Finite Group Theory

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1. Sylow game

Definition 1.1 (Sylow p-subgroup)**.** Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. A *Sylow p-subgroup* is a subgroup of order p^a. We are going to denote the set of Sylow p-subgroups by $\mathrm{Syl}_p(G)$ and the number of Sylow p-subgroups by $n_p(G)$.

Theorem 1.1 (The Sylow theorem)**.** *Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. Then,*

$$p \mid n_p - 1, \qquad n_p \mid m$$

for some $k \in \mathbb{N}$.

Proof. Step 1: Sylow p-subgroups exist. We apply mathematical induction. The base step is trivial. Suppose every finite group of order less than n possesses a Sylow p-subgroup.

By applying the orbit-stabilizer theorem for the action $G \curvearrowright G$ by conjugation, build the class equation

$$|G| = |Z(G)| + \sum_i |G : C_G(g_i)|.$$

There are two cases: $p \mid |Z(G)|$ or $p \nmid |Z(G)|$.

Case 1: $p \mid |Z(G)|$. The group G has a normal cyclic subgroup C of order p, because $Z(G)$ has a subgroup of order p by Cauchy's theorem. If we let P be a Sylow p-subgroup of G/C, then

$$|P| = p^{a-1}.$$

For the quotient map $\pi : G \to G/C$ we have

$$|\pi^{-1}(P)| = |C| \cdot |P| = p^a,$$

by applying the first isomorphism theorem to π restricted onto $\pi^{-1}(P)$.

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Case 2: $p \nmid |Z(G)|$. Since $p \mid n$, we have $p \nmid |G : C_G(g)|$ for some $g \in G$. It means $p^a \mid |C_G(g)|$, thereby, by the inductive assumption, there is a Sylow p-subgroup P of $|C_G(g)|$ such that
$$|P| = p^a,$$
which is also a Sylow p-subgroup of G.

Therefore, we are done for Step 1.

Step 2: A lemma. We prove a lemma: given a Sylow p-subgroup P of G the normalizer subgroup $N_G(P)$ has a unique Sylow p-subgroup, P.

Here is the proof. Note that P is normal in $N_G(P)$ and p does not divide the order of the quotient group. Let P' be a Sylow p-subgroup of $N_G(P)$. Since every element of P' has order that is a power of p, the image of P' under the quotient map $\pi : N_G(P) \to N_G(P)/P$ is trivial. Therefore, $P' = P$.

Step 3: Sylow p-subgroups get action by conjugation. Let P be a Sylow p-subgroup of G. We construct equations via the orbit-stabilizer theorm for various actions to extract information on n_p. Note that stabilizers in setwise conjugation action is represented by normalizer subgroups.

(1) The action $P \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = 1 + \sum_i |P : N_P(P_i)|.$$
Here we have $p \mid |P : N_P(P_i)|$ since $P = N_P(P_i) \subset N_G(P_i)$ if and only if $P = P_i$.

(2) Suppose the action $G \curvearrowright \mathrm{Syl}_p(G)$ is not transitive. Take another Sylow p-subgroup P' is not conjugate with P in G. The two actions $P \curvearrowright \mathrm{Orb}_G(P)$ and $P' \curvearrowright \mathrm{Orb}_G(P)$ gives
$$|\operatorname{Orb}_G(P)| = 1 + \sum_i |P : N_P(P_i)| = \sum_i |P' : N_{P'}(P_i)|.$$
It implies $|\operatorname{Orb}_G(P)| \equiv 0, 1 \pmod p$ simultaneously, which leas a contradiction.

(3) The action $G \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = |G : N_G(P_i)|$$
for all $P_i \in \mathrm{Syl}_p(G)$ because the action is transitive.

Then, (1) proves $p \mid n_p - 1$, and (3) proves $n_p \mid m$. □

Corollary 1.2. *Let G be a finite group. Then,*

(1) *every pair of two Sylow p-subgroup is conjugate.*
(2) *every p-subgroup is contained in a Sylow p-subgroup.*
(3) *a Sylow p-subgroup is normal if and only if $n_p = 1$.*

First, find a normal subgroup. Second, find a normal subgroup of a subgroup.

By Hölder program, normal subgroups always benefit:

(1) existence of subgroup of particular order(by extension),
(2) contradiction by n_p element counting

A normal subgroup of a subgroup makes *normalizer lifting* that results in:

(1) existence of subgroup of particular order(by normalizer),

(2) existence of normal subgroup,
(3) constraint of n_p by normalizer of Sylow subgroup.

Find a subgroup of nice order

2. Simple groups

2.1. **Symmetric groups.**

2.2. **Linear groups.**