VLASOV-POISSON SYSTEM

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1. Vlasov-Poisson system

Consider the following Cauchy problem for the *Valsov-Poisson system*:

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, & (t,x,v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\
E(t,x) = -\nabla_x \Phi, & \\
\Phi(t,x) = (-\Delta_x)^{-1} \rho, & \lim_{|x|\to\infty} \Phi(t,x) = 0, \\
\rho(t,x) = \int f \, dv, & \\
f(0,x,v) = f_0(x,v) \ge 0, &
\end{cases}
\tag{1}
$$

where $\gamma = \pm 1$. For example, we have *repulsive problem* $\gamma = +1$ for electrons in plasma theory and *attractive problems* $\gamma = -1$ for galactic dynamics. (ρ denotes mass density.)

This report is a review of Schaffer's paper [], and is written thanks to Glassey's book []. We mainly investigate the local and global existence problem for a classical solution of the Cauchy problem for the Vlasov-Poisson system. More precisely, we prove there is a unique global C_c^1 solution when given a C_c^1 initial data f_0. Let us define our solution space.

Definition. Let $f_0 : \mathbb{R}^6 \to [0,\infty]$ be a function. A function $f : [0,T] \times \mathbb{R}^6 \to \mathbb{R}$ is said to be a *classical solution* of the Cauchy problem for the Vlasov-Poisson system with initial data f_0 if $f \in C^1([0,T]; C_c^1(\mathbb{R}^6))$ and satisfies all equations in (1) on its domain. Further, if $f \in C^1(\mathbb{R}^+; C_c^1(\mathbb{R}^6))$, then the classical solution f is said to be *global*.

The precise statement of the global existence theorem is as follows:

Theorem 1.1. *Let $f_0 \in C_c^1(\mathbb{R}^6, [0,\infty))$. Then, there exists a unique global classical solution of the Cauchy problem for the Vlasov-Poisson system with initial data f_0.*

Results in 1.1 and 1.2 provide basic ingredients that will be used in the whole article. On the other hand, results in 1.3 cannot be used in any local existence proof because they assume the existence of solutions, but they help understand the fundamental nature of solutions of the Vlasov-Poisson system and are used in the proof of global existence.

Notation. We use the asymptotic notation

$$g(t) \lesssim h(t) \quad \iff \quad \exists\, c = c(f_0), \quad g(t) \le c\, h(t)$$

and

$$g(t) \simeq h(t) \quad \iff \quad \exists\, c, \quad g(t) = c\, h(t).$$

This report does not contain any other norms except the L^p norms so that double vertical bars always refer to the L^p norms. We also omit marginalized variables and the subscript L. For example,

$$\|f(t)\|_p = (\iint |f(t,x,v)|^p\, dv\, dx)^{1/p}, \quad \|\rho(t)\|_p = (\int |\rho(r,x)|^p\, dx)^{1/p}.$$

1.1. **Poisson equation.** For the three-dimensional boundaryless problem of the Poisson equation

$$-\Delta\Phi(x) = \rho(x)$$

in which the solution Φ vanishes at infinity, it is well-known that

$$\Phi = \tfrac{1}{4\pi|x|} * \rho,$$

so the electric field in the Vlasov-Poisson system is given by

$$E = -\nabla_x \Phi = -\nabla_x(\tfrac{1}{4\pi|x|} * \rho) = \frac{x}{4\pi|x|^3} * \rho.$$

It can be rewritten as

$$E(t,x) = \frac{1}{4\pi}\int \frac{(x-y)\rho(t,y)}{|x-y|^3}\,dy.$$

The nonlinearity of the system is originated from the force field E, so its estimates play a crucial role in study of the nonlinear system. Since it is given by the solution of the Poisson equation, estimates of the Riesz potential, the convolution with a kernel of the form $|x|^{-(d-\alpha)}$, are directly connected to estimates of the force field.

Lemma 1.2 (Estimates of Riesz potential)**.** *Let* $\rho \in C_c^1(\mathbb{R}^d)$*.*

i. *(Field estimate)*

$$\|\tfrac{1}{|x|^{d-1}} * \rho\|_\infty \lesssim \|\rho\|_\infty^{1-1/d}\|\rho\|_1^{1/d}$$

ii. *(Field derivative estimate) For* $\log^+(x) := \max\{0, \log x\}$*,*

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty + \|\rho\|_1.$$

Proof.

i. Let $0 \le \frac{1}{p} < \frac{\alpha}{d} < \frac{1}{q} \le 1$. Since $(d-\alpha)p < d < (d-\alpha)q$,

$$\begin{aligned}
|\tfrac{1}{|x|^{d-\alpha}} * \rho| &= \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\,dy + \int_{|x-y|\ge R} \frac{\rho(y)}{|x-y|^{d-\alpha}}\,dy \\
&\le \|\rho\|_{p'}\Big(\int_{|y|<R} \frac{dy}{|y|^{(d-\alpha)p}}\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_{|y|\ge R} \frac{dy}{|y|^{(d-\alpha)q}}\Big)^{1/q} \\
&\simeq \|\rho\|_{p'}\Big(\int_0^R r^{d-1-(d-\alpha)p}\,dr\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_R^\infty r^{d-1-(d-\alpha)q}\,dr\Big)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} + \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}.
\end{aligned}$$

By choosing R such that $\|\rho\|_{p'} R^{\frac{d}{p}-d+\alpha} = \|\rho\|_{q'} R^{\frac{d}{q}-d+\alpha}$, we get

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty \lesssim \|\rho\|_{p'}^{\frac{1-\frac{\alpha}{d}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho\|_{q'}^{\frac{\frac{1}{p}-1+\frac{\alpha}{d}}{\frac{1}{p}-\frac{1}{q}}},$$

so the inequality

$$\|\tfrac{1}{|x|^{d-\alpha}} * \rho\|_\infty^{\frac{1}{q}-\frac{1}{p}} \lesssim \|\rho\|_p^{\frac{1}{q}-\frac{\alpha}{d}} \|\rho\|_q^{\frac{\alpha}{d}-\frac{1}{p}}$$

is obtained by interchaning p and q with their conjugates. The desired result gets $p = \infty$, $\alpha = 1$, and $q = 1$.

ii. Let $0 < R_a \le R_b$ be constants which will be determined later. Divide the region radially

$$|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)| \lesssim \nabla \int_{|x-y|<R_a} + \nabla \int_{R_a \le |x-y| < R_b} + \nabla \int_{R_b \le |x-y|}.$$

For the first integral,

$$\int_{|y|<R_a} \frac{\nabla\rho(x-y)}{|y|^{d-1}}\,dy \le \|\nabla\rho\|_\infty \int_{|y|<R_a} \frac{1}{|y|^{d-1}}\,dy$$
$$\simeq \|\nabla\rho\|_\infty \int_0^{R_a} 1\,dr = R_a\|\nabla\rho\|_\infty.$$

For the second integral,

$$\int_{R_a\le|x-y|<R_b} \frac{\rho(y)}{|x-y|^d}\,dy \le \|\rho\|_\infty \int_{R_a\le|x-y|<R_b} \frac{1}{|x-y|^d}\,dy$$
$$\simeq \|\rho\|_\infty \int_{R_a}^{R_b} \frac{1}{r}\,dr = (\log \tfrac{R_b}{R_a})\|\rho\|_\infty.$$

For the third integral,

$$\int_{R_b\le|x-y|} \frac{\rho(y)}{|x-y|^d}\,dy \le R_b^{-d}\|\rho\|_1.$$

Thus,

$$|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)| \lesssim R_a\|\nabla\rho\|_\infty + (\log \tfrac{R_b}{R_a})\|\rho\|_\infty + R_b^{-d}\|\rho\|_1.$$

Assuming ρ is nonzero so that $\|\nabla\rho\|_\infty > 0$, let $R_a = \min\{1, \|\nabla\rho\|_\infty^{-1}\}$ and $R_b = 1$. Since

$$\log \tfrac{1}{R_a} \le \log^+ \|\nabla\rho\|_\infty \quad \text{and} \quad R_a \lesssim \|\nabla\rho\|_\infty,$$

we have

$$\|\nabla(\tfrac{1}{|x|^{d-1}} * \rho)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty + \|\rho\|_1. \qquad \square$$

1.2. **Characteristics and volume preservation.** The Vlasov-Poisson equation is quite hyperbolic. What we mean here is that the method of characteristic curves is useful, and it does not regularizes the solution directly. Although we do not have an explicit representation formula, solutions written by characteristic curves make appropriate estimates possible.

