# VLASOV-POISSON SYSTEM

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## Contents



## 1. Vlasov-Poisson system

Consider a Cauchy problem of the *Valsov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, \qquad (t,x,v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t,x) = -\nabla_x \Phi, \\ \Phi(t,x) = (-\Delta_x)^{-1} \rho, \\ \rho(t,x) = \int f \, dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $\gamma = \pm 1$. For example, we have *repulsive problem* $\gamma = +1$ for electrons in plasma theory and *attractive problems* $\gamma = -1$ for galactic dynamics. ($\rho$ denotes mass density.)

Results in 1.1 and 1.2 provide basic ingredients that will be used in the whole article. On the other hand, results in 1.3 and 1.4 cannot be used in any local existence proof because they assume the existence of solutions, but they help understand the fundamental nature of solutions of the Vlasov-Poisson system and are used in the proof of global existence.

We use the asymptotic notation

$$g(t) \lesssim h(t) \quad \iff \quad \exists\, c = c(f_0), \quad g(t) \le c\, h(t)$$

and

$$g(t) \simeq h(t) \quad \iff \quad \exists\, c, \quad g(t) = c\, h(t).$$

1.1. **Poisson equation.** For the boundaryless problem in which the potential function vanishes at infinity, we have

$$\Phi = \tfrac{1}{4\pi|x|} * \rho,$$

so

$$E = -\nabla_x \Phi = -\nabla_x(\tfrac{1}{4\pi|x|} * \rho) = \frac{x}{4\pi|x|^3} * \rho,$$

or it can be rewritten as

$$E(t,x) = \frac{1}{4\pi} \int \frac{(x-y)\rho(t,y)}{|x-y|^3}\, dy.$$

The nonlinearity of the system is originated from the force field $E$, so its estimates play the most important role in investigation of the nonlinear system. Since it is given by the solution of the Poisson equation, estimates of the Riesz potential is directly connected to estimates of the force field.

**Lemma 1.1** (Estimates of Riesz potential)**.** *Let* $\rho \in C_c^1(\mathbb{R}^d)$*.*

(1) *(Field estimate)*

$$\|\tfrac{1}{|x|^{d-1}} * \rho\|_\infty \lesssim \|\rho\|_\infty^{1-1/d} \|\rho\|_1^{1/d}$$

(2) *(Field derivative estimate) For* $\log^+(x) := \max\{0, \log x\}$,

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty + \|\rho\|_1.$$

*Proof.*

(1) Let $0 \le \frac{1}{p} < \frac{\alpha}{d} < \frac{1}{q} \le 1$. Since $(d-\alpha)p < d < (d-\alpha)q$,

$$\begin{aligned}
|\tfrac{1}{|x|^{d-\alpha}} * \rho| &= \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\, dy + \int_{|x-y|\ge R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\, dy \\
&\le \|\rho\|_{p'} (\int_{|y|<R} \frac{dy}{|y|^{(d-\alpha)p}})^{1/p} + \|\rho\|_{q'} (\int_{|y|\ge R} \frac{dy}{|y|^{(d-\alpha)q}})^{1/q} \\
&\simeq \|\rho\|_{p'} (\int_0^R r^{d-1-(d-\alpha)p}\, dr)^{1/p} + \|\rho\|_{q'} (\int_R^\infty r^{d-1-(d-\alpha)q}\, dr)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} + \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}.
\end{aligned}$$

By choosing $R$ such that $\|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} = \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}$, we get

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty \lesssim \|\rho\|_{p'}^{\frac{1-\frac{\alpha}{d}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho\|_{q'}^{\frac{\frac{1}{p}-1+\frac{\alpha}{d}}{\frac{1}{p}-\frac{1}{q}}},$$

so the inequality

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty^{\frac{1}{q}-\frac{1}{p}} \lesssim \|\rho\|_p^{\frac{1}{q}-\frac{\alpha}{d}} \|\rho\|_q^{\frac{\alpha}{d}-\frac{1}{p}}$$

is obtained by interchaning $p$ and $q$ with their conjugates. The desired result gets $p = \infty$, $\alpha = 1$, and $q = 1$.

