Analysis 6 : Harmonic Analysis

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CHAPTER 1

Basic techniques

1. Interpolation

1.1. The distribution function.

Definition 1.1. Let f be a measurable function on a measure space (X, μ). The *distribution function* $\lambda_f : [0, \infty) \to [0, \infty)$ is defined as:

$$\lambda_f(\alpha) := \mu(\{x : |f(x)| > \alpha\}).$$

Do not use $\mu(\{x : |f(x)| \geq \alpha\})$. The strict inequality implies the *lower semi-continuity* of λ_f.

Theorem 1.1 (Fubini). *For $p > 0$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^p \frac{d\alpha}{\alpha}.$$

Definition 1.2.

$$\|f\|_{L^{p,q}}^q := p \int_0^\infty \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right]^q \frac{d\alpha}{\alpha}.$$

Also,

$$\|f\|_{L^{p,\infty}} := \sup_{0<\alpha<\infty} \left[\alpha \cdot \mu(|f| > \alpha)^{\frac{1}{p}}\right].$$

Theorem 1.2. *For $p \geq 1$ we have $\|f\|_{p,\infty} \leq \|f\|_p$.*

Proof. By the Chebyshev inequality,

$$\sup_{0<\alpha<\infty} [\alpha^p \cdot \mu(|f| > \alpha)] \leq \int_0^\infty p\alpha^{p-1} \cdot \mu(|f| > \alpha)\, d\alpha = \|f\|_{L^p}^p.$$

□

1.2. Real interpolation.

Theorem 1.3 (Marcinkiewicz interpolation). *Let X be a σ-finite measure space and Y be a measure space. Let*

$$1 < p_0 < p < p_1 < \infty.$$

If a sublinear operator $T \colon L^{p_0}(X) + L^{p_1}(X) \to M(Y)$ has two weak-type estimates

$$\|T\|_{L^{p_0}(X) \to L^{p_0,\infty}(Y)} < \infty \quad \textit{and} \quad \|T\|_{L^{p_1}(X) \to L^{p_1,\infty}(Y)} < \infty,$$

then it has a strong-type estimate

$$\|T\|_{L^p(X) \to L^p(X)} < \infty.$$

Proof. Let $f \in L^p(X)$ and denote $f_h = \chi_{|f|>\alpha} f$ and $f_l = \chi_{|f|\le\alpha} f$. It is easy to show $f_h \in L^{p_0}$ and $f_l \in L^{p_1}$. Then,

$$\begin{aligned}
\|Tf\|_{L^p(Y)}^p &\sim \int \alpha^p \cdot \mu(|Tf| > \alpha)\,\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^p \cdot \mu(|T(f\cdot \mathbf{1}_{|f|>\alpha})| > \alpha)\,\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \mu(|Tf_l| > \alpha)\,\frac{d\alpha}{\alpha} \\
&\le \int \alpha^p \cdot \frac{1}{\alpha^{p_0}} \|Tf_h\|_{L^{p_0,\infty}}^{p_0}\,\frac{d\alpha}{\alpha} + \int \alpha^p \cdot \frac{1}{\alpha^{q_1}} \|Tf_l\|_{L^{p_1,\infty}}^{p_1}\,\frac{d\alpha}{\alpha} \\
&\lesssim \int \alpha^{p-p_0} \|f_h\|_{p_0}^{p_0}\,\frac{d\alpha}{\alpha} + \int \alpha^{p-p_1} \|f_l\|_{p_1}^{p_1}\,\frac{d\alpha}{\alpha} \\
&\sim \|f\|_p^p.
\end{aligned}$$

by (1) Fubini, (2) Sublinearlity, (3) Chebyshev, (4) Boundedness, (5) Fubini. □

Theorem 1.4 (Hadamard's three line lemma). *Let f be a bounded holomorphic function on the vertical unit stripe $\{z : 0 < \operatorname{Re} z < 1\}$. Then, for $0 < \theta < 1$,*

$$\|f\|_{L^\infty(\mathrm{Re}=\theta)} \le \|f\|_{L^\infty(\mathrm{Re}=0)}^{1-\theta} \|f\|_{L^\infty(\mathrm{Re}=1)}^{\theta}.$$

Proof. Define

$$g(z) := \frac{f(z)}{\|f\|_{L^\infty(\mathrm{Re}=0)}^{1-z} \|f\|_{L^\infty(\mathrm{Re}=1)}^{z}}, \qquad g_n(z) = g(z) e^{\frac{z^2-1}{n}}.$$

Then we have

(1) $g_n \to g$ pointwisely as $n \to \infty$,
(2) $g_n(z) \to 0$ uniformly as $\operatorname{Im} z \to \infty$.