Moreover, since the Vlasov-Poisson system is a Hamiltonian system on the phase space $\mathbb{R}^3_x \times \mathbb{R}^3_v$ with the Hamiltonian $H(x,v) = \frac{1}{2}v^2 + \gamma\Phi(x,v)$, it has the volume preserving propoerty. We, however, will show the volume preservation by computation of the Jacobian determinant for coordinates transformations through characteristic flows.

Lemma 1.3. *Let f be a classical solution of the Vlasov-Poisson system.*

i. *Fix t, x, v. The system of ordinary differential equations*

$$\dot{X}(s;t,x,v) = V(s;t,x,v), \quad \dot{V}(s;t,x,v) = \gamma E(t, X(s;t,x,v)),$$
$$X(t;t,x,v) = x, \qquad V(t;t,x,v) = v,$$

where the dot symbol denote the time derivative $\frac{d}{ds}$, has a solution (X,V) in $C^1([0,T],\mathbb{R}^6)$.

ii. *Fix t, x, v. Then, $f(s, X(s;t,x,v), V(s;t,x,v)) = \mathrm{const}$.*

iii. *Fix t, and let*

$$y(s;x,v) := X(s;t,x,v) \quad \text{and} \quad w(s;x,v) := V(s;t,x,v).$$

Then, the Jacobian of coordinates transform $(x,v) \mapsto (y,w)$ is 1 for all s.

Proof.

i. Note that we have

$$\rho \in C^1([0,T];C_c^1(\mathbb{R}^6)), \quad \Phi \in C^1([0,T];C^{2,\alpha}(\mathbb{R}^6))$$

so that

$$E \in C^1([0,T];C^{1,\alpha}(\mathbb{R}^6))$$

by the Hölder regularity of the Poisson equation. Since a map

$$(x,v) \mapsto (v, \gamma E(t,x))$$

is globally Lipschitz with respect to (x,v) for each t, we can apply the Picard Lindelöf theorem.

ii. Differentiate and use the chain rule to get

$$\begin{aligned}
&\frac{d}{ds} f(s,y,w) \\
&\quad = \partial_t f(s,y,w) + \dot{X}(s;s,y,w) \cdot \nabla_x f(s,y,w) + \dot{V}(s;s,y,w) \cdot \nabla_v f(s,y,w) \\
&\quad = \partial_t f(s,y,w) + w \cdot \nabla_x f(s,y,w) + \gamma E(s,y) \cdot \nabla_v f(s,y,w) = 0,
\end{aligned}$$

where we denote $y = X(s;t,x,v)$ and $w = V(s;t,x,v)$.

iii. Let $J(s) = \frac{\partial(y,w)}{\partial(x,v)}$ be the Jacobi matrix. Because when $s = t$ the Jacobian is

$$\det J(t) = \det \frac{\partial(x,v)}{\partial(x,v)} = 1,$$

We want to show

$$\det J(s) = \text{const}\,.$$

Since

$$J^{-1}(s) \frac{d}{ds} J(s) = \frac{\partial(x,v)}{\partial(y,x)} \frac{d}{ds} \frac{\partial(y,w)}{\partial(x,v)} = \frac{\partial(\dot{y},\dot{w})}{\partial(y,w)} = \begin{pmatrix} 0 & 1 \\ \gamma \frac{\partial E}{\partial y}(s,y) & 0 \end{pmatrix},$$

the Jacobi formula deduces that

$$\frac{d}{ds} \det J(s) = \det(s) \operatorname{tr} \left(J^{-1}(s) \frac{d}{ds} J(s) \right) = 0. \qquad \square$$

Corollary 1.4. *Let f be a classical solution of the Cauchy problem for the Vlasov-Poisson system. Then, for any measurable function $\beta : \mathbb{R} \to \mathbb{R}$ such that $\iint \beta \circ f_0(x,v)\,dv\,dx < \infty$, we have*

$$\iint \beta \circ f(t,x,v)\,dv\,dx = \text{const}\,.$$

In particular,

$$\|f(t)\|_p = \text{const}$$

for $1 \le p \le \infty$.

Proof. Fix $t, s[0,T]$ and denote $y = X(s;t,x,v)$ and $w = V(s;t,x,v)$. Then,

$$\begin{aligned}\iint \beta\circ f(t,x,v)\,dv\,dx &= \iint \beta\circ f(s, X(s;t,x,v), V(s;t,x,v))\,dv\,dx\\ &= \iint \beta\circ f(s,y,w)\,dw\,dy\end{aligned}$$

for $s \le T$. □

To sum up our weapons obtained so far, we have the following.

Corollary 1.5. *If a function $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ satisfies*

$$\iint f(t,x,v)\,dv\,dx = \text{const},$$

and if we let

$$\rho(t,x) = \int f(t,x,v)\,dv, \quad E(t,x) = \frac{1}{4\pi}\int \frac{(x-y)\rho(t,y)}{|x-y|^3}\,dy,$$

then

i. $\|\rho(t)\|_1 = \text{const}$,
ii. $\|E(t)\|_\infty \lesssim \|\rho(t)\|_\infty^{2/3}$,
iii. $\|\nabla E(t)\|_\infty \lesssim 1 + \|\rho\|_\infty \log^+ \|\nabla\rho\|_\infty$.

These estimates will be applied not only to the global existence proof, which assumes the local existence, but also to approximate solutions.

Remark. Note that the volume preservation is also yielded for a approximation scheme, which will be suggested in the next section, hence the same results in Corollary 1.4 for the approximate solutions in the same manner. The proof will be omitted.

1.3. **Conservation laws and moment propagation.** Usual algebraic computations with Stokes' theorem get several conservations laws, particularly including energy conservation.

Lemma 1.6. *Let f be a classical solution of the Vlasov-Poisson system.*

i. *(Continuity equation)*

$$\rho_t + \nabla_x\cdot j = 0, \qquad \textit{where} \quad j = \int vf\,dv.$$

ii. *(Energy conservation)*

$$\iint |v|^2 f\,dv\,dx + \gamma\int |E|^2\,dx = \text{const}\,.$$

Proof.

i. Integrate with respect to v to get

$$\begin{aligned} 0 &= \int f_t \, dv + \int v \cdot \nabla_x f \, dv \\ &= \rho_t + \nabla_x \cdot \int v f \, dv \\ &= \rho_t + \nabla_x \cdot j. \end{aligned}$$

ii. Multiply $|v|^2$ and integrate with respect to v and x to get

$$\begin{aligned} \frac{d}{dt} \iint |v|^2 f \, dv \, dx &= \iint |v|^2 f_t \, dv \, dx = -\iint |v|^2 \gamma E \cdot \nabla_v f \, dv \, dx \\ &= \iint 2v \cdot \gamma E f \, dv \, dx = -2\gamma \int \nabla_x \Phi \cdot j \, dx \\ &= 2\gamma \int \Phi \nabla_x \cdot j \, dx = 2\gamma \int \Phi \Delta_x \Phi_t \, dx \\ &= -\frac{d}{dt} \gamma \int |E|^2 \, dx. \end{aligned}$$

Thus

$$\iint |v|^2 f \, dv \, dx + \gamma \int |E|^2 \, dx = \text{const}. \qquad \square$$

Kinetic energy is a type of quantities which are called moments; we call the quantities of the form

$$\iint |v|^k f(t, x, v) \, dv \, dx$$

moments, with a positive real k. The energy conservation proves the bound of the 2-moment, kinetic energy,

$$\iint |v|^2 f(t, x, v) \, dv \, dx \lesssim 1$$

if $\gamma = +1$. In fact, a bound of kinetic energy exists even for $\gamma = -1$. As a corollary, the $L^{5/3}$ norm of mass density $\|\rho\|_{5/3}$ gets bounded.

Lemma 1.7 (Bound for kinetic energy)**.** *Let $f \in C^1([0, T], C_c^1(\mathbb{R}^6))$ be a solution of the Vlasov-Poisson system. For $t \in [0, T]$,*

i. $\|\rho(t)\|_{5/3} \lesssim \iint |v|^2 f \, dv \, dx$.
ii. $\iint |v|^2 f \, dv \, dx \lesssim 1$.

