Diachrony of Spectra

Ikhan Choi

Postech - Unist - Kaist Joint Seminar

August 4, 2019

Introduction

Introduction

Definition

Let R be a commutative ring. The *spectrum* of R is the set of prime ideals of R.

Introduction

Definition

Let R be a commutative ring. The *spectrum* of R is the set of prime ideals of R.

Example

$$\mathrm{Spec}(\mathbb{Z}) = \{ 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}, 11\mathbb{Z}, \cdots \}.$$

Introduction

Definition

Let R be a commutative ring. The *spectrum* of R is the set of prime ideals of R.

Example

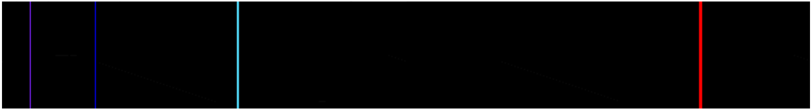
$$\operatorname{Spec}(\mathbb{Z}) = \{ 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}, 11\mathbb{Z}, \cdots \}.$$

Question

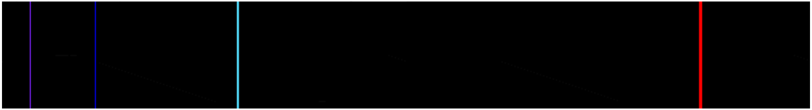
Why is it defined like this?

Contents

Hydrogen spectral series



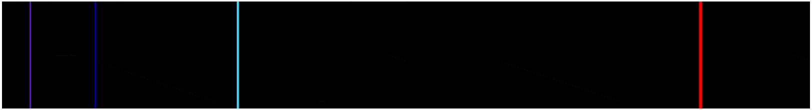
Hydrogen spectral series



410.2nm 468.1nm 656.3 nm

434.0nm

Hydrogen spectral series

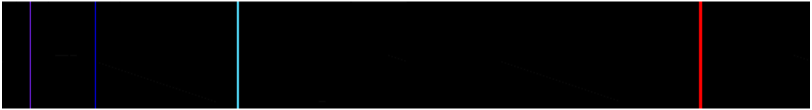


410.2nm 468.1nm 656.3 nm
434.0nm

Question

How can we explain and compute this phenomenon?

Hydrogen spectral series



410.2nm 468.1nm 656.3 nm
434.0nm

Question

How can we explain and compute this phenomenon?

A: By the following formula!

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \text{for} \quad n_1, n_2 \in \mathbb{N}.$$

Rydberg's formula (1): Bohr model

Bohr's postulates:

Rydberg's formula (1): Bohr model

Bohr's postulates:

- The electrons are on certian stable orbits.

Rydberg's formula (1): Bohr model

Bohr's postulates:

- The electrons are on certian stable orbits.
- The stationary orbits are computed by the old quantization assumption for angular momenta:

$$mvr = n\hbar.$$

Rydberg's formula (1): Bohr model

Bohr's postulates:

- The electrons are on certian stable orbits.
- The stationary orbits are computed by the old quantization assumption for angular momenta:

$$mvr = n\hbar.$$

- An electron absorbs or emits light frequency f when they jump from an orbit to another, satisfying

$$\Delta E = hf.$$

Rydberg's formula (1): Bohr model

Bohr's postulates:

- The electrons are on certian stable orbits.
- The stationary orbits are computed by the old quantization assumption for angular momenta:

$$mvr = n\hbar.$$

- An electron absorbs or emits light frequency f when they jump from an orbit to another, satisfying

$$\Delta E = hf.$$

The constant h is called the Planck constant and $\hbar := \frac{h}{2\pi}$.

Rydberg's formula (1): Bohr model

From the three relations

$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

Rydberg's formula (1): Bohr model

From the three relations

$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

we deduce

$$E = -\frac{k^2e^4m}{2\hbar^2}\frac{1}{n^2} \approx -13.6\frac{1}{n^2}\ (eV).$$

Rydberg's formula (1): Bohr model

From the three relations

$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

we deduce

$$E = -\frac{k^2 e^4 m}{2\hbar^2}\frac{1}{n^2} \approx -13.6\frac{1}{n^2} \ (eV).$$

Proposition (Rydberg formula)

The wavelengths λ of absorbed or emitted photons from a hydrogen atom is estimated by the following formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \textit{for} \quad n_1, n_2 \in \mathbb{N},$$

where $R := \frac{k^2 e^4 m}{4\pi\hbar^3 c}$ is the Rydberg constant.

Rydberg's formula (2): Schrödinger equation

More mathematically!

Rydberg's formula (2): Schrödinger equation

More mathematically!
In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

$$i\hbar \frac{\partial}{\partial t} \Psi(t, x) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(t, x) + V(x)\Psi(t, x),$$

Rydberg's formula (2): Schrödinger equation

More mathematically!

In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

$$i\hbar \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x),$$

energy kinetic energy potential energy

Rydberg's formula (2): Schrödinger equation

More mathematically!

In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

$$i\hbar \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x),$$

energy kinetic energy potential energy

where V is given by the Coulomb potential

$$V(x) = -k\frac{e^2}{|x|}.$$

Rydberg's formula (2): Schrödinger equation

More mathematically!
In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

$$i\hbar \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x),$$

energy kinetic energy potential energy

where V is given by the Coulomb potential

$$V(x) = -k\frac{e^2}{|x|}.$$

By solving it, we obtain the probability distribution $|\Psi(t,x)|^2$ of the electron at time t, hence the assumption $\forall t,\ \int |\Psi(t,x)|^2\, dx = 1 < \infty$.

Rydberg's formula (2): Schrödinger equation

More mathematically!
In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

$$i\hbar \frac{\partial}{\partial t}\Psi(t,x) = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x),$$

energy kinetic energy potential energy

where V is given by the Coulomb potential

$$V(x) = -k\frac{e^2}{|x|}.$$

By solving it, we obtain the probability distribution $|\Psi(t,x)|^2$ of the electron at time t, hence the assumption $\forall t,\ \int |\Psi(t,x)|^2\, dx = 1 < \infty$.
Let's solve.

Separation of variables and Eigenvalue problems

$$i\hbar \frac{\partial \Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(t,x) + V(x)\Psi(t,x).$$

Separation of variables and Eigenvalue problems

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x).$$

"Mathematization":

$$i\partial_t\Psi(t,x) = (-\Delta + V(x))\Psi(t,x), \qquad V(x) = -\frac{2}{|x|}.$$

Separation of variables and Eigenvalue problems

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x).$$

"Mathematization":

$$i\partial_t\Psi(t,x) = (-\Delta + V(x))\Psi(t,x), \qquad V(x) = -\frac{2}{|x|}.$$

Ansatz: if the solutions has the form $\Psi(t,x) = \phi(t)\psi(x)$,

Separation of variables and Eigenvalue problems

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x).$$

"Mathematization":

$$i\partial_t\Psi(t,x) = (-\Delta + V(x))\Psi(t,x), \qquad V(x) = -\frac{2}{|x|}.$$

Ansatz: if the solutions has the form $\Psi(t,x) = \phi(t)\psi(x)$, then

$$\frac{i\partial_t\phi(t)}{\phi(t)} = \frac{(-\Delta + V(x))\psi(x)}{\psi(x)}$$

Separation of variables and Eigenvalue problems

$$i\hbar \frac{\partial \Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \Psi(t,x) + V(x)\Psi(t,x).$$

"Mathematization":

$$i\partial_t \Psi(t,x) = (-\Delta + V(x))\Psi(t,x), \qquad V(x) = -\frac{2}{|x|}.$$

Ansatz: if the solutions has the form $\Psi(t,x) = \phi(t)\psi(x)$, then

$$\frac{i\partial_t \phi(t)}{\phi(t)} = \frac{(-\Delta + V(x))\psi(x)}{\psi(x)} = E$$

for some constant E, which is interpreted as the energy of electron.

