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Chapter 1

Analysis

1.1 Elementary analysis

1.1.1 Sequences

(1) Show that for a nonnegative sequence a_n if $\sum a_n$ diverges then $\sum \frac{a_n}{1+a_n}$ also diverges.

(2) Show that every real sequence has a monotonic subsequence that converges to the limit superior of the original sequence.

(3) Show that if a decreasing nonnegative sequence a_n converges to 0 and satisfies $S_n \leq 1 + na_n$ then S_n is bounded by 1.

1.1.2 Functions

Every function in here is real valued and defined on $\mathbb{R}$ if not mentioned.

(1) Show that if both limits of a function and its derivative at infinity exist then the function vanishes at infinity.

(2) Let f be C^2 with $f''(c) \neq 0$. Defined a function ξ such that $f(x) - f(c) = f'(\xi(x))(x - c)$ with $|\xi - c| \leq |x - c|$, show that $\xi'(c) = 1/2$.

(3) Let f be a C^2 function such that $f(0) = f(1) = 0$. Show that $\|f\| \leq \frac{1}{8}\|f''\|$.

(4) Show that the set of local minima of a convex function is connected.

(5) Show that a smooth function such that for each x there is n having the nth derivative vanish is a polynomial.

(6) Show that if a C^1 function f satisfies $f(x) \neq 0$ for x such that $f'(x) = 0$, then in a bounded set there are only finite points at which f vanishes.

(7) Let a function f be differentiable. For $a < a' < b < b'$ show that there exist $a < c < b$ and $a' < c' < b'$ such that $f(b) - f(a) = f'(c)(b-a)$ and $f(b') - f(a') = f'(c')(b'-a')$.

(8) Show that if a sequence of real functions $f_n\colon [0,1] \to [0,1]$ satisfies $|f(x) - f(y)| \leq |x-y|$ whenever $|x-y| \geq \frac{1}{n}$, then the sequence has a uniformly convergent subsequence.

(9) Let f be a differentiable function on the unit closed interval. Show that if $f(0) = 0$ there is c such that $cf'(c) = f(c)$. (Flett)

(10) Let f be a differentiable function on the unit closed interval. Show that if $f(0) = 0$ there is c such that $cf(c) = (1-c)f'(c)$.

(11) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $f(x) = c$ cannot have exactly two solutions for every constant $c \in \mathbb{R}$.

(12) Show that a continuous function that takes on no value more than twice takes on some value exactly once.

(13) Let f be a function that has the intermediate value property. Show that if the preimage of every singleton is closed, then f is continuous.

1.1.3 Integration

(1) Find the value of $\lim_{n\to\infty} \frac{1}{n}\left(\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x)\,dx\right)$.

(2) Show that if $xf'(x)$ is bounded and $x^{-1}\int_0^x f \to L$ then $f(x) \to L$ as $x \to \infty$.

1.2 Real analysis

1.2.1 Measure theory

(1) Show that a measurable subset of $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression.

1.2.2 Fourier analysis

1.3 Complex analysis

(1) Show that if a holomorphic function has positive real parts on the open unit disk then $|f'(0)| < 2\operatorname{Re} f(0)$.

(2) Show that if at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.

(3) Show that if a holomorphic function on a domain containing the closed unit disk is injective on the unit circle then so is on the disk.

(4) Show that for a holomorphic function f and every z_0 in the domain there are $z_1 \neq z_2$ such that $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$.

(5) For two linearly independent entire functions, show that one cannot dominate the other.

(6) Show that uniform limit of injective holomorphic function is either constant or injective.

(7) Suppose the set of points in a domain $U \subset \mathbb{C}$ at which a sequence of bounded holomorphic functions (f_n) converges has a limit point. Show that (f_n) compactly converges.

1.4 Functional analysis

(1) Show that if $A^\circ \subset B$ and B is closed, then $(A \cup B)^\circ \subset B$.

1.5 Proability theory

(1) Find the probability that arbitrarily chosen positive integers are coprime.

1.6 Differential equations

Chapter 2

Algebra

2.1 Elementary algebra

2.1.1 Linear algebra

(1) Show that normal nilpotent matrix equals zero.

(2) Show that two matrices AB and BA have same nonzero eigenvalues whose both multiplicities are coincide...

(3) Show that if A is a square matrix whose characteristic polynomial is minimal then a matrix commuting A is a polynomial in A.

(4) Show that the order of 2×2 integer matrices divide 12 if it is finite.

2.1.2 Groups

(1) Show that a finite symmetric group has two generators.

(2) Show that a group of order $2p$ for a prime p has exactly two isomorphic types.

(3) Show that a group G is abelian if $|G| = p^2$ for a prime p.

(4) Show that a group G is abelian if it has a surjective cube map.

(5) Let G be a finite group of order n and p the smallest prime divisor of n. Show that a subgroup of G of index p is normal in G.

(6) Find all n such that $(\mathbb{Z}/n\mathbb{Z})^\times$ is cyclic.

(7) Show that a nontrivial normalizer of a p-group meets its center out of identity.

(8) Show that a proper subgroup of a finite p-group is a proper subgroup of its normalizer. In particular, every finite p-group is nilpotent.

(9) Show that a finite group G satisfying $\sum_{g\in G} \operatorname{ord}(g) \leq 2n$ is abelian.

(10) Show that the order of a group with trivial automorphism group is either 1 or 2.

(11) Find all homomorphic images of A_4 up to isomorphism.

(12) Show that in a group of order 105 is a single Sylow p-subgroup for $p = 5, 7$.

(13) Show that the number of Sylow p-subgroups of $\mathrm{SL}_3(\mathbb{F}_p)$ is $(p^2 + p + 1)(p + 1)$.

2.1.3 Rings

(1) Show that a finite integral domain is a field.

(2) Show that every ring of order p^2 for a prime p is commutative.

(3) Show that a semiring with multiplicative identity and cancellative addtion has commutative addition.

(4) Show that the complement of a saturated monoid in a commutative ring is a union of prime ideals.

2.1.4 Galois theory

(1) Show that the Galois group of a quintic over $\mathbb{Q}$ having exactly three real roots is isomorphic to S_5.

Chapter 3

Geometry

3.1 Classical geometry

3.2 Differential geometry

(1) Show that the tangent space of the unitary group at the identity is identified with the space of skew Hermitian matrices.

(2) Prove the Jacobi formula for matrix.

(3) Show that S^3 and T^2 are parallelizable.

(4) Show that $\mathbb{R}P^n = S^n/Z_2$ is orientable if and only if n is odd.

3.3 Algebraic geometry

3.4 Algebraic topology