De Rham's Theorem

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We assume the reader is familiar with some algebraic topology: singular cohomology, the induced cohomology map, the zig-zag lemma, the five lemma, and the homotopy equivalence. In this report, smooth manifolds are called just manifolds and denoted by M.

1 Smooth Singular Cohomology

2 De Rham cohomology

In this section, we will prove three important theorems for de Rham cohomology: Poincare's lemma, the Mayer-Vietoris sequence, additivity condition.

Theorem 2.1 (Homotopy axiom)**.** *Let smooth maps $f, g\colon M_1 \to M_2$ be smoothly homotopic. Then, the induced cohomology maps $f^*, g^*\colon H^k_{dR}(M_2) \to H^k_{dR}(M_1)$ are same.*

Proof. Let $H\colon M_1 \times I \to M_2$ be a homotopy between f and g. Let $i_t\colon M_1 \to M_1 \times I$ be a map defined by $i_t(p) = (p, t)$ for $t \in I$. Then $f = H \circ i_0,\ g = H \circ i_1$.

Step 1. the pullbacks i_0^* and i_1^* are cochain homotpic:
Define $h\colon \Omega^k(M_1 \times I) \to \Omega^{k-1}(M_1)$ by

$$h\omega = \int_0^1 i_t^*(i_{\frac{\partial}{\partial t}}\omega)\,dt$$

where i_t^* and $i_{\frac{\partial}{\partial t}}$ denote the pullback and interior product respectively.

$$\begin{tikzcd}
\Omega^{k-1}(M_1) & \\
\Omega^{k-1}(M_1 \times I) \arrow[u, "i_t^*"] & \Omega^k(M_1 \times I) \arrow[l, "i_{\frac{\partial}{\partial t}}"'] \arrow[ul, "h"']
\end{tikzcd}$$

Then we have

$$\begin{aligned}
(dh+hd)\omega &= \int_0^1 di_t^*(i_{\frac{\partial}{\partial t}}\omega)\,dt + \int_0^1 i_t^*(i_{\frac{\partial}{\partial t}}d\omega)\,dt \\
&= \int_0^1 i_t^*((di_{\frac{\partial}{\partial t}} + i_{\frac{\partial}{\partial t}}d)\omega)\,dt \\
&= \int_0^1 i_0^*(\phi_t^{\frac{\partial}{\partial t}})^*(L_{\frac{\partial}{\partial t}}\omega)\,dt \\
&= \int_0^1 i_0^*\frac{d}{dt}((\phi_t^{\frac{\partial}{\partial t}})^*\omega)\,dt \\
&= \int_0^1 \frac{d}{dt}(i_0^*(\phi_t^{\frac{\partial}{\partial t}})^*\omega)\,dt \\
&= \int_0^1 \frac{d}{dt}(i_t^*\omega)\,dt \\
&= (i_1^* - i_0^*)\omega
\end{aligned}$$

for all $\omega \in H^k_{dR}(M_1 \times I)$. Thus i_0^* and i_1^* are cochain homotopic.

Step 2. the pullbacks f^* and g^* are cochain homotopic:
Let h be a cochain homotopy between i_0^* and i_1^*. Then we have

$$\begin{aligned}
d(hH^*) + (hH^*)d &= dhH^* + hdH^* \\
&= (dh+hd)H^* \\
&= (i_1^* - i_0^*)H^* \\
&= (Hi_1)^* - (Hi_0)^* \\
&= g^* - f^*.
\end{aligned}$$

Thus f^* and g^* are cochain homotopic.

$$\begin{tikzcd}
\Omega^{k-1}(M_1) \arrow[r, "d"] & \Omega^k(M_1) & \\
& \Omega^k(M_1 \times I) \arrow[ul, "h"] \arrow[u, shift left, "i_1^*"'] \arrow[u, shift right, "i_0^*"] \arrow[r, "d"'] & \Omega^{k+1}(M_1 \times I) \arrow[ul, "h"'] \\
& \Omega^k(M_2) \arrow[u, "H^*"'] &
\end{tikzcd}$$