Separation of variables and Eigenvalue problems

$$i\hbar\frac{\partial\Psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(t,x) + V(x)\Psi(t,x).$$

"Mathematization":

$$i\partial_t\Psi(t,x) = (-\Delta + V(x))\Psi(t,x), \qquad V(x) = -\frac{2}{|x|}.$$

Ansatz: if the solutions has the form $\Psi(t,x) = \phi(t)\psi(x)$, then

$$\frac{i\partial_t\phi(t)}{\phi(t)} = \frac{(-\Delta + V(x))\psi(x)}{\psi(x)} = E$$

for some constant E, which is interpreted as the energy of electron.
$\therefore$ We have two *eigenvalue problems* with *shared eigenvalue* E:

$$i\frac{d}{dt}\phi(t) = E\phi(t), \qquad (-\Delta + V(x))\psi(x) = E\psi(x).$$

(Solutions may or may not exist according to E!)

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

- Functions of the form $\Psi(t,x) = \phi_E(t)\psi_E(x)$ and linear combinations of them are solutions of the original Schrödinger equation.

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

- Functions of the form $\Psi(t,x) = \phi_E(t)\psi_E(x)$ and linear combinations of them are solutions of the original Schrödinger equation.
- It is known that *all solutions are found in this way*: general solution of the original Schrödinger equation is given by

$$\Psi(t,x) = \sum_E \phi_E(t)\psi_E(x) \quad \text{or} \quad \int_E \phi_E(t)\psi_E(x)\, dE.$$

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

- Functions of the form $\Psi(t,x) = \phi_E(t)\psi_E(x)$ and linear combinations of them are solutions of the original Schrödinger equation.
- It is known that *all solutions are found in this way*: general solution of the original Schrödinger equation is given by

$$\Psi(t,x) = \sum_E \phi_E(t)\psi_E(x) \quad \text{or} \quad \int_E \phi_E(t)\psi_E(x)\,dE.$$

- For a given E, ϕ_E and ψ_E are of course not unique. In fact they form a vector space which is called the eigenspace.

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

- Functions of the form $\Psi(t,x) = \phi_E(t)\psi_E(x)$ and linear combinations of them are solutions of the original Schrödinger equation.
- It is known that *all solutions are found in this way*: general solution of the original Schrödinger equation is given by

$$\Psi(t,x) = \sum_E \phi_E(t)\psi_E(x) \quad \text{or} \quad \int_E \phi_E(t)\psi_E(x)\, dE.$$

- For a given E, ϕ_E and ψ_E are of course not unique. In fact they form a vector space which is called the eigenspace.
- Note that for some E we probably cannot find the solution $\psi_E(x)$ that satisfies $\int |\psi_E(x)|^2\, dx = 1$: the eigenspace is trivial.

Separation of variables and Eigenvalue problems

Suppose we already have found the solutions $\phi_E(t)$, $\psi_E(x)$ of the eigenvalue problems for each complex number E.
Here are some facts:

- Functions of the form $\Psi(t,x) = \phi_E(t)\psi_E(x)$ and linear combinations of them are solutions of the original Schrödinger equation.
- It is known that *all solutions are found in this way*: general solution of the original Schrödinger equation is given by

$$\Psi(t,x) = \sum_E \phi_E(t)\psi_E(x) \quad \text{or} \quad \int_E \phi_E(t)\psi_E(x)\, dE.$$

- For a given E, ϕ_E and ψ_E are of course not unique. In fact they form a vector space which is called the eigenspace.
- Note that for some E we probably cannot find the solution $\psi_E(x)$ that satisfies $\int |\psi_E(x)|^2\, dx = 1$: the eigenspace is trivial.
- Since $\phi_E(t) \propto e^{-iEt}$ is easily solved, the main difficulty is ψ_E.

Separation of variables and Eigenvalue problems

So, here are what we need to investigate seriously:

Separation of variables and Eigenvalue problems

So, here are what we need to investigate seriously: *what are the eigenvalues and eigenvectors of the linear operator*

$$\mathcal{H} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined by

$$\mathcal{H}\psi(x) := (-\Delta - 2|x|^{-1})\psi(x)?$$

Also, how can we compute them?

Separation of variables and Eigenvalue problems

So, here are what we need to investigate seriously: *what are the eigenvalues and eigenvectors of the linear operator*

$$\mathcal{H} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined by

$$\mathcal{H}\psi(x) := (-\Delta - 2|x|^{-1})\psi(x)?$$

Also, how can we compute them?

$\star\star\star$ **The Beginning of Spectral Theory** $\star\star\star$

Separation of variables and Eigenvalue problems

So, here are what we need to investigate seriously: *what are the eigenvalues and eigenvectors of the linear operator*

$$\mathcal{H} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined by

$$\mathcal{H}\psi(x) := (-\Delta - 2|x|^{-1})\psi(x)?$$

Also, how can we compute them?

$\star\star\star$ **The Beginning of Spectral Theory** $\star\star\star$

Remark

Simply, $L^2(\mathbb{R}^3)$ is the space of $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\int |f|^2 < \infty$. In fact, there are many technical issues to formalize this problem, for example, L^2 function is in general not differentiable.

Separation of variables and Eigenvalue problems

So, here are what we need to investigate seriously: *what are the eigenvalues and eigenvectors of the linear operator*

$$\mathcal{H} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

defined by

$$\mathcal{H}\psi(x) := (-\Delta - 2|x|^{-1})\psi(x)?$$

Also, how can we compute them?

$\star\star\star$ **The Beginning of Spectral Theory** $\star\star\star$

Remark

Simply, $L^2(\mathbb{R}^3)$ is the space of $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\int |f|^2 < \infty$. In fact, there are many technical issues to formalize this problem, for example, L^2 function is in general not differentiable.
Don't be so pedantic in doing physics.

Eigenvalues of hydrogen Hamiltonian

Anyway, with long long calculations and hard hard mathematics, experts have found the following result:

Proposition

The eigenvalues of the operator $\mathcal{H} = -\Delta - 2|x|^{-1}$ *are*

Eigenvalues of hydrogen Hamiltonian

Anyway, with long long calculations and hard hard mathematics, experts have found the following result:

Proposition

The eigenvalues of the operator $\mathcal{H} = -\Delta - 2|x|^{-1}$ *are*

$$-1, -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}, \dots$$

Eigenvalues of hydrogen Hamiltonian

Anyway, with long long calculations and hard hard mathematics, experts have found the following result:

Proposition

The eigenvalues of the operator $\mathcal{H} = -\Delta - 2|x|^{-1}$ are

$$-1, -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}, \dots$$

with multiplicity 1, 4, 9, 16 and so on.

- Eigenvalues embody the possible energies of an electron, so we can give the Rydberg formula a reasonable explanation.