(2) Let $0 < R_a \le R_b$ be constants which will be determined later. Divide the region radially

$$|\nabla(\tfrac{1}{|x|^{d-\alpha}} * \rho)| \lesssim \nabla \int_{|x-y|<R_a} + \nabla \int_{R_a \le |x-y| < R_b} + \nabla \int_{R_b \le |x-y|}.$$

For the first integral,

$$\begin{aligned}\int_{|y|<R_a} \frac{\nabla\rho(x-y)}{|y|^{d-1}}\,dy &\le \|\nabla\rho\|_\infty \int_{|y|<R_a} \frac{1}{|y|^{d-1}}\,dy \\ &\simeq \|\nabla\rho\|_\infty \int_0^{R_a} 1\,dr = R_a \|\nabla\rho\|_\infty.\end{aligned}$$

For the second integral,

$$\begin{aligned}\int_{R_a \le |x-y| < R_b} \frac{\rho(y)}{|x-y|^d}\,dy &\le \|\rho\|_\infty \int_{R_a \le |x-y| < R_b} \frac{1}{|x-y|^d}\,dy \\ &\simeq \|\rho\|_\infty \int_{R_a}^{R_b} \frac{1}{r}\,dr = (\log \tfrac{R_b}{R_a}) \|\rho\|_\infty.\end{aligned}$$

For the third integral,

$$\int_{R_b \le |x-y|} \frac{\rho(y)}{|x-y|^d}\,dy \le R_b^{-d} \|\rho\|_1.$$

Thus,

$$|\nabla(\tfrac{1}{|x|^{d-\alpha}} * \rho)| \lesssim R_a \|\nabla\rho\|_\infty + (\log \tfrac{R_b}{R_a}) \|\rho\|_\infty + R_b^{-d} \|\rho\|_1.$$

Assuming $\rho$ is nonzero so that $\|\nabla\rho\|_\infty > 0$, let $R_a = \min\{1, \|\nabla\rho\|_\infty^{-1}\}$ and $R_b = 1$. Since

$$\log \tfrac{1}{R_a} \le \log^+ \|\nabla\rho\|_\infty \quad \text{and} \quad R_a \lesssim \|\nabla\rho\|_\infty,$$

we have

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty + \|\rho\|_1. \qquad \square$$

### 1.2. **Volume preservation.**

< To be written... >

To sum up our weapons obtained in 1.1 and 1.2,

**Corollary 1.2.** *Let $\rho, E \in C^1(\mathbb{R}^+ \times \mathbb{R}^3)$ be mass density and force field with relation*

$$E(t,x) = \frac{1}{4\pi} \int \frac{(x-y)\rho(t,y)}{|x-y|^3}\,dy.$$

*Then, for $t \in [0,\infty)$*

(1) $\|\rho(t)\|_1 = \text{const}$,
(2) $\|E(t)\|_\infty \lesssim \|\rho(t)\|_\infty^{2/3}$,
(3) $\|\nabla E(t)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty$.

1.3. **Conservative laws.**

**Lemma 1.3.** *Let $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system.*

(1) *(Continuity equation)*

$$\rho_t + \nabla_x \cdot j = 0, \qquad \textit{where} \quad j = \int vf\,dv.$$

(2) *(Energy conservation)*

$$\iint |v|^2 f\,dv\,dx + \gamma \int |E|^2\,dx = \text{const}\,.$$

*Proof.*

(1) Integrate with respect to $v$ to get

$$\begin{aligned} 0 &= \int f_t\,dv + \int v \cdot \nabla_x f\,dv \\ &= \rho_t + \nabla_x \cdot \int vf\,dv \\ &= \rho_t + \nabla_x \cdot j. \end{aligned}$$

