Propositions on Differential Geometry

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Contents

1. Smooth manifolds

Proposition 1.1. *Let $\gamma : I \to M$ be a curve with $\gamma(0) = p$. There is a chart (U, φ) of p such that $\varphi^{-1}|_I = \gamma$.*

Proposition 1.2. *Let M be a connected manifold. For distinct two points on M, there is a curve connecting them that is an embedding into M.*

Proof. Let $\gamma : I \to M$ be a curve connecting the given two points, say p, q.

Step 1: Constructing a piecewise linear curve. For $t \in I$, take a convex chart U_t at $\gamma(t)$. Since I is compact, we can choose a finite $\{t_i\}_i$ such that $\bigcup_i \gamma^{-1}(U_{t_i}) = I$. This implies $\operatorname{im}\gamma \subset \bigcup_i U_{t_i}$. Reorganize indices such that $\gamma(t_1) = p$, $\gamma(t_n) = q$, and $U_{t_i} \cap U_{t_{i+1}} \neq \varnothing$ for all $1 \leq i \leq n-1$. It is possible since the graph with $V = \{i\}_i$ and $E = \{(i, j) : U_{t_i} \cap U_{t_j} \neq \varnothing$ is connected. Choose $p_i \in U_{t_i} \cap U_{t_{i+1}}$ such that they are all dis for $1 \leq i \leq n-1$ and let $p_0 = p$, $p_n = q$.

How can we treat intersections?

Therefore, we get a piecewise linear curve which has no self intersection from p to q.

Step 2: Smoothing the curve. □

Proposition 1.3. *Let M is an embedded manifold with boundary in N. Any kind of sections on M can be extended on N.*

Proposition 1.4. *Every ring homomorphism $C^\infty(M) \to \mathbb{R}$ is obtained by an evaluation at a point of M.*

Proof. Suppose $\phi : C^\infty(M) \to \mathbb{R}$ is not an evaluation. Let h be a positive exhaustion function. Take a compact set $K := h^{-1}([0, \phi(h)])$. For every $p \in K$, we can find $f_p \in C^\infty(M)$ such that $\phi(f_p) \neq f_p(p)$ by the assumption. Summing $(f_p - \phi(f_p))^2$ finitely on K and applying the extreme value theorem, we obtain a function $f \in C^\infty(M)$ such that $f \geq 0$, $f|_K > 1$, and $\phi(f) = 0$. Then, the function $h + \phi(h)f - \phi(h)$ is in kernel of ϕ although it is strictly positive and thereby a unit. It is a contradiction. □

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2. Tangent bundle

Proposition 2.1. *The n-sphere S^n possesses a nonvanishing vector field iff n is odd.*

3. Geodesics

Proposition 3.1. *Let $p \in M$. The set of points that is geodesically connected to p is open.*