CURVED FOLDING AND PLANAR CUTTING OF SIMPLE CLOSED CURVE ON A CONICAL ORIGAMI

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Abstract. The fold-and-cut theorem states that we can find a flat folding of paper, so that one complete straight cut on folding creates any desired plane graph with straight sides. In this paper, we extended this problem to *curved origami* for piecewise C^1 simple closed curves. Especially, many curves on a paper can be cut by a straight plane after we fold the paper even if the folded paper has a *conical shape*—the surface consists of half-lines with a common vertex. Let $\gamma : I \to \mathbb{R}^2$ be a piecewise C^1 simple closed curve such that there exists a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in [0, 2\pi)$ for a Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$. We proved that there exists a way to fold conically so that γ can be cut by a plane after we fold if a simple condition described in (1) is satisfied.

1. Introduction

Origami is the art of folding paper, which also often refers to the mathematical study concerned with the paper folding. The fold-and-cut theorem proved in 1999 [3] states that we can find a flat folding of paper, so that one complete straight cut on folding creates any desired plane graph with straight sides. In this paper, we suggest a kind of the fold-and-cut theorem for not just flat folding but *curved folding* imbedded in three-dimensional Euclidean space $\mathbb{R}^3$. Once you have drawn a curve on a piece of paper, for many of the cases, we can do crumple the paper and cut it with one straight plane to cut out exactly the curve drawn. To study origamis in three-dimensional space, we consider the metric preserving map of $\Omega \subset \mathbb{R}^2$ called an *origami* to model the process of folding the paper. For the given curve $\gamma : I \to \mathbb{R}^2$ and origami $u : \Omega = \mathbb{R}^2 \to \mathbb{R}^3$, the curve γ is called a *cut on origami* u if there exists a plane $S \subset \mathbb{R}^3$ such that $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$. An *origami* is defined as a Lipschitz continuous piecewise C^1 map $u : \Omega \to \mathbb{R}^3$ for connected set $\Omega \subset \mathbb{R}^2$ such that the gradient Du is a 3×2 matrix with orthonormal columns for all points of $\Omega \setminus \Sigma_u$, where $\Sigma_u \subset \Omega$ is the set of the points at which u is *not* differentiable. In addition, to make our model more physically realizable, we require the paper not to intersect itself transversally. See [2].

We consider single vertex origamis in the curved fold-and-cut problem. The image of the single vertex origami becomes a (general) cone in $\mathbb{R}^3$, which means a surface generated by a continuously moving half-line cast from a common apex point. We call the origami whose image is a cone by the *conical origami.* This paper discusses the existence of the conical origami u such that the given *piecewise C^1 simple closed curve* is a cut on u.

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We set up Cartesian coordinates (x, y) on $\mathbb{R}^2$, and cylindrical coordinates (ρ, ψ, z) on $\mathbb{R}^3$. The origin of $\mathbb{R}^2$ is the preimage of the vertex under the conical origami and the $\rho\psi$-plane of $\mathbb{R}^3$ intersects the image of the origami u exactly at $u(\operatorname{im}\gamma)$ where γ is the given simple closed curve in $\mathbb{R}^2$. In Section 2, we give a proposition concerning the piecewise C^1 simple closed curve on $\mathbb{R}^2$. The proposition describes some necessary conditions for the existence of the conical origami u, such that the origin of $\mathbb{R}^2$ is the preimage of the vertex, and the given piecewise C^1 simple closed curve is a cut on u. For example, if such an origami u exists, then the interior of the given simple closed curve γ is required to be *star-shaped* (there exists a point that is called kernel or center such that any half-line cast from the point intersects γ only once). Exactly the kernel of the star-shaped curve is the origin. Namely, the proposition guarantees the existence of the Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$ that is periodic with period 2π, such that the piecewise C^1 simple closed curve γ is parameterized by $\psi \in I = [0, 2\pi)$ as $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$. In this article, I always denotes a half-open interval $[0, 2\pi)$.

Consider the piecewise C^1 simple closed curve γ in $\mathbb{R}^2$ that is a cut on a certain conical origami. Then, we have the Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$ that is used to parameterize γ. In fact, because the curve γ is piecewise C^1, the function r is also piecewise C^1. Let u be one of the conical origamis on which γ is cut, such that $u(O)$ is the vertex of the conical origami and the vertex has cylindrical coordinate $(0, 0, z)$ for a positive real number z. For the function r, a function $A_z : [0, 2\pi] \to [0, \infty)$ is defined as a strictly increasing function whose value is equal to the amount of the angle that the point $u(\gamma(\theta))$ has traveled over $\theta \in [0, \psi]$ in the cylindrical coordinates. In other words, the function A_z is the *total variation* of the angular coordinate of the polar parameterization u, so we call $A_z(\psi)$ the *total angular variation* of u up to ψ. Actually the function A_z is independent of u, it can be defined only by the function r and a real number z.

Our main results are summarized by a part of Theorem 3.1 as follows:

Theorem. *Let γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ such that there exists a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$ for a Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$ that has period 2π. If*

$$\sup_z A_z(2\pi) \geq 2\pi, \tag{1}$$

then there exists the conical origami u such that γ is cut on u.

We prove the theorem by directly presenting the concrete illustration of the conical origami on cylindrical coordinates, that cuts the given simple closed curve.

In Section 2, we define new terminologies such as conical origami and give the proposition describing the necessary condtions and existence of the star-shaped parameterization of the given piecewise C^1 simple closed curve. In Section 3, we define the function A_z and the main theorem. The ideas of proof will be also given. In section 4, we prove the main theorem that there exists the concial origami on which the given piecewise C^1 simple closed curve is cut through Theorem 3.1, if the curve has the parameterization with the Lipschitz continuous function r and satisfies the condition (1).

