Real Analysis I : Measure Theory

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Topological measures

1. Radon measures

In this section, we assume every base space X is locally compact Hausdorff (Why?). In locally compact Hausdorff spaces, compact finiteness and locally finiteness are equivalent. We will consider locally finite Borel measures as the minimally compatible measures with a given topology on X.

Definition 1.1. A *Radon measure* is a Borel measure on X which satisfies the following three conditions:

(1) locally finite,
(2) outer regular on all Borel sets,
(3) inner regular on all open sets.

Radon measures are rather simply characterized when the base space X is σ-compact. The following proposition proves the equivalence between regularity and Radonness of locally finite Borel measure on a σ-compact space.

Proposition 1.1. *A Radon measure is inner regular on all σ-finite Borel sets.*

Proof. First we prove for Borel sets of finite measure. Let E be a Borel set with $\mu(E) < \infty$ and U be an open set containing E. By outer regularity, there is an open set $V \supset U - E$ such that

$$\mu(V) < \mu(U - E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K) > \mu(U) - \frac{\varepsilon}{2}.$$

Then, we have a compact set $K - V \subset K - (U - E) \subset E$ such that

$$\begin{aligned}\mu(K - V) &\geq \mu(K) - \mu(V) \\ &> \left(\mu(U) - \frac{\varepsilon}{2}\right) - \left(\mu(U - E) + \frac{\varepsilon}{2}\right) \\ &\geq \mu(E) - \varepsilon.\end{aligned}$$

It implies that a Radon measure is inner regular on Borel sets of finite measures.

Suppose E is a σ-finite Borel set so that $E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(E_n) < \infty$. We may assume E_n are pairwise disjoint. Let K_n be a compact subset of E_n such that

$$\mu(K_n) > \mu(E_n) - \frac{\varepsilon}{2^n},$$

and define $K = \bigcup_{n=1}^{\infty} K_n \subset E$. Then,

$$\mu(K) = \sum_{n=1}^{\infty} \mu(K_n) > \sum_{n=1}^{\infty} \left(\mu(E_n) - \frac{\varepsilon}{2^n}\right) = \mu(E) - \varepsilon.$$

Therefore, a Radon measure is inner regular on all σ-finite Borel sets. $\square$

We get a corollary:

Corollary 1.2. *If X is σ-compact, then a locally finite Borel measure is Radon if and only if it is regular.*

Theorem 1.3. *If every open set in X is σ-compact, then every locally finite Borel measure is regular.*

Proposition 1.4. *In a second countable space, every open set is σ-compact.*

Corollary 1.5. *If X is second countable, then every locally finite Borel measure is regular and also Radon.*

Two corollaries are presented as follows:

$$\begin{tikzcd}
\begin{array}{c}\text{locally finite}\\ \text{Borel regular}\end{array} \arrow[r] & \text{Radon} \arrow[l, bend right, dashed, "X \text{ is } \sigma\text{-compact}"'] \arrow[r] & \begin{array}{c}\text{locally finite}\\ \text{Borel}\end{array} \arrow[ll, bend left, dashed, "X \text{ is second countable}"]
\end{tikzcd}$$

Many applications assume X is an open subset of a Euclidean space, so X is usually second countable. In this case, the followings will be synonym: A measure is

(1)

$$L^1_{\text{loc}} = \text{absolutely continuous measures} \subset \text{Radon measures} \subset \mathcal{D}'.$$

2. Riesz-Markov-Kakutani representation theorem

In this section, we always assume X is a locally compact Hausdorff space. Hence we can use the Urysohn lemma in the following way: If a compact subset K and a closed subset F are disjoint, then by applying the Urysohn lemma on a compact neighborhood of K, we can find a continuous function $f : X \to [0,1]$ such that $f|_K = 1$ and $f|_F = 0$. In particular, there always exists a "continuous characteristic function" $\phi \in C_c(X)$ with $\phi|_K = 1$.

There are two Riesz-Markov-Kakutani theorems: the first theorem describes the positive elements in $C_c(X)^*$ as Radon measures when the natural colimit topology is assumed, and the second theorem describes $C_c(X)^*$ as the space of finite Radon measures when uniform topology is assumed.

2.1. The first theorem. Positivity of linear functional itself implies a rather strong continuity property.

THEOREM 2.1. *Let X be a locally compact Hausdorff space. If a linear functional on $C_c(X)$ is positive, then it is continuous with respect to the colimit topology.*

PROOF. Let I be a positive linear functional on $C_c(X)$. We want to show that on every compact subset K of X we have $|I(f)| \lesssim_K \|f\|$ for all $f \in C_K(X)$. The proof idea comes from the Hölder inequality $|\int_K f\,d\mu| \le \mu(K)\|f\|$.

Choose $\phi \in C_c(X)$ such that $\phi|_K = 1$ using the Urysohn lemma. Since $f = \phi f \le I(\phi)\|f\|$, we have

$$I(f) \le I(\phi\|f\|) = I(\phi)\|f\|.$$

Putting $-f$ instead of f, we also get $-I(f) \le I(\phi)\|f\|$. Therefore, $|I(f)| \le I(\phi)\|f\|$. □

Jordan decomposition: $(C_0(X), \|\cdot\|)^* = (C_c(X), \|\cdot\|)^* \subset (C_c(X), \lim_\to)^*$ converse?

CHAPTER 4

Hmmmm

0.2. Convergence in measure. Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0. \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > n^{-1},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n(x) - f(x)| > n^{-1}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.\end{aligned}$$

Since for every k

$$\limsup_n \{x : |f_n(x) - f(x)| > k^{-1}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\},$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.$$

THEOREM 0.2. *Let f_n be a sequence of measurable functions on a measure space (X, μ). If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.*

PROOF. Since $d_{f_n - f}(1/k) \to 0$ as $n \to \infty$, we can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) > 2^{-k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. □