Eigenvalues of hydrogen Hamiltonian

Anyway, with long long calculations and hard hard mathematics, experts have found the following result:

Proposition

The eigenvalues of the operator $\mathcal{H} = -\Delta - 2|x|^{-1}$ *are*

$$-1, -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}, \dots$$

with multiplicity 1, 4, 9, 16 and so on.

- Eigenvalues embody the possible energies of an electron, so we can give the Rydberg formula a reasonable explanation.
- This result explains not only the discretized energy spectrum but also the number of orbitals in each electron shell!

Eigenvalues of hydrogen Hamiltonian

Anyway, with long long calculations and hard hard mathematics, experts have found the following result:

Proposition

The eigenvalues of the operator $\mathcal{H} = -\Delta - 2|x|^{-1}$ *are*

$$-1, -\frac{1}{4}, -\frac{1}{9}, -\frac{1}{16}, \ldots$$

with multiplicity 1, 4, 9, 16 and so on.

- Eigenvalues embody the possible energies of an electron, so we can give the Rydberg formula a reasonable explanation.
- This result explains not only the discretized energy spectrum but also the number of orbitals in each electron shell!
- We call the set of eigenvalues by **spectrum** of $\mathcal{H}$.

Eigenvalues of hydrogen Hamiltonian

The simultaneous equation is solved when $E = -\frac{1}{n^2}$ for some $n \in \mathbb{N}$:

$$i\frac{d}{dt}\phi(t) = E\phi(t), \qquad (-\Delta + V(x))\psi(x) = E\psi(x).$$

Eigenvalues of hydrogen Hamiltonian

The simultaneous equation is solved when $E = -\frac{1}{n^2}$ for some $n \in \mathbb{N}$:

$$i\frac{d}{dt}\phi(t) = E\phi(t), \qquad (-\Delta + V(x))\psi(x) = E\psi(x).$$

General solution of the Schrödinger equation is like

$$\begin{aligned}
\Psi(t,x) &= \sum_{n=1}^{\infty} \phi_n(t)\psi_n(x) \\
&= \sum_{n=1}^{\infty} e^{i\frac{1}{n^2}t}\left(\sum_{i=1}^{n^2} c_{n,i}\psi_{n,i}(x)\right) \\
&= \sum_{n=1}^{\infty} e^{i\frac{1}{n^2}t}\sum_{l=0}^{n-1}\sum_{m=-l}^{l} c_{nlm}\,\psi_{nlm}(x).
\end{aligned}$$

Conclusion of Section 1

$$
\begin{tikzcd}
\text{Partial Differential Equations with Time Evolution} \arrow[d, "\text{Separation of variables}"] \\
\text{Simultaneous Eigenvalue Problems} \arrow[d] \\
\text{Study of Eigenvalues} = \text{Study of Hydrogen Spectrum}
\end{tikzcd}
$$

Contents

Spectral theory?

So far, we have seen that spectral theory refers to the theory about eigenvalues and eigenvectors,

Spectral theory?

So far, we have seen that spectral theory refers to the theory about eigenvalues and eigenvectors, especially often for INFINITE dimensional linear operators.

Spectral theory?

So far, we have seen that spectral theory refers to the theory about eigenvalues and eigenvectors, especially often for INFINITE dimensional linear operators.
In this section, we

- review the spectral theory on finite dimensional vector spaces,
- introduce Hilbert spaces — a typical example of infinite dimensional vector spaces — to state some results which extend the spectral theory to infinite dimensional spaces,
- and give a precise definition of "spectrum" of an operator.

Spectral theory?

So far, we have seen that spectral theory refers to the theory about eigenvalues and eigenvectors, especially often for INFINITE dimensional linear operators.
In this section, we

- review the spectral theory on finite dimensional vector spaces,
- introduce Hilbert spaces — a typical example of infinite dimensional vector spaces — to state some results which extend the spectral theory to infinite dimensional spaces,
- and give a precise definition of "spectrum" of an operator.

From now, we basically assume the scalar field as $\mathbb{C}$.

Spectral theorem for matrices

The term “spectral theorem” is given to several theorems that show a condition for a linear operator to be diagonalizable.
(diagonalizability is the firstly considered “spectral property”!)

Spectral theorem for matrices

The term “spectral theorem” is given to several theorems that show a condition for a linear operator to be diagonalizable.
(diagonalizability is the firstly considered “spectral property”!)
In particular, spectral theorems state the relation

$$\text{condition related to adjoint} \quad \Longleftrightarrow \quad \text{a kind of diagonalizability.}$$

Spectral theorem for matrices

The term “spectral theorem” is given to several theorems that show a condition for a linear operator to be diagonalizable.
(diagonalizability is the firstly considered “spectral property”!)
In particular, spectral theorems state the relation

$$\text{condition related to adjoint} \quad \Longleftrightarrow \quad \text{a kind of diagonalizability.}$$

Vector space with inner product	Vector space $without$ additional structure

Spectral theorem for matrices

The term "spectral theorem" is given to several theorems that show a condition for a linear operator to be diagonalizable.
(diagonalizability is the firstly considered "spectral property"!)
In particular, spectral theorems state the relation

$$\text{condition related to adjoint} \iff \text{a kind of diagonalizability.}$$

Vector space with inner product	Vector space *without* additional structure

The most famous examples are for:

Definition

Let V be a finite dimensional complex inner product space and $A : V \to V$ be linear. (i.e., let A be a complex square matrix.) Then, A is said to be *normal* if $AA^* = A^*A$, and *Hermitian* if $A = A^*$

Spectral theorem for matrices

The term "spectral theorem" is given to several theorems that show a condition for a linear operator to be diagonalizable.
(diagonalizability is the firstly considered "spectral property"!)
In particular, spectral theorems state the relation

condition related to adjoint $\iff$ a kind of diagonalizability.

Vector space with inner product	Vector space *without* additional structure

The most famous examples are for:

Definition

Let V be a finite dimensional complex inner product space and $A : V \to V$ be linear. (i.e., let A be a complex square matrix.) Then, A is said to be *normal* if $AA^* = A^*A$, and *Hermitian* if $A = A^*$

Note that the conjugate transpose depends on the inner product structure: A^* is defined by

$$\langle x, Ay \rangle = \langle A^*x, y \rangle.$$

Spectral theorem of matrices

Theorem (Spectral theorem for normal matrices)

A complex square matrix A *is normal if unitarily diagonalizable.*

Theorem (Spectral theorem for Hermitian matrices)

A complex square matrix A *is Hermitian if unitarily diagonalizable and all eigenvalues are real.*

Spectral theorem of matrices

Theorem (Spectral theorem for normal matrices)

A complex square matrix A *is normal if unitarily diagonalizable.*

Theorem (Spectral theorem for Hermitian matrices)

A complex square matrix A *is Hermitian if unitarily diagonalizable and all eigenvalues are real.*

TFAE: a matrix is/has

- unitarily diagonalizable
- a set of eigenvectors forms an orthonormal basis of V.

Spectral theorem of matrices

Theorem (Spectral theorem for normal matrices)

A complex square matrix A is normal if unitarily diagonalizable.

Theorem (Spectral theorem for Hermitian matrices)

A complex square matrix A is Hermitian if unitarily diagonalizable and all eigenvalues are real.

TFAE: a matrix is/has

- unitarily diagonalizable
- a set of eigenvectors forms an orthonormal basis of V.

