VLASOV-POISSON SYSTEM

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Contents

1. Vlasov-Poisson system

Consider the following Cauchy problem for the *Valsov-Poisson system*:

$$\begin{cases} f_t + v\cdot\nabla_x f + \gamma E\cdot\nabla_v f = 0, \qquad (t,x,v)\in\mathbb{R}^+_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v, \\ E(t,x) = -\nabla_x\Phi, \\ \Phi(t,x) = (-\Delta_x)^{-1}\rho, \\ \rho(t,x) = \int f\,dv, \\ f(0,x,v) = f_0(x,v)\ge 0, \end{cases} \tag{1}$$

where $\gamma=\pm1$. For example, we have *repulsive problem* $\gamma=+1$ for electrons in plasma theory and *attractive problems* $\gamma=-1$ for galactic dynamics. (ρ denotes mass density.)

This report mainly investigates the local and global existence problem of the Cauchy problem for the Vlasov-Poisson system. Results in 1.1 and 1.2 provide basic ingredients that will be used in the whole article. On the other hand, results in

1.3 cannot be used in any local existence proof because they assume the existence of solutions, but they help understand the fundamental nature of solutions of the Vlasov-Poisson system and are used in the proof of global existence.

Notation. We use the asymptotic notation

$$g(t) \lesssim h(t) \quad \Longleftrightarrow \quad \exists\, c = c(f_0), \quad g(t) \le c\, h(t)$$

and

$$g(t) \simeq h(t) \quad \Longleftrightarrow \quad \exists\, c, \quad g(t) = c\, h(t).$$

This report does not contain any of Sobolev norms. We omit marginalized variables and the L character for subscript. For example,

$$\|f(t)\|_p = (\iint |f(t,x,v)|^p \, dv \, dx)^{1/p}, \quad \|\rho(t)\|_p = (\int |\rho(r,x)|^p \, dx)^{1/p}.$$

1.1. **Poisson equation.** For the three-dimensional boundaryless problem of the Poisson equation

$$-\Delta\Phi(x) = \rho(x)$$

in which the solution Φ vanishes at infinity, we have

$$\Phi = \tfrac{1}{4\pi|x|} * \rho,$$

so the electric field in the Vlasov-Poisson system is given by

$$E = -\nabla_x \Phi = -\nabla_x(\tfrac{1}{4\pi|x|} * \rho) = \frac{x}{4\pi|x|^3} * \rho.$$

It can be rewritten as

$$E(t,x) = \frac{1}{4\pi} \int \frac{(x-y)\rho(t,y)}{|x-y|^3} \, dy.$$

The nonlinearity of the system is originated from the force field E, so its estimates play the most important role in investigation of the nonlinear system. Since it is given by the solution of the Poisson equation, estimates of the Riesz potential is directly connected to estimates of the force field.

Lemma 1.1 (Estimates of Riesz potential)**.** *Let $\rho \in C_c^1(\mathbb{R}^d)$.*

i. *(Field estimate)*

$$\|\tfrac{1}{|x|^{d-1}} * \rho\|_\infty \lesssim \|\rho\|_\infty^{1-1/d} \|\rho\|_1^{1/d}$$

ii. *(Field derivative estimate) For* $\log^+(x) := \max\{0, \log x\}$,

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty + \|\rho\|_1.$$

Proof.

i. Let $0 \le \frac{1}{p} < \frac{\alpha}{d} < \frac{1}{q} \le 1$. Since $(d-\alpha)p < d < (d-\alpha)q$,

$$\begin{aligned}
|\tfrac{1}{|x|^{d-\alpha}} * \rho| &= \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\,dy + \int_{|x-y|\ge R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\,dy \\
&\le \|\rho\|_{p'} (\int_{|y|<R} \frac{dy}{|y|^{(d-\alpha)p}})^{1/p} + \|\rho\|_{q'} (\int_{|y|\ge R} \frac{dy}{|y|^{(d-\alpha)q}})^{1/q} \\
&\simeq \|\rho\|_{p'} (\int_0^R r^{d-1-(d-\alpha)p}\,dr)^{1/p} + \|\rho\|_{q'} (\int_R^\infty r^{d-1-(d-\alpha)q}\,dr)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} + \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}.
\end{aligned}$$

By choosing R such that $\|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} = \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}$, we get

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty \lesssim \|\rho\|_{p'}^{\frac{1-\frac{\alpha}{d}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho\|_{q'}^{\frac{\frac{1}{p}-1+\frac{\alpha}{d}}{\frac{1}{p}-\frac{1}{q}}},$$

so the inequality

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty^{\frac{1}{q}-\frac{1}{p}} \lesssim \|\rho\|_p^{\frac{1}{q}-\frac{\alpha}{d}} \|\rho\|_q^{\frac{\alpha}{d}-\frac{1}{p}}$$

is obtained by interchaning p and q with their conjugates. The desired result gets $p = \infty$, $\alpha = 1$, and $q = 1$.

ii. Let $0 < R_a \le R_b$ be constants which will be determined later. Divide the region radially

$$|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)| \lesssim \nabla \int_{|x-y|<R_a} + \nabla \int_{R_a \le |x-y| < R_b} + \nabla \int_{R_b \le |x-y|}.$$

For the first integral,

$$\begin{aligned}
\int_{|y|<R_a} \frac{\nabla\rho(x-y)}{|y|^{d-1}}\,dy &\le \|\nabla\rho\|_\infty \int_{|y|<R_a} \frac{1}{|y|^{d-1}}\,dy \\
&\simeq \|\nabla\rho\|_\infty \int_0^{R_a} 1\,dr = R_a \|\nabla\rho\|_\infty.
\end{aligned}$$

