Contents

1 Kinetic theory

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 1.1 (Velocity averaging)**.** *Let T be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. Then,*

$$\| \int u\varphi \, dv \|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|^{1/2}_{L^2_{t,x,v}} \|Tu\|^{1/2}_{L^2_{t,x,v}}$$

for $\varphi \in C^\infty_c(\mathbb{R}^n_v)$,

Proof. Let $m(t,x) = \int u\varphi \, dv$. By Fourier transform with respect to t and x, we have

$$\widehat{u}(\tau,\xi,v) = i\,\frac{\widehat{Tu}(\tau,\xi,v)}{\tau + v\cdot\xi}$$

and

$$\widehat{m}(\tau,\xi) = \int \widehat{u}(\tau,\xi,v)\varphi(v)\,dv.$$

Fixing τ, ξ, decompose the integral and use Hölder's inequality to get

$$\begin{aligned} |\widehat{m}(\tau,\xi)| &\le \int_{|\tau+v\cdot\xi|<\alpha} |\widehat{u}\varphi|\,dv + \int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\widehat{Tu}\varphi|}{|\tau+v\cdot\xi|}\,dv \\ &\le \|\widehat{u}\|^{1/2}_{L^2_v} \Big(\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\,dv\Big)^{1/2} + \|\widehat{Tu}\|^{1/2}_{L^2_v} \Big(\int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|^2}\,dv\Big)^{1/2} \end{aligned}$$

We are going to estimate the integrals as

$$\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\,dv \lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}, \qquad \int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|}\,dv \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

We may assume that $\sqrt{\tau^2+|\xi|^2} \gg 1$, that is, it is enough to show them for $\sqrt{\tau^2+|\xi|^2} \ge C$ with arbitrarily taken constant C, because the case that $\sqrt{\tau^2+|\xi|^2} \lesssim 1$ easily proves the inequality.

Define coordinates (v_1, v_2) on $\mathbb{R}_v$ as follows:

$$v_1 := \frac{\tau + v\cdot\xi}{|\xi|} \ \in \mathbb{R}\ , \qquad v_2 := v - \frac{v\cdot\xi}{|\xi|^2}\xi \ \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v - \frac{\tau}{|\xi|})^2 + |v_2|^2 \qquad \text{and} \qquad \int dv = \iint dv_2\,dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| < R$. Then,

$$\begin{aligned}\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\, dv &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \int_{|v_2|^2 \le R^2-(v_1-\frac{\tau}{|\xi|})^2} dv_2\, dv_1 \\ &\lesssim (R^2 - \frac{\tau^2}{|\xi|^2})^{\frac{n-1}{2}} \cdot \frac{2\alpha}{|\xi|},\end{aligned}$$

where we value the term $(R^2 - \frac{\tau^2}{|\xi|^2})$ as 0 when $R^2 < \frac{\tau^2}{|\xi|^2}$. Since

$$(R^2 - \frac{\tau^2}{|\xi|^2})^{\frac{n-1}{2}} \frac{2\alpha}{|\xi|} \cdot \sqrt{\tau^2 + |\xi|^2} \lesssim \begin{cases} 0, & |\tau| \gg 1 \\ C, & |\xi| \gg 1 \end{cases},$$

we have

$$\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2\, dv \lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}.$$

For the second integral, suppose that φ is supported on $|v| < C$ so that $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < C$. Then,

$$\begin{aligned}\int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau + v\cdot\xi|}\, dv &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \int_{|v_2|<C} \frac{1}{v_1^2|\xi|^2}\, dv_2\, dv_1 \\ &\simeq \int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \frac{dv_1}{v_1^2|\xi|^2}.\end{aligned}$$

If $|\xi| \gtrsim |\tau|$, then

$$\begin{aligned}\int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \frac{dv_1}{v_1^2|\xi|^2} &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \frac{dv_1}{v_1^2|\xi|^2} \\ &\simeq \frac{1}{\alpha|\xi|} \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.\end{aligned}$$

