Analysis II : General Topology

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Contents

Preface

The word topology is used in two different contexts: analytic sense and geometric sense. When we are talking the stories of doughnuts and coffee mugs, they are in fact involved in topology of geometric sense, which is also referred to as a branch of mathematics that studies spaces such as manifolds or CW complexes. In analysis, the topology is mentioned greatly unrelatedly to the stuffs like doughnuts, but it refers to the minimal structure that is required in order to define concepts of limit and continuity. More precisely, once a structure called topology is settled on a set, then we can expand some basic analytic theories using limit and continuity. Normed spaces are the first examples which possess a certain topology. With the topology, we could describe formally why a sequence converges to a point or whether a function is continuous. Simply, we say a norm induces a natural topology. This book is of course interested in the latter issues, as we have seen the title of the book.

Then, why should we study topologies for analysis? In Euclidean spaces or the space of continuout functions on a compact set that we have usually studied, we have a fixed standard norm to determine if a sequence converges. For different topologies in analysis provides a sort of different criteria, the convergence changes even for a same sequence when we judge it in different topologies so that we can argue in what sense a sequence converges. Especially in real analysis or functional analysis it is an extremely important way of recognizing the various convergence modes of functional sequences. A nonnegligible problem that occurs here is that some convergences cannot be formulated with neither norms nor metrics anymore, like pointwise convergence of functional sequences defined with uncountable domain. Thereby, the study of topologies necessarily made sophisticated connections with mathematical analysis.

General topology is the abstract study of topologies and topological spaces. Similarly as mentioned above, there are two large branches of general topology; both are contributed to build nice frameworks of other mathematics. One is for algebraic topology and studies the category of convinient spaces in which well-known constructions and computational tools are available, and the other is for doing abstract analysis. One interesting feature of general topology is that the basic topology in analysis is a preliminary of the abstract study of the spaces used in algebraic topology, so everyone starts to learn it from analysis.

In this book, we are going to assume the reader is already familiar to the theory of normed spaces. For instance, we can require the reader to know what the uniform convergence is because it can be regarded as just a convergence in a properly defined norm. Metrics are defined at the first of this book.

For the first, ... In Chapters 2 and 3, ...

This book is written in order to be used in my imaginary lectures for students who lost their chances to learn professional educations for various reasons. I tried to put convincing explanations at every newly defined concept as with clear logics as possible and cram supplementary stories that are not necessary, which might not be really satisfied. Therefore, I think this book would never be a good choice for a standard course text relative to the other existing great books. However, I sincerely organized the materials hoping there would be at least one student in the world affected affirmitively. I will be very satisfied if one of you just could enjoy math with this book.

CHAPTER 1

Topological structures

Firstly we discuss to what extent the definition of analytic notions such as limit and continuity can be extended. One of main interest in general topology is to make extended version of mathematical calculus on a set without algebraic operations. However, lay it up in the heart that several properties must be compromised when we try to make generalizations.

Recall that we measured how near the two points are by taking absolute value of algebraic subtraction of two position vectors in normed spaces. How can we dismiss the subtraction? Saying only the results, mathematicians succeeded to generalize limit of sequences and continuity of functions, but compromising the theory of differentiation and integration. Topology is the term for this successful solution. In other words, for the most part, wonderful statements purely related to limit and continuity were possible to be extended without big flaws even if we forget the vector space operations by introducing the concept of topology, but differentiation and integration could not on the other hand.

Topological structure refers to an additional function on a given set or a more complicated mathematical device which solves the problem by being put on a set. Norm is a typical example of topological structure, and so is "topology".

1. Metric

Metric is a generalization of norm and a special example of topology. For example, every subset of a normed vector space is equipped with a metric. Since general topology might be too abstract for novices to grasp, we will make a bridge from norms to topologies.

Metric was the first successful trial to find an abstract framework for studying limits. Later, we will find that metric provides a surprisingly appropriate and widely-applicable tool to understand the nature of mathematical analysis.

1.1. Metric spaces. A metric is a function which assigns a nonnegative real number, which has a meaning of distance, to a pair of two points. A metric space is just a set endowed with a metric.

Definition 1.1. Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x, y) = 0$ iff $x = y$, (nondegenracy)
(2) $d(x, y) = d(y, x)$, (symmetry)
(3) $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

A pair (X, d) of a set X and a metric on X is called a *metric space*. We often write it simply X.

