Real Analysis I : Measure Theory

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CHAPTER 1

Topological measures

1. Radon measures

In LCH, compact finiteness and locally finiteness are equivalent.

DEFINITION 1.1. A *Radon measure* is a Borel measure which is
(1) outer regular on all Borel sets,
(2) inner regular on all open sets,
(3) compact finite.

Radon measures are rather simply characterized when the base space is σ-compact.

THEOREM 1.1. *A Radon measure is inner regular on all σ-finite Borel sets.*

PROOF. Let E be a Borel set with $\mu(E) < \infty$. By outer regularity, there is an open set $U \supset E$ such that

$$\mu(U) < \mu(E) + \frac{\varepsilon}{2}.$$

Then,

$$\mu(U \setminus E) < \frac{\varepsilon}{2}.$$

By outer regularity, there is an open set $V \supset U \setminus E$ such that

$$\mu(V) < \mu(U \setminus E) + \frac{\varepsilon}{2}.$$

By inner regularity, there is a compact set $K \subset U$ such that

$$\mu(K)\frac{\varepsilon}{2} > \mu(U).$$

Then, we have $K \setminus V \subset E$ and

$$\mu(K \setminus V) = blabla$$

$\square$

cptfin Borel regular =¿ Radon =¿ cptfin Borel sigma-cpt cptfin Borel regular == Radon =¿ cptfin Borel second countable =¿ every open is sigma cpt cptfin Borel regular == Radon == cptfin Borel

COROLLARY 1.2. *If X is σ-compact, then a compact finite Borel measure is Radon if and only if it is regular.*

THEOREM 1.3. *If X is second countable, then every compact finite Borel measure is regular.*

2. The Riesz-Markov-Kakutani theorem

In this section, we always assume X is a locally compact Hausdorff space. Hence we can use the Urysohn lemma: If K is compact and F is closed, then we can find a continuous function $f : X \to [0, 1]$ such that $f|_K = 1$ and $f|_F = 0$.

2.1. The first theorem. Positivity of linear functional itself implies a rather strong continuity property.

THEOREM 2.1. *Let $C_c(X)$ be a space of compactly supported continuous functions on X. (Give an LF topology with a directed inductive family $C_K(X)$.) If a linear functional I is positive, then continuous with respect to the topology.*

PROOF. Let K be a compact subset. We want to show $|I(f)| \lesssim \|f\|$ for $f \in C_K(X)$. The proof idea comes from $|\int_K f\,d\mu| \le \mu(K)\|f\|$.
Choose $\phi \in C_c(X)$ such that $\phi|_K = 1$. □

Jordan decomposition: $(C_0(X), u)^* \subset (C_c(X), LF)^*$ converse?

CHAPTER 2

Hmmmm

0.2. Convergence in measure. Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0. \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > n^{-1},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n(x) - f(x)| > n^{-1}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.\end{aligned}$$

Since for every k

$$\limsup_n \{x : |f_n(x) - f(x)| > k^{-1}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\},$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.$$

THEOREM 0.2. *Let f_n be a sequence of measurable functions on a measure space (X, μ). If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.*

PROOF. Since $d_{f_n - f}(1/k) \to 0$ as $n \to \infty$, we can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) > 2^{-k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. □