Classical differential geometry

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Contents

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1. Introduction

1.1. **Parametrizations and coordinates.** For each text on classical differential geometry, the definitions frequently vary. In this note, we define as follows.

Definition 1.1. An *m-dimensional parametrization* is a smooth map $\alpha : U \to \mathbb{R}^n$ such that

(1) $U \subset \mathbb{R}^m$ is open,
(2) the Fréchet derivative $d\alpha : U \times \mathbb{R}^m \to \mathbb{R}^n$ is injective everywhere, (immersion)
(3) α is one-to-one, (immersed submanifold)
(4) α is a homomorphism onto its image. (embedded submanifold)

The Euclidean space $\mathbb{R}^n$ is called the *ambient space*.

The first condition is necessary to avoid differentiating at points that are not in the interior of domain. Of course, it is possible to generalize the definition of differentiation on boundary points, but we will not introduce the notion because it goes out of the scope of this note.

The second condition is the most important one. This condition is paraphrased as follows: the set of column vectors of $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$, which are exactly the partial derivatives $\{\partial_i \alpha(x)\}_{i=1}^m \subset \mathbb{R}^n$ of α with respect to each direction, is linearly independent at every point $x \in U$. Differential geoemtry do not consider parametrizations that fail this. This condition is vital for considering an appropriate and well-defined linear approximation of curves or surfaces. If it is not staisfied, every definition including tangent spaces in differential geometry can suffer.

Above the second condition, we call the image of a parametrization α a (immersed) submanifold if the third condition is satisfied. If α is not one-to-one, then two distinct ordered tuples of real numbers may represent the same point. Namely, this condition allows to use the inverse map $\alpha^{-1} : \operatorname{im} \alpha \to U$. By this, we can recognize a parametrization as the inverse of a coordinate map. To describe a geometric object that cannot be covered by a single injective parametrization, such as a circle or a sphere, we can admit several parametrizations.

The forth condition is stronger than the third, so we may assume just (1), (2), and (4). It is sometimes called a proper patch in some references. This condition is introduced to exclude some exceptional examples such as lemniscates: see Example 1.4.

Definition 1.2. A subset $M \subset \mathbb{R}^n$ is called a *regular curve* (resp. *regular surface*) if there exists a one-dimensional (resp. two-dimensional) parametrization whose image is exactly M.

All curves and surfaces in this note are assumed to be regular: all of the four conditions are satisfied. One can notice that this definition is exactly same as an embedded submanifold of $\mathbb{R}^n$ that can be covered by a single parametrization.

We also just often say that α is a regular curve (resp. regular surface) for *a particular parametrization* α. However, note that a curve or surface admits infinitely many parametrizations. We can solve many geometry or physics problems very easily by choosing an appropriate parametrization. Related to the choice of parametrizations, the following issues are always importantly considered when developing a theory of differential geometry:

- Well-definedness of a structure with respect to the dependency on parametrizations(coordinates).
- Existence of a parametrization(coordinates) that has nice properties we want.

Definition 1.3. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. The inverse $\varphi : M \to U$ of a parametrization is called a *coordinate map*.

Coordinates and parametrizations have equivalent information except that the direction is opposite (only if parametrization satisfies the injectivity). A parametrization tends to be taken for explicit computations while coordinates are more usefully taken in verifications of abstract propositions. We use the term *reparametrization* to refer to nothing but a choice of another parametrization for the same curve or surface. As said, the choice of coordinate(parametrization) is important in differential geometry.

Example 1.1.

(1) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a map given by

$$\alpha(t) = (\cos t, \sin t, t).$$

Since $d\alpha|_t(1) = \alpha'(t) = (-\sin t, \cos t, 1)$ is always nonzero so that $d\alpha$ is injective everywhere, α is a parametrization of the regular curve

$$\{\, (x, y, z) \in \mathbb{R}^3 : x = \cos z,\ y = \sin z \,\}.$$

Notice that it is enough to check $\alpha'(t) \neq 0$ for a curve parametrization α to show the injectivity of $d\alpha$. This curve is an example of circular helices.

(2) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a map given by

$$\alpha(t) = (t^3, t^6, t^9).$$

Since $d\alpha|_t(1) = \alpha'(t) = (3t^2, 6t^5, 9t^8)$ is zero when $t = 0$, *it would be better to avoid calling* α *a parametrization*. Instead, the restrictions $\alpha_+ : (0, \infty) \to \mathbb{R}^3$ and $\alpha_- : (-\infty, 0) \to \mathbb{R}^3$ satisfy the axioms of parametrization at the beginning.

However, by reparametrization, we can show the image of α is a regular curve, that is, we can find a parametrization that shares the image with α, even though *we sometimes say that* α *is not a regular curve* according to the fact α' can vanish. Consider $\beta : \mathbb{R} \to \mathbb{R}^3$ defined by

$$\beta(t) = (t, t^2, t^3).$$

This map has the same image $\operatorname{im}\alpha = \operatorname{im}\beta$, but $\beta'(t) = (1, 2t, 3t^2) \neq 0$ for all $t \in \mathbb{R}$.

(3) Let S^1 be the unit circle in $\mathbb{R}^2$, precisely

$$S^1 := \{\, (x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1 \,\}.$$

It cannot be covered by a single parametrization, so we can consider two different parametrizations $\alpha : (0, 2\pi) \to \mathbb{R}^2$ and $\beta : (\pi, 3\pi) \to \mathbb{R}^2$ for S^1:

$$\alpha(t) = (\cos t, \sin t), \qquad \beta(t) = (\cos t, \sin t).$$

Then, we have $S^1 = \operatorname{im}\alpha \cup \operatorname{im}\beta$. If we want to investigate the geometry of S^1 near the point $(1, 0)$, we can choose β rather than α because $(1, 0) \notin \operatorname{im}\alpha$.

(4) Let $\alpha, \beta : (0, 2\pi) \to \mathbb{R}^2$ be maps given by

$$\alpha(t) = (\sin t, \sin 2t), \qquad \beta(t) = (\sin t, -\sin 2t).$$

They are one-to-one smooth maps such that $d\alpha|_t(1) \neq 0 \neq d\beta|_t(1)$, and one can check that they have common images; it is shaped like the character '∞'. A problem occurs when we think tangent vectors at $(0, 0)$:

$$\alpha'(0) = (1, 2), \qquad \beta'(0) = (1, -2)$$

imply that the notion of tangent vectors at a point do depend on the choice of parametrizations.

Generally, since one philosophy of parametrizations is to view them as *identifications* between curved spaces and flat Euclidean spaces, we want for them to have the images of open sets be open with respect to subspace topology. Thus, we assume parametrizations are homeomorphisms onto its images.

(5) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be any smooth function and $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be a map given by

$$\alpha(x, y) = (x, y, f(x, y)).$$

Then, α is a two-dimensional parametrization because

$$\begin{aligned} d\alpha|_{(x,y)}(1,0) &= \frac{\partial\alpha}{\partial x}(x,y) = \Big(1, 0, \frac{\partial f}{\partial x}(x,y)\Big), \\ d\alpha|_{(x,y)}(0,1) &= \frac{\partial\alpha}{\partial y}(x,y) = \Big(0, 1, \frac{\partial f}{\partial y}(x,y)\Big) \end{aligned}$$

are linearly independent for every $(x, y) \in \mathbb{R}^2$. A parametrization of this form is called a *Monge patch*. Notice that it is enough to check that the two partial derivatives $\partial_x\alpha$ and $\partial_y\alpha$ are linearly independent for a surface parametrization α.