(2) Multiply $|v|^2$ and integrate with respect to $v$ and $x$ to get

$$\begin{aligned} \frac{d}{dt}\iint |v|^2 f\,dv\,dx &= \iint |v|^2 f_t\,dv\,dx = -\iint |v|^2 \gamma E \cdot \nabla_v f\,dv\,dx \\ &= \iint 2v \cdot \gamma E f\,dv\,dx = -2\gamma \int \nabla_x \Phi \cdot j\,dx \\ &= 2\gamma \int \Phi \nabla_x \cdot j\,dx = 2\gamma \int \Phi \Delta_x \Phi_t\,dx \\ &= -\frac{d}{dt}\gamma \int |E|^2\,dx. \end{aligned}$$

Thus

$$\iint |v|^2 f\,dv\,dx + \gamma \int |E|^2\,dx = \text{const}\,.$$ □

1.4. **Moment propagation.** We have a bound of kinetic energy even for $\gamma = -1$.

**Lemma 1.4** ($L^{5/3}$ estimate of $\rho$)**.** *Let $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system. For $t \in [0,T]$,*

(1) $\|\rho(t)\|_{L_x^{5/3}} \lesssim \iint |v|^2 f\,dv\,dx$.
(2) $\iint |v|^2 f\,dv\,dx \lesssim 1$.

*Proof.*

(1) Note

$$\rho(t,x) = \int f(t,x,v)\,dv \le \int_{|v|<R} f\,dv + \frac{1}{R^2}\int_{|v|\ge R} |v|^2 f\,dv$$
$$\lesssim R^3 + R^{-2}\int |v|^2 f\,dv.$$

Set $R^3 = R^{-2}\int |v|^2 f\,dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f\,dv.$$

(2) It is trivial for $\gamma = +1$. Suppose $\gamma = -1$. By the Hardy-Littlewood-Sobolev inequality,

$$\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$$

for $p = 2$, $d = 3$, and $\alpha = 1$ implies $q = 6/5$, hence the bound of $\|E(t)\|_2$

$$\|E(t)\|_2 \simeq \|\frac{1}{|x|^{d-\alpha}} *_x \rho(t,x)\|_{L^2_x} \lesssim \|\rho(t)\|_{6/5}.$$

So, interpolation with Hölder's inequality gives

$$\|E(t)\|_2 \lesssim \|\rho\|_1^{7/12}\|\rho\|_{5/3}^{5/12} \simeq \|\rho\|_{5/3}^{5/12}.$$

Thus (1) gives

$$\iint |v|^2 f\,dv\,dx = c + \|E(t)\|_2^2 \lesssim c + (\iint |v|^2 f\,dv\,dx)^{1/2},$$

so the kinetic energy $\iint f\,dv\,dx$ is bounded. As a corollary, $\|\rho(t)\|_{5/3}$ is also bounded. □

## 2. Local existence

2.1. **Approximate solution.** Suppose $f_0 \in C_c^1(\mathbb{R}^6)$. With initial term $f_0(t,x,v) = f_0(x,v)$, inductively define $f_{n+1}$ by letting the electric field $E$ be known:

$$\begin{cases} \partial_t f_{n+1} + v\cdot\nabla_x f_{n+1} + \gamma E_n \cdot \nabla_v f_{n+1} = 0, \\ E_n(t,x) = -\nabla_x \Phi_n, \\ \Phi_n(t,x) = (-\Delta_x)^{-1}\rho_n, \\ \rho_n(t,x) = \int f_n\,dv, \\ f_{n+1}(0,x,v) = f_0(x,v). \end{cases}$$

**Proposition 2.1.** *The sequence of approximate solutions $f_n$ uniquely exists globally and $C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$.*

*Proof.* Notice that $f_0$ is clearly in $C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$. Assume $f_n \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$. Then, we have

$$\rho_0 \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6)), \quad \Phi_n \in C^1(\mathbb{R}^+; C^{2,\alpha}(\mathbb{R}^6)), \text{ and } E_n \in C^1(\mathbb{R}^+; C^{1,\alpha}(\mathbb{R}^6))$$

by the Hölder regularity of the Poisson equation.