Remind what we did in the previous section. The purpose of separation of variables is to *construct an orthonormlal basis for the solution space.*

Hilbert space

Hilbert space

Definition

An inner product space, possibly infinite dimensional, is called a *Hilbert space* if it is complete; the metric

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

has the space become a complete metric space.

Hilbert space

Definition

An inner product space, possibly infinite dimensional, is called a *Hilbert space* if it is complete; the metric

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

has the space become a complete metric space.

- The vector space $\mathbb{C}^n \Leftrightarrow$ Finite dimensional Hilbert space.

Hilbert space

Definition

An inner product space, possibly infinite dimensional, is called a *Hilbert space* if it is complete; the metric

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

has the space become a complete metric space.

- The vector space $\mathbb{C}^n$ $\Leftrightarrow$ Finite dimensional Hilbert space.
- The space $L^2(X)$ is a Hilbert space with $\langle f, g \rangle := \int_X fg \, dx$.

Hilbert space

Definition

An inner product space, possibly infinite dimensional, is called a *Hilbert space* if it is complete; the metric

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

has the space become a complete metric space.

- The vector space $\mathbb{C}^n \Leftrightarrow$ Finite dimensional Hilbert space.
- The space $L^2(X)$ is a Hilbert space with $\langle f, g \rangle := \int_X fg \, dx$.
- Conversly, Hilbert space usually means the L^2 space of wave functions, by physicists.

Hilbert space

Definition

An inner product space, possibly infinite dimensional, is called a *Hilbert space* if it is complete; the metric

$$d(x, y) := \sqrt{\langle x - y, x - y \rangle}$$

has the space become a complete metric space.

- The vector space $C^n \Leftrightarrow$ Finite dimensional Hilbert space.
- The space $L^2(X)$ is a Hilbert space with $\langle f, g \rangle := \int_X fg \, dx$.
- Conversly, Hilbert space usually means the L^2 space of wave functions, by physicists.
- The space $\ell^2(\mathbb{C})$ of sqare summable sequeces is a Hilbert space with $\langle (a_n), (b_n) \rangle := \sum_n a_n b_n$.

Bounded operators

Theorem (?)

In finite dimensions, something linear is always continuous.

Bounded operators

Theorem (?)

In finite dimensions, something linear is always continuous.

- However, this may be wrong in infinite dimensions.
  We should find linear operators of nicer properties to state the generalized spectral theorem.

Bounded operators

Theorem (?)

In finite dimensions, something linear is always continuous.

- However, this may be wrong in infinite dimensions.
  We should find linear operators of nicer properties to state the generalized spectral theorem.

Definition

A linear operator $T : H \to H$ on a Hilbert space is called *bounded* if there is a constant $C > 0$ such that for all $x \in H$

$$\|Ax\| \leqslant C\|x\|.$$

The set of bounded operators on H is denoted by $B(H)$.

Bounded operators

Theorem (?)

In finite dimensions, something linear is always continuous.

- However, this may be wrong in infinite dimensions.
  We should find linear operators of nicer properties to state the generalized spectral theorem.

Definition

A linear operator $T : H \to H$ on a Hilbert space is called *bounded* if there is a constant $C > 0$ such that for all $x \in H$

$$\|Ax\| \leqslant C\|x\|.$$

The set of bounded operators on H is denoted by $B(H)$.

Theorem

A linear operator on a Hilbert space is bounded iff continuous.

Compact operators

Bounded operators are not enough.

Definition

A linear operator T on a Hilbert space is called *compact* if image of bounded set is relatively compact.

Compact operators

Bounded operators are not enough.

Definition

A linear operator T on a Hilbert space is called *compact* if image of bounded set is relatively compact.

Remark

The closed ball in infinite dimensional Hilbert space is not compact: we can find a sequence not having any convergent subsequence.

Compact operators

Bounded operators are not enough.

Definition

A linear operator T on a Hilbert space is called *compact* if image of bounded set is relatively compact.

Remark

The closed ball in infinite dimensional Hilbert space is not compact: we can find a sequence not having any convergent subsequence.

Example

An operator $T : \ell^2 \to \ell^2$ defined by

$$T(a_1, a_2, a_3, \cdots) = (a_1, \frac{a_2}{2}, \frac{a_3}{3}, \cdots)$$

is compact, but the identity $\mathrm{I} \to \ell^2 \to \ell^2$

$$\mathrm{I}(a_1, a_2, a_3, \cdots) = (a_1, a_2, a_3, \cdots),$$

which is clearly bounded, is not compact.

Spectral theorem for compact normal operators

Theorem (Spectral theorem for compact normal operators)

Let T *be a compact normal operator on a separable Hilbert space. Then, there exists a countable orthonormal basis consisting of eigenvectors, with corresponding eigenvalues that converges to 0.*

Theorem (Spectral theorem for compact self-adjoint operators)

Let T *be a compact self-adjoint operator on a separable Hilbert space. Then, there exists a countable orthonormal basis consisting of eigenvectors, with corresponding eigenvalues that are reals and converges to 0.*

Spectral theorem for compact normal operators

Theorem (Spectral theorem for compact normal operators)

Let T *be a compact normal operator on a separable Hilbert space. Then, there exists a countable orthonormal basis consisting of eigenvectors, with corresponding eigenvalues that converges to 0.*

Theorem (Spectral theorem for compact self-adjoint operators)

Let T *be a compact self-adjoint operator on a separable Hilbert space. Then, there exists a countable orthonormal basis consisting of eigenvectors, with corresponding eigenvalues that are reals and converges to 0.*

Remark

There are some concepts we will skip:

- we did not define "separable" space,
- we did not define "countable (Schauder) basis".

Discrete spectrum

The operator $-\Delta - 2|x|^{-1}$ is an example of what we call *elliptic operators* with discrete spectrum.
We will not deal with this in detail, but:

Discrete spectrum

The operator $-\Delta - 2|x|^{-1}$ is an example of what we call *elliptic operators* with discrete spectrum.
We will not deal with this in detail, but:

The operator $-\Delta - 2|x|^{-1}$ is unbounded,

Discrete spectrum

The operator $-\Delta - 2|x|^{-1}$ is an example of what we call *elliptic operators* with discrete spectrum.
We will not deal with this in detail, but:

The operator $-\Delta - 2|x|^{-1}$ is unbounded,

but it is known that we can view it as the inverse of a compact self-adjoint(positive) operator.

Discrete spectrum

The operator $-\Delta - 2|x|^{-1}$ is an example of what we call *elliptic operators* with discrete spectrum.
We will not deal with this in detail, but:

The operator $-\Delta - 2|x|^{-1}$ is unbounded,

but it is known that we can view it as the inverse of a compact self-adjoint(positive) operator.

$\downarrow$

The eigenvalues are distributed like

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

Continuous spectrum

For hydrogen atom $\Rightarrow$ we can define the spectrum of Hamiltonian $\mathcal{H} = -\Delta + 2|x|^{-1}$ as the set of *eigenvalues*.

Continuous spectrum

For hydrogen atom $\Rightarrow$ we can define the spectrum of Hamiltonian $\mathcal{H} = -\Delta + 2|x|^{-1}$ as the set of *eigenvalues*.

For a free particle $\Rightarrow$

Continuous spectrum

For hydrogen atom $\Rightarrow$ we can define the spectrum of Hamiltonian $\mathcal{H} = -\Delta + 2|x|^{-1}$ as the set of *eigenvalues*.