Let $S = \operatorname{im}\alpha$ be the regular surface determined by α, and let p be a point on the surface S so that we have $p = (x, y, f(x, y))$. Associated with α, a coordinate map $\varphi : S \to \mathbb{R}^2$ is defined as the inverse of α:

$$\varphi(p) := \alpha^{-1}(p) = (x, y).$$

This map φ consists of two real-valued functions on S,

$$x : S \to \mathbb{R} : p \mapsto x, \qquad y : S \to \mathbb{R} : p \mapsto y.$$

In this regard, we often write the coordinates φ as (x, y).

(6) Let

$$S = \{\, (x, y) \in \mathbb{R}^2 : x > 0 \text{ or } y \neq 0 \,\}.$$

The set S is a regular surface. Consider two different coordinates

$$(x,y) : S \to S : (x,y) \mapsto (x,y),$$
$$(r,\theta) : S \to \mathbb{R}_{>0} \times (-\pi,\pi) : (x,y) \mapsto \left(\sqrt{x^2+y^2},\ 2\tan^{-1}\frac{y}{\sqrt{x^2+y^2}+x}\right),$$

where $\tan^{-1}(t) := \int_0^t \frac{ds}{1+s^2}$. They are the inverses of parametrizations $\alpha : S \to \mathbb{R}^2$ and $\beta : (0,\infty) \times (-\pi,\pi) \to \mathbb{R}^2$ defined by

$$\alpha(x,y) = (x,y), \qquad \beta(r,\theta) = (r\cos\theta, r\sin\theta).$$

The coordinate maps (x,y) and (r,θ) are called *Cartesian coordinates* and *polar coordinates* respectively.

1.2. **Differentiation.** Differentiation in differential geometry can be understood in many different viewpoints. We, here, review the two kinds of main usages of differentiation: differentiation of parametrizations, and differentiation by directional vectors. Do not forget that all differentiations in this note will be done thanks to the structure of the ambient space $\mathbb{R}^n$.

1.2.1. *Differentiation of parametrizations.* We introduce the notion of tangent spaces, geometrically the spaces of vectors that starts from each base point, by differentiation of parametrization. In this note we define tangent spaces in several equivalent ways:

Definition 1.4. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. Let $p \in M$ be a point and $x = \alpha^{-1}(p) \in U$ be the coordinates of p. The *tangent space* of M at p, denoted by T_pM, can be defined as either one of the followings:

(1) the span of the linearly independent set of vectors $\{\partial_i\alpha(x)\}_{i=1}^m \subset \mathbb{R}^n$,
(2) the image of the Fréchet derivative $d\alpha|_x : \mathbb{R}^m \to \mathbb{R}^n$,
(3) the set of vectors $v \in \mathbb{R}^n$ such that there exists a curve $\gamma : I \to M$ satisfying $\gamma(0) = p$ and $\gamma'(0) = v$.

We can check the definitions are independent on the parametrization α, and that the tangent space T_pM is an m-dimensional linear subspace of $\mathbb{R}^n$.

Remark. We can show the three conditions are equivalent, but the proof will not be given; what is more important is to understand the role and meaning of tangent spaces because there is no agreed standard definition of tangent spaces in the level of this note. There exist a lot more neat but difficult characterizations for tangent spaces we will not cover.

Remark. We can easily check that $T_p\mathbb{R}^n = \mathbb{R}^n$ for any $p \in \mathbb{R}^n$. The notation $T_p\mathbb{R}^n$ will be used to emphasize that a vector in $\mathbb{R}^n$ is geometrically recognized to cast from the point p. Since $T_p\mathbb{R}^n = \mathbb{R}^n = T_q\mathbb{R}^n$ for every pair of points $p,q \in \mathbb{R}^n$, summation and inner product of a vector in $T_p\mathbb{R}^n$ and a vector in $T_q\mathbb{R}^n$ make sense. This identification of tangent spaces are allowed *only for the case of linear spaces* such as $\mathbb{R}^n$. (In fact, the identification $T_p\mathbb{R}^n = \mathbb{R}^n$ is *natural* in categorical language.)

Remark. One way to view tangent spaces is to see them as domains and codomains of Fréchet derivatives. For open sets $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$, the Fréchet derivative of a smooth map $F : U \to V$ at $x \in U$ is a linear transformation $dF|_x : T_xU \to T_{F(x)}V$. Since

$T_xU = \mathbb{R}^m$ and $T_{F(x)}V = \mathbb{R}^n$, the original definition on Euclidean spaces agrees with it. In this reason, the Fréchet derivative dF is also called a *tangent map*, *pushforward*, or *differential* in differential geoemtry.

Notation. Let α be a parametrization for a regular curve or surface M. For derivatives of α, we will use the following notations:

$$\partial_t\alpha = \alpha', \quad \partial_x\alpha = \alpha_x, \quad \partial_i\alpha = \alpha_i.$$

The set $\{\alpha_i\}_i$ will be used to denote a basis of tangent space T_pM.

1.2.2. *Differentiation by tangent vectors.*

Definition 1.5. Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $M = \operatorname{im}\alpha$.

(1) A *scalar field*, *smooth function*, or just a *function* is a function $f : M \to \mathbb{R}$ such that $f \circ \alpha : U \to \mathbb{R}$ is smooth.
(2) A *vector field* is a map $X : M \to \mathbb{R}^n$ such that $X \circ \alpha : U \to \mathbb{R}^n$ is smooth.
(3) A *tangent vector field* is a vector field $X : M \to \mathbb{R}^n$ such that $X|_p \in T_pM$.

The set of tangent vector fields is often denoted by $\mathfrak{X}(M)$. In general, vector fields are basically assumed to be tangent. However, we will distinguish them in this note.

Definition 1.6. Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization $M = \operatorname{im}\alpha$.

(1) The coordinate representation of a function $f : M \to \mathbb{R}$ is
$$f \circ \alpha : U \to \mathbb{R}.$$
(2) The (external) coordinate representation of a vector field $X : M \to \mathbb{R}^n$ is
$$X \circ \alpha : U \to \mathbb{R}^n.$$
(3) The coordinate representation of a tangent vector field $X : M \to \mathbb{R}^n$ is
$$(X^1 \circ \alpha, \cdots, X^m \circ \alpha) : U \to \mathbb{R}^m$$
where $X = \sum_i X^i\alpha_i$.

Definition 1.7. Let M be the image of a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$. Let $v = \sum_i v^i\alpha_i|_p \in T_pM$ be a tangent vector at $p = \alpha(x)$. For a function $f : M \to \mathbb{R}$, its partial derivative is defined by

$$\partial_v f(p) := \sum_{i=1}^m v^i\partial_i(f \circ \alpha)(x) \in \mathbb{R}.$$

For a vector field $X : M \to \mathbb{R}^n$, its partial derivative is defined by

$$\partial_v X|_p := \sum_{i=1}^m v^i\partial_i(X \circ \alpha)(x) \in \mathbb{R}^n.$$

This definition is not dependent on parametrization α.