For the second integral,

$$\begin{aligned}
\int_{R_a \le |x-y| < R_b} \frac{\rho(y)}{|x-y|^d}\,dy &\le \|\rho\|_\infty \int_{R_a \le |x-y| < R_b} \frac{1}{|x-y|^d}\,dy \\
&\simeq \|\rho\|_\infty \int_{R_a}^{R_b} \frac{1}{r}\,dr = (\log \tfrac{R_b}{R_a}) \|\rho\|_\infty.
\end{aligned}$$

For the third integral,

$$\int_{R_b \le |x-y|} \frac{\rho(y)}{|x-y|^d}\,dy \le R_b^{-d} \|\rho\|_1.$$

Thus,

$$|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)| \lesssim R_a \|\nabla\rho\|_\infty + (\log \tfrac{R_b}{R_a}) \|\rho\|_\infty + R_b^{-d} \|\rho\|_1.$$

Assuming ρ is nonzero so that $\|\nabla\rho\|_\infty > 0$, let $R_a = \min\{1, \|\nabla\rho\|_\infty^{-1}\}$ and $R_b = 1$. Since

$$\log\tfrac{1}{R_a} \leq \log^+\|\nabla\rho\|_\infty \quad \text{and} \quad R_a \lesssim \|\nabla\rho\|_\infty,$$

we have

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+\|\nabla\rho\|_\infty + \|\rho\|_1. \qquad \square$$

1.2. **Characteristics and volume preservation.** The Vlasov-Poisson equation is quite hyperbolic. What we mean here is that the method of characteristic curves is useful, and it does not regularizes the solution directly. Although we do not have an explicit representation formula, solutions given by characteristics make appropriate estimates possible.

Moreover, since the Vlasov-Poisson system is a Hamiltonian system on the phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ with the Hamiltonian $H(x,v) = \frac{1}{2}v^2 + \gamma\Phi(x,v)$, it has the volume preserving propoerty. We, however, will show the volume preservation by computation of the Jacobian determinant for transformations through characteristic flows.

Lemma 1.2. *Let $f \in C^1([0,T], C^1_c(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system.*

i. *Fix t, x, v. The ordinary differential equation*

$$\dot{X}(s;t,x,v) = V(s;t,x,v), \quad \dot{V}(s;t,x,v) = \gamma E(t, X(s;t,x,v)),$$
$$X(t;t,x,v) = x, \qquad V(t;t,x,v) = v$$

with time variable s has a solution (X,V) in $C^1([0,T],\mathbb{R}^6)$.

ii. *Fix t, x, v. Then, $f(s, X(s;t,x,v), V(s;t,x,v)) = \mathrm{const}.$*

iii. *Fix t and let*

$$y(s,x,v) := X(s;t,x,v) \quad \textit{and} \quad w(s,x,v) := V(s;t,x,v).$$

Then, the Jacobian of coordinates transform $(x,v) \mapsto (y,w)$ is 1 for all s.

Proof.

i. Note that we have

$$\rho \in C^1([0,T]; C^1_c(\mathbb{R}^6)), \quad \Phi \in C^1([0,T]; C^{2,\alpha}(\mathbb{R}^6))$$

so that

$$E \in C^1([0,T]; C^{1,\alpha}(\mathbb{R}^6))$$

by the Hölder regularity of the Poisson equation. Since a map

$$(x,v) \mapsto (v, \gamma E(t,x))$$

is globally Lipschitz with respect to (x,v) for each t, we can apply the Picard Lindelöf theorem.

ii. Differentiate and use the chain rule to get

$$\begin{aligned}
&\frac{d}{ds} f(s,y,w) \\
&\quad = \partial_t f(s,y,w) + \dot{X}(s;s,y,w)\cdot\nabla_x f(s,y,w) + \dot{V}(s;s,y,w)\cdot\nabla_v f(s,y,w) \\
&\quad = \partial_t f(s,y,w) + w\cdot\nabla_x f(s,y,w) + \gamma E(s,y)\cdot\nabla_v f(s,y,w) = 0,
\end{aligned}$$

where we denote $y = X(s;t,x,v)$ and $w = V(s;t,x,v)$.

iii. Let $J(s) = \frac{\partial(y,w)}{\partial(x,v)}$ be the Jacobi matrix. Because when $s = t$ the Jacobian is

$$\det J(t) = \det \frac{\partial(x,v)}{\partial(x,v)} = 1,$$

We want to show

$$\det J(s) = \text{const}\,.$$

Since

$$J^{-1}(s)\frac{d}{ds}J(s) = \frac{\partial(x,v)}{\partial(y,x)}\frac{d}{ds}\frac{\partial(y,w)}{\partial(x,v)} = \frac{\partial(\dot{y},\dot{w})}{\partial(y,w)} = \begin{pmatrix} 0 & 1 \\ \gamma\frac{\partial E}{\partial y}(s,y) & 0 \end{pmatrix},$$

the Jacobi formula deduces that

$$\frac{d}{ds}\det J(s) = \det(s)\,\mathrm{tr}\left(J^{-1}(s)\frac{d}{ds}J(s)\right) = 0. \qquad \square$$

Corollary 1.3. *Let $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ be a solution of the Cauchy problem for the Vlasov-Poisson system. Then, for any measurable function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\iint \beta \circ f_0(x,v)\,dv\,dx < \infty$, we have*

$$\iint \beta \circ f(t,x,v)\,dv\,dx = \text{const}\,.$$

In particular,

$$\|f(t)\|_p = \text{const}$$

for $1 \le p \le \infty$.