Proof.

i. Note

$$\begin{aligned} \rho(t, x) = \int f(t, x, v) \, dv &\leq \int_{|v|<R} f \, dv + \frac{1}{R^2} \int_{|v| \geq R} |v|^2 f \, dv \\ &\lesssim R^3 + R^{-2} \int |v|^2 f \, dv. \end{aligned}$$

Set $R^3 = R^{-2} \int |v|^2 f\,dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f\,dv.$$

ii. It is trivial for $\gamma = +1$ from the energy conservation. Suppose $\gamma = -1$. By the Hardy-Littlewood-Sobolev inequality,

$$\frac{1}{p} + \frac{\alpha}{d} = \frac{1}{q}$$

for $p = 2$, $d = 3$, and $\alpha = 1$ implies $q = 6/5$, hence the bound of $\|E(t)\|_2$

$$\|E(t)\|_2 \simeq \|\frac{1}{|x|^{d-\alpha}} *_x \rho(t,x)\|_{L^2_x} \lesssim \|\rho(t)\|_{6/5}.$$

So, interpolation with Hölder's inequality gives

$$\|E(t)\|_2 \lesssim \|\rho\|_1^{7/12} \|\rho\|_{5/3}^{5/12} \simeq \|\rho\|_{5/3}^{5/12}.$$

Thus (1) gives

$$\iint |v|^2 f\,dv\,dx = c + \|E(t)\|_2^2 \lesssim c + (\iint |v|^2 f\,dv\,dx)^{1/2},$$

so the kinetic energy $\iint f\,dv\,dx$ is bounded. □

Remark. If we had a bound of higher moment

$$\iint |v|^k f(t,x,v)\,dv\,dx \lesssim 1$$

for some $k > 6$ so that $\|\rho(t)\|_p \lesssim 1$ for some $p = \frac{k+3}{3} > 3$, then we would obtain

$$\|E(t)\|_\infty^{1-\frac{1}{p}} \lesssim \|\rho\|_p^{\frac{2}{3}} \|\rho\|_1^{\frac{1}{3}-\frac{1}{p}} \lesssim 1.$$

We will see that this estimate proves the global existence immediately; this is the idea of the paper of Lions and Perthame[]. We do not cover this in detail.

2. Local existence

The proof of local existence follows the following steps:

(1) construction of an approximate solution,
(2) establishment of a priori estimates,
(3) (subsequential) convergence of the approximate solution,
(4) verification of the solvability for the limit.

The Vlasov-Poisson system is good enough to show cdirect convergence of approximate solutions, not in the sense of subsequences.

2.1. **Approximate solution.**

Definition 2.1. We define an (global) *approximate solution* as a sequence of functions $f_n \in C^1(\mathbb{R}^+; C^1_c(\mathbb{R}^6))$ such that

$$\begin{cases} \partial_t f_{n+1} + v \cdot \nabla_x f_{n+1} + \gamma E_n \cdot \nabla_v f_{n+1} = 0, \\ E_n(t,x) = -\nabla_x \Phi_n, \\ \Phi_n(t,x) = (-\Delta_x)^{-1} \rho_n, \qquad \lim_{|x|\to\infty} \Phi(t,x) = 0, \\ \rho_n(t,x) = \int f_n \, dv, \\ f_{n+1}(0,x,v) = f_0(x,v). \end{cases}$$

This definition is made in order to let the force field E constant when solving f_{n+1}. Note that it assumes for f_0 to be automatically C^1_c by definition.

Proposition 2.1. *An approximate solution exists for given initial term $f_0 \in C^1_c(\mathbb{R}^6)$.*

Proof. Let $f_0(t,x,v) = f_0(x,v)$. Notice that f_0 is clearly in $C^1(\mathbb{R}^+; C^1_c(\mathbb{R}^6))$. Assume $f_n \in C^1(\mathbb{R}^+; C^1_c(\mathbb{R}^6))$ satisfies the approximate system. We want to show that there is f_{n+1} that satisfies the approximate system and $f_{n+1} \in C^1(\mathbb{R}^+; C^1_c(\mathbb{R}^6))$.

We have for $0 < \alpha < 1$ that

$$\rho_n \in C^1(\mathbb{R}^+; C^1_c(\mathbb{R}^6)), \quad \Phi_n \in C^1(\mathbb{R}^+; C^{2,\alpha}(\mathbb{R}^6)), \text{ and } E_n \in C^1(\mathbb{R}^+; C^{1,\alpha}(\mathbb{R}^6))$$

by the Hölder regularity of the Poisson equation. Since a map $(x,v) \mapsto (v, \gamma E_n(t,x))$ is globally Lipschitz with respect to (x,v) for each t, the classical Picard iteration uniquely solves the characteristic equation

$$\begin{cases} \dot{X}_{n+1}(s;t,x,v) = V_{n+1}(s,t,x,v) \\ \dot{V}_{n+1}(s;t,x,v) = \gamma E_n(s, X_{n+1}(s;t,x,v)) \end{cases}$$

with condition $(X_{n+1}(t;t,x,v), V_{n+1}(t;t,x,v)) = (x,v)$, and proves the uniqueness and regularity of the solution $s \mapsto (X_{n+1}, V_{n+1})(s;t,x,v) \in C^1(\mathbb{R}^+, \mathbb{R}^6)$.

Define

$$f_{n+1}(t,x,v) := f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)).$$

Then, f_{n+1} is clearly C^1, and we can show that

$$\begin{aligned} &f_{n+1}(s, X_{n+1}(s;t,x,v), V_{n+1}(s;t,x,v)) \\ &\quad = f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)) = \text{const} \end{aligned}$$

and that f_{n+1} satisfies the approximate system by the chain rule

$$\begin{aligned} 0 &= \frac{d}{ds} f_{n+1}(s, X_{n+1}(s;t,x,v), V_{n+1}(s;t,x,v)) \Big|_{s=t} \\ &= \partial_t f_{n+1}(t,x,v) + \dot{X}_{n+1}(t;t,x,v) \cdot \nabla_x f_{n+1}(t,x,v) \\ &\qquad + \dot{V}_{n+1}(t;t,x,v) \cdot \nabla_v f_{n+1}(t,x,v) \\ &= \partial_t f_{n+1}(t,x,v) + v \cdot \nabla_x f_{n+1}(t,x,v) + \gamma E_n(t,x) \cdot \nabla_v f_{n+1}(t,x,v). \end{aligned}$$

Also, f_{n+1} has compact support for each t since the characteristic does not blow up; finally we have $f_{n+1} \in C^1(\mathbb{R}^+, C^1_c(\mathbb{R}^6))$. □

Remark. Although the approximate solution is unique when given the initial term $f_0(t,x,v) = f_0(x,v)$, we do not care of its uniqueness, but only the existence.

In this section, we fix an approximate solution f_n.

2.2. **Local a priori estimates.** Recall that the characteristic curves of f_n are solutions of the system

$$\begin{cases} \dot{X}_{n+1}(s;t,x,v) = V_{n+1}(s,t,x,v) \\ \dot{V}_{n+1}(s;t,x,v) = \gamma E_n(s, X_{n+1}(s;t,x,v)). \end{cases}$$

Firstly, the volume preserving property still holds for our approximate system. Therefore, we have

$$\|\rho_n(t)\|_1 = \text{const}, \quad \|f_n(t)\|_p = \text{const}\,.$$

Next, we prove local-time bounds on fields E_n and its spatial derivative $\nabla_x E_n$. The bounds crucially act in the proof of convergence of f_n. Note that $\nabla_x E_n$ is a gradient of a vector field E_n, which is 9-dimensional.

Introduce the following quantity.

Definition 2.2. Define the *velocity support* or *maximal velocity*

$$Q_n(t) = \sup\{\, |v| : f_n(s,x,v) \neq 0,\ s \in [0,t]\,\}.$$

In particular, Q_0 is independent on t.

Lemma 2.2. *Let $T > 0$ be a constant such that*

$$T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}.$$

Then, we have the following bounds:

i. *For $t \le T$,*

$$\|\rho_n(t)\|_\infty + \|E_n(t)\|_\infty + Q_n(t) \lesssim 1$$

indendent on n. In addition, the support of X_n is also uniformly bounded in $t \le T$.

ii. *For $t \le T$,*

$$\|\nabla_x \rho_n(t)\|_\infty + \|\nabla_x E_n(t)\|_\infty \lesssim 1$$

independent on n.

The dynamics of controls among uniform norms of each quantity including ρ and E can be summarized as follows:

$$\log\|E(t)\|_\infty \quad \underset{\substack{\text{Elliptic regularity of} \\ \text{Poisson's eqn}}}{\lesssim} \quad \log\|\rho(t)\|_\infty \quad \lesssim \quad \log Q(t),$$

and

$$Q(t) \quad \underset{\text{By def}}{\lesssim} \quad |(X,V)(t)| \quad \underset{\substack{\text{Equations of} \\ \text{characteristics}}}{\lesssim} \quad \int_0^t (1 + \|E(s)\|_\infty)\, ds.$$

Then, Gronwall's inequality saves the game for the bound of Q. Also, we can observe that every norms are controlled by the velocity support Q. For detail explanations, see the following proof.