Proposition 1.1. *Let M be the image of a parametrization. Let X be a tangent vector field on M.*

(1) *If f is a function, then so is $\partial_X f$.*
(2) *If Y is a vector field, then so is $\partial_X Y$.*
(3) *If Y is a tangent vector field, then so is $\partial_X Y - \partial_Y X$.*

Proof. (1) and (2) are clear. For (3), if we let $X = \sum_i X^i \alpha_i$ and $Y = \sum_j Y^j \alpha_j$ for a parametrization $\alpha : U \subset \mathbb{R}^m \to \mathbb{R}^n$, then

$$\begin{aligned}\partial_X Y - \partial_Y X &= \partial_X (Y^j \alpha_j) - \partial_Y (X^i \alpha_i) \\ &= [(\partial_X Y^j)\alpha_j + Y^j \partial_X \alpha_j] - [(\partial_Y X^i)\alpha_i + X^i \partial_Y \alpha_i] \\ &= [(\partial_X Y^j)\alpha_j + Y^j X^i \partial_i \alpha_j] - [(\partial_Y X^i)\alpha_i + X^i Y^j \partial_i \alpha_j] \\ &= (\partial_X Y^j)\alpha_j - (\partial_Y X^i)\alpha_i. \qquad \square\end{aligned}$$

Notation. Let M be the image of a parametrization α. For derivatives of functions on M by tangent vectors, we will use

$$\partial_{\alpha_i} f = \partial_i f, \quad \partial_{\alpha_t} f = \partial_t f = f', \quad \partial_{\alpha_x} f = \partial_x f = f_x.$$

For derivatives of vector fields on M by tangent vectors, we will use

$$\partial_{\alpha_i} X = \partial_i X, \quad \partial_{\alpha_t} X = \partial_t X = X', \quad \partial_{\alpha_x} X = \partial_x X = X_x.$$

We will *not* use f_i or X_i for $\partial_i f$ and $\partial_i X$ because it is confusig with coordinate representations, and *not* use the nabula symbol ∇_v in this sense because it will be devoted to another kind of derivatives introduced in Section 4.

Example 1.2.

(1) Let α be an m-dimensional parametrization. The value of $\partial_i \alpha = \alpha_i : \operatorname{im} \alpha \to \mathbb{R}^3$ is always a tanget vector at each point $p = \alpha(x)$, and α_i becomes a vector field.

Let s be either a smooth function or vector field on α. Then, we can compute the directional derivative as

$$\partial_i s := \partial_i (s \circ \alpha) = \partial_t (s \circ \gamma)$$

by taking $\gamma(t) = \alpha(x + te_i)$, where e_i is the i-th standard basis vector for $\mathbb{R}^m$.

(2) Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^3$ be a regular surface given by

$$\alpha(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, 1 - \frac{2}{1 + x^2 + y^2} \right).$$

This map gives a parametrization for the sphere S^2 without the north pole $(0, 0, 1)$, and is called the *stereographic projection*. Let $f : S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}$ be the height function of α defined by

$$f(p) := z$$

for $p = (x, y, z) \in S^2 \setminus \{(0, 0, 1)\}$. Its coordinate representation is

$$f \circ \alpha(x, y) = 1 - \frac{2}{1 + x^2 + y^2}.$$

Then, the directional derivative is

$$\partial_x f = \frac{\partial (f \circ \alpha)}{\partial x} = \frac{\partial}{\partial x} \left(1 - \frac{2}{1 + x^2 + y^2} \right) = \frac{4x}{(1 + x^2 + y^2)^2}.$$

Note that $\partial_x f \neq \partial_{(1,0,0)} z = 0$.

1.3. **Linear algebra on tangent spaces.**

2. Local theory of curves

2.1. **Theory.**

2.1.1. *Parametrization.* By definition, a regular curve has at least one parametrization. However, a given parametrization may not have useful properties, so we often take a new parametrization. The existence of a parametrization with certain properties is one of the main problems in differential geometry. Practically, the existence proof is usually done by constructing a *diffeomorphism* between open sets in $\mathbb{R}^m$; a bijective smooth map whose inverse is also smooth.

We introduce the arc-length reparametrization. It is the most general choice for the local study of curves.

Definition 2.1. A parametrization α of a regular curve is called a *unit speed curve* or an *arc-length parametrization* when it satisfies $\|\alpha'\| = 1$.

Theorem 2.1. *Every regular curve may be assumed to have unit speed. Precisely, for every regular curve, there is a parametrization α such that $\|\alpha'\| = 1$.*

Proof. By the definition of regular curves, we can take a parametrization $\beta : I_t \to \mathbb{R}^d$ for a given regular curve. We will construct an arc-length parametrization from β.

Define $\tau : I_t \to I_s$ such that

$$\tau(t) := \int_0^t \|\beta'(s)\| \, ds.$$

Since τ is smooth and $\tau' > 0$ everywhere so that τ is strictly increasing, the inverse $\tau^{-1} : I_s \to I_t$ is smooth by the inverse function theorem; τ is a diffeomorphism. Define $\alpha : I_s \to \mathbb{R}^d$ by $\alpha := \beta \circ \tau^{-1}$. Then, by the chain rule,

$$\alpha' = \frac{d\alpha}{ds} = \frac{d\beta}{dt}\frac{d\tau^{-1}}{ds} = \beta' \left(\frac{d\tau}{dt}\right)^{-1} = \frac{\beta'}{\|\beta'\|}. \qquad \square$$

2.1.2. *Differentiation of Frenet-Serret frame.* The Frenet-Serret frame is a standard frame for a curve, and it is in particular effective when we assume the arc-length parametrization. It is defined for nondegenerate regular curves, i.e. nowhere straight curves. It provides with a useful orthonormal basis of $T_p\mathbb{R}^3 \supset T_pC$ for points p on a regular curve C.

Definition 2.2. We call a curve parametrized as $\alpha : I \to \mathbb{R}^3$ is *nondegenerate* if the normalized tangent vector $\alpha'/\|\alpha'\|$ is never locally constant everywhere. In other words, α is nowhere straight.

Definition 2.3 (Frenet-Serret frame)**.** Let α be a nondegenerate curve. The *tangent unit vector*, *normal unit vector*, *binormal unit vector* are $T_p\mathbb{R}^3$-valued vector fields on α defined by:

$$\mathrm{T}(t) := \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad \mathrm{N}(t) := \frac{\mathrm{T}'(t)}{\|\mathrm{T}'(t)\|}, \qquad \mathrm{B}(t) := \mathrm{T}(t) \times \mathrm{N}(t).$$

The set of vector fields $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$, which is called *Frenet-Serret frame*, forms an orthonormal basis of $T_p\mathbb{R}^3$ at each point p on α. The Frenet-Serret frame is uniquely determined up to sign as α changes.

We study the derivatives of the Frenet-Serret frame and their coordinate representations. In the coordinate representations on the Frenet-Serret frame, important geometric measurements such as curvatrue and torsion come out as coefficients.

Definition 2.4. Let α be a nondegenerate curve. The *curvature* and *torsion* are scalar fields on α defined by:

$$\kappa(t) := \frac{\langle \mathrm{T}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}, \quad \tau(t) := -\frac{\langle \mathrm{B}'(t), \mathrm{N}(t)\rangle}{\|\alpha'\|}.$$

Note that $\kappa > 0$ cannot vanish by definition of nondegenerate curve. This definition is independent on α.