Solve

$$\begin{cases} \dot{X}_{n+1}(s;t,x,v) = V_{n+1}(s,t,x,v) \\ \dot{V}_{n+1}(s;t,x,v) = \gamma E_n(s, X_n(s;t,x,v)) \end{cases}$$

and define

$$f_{n+1}(t,x,v) := f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)).$$

Then, we can easily check $f_{n+1} \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$ and it satisfies the approximate system by the chain rule.

Since the approximate system is a semilinear with respect to $f_{n+1}$, we can get that the $C^1(\mathbb{R}^+ \times \mathbb{R}^6)$ solution is unique. (This semilinearity can also prove the existence, but it is quite difficult to show its compact support of $f_{n+1}$.) $\square$

Note that the estimate of $\|\nabla E_{n-1}\|_\infty$ guarantees the existence of characteristics, defined as follows:

$$\begin{cases} \dot{X}_n(s;t,x,v) = V_n(s,t,x,v) \\ \dot{V}_n(s;t,x,v) = \gamma E_{n-1}(s, X_n(s;t,x,v)). \end{cases}$$

Then, for fixed $t, x, v$, the chain rule implies

$$\frac{d}{ds} f_n(s, X_n(s;t,x,v), V_n(s;t,x,v)) = \partial_t f_n + \dot{X}_n \cdot \nabla_x f_n + \dot{V}_n \cdot \nabla_v f_n = 0$$

so that we have

$$f_n(s, X_n(s;t,x,v), V_n(s;t,x,v)) = f_0(X_n(0;t,x,v), V_n(0;t,x,v)).$$

2.2. **Local A priori estimates.** Firstly, the volume preserving property still holds for our approximate system, so we have

$$\|\rho_n(t)\|_1 = \text{const}, \quad \|f_n(t)\|_p = \text{const}.$$

Next, we prove local-time bounds on fields $E_n$. Introduce the following quantity.

**Definition 2.1.** Define the *velocity support* or *maximal velocity*

$$Q_n(t) = \sup\{\, |v| : f_n(s,x,v) \neq 0,\ s \in [0,t] \,\}$$

**Lemma 2.2.**

(1) *For small $t < c(f_0)$,*

$$\rho_n(t) + E_n(t) + Q_n(t) \lesssim 1$$

*indendent on $n$.*

*Proof.*

(1) Since
$$\|\rho_n(t)\|_\infty \le Q^3(t)\|f_0\|_\infty \lesssim Q_n^3(t),$$
a rough estimate for $\|E\|_\infty$ gives
$$\|E_n(t)\|_\infty \le \|\rho_n(t)\|_\infty^{2/3}\|\rho_n(t)\|_1^{1/3} \lesssim Q_n^2(t).$$
Let $c = c(f_0)$ be a constant such that $\|E_n(t)\| \le cQ^2(t)$. We claim that
$$Q_n(t) \le \frac{Q_0}{1 - cQ_0 t}$$
for all $n$. Easily checked for $n = 0$; $Q_0(t) = Q_0 \le \frac{Q_0}{1-cQ_0t}$.

Assume $Q_n(t) \le \frac{Q_0}{1-cQ_0t}$. Then,
$$\begin{aligned}|V_{n+1}(s;0,x,v)| &\le |v| + \int_0^t |E_n(s;0,x,v)|\,ds\\ &\le Q_0 + c\int_0^t Q_n^2(s)\,ds\end{aligned}$$
implies
$$\begin{aligned}Q_{n+1}(t) &\le Q_0 + c\int_0^t Q_n^2(s)\,ds\\ &\le Q_0 + c\int_0^t \left(\frac{Q_0}{1-cQ_0s}\right)^2 ds = \frac{Q_0}{1-cQ_0t}.\end{aligned}$$
By induction, $Q_n(t) \lesssim 1$ for all $n$ and $t \in [0,T]$, where $T < (cQ_0)^{-1}$. □