Proof.

i. Since
$$\|\rho_n(t)\|_\infty \le Q_n^3(t)\|f_0\|_\infty,$$
a rough estimate for $\|E\|_\infty$ gives
$$\|E_n(t)\|_\infty \le \|\rho_n(t)\|_\infty^{2/3}\|\rho_n(t)\|_1^{1/3} \le Q_n^2(t)\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3}.$$
Let $c(f_0) = \|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3}$ be a constant such that $\|E_n(t)\| \le cQ_n^2(t)$. We claim that
$$Q_n(t) \le \frac{Q_0}{1 - cQ_0 t}$$
for all n. Easily checked for $n = 0$; $Q_0(t) \equiv Q_0 \le \frac{Q_0}{1-cQ_0t}$.

Assume $Q_n(t) \le \frac{Q_0}{1-cQ_0t}$ for $t \in [0,T]$. Let $f_0(x,v) \ne 0$. Since
$$\begin{aligned} X_{n+1}(t;0,x,v) &= x + \int_0^t V_{n+1}(s';t,x,v)\,ds', \\ V_{n+1}(t;0,x,v) &= v + \int_0^t \gamma E_n(s', X_{n+1}(s;t,x,v))\,ds', \end{aligned}$$
an inequality
$$\begin{aligned} |V_{n+1}(t;0,x,v)| &\le |v| + \int_0^t |E_n(s;0,x,v)|\,ds \\ &\le Q_0 + c\int_0^t Q_n^2(s)\,ds \end{aligned}$$
implies
$$\begin{aligned} Q_{n+1}(t) &\le Q_0 + c\int_0^t Q_n^2(s)\,ds \\ &\le Q_0 + c\int_0^t \left(\frac{Q_0}{1-cQ_0s}\right)^2 ds = \frac{Q_0}{1-cQ_0t}. \end{aligned}$$
By induction, $Q_n(t) \le \frac{Q_0}{1-cQ_0t} \lesssim 1$ for all n and $t \in [0,T]$, where $T < (cQ_0)^{-1}$. Furthermore,
$$\|\rho_n(t)\|_\infty \lesssim Q_n^3(t) \lesssim 1, \quad \|E_n(t)\|_\infty \lesssim Q_n^2(t) \lesssim 1.$$
The position support is bounded because
$$|X_n(t;0,x,v)| \le |x| + \int_0^t |V_n(s;0,x,v)|\,ds \le |x| + TQ_n(t) \lesssim 1.$$

ii. Since we already have bounds for $\|\rho_n\|_\infty$ and $\|\rho_n\|_1$, what we should estimate in order to bound $\|\nabla_x E_n\|_\infty$ is $\nabla_x \rho_n$. To do this, we will consider $\nabla_x X_n$ and $\nabla_x V_n$. In particular, we have

$$\begin{aligned}\nabla_x X_n(t;t,x,v) &= \nabla_x x = (1,0,0\,;0,1,0\,;0,0,1),\\ \nabla_x V_n(t;t,x,v) &= \nabla_x v = 0.\end{aligned}$$

Two inequalities

$$\begin{aligned}|\nabla_x X_{n+1}(s;t,x,v)| &= \Big|\underbrace{(1,0,\cdots,0,1)}_{9} - \int_s^t \nabla_x V_{n+1}(s';t,x,v)\,ds'\Big|\\ &\le \sqrt{3} + \int_s^t |\nabla_x V_{n+1}(s';t,x,v)|\,ds'\end{aligned}$$

and

$$\begin{aligned}|\nabla_x V_{n+1}(s;t,x,v)| &= |\int_s^t \nabla_x E_n(s', X_{n+1}(s';t,x,v))\,ds'|\\ &\le \int_s^t |\nabla_x X_{n+1}(s';t,x,v)| \cdot \|\nabla_x E_n(s')\|_\infty\,ds'\end{aligned}$$

are combined as

$$\begin{aligned}&|\nabla_x X_{n+1}(s;t,x,v)| + |\nabla_x V_{n+1}(s;t,x,v)|\\ \le{}& \sqrt{3} + \int_s^t (1+\|\nabla_x E_n(s')\|_\infty)(|\nabla_x X_{n+1}(s';t,x,v)| + |\nabla_x V_{n+1}(s';t,x,v)|)\,ds'.\end{aligned}$$

By the Gronwall inequality, we get

$$|\nabla_x X_{n+1}(s;t,x,v)| + |\nabla_x V_{n+1}(s;t,x,v)| \le \sqrt{3}\,e^{\int_s^t (1+\|\nabla_x E_n(s')\|_\infty)\,ds'}$$

for $0 \le s \le t$. Thus we have

$$\begin{aligned}|\nabla_x \rho_{n+1}(t,x)| &= |\int \nabla_x f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v))\,dv|\\ &\le \|\nabla_{x,v} f_0\|_\infty \int (|\nabla_x X_{n+1}(0;t,x,v)| + |\nabla_x V_{n+1}(0;t,x,v)|)\,dv\\ &\lesssim \|\nabla_{x,v} f_0\|_\infty Q_{n+1}^3(t) \cdot e^{\int_0^t (1+\|\nabla_x E_n(s)\|_\infty)\,ds}.\end{aligned}$$

Recall that

$$\begin{aligned}\|\nabla_x E_{n+1}(t)\| &\lesssim (1 + \|\rho_{n+1}(t)\|_\infty \log^+ \|\nabla_x \rho_{n+1}(t)\|_\infty + \|\rho_{n+1}(t)\|_1)\\ &\lesssim 1 + \log^+ \|\nabla_x \rho_{n+1}(t)\|_\infty\end{aligned}$$

for $t \le T$. By inserting the estimate for $|\nabla_x \rho_{n+1}(t,x)|$, we can find a constant $c = c(f_0)$ such that

$$1 + \|\nabla_x E_{n+1}(t)\|_\infty \le c(1 + \int_0^t (1 + \|\nabla_x E_n(s)\|_\infty)\,ds)$$

in $t \le T$, where $T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}$. Without loss of generality, we may assume that c satisfies

$$c \ge \sup_{s \in [0,T]} (1 + \|E_0(s)\|_\infty).$$

Then, induction obtains the bound

$$1 + \|E_n(t)\|_\infty \le ce^{ct} \le ce^{cT} \lesssim 1$$

for all n and $t \le T$. The derivative of mass density is bounded since

$$\|\nabla_x \rho_{n+1}(t)\|_\infty \lesssim e^{\int_0^t (1 + \|\nabla_x E_n(s)\|_\infty)\, ds} \lesssim 1. \qquad \square$$

2.3. **Convergence of approximate solution.** Although most of the nonlinear systems fail to have convergent approximate solutions so that compactness methods are often applied, the approximate solutions constructed and investigated in the previous subsections uniformly converges.

Lemma 2.3. *Let $T > 0$ be a constant such that*

$$T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}.$$

i. *For $t \le T$ and $n \ge 1$,*

$$\|f_{n+1}(t) - f_n(t)\|_\infty \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty \, ds.$$

ii. *For $s \le T$ and $n \ge 1$,*

$$\|E_n(s) - E_{n-1}(s)\|_\infty \lesssim \|f_n(s) - f_{n-1}(s)\|_\infty.$$

iii. *f_n converges to a function f uniformly in $C([0,T] \times \mathbb{R}^6)$.*

iv. *For each t, x, v a sequence of maps*

$$s \mapsto (X_n(s;t,x,v), V_n(s;t,x,v))$$

converges in $C^1([0,T], \mathbb{R}^6)$ so that its limit (X, V) satisfies the equations

$$\dot{X}(s;t,x,v) = V(s;t,x,v), \quad \dot{V}(s;t,x,v) = \gamma E(s, X(s;t,x,v)),$$

where

$$E(t,x) = \frac{1}{4\pi} \iint \frac{(x-y) f(t,y,v)}{|x-y|^3} \, dv \, dy.$$

Proof.

i. Denote

$$g(s) := |X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| + |V_{n+1}(s;t,x,v) - V_n(s;t,x,v)|$$

for given t, x, v. The C^1 regularity of f_0 gives

$$\begin{aligned}
&|f_{n+1}(t,x,v) - f_n(t,x,v)| \\
&\quad = |f_0(X_{n+1}(0;t,x,v), V_{n+1}(0;t,x,v)) - f_0(X_n(0;t,x,v), V_n(0;t,x,v))| \\
&\quad \lesssim |X_{n+1}(0;t,x,v) - X_n(0;t,x,v)| + |V_{n+1}(0;t,x,v) - V_n(0;t,x,v)| \\
&\quad = g(0).
\end{aligned}$$

If an inequality

$$\sup_{s\in[0,t]} g(s) \lesssim \int_0^t \|E_n(s) - E_{n-1}(s)\|_\infty \, ds$$

is obtained for $t \le T$, then we are done.