Theorem 2.2 (Frenet-Serret formula)**.** *Let α be a nondegenerate curve. Then,*

$$\begin{pmatrix} \mathrm{T}' \\ \mathrm{N}' \\ \mathrm{B}' \end{pmatrix} = \|\alpha'\| \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathrm{T} \\ \mathrm{N} \\ \mathrm{B} \end{pmatrix}.$$

Proof. Note that $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$ is an orthonormal basis. We first show the first and third rows, and the second row later.

Step 1: Show that $\mathrm{T}', \mathrm{B}', \mathrm{N}$ *are parallel.* Two vectors T' and N are parallel by definition of N. Since $\langle \mathrm{T}, \mathrm{B}\rangle = 0$ and $\langle \mathrm{B}, \mathrm{B}\rangle = 1$ are constant, we have

$$\langle \mathrm{B}', \mathrm{T}\rangle = \langle \mathrm{B}, \mathrm{T}\rangle' - \langle \mathrm{B}, \mathrm{T}'\rangle = 0, \qquad \langle \mathrm{B}', \mathrm{B}\rangle = \tfrac{1}{2}\langle \mathrm{B}, \mathrm{B}\rangle' = 0,$$

which show B' and N are parallel. By the definition of κ and τ, we get

$$\mathrm{T}' = \|\alpha'\|\kappa \mathrm{N}, \qquad \mathrm{B}' = -\|\alpha'\|\tau \mathrm{N}.$$

Step 2: Describe N'*.* Since

$$\begin{aligned} \langle \mathrm{N}', \mathrm{T}\rangle &= -\langle \mathrm{N}, \mathrm{T}'\rangle = -\|\alpha'\|\kappa, \\ \langle \mathrm{N}', \mathrm{N}\rangle &= \tfrac{1}{2}\langle \mathrm{N}, \mathrm{N}\rangle' = 0, \\ \langle \mathrm{N}', \mathrm{B}\rangle &= -\langle \mathrm{N}, \mathrm{B}'\rangle = \|\alpha'\|\tau, \end{aligned}$$

we have

$$\mathrm{N}' = \|\alpha'\|(-\kappa \mathrm{T} + \tau \mathrm{B}).$$ □

Remark. Let $\mathrm{X}(t)$ be the curve of orthogonal matrices $(\mathrm{T}(t), \mathrm{N}(t), \mathrm{B}(t))^T$. Then, the Frenet-Serret formula reads

$$\mathrm{X}'(t) = A(t)\mathrm{X}(t)$$

for a matrix curve $A(t)$ that is completely determined by $\kappa(t)$ and $\tau(t)$, if we let us only consider arc-length parametrized curves. This is a typical form of an ODE system, so we can apply the Picard-Lindelöf theorem to get the following proposition: if we know $\kappa(t)$ and $\tau(t)$ for all time t, and if $\mathrm{T}(0)$ and $\mathrm{N}(0)$ are given so that an initial condition

$$\mathrm{X}(0) = (\mathrm{T}(0),\ \mathrm{N}(0),\ \mathrm{T}(0) \times \mathrm{N}(0))$$

is established, then the solution $\mathrm{X}(t)$ exists and uniquely determined in a short time range. Furthermore, if $\alpha(0)$ is given in addition, the integration

$$\alpha(t) = \alpha(0) + \int_0^t \mathrm{T}(s)\, ds$$

provides a complete formula for unit speed parametrization α.

Remark. Skew-symmetry in the Frenet-Serret formula is not by chance. Let $\mathrm{X}(t) = (\mathrm{T}(t), \mathrm{N}(t), \mathrm{B}(t))^T$ and write $\mathrm{X}'(t) = A(t)\mathrm{X}(t)$ as we did in the above remark. Since $\mathrm{X}(t+h) = R_t(h)\mathrm{X}(t)$ for a family of special orthogonal matrices $\{R_t(h)\}_h$ with $R_t(0) = I$, we can describe $A(t)$ as

$$A(t) = \left.\frac{dR_t}{dh}\right|_{h=0}.$$

By differentiating the relation $R_t^T(h)R_t(h) = I$ with respect to h, we get to know that $A(t)$ is skew-symmetric for all t. In other words, the tangent space $T_I\mathrm{SO}(3)$ forms a skew symmetric matrix.

2.2. **Problems.**

2.2.1. *Computational problems.* The following proposition gives the most effective and shortest way to compute the Frenet-Serret apparatus in general case. If we try to reparametrize the given curve into a unit speed curve or find κ by differentiating T, then we must encounter the normalizing term of the form $\sqrt{(-)^2 + (-)^2 + (-)^2}^{-1}$, and it must be painful when time is limited. The Frenet-Serret frame is useful in proofs of interesting propositions, but not a good choice for practical computation. Instead, a computation from derivatives of parametrization is highly recommended.

Proposition 2.3. *Let α be a nondegenerate curve. Then,*

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\alpha' \times \alpha'' \cdot \alpha'''}{\|\alpha' \times \alpha''\|}$$

and

$$\mathrm{T} = \frac{\alpha'}{\|\alpha'\|}, \qquad \mathrm{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}, \qquad \mathrm{N} = \mathrm{B} \times \mathrm{T}.$$

Proof. If we let $s = \|\alpha'\|$, then

$$\begin{aligned}
\alpha' &= s\mathrm{T}, \\
\alpha'' &= s'\mathrm{T} + s^2\kappa\mathrm{N}, \\
\alpha''' &= (s'' - s^3\kappa^2)\mathrm{T} + (3ss'\kappa + s^2\kappa')\mathrm{N} + (s^3\kappa\tau)\mathrm{B}.
\end{aligned}$$

Now the formulas are easily derived. □

2.2.2. *General problems.* We are interested in regular curves, not a particular parametrization. By the Theorem 2.1, we may always assume that a parametrization α has unit speed. Let α be a nondegenerate unit speed space curve, and let $\{\mathrm{T}, \mathrm{N}, \mathrm{B}\}$ be the Frenet-Serret frame for α.

Consider a diagram as follows:

$$\begin{array}{ccccc}
\langle \alpha, \mathrm{T}\rangle = \,? & \longleftrightarrow & \langle \alpha, \mathrm{N}\rangle = \,? & \longleftrightarrow & \langle \alpha, \mathrm{B}\rangle = \,? \\
\downarrow & & \downarrow & & \downarrow \\
\langle \alpha', \mathrm{T}\rangle = 1 & & \langle \alpha', \mathrm{N}\rangle = 0 & & \langle \alpha', \mathrm{B}\rangle = 0.
\end{array}$$

Here the arrows indicate which term we are able to get by differentiation. For example, if we know a condition

$$\langle \alpha(t), \mathrm{T}(t)\rangle = f(t),$$

then we can obtain

$$\langle \alpha(t), \mathrm{N}(t) \rangle = \frac{f'(t) - 1}{\kappa(t)}$$

by direct differentiation since we have known $\langle \alpha', \mathrm{T} \rangle$ but not $\langle \alpha, \mathrm{N} \rangle$. Further, we get

$$\langle \alpha(t), \mathrm{B}(t) \rangle = \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t) f(t)}{\tau(t)}$$

since we have known $\langle \alpha, \mathrm{T} \rangle$ and $\langle \alpha', \mathrm{N} \rangle$ but not $\langle \alpha, \mathrm{B} \rangle$. Thus, $\langle \alpha, \mathrm{T} \rangle = f$ implies

$$\alpha(t) = f(t) \cdot \mathrm{T} + \frac{f'(t) - 1}{\kappa(t)} \cdot \mathrm{N} + \frac{\left(\frac{f'(t)-1}{\kappa(t)}\right)' + \kappa(t) f(t)}{\tau(t)} \cdot \mathrm{B},$$

when given $\tau(t) \neq 0$.

