VLASSOV-POISSON SYSTEM

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1. Vlasov-Poisson equation

Consider a Cauchy problem of the *Valsov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, \qquad (t,x,v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t,x) = -\nabla_x \Phi \\ \Phi(t,x) = (-\Delta_x)^{-1} \rho, \\ \rho(t,x) = \int f\,dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $\gamma = \pm 1$ denotes the charge of particles we are concerned with. For example, $\gamma = -1$ for electrons in plasma and $\gamma = +1$ for galaxies. For the boundaryless problem in which the potential function vanishes at infinity, we have

$$E = -\nabla_x \Phi = -\nabla_x \left(-\tfrac{1}{4\pi|x|} * \rho\right) = -\frac{x}{4\pi|x|^3} * \rho$$

for $\gamma = -1$. (ρ denotes mass density.)

1.1. A priori estimates.

Lemma 1.1.

$$\|\rho(t)\|_{L_x^{5/3}} \lesssim 1.$$

Proof.

$$\begin{aligned} \rho(t,x) = \int f(t,x,v)\,dv &\le \int_{|v|<R} f\,dv + \frac{1}{R^2}\int_{|v|\ge R} |v|^2 f\,dv \\ &\lesssim R^3 + R^{-2}\int |v|^2 f\,dv. \end{aligned}$$

Set $R^3 = R^{-2}\int |v|^2 f\,dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f\,dv.$$

Take $d=3$, $p=2$, and $\lambda=2$. Then, by the Hardy-Littlewood-Sobolev inequality,

$$\frac{1}{p}+1=\frac{1}{q}+\frac{\lambda}{d}$$

implies $q=6/5$ and we can bound the L^2-norm of the Riesz potential $\|E(t)\|_2$ by interpolation of $\|\rho(t)\|_{6/5}$ and $\|\rho(t)\|_1$:

$$\|E(t)\|_{L^2_x} \simeq \|\frac{1}{|x|^2} *_x \rho(t,x)\|_{L^2_x} \lesssim \|\rho(t)\|_{6/5} \le \|\rho\|_1^{7/12}\|\rho\|_{5/3}^{5/12}.$$

Thus

$$\|E(t)\|_2 \lesssim \|\rho(t)\|_{5/3}^{5/12} \lesssim \Big(\iint |v|^2 f\,dv\,dx\Big)^{1/4}.$$

It means $(\iint |v|^2 f\,dv\,dx)^{1/2}$ bounds $(\iint |v|^2 f\,dv\,dx)$, hence the total kinetic energy of the system remains bounded in any time even if $\gamma=+1$. As a corollary, $\|\rho\|_{5/3}$ is also bounded. □

Lemma 1.2. *For* $1 \le q < \frac{N}{N-2} = 3 < p \le \infty$,

$$\|E(t,x)\|_{L^\infty_x} \lesssim \|\rho(t,x)\|_{L^p_x}^{\frac{\frac{2}{N}-1+\frac{1}{q}}{\frac{1}{q}-\frac{1}{p}}} \|\rho(t,x)\|_{L^q_x}^{\frac{1-\frac{1}{p}-\frac{2}{N}}{\frac{1}{q}-\frac{1}{p}}}.$$

Proof. Fix time t. For $2p < N < 2q$,

$$\begin{aligned}
4\pi|E(t,x)| &= |\tfrac{1}{|x|^2} *_x \rho(t,x)| \\
&\le \int_{|x-y|<R} \frac{\rho(t,y)}{|x-y|^2}\,dy + \int_{|x-y|\ge R} \frac{\rho(t,y)}{|x-y|^2}\,dy \\
&\le \|\rho\|_{p'}\Big(\int_{|y|<R} \frac{dy}{|y|^{2p}}\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_{|y|\ge R} \frac{dy}{|y|^{2q}}\Big)^{1/q} \\
&\simeq \|\rho\|_{p'}\Big(\int_0^R r^{N-1-2p}\,dr\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_R^\infty r^{N-1-2q}\,dr\Big)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{N}{p}-2} + \|\rho\|_{q'} R^{\frac{N}{q}-2}.
\end{aligned}$$

By choosing R such that $\|\rho\|_{p'}R^{\frac{N}{p}-2} = \|\rho\|_{q'}R^{\frac{N}{q}-2}$, we get

$$\|E(t,x)\|_{L^\infty_x} \lesssim \|\rho(t,x)\|_{L^{p'}_x}^{\frac{\frac{2}{N}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho(t,x)\|_{L^{q'}_x}^{\frac{\frac{1}{p}-\frac{2}{N}}{\frac{1}{p}-\frac{1}{q}}},$$

hence the inequality by interchaning p and q with their conjugates. □

1.2. Schaeffer's global existence proof.

Theorem (Schaeffer, 1991)**.** *Let $f_0 \in C^1_{c,x,v}$ and $f_0 \geq 0$. Then, the Cauchy problem for the VP system has a unique C^1 global solution.*

Definition 1.1. For a local solution f,

$$Q(t) := 1 + \sup\{|v| : f(s,x,v) \neq 0 \text{ for some } s \in [0,t],\, x \in \mathbb{R}^3_x\}.$$

Decompose $[t-\Delta, t] \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ as

$$\begin{aligned} U &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq r \right\}, \\ B &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| \geq P, \quad |v| \geq P \right\} \setminus U, \\ G &= \left\{ (s,x,v) : \ |v - \widehat{V}(t)| < P \quad \text{or} \quad |v| < P \right\}. \end{aligned}$$

(We can let $U \mapsto U \cap \{|v| \geq P\}$ to make the decomposition disjoint.) Later we choose

$$P = Q^{4/11}, \quad r = R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}, \quad R = Q^{16/33} \log^{1/2} Q.$$

Also, later we choose $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

1.2.1. *Some observations.* Our goal is to obtain a priori estimate like

$$\|E(t)\|_\infty \lesssim Q(t)^a \qquad \text{for some } a < 1.$$

Since the force field E measures the maximal rate of changes in velocity, the estimate can be read very roughly as

$$Q'(t) \lesssim Q(t)^a,$$

which lead its polynomial growth.