For a free particle $\Rightarrow \mathcal{H} = -\Delta + V = -\Delta$, we cannot; eigenvectors exist for $E \geqslant 0$, and they are "linear combinations" of

$$\psi_E(x) = e^{ik\cdot x}, \qquad \text{for } k \in \mathbb{R}^3 \text{ s.t. } |k|^2 = E,$$

which are all not in L^2.

Continuous spectrum

For hydrogen atom $\Rightarrow$ we can define the spectrum of Hamiltonian $\mathcal{H} = -\Delta + 2|x|^{-1}$ as the set of *eigenvalues*.

For a free particle $\Rightarrow \mathcal{H} = -\Delta + V = -\Delta$, we cannot; eigenvectors exist for $E \geqslant 0$, and they are "linear combinations" of

$$\psi_E(x) = e^{ik\cdot x}, \qquad \text{for } k \in \mathbb{R}^3 \text{ s.t. } |k|^2 = E,$$

which are all not in L^2.

However, their integral may serve as a solution:

$$\Psi(t,x) = \int_E e^{-iEt} \int_{|k|=\sqrt{E}} a_E(k) e^{ik\cdot x}\, dk\, dE.$$

Continuous spectrum

For hydrogen atom $\Rightarrow$ we can define the spectrum of Hamiltonian $\mathcal{H} = -\Delta + 2|x|^{-1}$ as the set of *eigenvalues*.

For a free particle $\Rightarrow \mathcal{H} = -\Delta + V = -\Delta$, we cannot; eigenvectors exist for $E \geqslant 0$, and they are "linear combinations" of

$$\psi_E(x) = e^{ik\cdot x}, \qquad \text{for } k \in \mathbb{R}^3 \text{ s.t. } |k|^2 = E,$$

which are all not in L^2.

However, their integral may serve as a solution:

$$\Psi(t,x) = \int_E e^{-iEt} \int_{|k|=\sqrt{E}} a_E(k) e^{ik\cdot x} \, dk \, dE.$$

- Energy spectrum of free particle is not *quantized* = *discretized*.
- We want to say $-\Delta$ has the spectrum $[0,\infty)$.

Continuous spectrum

Definition

Let T be an operator on a Hilbert space H. The *spectrum* of T is defined by

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(H)\}.$$

Continuous spectrum

Definition

Let T be an operator on a Hilbert space H. The *spectrum* of T is defined by

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } B(H)\}.$$

In particular,

Definition

The *point spectrum* of T is defined by the set of eigenvalues:

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not injective }\}.$$

Definition

The *continuous spectrum* of T is defined by

$$\sigma_c(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is injective but has proper dense image }\}.$$

Conclusion of Section 2

- In infinite dimensional spaces, the spectral theorems are generalized for compact operators.
- The spectrum of an operator T is defined by

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertiable } \}.$$

Contents

C^*-algebras

The goal of this section is to state the Gelfand-Naimark theorem, which crucially affects to the definition of $\mathrm{Spec}(R)$.

C^*-algebras

The goal of this section is to state the Gelfand-Naimark theorem, which crucially affects to the definition of $\mathrm{Spec}(\mathbb{R})$. We give definitions:

Psudo-definition

An *algebra* is a vector space with vector multiplication.
Equivalently, a ring with scalar multiplication.

C^*-algebras

The goal of this section is to state the Gelfand-Naimark theorem, which crucially affects to the definition of $\mathrm{Spec}(R)$. We give definitions:

Psudo-definition

An *algebra* is a vector space with vector multiplication.
Equivalently, a ring with scalar multiplication.

Definition

A C^*-*algebra* is a complex associative algebra with involution $*$ and norm $\| \cdot \|$ such that the norm is complete and $\|x^*x\| = \|x^*\|\|x\|$.

C^*-algebras

The goal of this section is to state the Gelfand-Naimark theorem, which crucially affects to the definition of $\mathrm{Spec}(R)$. We give definitions:

Psudo-definition

An *algebra* is a vector space with vector multiplication.
Equivalently, a ring with scalar multiplication.

Definition

A C^*-*algebra* is a complex associative algebra with involution $*$ and norm $\|\cdot\|$ such that the norm is complete and $\|x^*x\| = \|x^*\|\|x\|$.

Example (1)

Let H be a complex Hilbert space. Then, $B(H)$ is a C^*-algebra.
C^*-algerbras are invented to learn the abstract study of $B(H)$.

C^*-algebras

The goal of this section is to state the Gelfand-Naimark theorem, which crucially affects to the definition of $\mathrm{Spec}(R)$. We give definitions:

Psudo-definition

An *algebra* is a vector space with vector multiplication.
Equivalently, a ring with scalar multiplication.

Definition

A C^**-algebra* is a complex associative algebra with involution $*$ and norm $\|\cdot\|$ such that the norm is complete and $\|x^*x\| = \|x^*\|\|x\|$.

Example (1)

Let H be a complex Hilbert space. Then, $B(H)$ is a C^*-algebra.
C^*-algerbras are invented to learn the abstract study of $B(H)$.

Example (2)

Let X be a compact space. Then, the set of complex-valued continuous functions $C(X, \mathbb{C})$ is a commutative C^*-algebra.

Gelfand theory (1): continuous function space

Assumption: unless mentioned otherwise, we will only discuss *commutative unital* C^*-algebras.

Gelfand theory (1): continuous function space

Assumption: unless mentioned otherwise, we will only discuss *commutative unital* C^*-algebras.

The following definition is natural:

Definition

Let A be (possibly non-commutative) C^*-algebra. A complex number λ is in the *spectrum* of $a \in A$ if $a - \lambda e$ is not invertible. The spectrum is denoted by $\sigma(a)$.

Gelfand theory (1): continuous function space

Assumption: unless mentioned otherwise, we will only discuss *commutative unital* C^*-algebras.

The following definition is natural:

Definition

Let A be (possibly non-commutative) C^*-algebra. A complex number λ is in the *spectrum* of $a \in A$ if $a - \lambda e$ is not invertible. The spectrum is denoted by $\sigma(a)$.

What can be told for a continuous function on a compact X? For $C(X)$,

Gelfand theory (1): continuous function space

Assumption: unless mentioned otherwise, we will only discuss *commutative unital* C^*-algebras.

The following definition is natural:

Definition

Let A be (possibly non-commutative) C^*-algebra. A complex number λ is in the *spectrum* of $a \in A$ if $a - \lambda e$ is not invertible. The spectrum is denoted by $\sigma(a)$.

What can be told for a continuous function on a compact X? For $C(X)$,

- $\sigma(f) = \mathrm{im}(f)$,

Gelfand theory (1): continuous function space

Assumption: unless mentioned otherwise, we will only discuss *commutative unital* C^*-algebras.

The following definition is natural:

Definition

Let A be (possibly non-commutative) C^*-algebra. A complex number λ is in the *spectrum* of $a \in A$ if $a - \lambda e$ is not invertible. The spectrum is denoted by $\sigma(a)$.

What can be told for a continuous function on a compact X? For $C(X)$,

- $\sigma(f) = \mathrm{im}(f)$,
- every maximal ideal is of the form $\{f : f(x) = 0\}$ for a single point $x \in X$.