Step 3. the induced cochain maps f^* and g^* are equal:
If ω is a closed form on M_2, then

$$(g^* - f^*)\omega = d(hH^*)\omega + (hH^*)d\omega = d(hH^*)\omega \in \operatorname{im} d.$$

Thus f^* and g^* are equal on the cohomology groups. □

Theorem 2.2 (Homotopy invariance)**.** *Let M and N be homotopy equivalent. Let a smooth map $f\colon M \to N$ be the homotopy equivalence. Then the induced cohomology map $f^*\colon H^k_{dR}(N) \to H^k_{dR}(M)$ is an isomorphism.*

Proof. Let g be the homotopy inverse. Since $g \circ f$ and $f \circ g$ are homotopic to id_M and id_N respectively, the induced cohomology maps satisfy $\mathrm{id}_M^* = (g \circ f)^* = f^* \circ g^*$ and $\mathrm{id}_N^* = (f \circ g)^* = g^* \circ f^*$ by Theorem 2.1,

$$\begin{tikzcd} H^k_{dR}(M) \arrow[loop left, "\mathrm{id}_M^*"] \arrow[r, shift right, "g^*"'] & H^k_{dR}(N) \arrow[l, shift right, "f^*"'] \arrow[loop right, "\mathrm{id}_N^*"] \end{tikzcd}$$

We can lead that f^* is surjective from $\mathrm{id}_M^* = f^* \circ g^*$, and f^* is injective from $\mathrm{id}_N^* = g^* \circ f^*$. Therefore, f^* is an isomorphism. ☐

Theorem 2.3 (Dimension axiom)**.** *Let M be a singleton. Then we have*

$$H^k_{dR}(M) \cong \begin{cases} \mathbb{R} & , k = 0 \\ 0 & , k > 0 \end{cases}.$$

Proof. **Case 1.** $k = 0$:

$$0 \longrightarrow \Omega^0(M) = C^\infty(M, \mathbb{R}) \xrightarrow{d_0} \Omega^1(M)$$

Since every function from a singleton is constant, $\Omega^0(M) = \ker d_0$. Thus the de Rham cohomology group is $H_dR^k(M) \cong C^\infty(M, \mathbb{R}) = \mathbb{R}$.

Case 2. $k > 0$:
All k-forms on a singleton is 0 for $k > 0$. It is obviously followed that $H^k_{dR}(M) \cong 0$. ☐

Theorem 2.2 and 2.3 show the Poincare lemma.

Theorem 2.4 (Poincare's lemma)**.** *Let M be a contractible manifold. Then we have*

$$H^k_{dR}(M) \cong \begin{cases} \mathbb{R} & , k = 0 \\ 0 & , k > 0 \end{cases}.$$

Proof. Let N be a singleton. Let $f \colon M \to N$ and $g \colon N \to M$ be maps. The map $g \circ f$ is a constant function that is homotopic to id_M, and the map $f \circ g$ is the identity map id_N. Thus M and N are homotopy equivalent. By Theorem 2.2 and 2.3, we are done. ☐

Following two theorems is proved without assuming the previous theorems.

Theorem 2.5 (Mayer-Vietoris Sequence)**.** *Let $\{U, V\}$ be an open cover of M. For any k, there is a homomorphism $\delta \colon H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(U \cup V)$ such that the following is an exact sequence:*

$$\begin{tikzcd} & H^k_{dR}(U \cup V) \arrow[d, "k^* \oplus l^*"] & H^{k+1}_{dR}(U \cup V). \\ & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[d, "i^* - j^*"'] & \\ H^{k-1}_{dR}(U \cap V) \arrow[uur, "\delta"] & H^k_{dR}(U \cap V) \arrow[uur, "\delta"'] & \end{tikzcd}$$

