

C^* -algebras

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Definition. In this note, an *algebra* refers to a vector space over \mathbb{C} that has a pseudo-ring structure; always associative but possibly nonunital.

1. BANACH ALGEBRAS

1.1. Basics.

Definition 1.1. A normed algebra \mathcal{A} is called a *Banach algebra* if the norm induces complete metric space.

1.2. Spectra.

1.3. Holomorphic functional calculus.

1.4. Gelfand theory.

2. C^* -ALGEBRAS

2.1. Basics.

Definition 2.1. A normed $*$ -algebra \mathcal{A} is called a *C^* -algebra* if

- (1) \mathcal{A} is Banach,
- (2) \mathcal{A} satisfies the C^* -identity: $\|x^*x\| = \|x\|^2$.

Theorem 2.1. *Every nonunital C^* -algebra is a maximal ideal of a unital C^* -algebra.*

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Proof. Let \mathcal{A} be a nonunital C^* -algebra. It is enough to show the existence of unital C^* -algebra $\tilde{\mathcal{A}}$ such that \mathcal{A} is a normed $*$ -subalgebra of $\tilde{\mathcal{A}}$ with codimension one. It is because a subalgebra is a maximal ideal if and only if the quotient can have a natural ring structure that makes a field.

Step 1: Construct a unital normed $$ -algebra.* Since \mathcal{A} is a Banach space, the space of bounded operators $B(\mathcal{A})$ is a Banach algebra. We can recognize \mathcal{A} as a normed subalgebra of $B(\mathcal{A})$ because the left multiplication $(y \mapsto xy) \in B(\mathcal{A})$ has the norm

$$\|(y \mapsto xy)\| = \sup_{y \in \mathcal{A}} \frac{\|xy\|}{\|y\|}$$

that is shown to be equal to $\|x\|$ by putting $y = x^*$ and applying the C^* -identity. Define an algebra $\tilde{\mathcal{A}}$ as the subalgebra:

$$\tilde{\mathcal{A}} := \{ (y \mapsto xy + \lambda y) \in B(\mathcal{A}) : x \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Since $\tilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$ as algebras, let us write the map $y \mapsto xy + \lambda y$ as (x, λ) . Then, $\tilde{\mathcal{A}}$ is a normed $*$ -algebra with induced norm and involution

$$\|(x, \lambda)\| = \sup_{y \in \mathcal{A}} \frac{\|xy + \lambda y\|}{\|y\|}, \quad (x, \lambda)^* = (x^*, \bar{\lambda}).$$

Then, \mathcal{A} is a normed $*$ -subalgebra of $\tilde{\mathcal{A}}$ because the norm and involution of \mathcal{A} agree with $\tilde{\mathcal{A}}$.

Step 2: $\tilde{\mathcal{A}}$ is Banach. Suppose (x_n, λ_n) is Cauchy in $\tilde{\mathcal{A}}$. Since \mathcal{A} is complete so that it is closed in $\tilde{\mathcal{A}}$, we can induce a norm on the quotient $\tilde{\mathcal{A}}/\mathcal{A}$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|x\| \leq \|(x, \lambda)\| + |\lambda|$ shows that x_n is Cauchy in \mathcal{A} .

Since a finite dimensional normed space is always Banach and \mathcal{A} is Banach, λ_n and x_n converge. Finally, the inequality $\|(x, \lambda)\| \leq \|x\| + |\lambda|$ implies that (x_n, λ_n) converges.

Step 3: $\tilde{\mathcal{A}}$ is C^ .* The C^* -identity easily follows from the following inequality:

$$\begin{aligned} \|(x, \lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\ &= \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\| \\ &= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y)\| \\ &\leq \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y\| \\ &= \|(x, \lambda)^*(x, \lambda)\|. \end{aligned} \quad \square$$

2.1.1. Continuous functional calculus.

Theorem 2.2 (Gelfand-Naimark). *For a commutative unital C^* -algebra \mathcal{A} , the Gelfand transform gives an isometric $*$ -isomorphism $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$.*

Proof. Step 1: Γ is a $$ -homomorphism.* We will show $h(x^*) = \overline{h(x)}$ for linear characters $h \in \sigma(\mathcal{A})$. First assume that $x \in \mathcal{A}$ is self-adjoint.

By the holomorphic functional calculus,

$$e^{itx} = \sum_{n=1}^{\infty} \frac{(itx)^n}{n!}.$$

Since the involution is continuous,

$$(e^{itx})^* = \sum_{n=1}^{\infty} \frac{(-itx)^n}{n!} = e^{-itx},$$

so we have $\|e^{itx}\|^2 = \|e^{itx}e^{-itx}\| = 1$. Then, the inequality

$$1 = \|e^{itx}\| \geq |h(e^{itx})| = |e^{ith(x)}| = e^{-t \operatorname{Im} h(x)}$$

proves $h(x) \in \mathbb{R}$.