Let $0 \le s \le t \le T$. Because

$$X_n(s;t,x,v) = x - \int_s^t V_n(s';t,x,v)\, ds',$$
$$V_n(s;t,x,v) = v - \int_s^t \gamma E_{n-1}(s', X_n(s;t,x,v))\, ds',$$

we have two inequalities

$$\begin{aligned}
&|V_{n+1}(s;t,x,v) - V_n(s;t,x,v)| \\
&\quad \le \int_s^t |E_n(s', X_{n+1}(s';t,x,v)) - E_{n-1}(s', X_n(s';t,x,v))|\, ds' \\
&\quad \le \int_s^t (|E_n(s', X_{n+1}) - E_n(s', X_n)| + |E_n(s', X_n) - E_{n-1}(s', X_n)|)\, ds' \\
&\quad \le \int_s^t (\|\nabla_x E_n(s')\|_\infty |X_{n+1}(s') - X_n(s')| + \|E_n(s') - E_{n-1}(s')\|_\infty)\, ds'
\end{aligned}$$

and

$$|X_{n+1}(s;t,x,v) - X_n(s;t,x,v)| \le \int_s^t |V_{n+1}(s';t,x,v) - V_n(s';t,x,v)|\, ds'$$

for $s \in [0,t]$. By combining the two inequalities above, we get

$$g(s) \le \int_s^t a(s')g(s')\, ds' + \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty \, ds', \tag{2}$$

where $a(s) := 1 + \|\nabla_x E_n(s)\|_\infty$.

Here we use a Gronwall-type inequality to bound $g(s)$. In more detail, multiplying

$$a(s)e^{-\int_s^t a(s')ds'}$$

on the both-hand-side of (2), and using $a \lesssim 1$ in $t \le T$, we have

$$\begin{aligned}
-\frac{d}{ds}&\left(e^{-\int_s^t a(s')\, ds'} \int_s^t a(s')g(s')\, ds'\right) \\
&\le a(s)e^{-\int_s^t a(s')ds'} \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty \, ds' \\
&\lesssim \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty \, ds'
\end{aligned}$$

Integrate from s to t and bound $(t-s) \le T \lesssim 1$ to get

$$e^{-\int_s^t a(s')\, ds'} \int_s^t a(s')g(s')\, ds' \lesssim \int_s^t \|E_n(s') - E_{n-1}(s')\|_\infty \, ds'. \tag{3}$$

Since $e^{\int_s^t a(s')\,ds'} \le e^{T \sup_{s\in[0,t]} a(s)} \lesssim 1$, the inequalities (2) and (3) implies

$$g(s) \lesssim \int_0^t \|E_n(s') - E_{n-1}(s')\|_\infty \, ds' \tag{4}$$

for arbitrary $s \in [0, t]$.

ii. Notice that

$$\|E_n(t) - E_{n-1}(t)\|_\infty \lesssim \|\rho_n(t) - \rho_{n-1}(t)\|_1^{1/3} \|\rho_n(t) - \rho_{n-1}(t)\|_\infty^{2/3}.$$

For L^∞-norm,

$$\begin{aligned}\|\rho_n(t) - \rho_{n-1}(t)\|_\infty &\le \max\{Q_n^3(t), Q_{n-1}^3(t)\}\|f_n(t) - f_{n-1}(t)\|_\infty \\ &\lesssim \|f_n(t) - f_{n-1}(t)\|_\infty.\end{aligned}$$

For L^1-norm, since the support of f_n, f_{n-1} is bounded in both directions x, v in finite time,

$$\|\rho_n(t) - \rho_{n-1}(t)\|_1 \le \|f_n(t) - f_{n-1}(t)\|_1 \lesssim \|f_n(t) - f_{n-1}(t)\|_\infty$$

for $t \le T$, where $T < \infty$ arbitrary.

iii. From (i) and (ii), there is a constant $c = c(f_0)$ such that for $t < T$,

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le c \int_0^t \|f_n(s) - f_{n-1}(s)\|_\infty \, ds.$$

We can easily get with induction

$$\|f_{n+1}(t) - f_n(t)\|_\infty \le M \frac{(ct)^n}{n!},$$

where $M = \sup_{s\in[0,T]} \|f_1(s) - f_0(s)\|_\infty$. Therefore,

$$\sum_{n=0}^\infty \|f_{n+1} - f_n\|_\infty \le \sup_{t\in[0,T]} M e^{ct} < \infty$$

implies f_n uniformly converges in $C([0, T] \times \mathbb{R}^6)$.

iv. Write

$$X(s) = X(s; t, x, v), \qquad V(s) = V(s; t, x, v)$$

for given t, x, v. Recall that g measures the difference between (X_{n+1}, V_{n+1}) and (X_n, V_n). By the inequality (4) and the result in (ii),

$$\sup_{s\in[0,T]} g(s) \lesssim \int_0^T \|E_n(s) - E_{n-1}(s)\|_\infty \lesssim T\|f_n - f_{n-1}\|_\infty.$$

Then, the uniform convergence of characteristics (X_n, V_n) is clear by the absolute convergence of the series $\sum \|f_{n+1} - f_n\|_\infty$.

Also for the uniform convergence of $(\dot{X}_n, \dot{V}_n)$, it is proved by the absolute convergence of the series $\sum \|f_{n+1} - f_n\|_\infty$ since

$$\begin{aligned}&\|\dot{X}_{n+1} - \dot{X}_n\|_\infty = \|V_{n+1} - V_n\|_\infty, \\ &\|\dot{V}_{n+1} - \dot{V}_n\|_\infty \le \|\nabla_x E_n\|_\infty \|X_{n+1} - X_n\|_\infty + \|E_n - E_{n-1}\|_\infty,\end{aligned}$$

yielding

$$\|\dot{X}_{n+1} - \dot{X}_n\|_\infty + \|\dot{V}_{n+1} - \dot{V}_n\|_\infty \lesssim \|f_n - f_{n-1}\|_\infty.$$

Then, by limiting the both-hand-side of equations

$$\dot{X}_n(s) = V_n(s), \qquad \dot{V}_n(s) = \gamma E_{n-1}(s, X_n(s)),$$

we easily get

$$\dot{X}(s) = V(s), \qquad \dot{V}(s) = \gamma E(s, X(s)). \qquad \square$$

Theorem 2.4 (Local existence)**.** *Let f_n be an approximate solution. Then, there is a constant $T = T(f_0) > 0$ be a constant such that the limit f of f_n is a classical solution of the Cauchy problem for the Vlasov-Poisson system with time domain $[0, T]$.*

Proof. Take T such that $T < (Q_0\|f_0\|_\infty^{2/3}\|f_0\|_1^{1/3})^{-1}$. Let $X(s; t, x, v)$ and $V(s; t, x, v)$ be the limits of $X_n(s; t, x, v)$ and $V_n(s; t, x, v)$ for given t, x, v. Notice that

$$\begin{aligned} f(t, x, v) = \lim_{n\to\infty} f_n(t, x, v) &= \lim_{n\to\infty} f_0(X_n(0; t, x, v), V_n(0; t, x, v)) \\ &= f_0(X(0; t, x, v), V(0; t, x, v)), \end{aligned}$$

which shows f is C^1 since f_0 and (X, V) are C^1. We can check it solves the system by expand the right-hand-side of

$$0 = \frac{d}{ds} f(s, X(s; t, x, v), V(s; t, x, v))|_{s=t}$$

using the chain rule. The compact support is by the fact that characteristic curves do not blow up. $\square$

2.4. **Uniqueness.**

Theorem 2.5 (Uniqueness)**.** *Suppose $f_1, f_2 \in C^1([0, T]; C_c^1(\mathbb{R}^6))$ are classical solutions of the Cauchy problem for the Vlasov-Poisson system with a common initial data f_0. Then, $f_1 = f_2$.*

Proof. As we did in (i) and (ii) of Lemma 2.3, we can obtain

$$\|f_1(t) - f_2(t)\|_\infty \lesssim \int_0^t \|f_1(s) - f_2(s)\|_\infty \, ds$$

for $t \le T$. By the Gronwall lemma, we get

$$\int_0^t \|f_1(s) - f_2(s)\|_\infty \le 0. \qquad \square$$

2.5. **Prolongation criterion.** We give in this last subsection a sufficient condition for a local classical solution f to be global.