2. Definition of Conical Origami and Cut

In this section, we propose the definitions of the words to present the problem, such as conical origami and cut. First, we define *conical origami* as a Lipschitz continuous piecewise C^1 map for analytical models of paper folding. For a connected set $\Omega \subset \mathbb{R}^n$, if a map $f : \Omega \to \mathbb{R}^m$ is piecewise C^1, of course, then f is almost everywhere differentiable. Here, we call *singular set* of the map f the set of points $\Sigma_f \subset \Omega$ at which f is not differentiable. The singular set Σ_f is closed in Ω and arbitrary compact set in Ω intersects the finite number of connected components of $\Omega \setminus \Sigma_f$. A conical origami is the origami whose image is a (general) cone, where the *origami* will be defined as a kind of rigid map that models the paper folding imbedded in $\mathbb{R}^3$. See [2]. Next, we suggest an extension to *curved origami* of the cutting paper as an expression, *cut on the origami.* To refer to usual geometric approach to origami, see [1, 4, 5, 6].

An *origami* is a Lipschitz continuous piecewise C^1 map $u : \mathbb{R}^2 \to \mathbb{R}^3$ such that the gradient Du has orthonormal columns and the intersecting itself is excluded. To make the model more physically realizable, we allow precise overlappings which can be approximated by injective maps, that means, it is possible to be tangent to itself but not to transverse. For example, the map $u(x, y) = (|x|, y, 0)$ is not injective but can be obtained as $k \to \infty$ of the injective maps $u_k(x, y) = (|x| \cos \frac{1}{k}, y, x \sin \frac{1}{k})$, which represent actual folding process along time (see [2]). In this paper, we only focus on the *conical origami*, that is the origami whose image is a cone.

Definition (Conical Origami)**.** Let $\Omega \subset \mathbb{R}^2$ be a connected set. A Lipschitz continuous piecewise C^1 map $u : \Omega \to \mathbb{R}^3$ is a *conical origami* if it satisfies: the gradient Du has orthonormal columns for all point of $\Omega \setminus \Sigma_u$; there exists a sequence of maps $u_k : \Omega \to \mathbb{R}^3$ that are Lipschitz continuous and injective, such that $u_k \to u$ in the uniform convergence; the image of u is a cone.

The *vertex of the conical origami* u is the vertex point of the image of u.

If the condition that the image of u is a cone is excluded, then the map u is called just an *origami.* See [2].

The fold-and-cut theorem states that we can find a flat folding of paper, so that one complete straight cut on folding creates any desired plane graph of cuts made up with straight sides. Similarly, within three-dimensional space, we ask that there is an (conical) origami map such that a certain planar straight cut on folding creates the given curve, especially *piecewise* C^1 *simple closed curve*, on the unfolded paper. If there is such an origami u, we call the curve *cut on* u. For more details on the fold-and-cut theorem, see [3].

Definition (Cut)**.** A curve $\gamma : I \to \mathbb{R}^2$ is a *cut on origami* $u : \Omega = \mathbb{R}^2 \to \mathbb{R}^3$ if there exists a plane $S \subset \mathbb{R}^3$ such that $S \cap \operatorname{im} u = u(\operatorname{im} \gamma)$.

We will approach the problem about the simple closed curve by a parameterization. Also the conical origami is parameterized by a metric preserving map whose codomain is presented in the cylindrical coordinates. The cylindrical coordinate system is set up by letting the plane S containing $u(\operatorname{im} \gamma)$ be $\rho\psi$-plane and the preimage of the vertex of u be on the z-axis.

The following proposition presents the conditions required for a piecewise C^1 simple closed curve to be cut on a conical origami, regarding the existence of the polar parameterization. Using the function r defined in the following proposition, we have the polar equation $\rho = r(\psi)$ represent the curve γ.

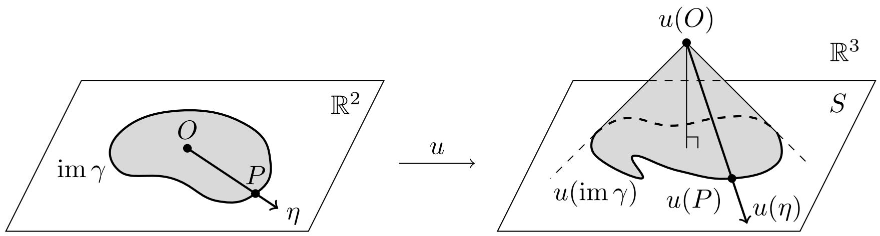

FIGURE 1. The point $u(O)$ cannot be in the plane S and the curve γ is star-shaped.

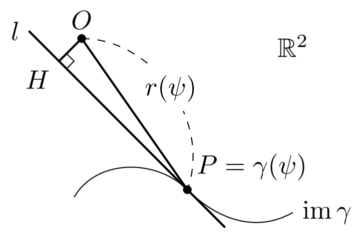

FIGURE 2. The function r is Lipschitz continuous.

Proposition 2.1. *Let γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ and u be a conical origami that has the origin of $\mathbb{R}^2$ be the preimage of its vertex. If γ is a cut on u, then there exists a Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$ which has period 2π, such that γ has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$.*

Proof. First, we prove that γ is star-shaped with kernel O, that is, any half-line cast from O intersects γ only once. Then, we can represent any point on γ by $(r(\psi)\cos\psi, r(\psi)\sin\psi)$ for a periodic piecewise C^1 function $r : \mathbb{R}^2 \to (0, \infty)$ with period 2π. After that, we prove that the function r is Lipschitz continuous if γ has the parameterization. Since the star-shaped curve implies that r is a function on I and positive valued, the proof is complete if we prove that γ is star-shaped and the function r is Lipschitz continuous. Let O be a point on $\mathbb{R}^2$ such that the point $u(O)$ is the vertex of the conical origami u. Before the proof, we show a lemma that $u(O)$ cannot be in the plane S, where the plane S satisfies $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$.

Assume that $u(O)$ is in the plane S. For a point $P \neq O$ which is on the curve γ, we have $u(P) \in S$. If we let η be a half-line on $\mathbb{R}^2$ such that $u(\eta)$ cast from $u(O)$ and passing through the point $u(P)$, then the plane S contains $u(\eta)$ because $u(O)$ is in S. The image $(\operatorname{im} u)$ also contains the half-line $u(\eta)$ because $(\operatorname{im} u)$ is a conical surface. Since $u(\eta)$ is included in both S and $(\operatorname{im} u)$, we obtain $u(\eta) \subset u(\operatorname{im}\gamma)$ from the relation $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$. So, the curve $u(\gamma)$ is unbounded and it is contradiction for γ to be a closed curve. Therefore $u(O)$ cannot be in the plane S.