If $|\xi| \ll |\tau|$ such that at least $|\tau| > C|\xi|$, then

$$\begin{aligned}\int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \frac{dv_1}{v_1^2|\xi|^2} &\lesssim \int_{|v_1-\frac{\tau}{|\xi|}|<C} \frac{dv_1}{v_1^2|\xi|^2} \\ &\simeq \frac{1}{|\xi|^2}\left(\frac{1}{\frac{\tau}{|\xi|} - C} - \frac{1}{\frac{\tau}{|\xi|} + C}\right) \\ &= \frac{2C}{\tau^2 - C^2|\xi|^2} \ll \frac{1}{\sqrt{\tau^2+|\xi|^2}},\end{aligned}$$

hence

$$\int_{|\tau+v\cdot\xi|\ge\alpha} \frac{|\varphi|^2}{|\tau + v\cdot\xi|}\, dv \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.$$

To sum up, we have

$$|\widehat{m}(\tau,\xi)| \lesssim \frac{1}{(\tau^2+|\xi|^2)^{1/4}}(\sqrt{\alpha}\cdot\|\widehat{u}\|_{L^2_v}^{1/2} + \frac{1}{\sqrt{\alpha}}\cdot\|\widehat{Tu}\|_{L^2_v}^{1/2}).$$

Letting $\alpha = \sqrt{\|\widehat{Tu}\|_{L^2_v}/\|\widehat{u}\|_{L^2_v}}$ and squaring,

$$(\tau^2+|\xi|^2)^{1/2}|\widehat{m}(\tau,\xi)|^2 \lesssim \|\widehat{u}\|_{L^2_v}^{1/2}\|\widehat{Tu}\|_{L^2_v}^{1/2}.$$

Therefore, the integration on $\mathbb{R}_\tau \times \mathbb{R}^n_\xi$ and Plancheral's theorem gives

$$\|m\|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|_{L^2_{t,x,v}}^{1/2}\|Tu\|_{L^2_{t,x,v}}^{1/2}.$$

$\square$

Corollary 1.2. *Let $\mathcal{F}$ be a family of functions on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ and $T\mathcal{F}$ are bounded in $L^2_{t,x,v}$, then $\int \mathcal{F}\varphi\, dv$ is bounded in $H^{1/2}_{t,x}$.*

Theorem 1.3. *Let $\mathcal{F}$ be a family of functions on $I_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ is weakly relatively compact and $T\mathcal{F}$ is bounded in $L^1_{t,x,v}$, then $\int \mathcal{F}\varphi\, dv$ is relatively compact in $L^1_{t,x}$.*

2 Peetre's theorem

Lemma 2.1. *Suppose a linear operator $L : C_c^\infty(M) \to C_c^\infty(M)$ satisfies*

$$\operatorname{supp}(Lu) \subset \operatorname{supp}(u) \quad \textit{for} \quad u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m, we can find $u \in C_c^\infty(U \setminus \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C. We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x.

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing \quad \text{for} \quad i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m:

$$\begin{aligned}
\sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\
&\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\
&< \infty.
\end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i\geq 0} 2^{-i}\frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i\geq 0} 2^{-i}\frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i\geq 0} 2^{-i}\cdot 4^i = \infty,$$

which leads a contradiction.

$\square$

3 Characteristic curve

Algorithm:

(1) Establish the associated vector field by substituting $u \mapsto y$.

(2) Find the integral curve.

(3) Eliminate the auxiliary variables to get an algebraic equation.

(4) Verify the computed solution is in fact the real solution.

Proposition 3.1. *Suppose that there exists a smooth solution $u : \Omega \to \mathbb{R}_y$ of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, & (t, x) \in \Omega \subset \mathbb{R}_{t\geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, & \text{at } x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t, x)$.

Let $\gamma : I \to \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$X = \frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\tilde{\gamma} : \tilde{I} \to M$ as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\tilde{\gamma}(0) = \gamma(0)$. Since X and $\tilde{X}$ coincide on M, the curve $\tilde{\gamma}$ is also an integral curve of X with $\tilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\tilde{I} \subset I$ and $\gamma(\theta) = \tilde{\gamma}(\theta)$ for all $\theta \in \tilde{I}$.