The most familiar metric comes from the standard norm on Euclidean space $\mathbb{R}^d$. Notice that the third axiom, the triangle inequality, is named after the one for norms. In this context, we can see metrics as a generalization of norms for spaces not admitting the vector space structure. When we use an analytic theory on Euclidean spaces or more generally normed spaces, the following metric is considered as the standard. Moreover, every subset of normed space is also an example of metric space because the metric function can be inherited to every subset of a given metric space.

Example 1.1. A normed space X is a metric space. Precisely, the norm structure naturally defines a real-valued function d on $X \times X$ defined by $d(x, y) := \|x - y\|$ and it satisfies the axioms of metric.

Proof. It is quite easy. Just recall the axioms of norm and deduce the conclusion for each axiom of metric. □

Example 1.2. Let (X, d) be a metric space. Every subset of X has a natural induced metric, just the restriction of original metric d.

Proof. Obvious. □

In fact, the converse holds; every metric space can be viewed as a subset of a normed space. This deeper result on the relation between normed spaces and metric spaces is discovered by Kuratowski. Since the theorem does not play any important role in the whole book, readers who want to read fast may skip. To state the theorem, we introduce isometry, a map preserving metrics.

Definition 1.2. Let X and Y be metric spaces. A map $\phi : X \to Y$ is called an *isometry* if $d(x, y) = d(\phi(x), \phi(y))$ for all $x, y \in X$. If there is a bijective isometry between X and Y, then we say the spaces are *isometric*.

Every isometry is clearly injective so that it is bijective if and only if it is surjective. Also, the inverse of bijective isometry is also an isometry. If two metric spaces are isometric, we can view them as virtually same, in the "category" of metric spaces. The following theorem tells another characterization of metric spaces.

Proposition 1.1 (Kuratowski embedding). *Every metric space is isometric to a subset of a normed space. In other words, for every metric space (X, d), there is an isometry ϕ from X to a normed space.*

Proof. Choose any point $p \in X$. Let Y be the space of bounded real-valued functions on X. It is a normed space with uniform norm. Define $\phi : X \to Y$ by $\phi(x)(t) = d(x,t) - d(p,t)$. Note that $\phi(x)$ is bounded with $\|\phi(x)\| = \sup_{t\in X} |d(x,t) - d(p,t)| = d(x,p)$. Then,

$$\|\phi(x) - \phi(y)\| = \sup_{t\in X} |\phi(x)(t) - \phi(y)(t)| = \sup_{t\in X} |d(x,t) - d(y,t)| = d(x,y).$$

This proves ϕ is a isometry. $\square$

Remark. The space Y is somtimes denoted by $\ell^\infty(X)$, and it is in fact a Banach space. In addition, the image of the isometry ϕ is in a closed subspace $C_b(X) \subset \ell^\infty(X)$, the space of bounded real-valued continuous functions.

We have seen metrics can be seen as the generalization of norms. However, there are also many examples of metrics that are not involved directly in the norms. Even if they are far from subsets of a normed space, we can apply our intuition of balls. The examples below are given without proofs.

Example 1.3. Let X be a set. Then, a function $d : X \times X \to \mathbb{R}_{\geq 0}$ defined by

$$d(x,y) := \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases}$$

is a metric on X. This metric is sometimes called *discrete metric* because balls can separate all single points out.

Example 1.4. Let $G = (V, E)$ be a connected graph. Define $d : V \times V \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the distance of two vertices; the length of shortest path connecting two vertices. Then, (V, d) is a metric space.

Example 1.5. Let $\mathcal{P}(X)$ be the power set of a finite set X. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the cardinality of the symmetric difference; $d(A,B) := |(A - B) \cup (B - A)|$. Then $(\mathcal{P}(X), d)$ is a metric space.

Example 1.6. Let C be the set of all compact subsets of $\mathbb{R}^n$. Recall that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Then, $d : C \times C \to \mathbb{R}_{\geq 0}$ defined by

$$d(A,B) := \max\{\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|\}$$

is a metric on C. It is a little special case of *Hausdorff metric.*

1.2. Limits and continuity. Many freshmen misunderstand the main role of metric for its name. They recognize metric as something measures a distance and belonging to the study of geoemtry. We cannot strongly affirm it is false, but I hope to mention that a metric is quite far from geometric structures, and is rather an analytic structure. Meaning, metric is in fact not interested in measuring a distance between two points; the main function of metric is to make balls. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuistic sense. Metric can be considered as a device to let someone accept the notion of neighborhoods more friendly, which is vital for analysis of limits and continuity.