Proof. Fix $t, s[0,T]$ and denote $y = X(s;t,x,v)$ and $w = V(s;t,x,v)$. Then,

$$\begin{aligned}\iint \beta \circ f(t,x,v)\,dv\,dx &= \iint \beta \circ f(s, X(s;t,x,v), V(s;t,x,v))\,dv\,dx \\ &= \iint \beta \circ f(s,y,w)\,dw\,dy\end{aligned}$$

for $s \le T$. $\square$

Remark. Note that this result can be obtained in the approximation scheme, which will be suggested in the next section.

To sum up our weapons obtained in 1.1 and 1.2,

Corollary 1.4. *If a function $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ satisfies*

$$\iint f(t,x,v)\,dv\,dx = \text{const},$$

and if we let

$$\rho(t,x) = \int f(t,x,v)\,dv, \quad E(t,x) = \frac{1}{4\pi}\int \frac{(x-y)\rho(t,y)}{|x-y|^3}\,dy,$$

then

i. $\|\rho(t)\|_1 = \text{const},$

ii. $\|E(t)\|_\infty \lesssim \|\rho(t)\|_\infty^{2/3}$,
iii. $\|\nabla E(t)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty$.

These estimates will be applied not only to the global existence proof, which assumes the local existence, but also to approximate solutions.

1.3. **Conservation laws and moment propagation.** Usual algebraic computations with Stokes' theorem get several conservations laws, particularly including energy conservation.

Lemma 1.5. *Let $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system.*

i. *(Continuity equation)*

$$\rho_t + \nabla_x \cdot j = 0, \qquad \textit{where} \quad j = \int v f \, dv.$$

ii. *(Energy conservation)*

$$\iint |v|^2 f \, dv \, dx + \gamma \int |E|^2 \, dx = \text{const}\,.$$

Proof.

i. Integrate with respect to v to get

$$\begin{aligned} 0 &= \int f_t \, dv + \int v \cdot \nabla_x f \, dv \\ &= \rho_t + \nabla_x \cdot \int v f \, dv \\ &= \rho_t + \nabla_x \cdot j. \end{aligned}$$

ii. Multiply $|v|^2$ and integrate with respect to v and x to get

$$\begin{aligned} \frac{d}{dt} \iint |v|^2 f \, dv \, dx &= \iint |v|^2 f_t \, dv \, dx = -\iint |v|^2 \gamma E \cdot \nabla_v f \, dv \, dx \\ &= \iint 2v \cdot \gamma E f \, dv \, dx = -2\gamma \int \nabla_x \Phi \cdot j \, dx \\ &= 2\gamma \int \Phi \nabla_x \cdot j \, dx = 2\gamma \int \Phi \Delta_x \Phi_t \, dx \\ &= -\frac{d}{dt} \gamma \int |E|^2 \, dx. \end{aligned}$$

Thus

$$\iint |v|^2 f \, dv \, dx + \gamma \int |E|^2 \, dx = \text{const}\,. \qquad \square$$

Moments are quantities of the form

$$\iint |v|^k f(t,x,v) \, dv \, dx$$

for a positive real k. The energy conservation proves the bound of the 2-moment, which is also called kinetic energy,

$$\iint |v|^2 f(t,x,v)\,dv\,dx \lesssim 1$$

if $\gamma = +1$. In fact, a bound of kinetic energy exists even for $\gamma = -1$. As a corollary, the $L^{5/3}$ norm of mass density $\|\rho\|_{5/3}$ gets bounded.

Lemma 1.6 (Bound for kinetic energy)**.** *Let $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system. For $t \in [0,T]$,*

i. $\|\rho(t)\|_{5/3} \lesssim \iint |v|^2 f\,dv\,dx$.
ii. $\iint |v|^2 f\,dv\,dx \lesssim 1$.

Proof.

i. Note

$$\begin{aligned}\rho(t,x) = \int f(t,x,v)\,dv &\le \int_{|v|<R} f\,dv + \frac{1}{R^2}\int_{|v|\ge R} |v|^2 f\,dv \\ &\lesssim R^3 + R^{-2}\int |v|^2 f\,dv.\end{aligned}$$

Set $R^3 = R^{-2}\int |v|^2 f\,dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f\,dv.$$

ii. It is trivial for $\gamma = +1$ from the energy conservation. Suppose $\gamma = -1$. By the Hardy-Littlewood-Sobolev inequality,

$$\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$$

for $p = 2$, $d = 3$, and $\alpha = 1$ implies $q = 6/5$, hence the bound of $\|E(t)\|_2$

$$\|E(t)\|_2 \simeq \|\frac{1}{|x|^{d-\alpha}} *_x \rho(t,x)\|_{L^2_x} \lesssim \|\rho(t)\|_{6/5}.$$

So, interpolation with Hölder's inequality gives

$$\|E(t)\|_2 \lesssim \|\rho\|_1^{7/12}\|\rho\|_{5/3}^{5/12} \simeq \|\rho\|_{5/3}^{5/12}.$$

Thus (1) gives

$$\iint |v|^2 f\,dv\,dx = c + \|E(t)\|_2^2 \lesssim c + (\iint |v|^2 f\,dv\,dx)^{1/2},$$

so the kinetic energy $\iint f\,dv\,dx$ is bounded. □

If we justify a bound of higher moment

$$\iint |v|^k f(t,x,v)\,dv\,dx \lesssim 1$$

for some $k > 6$ so that we have $\|\rho(t)\|_p \lesssim 1$ for some $p = \frac{k+3}{3} > 3$, then we obtain

$$\|E(t)\|_\infty^{1-\frac{1}{p}} \lesssim \|\rho\|_p^{\frac{2}{3}} \|\rho\|_1^{\frac{1}{3}-\frac{1}{p}} \lesssim 1.$$

We will see that this estimate proves the global existence immediately; this is the idea of the paper of Lions and Perthame[]. We do not cover this in detail.

