) + |k|^2 \hat{u}(t, k) = 0,$$

solved by

$$\hat{u}(t, k) = \hat{u}_0(k) e^{-i|k|^2 t}.$$

Taking inverse Fourier transform, the solution of the original equation is given by

$$u(t, x) = \int \hat{u}_0(k) e^{i(k \cdot x - |k|^2 t)} dk.$$

Define phase

$$\Phi(t, x, k) := k \cdot x - |k|^2 t$$

and an oscillatory integral

$$I(t, x) := \int \hat{u}_0(k) e^{i\Phi} dk,$$

which is exactly the same with the general solution.

The statement of our dispersive inequality is given as follows.

Theorem 4.1 (Dispersive estimate). *We have an asymptotic inequality*

$$\sup_x |I(t, x)| \lesssim t^{-\frac{d}{2}}$$

for $t \gtrsim 1$.

We can observe the inequality implies that the solution decays as time flows.

To prove this, we are going to make pointwise estimates for I .

To begin with, fix x . Note that the equation (??) has a rotational symmetry on physical space. Thus we assume $x_2 = \dots = x_d = 0$. Also, suppose the support of $a(\xi)$ is restricted to an annulus $\frac{1}{2} < |\xi| < 2$. This assumption is valid because a simple dyadic decomposition guarantees the generality.

4.2. Stationary and nonstationary phases.

Definition 4.1. Let Φ be the phase defined as above. A *stationary* point $\xi^o(t, x)$ is a point at which $\nabla_\xi \Phi$ vanishes.

The word “stationary” is not with respect to time flows, but change of ξ . I think it would be fantastic if here is a 3d image to explain how the principles of stationary and nonstationary phase work.

The idea is to divide the oscillatory integral.

$$I = I_{stat} + I_{nonstat}.$$

We have $height \times base \sim base$ for I_{stat} , and cancellation for $I_{nonstat}$. If one has a nice estimate, then the other must be bigger. Therefore, a natural question comes up with: how can we choose the suitable boundary to get an optimal estimate? Here, what we are going to use is a heuristic technique called “Linearizing phase”. In fact, it is not a linearization but a *Taylorization*, but never mind. By the technique, we can expect to read out a suitable estimate: this is exactly $-\frac{d}{2}$, which is mentioned at the statement of our theorem. Proof will begin after getting $-\frac{d}{2}$.

Let us give a toy example to catch the idea.

Example 4.2 (Fresnel type integral). Let

$$I(\lambda) = \int a(\xi) e^{i\lambda \xi^n} d\xi$$

and $\Phi(\lambda, \xi) = \lambda \xi^n$. In this problem, λ plays a similar role with t .

4.3. Heuristic method by linearizing phase. At first, the stationary point ξ^o is computed as

$$\xi_1^o = \left(\frac{|x_1|}{\alpha t} \right)^{\frac{1}{\alpha-1}}, \quad \xi_2^o = \dots = \xi_d^o = 0.$$

Note $|\xi^o| = \xi_1^o$. Although ξ^o depends on t , since the amplitude function $a(\xi)$ is supported on $\frac{1}{2} < |\xi|$, we can let $|\xi^o|$ as a constant asymptotically. Let $\xi' = \xi - \xi^o$ be an auxiliary variable for localization at ξ^o .

Intuitively, if $|\Phi(\xi) - \Phi(\xi^o)|$ is greater than 2π , then ξ belongs to the region of nonstationary. Thus, our plan is to find the region $\{|\xi'| : |\Phi(\xi) - \Phi(\xi^o)| \lesssim 1\}$ since 2π is same with 1. Apply the Taylor expansion. Since

$$\partial_{\xi_i} \partial_{\xi_j} \Phi(\xi) = \alpha |\xi|^{\alpha-2} t (\delta_{ij} + (\alpha-2) \frac{\xi_i \xi_j}{|\xi|^2}),$$

we have

$$\text{Hess}_{\xi^o}[\Phi] = \begin{pmatrix} \alpha(\alpha-1)|\xi^o|^{\alpha-2}t & 0 \\ 0 & \alpha|\xi^o|^{\alpha-2}t \cdot \text{id}_{d-1} \end{pmatrix}.$$

Therefore, by the Taylor expansion,

$$\begin{aligned} 1 &\gtrsim |\Phi(\xi) - \Phi(\xi^o)| \\ &\sim |\text{Hess}_{\xi^o}[\Phi](\xi', \xi')| \\ &\sim \alpha |\xi^o|^{\alpha-2} t [(\alpha-1)|\xi'_1|^2 + |\xi'_2|^2 + \dots + |\xi'_d|^2] \\ &\sim t |\xi'|^2. \end{aligned}$$

This lets us know the desired boundary $|\xi'| \lesssim t^{-\frac{1}{2}}$: say, $I_{stat} \sim \int \chi_{|\xi'| < t^{-1/2}}(\xi') a(\xi) e^{i\Phi} d\xi$.

4.4. Dyadic decomposition. This section begins the real proof. The stationary part is a piece of cake: since the area of base is asymptotically d -power of the radius $t^{-\frac{1}{2}}$,

$$|I_{stat}| \lesssim 1 \times t^{-\frac{d}{2}} = t^{-\frac{d}{2}}.$$

Now, what we have to do is to show $|I_{nonstat}| \lesssim t^{-\frac{d}{2}}$.

Both the height and the volume of the base are roughly 1 on the nonstationary region $|\xi'|$, we need to do something genius. The height cannot be reduced under 1, so we are going to decompose the supports of the amplitude a : take a partition of unity

$$\chi_{<1}(\xi') = \chi_{<t^{-1/2}}(\xi') + \sum_{\substack{t^{-1/2} < \mu t^{-1/2} \leq 1 \\ \log_2 \mu \in \mathbb{Z}}} \chi_{\mu t^{-1/2}}(\xi'),$$

where $\chi_{<k}(\xi')$ and $\chi_k(\xi')$ are smooth functions supported on $|\xi'| < 2k$ and $\frac{1}{2}k < |\xi'| < 2k$ respectively. If we define

$$I_\mu := \int \chi_{\mu t^{-1/2}}(\xi') a(\xi) e^{i\Phi} d\xi,$$

then we have

$$\begin{aligned} I &= \int a(\xi) e^{i\Phi} d\xi \\ &= \int \chi_{<1}(\xi') a(\xi) e^{i\Phi} d\xi \\ &= \int \chi_{<t^{-1/2}}(\xi') a(\xi) e^{i\Phi} d\xi + \sum_{\substack{t^{-1/2} < \mu t^{-1/2} \leq 1 \\ \log_2 \mu \in \mathbb{Z}}} I_\mu \\ &= I_{stat} + I_{nonstat}. \end{aligned}$$

Notice that $|\xi'| \sim \mu t^{-1/2} \Leftrightarrow t^{-1} |\xi'|^{-2} \sim \mu^{-2}$ and the base gets reduced to $(\mu t^{-1/2})^d \sim t^{-\frac{d}{2}}$. This is a reason why the dyadic decomposition is useful. For some reasons that will be seen, dyadic decomposition becomes a powerful tool to estimate an oscillatory integral for polynomial phase. However, even though the sum in $I_{nonstat}$ is finite, the number of μ 's that are summed is dependent on t , so we want to get rid of such dependency by summing up with a constant bound, which is impossible as of now since $\mu = 2, 4, 8, 16, \dots$ and $d > 0$. In this situation, we can compress the size of μ^d by the *repeated integral by parts*. The power of magic number $t^{-\frac{1}{2}}$ arises in this procedure.