Definition 1.3. Let X be a metric space. A set of the form

$$\{y \in X : d(x,y) < \varepsilon\}$$

for $\varepsilon > 0$ is called a *ball centered at* x and denoted by $B(x,\varepsilon)$ or $B_\varepsilon(x)$.

Now let us reformulate the definitions of limits and continuity with balls, which we actually use in the usual calculus on Euclidean spaces or generally on normed spaces. Compare the following definitions to what we remember.

Definition 1.4. Let $\{x_n\}_n$ be a sequence of points on a metric space (X,d). We say that a point x is a *limit* of the sequence or the sequence *converges to* x if for arbitrarily small ball of size ε cenetered at x, $B(x,\varepsilon)$, we can find n_0 such that $x_n \in B(x,\varepsilon)$ for $n > n_0$. If it is satisfied, then we write

$$\lim_{n\to\infty} x_n = x,$$

or simply

$$x_n \to x$$

as $n \to \infty$. If there is no such limit x, then we say the sequence *diverges*.

Definition 1.5. A function $f : X \to Y$ between metric spaces is called *continuous at* $x \in X$ if for any ball $B(f(x),\varepsilon) \subset Y$ centered at $f(x)$ there is a ball $B(x,\delta) \subset X$ centered at x such that $f(B(x,\delta)) \subset B(f(x),\varepsilon)$. The function f is called *continuous* if it is continuous at every point on X.

We have to emphasize that taking either ε or δ really means taking a ball of the very radius.

There are a lot of valuable propositions for limit and continuity, but we postpone to mention them to later because they are generalized to topological spaces: for example, the limit of a sequence is unique.....

Example 1.7. Let X be the discrete metric space in Example 1.3. A sequence $\{x_n\}_n$ converges to x if and only if it is eventually x; there is a positive integer n_0 such that $x_n = x$ for all $n > n_0$.

Example 1.8. Let X and Y be metric spaces. If X is equipped with the discrete metric in Example 1.3, then every function $f : X \to Y$ is continuous.

Example 1.9. An isometry is always continuous.

The sequence of real numbers defined by $x_n = \frac{1}{n}$ diverges in discrete metric. Even for the same sequence on a same set, the convergence depends on the attached metric. However, we cannot say different metrics always provide different criteria for convergence. In other words, when we consider a metric as a function that takes a sequence as input and outputs whether a sequence converges or diverges, there may be two different metrics which gives exactly same answer about convergence. Of course, the continuity of functions has the same issue. This allows us to think an equivalence relation on the set of metrics, that is, two equivalent metrics give a common criterion for convergence and continuity. In this situation, it can be paraphrased into that the two metrics induce exactly same topology. Some definitions and theorems for the equivalence checking will be given as follows.

Remark. There exist two different topologies that have same sequential convergence data. For example, This means the informations of sequence convergence are not sufficient to uniquely characterize a topology. Instead, convergence data of generalized sequences, which is called nets, recover the topology. For topologies having a property called the first countability, it is enough to consider only usual sequences. These problem will be profoundly treated in Chapter 3.

1.3. Pseudometrics.

Problems.

Problem 1.1. Show that there is a metric d on $\mathbb{R}$ such that a sequence $\{x_n\}_n$ defined by $x_n = x + \frac{1}{n}$ is convergent with respect to d if and only if $x \neq 0$.

Problem 1.2. Let d and d' be metrics on a set X. Suppose that a sequence $\{x_n\}_n$ in X converges to x in d and converges to x' in d'. Show that $x = x'$.

Problem 1.3.

2. Topology

We define topology and introduce some supplementary notions.