Definition 2.3. Let $f \in C^1([0, T]; C_c^1(\mathbb{R}^6))$. Define for $t \in [0, T]$

$$Q(t) := \sup\{\, |v| : f(s, x, v) \ne 0, \; s \in [0, t] \,\}.$$

Proposition 2.6. *Let f be a classical solution of the Cauchy problem for the Vlasov-Poisson system. If $Q(t)$ is bounded, then f is continued to a classical solution with a longer time interval.*

Proof. Suppose $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$. Since Q does not blow up, we may define

$$Q(T) := \lim_{t \uparrow T} Q(t).$$

We are going to apply the local existence result for a new problem, in which we write $\widetilde{f}$ for the solution, with initial condition $\widetilde{f}(0,x,v) := f(t_0,x,v)$ for some $t_0 < T$. In subsection 2.3, we have shown the length of time interval for existence T is given by the condition

$$T < (Q_0 \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}.$$

It means that, if we arrange it for the new solution $\widetilde{f}$, the interval of existence of $\widetilde{f}$ has in fact a lower bound $\widetilde{T} > 0$ that depends only on $Q(T)$ for any new initial time t_0; it is because the monotonicity of Q says $Q(T)^{-1} < Q(t_0)^{-1}$ and the volume preservation implies $\|f_0\|_\infty = \|f(t_0)\|_\infty$ and $\|f_0\|_1 = \|f(t_0)\|_1$. In other words, we can take any $\widetilde{T}$ such that

$$\widetilde{T} < (Q(T) \|f_0\|_\infty^{2/3} \|f_0\|_1^{1/3})^{-1}.$$

Note that the bound does not depend on t_0 but only on its upper bound T.

Set $t_0 = T - \frac{1}{2}\widetilde{T}$ so that $t_0 < T < t_0 + \widetilde{T}$. Then, we can construct a new solution in $C^1([0, t_0 + \widetilde{T}), C_c^1(\mathbb{R}^6))$ by pasting solutions $f \in C^1([0,T], C_c^1(\mathbb{R}^6))$ and $\widetilde{f} \in C^1([t_0, t_0 + \widetilde{T}], C_c^1(\mathbb{R}^6))$. $\square$

Corollary 2.7. *If the following statement holds, then Theorem 1.1 is true: every (local) classical solution f with a given initial data $f_0 \in C_c^1(\mathbb{R}^6)$ satisfies $Q(t) \le h(t)$ for a continuous function $h : [0,\infty) \to [0,\infty)$.*

Proof. Suppose $f \in C^1([0,T_{max}), C_c^1(\mathbb{R}^6))$ for $T_{max} < \infty$ is the maximal solution with initial data f_0. Since Q is bounded on $[0, T_{max}]$, we can apply the previous proposition, which contradicts to the maximality of $T_{\max}$. Hence $T_{max} = \infty$, and the solution f is prolonged forever. $\square$

3. Global existence

Theorem (Schaeffer, 1991)**.** *Let $f_0 \in C_c^1(\mathbb{R}^6, [0,\infty))$ and $p > \frac{33}{17}$. The classical solution f of the Cauchy problem for the Vlasov-Poisson system with an initial data f_0 has a constant $c = c(f_0, p)$ such that*

$$Q(t) \le c(1+t)^p.$$

3.1. **Time averaging.** Fix a (local) classical solution f. If we had an integral inequality

$$Q(t) - Q(t-\Delta) \lesssim \int_{t-\Delta}^{t} Q(s)^a \, ds$$

for some constant $0 \le a \le 1$, then we would be able to prove that

$$Q(t) \lesssim \begin{cases} (1+t)^{\frac{1}{1-a}} & , 0 \le a < 1 \\ e^{ct} & , a = 1 \end{cases} \tag{5}$$

using a nonlinear Gronwall inequality. To obtain this, we may try as follows: if we got an estimate on the field

$$\|E(t)\|_\infty \lesssim Q(t)^a,$$

then for any $t, \widehat{x}, \widehat{v}$ such that $f(t, \widehat{x}, \widehat{v}) \neq 0$ and for any $\Delta > 0$ we have

$$|v - V(t-\Delta; t, \widehat{x}, \widehat{v})| = |\int_{t-\Delta}^t \gamma E(s, X(s; t, \widehat{x}, \widehat{v}))\, ds| \lesssim \int_{t-\Delta}^t Q(s)^a\, ds,$$

so there would be a constant $c = c(f_0)$ such that

$$|v| \le |V(t-\Delta; t, \widehat{x}, \widehat{v})| + c \int_{t-\Delta}^t Q(s)^a\, ds \le Q(t-\Delta) + c \int_{t-\Delta}^t Q(s)^a\, ds,$$

which deduces

$$Q(t) \le Q(t-\Delta) + c \int_{t-\Delta}^t Q(s)^a\, ds.$$

However, an optimized version of the estimate in (i) of Lemma 1.2 that uses $\|\rho(t)\|_{5/3} \lesssim 1$ gives only

$$\|E(t)\|_\infty \lesssim \|\rho(t)\|_\infty^{4/9} \|\rho(t)\|_{5/3}^{5/9} \lesssim (Q(t)^3)^{4/9} \cdot 1^{5/9} = Q(t)^{4/3},$$

so we need another approach because $4/3 > 1$. Our strategy is to average in the time direction. Precisely, we estimate the averaged field

$$\frac{1}{\Delta} \int_{t-\Delta}^t |E(s, X(s; t, \widehat{x}, \widehat{v}))|\, ds \lesssim Q(t)^a$$

for arbitrary $t, \widehat{x}, \widehat{v}$ and for suitably chosen Δ. Then, we would get a weaker inequality

$$Q(t) - Q(t-\Delta) \lesssim \Delta \cdot Q(t)^a,$$

which is also able to deduce (5). The detailed proof of (5) will be presented in Section 3.5.

Notation. Fix $(t, \widehat{x}, \widehat{v}) \in \mathbb{R}^+ \times \mathbb{R}^6$. We will write

$$\widehat{X}(s) := X(s; t, \widehat{x}, \widehat{v}) \quad \text{and} \quad \widehat{V}(s) := V(s; t, \widehat{x}, \widehat{v}).$$

Also, we will use the notations

$$X(s) := X(s; t, x, v) \quad \text{and} \quad V(s) := V(s; t, x, v)$$

assuming x, v are given without any confusion. Symbols y and w are always used for $X(s)$ and $V(s)$ at time s especially when applying volume preserving coordinates transformation $(x, v) \mapsto (X(s), V(s)) = (y, w)$.

3.2. **Estimate on the force field.** Our ultimate goal is to bound the integral

$$\begin{aligned}\int_{t-\Delta}^{t} |E(s,\widehat{X}(s))|\,ds &\le \frac{1}{4\pi}\int_{t-\Delta}^{t}\iint \frac{f(s,y,w)}{|y-\widehat{X}(s)|^2}\,dw\,dy\,ds \\ &= \frac{1}{4\pi}\int_{t-\Delta}^{t}\iint \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx\,ds.\end{aligned}$$

The main difficulty of this integral is that $|y-\widehat{X}(s)|^{-2}$ is not integrable with respect to y on the region where $|y|$ is large; the inverse square has too slow decay rate to be integrable in three dimesional space $\mathbb{R}^3$.

Then, we want to find a lower bound of the relative position vector $|X(s)-\widehat{X}(s)|$ assuming it is large. When the distance between $X(s)$ and $\widehat{X}(s)$ is sufficiently large so that the interaction between particles at positions $X(s)$ and $\widehat{X}(s)$ is sufficiently weak, the distance will change almost linearly in such a way that both velocity and time. Intuitively, we can write

$$|X(s)-\widehat{X}(s)| \gg 1 \quad\Longrightarrow\quad X(s)-\widehat{X}(s) \approx (v-\widehat{v})(s-c_1)+c_2,$$

where c_1, c_2 are constants that depend on $(t,x,v,\widehat{x},\widehat{v})$.

Then, the time function $|X(s)-\widehat{X}(s)|^{-2} \approx |(v-\widehat{v})(s-c_1)+c_2|^{-2}$ is integrable in time direction, and the time integration on interval $|(v-\widehat{v})(s-c_1)+c_2| \ge r$ for a proper radius r will be approximately $(|v-\widehat{v}|r)^{-1}$. For details, see Proposition 3.3. It means that the singularity issue of a spatial function is changed to an estimate problem for a function of velocity. Finally by taking r such that $(|v-\widehat{v}|r)^{-1} \lesssim |v|^2$, the kinetic energy bound the quantity what we want.

3.3. **Lemmas.** The following lemma suggests an appropriate way to choose the time averaging interval Δ.