(*star-shapedness*) Assume that γ is not star-shaped. If a half-line η cast from O intersects $\operatorname{im}\gamma$ more than once, the image $u(\eta)$ also intersects $u(\gamma)$ more than once. Since $S \supset u(\operatorname{im}\gamma) \supset u(\operatorname{im}\gamma) \cap u(\eta)$ and the cardinality of $u(\operatorname{im}\gamma) \cap u(\eta)$ is greater than one, S contains the half line $u(\eta)$ and also the point O. By the previous lemma, each half-line η cast from O intersects $\operatorname{im}\gamma$ at most once. If there is a half-line cast from O that does not meet $\operatorname{im}\gamma$, the point O should be in the interior of the simple closed curve γ and there exists a half-line intersects $\operatorname{im}\gamma$ more than twice. Therefore all of the half-lines cast form O intersects $\operatorname{im}\gamma$ only once, and the γ is star-shaped.

(*Lipschitz continuity*) For a point P on the curve γ, there exists a piecewise C^1 function $r : \mathbb{R} \to (0, \infty)$ such that we have $P = \gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I$ because γ is star-shaped and the kernel is O. Let l be a tangent line at a point P, and H be the foot of the perpendicular from O to l. From $\frac{r'(\psi)}{r(\psi)} = \tan\left(\frac{\pi}{2} - \angle OPH\right)$, we have the length of the line segment OH as follows:

$$OH(\psi) = r(\psi)\sin\angle OPH = \frac{r(\psi)^2}{\sqrt{r(\psi)^2 + r'(\psi)^2}}.$$

Let z be the distance between the point $u(O)$ and the plane S in $\mathbb{R}^3$. Since S contains $u(l)$, we have $0 \le z \le OH(\psi)$ for all ψ. If the derivatve r is unbounded, then the value of $OH(\psi)$ can be arbitrarily small. So we have $z = 0$, it implies that $u(O)$ is on the plane S. The function r is piecewise C^1 and the derivative of r is bounded, hence the function r is Lipschitz continuous. □

Notice that the function r has positive lower bound, since O is in the interior of γ. In Theorem 3.1, a concrete illustration of the conical origami is presented on the cylindrical coordinates for the given piecewise C^1 simple closed curve, that has the star shaped parameterization with Lipschitz continuous function r and satisfies the condition (1). In fact, the coordinate system on $\mathbb{R}^2$ and the preimage of the vertex of the conical origami is given in advance.

3. The Main Theorem and The Idea of Proof

For given piecewise C^1 simple closed curve γ that satisfies the condition (1) and Proposition 2.1, we propose a map $u : \mathbb{R}^2 \to \mathbb{R}^3$ in Theorem 3.1. Let $r : \mathbb{R} \to (0, \infty)$ be a Lipschitz continuous function with period 2π such that γ has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$. To make the definition of the map u easier, we define *total angular variation*, $A_z(\psi)$, before the definition. After we propose Theorem 3.1, our plans to prove the theorem are introduced. We prove that this map u is a conical origami and γ is cut on u in Section 4.

For a while, assume that u is a conical origami such that $u(O)$ is the vertex of the conical origami and it has cylindrical coordinate $(0, 0, z)$ for a positive real number z. Suppose that the given curve γ is a cut on u. We suggest a function $A_z : [0, 2\pi] \to [0, \infty)$ defined as a strictly increasing function whose value is equal to the amount of the angle that the point $u(\gamma(\theta))$ has traveled over $\theta \in [0, \psi]$ in the cylindrical coordinates. In other words, the function A_z is total variation of the angular coordinate of the polar parameterization u, so we call $A_z(\psi)$ the *total angular variation* of u up to ψ. If the image $u(\operatorname{im}\gamma)$ is also star-shaped, then $A_z(\psi)$ is the angular coordinate of $u(\gamma(\psi))$.

Consider a infinitesimal triangle constructed by three points $u(\gamma(\psi + d\psi))$, $u(O)$, and $u(\gamma(\psi))$. Recall that the range of the map $u \circ \gamma$ belongs to $\rho\psi$-plane, which is the plane S in the definition of a cut. By projection of this triangle to $\rho\psi$-plane, we can say that the infinitesimal value of the total angular variation $dA_z(\psi)$ denotes the size of the angle constructed by $u(\gamma(\psi + d\psi))$, O, and $u(\gamma(\psi))$. If this infinitesimal

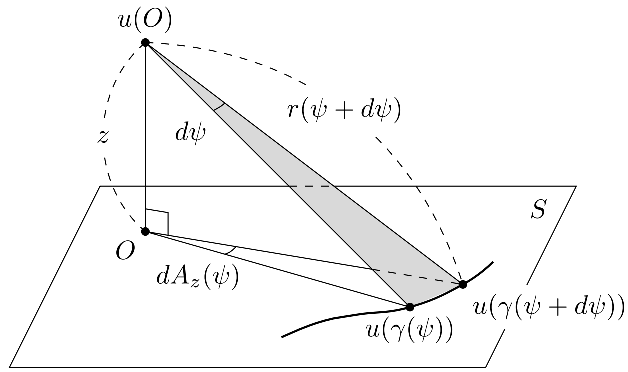

FIGURE 3. The infinitesimal triangle and its projection to the plane.

angle was integrated from 0 to ψ, then the value would be same with $A_z(\psi)$. Since u preserves the metric, the infinitesimal angle $d\psi$ denotes the angle $dA_z(\psi)$ before the projection. In addition, the differential coefficient of A_z does not need to be always greater than 1, but it is less than 1 if and only if $r'(\psi) > r(\psi)$ for sufficiently small positive number z where r' denotes the derivative of r. Actually the function A_z is independent of u, it can be defined only by the function r and a positive real number z.

