

Functional Analysis II : Operator Algebra

Lecture by Ikhan Choi

Notes by Ikhan Choi

Contents

Chapter 1. Banach algebras	5
1. Spectra	6
2. Gelfand theory	7
Chapter 2. Operators on Hilbert spaces	9
1. Spectral theory	10
Chapter 3. C^* -algebras	11
1. Basics	12
2. The Gelfand-Naimark theorems	13
2.1. Commutative Banach algebras	13
Chapter 4. Von Neumann algebras	15

CHAPTER 1

Banach algebras

1. Spectra

2. Gelfand theory

THEOREM 2.1. *If x, y commutes, then $\sigma(xy) \subset \sigma(x)\sigma(y)$.*

CHAPTER 2

Operators on Hilbert spaces

1. Spectral theory

CHAPTER 3

C^* -algebras

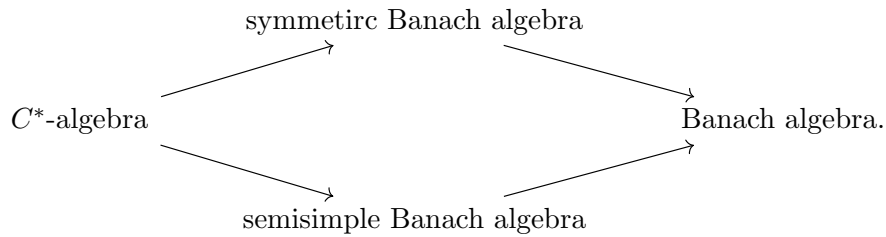
1. Basics

2. The Gelfand-Naimark theorems

2.1. Commutative Banach algebras. The Gelfand representation $A \rightarrow C_0(\widehat{A})$ can be defined for commutative Banach algebras.

DEFINITION 2.1. A *symmetric* Banach algebra is an involutive Banach algebra for which the Gelfand representation preserves the involution. We will not consider non-symmetric involutive Banach algebras in this section.

Notice the following implication:



Let A be a commutative Banach algebra.

THEOREM 2.1. *If A is semisimple, then the Gelfand representation is a monomorphism; it is injective.*

PROOF. It is because the kernel is given by the Jacobson radical. \square

THEOREM 2.2. *If A is symmetric, then the Gelfand representation is an epimorphism; it has a dense range.*

PROOF. The image is closed under all operations except involution, separates points, and vanishes nowhere. If A is symmetric, then the image is closed under involution. Thus, by the Stone-Weierstrass theorem, we get the result. \square

C^* -algebras are semisimple and symmetric (even if it is noncommutative).

THEOREM 2.3. *A C^* -algebra is semisimple.*

THEOREM 2.4. *A C^* -algebra is symmetric.*

PROOF 1. It is by Arens. \square

PROOF 2. It is by Fukamiya. \square

Furthermore,

THEOREM 2.5. *If A is a commutative C^* -algebra, then the Gelfand representation is isometric.*

Since an isometry is injective and has a closed range, therefore, it should be isometric * -isomorphism.

CHAPTER 4

Von Neumann algebras