Here i, j, k, l are inclusion maps as such

$$\begin{tikzcd}
 & M \arrow[d, equal] & \\
 & U \cup V & \\
U \arrow[ur, "k"] & & V \arrow[ul, "l"'] \\
 & U \cap V \arrow[ul, "i"] \arrow[ur, "j"'] &
\end{tikzcd}$$

and two maps $i^* - j^*,\ k^* \oplus l^*$ is defined from the pullback of the inclusions by $(i^* - j^*)(\alpha, \beta) = \alpha|_{U\cap V} - \beta|_{U\cap V}$ and $(k^* \oplus l^*)(\omega) = (\omega|_U, \omega|_V)$. The maps in the previous diagram are the induced cohomology maps.

Proof. By the Zigzag lemma, it is sufficient to prove that the following is a short exact sequence:

$$0 \longrightarrow \Omega^k(U \cup V) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap V) \longrightarrow 0.$$

We have to prove the exactness at $\Omega^k(U\cup V),\ \Omega^k(U)\oplus\Omega^k(V),\ \Omega^k(U\cap V)$. Let $\{\phi, \psi\}$ be a partition of unity subordinate to $\{U, V\}$ such that ϕ, ψ are defined on U, V respectively.

Step 1. The exactness at $\Omega^k(U \cup V)$:
It is equivalent to show that $k^* \oplus l^*$ is injective. Let $\omega \in \Omega^k(U \cup V)$. If $(k^* \oplus l^*)(\omega) = (0, 0)$, then we get $\omega|_U = 0,\ \omega|_V = 0$ and this implies $\omega = 0$.

Step 2. The exactness at $\Omega^k(U) \oplus \Omega^k(V)$:
Let $\omega \in \Omega^k(U \cup V)$. Then, $(i^* - j^*) \circ (k^* \oplus l^*)\omega = (i^* - j^*)(\omega|_U, \omega|_V) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0$. Thus $\mathrm{im}(k^* \oplus l^*) \subset \ker(i^* - j^*)$.

Let $(\alpha, \beta) \in \Omega^k(U) \oplus \Omega^k(V)$ be satisfy $(i^* - j^*)(\alpha, \beta) = \alpha|_{U\cap V} - \beta|_{U\cap V} = 0$. Then, if we define $\omega = \phi\alpha + \psi\beta$, then $(k^* \oplus l^*)(\omega) = (\omega|_U, \omega|_V) = (\alpha, \beta)$ since $\alpha|_{U\cap V} = \beta_{U\cap V}$. Thus $\ker(i^* - j^*) \subset \mathrm{im}(k^* \oplus l^*)$.

Step 3. The exactness of $\Omega^k(U \cap V)$:
It is equivalent to show that $i^* - j^*$ is surjective. Let $\omega \in \Omega^k(U \cap V)$. If we define $\alpha = \psi\omega \in \Omega^k(U)$ and $\beta = -\phi\omega \in \Omega^k(V)$, then $(i^* - j^*)(\alpha, \beta) = \psi\omega|_{U\cap V} - (-\phi\omega|_{U\cap V}) = \omega$. □

Theorem 2.6 (Additivity Axiom)**.** *Let $\{M_i\}_i$ be a collection of smooth manifolds. Then we have*

$$H^k_{dR}\left(\coprod_i M_i\right) = \prod_i H^k_{dR}(M_i).$$

Proof. Let $\iota_i \colon M_i \hookrightarrow \coprod_i M_i$ be the inclusions. Then the pullbak $\left(\prod_i \iota_i\right)^* \colon \omega \mapsto \prod_i \omega|_{M_i}$ induces the cohomology map that satisfies injectivity and surjectivity. □

In fact de Rham cohomology is a contravariant functor from manifolds to abelian groups.

Remark. These properties of de Rham cohomology is deeply related on the Eilenberg-Steenrod axioms. De Rham cohomology and singular cohomology satisfied the axioms. It means that singular cohomology also has the properties above. We will skip the proof of validations of the properties for singular cohomology.