2.1. Filters. Suppose we want to find a proper way to define limit and convergence. Recall how we define convergence of a sequence of real numbers: we say a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a number x if for each $\varepsilon > 0$ there is $n_0(\varepsilon) \in \mathbb{N}$ such that $|x - x_n| < \varepsilon$ whenever $n > n_0$. Simply, x_n is close to x if n is close to the infinity. Observe the two necessary structures to make this possible; the "system of neighborhoods" at each point x, and the total order on the index set $\mathbb{N}$ the set of natural numbers. Without the order structure, we would not be able to formulate the intuition of the direction toward which a sequence is converging. Even though the order on $\mathbb{N}$ is totally defined so that we can compare every pair of two elements, but it can be generalized to the case of partial orders.

Definition 2.1. A subset $\mathcal{D}$ of a poset is called *(upward) directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$. Similarly, $\mathcal{D}$ is called *downward directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $c \leq a$ and $c \leq b$.

The directedness of a partially ordered set is an essential notion to define limit.

Let X be a set and $x \in X$. Then, the power set $\mathcal{P}(X)$ is a poset with inclusion relation. The filter bases are defined abstractly:

Definition 2.2. A *filter base* is a nonempty and downward directed subset of a poset.

And concretely:

Definition 2.3. A collection $\mathcal{B}_x$ of subsets of X is called a *filter base at x* if every element of $\mathcal{B}_x$ contains x and it forms a nonempty downward directed subset; every $U \in \mathcal{B}_x$ contains x, and for all $U_1, U_2 \in \mathcal{B}_x$ there is $U \in \mathcal{B}_x$ such that $U \subset U_1 \cap U_2$.

Among filters, we can give a relation structure as follows.

Definition 2.4. Let $\mathcal{B}_x, \mathcal{B}'_x$ be filter bases at x. We say $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$, or a *refinement* of $\mathcal{B}_x$ if for every $U \in \mathcal{B}_x$ there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$.

As synonyms, all the following expressions tell the same thing.

(1) $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$,
(2) $\mathcal{B}'_x$ is *stronger than* $\mathcal{B}_x$,
(3) $\mathcal{B}_x$ is *coarser than* $\mathcal{B}'_x$,
(4) $\mathcal{B}_x$ is *weaker than* $\mathcal{B}'_x$.

The relation is a preorder so that we can consider the equivalence classes on which the natural partial order can be defined.

Proposition 2.1. *The refinement relation between filter bases is a preorder, and each equivalence class contains a unique maximal element.*

Proof. To show a relation is a preorder, we need to check transitivity. Suppose $\mathcal{B}''_x$ is finer than $\mathcal{B}'_x$ and $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$. For any $U \in \mathcal{B}_x$, there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$, and there is also $U'' \in \mathcal{B}''_x$ such that $U'' \subset U'$. Since $U'' \subset U$, we can conclude $\mathcal{B}''_x$ is finer than $\mathcal{B}_x$.

We can say two filter bases are equivalent if they are both finer than each other. Consider an equivalence class of filter bases and just denote it by A. Then, $\bigcup_{\mathcal{B}_x \in A} \mathcal{B}_x$ is also contained in A since it is equivalent to an arbitrary filter base $\mathcal{B}_x$ in A. It is also easy to check that this is maximal. □

Now we define filters.

DEFINITION 2.5. A *filter at x* is the maximal element of an equivalence class of filter bases at x.

In other words, filters have one-to-one correspondence to the equivalence classes of filter bases. A filter is identified to an equivalence class of filter bases. They can be also characterized by three axioms.

THEOREM 2.2. *A collection $\mathcal{F}_x$ of subsets of X is a filter at x if and only if every element contains x and it is closed under supersets and finite intersections;*

(1) $x \in U$ *for* $U \in \mathcal{F}_x$,
(2) *if* $U \subset V$ *and* $U \in \mathcal{F}_x$, *then* $V \in \mathcal{F}_x$,
(3) *if* $U, V \in \mathcal{F}_x$, *then* $U \cap V \in \mathcal{F}_x$.

PROOF. □

Many references use the above theorem as the definition of filter because it is useful for someone who wants to check whether a given family is a filter.

THEOREM 2.3. *A filter $\mathcal{F}'_x$ is finer than another filter $\mathcal{F}_x$ if and only if $\mathcal{F}'_x \supset \mathcal{F}_x$.*

PROOF. □

The following examples will be helpful to catch the intuition.

EXAMPLE 2.1. Let x be a point in a metric space. The set of all open balls cenetered at x is a filter base at x. The set of all open balls containing x is aslo a filter base and they are equivalent. A filter equivalent to these filter bases are called *neighborhood filter at x*.