The function A_z can be calculated through the projection of the angle mapped by u on the $\rho\psi$-plane. Let z be the altitude of the vertex of the conical origami. According to the law of cosines, we have a relation between the infinitesimal $d\psi$ and $dA_z(\psi)$ as follows:

$$\begin{aligned} r(\psi)^2 + r(\psi + d\psi)^2 - 2r(\psi)r(\psi + d\psi)\cos d\psi \\ = \big(r(\psi)^2 - z^2\big) + \big(r(\psi + d\psi)^2 - z^2\big) \\ - 2\sqrt{r(\psi)^2 - z^2}\sqrt{r(\psi + d\psi)^2 - z^2}\cos dA_z(\psi). \end{aligned}$$

By the half-angle formula for sine and some calculations, we obtain the square of the derivative of A_z:

$$\left(\frac{dA_z(\psi)}{d\psi}\right)^2 = \left(\frac{r(\psi)}{\sqrt{r(\psi)^2 - z^2}}\right)^2 - \left(\frac{zr'(\psi)}{r(\psi)^2 - z^2}\right)^2.$$

Recall that A_z can be determined only by r and z. Since the sign of the derivative of A_z with respect to ψ cannot be determined in here, we define the angular coordinate of u as the integral of the product of $dA_z/d\psi$ and either 1 or (-1) in Theorem 3.1.

Before giving the definition of the function A_z, the domain of z is defined as the open interval U_r that depends on the function r. If the nonnegative real number z does not belong to U_r, then the image of u is the plane if $z = 0$; there exist a point such that A_z is not strictly increasing if $z = \sup U_r$; the integrand of the integral A_z is not real number if $z > \sup U_r$.

Definition. Let γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ that has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$, for a Lipschitz continuous function $r : \mathbb{R} \to (0, \infty)$ with period 2π. The open interval U_r is defined such that:

$$U_r := \left(0 \,,\, \inf_{\psi \in I \setminus \Sigma_r} \frac{r(\psi)^2}{\sqrt{r(\psi)^2 + r'(\psi)^2}} \right),$$

where r' denotes the derivative of the function r and Σ_r denotes the singular set. For each $z \in U_r$, an injective function $A_z : [0, 2\pi] \to [0, \infty)$ is defined by the following integral:

$$A_z(\psi) := \int_0^\psi \sqrt{\left(\frac{r(\theta)}{\sqrt{r(\theta)^2 - z^2}} \right)^2 - \left(\frac{z r'(\theta)}{r(\theta)^2 - z^2} \right)^2} \, d\theta.$$

The value $A_z(\psi)$ is called the *total angular variation* of u up to ψ.

In particular, we define $A_0(\psi)$ to be equal to ψ. Because the total angular variation A_z does not allude the sign of the infinitesimal change of the angular coordiate, something to indicate the sign is required. For a subset κ of the interval I, let $A_z^\kappa(\psi)$ be a abbreviation of the integral

$$A_z^\kappa(\psi) = \int_0^\psi (-1)^{\mathbf{1}_\kappa(\theta)} A_z'(\theta) \, d\theta,$$

where A_z' denotes the derivative of A_z with respect to ψ and $\mathbf{1}_\kappa$ is the indicator function of κ. The value of $A_z^\kappa(\psi)$ can represent the exact angular coordinate of a conical origami in cylindrical coordinates. If ψ belongs to κ, then the infinitesimal change of the angular coordinate becomes negative.

Since the derivative of A_z is positive, the function A_z is strictly increasing for ψ and thereby it has the inverse function. The notations A_z, A_z^κ and the property that A_z is strictly increasing will be frequently used without the definition. The curve γ is piecewise C^1, so the function r is C^1 on the set $\mathbb{R} \setminus \Sigma_r$. The derivative r' is bounded since the function r is Lipschitz continuous, and the function r has a positive lower bound. So, the open interval U_r is non-empty.

Theorem 3.1. *Let $r : \mathbb{R} \to (0, \infty)$ be a Lipschitz continuous function with period 2π and γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ that has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$. If*

$$\sup_{z \in U_r} A_z(2\pi) \geq 2\pi,$$

then there exist a closed interval $\kappa \subset I$ and a real number $z \in U_r$ such that a map $u : \mathbb{R}^2 \to \mathbb{R}^3$ is a conical origami and γ is a cut on u; the map u is defined as follows:

$$u(\rho, \psi)_{polar} = \left(\rho \sqrt{1 - \frac{z^2}{r(\psi)^2}} \,,\, A_z^\kappa(\psi) \,,\, z\left(1 - \frac{\rho}{r(\psi)}\right) \right)_{cylindrical} . \tag{2}$$

The proof of Theorem 3.1 will be given at the end of Section 4. Notice that the map u defined in Theorem 3.1 as (2) depends only on r and κ and z, because we defined A_z only with r and z. Let u be the map defined in Theorem 3.1 as (2). Then, the altitude of a point $u(\rho\cos\psi, \rho\sin\psi) = u(\rho, \psi)_{polar}$ in cylindrical coordinate is 0 if and only if $\rho = r(\psi)$, that is, the point $(\rho\cos\psi, \rho\sin\psi)$ is on the curve γ. So we get $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$ where S is the plane $z = 0$, and the curve γ is a cut on u if u is an origami.

Recall the definition of the conical origami. To prove that the map u is the conical origami, we should show the four following propositions are true:

- The gradient Du has orthonormal columns for all points of $\mathbb{R}^2 \setminus \Sigma_u$;
- the image of u is a cone.
- there exists a sequence of maps u_k that are injective and uniformly converges to u;
- the maps u_k and u are Lipschitz continuous, and u is piecewise C^1;

We give the rigorous proof of Theorem 3.1 throughout Section 4. For the map u defined in Theorem 3.1, Theorem 4.1 shows that u preserves metric and Theorem 4.2 shows that the image of u is a cone. The proof of the other two conditions for conical origami is devided into two cases: the function r is a constant function or not. The case of the nonconstant function is treated in Theorem 4.3, 4.4, and 4.5; the case of the constant function is treated in Theorem 4.6.

In the case of nonconstant function, it is proved that we can find an injective u, so that we can let u_k be same with u. For an interval $\kappa = [\alpha, \beta] \subset I$ and a real number $z \in U_r$, the following three conditions play an important role in the proof.

(1) both $2A_z(\alpha) - A_z(\beta)$ and $2A_z(\beta) - A_z(\alpha)$ belong to the interval $[0, A_z(2\pi))$;
(2) the function r is strictly increasing over the interval $J = [A_z^{-1}(2A_z(\alpha) - A_z(\beta)), A_z^{-1}(2A_z(\beta) - A_z(\alpha))]$;
(3) the value of $A_z^\kappa(2\pi)$ is equal to 2π.