Gelfand theory (2): generated C^*-subalgebra

The following is also an important exmple to formulate functional calculus:

Definition

Let A be a (possibly non-commutative) C^*-algebra. An element $a \in A$ is called *normal* if $a^* a = a a^*$. For a normal element, a C^*-subalgebra defined by

$$C^*(a) := \overline{\{ p(a, a^*) : p \in \mathbb{C}[x, y] \}} \subset A$$

is said to be *generated by* a.

Gelfand theory (2): generated C^*-subalgebra

The following is also an important exmple to formulate functional calculus:

Definition

Let A be a (possibly non-commutative) C^*-algebra. An element $a \in A$ is called *normal* if $a^*a = aa^*$. For a normal element, a C^*-subalgebra defined by

$$C^*(a) := \overline{\{p(a, a^*) : p \in \mathbb{C}[x, y]\}} \subset A$$

is said to be *generated by* a.

Note that $C^*(a)$ is commutative since a is normal! For $C^*(a)$,

Gelfand theory (2): generated C^*-subalgebra

The following is also an important exmple to formulate functional calculus:

Definition

Let A be a (possibly non-commutative) C^*-algebra. An element $a \in A$ is called *normal* if $a^*a = aa^*$. For a normal element, a C^*-subalgebra defined by

$$C^*(a) := \overline{\{p(a, a^*) : p \in \mathbb{C}[x, y]\}} \subset A$$

is said to be *generated by* a.

Note that $C^*(a)$ is commutative since a is normal! For $C^*(a)$,

- if $a - \lambda$ is not invertible, there we can assign a maximal ideal containing it,

Gelfand theory (2): generated C^*-subalgebra

The following is also an important exmple to formulate functional calculus:

Definition

Let A be a (possibly non-commutative) C^*-algebra. An element $a \in A$ is called *normal* if $a^*a = aa^*$. For a normal element, a C^*-subalgebra defined by

$$C^*(a) := \overline{\{p(a, a^*) : p \in \mathbb{C}[x, y]\}} \subset A$$

is said to be *generated by* a.

Note that $C^*(a)$ is commutative since a is normal! For $C^*(a)$,

- if $a - \lambda$ is not invertible, there we can assign a maximal ideal containing it,
- conversely, for a maximal ideal $\mathfrak{m}$, the image of a under the projection $C^*(a) \to C^*(a)/\mathfrak{m} \cong \mathbb{C}$ is in a spectrum of a $^{(*)}$,

Gelfand theory (2): generated C^*-subalgebra

The following is also an important exmple to formulate functional calculus:

Definition

Let A be a (possibly non-commutative) C^*-algebra. An element $a \in A$ is called *normal* if $a^*a = aa^*$. For a normal element, a C^*-subalgebra defined by

$$C^*(a) := \overline{\{p(a, a^*) : p \in \mathbb{C}[x, y]\}} \subset A$$

is said to be *generated by* a.

Note that $C^*(a)$ is commutative since a is normal! For $C^*(a)$,

- if $a - \lambda$ is not invertible, there we can assign a maximal ideal containing it,
- conversely, for a maximal ideal $\mathfrak{m}$, the image of a under the projection $C^*(a) \to C^*(a)/\mathfrak{m} \cong \mathbb{C}$ is in a spectrum of a $^{(*)}$,
- the maps above are inverses of each other.

Therefore,

Gelfand theory (2): generated C^*-subalgebra

Proposition

There is a 1-1 correspondence between

$$\textit{spectrum of } a \quad \Longleftrightarrow \quad \textit{maximal ideal space of } C^*(a).$$

Gelfand theory (2): generated C^*-subalgebra

Proposition

There is a 1-1 correspondence between

$$\textit{spectrum of } a \quad \Longleftrightarrow \quad \textit{maximal ideal space of } C^*(a).$$

Consider $C(X)$. The above proposition states that image is corresponded to its domain: the element a acts like an "injective function" in $C^*(a)$.

Gelfand theory (2): generated C^*-subalgebra

Proposition

There is a 1-1 correspondence between

$$\textit{spectrum of } a \quad \Longleftrightarrow \quad \textit{maximal ideal space of } C^*(a).$$

Consider $C(X)$. The above proposition states that image is corresponded to its domain: the element a acts like an "injective function" in $C^*(a)$.

Definition

Let us define the *spectrum* of general commutative unital C^*-algebra A as the **set of maximal ideals** and denote it by $\sigma(A)$.

Spectrum as a topological space

Let $\sigma(A)$ be the spectrum of a comm. unit. C^*-algebra A.

Spectrum as a topological space

Let $\sigma(A)$ be the spectrum of a comm. unit. C^*-algebra A.

We set up the topology of pointwise convergence on $\sigma(A)$:
for example, consider $C(X)$. Since X is naturally corresponded to $\sigma(C(X))$ as a set,

$$x \to y \in X \quad \Longleftrightarrow \quad f(x) \to f(y) \quad \text{for all } f \in C(X).$$

Spectrum as a topological space

Let $\sigma(A)$ be the spectrum of a comm. unit. C^*-algebra A.

We set up the topology of pointwise convergence on $\sigma(A)$:
for example, consider $C(X)$. Since X is naturally corresponded to $\sigma(C(X))$ as a set,

$$x \to y \in X \quad \Longleftrightarrow \quad f(x) \to f(y) \quad \text{for all } f \in C(X).$$

Then,

Proposition

Let A be a comm. unit. C^*-algebra. Then, the spectrum $\sigma(A) \subset \mathbb{C}$ is compact (by the Banach-Alaoglu).

Spectrum as a topological space

Let $\sigma(A)$ be the spectrum of a comm. unit. C^*-algebra A.

We set up the topology of pointwise convergence on $\sigma(A)$:
for example, consider $C(X)$. Since X is naturally corresponded to $\sigma(C(X))$ as a set,

$$x \to y \in X \quad \Longleftrightarrow \quad f(x) \to f(y) \quad \text{for all } f \in C(X).$$

Then,

Proposition

Let A be a comm. unit. C^*-algebra. Then, the spectrum $\sigma(A) \subset \mathbb{C}$ is compact (by the Banach-Alaoglu).

Example

For the space ℓ^1 of summable sequences, the spectrum is the unit circle: $\sigma(\ell^1) = \mathbb{T}$.

Example

Let X be compact Hausdorff. Then, $\sigma(C(X))$ is homeomorphic to X.

Gelfand-Naimark theorem

Finally, we state the Gelfand-Naimark theorem.

Gelfand-Naimark theorem

Finally, we state the Gelfand-Naimark theorem.

Theorem (Gelfand-Naimark, 1943)

Let A *be a commutative unital* C^**-algebra. Then, we have a* C^**-algebra isomorphism*

$$A \to C(\sigma(A)).$$

This map is called Gelfand representation.

Conclusion of Section 3

Transitions of definition:

- Spectrum of $a \in A \to$ generalized eigenvalues;
- Spectrum of $a \in C(X) \to$ image of function;
- Spectrum of $a \in C^*(a) \to$ maximal ideals;

$$\Downarrow$$

- Spectrum of $C(X) :=$ maximal ideals $=$ domain $=$ image of inj,
- Spectrum of $A :=$ maximal ideals ($\approx$ domain(?)).

$$\text{Spectrum} \quad \Longleftrightarrow \quad \{\text{Maximal ideals}\} \quad \Longleftrightarrow \quad \{\text{Points}\}$$

Contents

Algebraic geometry

In algebriac geometry, we...

Algebraic geometry

In algebriac geometry, we...

