Binary Quadratic Forms

IKHAN CHOI

1. Equivalence

Definition 1.1. Two forms are called *equivalent* if they are in a same oribit with respect to $\mathrm{GL}_2(\mathbb{Z})$-action.

Definition 1.2. Two forms are called *properly equivalent* if they are in a same oribit with respect to $\mathrm{SL}_2(\mathbb{Z})$-action.

For representation problems, $\mathrm{GL}_2(\mathbb{Z})$-action is important. For the correspondence with the theory of qudratic fields, $\mathrm{SL}_2(\mathbb{Z})$ is rather important. From now, all equivalence relations are by $\mathrm{SL}_2(\mathbb{Z})$.

Example 1.1. Two forms (a, b, c) and $(a, -b, c)$ are equivalent but not properly equivalent in general.

Lemma 1.2. *For a form (a, b, c) and an integer n, we have*

(1) $(a, b, c) \sim (a,\ 2an + b,\ an^2 + bn + c)$
(2) $(a, b, c) \sim (cn^2 + bn + a,\ 2cn + b,\ c)$
(3) $(a, b, c) \sim (c, -b, a)$

2. Definite forms

Proposition 2.1. *The $\mathrm{SL}_2(\mathbb{Z})$-action on the definite forms is not faithful, i.e. the kernel is given by a nontrivial group $\{\pm I\}$.*

Proposition 2.2. *The $\mathrm{PSL}_2(\mathbb{Z})$-action on the definite forms is faithful.*

The faithfulness is not important though, so we choose $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ as the modular group instead of $\mathrm{PSL}_2(\mathbb{Z})$.

Definition 2.1. A positive definite form (a, b, c) is *reduced* if it satisfies

(1) $|b| \le a \le c$,
(2) if $|b| = a$ or $a = c$, then $b \ge 0$.

The term "reduced" means that it is considered as the unique representative of each orbit, under the action of $\mathrm{SL}_2(\mathbb{Z})$.

2.1. **Positive definite forms.**

Proposition 2.3. *The set of positive definite forms admits the $\mathrm{SL}_2(\mathbb{Z})$-action.*

Proposition 2.4. *The $\mathrm{SL}_2(\mathbb{Z})$-actions on positive definite forms and negative definite forms are isomorphic.*

Last Update: July 26, 2019.

3. Indefinite forms

4. Class group