EXAMPLE 2.2. Let S be a subset of a set. The set of all subsets containing S is a filter at x for every $x \in S$. If $S = \{x\}$, then it is called a *principal filter* at x.

EXAMPLE 2.3. The set of all subsets of $\mathbb{N}$ whose complement is finite is a filter, but it is not a filter at a point. However, it is intuitively a filter at infinity.

2.2. Topologies. Before defining topology, recall that it plays the most important role in the definition of continuous functions to deal with neighborhoods of a point. We want a structure to give a notion of neighborhoods of a point such as metrics, in other words, we want to generalize metric in a suitable way. There is a conventional definition of topology: topology is defined as a subset of the power set of underlying space satisfying some axioms, and it is said to consist of open sets so that a topology indicates that which subsets are open or not. However, this definition is so abstract that it might allow first-readers to lose its intuitions. Thereby, we attempt to take another way. Before introducing topology, we shall define a topological basis. Topological bases are often used to describe a particular topology as bases of vector spaces do. The main definition of topology will follow.

Let X be a set.

Definition 2.6. A collection $\mathcal{B}$ of subsets of X is called a *topological base* or simply a *base on* X if
$$\{U : x \in U \in \mathcal{B}\}$$
is a filter base at x for every $x \in X$.

A topological base is a kind of global version of filter base.

Definition 2.7. Let $\mathcal{B}$ be a topological base on X and $x \in X$. A filter base at x is called a *local base* of x if it is equivalent to the filter base $\{U : x \in U \in \mathcal{B}\}$.

As we have done in the previous section, we can settle the refinement order on the set of topological bases.

Definition 2.8. Let $\mathcal{B}, \mathcal{B}'$ be topological bases on X. We say $\mathcal{B}$ is *coarser* or *weaker* than $\mathcal{B}'$, and $\mathcal{B}'$ is *finer*, *stronger* than $\mathcal{B}$, or a *refinement* of $\mathcal{B}$ if every local base $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$ at every point $x \in X$.

Proposition 2.4. *The refinement relation between topological bases is a preorder, and each equivalence class contains a unique maximal element.*

Proof. □

A topology is defined to be the maximal element, which means in fact an equivalence class of topological bases.

Definition 2.9. A *topology on* X is the maximal element of an equivalence class of topological bases on X.

There is also a criterion for topology.

Theorem 2.5. *A collection $\mathcal{T}$ of subsets of X is a topology on X if and only if*

(1) $\varnothing, X \in \mathcal{T}$,
(2) *if* $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{T}$, *then* $\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$,
(3) *if* $U_1, U_2 \in \mathcal{T}$, *then* $U_1 \cap U_2 \in \mathcal{T}$.

Proof. □

Theorem 2.5 is usually used as a definition of topology because it allows us to check without difficulty whether a collection of subsets is a topology.

Since all topological structures are made to generalize the standard metric of Euclidean space, so drawing balls for representing base elements is always helpful in the whole story of general topology.

2.3. Bases and subbases.

Definition 2.10. Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. If $\mathcal{T}$ is the coarsest topology containing $\mathcal{B}$, then we say the topology $\mathcal{T}$ is *generated by* $\mathcal{B}$.

Theorem 2.6. *Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. The followings are equivalent:*

(1) *$\mathcal{B}$ generates $\mathcal{T}$,*
(2) *$\mathcal{B}$ and $\mathcal{T}$ are equivaent bases,*
(3) *$\mathcal{T}$ is the set of all arbitrary unions of elements of $\mathcal{B}$.*

Definition 2.11. Let $\mathcal{S} \subset \mathcal{P}(X)$. If a topology $\mathcal{T}$ is the coarsest topology containing $\mathcal{S}$, then we say $\mathcal{S}$ is called a *subbase* of $\mathcal{T}$.

Proposition 2.7. *Let $\mathcal{S} \subset \mathcal{P}(X)$. The set of finite intersections of elements of $\mathcal{S}$ is a basis.*

Here is the metric space example.

