Interchanging Limits, Derivatives, and Integrals

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1. Limit and derivative

f_n pointwisely, Df_n uniformly

2. Limit and integral

We want to find a criterion for This question asks the convergence

$$f_n \to f \quad \text{in} \quad L^1.$$

Theorem 2.1 (Lebesgue dominated convergence theorem)**.** *Let $\{f_\alpha\}_\alpha$ be a net of measurable functions $(X, \mu) \to \mathbb{R}$. Define a maximal function*

$$Mf(x) = \sup_\alpha |f_\alpha(x)|.$$

If $\|Mf\|_1 < \infty$, then

$$\lim_\alpha |f_\alpha(x) - f(x)| = 0 \quad a.e. \quad \implies \quad \lim_\alpha \|f_\alpha - f\|_{L^1} = 0.$$

continuity application

3. Derivative and integral

Theorem 3.1 (Scheffe)**.** *Let $\{f_n\}_n$ be a sequence of nonnegative functions in L^1. Suppose it converges to f pointwisely. Then,*

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1 \quad \iff \quad \lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

Define the Newton quotient as

$$D_h f(t, x) := \frac{f(t+h, x) - f(t, x)}{h}$$

for $h \neq 0$. We mainly recognize D_h as an operator that maps $f(0, x)$ to a function of x. Then, we can say that the partial derivative $\partial_t f(0, x)$ is well-defined a.e. x if and only if

$$\lim_{h\to 0} D_h f(0, x) = \partial_t f(0, x) \qquad \text{a.e. } x.$$

We may ask about conditions for the following to hold:

$$\lim_{h\to 0} D_h f(0, x) = \partial_t f(0, x) \qquad \text{in } L^1_x(X).$$

This question naturally arise because it implies the commutability

$$\frac{d}{dt}\int f(t, x)\, dx = \int \frac{\partial}{\partial t} f(t, x)\, dx$$

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at $t = 0$. As necessary conditions to formalize the statement, we must basically assume that $f(t, x) \in L^1_x$ for $|t| < \varepsilon$, and $\partial_t f(0, x) \in L^1_x$. Above this, if we give a stronger condition $\operatorname{ess\,sup}_{|t|<\varepsilon} |\partial_t f(t, x)| \in L^1_x$ than $\partial_t f(0, x) \in L^1_x$, then the L^1_x convergence is obtained.

Theorem 3.2 (Leibniz rule)**.** *Let $f : (-\varepsilon, \varepsilon) \times X \to \mathbb{R}$ be a curve of integrable functions such that $f(t, x)$ is absolutely continuous in t for a.e. x. If*

$$\operatorname*{ess\,sup}_{|t|<\varepsilon} |\partial_t f(t, x)|$$

is in L^1_x, then

$$\lim_{h\to 0} D_h f(0, x) = \partial_t f(0, x) \qquad \text{in } L^1_x(X).$$

Proof. Our strategy is Define a maximal function

$$Mf(x) := \sup_{|h|<\varepsilon} |D_h f(0, x)|.$$

Since

$$D_h f(0, x) = \frac{f(h, x) - f(0, x)}{h} = \frac{1}{h} \int_0^h \partial_t f(t, x)\, dt,$$

we have

$$\begin{aligned} |D_h f(0, x)| &\le \frac{1}{h} \int_0^h |\partial_t f(t, x)|\, dt \\ &\le \operatorname*{ess\,sup}_{|t|\le|h|} |\partial_t f(t, x)| \left[\frac{1}{h}\int_0^h dt\right] \\ &\le \operatorname*{ess\,sup}_{|t|<\varepsilon} |\partial_t f(t, x)|. \end{aligned}$$

Since the right hand side is constant with respect to h, we can deduce that Mf is in L^1_x. The pointwise Applying the Lebesgue dominated convergence theorem, we get the desired result...??? □

Remark. If f is assumed to be differentiable everywhere on $t \in (-\varepsilon, \varepsilon)$, then we may use the mean value theorem to prove the theorem instead of the differentiation theorem: we directly get

$$|D_h f(0, x)| \le \sup_{|t|<\varepsilon} |\partial_t f(t, x)|.$$

If f is assumed to be continuously differentiable on $t \in (-\varepsilon, \varepsilon)$, then

F is absolutely continuous,

$$\partial_t F = f \quad \iff \quad F(x, t) = \int_c^t f(x, s)\, dx.$$

Then, for

$$T_h f(x, 0) := \frac{1}{h} \int_0^h f(x, s)\, ds,$$

For $\|f\|_{L^1_x L^\infty_t} = \|\sup_t |f(x,t)|\|_{L^1_x} < \infty$ we have

$$\begin{aligned} |T_h f(x,0)| &\leq \frac{1}{h}\int_0^h |f(x,s)|\,ds \\ &\leq \left[\frac{1}{h}\int_0^h ds\right] \cdot \sup_t |f(x,t)| \\ &= \sup_t |f(x,t)|. \end{aligned}$$

Thus,

$$Mf(x,0) = \sup_h |T_h f(x,0)| \leq \sup_t |f(x,t)| \in L^1_x.$$

Since $f(x,0) \in L^1_x$, by the Lebesgue differentiation theorem, we get

$$\lim_{h\to 0} T_h f(x,0) = f(x,0)$$

for a.e. x.