Finite Group Theory

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1. Sylow game

Definition 1.1 (Sylow p-subgroup)**.** Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. A *Sylow p-subgroup* is a subgroup of order p^a. We are going to denote the set of Sylow p-subgroups by $\mathrm{Syl}_p(G)$ and the number of Sylow p-subgroups by $n_p(G)$.

Theorem 1.1 (The Sylow theorem)**.** *Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. Then,*

$$p \mid n_p - 1, \qquad n_p \mid m$$

for some $k \in \mathbb{N}$.

Proof. Step 1: Sylow p-subgroups exist. We apply mathematical induction. The base step is trivial. Suppose every finite group of order less than n possesses a Sylow p-subgroup.

By applying the orbit-stabilizer theorem for the action $G \curvearrowright G$ by conjugation, build the class equation

$$|G| = |Z(G)| + \sum_i |G : C_G(g_i)|.$$

There are two cases: $p \mid |Z(G)|$ or $p \nmid |Z(G)|$.

Case 1: $p \mid |Z(G)|$. The group G has a normal subgroup of order p by applying Cauchy's theorem for abelian groups on the center. Then, the inverse image of a Sylow p-subgroup of the quotient group is also a Sylow p-subgroup of G.

Case 2: $p \nmid |Z(G)|$. Since $p \mid n$, we have $p \nmid |G : C_G(g)|$ for some $g \in G$. Then, a Sylow p-subgroup of the centralizer is also a Sylow p-subgroup of G.

Therefore, we are done for Step 1.

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Step 2: Sylow p-subgroup that is normal is unique. Note that p does not divide the order of the quotient group. Every p-subgroup should be contained in the Sylow p-subgroup, the kernel of the quotient map. The Sylow p-subgroup is clearly unique.

Step 3: Sylow p-subgroups get action by conjugation. Let P be a Sylow p-subgroup of G. We construct class equations via the orbit-stabilizer theorm for various actions to extract information on n_p. Note that stabilizers in any setwise conjugation action is exactly normalizers.

(1) The action $P \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = 1 + \sum_i |P : N_P(P_i)|$$
since $P = N_P(P_i)$ implies $P \trianglelefteq N_G(P_i)$ and $P = P_i$.

(2) Suppose the action $G \curvearrowright \mathrm{Syl}_p(G)$ is not transitive. Take another Sylow p-subgroup P' is not conjugate with P in G. The two actions $P \curvearrowright \mathrm{Orb}_G(P)$ and $P' \curvearrowright \mathrm{Orb}_G(P)$ gives
$$|\mathrm{Orb}_G(P)| = 1 + \sum_i |P : N_P(P_i)| = \sum_i |P' : N_{P'}(P_i)|.$$
It deduces $|\mathrm{Orb}_G(P)| \equiv 0, 1 \pmod p$ simultaneously, which leas a contradiction.

(3) The action $G \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = |G : N_G(P_i)|$$
for all $P_i \in \mathrm{Syl}_p(G)$ because the action is transitive.

Then, (1) proves $p \mid n_p - 1$, and (3) proves $n_p \mid m$. □

Corollary 1.2. *Let G be a finite group. Then,*

(1) *every pair of two Sylow p-subgroup is conjugate.*
(2) *every p-subgroup is contained in a Sylow p-subgroup.*
(3) *a Sylow p-subgroup is normal if and only if $n_p = 1$.*

Theorem 1.3. *Alternative proof for existence of p-groups.*

Proof. Let $|G| = p^{a+b}m$. Let $\mathcal{P}_{p^a}$ be the set of all p^a-sets in G. Give $G \curvearrowright \mathcal{P}_{p^a}$ by left multiplication. Since $v_p(|\mathcal{P}_{p^a}|) = v_p(\binom{p^a(p^b m)}{p^a}) = b$, there is an orbit $\mathcal{O}$ such that $v_p(|\mathcal{O}|) \le b$. We have transitive action $G \curvearrowright \mathcal{O}$ and the stabilizer H satisfies $p^a \mid |G|/|\mathcal{O}| = |H|$. Since $H \curvearrowright \mathcal{O}$ trivially, $H \curvearrowright A$ for $A \in \mathcal{O} \subset \mathcal{P}_{p^a}$. It is only possible when $H \subset A$, hence $|H| = p^a$. □

By Hölder program, normal subgroups always benefit:

(1) existence of subgroup of particular order(by extension),
(2) contradiction by n_p element counting

A normal subgroup of a subgroup makes *normalizer lifting* that results in:

(1) existence of subgroup of particular order(by normalizer),
(2) existence of normal subgroup,
(3) constraint of n_p by normalizer of Sylow subgroup.

Find a subgroup of nice order

What we want to find is the *distribution of subgroups* such as their orders, numbers, itersections, normality, and isomorphic types.

In order to show the existence of subgroups of paricular orders:

(1) use induction and expand class equation
(2) extend subgroups of a quotient group
(3) get normalizers of Sylow p-subgroups

(1) get a permutation representation by left multiplication
(2) get normalizers or centralizers by class equation

(1) count elements
(2) compare Sylow p-subgroups of prime dominating subgroups

2. Simple groups

2.1. **Symmetric groups.**

2.2. **Linear groups.**

3. Extensions

outer semidirect product and inner semidirect product

Proposition 3.1. *Let N be a normal subgroup of G.*

(1) *there is $H < G$ such that $G = NH$ and $N \cap H = 1$,*
(2)
(3)