3 De Rham Homomorphism

Definition 3.1. The *de Rham homomorphism* of M is the map $I_M\colon H^k_{dR}(M) \to H^k_\infty(M;\mathbb{R})$ defined as the cohomology map induced by the integration:

$$I_M[\omega][c] = \int_c \omega$$

where $[\omega]$ and $[c]$ is the equivalence class of $\omega \in \ker d$ and $c \in \ker \partial$.

Note that the codomain of the de Rham homomorphism is not singular cohomology, but *smooth* singular cohomology.

The de Rham homomorphism is well-defined with respect to the underlying manifold M.

Lemma 3.2 (Well-definedness)**.** *The de Rham homomorphism is a well-defined homomorphism.*

Proof. The map I_M obviously preserves the addition, so it is a homomorphism if well-defined.

Let ω_1, ω_2 be representatives of $[\omega] \in H^k_{dR}(M)$. Then there is a form $\omega' \in \Omega^{k-1}(M)$ such that $\omega_1 - \omega_2 = d\omega'$, so we get

$$\int_c \omega_1 = \int_c \omega_2 + d\omega' = \int_c \omega_2 + \int_c d\omega' = \int_c \omega_2 + \int_{\partial c} \omega' = \int_c \omega_2$$

since $\partial c = 0$.

Let c_1, c_2 be representatives of $[c] \in H^\infty_k(M)$. Then there is a chain $c' \in C^\infty_{k+1}(M)$ such that $c_1 - c_2 = \partial c'$, so we get

$$\int_{c_1} \omega = \int_{c_2 + \partial c'} \omega = \int_{c_2} \omega + \int_{\partial c'} \omega = \int_{c_2} \omega + \int_{c'} d\omega = \int_{c_2} \omega$$

since $d\omega = 0$. □

Lemma 3.3 (Naturality)**.** *The de Rham homomorphism is a natural tranformation, i.e. for a smooth function $f\colon M \to N$, the following diagram commutes:*

$$\begin{tikzcd}
H^k_{dR}(N) \arrow[r, "f^*"] \arrow[d, "I_N"] & H^k_{dR}(M) \arrow[d, "I_M"] \\
H^k_\infty(N;\mathbb{R}) \arrow[r, "f^*"] & H^k_\infty(M;\mathbb{R})
\end{tikzcd}$$

Proof. What we should prove is $I_M f^*[\omega] = f^* I_N[\omega]$, and it is equivalent to prove $I_M[f^*\omega][c] = I_N[\omega][f \circ c]$ for every closed form ω on N and cycle c on M.

This is shown directly by the definition

$$I_M[f^*\omega][c] = \int_c f^*\omega = \int_{\Delta^k} c^* f^* \omega = \int_{\Delta^k} (f \circ c)^* \omega = \int_{f \circ c} \omega = I_N[\omega][f \circ c].$$

□

4 De Rham Manifold

Definition 4.1 (De Rham Manifold)**.** A manifold is called *de Rham* if the de Rham homomorphism is an isomorphism.

Definition 4.2 (De Rham Basis)**.** A basis of a manifold is called *de Rham* if the basis elements and their finite intersections are de Rham.

In fact, de Rham's theorem states that every manifold is a de Rham manifold. There are four important property of de Rham manifolds:

- A manifold diffeomorphic to a de Rham manifold is de Rham.
- A contractible submanifold of $\mathbb{R}^n$ is de rham.
- The union of two de Rham manifolds is de Rham if the intersection of the two is de Rham.
- The disjoint union of de Rham manifolds is de Rham.

Let us prove them with the properties we proved in Section 2.