We suggest a strategy for space curve problems:

- Build and differentiate equations of the following form:

$$\langle \text{ (interesting vector), (Frenet-Serret basis) } \rangle \; = \; \text{(some function)}.$$

- Aim for finding the coefficients of the position vector in the Frenet-Serret frame, and obtain relations of κ and τ by comparing with assumptions.
- Heuristically find a constant vector and show what you want directly.

Here we give example solutions of several selected problems. Always α denotes a reparametrized unit speed nondegenerate curve in $\mathbb{R}^3$.

Problem 2.1. A curve whose normal lines always pass through a fixed point lies in a circle.

Solution. Step 1: Formulate conditions. By the assumption, there is a constant point $p \in \mathbb{R}^3$ such that the vectors $\alpha - p$ and N are parallel so that we have

$$\langle \alpha - p, \mathrm{T} \rangle = 0, \qquad \langle \alpha - p, \mathrm{B} \rangle = 0.$$

Our goal is to show that $\|\alpha - p\|$ is constant and there is a constant vector v such that $\langle \alpha - p, v \rangle = 0$.

Step 2: Collect information. Differentiate $\langle \alpha - p, \mathrm{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{B} \rangle = 0$ to get

$$\tau = 0.$$

Step 3: Complete proof. We can deduce that $\|\alpha - p\|$ is constant from

$$(\|\alpha - p\|^2)' = \langle \alpha - p, \alpha - p \rangle' = 2 \langle \alpha - p, \mathrm{T} \rangle = 0.$$

Also, if we heuristically define a vector $v := \mathrm{B}$, then v is constant since

$$v' = -\tau \mathrm{N} = 0,$$

and clearly $\langle \alpha - p, v \rangle = 0$ □

Problem 2.2. A spherical curve of constant curvature lies in a circle.

Solution. Step 1: Formulate conditions. The condition that α lies on a sphere can be given as follows: for a constant point $p \in \mathbb{R}^3$,

$$\|\alpha - p\| = \mathrm{const}\,.$$

Also we have

$$\kappa = \mathrm{const}\,.$$

Step 2: Collect information. Differentiate $\|\alpha - p\|^2 = \mathrm{const}$ to get

$$\langle \alpha - p, \mathrm{T} \rangle = 0.$$

Differentiate $\langle \alpha - p, \mathrm{T} \rangle = 0$ to get

$$\langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}.$$

Differentiate $\langle \alpha - p, \mathrm{N} \rangle = -1/\kappa = \mathrm{const}$ to get

$$\tau \langle \alpha - p, \mathrm{B} \rangle = 0.$$

There are two ways to show that $\tau = 0$.

Method 1: Assume that there is t such that $\tau(t) \neq 0$. By the continuity of τ, we can deduce that τ is locally nonvanishing. In other words, we have $\langle \alpha - p, \mathrm{B} \rangle = 0$ on an open interval containing t. Differentiate $\langle \alpha - p, \mathrm{B} \rangle = 0$ at t to get $\langle \alpha - p, \mathrm{N} \rangle = 0$ near t, which is a contradiction. Therefore, $\tau = 0$ everywhere.

Method 2: Since $\langle \alpha - p, \mathrm{B} \rangle$ is continuous and

$$\langle \alpha - p, \mathrm{B} \rangle = \pm\sqrt{\|\alpha - p\|^2 - \langle \alpha - p, \mathrm{T} \rangle^2 - \langle \alpha - p, \mathrm{N} \rangle^2} = \pm\,\mathrm{const},$$

we get $\langle \alpha - p, \mathrm{B} \rangle = \mathrm{const}$. Differentiate to get $\tau \langle \alpha - p, \mathrm{N} \rangle = 0$. Finally we can deduce $\tau = 0$ since $\langle \alpha - p, \mathrm{N} \rangle \neq 0$.

Step 3: Complete proof. The zero torsion implies that the curve lies on a plane. A planar curve in a sphere is a circle. □

Problem 2.3. A curve such that $\tau/\kappa = (\kappa'/\tau\kappa^2)'$ lies on a sphere.

Solution. Step 1: Find the center heuristically. If we assume that α is on a sphere so that we have $\|\alpha - p\| = r$ for constants $p \in \mathbb{R}^3$ and > 0, then by the routine differentiations give

$$\langle \alpha - p, \mathrm{T} \rangle = 0, \qquad \langle \alpha - p, \mathrm{N} \rangle = -\frac{1}{\kappa}, \qquad \langle \alpha - p, \mathrm{B} \rangle = -\left(\frac{1}{\kappa}\right)' \frac{1}{\tau},$$

that is,

$$\alpha - p = -\frac{1}{\kappa}\mathrm{N} - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\mathrm{B}.$$

Step 2: Complete proof. Let us get started the proof. Define

$$p := \alpha + \frac{1}{\kappa}\mathrm{N} + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\mathrm{B}.$$

We can show that it is constant by differentiation. Also we can show that

$$\langle \alpha - p, \alpha - p \rangle$$

is constant by differentiation. So we are done. □

Problem 2.4. A curve with more than one Bertrand mates is a circular helix.

Solution. Step 1: Formulate conditions. Let β be a Bertrand mate of α so that we have

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \mathrm{N}_\beta = \pm \mathrm{N},$$

where λ is a function not vanishing somewhere and $\{\mathrm{T}_\beta, \mathrm{N}_\beta, \mathrm{B}_\beta\}$ denotes the Frenet-Serret frame of β. We can reformulate the conditions as follows:

$$\begin{array}{ccccc}
\langle \beta - \alpha, \mathrm{T} \rangle = 0 & \longleftrightarrow & \langle \beta - \alpha, \mathrm{N} \rangle = \lambda & \longleftrightarrow & \langle \beta - \alpha, \mathrm{B} \rangle = 0 \\
\downarrow & & \downarrow & & \downarrow \\
\langle \mathrm{T}_\beta, \mathrm{T} \rangle = ? & \longleftrightarrow & \langle \mathrm{T}_\beta, \mathrm{N} \rangle = 0 & \longleftrightarrow & \langle \mathrm{T}_\beta, \mathrm{B} \rangle = ? \\
\downarrow & & \downarrow & & \downarrow \\
\langle \mathrm{N}_\beta, \mathrm{T} \rangle = 0 & \longleftrightarrow & \langle \mathrm{N}_\beta, \mathrm{N} \rangle = \pm 1 & \longleftrightarrow & \langle \mathrm{N}_\beta, \mathrm{B} \rangle = 0.
\end{array}$$

Note that β is not unit speed.

Step 2: Collect information. Differentiate $\langle \beta - \alpha, \mathrm{N} \rangle = \lambda$ to get

$$\lambda = \text{const} \neq 0.$$

Differentiate $\langle \beta - \alpha, \mathrm{T} \rangle = 0$ and $\langle \beta - \alpha, \mathrm{B} \rangle = 0$ to get

$$\langle \mathrm{T}_\beta, \mathrm{T} \rangle = \frac{1 - \lambda\kappa}{\|\beta'\|}, \qquad \langle \mathrm{T}_\beta, \mathrm{B} \rangle = \frac{\lambda\tau}{\|\beta'\|}.$$

Differentiate $\langle \mathrm{T}_\beta, \mathrm{T} \rangle$ and $\langle \mathrm{T}_\beta, \mathrm{B} \rangle$ to get

$$\frac{1 - \lambda\kappa}{\|\beta'\|} = \text{const}, \qquad \frac{\lambda\tau}{\|\beta'\|} = \text{const}\,.$$

Thus, there exists a constant μ such that

$$1 - \lambda\kappa = \mu\lambda\tau$$

if α is not planar so that $\tau \neq 0$.