The second one is because g is bounded and for $z = x + yi$ we have

$$|g_n(z)| \lesssim |e^{\frac{z^2-1}{n}}| = e^{\mathrm{Re}\,\frac{z^2-1}{n}} = e^{\frac{x^2-y^2-1}{n}} \le e^{\frac{-y^2}{n}}.$$

By (1), it is enough to bound g_n for each n. Truncating the stripe, the outer region is controlled by (2) and the interior region is controlled by the maximum modulus principle. □

1.3. Complex interpolation.

Theorem 1.5 (Riesz-Thorin interpolation). *Let X, Y be σ-finite measure spaces. Let*

$$\frac{1}{p_\theta} = (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1}, \qquad \frac{1}{q_\theta} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

Then,

$$\|T\|_{p_\theta \to q_\theta} \le \|T\|_{p_0\to q_0}^{1-\theta} \|T\|_{p_1\to q_1}^{\theta}.$$

Proof. Note that

$$\|T\|_{p_\theta\to q_\theta} = \sup_f \frac{\|Tf\|_{q_\theta}}{\|f\|_{p_\theta}} = \sup_{f,g} \frac{|\langle Tf, g\rangle|}{\|f\|_{p_\theta}\|g\|_{q_\theta'}}.$$

Consider a holomorphic function

$$z \mapsto \langle Tf_z, g_z\rangle = \int \overline{g_z(y)}\, Tf_z(y)\,dy,$$

where f_z and g_z are defined as

$$f_z = |f|^{\frac{p_\theta}{p_0}(1-z)+\frac{p_\theta}{p_1}z}\frac{f}{|f|}$$

so that we have $f_\theta = f$ and

$$\|f\|_{p_\theta}^{p_\theta} = \|f_z\|_{p_x}^{p_x}$$

for $\operatorname{Re} z = x$.

Then,

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0\to q_0}\|f_z\|_{p_0}\|g_z\|_{q_0'} = \|T\|_{p_0\to q_0}\|f\|_{p_\theta}^{p_\theta/p_0}\|g\|_{q_\theta'}^{q_\theta'/q_0'}$$

for $\operatorname{Re} z = 0$, and

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_1\to q_1}\|f_z\|_{p_1}\|g_z\|_{q_1'} = \|T\|_{p_1\to q_1}\|f\|_{p_\theta}^{p_\theta/p_1}\|g\|_{q_\theta'}^{q_\theta'/q_1'}$$

for $\operatorname{Re} z = 1$. By Hadamard's three line lemma, we have

$$|\langle Tf_z, g_z\rangle| \le \|T\|_{p_0\to q_0}^{1-\theta}\|T\|_{p_1\to q_1}^{\theta}\|f\|_{p_\theta}\|g\|_{q_\theta'}$$

for $\operatorname{Re} z = \theta$. Putting $z = \theta$ in the last inequality, we get the desired result. $\square$

2. Maximal function

We often want to show a net of linear operators $\{T_t\}_t$ is an "approximate identity" in the sense of pointwise convergence, not a certain norm; in other words, say, we want to show

$$\lim_{t\to 0} T_t f(x) = f(x).e.$$

Suppose $T = \lim_t T_t$ is defined on L^1 and let $I : L^1 \hookrightarrow X$ be a canonical embedding. Assume that we have proved $T - I$ is continuous operator $L^1 \to X$ and $\ker(T - I)$ is dense in L^1. Then, $T - I$ must vanish at entire space L^1. It implies Tf and f are equal almost everywhere.

We introduce maximal function Mf defined by

$$Mf(x) = \sup_t |T_t f(x)|.$$

If it satisfies a boundedness, for example, if it satisfies something we call the weak-type estimate $\|Mf\|_{1,\infty} \lesssim \|f\|_1$, then

$$\|(T - I)f\|_{1,\infty} \le \|Tf\|_{1,\infty} + \|f\|_{1,\infty} \le \|Mf\|_{1,\infty} + \|f\|_1 \lesssim \|f\|_1$$

implies the continuity of $T - I$. If $\ker(T - I)$ contains test function space above this, then we get the desired result. This density argument can also be explained using approximation by g such that $Tg = g$:

$$\begin{aligned}
\|Tf - f\|_{1,\infty} &\le \|T(f - g)\|_{1,\infty} + \|Tg - g\|_{1,\infty} + \|g - f\|_{1,\infty} \\
&\le \|M(f - g)\|_{1,\infty} + \|g - f\|_1 \\
&\lesssim \|f - g\|_1 \to 0.
\end{aligned}$$

2.1. The Hardy-Littlewood maximal function. Hardy-Littlewood maximal function is the most famous maximal function.

Theorem 2.1 (Hardy-Littlewoord).

$$\|Mf\|_{1,\infty} \le 3^d \|f\|_1.$$

Proof. By the inner regularity of μ, there is a compact subset K of $\{|Mf| > \alpha\}$ such that

$$\mu(K) > \mu(\{|Mf| > \alpha\}) - \varepsilon.$$

For every $x \in K$, since $|Mf(x)| > \alpha$, we can choose an open ball B_x such that

$$\frac{1}{\mu(B_x)} \int_{B_x} |f| > \alpha \quad \iff \quad \mu(B_x) < \frac{1}{\alpha} \int_{B_x} |f|.$$