### 2.3. A priori estimates of $\nabla_x \rho_n$ and $\nabla_x E_n$.

### 2.4. Convergence of $f_n$.

**Lemma 2.3.** *In short time,*

(1)
$$\|f_{n+1}(t) - f_n(t)\|_\infty \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds.$$
(2)
$$\|E_n(s) - E_{n-1}(s)\|_\infty \lesssim \|f_n(s) - f_{n-1}(s)\|_\infty^{1/3}.$$
(3) *$f_n$ converges uniformly in $C([0,T]\times\mathbb{R}^6)$.*

*Proof.*

(1) Take $T$ such that
$$T < \min\{\|\nabla_x E_n(t)\|_\infty^{-1}, 1\} \quad \text{and} \quad Q_n(t) \lesssim 1$$
for all $n$ and $t < T$. This can be done thanks for the local a priori bound for $\nabla_x E_n$.

The $C^1$ regularity of $f_0$ lets

$$\begin{aligned}&|f_{n+1}(t,x,v)-f_n(t,x,v)|\\&\quad=|f_0(X_{n+1}(0;t,x,v),V_{n+1}(0;t,x,v))-f_0(X_n(0;t,x,v),V_n(0;t,x,v))|\\&\quad\lesssim|X_{n+1}(0;t,x,v)-X_n(0;t,x,v)|+|V_{n+1}(0;t,x,v)-V_n(0;t,x,v)|.\end{aligned}$$

Because

$$\begin{aligned}X_n(s;t,x,v)&=x-\int_s^t V_n(s';t,x,v)\,ds',\\V_n(s;t,x,v)&=v-\int_s^t E_{n-1}(s',X_n(s;t,x,v))\,ds',\end{aligned}$$

we have

$$\begin{aligned}&|V_{n+1}(s;t,x,v)-V_n(s;t,x,v)|\\&\quad\le\int_0^t|E_n(s,X_{n+1}(s;t,x,v))-E_{n-1}(s,X_n(s;t,x,v))|\,ds\\&\quad\le\int_0^t|E_n(s,X_{n+1})-E_n(s,X_n)|+|E_n(s,X_n)-E_{n-1}(s,X_n)|\,ds\\&\quad\le T\sup_{s\in[0,t]}\|\nabla_xE_n(s)\|_\infty|X_{n+1}(s)-X_n(s)|+\int_0^t\|E_n(s)-E_{n-1}(s)\|_\infty\,ds\end{aligned}$$

and

$$\begin{aligned}|X_{n+1}(s;t,x,v)-X_n(s;t,x,v)|&\le\int_0^t|V_{n+1}(s;t,x,v)-V_n(s;t,x,v)|\,ds\\&\le T\sup_{s\in[0,t]}|V_{n+1}(s)-V_n(s)|.\end{aligned}$$

Since $T$ is taken such that

$$T<\min\{\|\nabla E_n(t)\|_\infty^{-1},1\},$$

we get by combining the above two inequality,

$$\begin{aligned}&\sup_{s\in[0,t]}|X_{n+1}(s;t,x,v)-X_n(s;t,x,v)|+|V_{n+1}(s;t,x,v)-V_n(s;t,x,v)|\\&\qquad\lesssim\int_0^t\|E_n(s)-E_{n-1}(s)\|_\infty\,ds.\end{aligned}$$

(2) Notice that

$$\|E_n(t)-E_{n-1}(t)\|_\infty\lesssim\|\rho_n(t)-\rho_{n-1}(t)\|_1^{1/3}\|\rho_n(t)-\rho_{n-1}(t)\|_\infty^{2/3}.$$

