Diachrony of Spectra

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Postech - Unist - Kaist Joint Seminar

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Introduction

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Question

Why is it defined like this?

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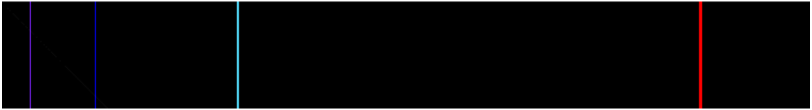
Hydrogen spectral series



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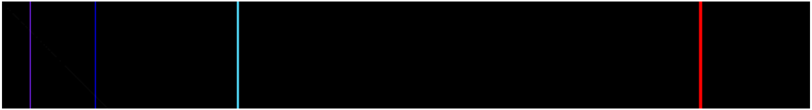

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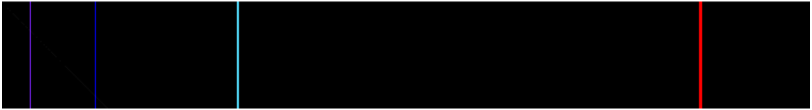

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How can we explain and compute this phenomenon?

Rydberg's formula : Bohr model

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The constant h is called the Planck constant and $\hbar := \frac{h}{2\pi}$.

Rydberg's formula : Bohr model

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$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

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Proposition (Rydberg formula)

The wavelengths λ of absorbed or emitted photons from a hydrogen atom is estimated by the following formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \textit{for} \quad n_1, n_2 \in \mathbb{N},$$

where $R := \frac{k^2e^4m}{4\pi\hbar^3c}$ is the Rydberg constant.

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In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t,x) \in \mathbb{R}^{1+3}$

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Remark

The first one is not mathematically correct statement because we should resolve some technical issues on convergence.

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The Beginning of Spectral Theory

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Since the eigenvalues embody the possible energy values of an electron, we just gave the Rydberg formula a reasonable explanation. This result explains not only the discretized energy spectrum but also the number of each orbitals!

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