Number Theory I : Classical therory

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Chapter 1

Diophantine equations

1.1 Quadratic equations

Problem 1.1. Consider a family of diophantine equations:

$$x^2 + y^2 - kxy - k = 0$$

for $k \in \mathbb{Z}$.

(1) Find the smallest three solutions such that $x > y \geq 0$ when $k = 4$.

(2) Show that if (x, y) is a solution, then $(y, ky - x)$ is also a solution.

(3) Show that if it has a solution, then at least one solution satisfies $x > 0 \geq y$.

(4) Show that the equation does not have a solution in the region $xy < 0$.

(5) Show that the equation has a solution if and only if k is a perfect square.

(6) Let a and b be integers. Deduce that if $ab + 1$ divides $a^2 + b^2$, then

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

Solution.

(1) Try for $y = 0, 1, \cdots, 8$. Then we get $(2, 0)$, $(8, 2)$, and $(30, 8)$.

(2) By substitution, we have

$$y^2 + (ky - x)^2 - ky(ky - x) - k = y^2 - 2kxy + x^2 + kxy - k = 0.$$

(3) Suppose not. By symmetry, we may assume we have a solution with $x > y > 0$. Take the solution such that $x + y$ is minimal. Note that we have

$$0 \leq x^2 + y^2 = k(xy + 1) \quad \Longrightarrow \quad k \geq 0,$$

and

$$2x^2 > x^2 + y^2 = kxy + k \geq kxy \quad \Longrightarrow \quad 2x > ky.$$

As we have seen, $(y, ky - x)$ is a solution, and $ky - x > 0$ by the assumption. Since $x + y > y + (ky - x)$, we obtain a contradiction for the minimality.

(4) Suppose $x, y \in \mathbb{Z}$ satisfy $xy < 0$. Since $xy \leq -1$,

$$x^2 + y^2 - kxy - k \geq x^2 + y^2 + k - k > 0.$$

(5)

□

Note. In general, the transformation $(x, y) \mapsto (y, ky - x)$ preserving the image of hyperbola is not easy to find. A strategy to find it in this problem is called the *Vieta jumping* or *root flipping*. It gets the name by the following reason: If (a, b) is a solution with $a > b$, then a quadratic equation

$$x^2 - kbx + b^2 - k = 0$$

has a root a, and the other root is $kb - a$ so that $(b, kb - a)$ is also a solution. The last problem is from the International Mathematical Olympiad 1988, and the Vieta jumping technique was firstly used to solve it.

Problem 1.2. Consider a diophantine equation:

$$y^2 = x^3 + 7.$$

Suppose (x, y) is a solution.

(1) Show that x is even and y is odd.

(2) Show that $x^3 + 8$ is divided by a prime p such that $p \equiv 3 \pmod 4$.

(3) Show that the equation has no solutions.