

# Real Analysis II : Fourier Analysis

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## CHAPTER 1

# Basic techniques

## 1. Interpolation

### 1.1. The distribution function.

DEFINITION 1.1. Let  $f$  be a measurable function on a measure space  $(X, \mu)$ . The *distribution function*  $\lambda_f : [0, \infty) \rightarrow [0, \infty)$  is defined as:

$$\lambda_f(\alpha) = \mu(\{x : |f(x)| > \alpha\}).$$

Do not use  $\mu(\{x : |f(x)| \geq \alpha\})$ . The strict inequality implies the *lower semi-continuity* of  $\lambda_f$ .

THEOREM 1.1 (Fubini). Denote  $f_h = \chi_{|f|>\alpha}f$  and  $f_l = \chi_{|f|\leq\alpha}f$ . For  $p > 0$ , we have

$$\|f\|_p^p = \int p\alpha^{p-1}\lambda_f(\alpha) d\alpha.$$

For  $p > 0$  and  $n > 0$ , we have

$$\|f\|_{p+n}^{p+n} = \int p\alpha^{p-1}\|f_h\|_n^n d\alpha.$$

For  $p < 0$  and  $n > 0$ , we have

$$\|f\|_{p+n}^{p+n} = \int |p|\alpha^{p-1}\|f_l\|_n^n d\alpha.$$

THEOREM 1.2. For  $p > 1$ , by the Chebyshev inequality, we have

$$\sup_{\alpha} \alpha^p \lambda_f(\alpha) \leq \int p\alpha^{p-1}\lambda_f(\alpha) d\alpha.$$

In other words,  $\|f\|_{p,\infty} \leq \|f\|_p$ .

### 1.2. Real interpolation.

THEOREM 1.3 (Marcinkiewicz interpolation). Let  $X$  be a  $\sigma$ -finite measure space and  $Y$  be a measure space. Let  $1 < p_0 < p < p_1 < \infty$ . Let  $T : L^{p_0}(X) + L^{p_1}(X) \rightarrow M(Y)$  be a sublinear operator. If  $T$  has a weak type estimate

$$\|T\|_{p_0 \rightarrow p_0, \infty}, \|T\|_{p_1 \rightarrow p_1, \infty} < \infty,$$

then

$$\|T\|_{p \rightarrow p} < \infty.$$

PROOF. Let  $f \in L^p$  and denote  $f_h = \chi_{|f|>\alpha}f$  and  $f_l = \chi_{|f|\leq\alpha}f$ . It is easy to show  $f_h \in L^{p_0}$  and  $f_l \in L^{p_1}$ . Then,

$$\begin{aligned} \|Tf\|_p^p &\sim \int \alpha^{p-1} \lambda_{Tf} d\alpha \\ &\lesssim \int \alpha^{p-1} \lambda_{Tf_h} d\alpha + \int \alpha^{p-1} \lambda_{Tf_l} d\alpha \\ &\leq \int \alpha^{p-1} \frac{1}{\alpha^{p_0}} \|Tf_h\|_{p_0, \infty}^{p_0} d\alpha + \int \alpha^{p-1} \frac{1}{\alpha^{p_1}} \|Tf_l\|_{p_1, \infty}^{p_1} d\alpha \\ &\lesssim \int \alpha^{p-p_0-1} \|f_h\|_{p_0}^{p_0} d\alpha + \int \alpha^{p-p_1-1} \|f_l\|_{p_1}^{p_1} d\alpha \\ &\sim \|f\|_p^p. \end{aligned}$$

by (1) Fubini, (2) Sublinearity, (3) Chebyshev, (4) Boundedness, (5) Fubini.  $\square$

**THEOREM 1.4** (Hadamard's three line lemma). *Let  $f$  be a bounded holomorphic function on the vertical unit stripe  $\{z : 0 < \operatorname{Re} z < 1\}$ . Then, for  $0 < \theta < 1$ ,*

$$\|f\|_{L^\infty(\operatorname{Re}=\theta)} \leq \|f\|_{L^\infty(\operatorname{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\operatorname{Re}=1)}^\theta.$$

**PROOF.** Define

$$g(z) := \frac{f(z)}{\|f\|_{L^\infty(\operatorname{Re}=0)}^{1-z} \|f\|_{L^\infty(\operatorname{Re}=1)}^z}, \quad g_n(z) = g(z) e^{\frac{z^2-1}{n}}.$$

Then we have

- (1)  $g_n \rightarrow g$  pointwisely as  $n \rightarrow \infty$ ,
- (2)  $g_n(z) \rightarrow 0$  uniformly as  $\operatorname{Im} z \rightarrow \infty$ .