Recall that the map u is determined by only κ and z if the curve γ and the function r were already given. Theorem 4.3 shows that there exist an interval $\kappa = [\alpha, \beta] \subset I$ and a real number $z \in U_r$ satisfying the three conditions if r is nonconstant. Theorem 4.4 shows that the map u is injective if the three conditions are satisfied. Theorem 4.5 shows that the map u is Lipschitz continuous and piecewise C^1 if the third condition of the three conditions is satisfied.

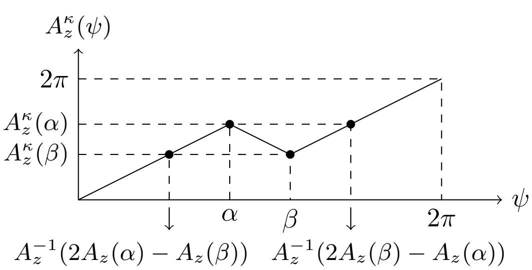

FIGURE 4. The rough shape of the graph of the function A_z^κ for a closed interval $\kappa = [\alpha, \beta]$.

The first condition is for the existence of the interval J in the second condition, where A_z^{-1} denotes the inverse function of A_z. The interval J in the second condition is used to prove that u is injective. To prove the injectivity of u, it is enough to show that the curve $u \circ \gamma$ is simple because the image of u is a cone. A point $u(\gamma(\psi))$ traces the curve $u \circ \gamma$ counter-clockwise, but clockwise when ψ is in the interval κ. So, the curve $u \circ \gamma$ can intersects itself only if the angular coordinate of u belongs to between $A_z^\kappa(\beta)$ and $A_z^\kappa(\alpha)$. From the definition of A_z^κ, we have

$$A_z^\kappa\Big(A_z^{-1}\big(2A_z(\alpha) - A_z(\beta)\big)\Big) = A_z^\kappa(\beta)\,,\ A_z^\kappa\Big(A_z^{-1}\big(2A_z(\beta) - A_z(\alpha)\big)\Big) = A_z^\kappa(\alpha).$$

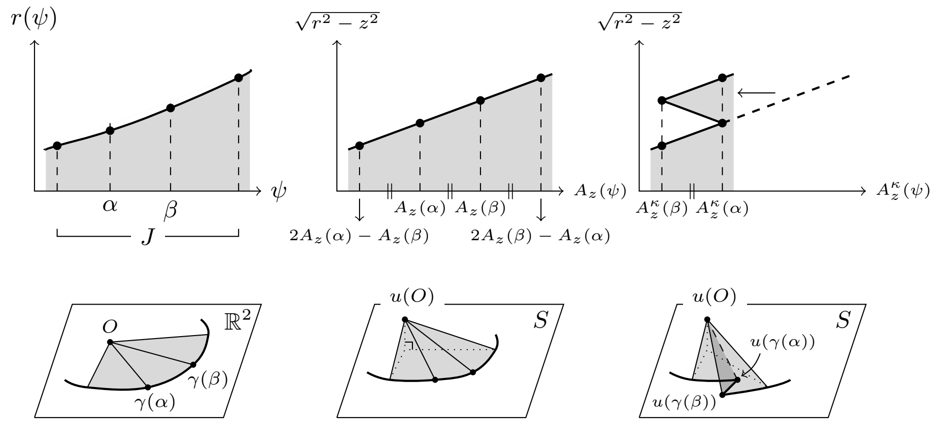

FIGURE 5. If the function r is strictly increasing over J, then the map u is injective.

It implies that we do not have to check whether $u \circ \gamma$ intersects itself for $\psi \notin J$. The condition that r is strictly increasing over J lets $u \circ \gamma$ be injective. The details of this idea will be presented rigorously in Theorem 4.4. The third condition is necessary for u to be continuous because the map u is given on the polar coordinates. By these conditions, we can conclude that there exists an conical origami u on which the given simple closed curve γ is a cut if the function r is not a constant function.

If the function r is constant, the curve γ is a circle such that the center is the origin of $\mathbb{R}^2$ and the radius is r. Intuitively, we can see any circle is a cut on a certain conical origami, which is folded like a filter paper. In this case, the sequence of maps u_k is required to be different from u since the limit point u is not injective, so we define a sequence u_k separately from u. Theorem 4.6 shows that folding as if we fold a filter paper satisfies the third and fourth condition to be a conical origami.

4. The proof of The Main Theorem

Theorem 4.1. *Let $r : \mathbb{R} \to (0, \infty)$ be a function that is Lipschitz continuous and piecewise C^1. For a closed interval $\kappa = [\alpha, \beta] \subset I = [0, 2\pi)$ and a real number $z \in U_r$, let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 3.1 as (2). The map u is a local isometric immersion at the point in $\mathbb{R}^2 \setminus \Sigma_u$, so the gradient Du has orthonormal columns for all points of $\mathbb{R}^2 \setminus \Sigma_u$.*

Proof. The partial derivatives of u in cylindrical coordinates are

$$\frac{\partial u}{\partial \rho} = \left(\sqrt{1 - \frac{z^2}{r(\psi)^2}}\, ,\, 0\, ,\, -\frac{z}{r(\psi)} \right),$$

and

$$\frac{\partial u}{\partial \psi} = \left(\frac{\rho z^2 r'(\psi)}{r(\psi)^2 \sqrt{r(\psi)^2 - z^2}}\, ,\, (-1)^{\mathbf{1}_\kappa(\psi)} A_z'(\psi)\, ,\, \frac{\rho z r'(\psi)}{r(\psi)^2} \right),$$

where the derivative of the function A_z is

$$A_z'(\psi) = \sqrt{\left(\frac{r(\psi)}{\sqrt{r(\psi)^2 - z^2}}\right)^2 - \left(\frac{zr'(\psi)}{r(\psi)^2 - z^2}\right)^2}.$$

The metric g on the image of u is

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2\left(1 - \frac{z^2}{r(\psi)^2}\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

because the codomain of u was given in the cylindrical coordinate system. Then, we have a calculatation of the first fundamental form with the inner product in the metric g as follows:

$$E = \left\langle \frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \rho} \right\rangle = 1\,,\ F = \left\langle \frac{\partial u}{\partial \rho}, \frac{\partial u}{\partial \psi} \right\rangle = 0\,,\ G = \left\langle \frac{\partial u}{\partial \psi}, \frac{\partial u}{\partial \psi} \right\rangle = \rho^2\,.$$