- Want to describe geometry with polynomials:
  The circle is described by the polynomial $x^2 + y^2 - 1$.

Algebraic geometry

In algebriac geometry, we...

- Want to describe geometry with polynomials:
  The circle is described by the polynomial $x^2 + y^2 - 1$.
- Want to solve geometric problems with properties of polynomials: for example,
  - to compute the number of singularity,
  - to determine the topological shape,
  - to investigate the dimension of intersection, etc.

Algebraic geometry

In algebriac geometry, we...

- Want to describe geometry with polynomials:
  The circle is described by the polynomial $x^2 + y^2 - 1$.
- Want to solve geometric problems with properties of polynomials: for example,
  - to compute the number of singularity,
  - to determine the topological shape,
  - to investigate the dimension of intersection, etc.
- Want to make a correspondence between geometry and algebra:
  ($\rightarrow$) use algebraic computations to analyze geometry,
  ($\leftarrow$) use the geometric intuition to study algebra.

Algebraic geometry

In algebriac geometry, we...

- Want to describe geometry with polynomials:
  The circle is described by the polynomial $x^2 + y^2 - 1$.
- Want to solve geometric problems with properties of polynomials: for example,
  - to compute the number of singularity,
  - to determine the topological shape,
  - to investigate the dimension of intersection, etc.
- Want to make a correspondence between geometry and algebra:
  ($\rightarrow$) use algebraic computations to analyze geometry,
  ($\leftarrow$) use the geometric intuition to study algebra.

In this section, assume that a "ring" is a commutative and unital one.

Algebraic variety

Remark

We give basic definitions here. For simplicity, we will do everything in the three dimension $\mathbb{C}^3$. Every concept is directly generalized to arbitrary dimensions.

Algebraic variety

Remark

We give basic definitions here. For simplicity, we will do everything in the three dimension $\mathbb{C}^3$. Every concept is directly generalized to arbitrary dimensions.

Definition

Let $T \subset \mathbb{C}[x, y, z]$ and define

$$\mathcal{Z}(T) := \{p \in \mathbb{C}^3 : f(p) = 0 \text{ for all } f \in T\}.$$

An *algebraic set* is a subset V of $\mathbb{C}^n$ satisfying $V = Z(T)$ for some T.

Algebraic variety

Remark

We give basic definitions here. For simplicity, we will do everything in the three dimension $\mathbb{C}^3$. Every concept is directly generalized to arbitrary dimensions.

Definition

Let $T \subset \mathbb{C}[x, y, z]$ and define

$$\mathcal{Z}(T) := \{p \in \mathbb{C}^3 : f(p) = 0 \text{ for all } f \in T\}.$$

An *algebraic set* is a subset V of $\mathbb{C}^n$ satisfying $V = Z(T)$ for some T.

Definition

Let $Y \subset \mathbb{C}^n$ and define

$$\mathcal{I}(Y) := \{f \in \mathbb{C}[x, y, z] : f(p) = 0 \text{ for all } p \in Y\}.$$

This is always a radical(square-free) ideal.

Algebraic variety

Proposition

- *For an algebraic set* $V \subset \mathbb{C}^3$*, we have* $\mathcal{V}(\mathcal{I}(V)) = V$*.*
- *For a radical ideal* $I \subset \mathbb{C}[x, y, z]$*, we have* $\mathcal{I}(\mathcal{V}(I)) = I$*.*

Algebraic variety

Proposition

- *For an algebraic set* $V \subset \mathbb{C}^3$*, we have* $\mathcal{V}(\mathcal{I}(V)) = V$*.*
- *For a radical ideal* $I \subset \mathbb{C}[x, y, z]$*, we have* $\mathcal{I}(\mathcal{V}(I)) = I$*.*

Definition

If an algebraic set is not a union of two proper algebraic subsets, then it is called *algebraic variety*.

Proposition

An algebraic set V *is an algebraic variety iff* $\mathcal{I}(V)$ *is prime.*

Algebraic variety

Proposition

- *For an algebraic set* $V \subset \mathbb{C}^3$*, we have* $\mathcal{V}(\mathcal{I}(V)) = V$*.*
- *For a radical ideal* $I \subset \mathbb{C}[x, y, z]$*, we have* $\mathcal{I}(\mathcal{V}(I)) = I$*.*

Definition

If an algebraic set is not a union of two proper algebraic subsets, then it is called *algebraic variety*.

Proposition

An algebraic set V *is an algebraic variety iff* $\mathcal{I}(V)$ *is prime.*

$$\text{algebraic sets} \quad \Longleftrightarrow \quad \text{radical ideals},$$

$$\text{algebraic varieties} \quad \Longleftrightarrow \quad \text{prime ideals}.$$

Coordinate ring

Consider an algebraic variety $S^2 = \{x^2 + y^2 + z^2 = 1\}$. The following two functions are same on S^2:

$$f = x + 1, \qquad g = x + x^2 + y^2 + z^2.$$

In other words, $f - g \in \mathcal{I}(S^2)$.

Coordinate ring

Consider an algebraic variety $S^2 = \{x^2 + y^2 + z^2 = 1\}$. The following two functions are same on S^2:

$$f = x + 1, \qquad g = x + x^2 + y^2 + z^2.$$

In other words, $f - g \in \mathcal{I}(S^2)$.

Definition

A *coordinate ring* or *structure ring* of an algebraic set V is the ring $\mathbb{C}[x, y, z]/\mathcal{I}(V)$.

Every ring of the form $\mathbb{C}[x, y, z]/I$, where I is an ideal, we can interpret the ring as the set of functions on the algebraic set $\mathcal{V}(I)$. Why do we introduce this?

Coordinate ring

Consider an algebraic variety $S^2 = \{x^2 + y^2 + z^2 = 1\}$. The following two functions are same on S^2:

$$f = x + 1, \qquad g = x + x^2 + y^2 + z^2.$$

In other words, $f - g \in \mathcal{I}(S^2)$.

Definition

A *coordinate ring* or *structure ring* of an algebraic set V is the ring $\mathbb{C}[x, y, z]/\mathcal{I}(V)$.

Every ring of the form $\mathbb{C}[x, y, z]/I$, where I is an ideal, we can interpret the ring as the set of functions on the algebraic set $\mathcal{V}(I)$. Why do we introduce this?

Main Philosophy of AG:

analyze the *function space* to study geometric objects!

Krull dimension

Let us see an example that shows the power of structure rings.

Equation	Ideal	Dimension
$\varnothing$	$\varnothing$	3
$x = 1$	$(x - 1)$	2 (plane)
$x = 1,\ y = 2$	$(x - 1, y - 2)$	1 (line)
$x = 1,\ y = 2,\ z = 3,$	$(x - 1, y - 2, z - 3)$	0 (point)

Krull dimension

Let us see an example that shows the power of structure rings.

Equation	Ideal	Dimension
$\varnothing$	$\varnothing$	3
$x = 1$	$(x-1)$	2 (plane)
$x = 1,\ y = 2$	$(x-1, y-2)$	1 (line)
$x = 1,\ y = 2,\ z = 3,$	$(x-1, y-2, z-3)$	0 (point)

Coordinate rings are

$$\mathbb{C}[x,y,z],\ \mathbb{C}[x,y,z]/(x-1),$$
$$\mathbb{C}[x,y,z]/(x-1,y-2),\ \mathbb{C}[x,y,z]/(x-1,y-2,z-3).$$

Definition

The *Krull dimension* of a ring R is the maximal length of chain of prime ideals.