Lemma 4.3. *A manifold diffeomorphic to a de Rham manifold is de Rham.*

Proof. Suppose that N is a manifold diffeomorphic to a de Rham manifold M and let $f\colon M \to N$ be the diffeomorphism. The three maps I_M, $f^*\colon H^k_{dR}(N) \to H^k_{dR}(M)$, and $f^*\colon H^k_\infty(N) \to H^k_\infty(M)$ are isomorphisms. By Lemma 3.3, we have $I_N = (f^*)^{-1} \circ I_M \circ f^*$. Thus I_N is an isomorphism. $\square$

Lemma 4.4. *A contractible submanifold of $\mathbb{R}^n$ is de rham.*

Proof. Let M be a contractible submanifold of $\mathbb{R}^n$. By Theorem 2.4, we have

$$H^k_{dR}(M) \cong H^k_\infty(M) \cong \begin{cases} \mathbb{R} & , k = 0 \\ 0 & , k > 0 \end{cases}.$$

For $k > 0$, the de rham homomorphism is obviously an isomorhpism. For $k = 0$, it is sufficient to show that the de Rham homomorphism is not trivial since $\mathbb{R}$ is free module with a unique generator. Let ω be a 0-form such that $\omega = 1$. Then, the function $I_M[\omega]$ is not triviral because

$$I_M[\omega][\sigma] = \int_\sigma \omega = \int_{\Delta^0} \sigma^*\omega = f(\sigma(0)) = 1$$

for a simplex $\sigma\colon \Delta^0 = \{0\} \to M$. $\square$

Lemma 4.5. *The union of two de Rham manifolds is de Rham if the intersection of the two is de Rham.*

Proof. Let U, V and $U \cap V$ be de rham manifolds. By Theorem 2.5, the rows of the following commutative diagram are exact.

$$\begin{tikzcd}
H^{k-1}_{dR}(U) \oplus H^{k-1}_{dR}(V) \arrow[r] \arrow[d, "I_U \oplus I_V"] & H^{k-1}_{dR}(U \cap V) \arrow[r, "\delta"] \arrow[d, "I_{U\cap V}"] & H^k_{dR}(U \cup V) \arrow[r] \arrow[d, "I_{U\cup V}"] & H^k_{dR}(U) \oplus H^k_{dR}(V) \arrow[r] \arrow[d, "I_U \oplus I_V"] & H^k_{dR}(U \cap V) \arrow[d, "I_{U\cap V}"] \\
H^{k-1}_\infty(U) \oplus H^{k-1}_\infty(V) \arrow[r] & H^{k-1}_\infty(U \cap V) \arrow[r, "\delta"] & H^k_\infty(U \cup V) \arrow[r] & H^k_\infty(U) \oplus H^k_\infty(V) \arrow[r] & H^k_\infty(U \cap V)
\end{tikzcd}$$

The homomorphisms $I_U \oplus I_V$ and $I_{U\cap V}$ are isomorphisms. By the five lemma, $I_{U\cup V}$ is an isomorphism. $\square$

Lemma 4.6. *The disjoint union of de Rham manifolds is de Rham.*

Proof. Let $\{M_i\}_i$ be a collection of de Rham manifolds. By Theorem 2.6 and Lemma 3.3, the de Rham homomorphism of the disjoint union $I_{\coprod_i M_i}$ is an isomorphism with the same logic of the proof of Lemma 4.3.

$$\begin{array}{ccc} H^k_{dR}(\coprod_i M_i) & \xrightarrow{(\prod \iota)^*} & \prod_i H^k_{dR}(M_i) \\ \downarrow{\scriptstyle I_{\coprod_i M_i}} & & \downarrow{\scriptstyle \prod_i I_{M_i}} \\ H^k_{\infty}(\coprod_i M_i) & \xrightarrow{(\prod \iota)^*} & \prod_i H^k_{\infty}(M_i) \end{array}$$

$\square$

5 De Rham's Theorem

Theorem 5.1 (De Rham's Theorem)**.** *The de Rham homomorphism is an isomorphism.*

Proof. The proof is divided into two parts.