It is same with the metric of polar coordinates, hence u is a local isometric immersion at all points, at which u is differentiable. □

Theorem 4.2. *Let $r : \mathbb{R} \to (0, \infty)$ be a function that is Lipschitz continuous and piecewise C^1. For a closed interval $\kappa = [\alpha, \beta] \subset I = [0, 2\pi)$ and a real number $z \in U_r$, let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 3.1 as (2). The image of u is a cone, not a plane, such that $u(O)$ is vertex of the cone.*

Proof. The equality $\partial^2 u/\partial\rho^2 = 0$ implies u is linear for ρ. The image $u(O)$ is $(0, 0, z)$ regardless of ψ. It menas that the image of u is a cone, not a plane because $z > 0$, such that $u(O)$ is vertex of the cone, where O is the origin of $\mathbb{R}^2$. □

Theorem 4.3. *Let $r : \mathbb{R} \to (0, \infty)$ be a Lipschitz continuous function with period 2π and γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ that has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$. If r is not a constant function and $\sup_{z \in U_r} A_z(2\pi) \geq 2\pi$, then there exists an interval $\kappa = [\alpha, \beta] \subset I$ and a real number $z \in U_r$ that satisfy:*

1. *both $2A_z(\alpha) - A_z(\beta)$ and $2A_z(\beta) - A_z(\alpha)$ belong to the interval $[0, A_z(2\pi))$;*
2. *the function r is strictly increasing over the interval $J = [A_z^{-1}(2A_z(\alpha) - A_z(\beta)), A_z^{-1}(2A_z(\beta) - A_z(\alpha))]$;*
3. *the value of $A_z^\kappa(2\pi)$ is equal to 2π.*

Proof. Assume that there is z such that $A_z(2\pi) = 2\pi$. For this z, let $\alpha = \beta$. Then, we can say that r is strictly increasing over $J = [\alpha, \beta] = \{\alpha\}$ since $2A_z(\alpha) - A_z(\beta) = 2A_z(\beta) - A_z(\alpha) = \alpha = \beta$. Furthermore, the measure of $\kappa = [\alpha, \beta]$ is 0, so we get

$$A_z(2\pi) = A_z^\kappa(2\pi) = 2\pi.$$

The set $\kappa = \{\alpha\}$ and this z satisfy the three conditions.

Otherwise if there is no z such that $A_z(2\pi) = 2\pi$, then $A_z(2\pi) > 2\pi$ for all $z \in U_r$ because A_z is continuous and

$$\sup_{z \in U_r} A_z(2\pi) \geq 2\pi.$$

Throughout the rest of the proof, assume that $A_z(2\pi) > 2\pi$ for all z.

Suppose that the function r is strictly increasing on a positive length interval $[a,b] \subset I$. Because the function r is piecewise C^1 and periodic, such interval $[a,b]$ exists. Let $\alpha_0(t)$ and $\beta_0(t)$ be a function defined on U_r such that:

$$\alpha_0(t) = A_t^{-1}\left(\frac{2}{3}A_t(a) + \frac{1}{3}A_t(b)\right), \beta_0(t) = A_t^{-1}\left(\frac{1}{3}A_t(a) + \frac{2}{3}A_t(b)\right).$$

From $A_t(a) < A_t(b)$, we get $\alpha_0(t) < \beta_0(t)$ for a sufficiently small t. Let z_0 be a real number in U_r satisfying

$$\sup_{t<z_0} \alpha_0(t) < \inf_{t<z_0} \beta_0(t).$$

Let us define κ as an interval $[\alpha, \beta] \subset \left[\sup_{t<z_0} \alpha_0(t), \inf_{t<z_0} \beta_0(t)\right]$ such that

$$0 < \beta - \alpha < \frac{A_{z_0}(2\pi) - 2\pi}{2\sup_{\psi \in I\setminus\Sigma_r} A'_{z_0}(\psi)}.$$

For the fixed κ, consider the value of $A_z^\kappa(2\pi)$ as a fuction of z. This function is equal to $A_z(2\pi) - 2\big(A_z(\beta) - A_z(\alpha)\big)$ and continuous with respect to z. By letting $z = z_0$, we obtain

$$\begin{aligned} A_{z_0}^\kappa(2\pi) &= A_{z_0}(2\pi) - 2\int_\alpha^\beta A'_{z_0}(\theta)\, d\theta \\ &\geq A_{z_0}(2\pi) - 2(\beta - \alpha) \sup_{\psi \in I\setminus\Sigma_r} A'_{z_0}(\psi) > 2\pi. \end{aligned}$$

On the other hand, letting $z = 0$, we get

$$A_0^\kappa(2\pi) = 2\pi - 2(\beta - \alpha) < 2\pi.$$

By the intermediate value theorem, there exists $z \in (0, z_0)$ satisfying the third condition. We will show that these κ and z satisfy the other two conditions.

From the definition of κ, we get

$$\alpha_0(z) \leq \sup_{t<z_0} \alpha_0(t) \leq \alpha\,,\ \beta \leq \inf_{t<z_0} \beta_0(t) \leq \beta_0(z).$$

It implies that $A_z(\alpha_0(z)) \leq A_z(\alpha)$, $A_z(\beta) \leq A_z(\beta_0(z))$ and

$$\begin{aligned} A_z(a) = 2A_z(\alpha_0(z)) - A_z(\beta_0(z)) &\leq 2A_z(\alpha) - A_z(\beta) \\ &\leq 2A_z(\beta) - A_z(\alpha) \leq 2A_z(\beta_0(z)) - A_z(\alpha_0(z)) = A_z(b). \end{aligned}$$

Since $2A_z(\alpha) - A_z(\beta)$ and $2A_z(\beta) - A_z(\alpha)$ belong to the interval $[A_z(a), A_z(b)]$, the two values are in the range of A_z. Also, we have the interval $J = [A_z^{-1}(2A_z(\alpha) - A_z(\beta)), A_z^{-1}(2A_z(\beta) - A_z(\alpha))]$ be a subset of $[a,b]$, which the function r is strictly increasing on. Hence, the first and second conditions are satisfied. □