$\therefore$ The Krull dimension of coordinate ring is same with the "dimension" of the corresponding algebraic sets.

Maximal ideal is a point

Second example.

Theorem

The ideal $(x - a, y - b, z - c)$ *is maximal in* $\mathbb{C}[x, y, z]$.

Maximal ideal is a point

Second example.

Theorem

The ideal $(x-a, y-b, z-c)$ *is maximal in* $\mathbb{C}[x, y, z]$.

Conversely,

Theorem (Weak Hilbert Nullstellensatz)

Every maximal ideal in $\mathbb{C}[x, y, z]$ *has the form*

$$(x-a, y-b, z-c).$$

Maximal ideal is a point

Second example.

Theorem

The ideal $(x - a, y - b, z - c)$ *is maximal in* $\mathbb{C}[x, y, z]$.

Conversely,

Theorem (Weak Hilbert Nullstellensatz)

Every maximal ideal in $\mathbb{C}[x, y, z]$ *has the form*

$$(x - a, y - b, z - c).$$

This theorem implies that we have the following correspondence:

$$\text{points} \quad \Longleftrightarrow \quad \text{maximal ideals.}$$

Alexander Grothendieck

Alexander Grothendieck (1928 - 2014)

"Every ring should be recognized as the set of functions on a space."

Alexander Grothendieck

Alexander Grothendieck (1928 - 2014)

"Every ring should be recognized as the set of functions on a space."

What space?

Alexander Grothendieck

Alexander Grothendieck (1928 - 2014)

"Every ring should be recognized as the set of functions on a space."

What space? Spectrum!

Alexander Grothendieck

Alexander Grothendieck (1928 - 2014)
original major: functional analysis!!!
"Every ring should be recognized as the set of functions on a space."

What space? Spectrum!

Problems of maximal ideals

First trial:

Definition

Let R be a ring. Define the *spectrum* of R as the set of maximal ideals.

Problems of maximal ideals

First trial:

Definition

Let R be a ring. Define the *spectrum* of R as the set of maximal ideals.

But it had two problems:

1. The codomain is not unified;
2. We want the spectrum to have a *functoriality*.

Gelfand-Mazur theorem

Let $\mathfrak{m}$ be a maximal ideal(=point) of the ring R. Then, codomian field of functions at the point is characterized by the quotient $R/\mathfrak{m}$.

Gelfand-Mazur theorem

Let $\mathfrak{m}$ be a maximal ideal(=point) of the ring R. Then, codomian field of functions at the point is characterized by the quotient $R/\mathfrak{m}$.
For C^*-algebras, we could just use the maximal ideals becuase the field $R/\mathfrak{m}$ is always isomorphic to $\mathbb{C}$:

Theorem (Gelfand-Mazur)

A C^*-algebra that is a field, is isormophic to $\mathbb{C}$.

Gelfand-Mazur theorem

Let $\mathfrak{m}$ be a maximal ideal(=point) of the ring R. Then, codomian field of functions at the point is characterized by the quotient $R/\mathfrak{m}$.
For C^*-algebras, we could just use the maximal ideals becuase the field $R/\mathfrak{m}$ is always isomorphic to $\mathbb{C}$:

Theorem (Gelfand-Mazur)

A C^*-algebra that is a field, is isormophic to $\mathbb{C}$.

However, when we are concerned with general ring R, we cannot guarantee such results.
There are two choices about the codomain problem:

Gelfand-Mazur theorem

Let $\mathfrak{m}$ be a maximal ideal(=point) of the ring R. Then, codomian field of functions at the point is characterized by the quotient $R/\mathfrak{m}$.
For C^*-algebras, we could just use the maximal ideals becuase the field $R/\mathfrak{m}$ is always isomorphic to $\mathbb{C}$:

Theorem (Gelfand-Mazur)

A C^*-algebra that is a field, is isormophic to $\mathbb{C}$.

However, when we are concerned with general ring R, we cannot guarantee such results.
There are two choices about the codomain problem:

1. Restrict the condition for maximal ideals,
2. Compromise the unifies codomain.

Gelfand-Mazur theorem

Let $\mathfrak{m}$ be a maximal ideal(=point) of the ring R. Then, codomian field of functions at the point is characterized by the quotient $R/\mathfrak{m}$.
For C^*-algebras, we could just use the maximal ideals becuase the field $R/\mathfrak{m}$ is always isomorphic to $\mathbb{C}$:

Theorem (Gelfand-Mazur)

A C^*-algebra that is a field, is isormophic to $\mathbb{C}$.

However, when we are concerned with general ring R, we cannot guarantee such results.
There are two choices about the codomain problem:

1. Restrict the condition for maximal ideals,
2. Compromise the unifies codomain.(v)

Functoriality

We want $\operatorname{Spec} : \mathbf{CRing} \to \mathbf{Set}$ to be a (contravariant) functor: functoriality allows to define schemes and apply the powerful theory of sheaves to AG.

Functoriality

We want $\mathrm{Spec} : \mathbf{CRing} \to \mathbf{Set}$ to be a (contravariant) functor: functoriality allows to define schemes and apply the powerful theory of sheaves to AG. Under what definitions is a ring homomorphism $R \to S$ able to induce the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$?

Functoriality

We want Spec : $\mathbf{CRing} \to \mathbf{Set}$ to be a (contravariant) functor: functoriality allows to define schemes and apply the powerful theory of sheaves to AG. Under what definitions is a ring homomorphism $R \to S$ able to induce the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$?

Example

Consider the ring homomorphism $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. The most reasonable choice of the naturally induced map is

$$\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z}) : p \mapsto i^{-1}(p).$$

Functoriality

We want Spec : $\mathbf{CRing} \to \mathbf{Set}$ to be a (contravariant) functor: functoriality allows to define schemes and apply the powerful theory of sheaves to AG. Under what definitions is a ring homomorphism $R \to S$ able to induce the map $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$?

Example

Consider the ring homomorphism $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. The most reasonable choice of the naturally induced map is

$$\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z}) : p \mapsto i^{-1}(p).$$

If Spec is defined to be the set of maximal ideals, then the functoriality is not satisfied: (0) is maximal in $\mathbb{Q}$, but $i^{-1}((0)) = (0)$ is not maximal in $\mathbb{Q}$.

Functoriality

We want $\text{Spec} : \mathbf{CRing} \to \mathbf{Set}$ to be a (contravariant) functor: functoriality allows to define schemes and apply the powerful theory of sheaves to AG. Under what definitions is a ring homomorphism $R \to S$ able to induce the map $\text{Spec}(S) \to \text{Spec}(R)$?

Example

Consider the ring homomorphism $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$. The most reasonable choice of the naturally induced map is

$$\text{Spec}(\mathbb{Q}) \to \text{Spec}(\mathbb{Z}) : p \mapsto i^{-1}(p).$$

If Spec is defined to be the set of maximal ideals, then the functoriality is not satisfied: (0) is maximal in $\mathbb{Q}$, but $i^{-1}((0)) = (0)$ is not maximal in $\mathbb{Q}$.

Proposition

Inverse image of prime ideal under a ring homomorphism is prime.

Conclusion

When Grothendieck transplanted the idea of spectrum from functional analysis to algebraic geometry, the following definition comes up:

Definition

Let R be a ring. The spectrum of R is the set of prime ideals.