PART 1. A manifold with a de Rham basis is de Rham:
Let U be a manifold with a de Rham basis.

Step 1-1. The existence of the exhaustion function:
A function $f\colon U \to \mathbb{R}$ is called an *exhaustion function* if a set $\{p : f(p) < c\}$ is relatively compact for every $c \in \mathbb{R}$. The existence of this function will be used to cover U by compact subsets.

Let $\{U_\alpha\}_{\alpha\in\mathbb{N}}$ be a countable open cover of U such that each U_α is relatively compact. Let $\{\psi_\alpha\}_{\alpha\in\mathbb{N}}$ be a partition of unity subordinate to $\{U_\alpha\}$. Define $f = \sum_{\alpha\in\mathbb{N}} \alpha\psi_\alpha$ that is smooth since it is the finite sum of nonzero smooth functions at each point on U. For an integer k, let p be a point on U such that $p \notin \bigcup_{\alpha=1}^k U_\alpha$. Then, we have $f(p) \geq \sum_{\alpha=k+1}^\infty \alpha\psi_\alpha(p) > k\sum_{\alpha=k+1}^\infty \psi_\alpha(p) \geq k$. For any real number c the set $\{f(p) < c\}$ is an open subset of $\bigcup_{\alpha=1}^{\lfloor c\rfloor+1} U_\alpha$ that is relatively compact. So f is an exhaustion function.

Step 1-2. A finite union of de Rham basis elements is de Rham:
For induction, assume that the union of any j elements of the de Rham basis is de Rham. Let $\{D_i\}_{i=1}^{j+1}$ be arbitrary elements of the de Rham basis. By the assumption, $\bigcup_{i=1}^j D_i$, D_{j+1}, and $(\bigcup_{i=1}^j D_i) \cap D_{j+1} = \bigcup_{i=1}^j (D_i \cap D_{j+1})$ are de Rham. By Lemma 4.5, $\bigcup_{i=1}^{j+1} D_i$ is de Rham.

Step 1-3. A manifold with a de Rham basis is de Rham:
Let m be a positive integer. We can pick a finite open cover $\{D_i^m\}_{i=1}^k$ of $f^{-1}([m, m+1])$ such that each D_i^m is de Rham and $D_i^m \subset f^{-1}[m - \frac{1}{2}, m + \frac{3}{2}]$ because of the compactness. Define $D^m = \bigcup_{i=1}^k D_i^m$. Since D^m and $D^m \cap D^{m+1}$ are all finite union of de Rham basis elements, they are also de Rham.

Define D_{even} and D_{odd} as disjoint unions of de Rham manifolds by $D_{even} = \bigcup_{m\in\mathbb{N}} D^{2m}$ and $D_{odd} = \bigcup_{m\in\mathbb{N}} D^{2m+1}$. The intersection $D_{even} \cap D_{odd} = \bigcup_{m\in\mathbb{N}}(D^m \cap D^{m+1})$ is also a disjoint union of de Rham manifolds. By Lemma 4.6, D_{even}, D_{odd} and $D_{even} \cap D_{odd}$ are de Rham. By Lemma 4.5, $D_{even} \cup D_{odd} = U$ is de Rham.

PART 2. Every manifold is de Rham:

Step 2-1. An open subset of $\mathbb{R}^n$ is de Rham:

Let $\{B_\alpha\}_\alpha$ be a collection of all open balls contatined in an open set $U \subset \mathbb{R}^n$. By Lemma 4.4, $\{B_\alpha\}_\alpha$ is a de Rham basis of U since open balls and their intersections are convex.

Step 2-2. Every manifold is de Rham:
Let $\{U_\alpha\}_\alpha$ be a smooth atlas of M. By Lemma 4.3, $\{U_\alpha\}_\alpha$ is a de Rham basis of M since each U_α and their intersections are diffeomorphic to open subsets of $\mathbb{R}^n$. $\square$