We have shown that the torsion is either always zero or never zero at every point: $\lambda\tau/\|\beta'\| = \text{const}$. The problem can be solved by dividing the cases, but in this solution we give only for the case that α is not planar; the other hand is not difficult.

Step 3: Complete proof. If

$$\beta = \alpha + \lambda \mathrm{N}, \qquad \widetilde{\beta} = \alpha + \widetilde{\lambda} \mathrm{N}$$

are different Bertrand mates of α with $\lambda \neq \widetilde{\lambda}$, then (κ, τ) solves a two-dimensional linear system

$$\begin{cases} \kappa + \mu\tau = \lambda^{-1}, \\ \kappa + \widetilde{\mu}\tau = \widetilde{\lambda}^{-1}. \end{cases}$$

It is nonsingular since $\mu = \widetilde{\mu}$ implies $\lambda = \widetilde{\lambda}$, which means we can represent κ and τ in terms of constants $\lambda, \widetilde{\lambda}, \mu$, and $\widetilde{\mu}$. Therefore, κ and τ are constant. $\square$

Here is a well-prepared problem set for exercises.

Problem 2.5 (Plane curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:

(1) the curve α lies on a plane,

(2) $\tau = 0$,
(3) the osculating plane constains a fixed point.

Problem 2.6 (Helices)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:
(1) the curve α is a helix,
(2) $\tau/\kappa = \text{const}$,
(3) normal lines are parallel to a plane.

Problem 2.7 (Sphere curves)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:
(1) the curve α lies on a sphere,
(2) $(1/\kappa)^2 + ((1/\kappa)'/\tau)^2 = \text{const}$,
(3) $\tau/\kappa = (\kappa'/\tau\kappa^2)'$,
(4) normal planes contain a fixed point.

Problem 2.8 (Bertrand mates)**.** Let α be a nondegenerate curve in $\mathbb{R}^3$. TFAE:
(1) the curve α has a Bertrand mate,
(2) there are two constants $\lambda \neq 0, \mu$ such that $1/\lambda = \kappa + \mu\tau$.

3. Local theory of surfaces

3.1. **Theory.**

3.1.1. *Parametrization.*

Theorem 3.1. *Let S be a regular surface. Let v, w be linearly independent tangent vectors in T_pS for a point $p \in S$. Then, S admits a parametrization α such that $\alpha_x|_p = v$ and $\alpha_y|_p = w$.*

Theorem 3.2. *Let X, Y be linearly independent tangent vector fields on a regular surface S. Then, S admits a parametrization α such that $\alpha_x|_p$ and $\alpha_y|_p$ are parallel to $X|_p, Y|_p$ respectively for each $p \in S$.*

Theorem 3.3. *Let X, Y be linearly independent tangent vector fields on a regular surface S. If $\partial_X Y f = \partial_Y X$, then S admits a parametrization α such that $\alpha_x|_p = X|_p$ and $\alpha_y| = Y|_p$ for each $p \in S$.*

Let S be a regular surface embedded in $\mathbb{R}^3$. The inner product on T_pS induced from the standard inner product of $\mathbb{R}^3$ can be represented not only as a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the basis $\{(1,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{R}^3$, but also as a matrix

$$\begin{pmatrix} \langle \alpha_x, \alpha_x \rangle & \langle \alpha_x, \alpha_y \rangle \\ \langle \alpha_y, \alpha_x \rangle & \langle \alpha_y, \alpha_y \rangle \end{pmatrix}$$

in the basis $\{\alpha_x|_p, \alpha_y|_p\} \subset T_pS$.

Definition 3.1. *Metric coefficients*

$$\langle \alpha_x, \alpha_x \rangle =: g_{11} \qquad \langle \alpha_x, \alpha_y \rangle =: g_{12}$$
$$\langle \alpha_y, \alpha_x \rangle =: g_{21} \qquad \langle \alpha_y, \alpha_y \rangle =: g_{22}$$

Theorem 3.4 (Normal coordinates)**.** *...?*

3.1.2. *Differentiation of tangent vectors.*

Definition 3.2. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. The *Gauss map* or *normal unit vector* $\nu : U \to \mathbb{R}^3$ is a vector field on α defined by:

$$\nu(x,y) := \frac{\alpha_x \times \alpha_y}{\|\alpha_x \times \alpha_y\|}(x,y).$$

The set of vector fields $\{\alpha_x|_p, \alpha_y|_p, \nu|_p\}$ forms a basis of $T_p\mathbb{R}^3$ at each point p on α. The Gauss map is uniquely determined up to sign as α changes.

Definition 3.3 (Gauss formula, Γ^k_{ij}, L_{ij})**.** Let $\alpha : U \to \mathbb{R}^3$ be a regular surface. Define indexed families of smooth functions $\{\Gamma^k_{ij}\}^2_{i,j,k=1}$ and $\{L_{ij}\}^2_{i,j=1}$ by the Gauss formula

$$\begin{aligned} \alpha_{xx} &=: \Gamma^1_{11}\alpha_x + \Gamma^2_{11}\alpha_y + L_{11}\nu, & \alpha_{xy} &=: \Gamma^1_{12}\alpha_x + \Gamma^2_{12}\alpha_y + L_{12}\nu, \\ \alpha_{yx} &=: \Gamma^1_{21}\alpha_x + \Gamma^2_{21}\alpha_y + L_{21}\nu, & \alpha_{yy} &=: \Gamma^1_{22}\alpha_x + \Gamma^2_{22}\alpha_y + L_{22}\nu. \end{aligned}$$

The *Christoffel symbols* refer to eight functions $\{\Gamma^k_{ij}\}^2_{i,j,k=1}$. The Christoffel symbols and L_{ij} *do depend* on α.

We can easily check the symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ and $L_{ij} = L_{ji}$. Also,

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i (\partial_i Y^k)\alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma^k_{ij} \right) \alpha_k + X^i Y^j L_{ij} \nu. \end{aligned}$$

3.1.3. *Differentiation of normal vector.* The partial derivative $\partial_X \nu$ is a tangent vector field since

$$\langle \partial_X \nu, \nu \rangle = \frac{1}{2} \partial_X \langle \nu, \nu \rangle = 0.$$

Therefore, we can define the following useful operator.

