Analysis 3 : Measure Theory

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CHAPTER 1

Carathéodory's theory

1. σ-algebras

Definition 1.1. Let X be a set. A *ring of sets* is a family of subsets of X that is closed under finite union and finite relative complement; in other words, $\mathcal{R} \subset \mathcal{P}(X)$ is called a ring of sets if the following two conditions are satisfied:

(1) if $A, B \in \mathcal{R}$, then $A \cup B \in \mathcal{R}$,
(2) if $A, B \in \mathcal{R}$, then $A \setminus B \in \mathcal{R}$.

Proposition 1.1. *Let X be a set and $\mathcal{R} \in \mathcal{P}(X)$. Then, the followings are equivalent:*

(1) *$\mathcal{R}$ is a ring of sets,*
(2) *$\mathcal{R}$ is closed under symmetric difference and finite intersection,*
(3) *$\mathcal{R}$ is a ring,*
(4) *$\mathcal{R}$ is a Boolean ring.*

For the ring structure, we take the symmetric difference as addition and the intersection as multiplication.

Proposition 1.2. *A ring of sets is a distributive lattice.*

If a ring of sets contains a multiplicative identity, the entire set, then we call the ring of sets as follows:

Definition 1.2. An *algebra of sets* is a ring of sets with the entire set.

Proposition 1.3. *Let X be a set and $\mathcal{R} \in \mathcal{P}(X)$. Then, the followings are equivalent:*

(1) *$\mathcal{R}$ is an algebra of sets,*
(2) *$\mathcal{R}$ is closed under finite union, finite intersection, and complement,*
(3) *$\mathcal{R}$ is a Boolean algebra.*

An algebra of sets is sometimes called a field of sets.

2. Carathéodory's extension theorem

THEOREM 2.1 (Carathéodory's extension theorem). *Let $\mathcal{R}$ be a ring of sets over X. Let $\sigma(\mathcal{R})$ be the σ-algebra generated by $\mathcal{R}$. A set function $\mu : \mathcal{R} \to [0, \infty]$ is extended to a measure on $\sigma(\mathcal{R})$ if and only if it is a premeasure.*

CHAPTER 2

CHAPTER 3

Topological measures

1. Descriptive set theory

2. Borel measures

CHAPTER 4

Hmmmm

0.1. Convergence in measure. Since $\{f_n(x)\}_n$ diverges if and only if

$$\exists k > 0, \quad \forall n_0 > 0. \quad \exists n > n_0 : \quad |f_n(x) - f(x)| > n^{-1},$$

we have

$$\begin{aligned}\{x : \{f_n(x)\}_n \text{ diverges}\} &= \bigcup_{k>0} \bigcap_{n_0>0} \bigcup_{n>n_0} \{x : |f_n(x) - f(x)| > n^{-1}\} \\ &= \bigcup_{k>0} \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.\end{aligned}$$

Since for every k

$$\limsup_n \{x : |f_n(x) - f(x)| > k^{-1}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\},$$

we have

$$\{x : \{f_n(x)\}_n \text{ diverges}\} \subset \limsup_n \{x : |f_n(x) - f(x)| > n^{-1}\}.$$

THEOREM 0.1. *Let f_n be a sequence of measurable functions on a measure space (X, μ). If f_n converges to f in measure, then f_n has a subsequence that converges to f μ-a.e.*

PROOF. Since $d_{f_n - f}(1/k) \to 0$ as $n \to \infty$, we can extract a subsequence f_{n_k} such that

$$\mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) > 2^{-k}.$$

Since

$$\sum_{k=1}^{\infty} \mu(\{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) < \infty,$$

by the Borel-Canteli lemma, we get

$$\mu(\limsup_k \{x : |f_{n_k}(x) - f(x)| > k^{-1}\}) = 0.$$

Therefore, f_{n_k} converges μ-a.e. $\square$