2. Local existence

The proof of local existence follows the following steps:

(1) construction of an approximate solution,
(2) establishment of a priori estimates,
(3) (subsequential) convergence of the approximate solution,
(4) verification of the solvability for the limit.

The Vlasov-Poisson system is good enough to show cdirect convergence of approximate solutions, not in the sense of subsequences.

2.1. Approximate solution.

Definition 2.1. We define an (global) *approximate solution* as a sequence of functions $f_n \in C^1(\mathbb{R}^+, C_c^1(\mathbb{R}^6))$ such that

$$\begin{cases} \partial_t f_{n+1} + v \cdot \nabla_x f_{n+1} + \gamma E_n \cdot \nabla_v f_{n+1} = 0, \\ E_n(t,x) = -\nabla_x \Phi_n, \\ \Phi_n(t,x) = (-\Delta_x)^{-1} \rho_n, \\ \rho_n(t,x) = \int f_n \, dv, \\ f_{n+1}(0,x,v) = f_0(x,v). \end{cases}$$

This definition is made in order to let the force field E constant when solving f_{n+1}.

Proposition 2.1. *An approximate solution exists.*

Proof. Let $f_0(t,x,v) = f_0(x,v)$. Notice that f_0 is clearly in $C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$. Assume $f_n \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$ satisfies the approximate system. We want to show that there is f_{n+1} that satisfies the approximate system and $f_{n+1} \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$.

We have

$$\rho_n \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6)), \quad \Phi_n \in C^1(\mathbb{R}^+; C^{2,\alpha}(\mathbb{R}^6)), \text{ and } E_n \in C^1(\mathbb{R}^+; C^{1,\alpha}(\mathbb{R}^6))$$

by the Hölder regularity of the Poisson equation. Since a map $(x,v) \mapsto (v, \gamma E_n(t,x))$ is globally Lipschitz with respect to (x,v) for each t, the classical Picard iteration uniquely solves the characteristic equation

$$\begin{cases} \dot{X}_{n+1}(s;t,x,v) = V_{n+1}(s,t,x,v) \\ \dot{V}_{n+1}(s;t,x,v) = \gamma E_n(s, X_{n+1}(s;t,x,v)) \end{cases}$$

with condition $(X_{n+1}(t;t,x,v), V_{n+1}(t;t,x,v)) = (x,v)$ and proves the uniqueness and regularity of the solution $s \mapsto (X_{n+1}, V_{n+1})(s;t,x,v) \in C^1(\mathbb{R}^+, \mathbb{R}^6)$.

Define

$$f_{n+1}(t,x,v) := f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)).$$

Then, we can show that

$$\begin{aligned} &f_{n+1}(s, X_{n+1}(s;t,x,v), V_{n+1}(s;t,x,v)) \\ &\quad = f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)) = \text{const} \end{aligned}$$

and that f_{n+1} satisfies the approximate system by the chain rule

$$\begin{aligned} 0 &= \frac{d}{ds} f_{n+1}(s, X_{n+1}(s;t,x,v), V_{n+1}(s;t,x,v))\bigg|_{s=t} \\ &= \partial_t f_{n+1}(t,x,v) + \dot{X}_{n+1}(t;t,x,v)\cdot\nabla_x f_{n+1}(t,x,v) \\ &\qquad + \dot{V}_{n+1}(t;t,x,v)\cdot\nabla_v f_{n+1}(t,x,v) \\ &= \partial_t f_{n+1}(t,x,v) + v\cdot\nabla_x f_{n+1}(t,x,v) + \gamma E_n(t,x)\cdot\nabla_v f_{n+1}(t,x,v). \end{aligned}$$

Also, f_{n+1} has compact support for each t since the characteristic does not blow up; finally we have $f_{n+1} \in C^1(\mathbb{R}^+, C^1_c(\mathbb{R}^6))$. □

Remark. Although the approximate solution is unique when given the initial term $f_0(t,x,v) = f_0(x,v)$, we do not care of the uniqueness, but only the existence.

2.2. **Local a priori estimates.** Firstly, the volume preserving property still holds for our approximate system, so we have

$$\|\rho_n(t)\|_1 = \text{const}, \quad \|f_n(t)\|_p = \text{const}\,.$$

Next, we prove local-time bounds on fields E_n. Introduce the following quantity.

Definition 2.2. Define the *velocity support* or *maximal velocity*

$$Q_n(t) = \sup\{\,|v| : f_n(s,x,v) \neq 0,\ s\in[0,t]\,\}.$$

Lemma 2.2. *Let f_n be the sequence of approximate solutions, and let $T > 0$ be a constant such that*

$$T < (Q_0\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3})^{-1}.$$

Then, we have the following bounds:

i. *For $t \le T$,*

$$\|\rho_n(t)\|_\infty + \|E_n(t)\|_\infty + Q_n(t) \lesssim 1$$

indendent on n. In addition, the support of X_n is also uniformly bounded in $t \le T$.

ii. *For $t \le T$*

$$\|\nabla_x \rho_n(t)\|_\infty + \|\nabla_x E_n(t)\|_\infty \lesssim 1$$

independent on n.

