# Real Analysis I : Measure Theory

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## 1. Radon measures

In this section, we assume every base space is locally compact Hausdorff. In locally compact Hausdorff spaces, compact finiteness and locally finiteness are equivalent.

Definition 1.1. A *Radon measure* is a Borel measure which satisfies the following three conditions:

(1) outer regular on all Borel sets,
(2) inner regular on all open sets,
(3) locally finite.

Radon measures are rather simply characterized when the base space is $\sigma$-compact.

Theorem 1.1. *A Radon measure is inner regular on all $\sigma$-finite Borel sets.*

Proof. Let $E$ be a Borel set with $\mu(E) < \infty$. By outer regularity, there is an open set $U \supset E$ such that

$$\mu(U) < \mu(E) + \frac{\varepsilon}{2}.$$

Then,

$$\mu(U \setminus E) < \frac{\varepsilon}{2}.$$

By outer regularity, there is an open set $V \supset U \setminus E$ such that

$$\mu(V) < \mu(U \setminus E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K)\frac{\varepsilon}{2} > \mu(U).$$

Then, we have $K \setminus V \subset E$ and

$$\mu(K \setminus V) = blabla$$

□

$$\begin{tikzcd}
\begin{matrix}\text{locally finite}\\ \text{Borel regular}\end{matrix} \arrow[r] & \text{Radon} \arrow[l, bend right, dashed, "X \text{ is } \sigma\text{-compact}"'] \arrow[r] & \begin{matrix}\text{locally finite}\\ \text{Borel}\end{matrix} \arrow[ll, bend left, dashed, "X \text{ is second countable}"]
\end{tikzcd}$$

$$L^1_{\mathrm{loc}} = \text{absolutely continuous measures} \subset \text{Radon measures} \subset \mathcal{D}'.$$

Corollary 1.2. *If $X$ is $\sigma$-compact, then a compact finite Borel measure is Radon if and only if it is regular.*

Theorem 1.3. *If every open set in $X$ is $\sigma$-compact, then every locally finite Borel measure is regular.*

Proposition 1.4. *In a second countable space, every open set is $\sigma$-compact.*

## 2. The Riesz-Markov-Kakutani theorem

In this section, we always assume $X$ is a locally compact Hausdorff space. Hence we can use the Urysohn lemma: If $K$ is compact and $F$ is closed, then we can find a continuous function $f : X \to [0,1]$ such that $f|_K = 1$ and $f|_F = 0$.

There are two Riesz-Markov-Kakutani theorems: the first theorem describes the positive elements in $C_c(X)^*$ as Radon measures when LF topology is assumed, and the second theorem describes $C_c(X)^*$ as the space of finite Radon measures when uniform topology is assumed.

**2.1. The first theorem.** Positivity of linear functional itself implies a rather strong continuity property.

THEOREM 2.1. *Let $C_c(X)$ be a space of compactly supported continuous functions on $X$. (Give an LF topology with a directed inductive family $C_K(X)$.) If a linear functional $I$ is positive, then continuous with respect to the topology.*

PROOF. Let $K$ be a compact subset. We want to show $|I(f)| \lesssim \|f\|$ for $f \in C_K(X)$. The proof idea comes from $|\int_K f \, d\mu| \leq \mu(K)\|f\|$.

Choose $\phi \in C_c(X)$ such that $\phi|_K = 1$. □

Jordan decomposition: $(C_0(X), u)^* \subset (C_c(X), LF)^*$ converse?

CHAPTER 4

# Hmmmm

**0.2. Convergence in measure.** Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0. \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > n^{-1},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n(x) - f(x)| > n^{-1}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.\end{aligned}$$

Since for every $k$

$$\limsup_n \{x : |f_n(x) - f(x)| > k^{-1}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\},$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.$$

THEOREM 0.2. *Let $f_n$ be a sequence of measurable functions on a measure space $(X, \mu)$. If $f_n$ converges to $f$ in measure, then $f_n$ has a subsequence that converges to $f$ $\mu$-a.e.*

PROOF. Since $d_{f_n - f}(1/k) \to 0$ as $n \to \infty$, we can extract a subsequence $f_{n_k}$ such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) > 2^{-k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) = 0.$$

Therefore, $f_{n_k}$ converges $\mu$-a.e. $\square$