For arbitrary $x \in \mathcal{A}$, if we define self-adjoints

$$\operatorname{Re} x := \frac{x + x^*}{2}, \quad \operatorname{Im} x := \frac{x - x^*}{2i},$$

then

$$h(x^*) = h(\operatorname{Re} x) - ih(\operatorname{Im} x) = \overline{h(\operatorname{Re} x)} - i\overline{h(\operatorname{Im} x)} = \overline{h(\operatorname{Re} x) + ih(\operatorname{Im} x)} = \overline{h(x)}$$

for all $h \in \sigma(\mathcal{A})$.

Step 2: Γ is isometric. Note that we have

$$\|\widehat{x}\| = \sup_{h \in \sigma(\mathcal{A})} |\widehat{x}(h)| = \sup_{h \in \sigma(\mathcal{A})} |h(x)| = r(x).$$

For self adjoint $x \in \mathcal{A}$, since we have $\|x\|^2 = \|x^*x\| = \|x^2\|$, the spectral radius coincides with the norm by the Gelfand formula for spectral radius in Banach algebras:

$$r(x) = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

Hence

$$\|x\|^2 = \|x^*x\| = \|\widehat{x^*x}\| = \|\widehat{x^*}\widehat{x}\| = \|\widehat{x}\|$$

for arbitrary $x \in \mathcal{A}$.

Step 3: Γ is surjective. The step 1 shows that $\Gamma(\mathcal{A})$ is a unital *-subalgebra of $C(\sigma(\mathcal{A}))$, and it separates points by definition. By the Stone-Weierstrass theorem, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$. The step 2 shows that $\Gamma(\mathcal{A})$ is complete and hence closed so that $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A}))$. \square

Theorem 2.3 (Gelfand-Naimark). *For a commutative C*-algebra \mathcal{A} , the Gelfand transform gives an isometric *-isomorphism $\Gamma : \mathcal{A} \rightarrow C_0(\sigma(\mathcal{A}))$.*

2.2. Positive elements.

2.3. Gelfand-Naimark-Segal construction.

2.4. Operator topologies.

Theorem 2.4. *Let f be a linear functional on $B(H)$ for a Hilbert space H . Then, TFAE:*

- (1) f is WOT-continuous,
- (2) f is SOT-continuous,
- (3) $f(T) = \sum_{i=1}^n \langle Tx_i, y_i \rangle$ for some x_i, y_i .

Proof. (2) \Rightarrow (3) is the only nontrivial implication. By the definition of SOT, there exists $v \in \mathcal{H}^n$ such that

$$|f(T)| \leq \|T^{\oplus n} v\|.$$

The functional $f : \mathcal{A} \rightarrow \mathbb{C}$ factors through \mathcal{H}^n such that

$$\mathcal{A} \xrightarrow{v} \mathcal{H}^n \rightarrow \mathbb{C}.$$

□

3. VON NEUMANN ALGEBRAS

Theorem 3.1 (Double commutant theorem). *Let \mathcal{A} be a C^* -algebra in $\mathcal{B}(\mathcal{H})$. Then, $\mathcal{A}'' = \overline{\mathcal{A}}^{\text{WOT}}$.*

Proof. Since $\overline{\mathcal{A}}^{\text{WOT}}$ is closed convex, $\overline{\mathcal{A}}^{\text{SOT}} = \overline{\mathcal{A}}^{\text{WOT}}$. Also, \mathcal{A}'' is weakly closed, $\overline{\mathcal{A}}^{\text{WOT}} \subset \mathcal{A}''$. Let $T \in \mathcal{A}''$ and $v \in \mathcal{H}^n$. We want to show $Tv \in \overline{\mathcal{A}v}$.

Let P be the orthogonal projection on $\overline{\mathcal{A}v} \subset \mathcal{H}^n$. If $a \in \mathcal{A}$, then $a\overline{\mathcal{A}v} \subset \overline{\mathcal{A}v}$ because the multiplication by a is continuous on \mathcal{H}^n . It implies that $\text{ran}(aP) \subset \text{ran}(P)$ and $P(aP) = aP$ for all $a \in \mathcal{A}$. Because \mathcal{A} is closed under the involution, P commutes with a , so $P \in \mathcal{A}'$.

Then, $TP = PT$ implies $Tv = T(Pv) = P(Tv) \in \text{ran}(P) = \overline{\mathcal{A}v}$.

□