Since M is closed in E, the open interval $\tilde{I} = \gamma^{-1}(M)$ is closed in I, hence $\tilde{I} = I$ by the connectedness of I. □

Definition 3.1. The projection of the integral curve γ onto Ω is called a *characteristic.*

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X. Once we find a necessary condition of the form of algebraic equation, we can demostrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u, we can solve the ODE: let $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$ be the integral curve of X such that $\gamma(0) = (0, \xi, \xi)$. Then, the system of ODEs

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi \end{aligned}$$

is solved as

$$t(\theta) = \theta, \qquad y(\theta) = \xi, \qquad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t,x) = \frac{-1 + \sqrt{1+4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega = \{ (t,x) : tx > -\tfrac{1}{4} \}.$$

3.1 Wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad t, x > 0, \\ &u(0,x) = g(x), \qquad u(0,x) = h(x), \qquad u_x(t,0) = \alpha(t). \end{aligned}$$

Define $v := u_t - cu_x$. Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ \quad v(0,x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t,x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ \quad u(0,x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ \quad u_x(t,0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter $\xi > 0$,

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t,x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi) \end{aligned}$$

is solved as

$$t(\theta) = \theta, \qquad x(\theta) = -c\theta + \xi, \qquad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta')\, d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned} u(t,x) &= g(\xi) - \int_0^\theta v(s, \xi - cs)\, ds \\ &= g(x+ct) \\ &= \frac{3g(x+ct) - g(x-ct)}{2} - \int_0^t h(x + c(t-2s))\, ds \end{aligned}$$

3.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

(1) Suppose $u(0, x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(2) Suppose $u(0, x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(3) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and "paste" the solution together.

4 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0,1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every seperable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem -> continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable stict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by $\mathcal{T}_\beta$ coincides with $\mathcal{T}_\alpha$)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- Th integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x} \colon D' \to D'$ is continuous. They commute (Schwarz theorem holds).
- $D \to S \to L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

5 Analysis problems

Problem 5.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \le x - x_0 \le x \le 2^k$, $x - x_0 \in A_k$.

$$|A_k| \ge \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\ge \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\
&\ge \sum_{k=1}^{N} (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\
&= \sum_{k=1}^{N} \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&> \sum_{k=1}^{N} \frac{2^k}{2^{k+2}} \frac{1}{7k} \\
&= \frac{1}{28} \sum_{k=1}^{N} \frac{1}{k} \\
&\to \infty.
\end{aligned}$$

□

Problem 5.2. If $|xf'(x)| \le M$ and $\frac{1}{x}\int_0^x f(y)\,dy \to L$, then $f(x) \to L$ as $x \to \infty$.

Solution. Since

$$\begin{aligned}
\left| f(x) - \frac{F(x)-F(a)}{x-a} \right| &\le \frac{1}{x-a}\int_a^x |f(x)-f(y)|\,dy \\
&= \frac{1}{x-a}\int_a^x (x-y)|f'(c)|\,dy \\
&\le \frac{M}{x-a}\int_a^x \frac{x-y}{c}\,dy \\
&\le M\frac{x-a}{a}
\end{aligned}$$

by the mean value theorem and

$$f(x) - L = \left[f(x) - \frac{F(x)-F(a)}{x-a} \right] + \frac{x}{x-a}\left[\frac{F(x)}{x} - L \right] + \frac{a}{x-a}\left[\frac{F(a)}{a} - L \right],$$

we have for any $\varepsilon > 0$

$$\limsup_{x\to\infty} |f(x) - L| \le \varepsilon$$

where a is defined by $\frac{x-a}{a} = \frac{\varepsilon}{M}$. □

Problem 5.3. Let $f_n : I \to I$ be a sequence of real functions that satisfies $|f_n(x) - f_n(y)| \le |x - y|$ whenever $|x - y| \ge \frac{1}{n}$, where $I = [0, 1]$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap I \to I$ pointwisely.

Step [.1] For $n \ge 4$, we claim

$$|x - y| \le \frac{1}{n} \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{5}{n}. \tag{1}$$

Fix $x \in I$ and take $z \in I$ such that $|x - z| = \frac{2}{n}$ so that

$$|f_n(x) - f_n(z)| \le |x - z| = \frac{2}{n}.$$

If y satisfies $|x - y| \le \frac{1}{n}$, then we have $|y - z| \ge |x - z| - |x - y| \ge \frac{1}{n}$, so we get

$$|f_n(y) - f_n(z)| \le |y - z| \le |y - x| + |x - z| \le \frac{3}{n}.$$

Combining these two inequalities proves what we want.