Lemma 3.1. *Let $P>0$. Suppose $s \le [t-\Delta, t]$, where*

$$\Delta \cdot \sup_{s\in[0,t]} \|E(s)\|_\infty \le \frac{P}{4}.$$

i. *If $|v|<P$, then $|V(s)|<2P$.*
ii. *If $|v|\ge P$, then $\frac{1}{2}|v| \le |V(s)| \le 2|v|$.*
iii. *If $|v-\widehat{v}|<P$, then $|V(s)-\widehat{V}(s)|<2P$.*
iv. *If $|v-\widehat{v}|\ge P$, then $\frac{1}{2}|v-\widehat{v}| \le |V(s)-\widehat{V}(s)| \le 2|v-\widehat{v}|$.*

Proof. Note that

$$|v-V(s)| \le \int_s^t |E(s',X(s'))|\,ds' \le \Delta\cdot\sup_{s'\in[0,t]}\|E(s')\|_\infty \le \frac{P}{4},$$

and similarly

$$|\widehat{v}-\widehat{V}(s)| \le \frac{P}{4}.$$

For (i),

$$|V(s)| \le |v| + |v-V(s)| < P + \frac{P}{4} < 2P.$$

For (ii),

$$|V(s)| \geq |v| - |v - V(s)| \geq |v| - \frac{P}{4} \geq \frac{1}{2}|v|$$

and

$$|V(s)| \leq |v| + |v - V(s)| \leq |v| + \frac{P}{4} \leq 2|v|.$$

For (iii)

$$|V(s) - \widehat{V}(s)| \leq |V(s) - v| + |v - \widehat{v}| + |\widehat{v} - \widehat{V}(s)| < \frac{P}{4} + P + \frac{P}{4} < 2P.$$

For (iv),

$$|V(s) - \widehat{V}(s)| \geq -|V(s) - v| + |v - \widehat{v}| - |\widehat{v} - \widehat{V}(s)| \geq -\frac{P}{4} + |v - \widehat{v}| - \frac{P}{4} \geq \frac{1}{2}|v - \widehat{v}|$$

and

$$|V(s) - \widehat{V}(s)| \leq |V(s) - v| + |v - \widehat{v}| + |\widehat{v} - \widehat{V}(s)| \leq \frac{P}{4} + |v - \widehat{v}| + \frac{P}{4} \leq 2|v - \widehat{v}|.$$

□

From now for $0 \leq \Delta \leq t$, let it be such that

$$\Delta \cdot \sup_{s \in [0,t]} \|E(s)\|_\infty \leq \frac{P}{4}.$$

Lemma 3.2 (Lower bound of relative position vector)**.** *If v satisfies $|v - \widehat{v}| \geq P$, then there is $s_0 \in [t - \Delta, t]$ such that*

$$|X(s) - \widehat{X}(s)| \geq \frac{1}{4}|v - \widehat{v}||s - s_0|$$

for all $s \in [t - \Delta, t]$ and $x \in \mathbb{R}^3$.

Proof. Let $Z(s) := X(s) - \widehat{X}(s)$ be the relative position vector. Then,

$$Z'(s) = V(s) - \widehat{V}(s),$$
$$Z''(s) = \gamma[E(s, X(s), V(s)) - E(s, \widehat{X}(s), \widehat{V}(s))],$$

so the Taylor expansion at $s_0 \in [t - \Delta, t]$ gives

$$Z(s) = \big[Z(s_0) + Z'(s_0)(s - s_0)\big] + \left[\frac{Z''(\sigma)}{2}(s - s_0)^2\right]$$

for some σ between s, s_0.

Choose

$$s_0 = \operatorname*{arg\,min}_{s \in [t-\Delta,t]} |Z(s)|.$$

If $s_0 = t$ or $s_0 = t - \Delta$, then $\frac{d}{ds}|Z(s_0)|^2 \leq 0$ or $\frac{d}{ds}|Z(s_0)|^2 \geq 0$ respectively. Otherwise, $s_0 \in (t - \Delta, t)$, and $\frac{d}{ds}|Z(s_0)|^2 = 0$. Hence

$$Z(s_0) \cdot Z'(s_0)(s - s_0) = \frac{1}{2}\frac{d}{ds}|Z(s_0)|^2(s - s_0) \geq 0$$

for $s \in [t-\Delta, t]$, and

$$|Z(s_0) + Z'(s_0)(s-s_0)|^2 \geq |Z'(s_0)(s-s_0)|^2.$$

The condition $|v - \widehat{v}| \geq P$ implies

$$|Z'(s)| \geq \frac{1}{2}|v - \widehat{v}|$$

for $s \in [t-\Delta, t]$. Therefore,

$$|Z(s_0) + Z'(s_0)(s-s_0)| \geq |Z'(s_0)(s-s_0)| \geq \frac{1}{2}|v-\widehat{v}||s-s_0|,$$

and

$$\begin{aligned} \left|\frac{Z''(\sigma)}{2}(s-s_0)^2\right| &\leq \|E(t)\|_\infty (s-s_0)^2 \leq \|E(t)\|_\infty \Delta |s-s_0| \\ &\leq \frac{P}{4}|s-s_0| \leq \frac{1}{4}|v-\widehat{v}||s-s_0| \end{aligned}$$

yields

$$|X(s) - \widehat{X}(s)| = |Z(s)| \geq \frac{1}{4}|v-\widehat{v}||s-s_0|. \qquad \square$$

3.4. **Divide and conquer.** We estimate the integral of $|E(s, \widehat{X}(s))|$ by dividing the integral region $[t-\Delta, t] \times \mathbb{R}^6$ into three regions as:

$$\begin{aligned} U := \{\, (s,x,v) :\ & s \in [t-\Delta, t], \quad |v-\widehat{v}| \geq P, \\ & |X(s) - \widehat{X}(s)| \geq R \max\{|v|^{-3}, |v-\widehat{v}|^{-3}\}\,\}, \\ B := \{\, (s,x,v) :\ & s \in [t-\Delta, t], \quad |v-\widehat{v}| \geq P, \quad |v| \geq P, \\ & |X(s) - \widehat{X}(s)| < R \max\{|v|^{-3}, |v-\widehat{v}|^{-3}\}\,\}, \\ G := \{\, (s,x,v) :\ & s \in [t-\Delta, t], \quad \min\{|v-\widehat{v}|, |v|\} < P\,\} \\ = [t-\Delta&, t] \times \mathbb{R}^6 \setminus (U \cup B). \end{aligned}$$

The constants $P > 0$ and $R > 0$ will be determined later.

Proposition 3.3 (Ugly set estimate)**.**

$$\iiint_U \lesssim R^{-1}.$$

Proof. Write

$$r := R \max\{|v|^{-3}, |v-\widehat{v}|^{-3}\}.$$

Then,

$$U = \{\, (s,x,v) : s \in [t-\Delta, t], \quad |v-\widehat{v}| \geq P, \quad |X(s) - \widehat{X}(s)| \geq r\,\}.$$

Since $|X(s) - \widehat{X}(s)| \geq r$ on U,

$$\int_{|s-s_0| < \frac{4r}{|v-\widehat{v}|}} \frac{\chi_U(s,x,v)}{|X(s) - \widehat{X}(s)|^2}\, ds \leq \frac{1}{r^2} \int_{|s-s_0| < \frac{4r}{|v-\widehat{v}|}} ds = 8\frac{1}{|v-\widehat{v}|r}.$$

Since $|v-\widehat{v}| \geq P$ on U so that $|X(s)-\widehat{X}(s)| \geq \frac{1}{4}|v-\widehat{v}||s-s_0|$ by the previous lemma,

$$\begin{aligned}\int_{|s-s_0|\geq \frac{4r}{|v-\widehat{v}|}} \frac{\chi_U(s,x,v)}{|X(s)-\widehat{X}(s)|^2}\,ds &\leq 16\int_{|s-s_0|\geq \frac{4r}{|v-\widehat{v}|}} \frac{1}{|v-\widehat{v}|^2|s-s_0|^2}\,ds \\ &\leq 32\int_{4r}^{\infty} \frac{1}{|v-\widehat{v}|^3|s-s_0|^2} d(|v-\widehat{v}||s-s_0|) \\ &= 8\frac{1}{|v-\widehat{v}|r}.\end{aligned}$$

Therefore,

$$\int \frac{\chi_U(s,x,v)}{|X(s)-\widehat{X}(s)|^2}\,ds \lesssim \frac{1}{|v-\widehat{v}|r} = R^{-1}\frac{\min\{|v|^3,|v-\widehat{v}|^3\}}{|v-\widehat{v}|} \leq R^{-1}|v|^2$$

so that we have

$$\begin{aligned}\iiint_U \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx\,ds &= \iint f(t,x,v)\left(\int \frac{\chi_U(s,x,v)}{|X(s)-\widehat{X}(s)|^2}\,ds\right)dv\,dx \\ &\lesssim R^{-1}\iint |v|^2 f(t,x,v)\,dv\,dx \lesssim R^{-1}. \qquad \square\end{aligned}$$

Proposition 3.4 (Bad set estimate)**.**

$$\iiint_B \lesssim \Delta\cdot R\log\frac{4Q(t)}{P}.$$

Proof. Write

$$r := R\max\{|v|^{-3},|v-\widehat{v}|^{-3}\}.$$

Then,

$$B = \{\,(s,x,v) : s\in[t-\Delta,t],\quad |v|\geq P,\quad |v-\widehat{v}|\geq P,\quad |X(s)-\widehat{X}(s)|<r\,\}.$$

We need to control the union of two regions

$$|X(s)-\widehat{X}(s)| < R|v|^{-3} \quad\text{and}\quad |X(s)-\widehat{X}(s)| < R|v-\widehat{v}|^{-3}.$$

If we integrate $|X(s)-\widehat{X}(s)|^{-2}$ with respect to y on these regions, then we get integrands $|v|^{-3}$ and $|v-\widehat{v}|^{-3}$, which has singularities on regions at which $|v|$, $|v-\widehat{v}|$ are respectively small and large; an inverse cubic function is both sharp and broad in three dimensional space $\mathbb{R}^3$. The integral of inverse cube on the region $|v|\sim\infty$ is bounded by Q, and the region $|v|\sim 0$ is bounded by P.