Example 2.4. Let X be a metric space. A set of all balls $\mathcal{B} = \{B(x, \varepsilon) : x \in X, \varepsilon > 0\}$ is a base on X because for every point $x \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$, we have $x \in B(x, \varepsilon) \subset B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$ where $\varepsilon = \min\{\varepsilon_1 - d(x, x_1), \varepsilon_2 - d(x, x_2)\}$.

In metric spaces, of course, there can exist infinitely many bases, but they are hardly considered except $\mathcal{B}$. Sometimes in the context of metric spaces, the term neighborhood or basis are used to say $\mathcal{B}$. As we have seen, balls in metric spaces are the main concept to state ε-δ argument. This example would show a basis is fundamental language to describe the nature of limits in metric spaces.

2.4. Open sets and neighborhoods. def:nbhd and neighborhood filter

2.5. Closed sets and limit points.

2.6. Interior and closure.

3. Uniformity

3.1. Uniform spaces. uniformness of metric

3.2. Entourages.

3.3. Pseudometrics. Metric can be regarded as the "countably" uniform structure in some sense. In other texts, for this reason, one frequently introduces metric instead of uniformity in order to avoid superfluously complicated and less intuitive notions of uniform structures, when only doing elementary analysis not requiring uncountable local bases.

CHAPTER 2

Continuity

1. Continuous maps

- continuity, Cauchy, uniform,
Lipschitz

EXAMPLE 1.1. An isometry between metric spaces is Lipschitz continuous with costant 1.

1.1. Sequential continuity.

2. Homeomorphisms

- topological property: connected, compact

3. Connectedness

- connected, path-connected, locally path-connected - component - homotopy

CHAPTER 3

Convergence

– sequences - density and approximation - sequential spaces, first countable – nets and filters – completeness - completion

CHAPTER 4

Compactness

Definition 0.1. Let X be a topological space. A *cover* of a subset $A \subset X$ is a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X such that $A \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. If U_α are all open, then it is called *open cover.*

Definition 0.2. Let X be a topological space. A subset $K \subset X$ is called *compact* if every open cover of K has a finite subcover.

Proposition 0.1. *Let X be a topological space with a basis $\mathcal{B}$. A subset $K \subset X$ is compact if and only if every cover of the form $\{B_x \in \mathcal{B}\}_{x \in K}$ has a finite subcover.*

Remark. Let $\mathcal{P}$ be a property of a fuction $f \colon X \to Y$, such as continuity If we say f has $\mathcal{P}$ at a point x, then it would implies that x has a neighborhood U such that

0.1. Properties of compactness.

Theorem 0.2. *Let X and Y be topological spaces. For a continuous map $f \colon X \to Y$, the image $f(K)$ is compact for compact $K \subset X$.*

Remark. This is why the term "compact space" is widely used.

Corollary 0.3 (The extreme value theorem). *A continuous function on a closed interval has a global maximum and*

Heine-Cantor,

0.2. Characterizations of compactness.

$$\begin{tikzcd}
\text{net compact} \arrow[d, leftrightarrow] & \text{sequentially compact} \arrow[d, shift right] & \\
\text{compact} \arrow[r] & \text{countably compact} \arrow[u, dashed, shift right, "\text{sequential}"'] \arrow[l, dashed, bend left, "\substack{\text{metrizable}\\ \text{or 2nd countable}}"] \arrow[r] & \text{limit point compact} \arrow[l, dashed, bend left, "T_1"]
\end{tikzcd}$$

CHAPTER 5

Separation axioms

1. Separation axioms

2. Metrization theorems

CHAPTER 6

Function spaces

1. Continuous function spaces

Definition 1.1. Let X and Y be topological spaces. The *continuous functions space* $C(X,Y)$ is the set of continuous functions from X to Y. If $Y = \mathbb{R}$ or $\mathbb{C}$, then the continuous function space is denoted by $C(X)$.

In considering the continuous function space, Y will be assumed to be a metric space because of its usefulness in most applications. Then, there are two useful topologies on $C(X,Y)$. Since there is a difficulty to deal with open sets or basis directly in a function space, the convergence will be a reliable alternative to describe the topologies. Before giving definition of the topologies, define pseudometrics ρ_K on $C(X,Y)$ by

$$\rho_K(f,g) = \sup_{x\in K} d(f(x), g(x))$$

for $K \subset X$ compact.

Definition 1.2. Let X and Y be topological spaces. The *topology of pointwise convergence* on $C(X,Y)$ is a subspace topology inherited from the product topology on Y^X.

