

Interchanging limits, derivatives, and integrals

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1. LIMIT AND DERIVATIVE

f_n pointwisely, Df_n uniformly

2. LIMIT AND INTEGRAL

We want to find a criterion for This question asks the convergence

$$f_n \rightarrow f \quad \text{in} \quad L^1.$$

Theorem 2.1 (Lebesgue dominated convergence theorem). *Let $\{f_\alpha\}_\alpha$ be a net of measurable functions $(X, \mu) \rightarrow \mathbb{R}$. Define a maximal function*

$$Mf(x) = \sup_\alpha |f_\alpha(x)|.$$

If $\|Mf\|_1 < \infty$, then

$$\lim_\alpha |f_\alpha(x) - f(x)| = 0 \quad \text{a.e.} \quad \implies \quad \lim_\alpha \|f_\alpha - f\|_{L^1} = 0.$$

continuity application

3. DERIVATIVE AND INTEGRAL

This question asks under what conditions the following convergence holds: for fixed t_0 ,

$$\lim_{t \rightarrow t_0} \|N_t f(x, t_0) - \partial_t f(x, t_0)\|_{L_x^1} = 0,$$

where the Newton quotient is defined as

$$N_t f(x, t_0) := \frac{f(x, t) - f(x, t_0)}{t - t_0}$$

for $t \neq t_0$

Theorem 3.1 (Leibniz rule). *Let I be an interval in \mathbb{R} that containing t_0 . Let $f : X \times I \rightarrow \mathbb{R}$ be a measurable function such that for a.e. x the function $t \mapsto f(x, t)$ is continuous on I and differentiable on I° .*

If $\|\partial_t f\|_{L_x^1(L_t^\infty)} < \infty$, then

$$\frac{d}{dt} \int f(x, t) dx = \int \frac{\partial}{\partial t} f(x, t) dx.$$

Proof. Define a maximal function

$$Mf(x, t_0) = \sup_{t \in I \setminus \{t_0\}} |N_t f(x, t_0)|.$$

By the mean value theorem, we get

$$\|\partial_t f\|_{L_x^1(L_t^\infty)} < \infty \implies \|Mf(x, t_0)\|_{L_x^1} < \infty.$$

Apply the LDCT. □

F is absolutely continuous,

$$\partial_t F = f \iff F(x, t) = \int_c^t f(x, s) dx.$$

Then, for

$$T_h f(x, 0) := \frac{1}{h} \int_0^h f(x, s) ds,$$

For $\|f\|_{L_x^1 L_t^\infty} = \|\sup_t |f(x, t)|\|_{L_x^1} < \infty$ we have

$$\begin{aligned} |T_h f(x, 0)| &\leq \frac{1}{h} \int_0^h |f(x, s)| ds \\ &\leq \left[\frac{1}{h} \int_0^h ds \right] \cdot \sup_t |f(x, t)| \\ &= \sup_t |f(x, t)|. \end{aligned}$$

Thus,

$$Mf(x, 0) = \sup_h |T_h f(x, 0)| \leq \sup_t |f(x, t)| \in L_x^1.$$

Since $f(x, 0) \in L_x^1$, by the Lebesgue differentiation theorem, we get

$$\lim_{h \rightarrow 0} T_h f(x, 0) = f(x, 0)$$

for a.e. x .