For each $s\in[t-\Delta,t]$, if we apply the coordinates transformation $(x,v)\mapsto(y,w)=(X(s),V(s))$, then since $|v|\geq P$ implies

$$\frac{1}{2}P \leq \frac{1}{2}|v| \leq |w| \leq Q(s) \quad\text{and}\quad |w|\leq 2|v|,$$

we have

$$\begin{aligned}
&\int_{|v|\geq P}\int_{|X(s)-\widehat{X}(s)|<R|v|^{-3}} \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dx\,dv \\
&\qquad \lesssim \int_{\frac{1}{2}P\leq|w|\leq Q(s)}\int_{|y-\widehat{X}(s)|<R|V(t;s,y,w)|^{-3}} \frac{1}{|y-\widehat{X}(s)|^2}\,dy\,dw \\
&\qquad \simeq R\int_{\frac{1}{2}P\leq|w|\leq Q(t)} |V(t;s,y,w)|^{-3}\,dw \\
&\qquad \leq 8R\int_{\frac{1}{2}P\leq|w|\leq Q(t)} |w|^{-3}\,dw \\
&\qquad \simeq R\log\frac{2Q(t)}{P}.
\end{aligned}$$

Similarly but using the fact that $|v-\widehat{v}|\leq P$ implies

$$\frac{1}{2}P \leq \frac{1}{2}|v-\widehat{v}| \leq |w-\widehat{V}(s)| \leq 2Q(s) \quad \text{and} \quad |w-\widehat{V}(s)| \leq 2|v-\widehat{v}|,$$

we have

$$\int_{|v-\widehat{v}|\geq P}\int_{|X(s)-\widehat{X}(s)|<R|v-\widehat{v}|^{-3}} \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dx\,dv \lesssim R\log\frac{4Q(t)}{P}.$$

Therefore,

$$\iiint_B \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx\,ds \lesssim \Delta\cdot R\log\frac{4Q(t)}{P}.$$

□

Proposition 3.5 (Good set estimate)**.**

$$\iiint_G \lesssim \Delta\cdot P^{4/3}.$$

Proof. Note

$$\begin{aligned}
G = &\{\,(s,x,v) : s\in[t-\Delta,t], \quad |v|<P\} \\
&\cup \{\,(s,x,v) : s\in[t-\Delta,t], \quad |v-\widehat{v}|<P\}.
\end{aligned}$$

Since $|v|<P$ implies $|V(s;t,x,v)|<2P$, the coordinates transformation $(x,v)\mapsto (y,w)=(X(s),V(s))$ gives

$$\begin{aligned}
&\iint_{|v|<P} \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx \\
&\qquad \leq \int \frac{1}{|y-\widehat{X}(s)|^2}\int_{|w|<2P} f(s,y,w)\,dw\,dy \\
&\qquad \lesssim \|\int_{|w|<2P} f(s,y,w)\,dw\|_{L^\infty_y}^{4/9}\cdot\|\int_{|w|<2P} f(s,y,v)\,dv\|_{L^{5/3}_y}^{5/9} \\
&\qquad \lesssim ((2P)^3)^{4/9}\cdot 1^{5/9} \simeq P^{4/3}.
\end{aligned}$$

Similarly
$$\iint_{|v-\widehat{v}|<P} \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx \lesssim P^{4/3},$$
so we are done. $\square$

3.5. Bound on the velocity support.

3.5. Bound on the velocity support. Finally, with above estimates, we prove that Q does not blow up.

Corollary 3.6. *Let $c = c(f_0)$ be a constant such that*
$$\|E(s)\|_\infty \le cQ(s)^{4/3}$$
for all $s \in \mathbb{R}^+$, and define
$$\Delta := \frac{Q(t)^{4/11}}{4} \cdot \frac{1}{cQ(t)^{4/3}}.$$
If $\Delta < t$, then for any $a > 16/33$
$$Q(t) - Q(t-\Delta) \lesssim_a \Delta \cdot Q(t)^a.$$

Proof. Let $(d,e) = (4/11, 16/33)$ and
$$P = Q(t)^d \quad \text{and} \quad R = Q(t)^e(\log\frac{4Q(t)}{P})^{-1/2}.$$
Then, $\Delta \cdot cQ(t)^{4/3} = \frac{P}{4}$. Since
$$\Delta \cdot \sup_{s\in[0,t]} \|E(s)\|_\infty = \frac{P}{4} \cdot \frac{\sup_{s\in[0,t]} \|E(s)\|_\infty}{cQ(t)^{4/3}} \le \frac{P}{4},$$
we can use the estimates on U, B, and G :
$$\begin{aligned}\int_{t-\Delta}^t |E(s,\widehat{X}(s))|\,ds &\le \int_{t-\Delta}^t \iint \frac{f(t,x,v)}{|X(s)-\widehat{X}(s)|^2}\,dv\,dx\,ds \\ &\lesssim R^{-1} + \Delta\cdot R\log\frac{4Q(t)}{P} + \Delta\cdot P^{4/3} \\ &\simeq \Delta\cdot\left(Q(t)^{4/3}P^{-1}R^{-1} + R\log\frac{4Q(t)}{P} + P^{4/3}\right) \\ &= \Delta\cdot\left(Q(t)^{4/3-d-e}\sqrt{\log\frac{4Q(t)}{P}} + Q(t)^e\sqrt{\log\frac{4Q(t)}{P}} + Q(t)^{4d/3}\right).\end{aligned}$$
Because $d = 4/11$ and $e = 16/33$ satisfy
$$\frac{4}{3} - d - e = e = \frac{4}{3}d = \frac{16}{33},$$
we get
$$\int_{t-\Delta}^t |E(s,\widehat{X}(s))|\,ds \lesssim \Delta\cdot Q(t)^{16/33}\log^{1/2} Q(t)$$
and the desired result by setting $\widehat{x}$ and $\widehat{v}$ to be arbitrarily but $f(t,\widehat{x},\widehat{v}) \neq 0$. $\square$

Remark. Suppose $\Delta > 0$ had no lower bound. If we define an increasing function

$$j(q) := e^{\frac{1}{1-a}q^{1-a}},$$

then the inequality in the above corollary

$$Q(t) - Q(t-\Delta) \le c\Delta \cdot Q(t)^a$$

with $c = c(f_0, a)$ would give

$$\begin{aligned}\widetilde{Q}(t) - \widetilde{Q}(t-\Delta) &= j(Q(t)) - j(Q(t-\Delta))\\ &\le j(Q(t)) - j(Q(t) - c\Delta \cdot Q(t)^a)\\ &\le c\Delta \cdot Q(t)^a\ j'(Q(t))\\ &= c\Delta \cdot j(Q(t)) = c\Delta \cdot \widetilde{Q}(t),\end{aligned}$$

where $\widetilde{Q}(t) := j(Q(t))$. It derives a Gronwall-type differential inequality including the left lower Dini's derivative

$$D_-\widetilde{Q}(t) \lesssim_a \widetilde{Q}(t),$$

and this proves $\widetilde{Q}(t) \le \widetilde{Q}(0)e^{ct}$, which implies $Q(t) \lesssim (1+t)^{\frac{1}{1-a}}$. However, since there is a lower bound for Δ, we use another method to justify $Q(t) \lesssim (1+t)^{\frac{1}{1-a}}$.

Theorem (Schaeffer, 1991, restatement)**.** *For* $\frac{16}{33} < a < 1$*,*

$$Q(t) \lesssim_a (1+t)^{\frac{1}{1-a}}.$$

Proof. Let $c = c(f_0)$ be a constant such that $\|E(s)\|_\infty$ We do not fix t in this proof.

Define a function $\Delta : [0,\infty) \to [0,\infty)$ such that

$$\Delta(t)$$

$\square$