Compact sets

1. Let $X \subset \mathbb{R}^d$. Show that if X is bounded then every sequence in X has a convergent subsequence. (Bolzano-Weierstrass)

2. Let $X \subset \mathbb{R}^d$. Show that if every sequence in X has a convergent subsequence, then X is closed and bounded.

3. Let $X \subset \mathbb{R}^d$ be compact. Suppose an infinite set $\mathcal{C} \subset \mathcal{P}(X)$ only contains closed subsets of X. Show that if $\bigcap_{C \in A} C$ is nonempty for all finite subset $A \subset \mathcal{C}$, then $\bigcap_{C \in \mathcal{C}} C$ is nonempty.

Continuous functions

1. Let X be a set. Let $f_n : X \to \mathbb{R}$ be a sequence of functions. Show that f_n converges to $f : X \to \mathbb{R}$ uniformly if and only if $\lim_{n\to\infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$.

2. Let $X \subset \mathbb{R}^d$. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. Show that if f_n converges to $f : X \to \mathbb{R}$ uniformly, then f is also continuous. (In other words, the set of real-valued continuous functions $C(X)$ is always closed under the topology of uniform convergence.)

3. Let $X \subset \mathbb{R}^d$ be compact. Show that is $f : X \to \mathbb{R}$ is continuous then it is uniformly continuous.

4. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions. Show that if $f_n \to f$ pointwisely and $f_n' \to g$ uniformly, then $g = f'$.

Measures

Let X be a set and $\mathcal{F}$ be a σ-algebra on X. A *measure* on $\mathcal{F}$ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that

- $\mu(\varnothing) = 0$,
- $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for a sequence of disjoint sets $E_i \in \mathcal{F}$. (countable-additivity)

We call an element in $\mathcal{F}$ *measurable* (when we are known $\mathcal{F}$).

1. Show that if E_i is a monotonically increasing sequence of measurable subsets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$. (Continuity from below)
2. Show that if E_i is a monotonically decreasing sequence of measurable subsets, then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$ when given $\mu(E_1) < \infty$. (Continuity from above)
3. Show that there is no measure μ defined on the entire power set $\mathcal{P}(\mathbb{R})$ such that $\mu([a, b]) = b - a$ and $\mu(x + E) = \mu(E)$ for $x \in \mathbb{R}$, $E \subset \mathbb{R}$. (Hint: Define an equivalence relation on $\mathbb{R}$ such that $x \sim y$ iff $x - y \in \mathbb{Q}$. Take $N \subset [0, 1)$ such that N contains precisely one member of each equivalence class. Show $1 \leq \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(N) \leq 3$ to lead a contradiction.)

Measurable functions

Let X be a set. A σ-algebra $\mathcal{F}$ on X is also called a *measurable structure* and X with $\mathcal{F}$ is called a *measurable space*. A function $f : X \to Y$ between measurable spaces is called *measurable* if the measurability of $E \subset Y$ implies the measurability of $f^{-1}(E)$.

On $\mathbb{R}$, the smallest σ-algebra containing open sets is called *Borel σ-algebra* and its elements are called *Borel sets*. We will denote it by $\mathcal{B}(\mathbb{R})$. For a function $f : X \to \mathbb{R}$ where X is a measurable space, we call f just measurable if $f^{-1}(E)$ is measurable for all Borel sets E.

1. Let X be a measurable space. Show that if $f, g : X \to \mathbb{R}$ is measurable, then $f + g$, $|f|$, f^2, and fg are all measurable.
2. Let X be a measurable space and f_n be a sequence of bounded measurable functions. Show that $g = \sup_n f_n$ and $h = \limsup_n f_n$ are measurable.

Week 3

Simple functions

Let $(X, \mathcal{F})$ be a measurable space. A *characteristic function* or *indicator function* of a measurable set E is a function $\chi_E : X \to \mathbb{R}$ defined by

$$\chi_E(x) = \begin{cases} 1 & , x \in E \\ 0 & , x \notin E \end{cases}.$$

A finite linear combination of characteristic functions is called *simple function*, and it is a slight generalization of step functions used in the Riemann integral.

1. Show that a subset E is measurable iff its characteristic function χ_E is measurable.
2. Let $f : X \to [0, \infty]$ be a measurable function. Construct a monotonically increasing sequence of simple functions ϕ_n such that $\phi_n \to f$ pointwise. (Hint: $E_n^k = f^{-1}((k^{-1}2^{-n}, (k+1)2^{-n}])$, $F_n = f^{-1}((2^n, \infty])$.)
3. Show that $\phi_n \to f$ uniformly if f is bounded.

Almost everywhere convergence

Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. We say f_n converges to f μ-*almost everwhere* if $E = \{x : f_n(x) \text{ does not converges to } f(x)\}$ satisfies $\mu(E) = 0$.

1. Let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. Let μ be a complete measure(in other words, $\mu(E) = 0$ implies all subsets of E are measurable). Show that if $f_n \to f$ μ-a.e., then f is measurable.
2. (Optional!) Prove the Egorov's theorem: Let X be a probability space, and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. If $f_n \to f$ a.e., then for every $\varepsilon > 0$ there is a measurable subset $E \subset X$ such that $\mu(E) > 1 - \varepsilon$ and $f_n \to f$ uniformly on E.

Week 4

Integration of positive functions

Let $f : (X, \mu) \to [0, \infty]$ be a measurable function. Define

$$\int f \, d\mu := \sup \Big\{ \int \phi \, d\mu : 0 \le \phi \le f, \ \phi \text{ simple } \Big\}$$

where

$$\int (\sum_{i=1}^{n} c_i \chi_{E_i}) \, d\mu := \sum_{i=1}^{n} c_i \mu(E_i).$$

Here, the symbol X will always denote a measure space.

1. Let $f : X \to [0, \infty]$ be a measurable function. Show that if $\int f < \infty$, then $f(x) < \infty$ a.e. x.

2. Show that if $\{f_n : X \to [0, \infty]\}$ is a sequence of monotonically increasing measurable functions, and if $f = \lim_{n\to\infty} f_n$ pointwise, then $\int f = \lim_{n\to\infty} \int f_n$. (Monotone convergence theorem. Hint: take simple ϕ such that $\int f - \varepsilon < \int \phi$ and let $E_n := \{f_n \ge (1 - \varepsilon)\phi\}$. Then, $\int f_n \ge (1 - \varepsilon) \int_{E_n} \phi$ implies $\lim \int f_n \ge (1 - \varepsilon) \int \phi$ by the continuity of measure.)

3. Let $f : X \to [0, \infty]$ be a measurable function. Show that $\int f = 0$ iff $f = 0$ a.e.

4. Combine above two results to obtain the following: $\{f_n : X \to [0, \infty]\}$ is a sequence of monotonically increasing measurable functions, and if $f = \lim_{n\to\infty} f_n$ a.e., then $\int f = \lim_{n\to\infty} \int f_n$. (Monotone convergence theorem: a.e. version)

5. Show if $\{f_n : X \to [0, \infty]\}$ is a sequence of measurable functions, then

   $$\int \liminf_n f_n \le \liminf \int f_n.$$

   Give an example such that the equality does not hold. (Fatou's lemma)