Theorem 4.4. *Let $r : \mathbb{R} \to (0,\infty)$ be a Lipschitz continuous function with period 2π and γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ that has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$. For a closed interval $\kappa = [\alpha, \beta] \subset I$ and a real number $z \in U_r$, let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 3.1 as (2). The map u is injective if κ and z satisfy the following conditions:*

1. *both $2A_z(\alpha) - A_z(\beta)$ and $2A_z(\beta) - A_z(\alpha)$ belong to the interval $[0, A_z(2\pi))$;*
2. *the function r is strictly increasing over the interval $J = [A_z^{-1}(2A_z(\alpha) - A_z(\beta)), A_z^{-1}(2A_z(\beta) - A_z(\alpha))]$;*
3. *the value of $A_z^\kappa(2\pi)$ is equal to 2π.*

Proof. Assume that u is not injective, that is, there exist two distinct points $(\rho_1, \psi_1), (\rho_2, \psi_2)$ on polar coordinates such that $u(\rho_1 \cos\psi_1, \rho_1 \sin\psi_1) = u(\rho_2 \cos\psi_2, \rho_2 \sin\psi_2)$. From

$$\rho_1 \sqrt{1 - \frac{z^2}{r(\psi_1)^2}} = \rho_2 \sqrt{1 - \frac{z^2}{r(\psi_2)^2}}\,,\ z\left(1 - \frac{\rho_1}{r(\psi_1)}\right) = z\left(1 - \frac{\rho_2}{r(\psi_2)}\right),$$

we obtain $\rho_1 = \rho_2$ and $r(\psi_1) = r(\psi_2)$; suppose that $0 \le \psi_1 < \psi_2 < 2\pi$. Also, we have $A_z^\kappa(\psi_1) \equiv A_z^\kappa(\psi_2) \pmod{2\pi}$. The function $A_z^\kappa : [0, 2\pi] \to [0, \infty)$ has local minimums at $\psi = 0, \beta$, and local maximums at $\psi = \alpha, 2\pi$. By the third condition, we have

$$A_z^\kappa(2\pi) = A_z(2\pi) - 2\big(A_z(\beta) - A_z(\alpha)\big) = 2\pi,$$

and by the first condition, the following inequalities hold:

$$A_z^\kappa(\alpha) = A_z(\alpha) = 2\pi - \Big(A_z(2\pi) - \big(2A_z(\beta) - A_z(\alpha)\big)\Big) < 2\pi,$$
$$A_z^\kappa(\beta) = A_z(\alpha) - \big(A_z(\beta) - A_z(\alpha)\big) \ge 0.$$

Therefore, the value of $A_z^\kappa(\psi)$ for $\psi \in I$ belongs to the interval I and we obtain

$$A_z^\kappa(\psi_1) = A_z^\kappa(\psi_2).$$

If we suppose $[\psi_1, \psi_2] \cap \kappa = \varnothing$, then $A_z^\kappa(\psi_2) - A_z^\kappa(\psi_1) = A_z(\psi_2) - A_z(\psi_1) > 0$. So we get $\psi_2 \ge \alpha$ and $\psi_1 \le \beta$. Since $A_z^\kappa(\alpha)$ and $A_z^\kappa(\beta)$ are the local extrema of A_z^κ, we have

$$A_z^\kappa(\alpha) - A_z^\kappa(\psi_1) = |A_z(\alpha) - A_z(\psi_1)| \ge A_z(\alpha) - A_z(\psi_1)$$

and

$$A_z^\kappa(\psi_2) - A_z^\kappa(\alpha) \ge A_z^\kappa(\beta) - A_z^\kappa(\alpha) = -\big(A_z(\beta) - A_z(\alpha)\big).$$

Combining these two inequalities, we obtain $A_z(\psi_1) \ge 2A_z(\alpha) - A_z(\beta)$ by

$$0 = A_z^\kappa(\psi_2) - A_z^\kappa(\psi_1) \ge 2A_z(\alpha) - A_z(\beta) - A_z(\psi_1).$$

Similarly, we can also obtain $A_z(\psi_2) \le 2A_z(\beta) - A_z(\alpha)$, and these implies that ψ_1 and ψ_2 belong to the interval $J = [A_z^{-1}\big(2A_z(\alpha) - A_z(\beta)\big), A_z^{-1}\big(2A_z(\beta) - A_z(\alpha)\big)]$. Because the function r is strictly increasing over J, $r(\psi_1)$ cannot be equal to $r(\psi_2)$. Hence, the assumption that u is not injective is false. $\square$

Theorem 4.5. *Let $r : \mathbb{R} \to (0, \infty)$ be a Lipschitz continuous function with period 2π and γ be a piecewise C^1 simple closed curve in $\mathbb{R}^2$ that has a parameterization $\gamma(\psi) = (r(\psi)\cos\psi, r(\psi)\sin\psi)$ on $\psi \in I = [0, 2\pi)$. For a closed interval $\kappa \subset I$ and a real number $z \in U_r$, let $u : \mathbb{R}^2 \to \mathbb{R}^3$ be the map defined in Theorem 3.1 as (2). The map u is Lipschitz continuous and piececwise C^1 if $A_z^\kappa(2\pi) = 2\pi$.*

Proof. If we prove the continuity of u, the Lipschitz continuity is proved since each component of the derivatives of u is bounded for each variable. Also, u is piecewise C^1 since each component of u is piecewise C^1. Let us show the continuity of u.