So we need to bound the Riesz potential E. The following observation suggests a lower bound of relative velocity.

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$|y - \widehat{X}(s)| \geq \frac{1}{4} |v - \widehat{V}(t)| |s - s_0|$$

for some $s_0 \in [t-\Delta, t]$, where $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

Proof. Since $\Delta \|E(s)\|_\infty < \frac{P}{4}$, we have

$$|v - w| < \frac{P}{4} \quad \text{and} \quad |\widehat{V}(t) - \widehat{V}(s)| < \frac{P}{4}.$$

The condition $|v - \widehat{V}(t)| \geq P$ implies

$$\frac{1}{2}|v - \widehat{V}(t)| \leq |v - \widehat{V}(t)| - \frac{P}{4} - \frac{P}{4} < |w - \widehat{V}(s)|.$$

Let $Z(s) := y - \widehat{X}(s)$. Then,

$$\begin{aligned} Z'(s) &= w - \widehat{V}(s), \\ Z''(s) &= \gamma[E(s,y,w) - E(s, \widehat{X}(s), \widehat{V}(s))]. \end{aligned}$$

Let $s_0 \in [t-\Delta, t]$ minimize $s \mapsto |Z(s)|$ and expand Z as

$$Z(s) = Z(s_0) + Z'(s_0)(s-s_0) + \frac{Z''(\sigma)}{2}(s-s_0)^2$$

for some σ between s and s_0. Then,

$$|Z(s_0) + Z'(s_0)(s-s_0)| \geq |Z'(s_0)(s-s_0)| \geq \frac{1}{2}|v - \widehat{V}(t)||s-s_0|$$

and

$$\begin{aligned} |\frac{Z''(\sigma)}{2}(s-s_0)^2| &\leq \|E(t)\|_\infty (s-s_0)^2 \leq \|E(t)\|_\infty \Delta |s-s_0| \\ &\leq \frac{P}{4}|s-s_0| \leq \frac{1}{4}|v-\widehat{V}(t)||s-s_0| \end{aligned}$$

proves

$$|y - \widehat{X}(s)| = |Z(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|. \qquad \square$$

We introduce time averaging to use the above lower bound.

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$\int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim \frac{r^{-1}}{|v-\widehat{V}(t)|},$$

where $A = \{s : |y - \widehat{X}(s)| \geq r\}$.

Proof. Since $|y - \widehat{X}(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|$,

$$\begin{aligned} \int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds &\leq 16\int_{t-\Delta}^t \frac{1}{|v-\widehat{V}(t)|^2|s-s_0|^2}\chi_A(s)\,ds \\ &\leq 32\int_r^\infty \frac{1}{|v-\widehat{V}(t)|^3|s-s_0|^2}\,d(|v-\widehat{V}(t)||s-s_0|) \\ &= 32\,\frac{r^{-1}}{|v-\widehat{V}(t)|}. \end{aligned} \qquad \square$$

1.2.2. *Ugly set.* Therefore, if we let $r^{-1} \simeq \min\{|v|^3, |v-\widehat{V}(t)|^3\}$, then

$$\int_{t-\Delta}^t \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim |v|^2$$

so that we have

$$\iiint_U \frac{f(s,y,w)}{|y-\widehat{X}(s)|^2}\,dw\,dy\,ds \lesssim R^{-1}\int |v|^2 f(t,x,v)\,dv\,dx \lesssim R^{-1}$$

when

$$U \subset \{\,(s,x,v) :\ |v-\widehat{V}(t)| \geq P, \quad |y-\widehat{X}(s)| \geq R\max\{|v|^{-3}, |v-\widehat{V}(t)|^{-3}\}\,\}.$$

1.2.3. *Bad set.* Consider U^c. We need to control the union of two regions

$$|y - \widehat{X}(s)| < R|v|^{-3} \quad \text{and} \quad |y - \widehat{X}(s)| < R|v - \widehat{V}(t)|^{-3}.$$

Without any conditions, the integration of fundamental solution with respect to y gives

$$\int_{|y-\widehat{X}(s)|<r} \frac{1}{|y - \widehat{X}(s)|^2}\, dy \simeq r.$$

Claim. *If $|v| \geq P$ and $|v - \widehat{V}(t)| \geq P$, then*

$$\int_{U^c} \frac{1}{|y - \widehat{X}(s)|^2}\, dy \lesssim \max\{|w|^{-3}, |w - \widehat{V}(s)|^{-3}\}$$

for $s \in [t - \Delta, t]$.

Proof. It follows from

$$|w| \simeq |v|, \quad |w - \widehat{V}(s)| \simeq |v - \widehat{V}(t)|$$

for $|v| \geq P$ and $|v - \widehat{V}(t)| \geq P$. □

1.2.4. *Polynomial decay.*

Lemma 1.3. *Along the time of existence we have*

$$\|E(t)\|_{L^\infty_x} \lesssim Q(t)^{4/3}.$$

Proof. Note that we have

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

Since the velocity support of f is bounded by finite $Q(t)$,

$$\rho(t, x) = \int_{|v|<Q(t)} f(t, x, v)\, dv \lesssim Q(t)^3 \|f_0(x)\|_{L^\infty_v} \lesssim Q(t)^3,$$

so

$$\|E(t)\|_{L^\infty_x} \lesssim \|\rho(t)\|_{L^\infty_x}^{4/9} \lesssim Q(t)^{4/3}.$$ □