For $L^\infty$-norm,

$$\|\rho_n(t)-\rho_{n-1}(t)\|\lesssim Q_n^3(t)+Q_{n-1}^3(t)\lesssim1.$$

For $L^1$-norm, since the support of $f_n,f_{n-1}$ is bounded in both directions $x,v$,

$$\|\rho_n(t)-\rho_{n-1}(t)\|_1\le\|f_n(t)-f_{n-1}(t)\|_1\lesssim\|f_n(t)-f_{n-1}(t)\|_\infty.$$

$\square$

### 2.5. **Uniqueness.**

## 3. Global existence

### 3.1. **Prolongation criterion.**

**Theorem** (Schaeffer, 1991)**.** *Let $f_0 \in C^1_{c,x,v}$ and $f_0 \geq 0$. Then, the Cauchy problem for the VP system has a unique $C^1$ global solution.*

**Definition 3.1.** For a local solution $f$,

$$Q(t) := 1 + \sup\{|v| : f(s,x,v) \neq 0 \text{ for some } s \in [0,t],\, x \in \mathbb{R}^3_x\}.$$

Decompose $[t-\Delta, t] \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ as

$$\begin{aligned} U &= \Big\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq r \Big\}, \\ B &= \Big\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |v| \geq P \Big\} \setminus U, \\ G &= \Big\{ (s,x,v) : \ |v - \widehat{V}(t)| < P \quad \text{or} \quad |v| < P \Big\}. \end{aligned}$$

(We can let $U \mapsto U \cap \{|v| \geq P\}$ to make the decomposition disjoint.) Later we choose

$$P = Q^{4/11}, \quad r = R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}, \quad R = Q^{16/33} \log^{1/2} Q.$$

Also, later we choose $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

### 3.2. **Some observations.**

Our goal is to obtain a priori estimate like

$$\|E(t)\|_\infty \lesssim Q(t)^a \qquad \text{for some } a < 1.$$

Since the force field $E$ measures the maximal rate of changes in velocity, the estimate can be read very roughly as

$$Q'(t) \lesssim Q(t)^a,$$

which lead its polynomial growth.

So we need to bound the Riesz potential $E$. The following observation suggests a lower bound of relative velocity.

**Claim.** *Fix $t, x, v$. If $|v - \widehat{V}(t)| \geq P$, then*

$$|y - \widehat{X}(s)| \geq \frac{1}{4} |v - \widehat{V}(t)| |s - s_0|$$

*for some $s_0 \in [t-\Delta, t]$, where $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.*

*Proof.* Since $\Delta \|E(s)\|_\infty < \frac{P}{4}$, we have

$$|v - w| < \frac{P}{4} \quad \text{and} \quad |\widehat{V}(t) - \widehat{V}(s)| < \frac{P}{4}.$$

The condition $|v - \widehat{V}(t)| \geq P$ implies

$$\frac{1}{2} |v - \widehat{V}(t)| \leq |v - \widehat{V}(t)| - \frac{P}{4} - \frac{P}{4} < |w - \widehat{V}(s)|.$$

Let $Z(s) := y - \widehat{X}(s)$. Then,

$$\begin{aligned} Z'(s) &= w - \widehat{V}(s), \\ Z''(s) &= \gamma[E(s,y,w) - E(s,\widehat{X}(s),\widehat{V}(s))]. \end{aligned}$$

Let $s_0 \in [t-\Delta, t]$ minimize $s \mapsto |Z(s)|$ and expand $Z$ as

$$Z(s) = Z(s_0) + Z'(s_0)(s-s_0) + \frac{Z''(\sigma)}{2}(s-s_0)^2$$

for some $\sigma$ between $s$ and $s_0$. Then,

$$|Z(s_0) + Z'(s_0)(s-s_0)| \ge |Z'(s_0)(s-s_0)| \ge \frac{1}{2}|v - \widehat{V}(t)||s-s_0|$$

and

$$\begin{aligned} |\frac{Z''(\sigma)}{2}(s-s_0)^2| &\le \|E(t)\|_\infty (s-s_0)^2 \le \|E(t)\|_\infty \Delta |s-s_0| \\ &\le \frac{P}{4}|s-s_0| \le \frac{1}{4}|v-\widehat{V}(t)||s-s_0| \end{aligned}$$

proves

$$|y - \widehat{X}(s)| = |Z(s)| \ge \frac{1}{4}|v - \widehat{V}(t)||s-s_0|. \qquad \square$$

We introduce time averaging to use the above lower bound.