Definition 3.4. Let S be a regular surface embedded in $\mathbb{R}^3$. The *shape operator* is $\mathcal{S} : \mathfrak{X}(S) \to \mathfrak{X}(S)$ defined as

$$\mathcal{S}(X) := -\partial_X \nu.$$

Proposition 3.5. *The shape operator is self-adjoint, i.e. symmetric.*

Proof. Recall that $\partial_X Y - \partial_Y X$ is a tangent vector field. Then,

$$\langle X, \mathcal{S}(Y) \rangle = \langle X, -\partial_Y \nu \rangle = \langle \partial_Y X, \nu \rangle = \langle \partial_X Y, \nu \rangle = \langle \mathcal{S}(X), Y \rangle. \qquad \square$$

Theorem 3.6. *Let $\alpha : U \to \mathbb{R}^3$ be a regular surface and $\mathcal{S}$ be the shape operator. Then $\mathcal{S}$ has the coordinate representation*

$$\mathcal{S} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

with respect to the frame $\{\alpha_x, \alpha_y\}$ for tangent spaces. In other words, if we let $X = X^i \alpha_i$ and $\mathcal{S}(X) = \mathcal{S}(X)^j \alpha_j$, then

$$\begin{pmatrix} \mathcal{S}(X)^1 \\ \mathcal{S}(Y)^2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

Proof. Let $\mathcal{S}(X)^j = \mathcal{S}^j_i X_i$. Then, $g_{ik}\,\mathcal{S}^k_j = L_{ij}$ since

$$g_{ik}X^i\mathcal{S}^k_jY^j = \langle X, \mathcal{S}(Y)\rangle = \langle \partial_X Y, \nu\rangle = X^iY^jL_{ij}. \qquad \square$$

3.2. **Problems.**

3.2.1. *Computational problems.*

Definition 3.5. Let $\alpha : U \to \mathbb{R}^3$ be a regular surface.

$$E := \langle \alpha_x, \alpha_x\rangle = g_{11}, \qquad F := \langle \alpha_x, \alpha_y\rangle = g_{12}, \qquad G := \langle \alpha_y, \alpha_y\rangle = g_{22},$$
$$L := \langle \alpha_{xx}, \nu\rangle = L_{11}, \qquad M := \langle \alpha_{xy}, \nu\rangle = L_{12}, \qquad N := \langle \alpha_{yy}, \nu\rangle = L_{22}.$$

Corollary 3.7. *We have $GM - FN = EM - FL$, and the* Weingarten equations*:*

$$\nu_x = \frac{FM - GL}{EG - F^2}\alpha_x + \frac{FL - EM}{EG - F^2}\alpha_y,$$
$$\nu_y = \frac{FN - GM}{EG - F^2}\alpha_x + \frac{FM - EN}{EG - F^2}\alpha_y.$$

Theorem 3.8.

$$\Gamma^l_{ij} = \frac{1}{2}g^{kl}(g_{ik,j} - g_{ij,k} + g_{kj,i}).$$

$$\frac{1}{2}(\log g)_x = \Gamma^1_{11}.$$

$$\nu_x \times \nu_y = K\sqrt{\det g}\ \nu.$$
$$\alpha_x \times \alpha_y = \sqrt{\det g}\ \nu$$
$$\langle \nu_x \times \nu_y, \alpha_x \times \alpha_y\rangle = \det\begin{pmatrix} \langle \nu_x, \alpha_x\rangle & \langle \nu_x, \alpha_y\rangle \\ \langle \nu_y, \alpha_x\rangle & \langle \nu_y, \alpha_y\rangle \end{pmatrix} = \det\begin{pmatrix} -L & -M \\ -M & -N \end{pmatrix} = K\det g$$

Theorem 3.9 (Gaussian curvature formula)**.**

(1) *In general,*
$$K = \frac{LN - M^2}{EG - F^2}.$$
(2) *For orthogonal coordinates such that $F \equiv 0$,*
$$K = -\frac{1}{2\sqrt{\det g}}\left((\frac{1}{\sqrt{\det g}}E_y)_y + (\frac{1}{\sqrt{\det g}}G_x)_x \right).$$
(3) *For $f(x, y, z) = 0$,*
$$K = -\frac{1}{|\nabla f|^4}\begin{vmatrix} 0 & \nabla f \\ \nabla f^T & \mathrm{Hess}(f) \end{vmatrix},$$
where ∇f denotes the gradient $\nabla f = (f_x, f_y, f_z)$.
(4) *(Beltrami-Enneper) If τ is the torsion of an asymptotic curve, then*
$$K = -\tau^2.$$
(5) *(Brioschi) E, F, G describes K.*

Proof.

(1) Clear.
(2) We have $GM = EM$ and

$$\nu_x = -\frac{L}{E}\alpha_x - \frac{M}{G}\alpha_y, \qquad \nu_y = -\frac{M}{E}\alpha_x - \frac{N}{G}\alpha_y.$$

$$\nu_x \times \nu_y = \frac{LN - M^2}{EG}\alpha_x \times \alpha_y$$

After curvature tensors...

□

Example 3.1.

(1) (Monge's patch) For $(x, y, f(x, y))$,

$$K = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.$$

(2) (Surface of revolution). Let $\gamma(t) = (r(t), z(t))$ be a plane curve with $r(t) > 0$. Let

$$\alpha(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, z(t))$$

be a parametrization of a surface of revolution.

Then,

$$\begin{aligned} \alpha_\theta &= (-r(t)\sin\theta, r(t)\cos\theta, 0) \\ \alpha_t &= (r'(t)\cos\theta, r'(t)\sin\theta, z'(t)) \\ \nu &= \frac{1}{\sqrt{r'(t)^2 + z'(t)^2}}(z'(t)\cos\theta, z'(t)\sin\theta, -r'(t)), \end{aligned}$$

and

$$\begin{aligned} \alpha_{\theta\theta} &= (-r(t)\cos\theta, -r(t)\sin\theta, 0) \\ \alpha_{\theta t} &= (-r'(t)\sin\theta, -r'(t)\cos\theta, 0) \\ \alpha_{tt} &= (r''(t)\cos\theta, r''(t)\sin\theta, z''(t)). \end{aligned}$$

Thus we have

$$E = r(t)^2, \quad F = 0, \quad G = r'(t)^2 + z'(t)^2,$$

and

$$L = -\frac{r(t)z'(t)}{\sqrt{r'(t)^2 + z'(t)^2}}, \quad M = 0, \quad N = \frac{r''(t)z'(t) - r'(t)z''(t)}{\sqrt{r'(t)^2 + z'(t)^2}}.$$

Therefore,

$$K = \frac{LN - M^2}{EG - F^2} = \frac{z'(r'z'' - r''z')}{r(r'^2 + z'^2)^2}.$$

In particular, if $t \mapsto (r(t), z(t))$ is a unit-speed curve, then

$$K = -\frac{r''}{r}.$$

(3) (Models of hyperbolic planes)

3.2.2. *Constant curvature.*

Theorem 3.10. *Surfaces of the same constant Gaussian curvature are locally isomorphic.*

Proof. Let

$$\begin{pmatrix} \|\alpha_r\|^2 & \langle \alpha_r, \alpha_t \rangle \\ \langle \alpha_t, \alpha_r \rangle & \|\alpha_t\|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & h(r,t)^2 \end{pmatrix}$$

be the first fundamental form for a geodesic coordinate chart along a geodesic curve so that α_{tt} and α_{rr} are normal to the surface. Then,

$$K = -\frac{h_{rr}}{h}$$

is constant. Also, since

$$\frac{1}{2}(h^2)_r + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_{rt}, \alpha_t \rangle + \langle \alpha_r, \alpha_{tt} \rangle = \langle \alpha_r, \alpha_t \rangle_t = 0$$

implies $h_r = 0$ at $r = 0$, the function $f : r \mapsto h(r,t)$ satisfies the following initial value problem

$$f_{rr} = -Kf, \quad f(0) = 1, \quad f'(0) = 0.$$

Therefore, h is uniquely determined by K. □

4. Intrinsic geometry

Notations: Einstein summation convention, set of vector fields.
To n-dimensional.

