The computation of Galois group

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In this section, we assume the following setting:

- F is a perfect field,
- f is an irreducible quartic over F,
- E is the splitting of f over F,
- $G = \mathrm{Gal}(E/F)$,
- $H = G \cap V_4$.

Theorem 1.1. *There are only five isomorphic types of transitive subgroups of the symmetric group S_4.*

Corollary 1.2. $G \cong S_4,\ A_4,\ D_4,\ V_4,$ *or* C_4.

Proposition 1.3. *Two groups A_4 and V_4 are only transitive normal subgroups of S_4.*

Now we define our resolvent polynomial.

Proposition 1.4. *Let K be the fixed field of H. Then,*

$$K = F(\alpha_1\alpha_2 + \alpha_3\alpha_4,\ \alpha_1\alpha_3 + \alpha_2\alpha_4,\ \alpha_1\alpha_4 + \alpha_2\alpha_3).$$

Definition. Let K be the fixed field of H. A *resolvent cubic* is a cubic R_3 that has K as the splitting field over F.

Theorem 1.5. *We have*

1. $G \cong S_4$ *if* R_3 *is irreducible and ,*
2. $G \cong A_4$ *if* R_3 *is irreducible and ,*
3. $G \cong D_4$ *if* R_3 *has only one root in* K *and* f *is irreducible over* K*,*
4. $G \cong C_4$ *if* R_3 *has only one root in* K *and* f *is reducible over* K*,*
5. $G \cong V_4$ *if* R_3 *splits in* K*.*

Proof. There are five possible cases:

$$(G, H) = (S_4, V_4),\ (A_4, V_4),\ (D_4, V_4),\ (V_4, V_4),\ (C_4, C_2).$$

We have

$$[K : F] = |G/H|, \qquad [E : K] = |H|.$$

If f is reducible over K, then $\mathrm{Gal}(E/K)$ is no more a transitive subgroup of S_4 so that $H \neq V_4$ and $G \cong C_4$. □

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\begin{tikzcd}
E \arrow[rrrr] \arrow[dd, no head] & & & & 1 \arrow[d, no head, "2"] & \\
& & & & C_2 \arrow[dl, no head, "2"'] \arrow[dr, no head, "2"] & \\
K = F_{V_4 \cap G} \arrow[d, no head] & & & V_4 \arrow[dl, no head, "3"'] \arrow[dr, no head, "2"] & & C_4 \arrow[dl, no head, "2"] \\
F \arrow[r] & ? & A_4 \arrow[dr, no head, "2"'] & & D_4 \arrow[dl, no head, "3"] & \\
& & & S_4 & &
\end{tikzcd}
$$