With these balls, extract a finite open cover $\{B_i\}_i$ of K. Since the diameter of elements in this cover is clearly bounded, so the Vitali covering lemma can be applied to obtain a disjoint subcollection $\{B_k\}_k$ such that

$$K \subset \bigcup_i Bi \subset \bigcup_k 3B_k.$$

Therefore,

$$\begin{aligned}\mu(\{|Mf| > \alpha\}) - \varepsilon < \mu(K) &\\ &\le \sum_k 3^d \mu(B_k) \\ &\le 3^d \frac{1}{\alpha} \sum_k \int_{B_k} |f| \\ &\le 3^d \frac{\|f\|_1}{\alpha}.\end{aligned}$$

The disjointness is important in the last inequality which shows the constant does not depend on the number of B_k's. □

DEFINITION 2.1.

$$f^*(x) := \lim_{r \to 0+} \frac{1}{\mu(B)} \int_B |f(y) - f(x)|\, dy.$$

THEOREM 2.2 (Lebesgue differentiation). *$f^* = 0$ a.e.*

PROOF. Note that $f^* \le Mf + |f|$ implies

$$\|f^*\|_{1,\infty} \le \|Mf\|_{1,\infty} + \|f\|_{1,\infty} \lesssim \|f\|_1.$$

Note that $g^* = 0$ for $g \in C_c$. Approximate using $f^* = (f - g)^*$. □

3. Convergence of Fourier series

DEFINITION 3.1. The *Dirichlet kernel* is a function $D_n\colon \mathbf{T} \to \mathbb{R}$ defined by

$$D_n = \widehat{\mathbf{1}_{|k|\le n}}, \quad \text{or equivalently,} \quad \widehat{D_n} = \mathbf{1}_{|k|\le n}.$$

This is because they are invariant under inverse, in other words, they are even.

THEOREM 3.1.

$$D_n(x) = \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}.$$

PROOF.

$$\begin{aligned} D_n(x) &= \sum_{k=-n}^{n} e^{ikx} \\ &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x}. \end{aligned}$$

□

THEOREM 3.2. *If $f \in \mathrm{Lip}(\mathbf{T})$, then $D_n * f \to f$ pointwisely as $n \to \infty$.*

THEOREM 3.3.

$$\|D_n\|_{L^1(\mathbf{T})} \gtrsim \log n.$$

PROOF. By (2) $\sin x \le x$ for $x \in [0, \pi/2]$, (3) change of variable,

$$\begin{aligned} \|D_n\|_{L^1(\mathbf{T})} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin \frac{2n+1}{2}x}{\sin \frac{1}{2}x} \right| dx \\ &\ge \frac{2}{\pi} \int_0^{\pi} \frac{|\sin \frac{2n+1}{2}x|}{x}\, dx \\ &= \frac{2}{\pi} \int_0^{\frac{2n+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &= \frac{2}{\pi} \sum_{k=0}^{2n} \int_{\frac{k}{2}\pi}^{\frac{k+1}{2}\pi} \frac{|\sin x|}{x}\, dx \\ &\ge \frac{2}{\pi} \sum_{k=0}^{2n} \int_0^{\frac{1}{2}\pi} \frac{\sin x}{\frac{k+1}{2}\pi}\, dx \\ &\ge \frac{4}{\pi^2} \sum_{k=0}^{2n} \frac{1}{1+k} \\ &\ge \frac{4}{\pi^2} \log(2n+2). \end{aligned}$$

....

□

Definition 3.2. The *Fejér kernel* is

Theorem 3.4.

$$K_n(x) = \frac{1}{n+1}\frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}.$$

Proof. Since

$$\begin{aligned} D_n(x) = &= \frac{e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}}{e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}} \\ &= \frac{[e^{i\frac{2n+1}{2}x} - e^{-i\frac{2n+1}{2}x}][e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{inx} + e^{-inx}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2}, \end{aligned}$$

by telescoping, we get

$$\begin{aligned} \sum_{k=0}^{n} D_k(x) &= \frac{[e^{i(n+1)x} + e^{-i(n+1)x}] - [e^{i0x} + e^{-i0x}]}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{[e^{i\frac{n+1}{2}x} - e^{-i\frac{n+1}{2}x}]^2}{[e^{i\frac{1}{2}x} - e^{-i\frac{1}{2}x}]^2} \\ &= \frac{\sin^2 \frac{n+1}{2}x}{\sin^2 \frac{1}{2}x}. \end{aligned}$$

□

Two important results from Fejér kernel:

(1) If $f(x-)$, $f(x+)$ exist and $S_n f(x)$ converges, then $S_n f(x) \to \frac{1}{2}(f(x-)+f(x+))$.
(2) (If $f \in L^1(\mathbf{T})$, then $\sigma_n f \to f$ a.e.)
(3) If $f \in L^1(\mathbf{T})$, then $S_n f \to f$ in L^1 and L^2.
(4) If f is continuous and $\widehat{f} \in L^1(\mathbb{Z})$, then $S_n f \to f$ uniformly.
(5) Since $\sigma_n f$ is a trigonometric polynomial, the set of trigonometric polynomials are dense in $L^1(\mathbf{T})$ and $L^2(\mathbf{T})$.

CHAPTER 2

Differentiation theory