Binary Quadratic Forms

IKHAN CHOI

1. Equivalence

Definition 1.1. Two forms are called *equivalent* if they are in a same oribit with respect to $\mathrm{GL}_2(\mathbb{Z})$-action.

Definition 1.2. Two forms are called *properly equivalent* if they are in a same oribit with respect to $\mathrm{SL}_2(\mathbb{Z})$-action.

For representation problems, $\mathrm{GL}_2(\mathbb{Z})$-action is important. For the correspondence with the theory of qudratic fields, $\mathrm{SL}_2(\mathbb{Z})$ is rather important. From now, all equivalence relations are by $\mathrm{SL}_2(\mathbb{Z})$.

Example 1.1. Two forms (a, b, c) and $(a, -b, c)$ are equivalent but not properly equivalent.

The term "reduced" means that it is considered as the unique representative of each orbit. The thing is that the group action may differ in the context.

2. Definite forms

Proposition 2.1. *The* $\mathrm{SL}_2(\mathbb{Z})$*-action on the definite forms is not faithful, i.e. the kernel is given by a nontrivial group* $\{\pm I\}$*.*

Proposition 2.2. *The* $\mathrm{PSL}_2(\mathbb{Z})$*-action on the definite forms is faithful.*

The faithfulness is not important though, so we choose $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ as the modular group instead of $\mathrm{PSL}_2(\mathbb{Z})$.

2.1. Positive definite forms.

Proposition 2.3. *The set of positive definite forms admits the* $\mathrm{SL}_2(\mathbb{Z})$*-action.*

Proposition 2.4. *The* $\mathrm{SL}_2(\mathbb{Z})$*-actions on positive definite forms and negative definite forms are isomorphic.*

3. Indefinite forms

4. Class group

Last Update: July 25, 2019.