Step [.2] For $\varepsilon > 0$ and $N := \lceil \frac{15}{\varepsilon} \rceil$ we claim

$$|x - y| \le \frac{1}{N} \quad \text{and} \quad n > N \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{\varepsilon}{3} \tag{2}$$

when $N \ge 4$. It is allowed for $|x - y|$ to have the following two cases:

$$|x - y| \le \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x - y| \le \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \le \frac{5}{n} < \frac{5}{N} \le \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \le |x - y| \le \frac{1}{N} \le \frac{\varepsilon}{15}.$$

Hence the claim is proved.

Step [.3] We will prove f is uniformly continuous. For $\varepsilon > 0$, take $\delta := \frac{1}{N}$, where $N := \lceil \frac{15}{\varepsilon} \rceil$. We will show

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

for $x, y \in \mathbb{Q} \cap I$ and $N \geq 4$. Fix rational numbers x and y in I which satisfy $|x - y| < \delta$. Since $f_n(x)$ and $f_n(y)$ converges to $f(x)$ and $f(y)$ respectively, we may take an integer n_x and n_y, such that

$$n > n_x \quad \Longrightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{3}$$

and

$$n > n_y \quad \Longrightarrow \quad |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \tag{4}$$

Choose an integer n such that $n > \max\{n_x, n_y, N\}$. Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since f is continuous on a dense subset $\mathbb{Q} \cap I$, it has a unique continuous extension on the whole I. Let it denoted by the same notation f.

Step [.4] Finally, we are going to show $f_n \to f$ uniformly. For $\varepsilon > 0$, let $N := \lceil \frac{15}{\varepsilon} \rceil$. The uniform continuity of f allows to have $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{2}{3}\varepsilon. \tag{5}$$

Take a rational $r \in I$, depending on $x \in I$, such that $|x - r| < \min\{\frac{1}{N}, \delta\}$. Then, by (2) and (5), given $n > N \geq 4$, we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any $x \in I$. By limiting $n \to \infty$, we obtain

$$\lim_{n \to \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since ε and x are arbitrary, we can deduce the uniform convergence of f_n as $n \to \infty$. $\square$

Problem 5.4. A measurable subset of $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

Solution. Let $E \subset \mathbb{R}$ be measurable with $\mu(E) > 0$. We may assume E is bounded so that we have $E \subset I$ for a closed bounded interval since $\mathbb{R}$ is σ-compact. Let n be a positive integer arbitrarily taken. Then, we can find N such that $\sum_{k=1}^{N} \frac{1}{k} > (n-1)\frac{\mu(I)}{\mu(E)}$.

Assume that every point x in E is contained in at most $n-1$ sets among

$$E,\ \frac{1}{2}E,\ \frac{1}{3}E,\ \cdots,\ \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k\in A} \frac{1}{k}E = \varnothing$$

for any subset $A \subset \{1, \cdots, N\}$ with $|A| \geq n$. Define

$$E_A := \bigcap_{k\in A} \frac{1}{k}E \cap \bigcap_{k'\in A} \left(\frac{1}{k'}E\right)^c$$

for $A \subset \{1, \cdots, N\}$. Then, $\mu(E_A) = 0$ for $|A| \geq n$.

Note that we have

$$\mu(\tfrac{1}{k}E) = \sum_{k\in A} \mu(E_A) = \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) = \sum_{k=1}^{N} \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A) = \sum_{|A|<n} |A|\mu(E_A)$$

by double counting, and since E_A are dijoint, we have

$$\sum_{|A|<n} |A|\mu(E_A) = (n-1) \sum_{0<|A|<n} \mu(E_A) \leq (n-1)\mu(I),$$

hence a contradiction to

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) > (n-1)\mu(I).$$

Therefore, we may find an element x that belongs to $\frac{1}{k}E$ for $k \in A$, where $A \subset \{1, \cdots, N\}$ with $|A| = n$. Then, $ax \in E$ for all $a \in A \subset \mathbb{Z}$.

$\square$