Interchanging Limits, Derivatives, and Integrals

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1. Limit and derivative

Theorem 1.1. *f_n pointwisely, Df_n uniformly*

Corollary 1.2. *If $f_n \to f$ in C^1, then $Df_n \to Df$.*

2. Limit and integral

We want to find a criterion for This question asks the convergence

$$f_n \to f \quad \text{in} \quad L^1.$$

For a sequence of measurable functions $f_n : (X, \mu) \to \mathbb{R}$, define the maximal function

$$Mf(x) := \sup_n |f_n(x)|.$$

Theorem 2.1 (LDCT)**.** *If $\|Mf\|_{L^1} < \infty$ and $f_n \to f$ a.e., then $f_n \to f$ in L^1.*

continuity application

Theorem 2.2 (Scheffe)**.** *Let $\{f_n\}_n$ be a sequence of nonnegative functions in L^1. Suppose it converges to f pointwisely. Then,*

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1 \quad \Longrightarrow \quad \lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

3. Derivative and integral

Define the Newton quotient as

$$D_k f(t,x) := \frac{f(t+k,x) - f(t,x)}{k}$$

for $k \neq 0$. We mainly recognize D_k as an operator that maps $f(0,x)$ to a function of x. Then, we can say that the partial derivative $\partial_t f(0,x)$ is well-defined a.e. x if and only if

$$\lim_{k\to 0} D_k f(0,x) = \partial_t f(0,x) \qquad \text{a.e. } x.$$

We may ask about conditions for the following to hold:

$$\lim_{h\to 0} D_k f(0,x) = \partial_t f(0,x) \qquad \text{in } L^1_x(X).$$

This question naturally arise because it implies the commutability

$$\frac{d}{dt}\int f(t,x)\,dx = \int \frac{\partial}{\partial t} f(t,x)\,dx$$

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at $t = 0$. As necessary conditions to formalize the statement, we must basically assume that $f(t,x) \in L^1_x$ for $|t| < \varepsilon$, and $\partial_t f(0,x) \in L^1_x$. Above this, if we give a stronger condition $\operatorname{ess\,sup}_{|t|<\varepsilon} |\partial_t f(t,x)| \in L^1_x$ than $\partial_t f(0,x) \in L^1_x$, then the L^1_x convergence is obtained.

Theorem 3.1 (Leibniz rule). *Let $f : (-\varepsilon,\varepsilon) \times X \to \mathbb{R}$ be a curve of integrable functions such that $f(t,x)$ is absolutely continuous in t for a.e. x. If $\partial_t f \in L^1_x(L^\infty_t)$, then $D_k f(0,x) \to \partial_t f(0,x)$ in L^1_x.*

Proof. Our strategy is to apply the Lebesgue dominated convergence theorem. In order to do this, we should control $D_k f(0,x)$ uniformly on k.

The fundamental theorem of calculus for absolute continuous functions implies

$$D_k f(0,x) = \frac{1}{k}\int_0^k \partial_t f(t,x)\,dt,$$

so we have

$$|D_k f(0,x)| \le \frac{1}{k}\int_0^k |\partial_t f(t,x)|\,dt \le \|\partial_t f(t,x)\|_{L^\infty_t}.$$

Since the right hand side does not depend on k, the main condition for LDCT is satisfied.

The pointwise convergence (in a.e. sense) is satisfied due to the absolute continuity. By the Lebesgue dominated convergence theorem, we get the desired result. $\square$

Corollary 3.2. *Let*

$$Tf(x) := \int k(x,y)f(y)\,dy.$$

If $|k_x(x,y)|$ is monotone in x and $k_x(x,y)f(y) \in L^1_y$ for all x, then

$$\frac{d}{dx}Tf(x) = \int k_x(x,y)f(y)\,dy.$$