Proof.

i. Since

$$\|\rho_n(t)\|_\infty \le Q_n^3(t)\|f_0\|_\infty,$$

a rough estimate for $\|E\|_\infty$ gives

$$\|E_n(t)\|_\infty \le \|\rho_n(t)\|_\infty^{2/3}\|\rho_n(t)\|_1^{1/3} \le Q_n^2(t)\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3}.$$

Let $c(f_0)=\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3}$ be a constant such that $\|E_n(t)\|\le cQ_n^2(t)$. We claim that

$$Q_n(t)\le\frac{Q_0}{1-cQ_0t}$$

for all n. Easily checked for $n=0$; $Q_0(t)\equiv Q_0\le\frac{Q_0}{1-cQ_0t}$.

Assume $Q_n(t)\le\frac{Q_0}{1-cQ_0t}$. Then,

$$\begin{aligned}|V_{n+1}(t;0,x,v)|&\le|v|+\int_0^t|E_n(s;0,x,v)|\,ds\\&\le Q_0+c\int_0^tQ_n^2(s)\,ds\end{aligned}$$

implies

$$\begin{aligned}Q_{n+1}(t)&\le Q_0+c\int_0^tQ_n^2(s)\,ds\\&\le Q_0+c\int_0^t\left(\frac{Q_0}{1-cQ_0s}\right)^2ds=\frac{Q_0}{1-cQ_0t}.\end{aligned}$$

By induction, $Q_n(t)\le\frac{Q_0}{1-cQ_0t}\lesssim 1$ for all n and $t\in[0,T]$, where $T<(cQ_0)^{-1}$. Furthermore,

$$\|\rho_n(t)\|_\infty\lesssim Q_n^3(t)\lesssim 1,\quad \|E_n(t)\|_\infty\lesssim Q_n^2(t)\lesssim 1.$$

For the position support, we can bound it because

$$|X_n(t;0,x,v)|\le|x|+\int_0^t|V_n(s;0,x,v)|\,ds\le|x|+TQ_n(t)\lesssim 1.$$

ii. Two inequalities

$$\begin{aligned}|\nabla_xX_{n+1}(s;t,x,v)|&=\Big|\underbrace{(1,\cdots,1)}_{9}-\int_s^t\nabla_xV_{n+1}(s';t,x,v)\,ds'\Big|\\&\le 3+\int_s^t|\nabla_xV_{n+1}(s';t,x,v)|\,ds'\end{aligned}$$

and

$$\begin{aligned}|\nabla_xV_{n+1}(s;t,x,v)|&=|\int_s^t\nabla_xE_n(s',X_{n+1}(s';t,x,v))\,ds'|\\&\le\int_s^t|\nabla_xX_{n+1}(s';t,x,v)|\cdot\|\nabla_xE_n(s')\|_\infty\,ds'\end{aligned}$$

are combined as

$$\begin{aligned}&|\nabla_xX_{n+1}(s;t,x,v)|+|\nabla_xV_{n+1}(s;t,x,v)|\\\le{}&3+\int_s^t(1+\|\nabla_xE_n(s')\|_\infty)(|\nabla_xX_{n+1}(s';t,x,v)|+|\nabla_xV_{n+1}(s';t,x,v)|)\,ds'.\end{aligned}$$

By the Gronwall inequality, we get

$$|\nabla_x X_{n+1}(s;t,x,v)| + |\nabla_x V_{n+1}(s;t,x,v)| \le e^{\int_s^t (1+\|\nabla_x E_n(s')\|_\infty)\, ds'}$$

for $0 \le s \le t$.

Note that

$$\begin{aligned}|\nabla_x \rho_{n+1}(t,x)| &= |\int \nabla_x f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v))\, dv| \\ &\le \|\nabla_{x,v} f_0\|_\infty \int (|\nabla_x X_{n+1}(0;t,x,v)| + |\nabla_x V_{n+1}(0;t,x,v)|)\, dv \\ &\le \|\nabla_{x,v} f_0\|_\infty Q_{n+1}^3(t) \cdot e^{\int_0^t (1+\|\nabla_x E_n(s)\|_\infty)\, ds}.\end{aligned}$$

Recall that

$$\|\nabla_x E_{n+1}(t)\| \lesssim (1 + \|\rho_{n+1}(t)\|_\infty \log^+ \|\nabla_x \rho_{n+1}(t)\|_\infty + \|\rho_{n+1}(t)\|_1).$$

By inserting the estimate for $|\nabla_x \rho_{n+1}(t,x)|$, we can find a constant $c = c(f_0)$ such that

$$1 + \|\nabla_x E_{n+1}(t)\|_\infty \le c(1 + \int_0^t (1 + \|\nabla_x E_n(s)\|_\infty)\, ds)$$

in $t \le T$, where $T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}$. Without loss of generality, we may assume that c satisfies

$$c \ge \sup_{s\in[0,T]} (1 + \|E_0(s)\|_\infty).$$

Then, induction obtains the bound

$$1 + \|E_n(t)\|_\infty \le ce^{ct} \le ce^{cT} \lesssim 1$$

for all n and $t \le T$. The derivative of mass density is bounded since

$$\|\nabla_x \rho_{n+1}(t)\|_\infty \lesssim e^{\int_0^t (1+\|\nabla_x E_n(s)\|_\infty)\, ds}.$$

□

2.3. **Convergence of approximate solution.** Although most of the nonlinear systems fail to have convergent approximate solutions so that compactness methods are often applied, the constructed and investigated approximate solutions in the previous subsections uniformly converges.