The second one is because  $g$  is bounded and for  $z = x + yi$  we have

$$|g_n(z)| \lesssim |e^{\frac{z^2-1}{n}}| = e^{\operatorname{Re} \frac{z^2-1}{n}} = e^{\frac{x^2-y^2-1}{n}} \leq e^{\frac{-y^2}{n}}.$$

By (1), it is enough to bound  $g_n$  for each  $n$ . Truncating the stripe, the outer region is controlled by (2) and the interior region is controlled by the maximum modulus principle.  $\square$

### 1.3. Complex interpolation.

**THEOREM 1.5** (Riesz-Thorin interpolation). *Let  $X, Y$  be  $\sigma$ -finite measure spaces. Let*

$$\frac{1}{p_\theta} = \frac{1}{p_0}(1-\theta) + \frac{1}{p_1}\theta, \quad \frac{1}{q_\theta} = \frac{1}{q_0}(1-\theta) + \frac{1}{q_1}\theta.$$

*Then,*

$$\|T\|_{p_\theta \rightarrow q_\theta} \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta.$$

**PROOF.** Note that

$$\|T\|_{p_\theta \rightarrow q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g \rangle|}{\|f\|_{p_\theta} \|g\|_{q'_\theta}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z \rangle = \int \overline{g_z(y)} Tf_z(y) dy,$$

where  $f_z$  and  $g_z$  are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z) + \frac{p_\theta}{p_1}z} \frac{f}{|f|}$$

so that we have  $f_\theta = f$  and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for  $\operatorname{Re} z = x$ .

Then,

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0} \|f_z\|_{p_0} \|g_z\|_{q'_0} = \|T\|_{p_0 \rightarrow q_0} \|f\|_{p_\theta}^{p_\theta/p_0} \|g\|_{q'_\theta}^{q'_\theta/q'_0}$$

for  $\operatorname{Re} z = 0$ , and

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_1 \rightarrow q_1} \|f_z\|_{p_1} \|g_z\|_{q'_1} = \|T\|_{p_1 \rightarrow q_1} \|f\|_{p_\theta}^{p_\theta/p_1} \|g\|_{q'_\theta}^{q'_\theta/q'_1}$$

for  $\operatorname{Re} z = 1$ . By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z \rangle| \leq \|T\|_{p_0 \rightarrow q_0}^{1-\theta} \|T\|_{p_1 \rightarrow q_1}^\theta \|f\|_{p_\theta} \|g\|_{q'_\theta}$$

for  $\operatorname{Re} z = \theta$ . Putting  $z = \theta$  in the last inequality, we get the desired result.  $\square$



## 2. Maximal function

We often want to show a net of linear operators  $\{T_t\}_t$  is an “approximate identity” in the sense of pointwise convergence, not a certain norm; in other words, say, we want to show

$$\lim_{t \rightarrow 0} T_t f(x) = f(x) \quad a.e.$$

Suppose  $T = \lim_t T_t$  is defined on  $L^1$  and let  $I : L^1 \hookrightarrow X$  be a canonical embedding. Assume that we have proved  $T - I$  is continuous operator  $L^1 \rightarrow X$  and  $\ker(T - I)$  is dense in  $L^1$ . Then,  $T - I$  must vanish at entire space  $L^1$ . It implies  $Tf$  and  $f$  are equal almost everywhere.

We introduce maximal function  $Mf$  defined by

$$Mf(x) = \sup_t |T_t f(x)|.$$

If it satisfies a boundedness, for example, if it satisfies something we call the weak-type estimate  $\|Mf\|_{1,\infty} \lesssim \|f\|_1$ , then

$$\|(T - I)f\|_{1,\infty} \leq \|Tf\|_{1,\infty} + \|f\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_1 \lesssim \|f\|_1$$

implies the continuity of  $T - I$ . If  $\ker(T - I)$  contains test function space above this, then we get the desired result. This density argument can also be explained using approximation by  $g$  such that  $Tg = g$ :

$$\begin{aligned} \|Tf - f\|_{1,\infty} &\leq \|T(f - g)\|_{1,\infty} + \|Tg - g\|_{1,\infty} + \|g - f\|_{1,\infty} \\ &\leq \|M(f - g)\|_{1,\infty} + \|g - f\|_1 \\ &\lesssim \|f - g\|_1 \rightarrow 0. \end{aligned}$$

**2.1. The Hardy-Littlewood maximal function.** Hardy-Littlewood maximal function is the most famous maximal function.

THEOREM 2.1 (Hardy-Littlewood).

