

## Compact sets

1. Let  $X \subset \mathbb{R}^d$ . Show that if  $X$  is bounded then every sequence in  $X$  has a convergent subsequence. (Bolzano-Weierstrass)
2. Let  $X \subset \mathbb{R}^d$ . Show that if every sequence in  $X$  has a convergent subsequence, then  $X$  is closed and bounded.
3. Let  $X \subset \mathbb{R}^d$  be compact. Suppose an infinite set  $\mathcal{C} \subset \mathcal{P}(X)$  only contains closed subsets of  $X$ . Show that if  $\bigcap_{C \in A} C$  is nonempty for all finite subset  $A \subset \mathcal{C}$ , then  $\bigcap_{C \in \mathcal{C}} C$  is nonempty.

## Continuous functions

1. Let  $X$  be a set. Let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of functions. Show that  $f_n$  converges to  $f : X \rightarrow \mathbb{R}$  uniformly if and only if  $\lim_{n \rightarrow \infty} \sup_{x \in X} |f_n(x) - f(x)| = 0$ .
2. Let  $X \subset \mathbb{R}^d$ . Let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of continuous functions. Show that if  $f_n$  converges to  $f : X \rightarrow \mathbb{R}$  uniformly, then  $f$  is also continuous. (In other words, the set of real-valued continuous functions  $C(X)$  is always closed under the topology of uniform convergence.)
3. Let  $X \subset \mathbb{R}^d$  be compact. Show that if  $f : X \rightarrow \mathbb{R}$  is continuous then it is uniformly continuous.
4. Let  $f_n : [a, b] \rightarrow \mathbb{R}$  be a sequence of continuous functions. Show that if  $f_n \rightarrow f$  pointwisely and  $f'_n \rightarrow g$  uniformly, then  $g = f'$ .

## Measures

Let  $X$  be a set and  $\mathcal{F}$  be a  $\sigma$ -algebra on  $X$ . A *measure* on  $\mathcal{F}$  is a function  $\mu : \mathcal{F} \rightarrow [0, \infty]$  such that

- $\mu(\emptyset) = 0$ ,
- $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$  for a sequence of disjoint sets  $E_i \in \mathcal{F}$ . (countable-additivity)

We call an element in  $\mathcal{F}$  *measurable* (when we are known  $\mathcal{F}$ ).

1. Show that if  $E_i$  is a monotonically increasing sequence of measurable subsets, then  $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i \rightarrow \infty} \mu(E_i)$ . (Continuity from below)
2. Show that if  $E_i$  is a monotonically decreasing sequence of measurable subsets, then  $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i \rightarrow \infty} \mu(E_i)$  when given  $\mu(E_1) < \infty$ . (Continuity from above)
3. Show that there is no measure  $\mu$  defined on the entire power set  $\mathcal{P}(\mathbb{R})$  such that  $\mu([a, b]) = b - a$  and  $\mu(x + E) = \mu(E)$  for  $x \in \mathbb{R}$ ,  $E \subset \mathbb{R}$ . (Hint: Define an equivalence relation on  $\mathbb{R}$  such that  $x \sim y$  iff  $x - y \in \mathbb{Q}$ . Take  $N \subset [0, 1)$  such that  $N$  contains precisely one member of each equivalence class. Show  $1 \leq \sum_{r \in \mathbb{Q} \cap [0, 1)} \mu(N) \leq 3$  to lead a contradiction.)

## Measurable functions

Let  $X$  be a set. A  $\sigma$ -algebra  $\mathcal{F}$  on  $X$  is also called a *measurable structure* and  $X$  with  $\mathcal{F}$  is called a *measurable space*. A function  $f : X \rightarrow Y$  between measurable spaces is called *measurable* if the measurability of  $E \subset Y$  implies the measurability of  $f^{-1}(E)$ .

On  $\mathbb{R}$ , the smallest  $\sigma$ -algebra containing open sets is called *Borel  $\sigma$ -algebra* and its elements are called *Borel sets*. We will denote it by  $\mathcal{B}(\mathbb{R})$ . For a function  $f : X \rightarrow \mathbb{R}$  where  $X$  is a measurable space, we call  $f$  just measurable if  $f^{-1}(E)$  is measurable for all Borel sets  $E$ .

1. Let  $X$  be a measurable space. Show that if  $f, g : X \rightarrow \mathbb{R}$  is measurable, then  $f + g$ ,  $|f|$ ,  $f^2$ , and  $fg$  are all measurable.
2. Let  $X$  be a measurable space and  $f_n$  be a sequence of bounded measurable functions. Show that  $g = \sup_n f_n$  and  $h = \limsup_n f_n$  are measurable.

## Simple functions

Let  $(X, \mathcal{F})$  be a measurable space. A *characteristic function* or *indicator function* of a measurable set  $E$  is a function  $\chi_E : X \rightarrow \mathbb{R}$  defined by

$$\chi_E(x) = \begin{cases} 1 & , x \in E \\ 0 & , x \notin E \end{cases}.$$

A finite sum of characteristic functions is called *simple function*, and it is a slight generalization of step functions used in the Riemann integral.

1. Show that a subset  $E$  is measurable iff its characteristic function  $\chi_E$  is measurable.
2. Let  $f : X \rightarrow [0, \infty]$  be a measurable function. Construct a monotonically increasing sequence of simple functions  $\phi_n$  such that  $\phi_n \rightarrow f$  pointwise. (Hint:  $E_n^k = f^{-1}((k^{-1}2^{-n}, (k+1)2^{-n}])$ ,  $F_n = f^{-1}((2^{-n}, \infty])$ .)
3. Show that  $\phi_n \rightarrow f$  uniformly if  $f$  is bounded.

## Almost everywhere convergence

Let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of measurable functions. We say  $f_n$  converges to  $f$   *$\mu$ -almost everywhere* if  $E = \{x : f_n(x) \text{ does not converge to } f(x)\}$  satisfies  $\mu(E) = 0$ .

1. Let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of measurable functions. Let  $\mu$  be a complete measure (in other words,  $\mu(E) = 0$  implies all subsets of  $E$  are measurable). Show that if  $f_n \rightarrow f$   $\mu$ -a.e., then  $f$  is measurable.
2. (Optional!) Prove the Egorov's theorem: Let  $X$  be a probability space, and let  $f_n : X \rightarrow \mathbb{R}$  be a sequence of measurable functions. If  $f_n \rightarrow f$  a.e., then for every  $\varepsilon > 0$  there is a measurable subset  $E \subset X$  such that  $\mu(E) > 1 - \varepsilon$  and  $f_n \rightarrow f$  uniformly on  $E$ .