Lemma 2.3. *Let f_n be the sequence of approximate solutions, and let $T > 0$ be a constant such that*

$$T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}.$$

i. *for $t \le T$ and $n \ge 1$,*

$$\|f_{n+1}(t) - f_n(t)\|_\infty \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\, ds.$$

ii. *for $t \le T$ and $n \ge 1$,*

$$\|E_n(s) - E_{n-1}(s)\|_\infty \lesssim \|f_n(s) - f_{n-1}(s)\|_\infty.$$

iii. *f_n converges to a function f uniformly in $C([0,T] \times \mathbb{R}^6)$.*

iv. *(X_n, V_n) converges uniformly in $C([0,T]\times\mathbb{R}^6)$, and its limit (X,V) satisfies the characteristic equation*

$$\dot X = V, \quad \dot V = \gamma E,$$

where

$$E(t,x) = \frac{1}{4\pi}\iint \frac{(x-y)f(t,x,v)}{|x-y|^3}\,dv\,dx.$$

Proof.

i. Denote

$$g(s) := |X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| + |V_{n+1}(s;t,x,v) - V_n(s;t,x,v)|.$$

The C^1 regularity of f_0 gives

$$\begin{aligned}
&|f_{n+1}(t,x,v) - f_n(t,x,v)| \\
&\quad = |f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)) - f_0(X_n(0;t,x,v), V_n(0;t,x,v))| \\
&\quad \lesssim |X_{n+1}(0;t,x,v) - X_n(0;t,x,v)| + |V_{n+1}(0;t,x,v) - V_n(0;t,x,v)| \\
&\quad = g(0).
\end{aligned}$$

If an inequality

$$\sup_{s\in[0,t]} g(s) \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds$$

is obtained for $t \le T$, then we are done.

Let $0 \le s \le t \le T$. Because

$$\begin{aligned}
X_n(s;t,x,v) &= x - \int_s^t V_n(s';t,x,v)\,ds', \\
V_n(s;t,x,v) &= v - \int_s^t E_{n-1}(s', X_n(s;t,x,v))\,ds',
\end{aligned}$$

we have two inequalities

$$\begin{aligned}
&|V_{n+1}(s;t,x,v) - V_n(s;t,x,v)| \\
&\quad \le \int_s^t |E_n(s', X_{n+1}(s';t,x,v)) - E_{n-1}(s', X_n(s';t,x,v))|\,ds' \\
&\quad \le \int_s^t (|E_n(s', X_{n+1}) - E_n(s', X_n)| + |E_n(s', X_n) - E_{n-1}(s', X_n)|)\,ds' \\
&\quad \le \int_s^t (\|\nabla_x E_n(s')\|_\infty |X_{n+1}(s') - X_n(s')| + \|E_n(s') - E_{n-1}(s')\|_\infty)\,ds'
\end{aligned}$$

and

$$|X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| \le \int_s^t |V_{n+1}(s';t,x,v) - V_n(s';t,x,v)|\,ds'$$

for $s \in [0,t]$. By combining the two inequalities above, we get

$$g(s) \le \int_s^t a(s')g(s')\,ds' + \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty\,ds', \tag{2}$$

where $a(s) := 1 + \|\nabla_x E_n(s)\|_\infty$.

Here we use a Gronwall-type inequality. In more detail, multiplying

$$a(s)e^{-\int_s^t a(s')ds'}$$

on the both-hand-side of (2), and using $a \lesssim 1$ in $t \le T$, we have

$$\begin{aligned}
-\frac{d}{ds}&\left(e^{-\int_s^t a(s')\,ds'}\int_s^t a(s')g(s')\,ds'\right)\\
&\le a(s)e^{-\int_s^t a(s')ds'}\int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty\,ds'\\
&\lesssim \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty\,ds'
\end{aligned}$$

Integrate from s to t and bound $(t-s) \le T \lesssim 1$ to get

$$e^{-\int_s^t a(s')\,ds'}\int_s^t a(s')g(s')\,ds' \lesssim \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty\,ds'. \tag{3}$$

Since $e^{\int_s^t a(s')\,ds'} \le e^{T\sup_{s\in[0,t]}a(s)} \lesssim 1$, the inequalities (2) and (3) implies

$$g(s) \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty\,ds. \tag{4}$$

ii. Notice that

$$\|E_n(t) - E_{n-1}(t)\|_\infty \lesssim \|\rho_n(t) - \rho_{n-1}(t)\|_1^{1/3}\|\rho_n(t) - \rho_{n-1}(t)\|_\infty^{2/3}.$$

For L^∞-norm,

$$\begin{aligned}
\|\rho_n(t) - \rho_{n-1}(t)\|_\infty &\le \max\{Q_n^3(t), Q_{n-1}^3(t)\}\|f_n(t) - f_{n-1}(t)\|_\infty\\
&\lesssim \|f_n(t) - f_{n-1}(t)\|_\infty.
\end{aligned}$$

For L^1-norm, since the support of f_n, f_{n-1} is bounded in both directions x, v in finite time,

$$\|\rho_n(t) - \rho_{n-1}(t)\|_1 \le \|f_n(t) - f_{n-1}(t)\|_1 \lesssim \|f_n(t) - f_{n-1}(t)\|_\infty$$

for $t \le T$, where $T < \infty$ arbitrary.

iii. From (i) and (ii), there is a constant $c = c(f_0)$ such that for $t < T$,

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le c\int_0^t \|f_n(s) - f_{n-1}(s)\|_\infty\,ds.$$