$$\|Mf\|_{1,\infty} \leq 3^d \|f\|_1.$$

PROOF. Let  $E_\alpha = \{|f| > \alpha\}$ . By the inner regularity of  $\mu$ , we may assume  $E_\alpha$  is compact. For every  $x \in E_\alpha$ , we can choose  $B_x$  such that

$$\frac{1}{B_x} \int_{B_x} |f| > \alpha \quad \Rightarrow \quad \mu(B_x) < \alpha \int_{B_x} |f|.$$

By the Vitali covering and by the compactness of  $E_\alpha$ ,

$$\lambda_f(\alpha) = \mu(E_\alpha) \leq 3^d \sum \mu(B_k) \leq \frac{3^d}{\alpha} \sum_k \int_{B_k} |f| \leq \frac{3^d}{\alpha} \|f\|_1.$$

The disjointness is important in showing the constant  $3^d$  does not depend on the number of  $B_k$ 's.  $\square$

DEFINITION 2.1.

$$f^*(x) := \lim_{r \rightarrow 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)| dy.$$

THEOREM 2.2 (Lebesgue differentiation).  $f^* = 0$  a.e.

PROOF. Note that  $f^* \leq Mf + |f|$  implies

$$\|f^*\|_{1,\infty} \leq \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that  $g^* = 0$  for  $g \in C_c$ . Approximate using  $f^* = (f - g)^*$ .

□

### 3. Convergence of Fourier series

DEFINITION 3.1. The *Dirichlet kernel* is a function  $D_n: \mathbf{T} \rightarrow \mathbb{R}$  defined by

$$D_n = \widehat{\mathbf{1}_{|k| \leq n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = \mathbf{1}_{|k| \leq n}.$$

This is because they are invariant under inverse, in other words, they are even.

THEOREM 3.1.

$$D_n(x) = \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.$$

PROOF.

$$\begin{aligned} D_n(x) &= \sum_{k=-n}^n e^{ikx} \\ &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}. \end{aligned}$$

□

THEOREM 3.2. If  $f \in \text{Lip}(\mathbf{T})$ , then  $D_n * f \rightarrow f$  pointwisely as  $n \rightarrow \infty$ .

THEOREM 3.3.

$$\|D_n\|_{L^1(\mathbf{T})} \gtrsim \log n.$$

PROOF. By (2)  $\sin x \leq x$  for  $x \in [0, \pi/2]$ , (3) change of variable,

$$\begin{aligned} \|D_n\|_{L^1(\mathbf{T})} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x} \right| dx \\ &\geq \frac{2}{\pi} \int_0^{\pi} \frac{|\sin \frac{2n+1}{2}x|}{x} dx \\ &= \frac{2}{\pi} \int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x} dx \\ &= \frac{2}{\pi} \sum_{k=0}^{2n} \int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x} dx \\ &\geq \frac{2}{\pi} \sum_{k=0}^{2n} \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi} dx \\ &\geq \frac{4}{\pi^2} \sum_{k=0}^{2n} \frac{1}{1+k} \\ &\geq \frac{4}{\pi^2} \log(2n+2). \end{aligned}$$

....

□

DEFINITION 3.2. The *Fejér kernel* is

THEOREM 3.4.

$$K_n(x) = \frac{1}{n+1} \frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}.$$

PROOF. Since

$$\begin{aligned} D_n(x) &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2}, \end{aligned}$$

by telescoping, we get

$$\begin{aligned} \sum_{k=0}^n D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}. \end{aligned}$$

□

Two important results from Fejér kernel:

- (1) If  $f(x-)$ ,  $f(x+)$  exist and  $S_n f(x)$  converges, then  $S_n f(x) \rightarrow \frac{1}{2}(f(x-) + f(x+))$ .
- (2) (If  $f \in L^1(\mathbf{T})$ , then  $\sigma_n f \rightarrow f$  a.e.)
- (3) If  $f \in L^1(\mathbf{T})$ , then  $S_n f \rightarrow f$  in  $L^1$  and  $L^2$ .
- (4) If  $f$  is continuous and  $\hat{f} \in L^1(\mathbb{Z})$ , then  $S_n f \rightarrow f$  uniformly.
- (5) Since  $\sigma_n f$  is a trigonometric polynomial, the set of trigonometric polynomials are dense in  $L^1(\mathbf{T})$  and  $L^2(\mathbf{T})$ .

## CHAPTER 2

### **Differentiation theory**



## CHAPTER 3

# Calderon-Zygmund theory





## CHAPTER 4

### **Littlewood-Paley theory**