

Interchanging Limits, Derivatives, and Integrals

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1. LIMIT AND DERIVATIVE

f_n pointwisely, Df_n uniformly

2. LIMIT AND INTEGRAL

We want to find a criterion for This question asks the convergence

$$f_n \rightarrow f \quad \text{in } L^1.$$

Theorem 2.1 (Lebesgue dominated convergence theorem). *Let $\{f_\alpha\}_\alpha$ be a net of measurable functions $(X, \mu) \rightarrow \mathbb{R}$. Define a maximal function*

$$Mf(x) = \sup_\alpha |f_\alpha(x)|.$$

If $\|Mf\|_1 < \infty$, then

$$\lim_\alpha |f_\alpha(x) - f(x)| = 0 \quad \text{a.e.} \implies \lim_\alpha \|f_\alpha - f\|_{L^1} = 0.$$

continuity application

3. DERIVATIVE AND INTEGRAL

Theorem 3.1 (Scheffe). *Let $\{f_n\}_n$ be a sequence of nonnegative functions in L^1 . Suppose it converges to f pointwisely. Then,*

$$\lim_{n \rightarrow \infty} \|f_n\|_1 = \|f\|_1 \iff \lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0.$$

Define the Newton quotient as

$$D_h f(t, x) := \frac{f(t+h, x) - f(t, x)}{h}$$

for $h \neq 0$. We mainly recognize D_h as an operator that maps $f(0, x)$ to a function of x . Then, we can say that the partial derivative $\partial_t f(0, x)$ is well-defined a.e. x if and only if

$$\lim_{h \rightarrow 0} D_h f(0, x) = \partial_t f(0, x) \quad \text{a.e. } x.$$

We may ask about conditions for the following to hold:

$$\lim_{h \rightarrow 0} D_h f(0, x) = \partial_t f(0, x) \quad \text{in } L^1_x(X).$$

This question naturally arise because it implies the commutability

$$\frac{d}{dt} \int f(t, x) dx = \int \frac{\partial}{\partial t} f(t, x) dx$$

at $t = 0$. As necessary conditions to formalize the statement, we must basically assume that $f(t, x) \in L_x^1$ for $|t| < \varepsilon$, and $\partial_t f(0, x) \in L_x^1$. Above this, if we give a stronger condition $\text{ess sup}_{|t| < \varepsilon} |\partial_t f(t, x)| \in L_x^1$ than $\partial_t f(0, x) \in L_x^1$, then the L_x^1 convergence is obtained.

Theorem 3.2 (Leibniz rule). *Let $f : (-\varepsilon, \varepsilon) \times X \rightarrow \mathbb{R}$ be a curve of integrable functions such that $f(t, x)$ is absolutely continuous in t for a.e. x . If*

$$\text{ess sup}_{|t| < \varepsilon} |\partial_t f(t, x)|$$

is in L_x^1 , then

$$\lim_{h \rightarrow 0} D_h f(0, x) = \partial_t f(0, x) \quad \text{in } L_x^1(X).$$

Proof. Our strategy is to apply the Lebesgue dominated convergence theorem. The pointwise convergence (in a.e. sense) is satisfied due to the absolute continuity.

Define the maximal function

$$Mf(x) := \sup_{|h| < \varepsilon} |D_h f(0, x)|.$$

The fundamental theorem of calculus for absolute continuous functions implies

$$D_h f(0, x) = \frac{f(h, x) - f(0, x)}{h} = \frac{1}{h} \int_0^h \partial_t f(t, x) dt,$$

so we have

$$\begin{aligned} |D_h f(0, x)| &\leq \frac{1}{h} \int_0^h |\partial_t f(t, x)| dt \\ &\leq \text{ess sup}_{|t| \leq |h|} |\partial_t f(t, x)| \left[\frac{1}{h} \int_0^h dt \right] \\ &\leq \text{ess sup}_{|t| < \varepsilon} |\partial_t f(t, x)|. \end{aligned}$$

Since the right hand side is constant with respect to h , we can deduce that Mf is in L_x^1 . By the Lebesgue dominated convergence theorem, we get the desired result. \square

Remark. If f is assumed to be differentiable everywhere on $t \in (-\varepsilon, \varepsilon)$, then we may use the mean value theorem to prove the theorem instead of the fundamental theorem of calculus: we directly get

$$|D_h f(0, x)| \leq \sup_{|t| < \varepsilon} |\partial_t f(t, x)|.$$

If f is assumed to be continuously differentiable on $t \in (-\varepsilon, \varepsilon)$, then