We can easily get with induction

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le M\frac{(ct)^n}{n!},$$

where $M = \sup_{s\in[0,T]} \|f_1(s) - f_0(s)\|_\infty$. Therefore,

$$\sum_{n=0}^{\infty} \|f_{n+1}(t) - f_n(t)\|_\infty \leq Me^{ct} \leq Me^{cT} < \infty$$

implies f_n uniformly converges.

iv. The convergence of characteristics is clear by the inequality (4) and the convergence of f_n. □

Proposition 2.4 (Local existence)**.** *Let f_n be the sequence of approximate solutions. Then, there is a constant $T = T(f_0)$ be a constant such that the limit f of f_n is in $C^1([0,T], C_c^1(\mathbb{R}^6))$, and solves the Cauchy problem for the Vlasov-Poisson system.*

Proof. Take T such that $T < (Q_0\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3})^{-1}$. Let $X(s;t,x,v)$ and $V(s;t,x,v)$ be the limits of X_n and V_n. Notice that

$$\begin{aligned} f(t,x,v) = \lim_{n\to\infty} f_n(t,x,v) &= \lim_{n\to\infty} f_0(X_n(0;t,x,v), V_n(0;t,x,v)) \\ &= f_0(X(0;t,x,v), V(0;t,x,v)). \end{aligned}$$

We can check it solves the system by expand the right-hand-side of

$$0 = \frac{d}{ds} f(s, X(s;t,x,v), V(s;t,x,v))|_{s=t}$$

using the chain rule. □

2.4. **Uniqueness.**

2.5. **Prolongation criterion.**

Proposition 2.5. *If $Q(t)$ does not blow up, then the solution f of the Vlasov-Poisson system is continued globally to the entire $\mathbb{R}^+$.*

Proof. Suppose $f \in C^1([0,T_{max}), C_c^1(\mathbb{R}^6))$ for $T_{max} < \infty$ is the maximal solution. Since Q does not blow up, we may define

$$Q(T_{max}) := \lim_{t\to T_{max}+} Q(t).$$

We are going to apply the local existence result for the new system with initial condition $\widetilde{f}(0,x,v) = f(t_0,x,v)$ for some $t_0 < T_{max}$. In subsection 2.3, we have shown the length of time interval for existence T is given by the condition

$$T < (Q_0\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3})^{-1}.$$

It means that, if we arrange it for the new solution $\widetilde{f}$, the interval of existence of $\widetilde{f}$ has in fact a lower bound $\widetilde{T} > 0$ that depends only on $Q(T_{max})$ for any new initial time t_0 since Q is monotonically increasing and the volume preservation implies $\|f_0\|_\infty = \|f(t_0)\|_\infty$ and $\|f_0\|_1 = \|f(t_0)\|_1$.

By setting $t_0 = T_{max} - \frac{1}{2}\widetilde{T}$ we can show there exists a solution $f \in C^1([0, T_{\max} + \frac{1}{2}\widetilde{T}), C_c^1(\mathbb{R}^6))$, which contradicts to the maximality of $T_{\max}$. Hence $T_{max} = \infty$, and the solution f is prolonged forever. □

3. Global existence

Theorem (Schaeffer, 1991)**.** *Let $f_0 \in C_c^1(\mathbb{R}^6)$ and $f_0 \geq 0$. Then, the Cauchy problem for the Vlasov-Poisson system has a unique C_c^1 global solution.*

3.1. **Estimate on field.**

3.2. **Lower bound on relative position vectors.**

Our goal is to obtain a priori estimate like

$$\|E(t)\|_\infty \lesssim Q(t)^a \qquad \text{for some } a < 1.$$

Since the force field E measures the maximal rate of changes in velocity, the estimate can be read very roughly as

$$Q'(t) \lesssim Q(t)^a,$$

which leads its polynomial growth. So we need to bound E.

Fix a time of existence t and a point $(t, \widehat{x}, \widehat{v})$ and let

$$\widehat{X}(s) := X(s; t, \widehat{x}, \widehat{v}), \qquad \widehat{V}(s) := V(s; t, \widehat{x}, \widehat{v}).$$

Decompose $[t - \Delta, t] \times \mathbb{R}_x^3 \times \mathbb{R}_v^3$ as

$$\begin{aligned} U &= \left\{ (s, x, v) : \ |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq r \right\}, \\ B &= \left\{ (s, x, v) : \ |v - \widehat{V}(t)| \geq P, \quad |v| \geq P \right\} \setminus U, \\ G &= \left\{ (s, x, v) : \ |v - \widehat{V}(t)| < P \quad \text{or} \quad |v| < P \right\}. \end{aligned}$$

(We can let $U \mapsto U \cap \{|v| \geq P\}$ to make the decomposition disjoint.) Later we choose

$$P = Q^{4/11}, \quad r = R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}, \quad R = Q^{16/33}(\log^+ Q)^{1/2}.$$

Also, later we choose $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

Reading the proof, letting $y = X(s; t, x, v)$ and $w = V(s; t, x, v)$ be functions of time variable s, trace carefully the following four quantities:

$$|x - \widehat{X}(t)|, \ |y - \widehat{X}(s)|, \ |v - \widehat{V}(t)|, \ |w - \widehat{V}(s)|.$$

The following observation suggests a lower bound of relative position.

Proposition 3.1. *Fix x, v. Let $P > 0$ and $0 < \Delta < t$ be constants such that*

$$\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}.$$

If v satisfies $|v - \widehat{V}(t)| \geq P$, then there is $s_0 \in [t - \Delta, t]$ such that

$$|y - \widehat{X}(s)| \geq \frac{1}{4} |v - \widehat{V}(t)| |s - s_0|$$

for all $s \in [t - \Delta, t]$.