Proposition 1.1. *Let X be a topological space and Y be a metric space. The topology of pointwise convergence on $C(X,Y)$ is generated by pseudometrics $\rho_{\{x\}}$, namely all $\{g : d(f(x), g(x)) < \varepsilon\}$ for $f \in C(X,Y)$, $\varepsilon > 0$, and $x \in X$.*

Definition 1.3. Let X be a topological space and Y be a metric space. The *topology of compact convergence* on $C(X,Y)$ is a topology generated by pseudometrics ρ_K, namely all $\{g : \rho_K(f,g) < \varepsilon\}$ for $f \in C(X,Y)$, $\varepsilon > 0$, and compact $K \subset X$.

Proposition 1.2. *Let $C(X,Y)$ be a continuous function space for a topological space X and a metric space Y. A functional sequence in $C(X,Y)$ converges in the topology of compact convergence if and only if the functional sequence converges compactly.*

Theorem 1.3. *Let X be a topological space and Y be a metric space. If X is hemicompact, in other words, X has a sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that every compact subset of X is contained in K_n for some $n \in \mathbb{N}$, then the topology of compact convergence on $C(X,Y)$ is metrizable.*

Proof. bounding and merging pseudometrics □

$\frac{\varepsilon}{3}$ argument

1.1. The Arzela-Ascoli theorem. The Arzela-Ascoli theorem is a main technique to verify compactness of a subspace of continuous function space. The theorem requires the notion of equicontinuity, which lifts pointwise compactness up onto compactness in topology of compact convergence.

Definition 1.4. Let X be a topological space and Y be a metric space. A subset $\mathcal{F} \subset C(X,Y)$ is called *(pointwise) equicontinuous* if for every $\varepsilon > 0$ and each $x_0 \in X$, there is an open neighborhood U of x_0 such that $x \in U \Rightarrow d(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$.

Theorem 1.4 (Arzela-Ascoli, conventional version). *Let X be a compact space. For $(f_n)_{n\in\mathbb{N}} \subset C(X)$, if it is equicontinuous and pointwisely bounded, then there is a subsequence that uniformly converges.*

Theorem 1.5 (Arzela-Ascoli, metrized version). *Let X be a hemicompact space and Y be a metric space. Let $\mathcal{T}_p$ and $\mathcal{T}_c$ be the topology of pointwise and compact convergence on $C(X,Y)$ relatively. For $\mathcal{F} \subset C(X,Y)$, if $\mathcal{F}$ is equicontinuous and relatively compact in $\mathcal{T}_p$, then $\mathcal{F}$ is relatively compact in $\mathcal{T}_c$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and $K \subset X$ be a compact. By equicontinuity, for each $k \in \mathbb{N}$ a finite open cover $\{U_s\}_{s\in S_k}$ with a finite set $S_k \subset K$ can be taken such that $x \in U_s \Rightarrow d(f(x), f(s)) < \frac{1}{k}$ for all $f \in \mathcal{F}$. By the pointwise relative compactness, we can extract a subsequence $\{f_m\}_{m\in\mathbb{N}}$ of $\{f_n\}_n$ such that $\{f_m(s)\}_m$ is Cauchy for each $s \in \bigcup_{k\in\mathbb{N}} S_k$ by the diagonal argument.

For every $\varepsilon > 0$, let $k = \lceil (\frac{\varepsilon}{3})^{-1} \rceil$ and $m_0 = \max\{m_{0,s} : s \in S_k\}$ where $m_{0,s}$ satisfies that $m, m' > m_{0,s} \Rightarrow d(f_m(s), f_{m'}(s)) < \frac{\varepsilon}{3}$. By taking $s \in S_k$ such that $x \in U_s$ for arbitrary $x \in K$, we obtain, for $m, m' > m_0$,

$$d(f_m(x), f_{m'}(x)) \le d(f_m(x), f_m(s)) + d(f_m(s), f_{m'}(s)) + d(f_{m'}(s), f_{m'}(x)) < \varepsilon.$$

Thus, $\{f_m\}_m$ is a subsequence of $\{f_n\}_n$ that is uniformly Cauchy on K. □

converse of Arzela-Ascoli

1.2. The Stone-Weierstrass theorem.