Since each component of u is continuous, it is suffice to prove continuity in polar coordinates, that is to show the following coniditions:

$$\begin{cases} \text{for all } \psi & , \lim_{\rho \to 0} u(\rho, \psi) = u(0, \psi) \\ \text{for all } \rho & , u(\rho, 0) = u(\rho, 2\pi) \end{cases}$$

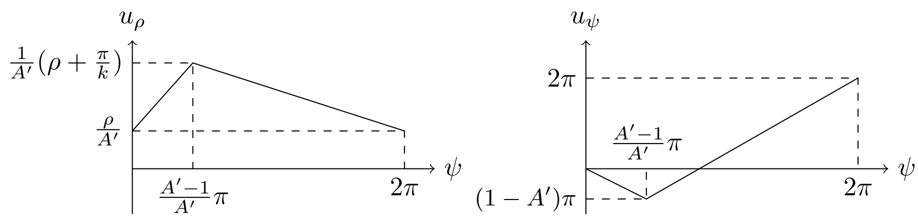

FIGURE 6. The radial and angular coordinates of u_k.

where $u(\rho, \psi)$ is defined on a polar coordinate system. Recall that the map u is defined as

$$u(\rho,\psi)_{polar} = \left(\rho\sqrt{1-\frac{z^2}{r(\psi)^2}}\ ,\ A_z^\kappa(\psi)\ ,\ z\left(1-\frac{\rho}{r(\psi)}\right)\right)_{cylindrical}.$$

The first condition is clearly true. The second condition is also true because $r(2\pi) = r(0)$ implies that the radial and axial components of $u(\rho, 0)_{polar}$ and $u(\rho, 2\pi)_{polar}$ are respecticely same, and the difference of the angular coordinates $A_z^\kappa(2\pi) - A_z^\kappa(0)$, which is equal to 2π, is an integer multiple of 2π. Therefore, the map u is continuous. □

In Theorem 4.6, we deal with the case that the function r is a constant function. Since the derivative of r is 0, we have $U_r = (0, r)$ and $A'_z = r/\sqrt{r^2 - z^2}$. For the simple statement of the proof, the notation A' still denotes $r/\sqrt{r^2 - z^2}$ for given z.

Theorem 4.6. *Let r be a positive real number and γ be a circle parameterized by angle; $\gamma(\psi) = (r\cos\psi, r\sin\psi)$ on $\psi \in I = [0, 2\pi)$. There exists a closed interval κ for every $z \in U_r$ such that the curve γ is a cut on $u : \mathbb{R}^2 \to \mathbb{R}^3$, which is defined in Theorem 3.1 as (2).*

Proof. Let z be a real number in $(0, r)$ and $A' = r/\sqrt{r^2 - z^2}$. Let $\kappa = \left[0, \frac{A'-1}{A'}\pi\right]$. Let $u_k : \mathbb{R}^2 \to \mathbb{R}^3$ be a sequence of maps such that: If $\psi \in \kappa$,

$$u_k(\rho,\psi)_{polar} = \left(\frac{\rho}{A'} + \frac{1}{k}\cdot\frac{\psi}{A'-1}\ ,\ -A'\psi\ ,\ z\left(1-\frac{\rho}{r}\right)\right)_{cylindrical};$$

if $\psi \in [0, 2\pi] \setminus \kappa$,

$$u_k(\rho,\psi)_{polar} = \left(\frac{\rho}{A'} + \frac{1}{k}\cdot\frac{2\pi-\psi}{A'+1}\ ,\ A'\psi - 2\pi(A'-1)\ ,\ z\left(1-\frac{\rho}{r}\right)\right)_{cylindrical}.$$

The sequence u_k converges uniformly to a map u as $k \to \infty$, which coincides with what we define in Theorem 3.1 as (2) for the interval κ. So, the gradient Du has orthonormal columns for all points of $\mathbb{R}^2 \setminus \Sigma_u$ by Theorem 4.1, and the image of u is a cone by Theorem 4.2. Assume that u_k is not injective, that is, there exist two distinct points $(\rho_1, \psi_1), (\rho_2, \psi_2)$ on polar coordinates such that $u(\rho_1\cos\psi_1, \rho_1\sin\psi_1) = u(\rho_2\cos\psi_2, \rho_2\sin\psi_2)$. Then, we obtain $\rho_1 = \rho_2$, and $0 \le \psi_1 \le \frac{A'-1}{A'}\pi < \psi_2 < 2\pi$ without loss of generality. Since simultaneous equations

$$\frac{\psi_1}{A'-1} = \frac{2\pi-\psi_2}{A'+1}, \quad -A'\psi_1 = A'\psi_2 - 2\pi(A'-1)$$

has a solution $\psi_1 = \psi_2 = \frac{A'-1}{A'}\pi$, the map u_k is injective. From $u_k(\rho,0)_{polar} = u_k(\rho,2\pi)_{polar}$, we can also prove that u_k and u are Lipschitz continuous and piecewise C^1 with the same logic with Theorem 4.5. Therefore, the map u is a conical origami if $\kappa = \left[0, \frac{A'-1}{A'}\pi\right]$.

Let S be the plane $z=0$ in the cylindrical coordinate system. A point $u(\rho\cos\psi, \rho\sin\psi)$ is in S if and only if $\rho = r$, which means the point is on the circle γ. So, we have $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$. Hence, the circle γ is cut on the conical origami u. □

Proof of Theorem 3.1. By Theorem 4.6, Theroem 3.1 is true if the function r is a constant function.

Consider the function r is not a constant function. By Theorem 4.1 and 4.2, the gradient Du has orthonormal columns for all points of $\mathbb{R}^2 \setminus \Sigma_u$ and the image of u is a cone. There exist a closed interval $\kappa \subset I$ and a real number U_r such that:

(1) both $2A_z(\alpha) - A_z(\beta)$ and $2A_z(\beta) - A_z(\alpha)$ belong to the interval $[0, A_z(2\pi))$;
(2) the function r is strictly increasing over the interval $J = [A_z^{-1}(2A_z(\alpha) - A_z(\beta)), A_z^{-1}(2A_z(\beta) - A_z(\alpha))]$;
(3) the value of $A_z^{\kappa}(2\pi)$ is equal to 2π

by Theorem 4.3. By Theorem 4.4 and 4.5, the map u is injective, Lipschitz continuous, and piecewise C^1. If we let u_k be a sequence of maps such that $u_k = u$ for all positive integer k, then u_k is uniformly converges to u and the map u is a conical origami. Let S be the plane $z=0$ in the cylindrical coordinate system. A point $u(\rho\cos\psi, \rho\sin\psi)$ is in S if and only if $\rho = r(\psi)$, which means the point is on the curve γ. So, we have $S \cap \operatorname{im} u = u(\operatorname{im}\gamma)$. The curve γ is a cut on the conical origami u for the κ and z satisfying the above three conditions.

Hence, the statement of Theorem 3.1 is true. □

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