**Claim.** *Fix $t, x, v$. If $|v - \widehat{V}(t)| \ge P$, then*

$$\int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim \frac{r^{-1}}{|v-\widehat{V}(t)|},$$

*where $A = \{s : |y - \widehat{X}(s)| \ge r\}$.*

*Proof.* Since $|y - \widehat{X}(s)| \ge \frac{1}{4}|v-\widehat{V}(t)||s-s_0|$,

$$\begin{aligned} \int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds &\le 16\int_{t-\Delta}^t \frac{1}{|v-\widehat{V}(t)|^2|s-s_0|^2}\chi_A(s)\,ds \\ &\le 32\int_r^\infty \frac{1}{|v-\widehat{V}(t)|^3|s-s_0|^2}\,d(|v-\widehat{V}(t)||s-s_0|) \\ &= 32\,\frac{r^{-1}}{|v-\widehat{V}(t)|}. \end{aligned} \qquad \square$$

### 3.3. Divide and conquer.

3.3.1. *Ugly set estimate.* Therefore, if we let $r^{-1} \simeq \min\{|v|^3, |v-\widehat{V}(t)|^3\}$, then

$$\int_{t-\Delta}^{t} \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim |v|^2$$

so that we have

$$\iiint_U \frac{f(s,y,w)}{|y-\widehat{X}(s)|^2}\,dw\,dy\,ds \lesssim R^{-1}\int |v|^2 f(t,x,v)\,dv\,dx \lesssim R^{-1}$$

when

$$U \subset \{\,(s,x,v):\ |v-\widehat{V}(t)| \geq P, \quad |y-\widehat{X}(s)| \geq R\max\{|v|^{-3}, |v-\widehat{V}(t)|^{-3}\}\,\}.$$

3.3.2. *Bad set estimate.* Consider $U^c$. We need to control the union of two regions

$$|y-\widehat{X}(s)| < R|v|^{-3} \quad \text{and} \quad |y-\widehat{X}(s)| < R|v-\widehat{V}(t)|^{-3}.$$

Without any conditions, the integration of fundamental solution with respect to $y$ gives

$$\int_{|y-\widehat{X}(s)|<r} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \simeq r.$$

**Claim.** *If $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$, then*

$$\int_{U^c} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \lesssim \max\{|w|^{-3}, |w-\widehat{V}(s)|^{-3}\}$$

*for $s \in [t-\Delta, t]$.*

*Proof.* It follows from

$$|w| \simeq |v|, \quad |w-\widehat{V}(s)| \simeq |v-\widehat{V}(t)|$$

for $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$. □

3.3.3. *Good set estimate.*

3.4. **Polynomial decay.**

**Lemma 3.1.** *Along the time of existence we have*

$$\|E(t)\|_{L^\infty_x} \lesssim Q(t)^{4/3}.$$

*Proof.* Note that we have

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{4/9}\|\rho\|_{5/3}^{5/9}.$$

Since the velocity support of $f$ is bounded by finite $Q(t)$,

$$\rho(t,x) = \int_{|v|<Q(t)} f(t,x,v)\,dv \lesssim Q(t)^3\|f_0(x)\|_{L^\infty_v} \lesssim Q(t)^3,$$

so

$$\|E(t)\|_{L^\infty_x} \lesssim \|\rho(t)\|_{L^\infty_x}^{4/9} \lesssim Q(t)^{4/3}.$$ □