Number Theory I : Classical therory

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May 23, 2019

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Chapter 1

Diophantine equations

1.1 Quadratic equations

Problem 1.1. Consider a family of diophantine equations:

$$x^2 + y^2 - kxy - k = 0$$

for $k \in \mathbb{Z}$.

(1) Show that if (a, b) is a solution, then $(b, b - ka)$ is also a solution.

(2) Show that the equation does not have a solution in the region $xy < 0$.

(3) Show that if it has a solution, then there is a solution on the x-axis.

(4) Let a and b be integers. Conclude that if $ab + 1$ divides $a^2 + b^2$, then

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square.

Solution.

(1)

(2) Suppose $x, y \in \mathbb{Z}$ satisfy $xy < 0$. Since $xy \leq -1$,

$$x^2 + y^2 - kxy - k \geq x^2 + y^2 + k - k > 0.$$

(3)

□

Problem 1.2. Consider a diophantine equation:

$$y^2 = x^3 + 7.$$

Suppose (x, y) is a solution.

(1) Show that x is even and y is odd.

(2) Show that $x^3 + 8$ is divided by a prime p such that $p \equiv 3 \pmod 4$.

(3) Show that the equation has no solutions.