Proof. Since $\Delta\|E(s)\|_\infty < \frac{P}{4}$, we have

$$|v - w| < \frac{P}{4} \quad \text{and} \quad |\widehat{V}(t) - \widehat{V}(s)| < \frac{P}{4}.$$

The condition $|v - \widehat{V}(t)| \geq P$ implies

$$\frac{1}{2}|v - \widehat{V}(t)| \leq |v - \widehat{V}(t)| - \frac{P}{4} - \frac{P}{4} < |w - \widehat{V}(s)|.$$

Let $Z(s) := y - \widehat{X}(s)$ be the relative position vector. Then,

$$\begin{aligned} Z'(s) &= w - \widehat{V}(s), \\ Z''(s) &= \gamma[E(s, y, w) - E(s, \widehat{X}(s), \widehat{V}(s))]. \end{aligned}$$

Let $s_0 \in [t - \Delta, t]$ minimize $s \mapsto |Z(s)|$ and expand Z as

$$Z(s) = Z(s_0) + Z'(s_0)(s - s_0) + \frac{Z''(\sigma)}{2}(s - s_0)^2$$

for some σ between s and s_0. Then,

$$|Z(s_0) + Z'(s_0)(s - s_0)| \geq |Z'(s_0)(s - s_0)| \geq \frac{1}{2}|v - \widehat{V}(t)||s - s_0|$$

and

$$\begin{aligned} |\frac{Z''(\sigma)}{2}(s - s_0)^2| &\leq \|E(t)\|_\infty (s - s_0)^2 \leq \|E(t)\|_\infty \Delta |s - s_0| \\ &\leq \frac{P}{4}|s - s_0| \leq \frac{1}{4}|v - \widehat{V}(t)||s - s_0| \end{aligned}$$

proves

$$|y - \widehat{X}(s)| = |Z(s)| \geq \frac{1}{4}|v - \widehat{V}(t)||s - s_0|.$$

□

We introduce time averaging to use the above lower bound.

Proposition 3.2. *Fix x, v. Let $P > 0$ and $0 < \Delta < t$ be constants such that*

$$\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}.$$

If v satisfies $|v - \widehat{V}(t)| \geq P$, then

$$\int_{t-\Delta}^{t} \frac{1}{|y - \widehat{X}(s)|^2} \chi_A(s)\, ds \lesssim \frac{r^{-1}}{|v - \widehat{V}(t)|},$$

where $A = \{s : |y - \widehat{X}(s)| \geq r\}$.

Proof. Since $|y-\widehat{X}(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|$,

$$\begin{aligned}
\int_{t-\Delta}^{t} \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds &\leq 16\int_{t-\Delta}^{t} \frac{1}{|v-\widehat{V}(t)|^2|s-s_0|^2}\chi_A(s)\,ds \\
&\leq 32\int_{r}^{\infty} \frac{1}{|v-\widehat{V}(t)|^3|s-s_0|^2}\,d(|v-\widehat{V}(t)||s-s_0|) \\
&= 32\,\frac{r^{-1}}{|v-\widehat{V}(t)|}.
\end{aligned}$$

□

3.3. **Divide and conquer.**

3.3.1. *Ugly set estimate.* Therefore, if we let $r^{-1} \simeq \min\{|v|^3, |v-\widehat{V}(t)|^3\}$, then

$$\int_{t-\Delta}^{t} \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim |v|^2$$

so that we have

$$\iiint_U \frac{f(s,y,w)}{|y-\widehat{X}(s)|^2}\,dw\,dy\,ds \lesssim R^{-1}\int |v|^2 f(t,x,v)\,dv\,dx \lesssim R^{-1}$$

when

$$U \subset \{\,(s,x,v):\ |v-\widehat{V}(t)| \geq P, \quad |y-\widehat{X}(s)| \geq R\max\{|v|^{-3}, |v-\widehat{V}(t)|^{-3}\}\,\}.$$

3.3.2. *Bad set estimate.* Consider U^c. We need to control the union of two regions

$$|y-\widehat{X}(s)| < R|v|^{-3} \quad \text{and} \quad |y-\widehat{X}(s)| < R|v-\widehat{V}(t)|^{-3}.$$

Without any conditions, the integration of fundamental solution with respect to y gives

$$\int_{|y-\widehat{X}(s)|<r} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \simeq r.$$

Claim. *If $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$, then*

$$\int_{U^c} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \lesssim \max\{|w|^{-3}, |w-\widehat{V}(s)|^{-3}\}$$

for $s \in [t-\Delta, t]$.

Proof. It follows from

$$|w| \simeq |v|, \quad |w-\widehat{V}(s)| \simeq |v-\widehat{V}(t)|$$

for $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$. □

3.3.3. *Good set estimate.*

3.4. **Polynomial decay.**

Lemma 3.3. *Along the time of existence we have*

$$\|E(t)\|_{L^\infty_x} \lesssim Q(t)^{4/3}.$$

Proof. Note that we have

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

Since the velocity support of f is bounded by finite $Q(t)$,

$$\rho(t,x) = \int_{|v|<Q(t)} f(t,x,v)\,dv \lesssim Q(t)^3 \|f_0(x)\|_{L^\infty_v} \lesssim Q(t)^3,$$

so

$$\|E(t)\|_{L^\infty_x} \lesssim \|\rho(t)\|_{L^\infty_x}^{4/9} \lesssim Q(t)^{4/3}.$$

□

Acknowledgement

This report is written in Undergraduate Research Program of Postech, under the support and advice of professor Donghyun Lee.