4.1. **Intrinsicness.**

4.1.1. *Coordinates intrinsicness.*

4.1.2. *Metric intrinsicness.* Isometry

Example 4.1. Let $\alpha : (-\log 2, \log 2) \times (0, 2\pi) \to \mathbb{R}^3$ and $\beta : (-\frac{3}{4}, \frac{3}{4}) \times (0, 2\pi) \to \mathbb{R}^3$ be regular surfaces given by

$$\alpha(x,\theta) = (\cosh x \cos\theta,\ \cosh x \sin\theta,\ x), \qquad \beta(r,z) = (r\cos z,\ r\sin z,\ z).$$

Their Riemannian metrics are

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix}_{(\alpha_x, \alpha_\theta)}, \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 + r^2 \end{pmatrix}_{(\beta_r, \beta_z)}.$$

Define a map $f : \operatorname{im}\alpha \to \operatorname{im}\beta$ by

$$f : \alpha(x,\theta) \mapsto \beta(\sinh x, \theta) = (r(x,\theta), z(x,\theta)).$$

The Jacobi matrix of f is computed

$$df|_{\alpha(x,\theta)} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}_{(\alpha_x, \alpha_\theta) \to (\beta_r, \beta_z)}.$$

Since f is a diffeomorphism and

$$\begin{pmatrix} \cosh^2 x & 0 \\ 0 & \cosh^2 x \end{pmatrix} = \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & 1 + r^2 \end{pmatrix} \begin{pmatrix} \cosh x & 0 \\ 0 & 1 \end{pmatrix},$$

the map f is an isometry.

4.2. **Covariant derivatives.**

4.2.1. *Orthogonal projection.* We are going to think about "intrinsic" derivatives for tangent vectors. For coordinate independence, directional derivatives of a tangent vector field should be at least a tangent vector field, which is false for the obvious partial derivatives in the embedded surface setting; for example, T is a tangent vector, but $\mathrm{N} = \kappa \mathrm{T}'$ is not tangent.

Recall that the Gauss formula reads

$$\partial_i \alpha_j = \Gamma_{ij}^k \alpha_k + L_{ij} \nu$$

so that we have

$$\begin{aligned} \partial_X Y &= X^i \partial_i (Y^j \alpha_j) \\ &= X^i (\partial_i Y^k) \alpha_k + X^i Y^j \partial_i \alpha_j \\ &= \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k + X^i Y^j L_{ij} \nu. \end{aligned}$$

If we write $\nabla_X Y = \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k$, then it embodies the orthogonal projection of $\partial_X Y$ onto its tangent space, and we have

$$\partial_X Y = \nabla_X Y + \mathrm{II}(X, Y) \nu.$$

Definition 4.1. Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $\operatorname{im} \alpha = M$. Let $X = X^i \alpha_i$ and $Y = Y^j \alpha_j$ be tangent vector fields on M. The *covariant derivative* of Y along X is defined as the orthogonal projection of the partial derivative $\partial_X Y$ onto the tangent space:

$$\nabla_X Y := \left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k.$$

Proposition 4.1. *Covariant derivatives are intrinsic. In other words, the above definition does not depend on the choice of parametrizations.*

Proof. Recall that the Christoffel symbols transform as follows:

$$X^i Y^j \Gamma_{ij}^k = X^a Y^b \left(\Gamma_{ab}^c + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \, \partial x^j} \right) \frac{\partial x^k}{\partial x^c}.$$

Thus, we have

$$\begin{aligned} &\left(X^i \partial_i Y^k + X^i Y^j \Gamma_{ij}^k \right) \alpha_k \\ &= X^a \frac{\partial}{\partial x^a} \left(Y^c \frac{\partial x^k}{\partial x^c} \right) \alpha_k + X^a Y^b \left(\frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \, \partial x^j} + \Gamma_{ab}^c \right) \frac{\partial x^k}{\partial x^c} \alpha_k \\ &= X^a \frac{\partial Y^c}{\partial x^a} \alpha_c + X^a Y^b \left(\frac{\partial^2 x^k}{\partial x^a \, \partial x^b} \frac{\partial x^c}{\partial x^k} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \, \partial x^j} \right) \alpha_c + X^a X^b \Gamma_{ab}^c \alpha_c \\ &= \left(X^a \partial_a Y^c + X^a Y^b \Gamma_{ab}^c \right) \alpha_c \end{aligned}$$

since

$$\frac{\partial^2 x^j}{\partial x^a \, \partial x^b} \frac{\partial x^c}{\partial x^j} + \frac{\partial x^i}{\partial x^a} \frac{\partial x^j}{\partial x^b} \frac{\partial^2 x^c}{\partial x^i \, \partial x^j} = \frac{\partial}{\partial x^a} \left(\frac{\partial x^j}{\partial x^b} \frac{\partial x^c}{\partial x^j} \right) = \partial_a \delta_b^c = 0. \qquad \square$$

4.2.2. *Connections.* We will give a coordinate-free axiomatic definition of covariant derivatives and show that they coincide. By doing this, we obtain an alternative proof for the statement that covariant derivatives are intrinsic.

Definition 4.2 (Affine connection)**.** Let M be the image of a parametrization. An *affine connection* is a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that

(1) $\nabla_{(-)}Y : \mathfrak{X}(M) \to \mathfrak{X}(M) : X \mapsto \nabla_X Y$ is $C^\infty(M)$-linear;
(2) $\nabla_X(-) : \mathfrak{X}(M) \to \mathfrak{X}(M) : Y \mapsto \nabla_X Y$ is $\mathbb{R}$-linear;
(3) the Leibniz rule
$$\nabla_X(fY) = (\partial_X f)Y + f\nabla_X Y$$
is satisfied.

Definition 4.3 (Metric connection)**.** Let M be the image of a parametrization and $\langle\, ,\rangle$ be a Riemannian metric on M. A *metric connection* is an affine connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:
$$\partial_Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle.$$

Definition 4.4 (Levi-Civita connection)**.** Let M be the image of a parametrization and $\langle\, ,\rangle$ be a Riemannian metric on M. A *Levi-Civita connection* is a metric connection $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ such that:
$$\nabla_X Y - \nabla_Y X = \partial_X Y - \partial_Y X.$$

Theorem 4.2. *Let $\alpha : U \to \mathbb{R}^n$ be an m-dimensional parametrization with $M = \operatorname{im}\alpha$. Then, there is a unique Levi-Civita connection on M.*

Proof. (Uniqueness) Suppose ∇ is a Levi-Citiva connection on M.
$$\begin{aligned} 2\langle \nabla_X Y, Z\rangle = \partial_X\langle Y, Z\rangle + \partial_Y\langle X, Z\rangle - \partial_Z\langle X, Y\rangle \\ - \langle [X, Z], Y\rangle - \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle. \end{aligned}$$

(Existence) □

Our claim is that this definition is equivalent to the above coordinate dependent definition, the Levi-Civita connection, of the covariant derivative.

Proposition 4.3. *Let S be a regular surface embedded in $\mathbb{R}^3$. If we define Christoffel symbols as the Gauss formula, then*
$$\mathfrak{X}(S) \times \mathfrak{X}(S) \to \mathfrak{X}(S) : (X^i\alpha_i, Y^j\alpha_j) \mapsto \left(X^i\partial_i Y^k + X^i Y^j \Gamma^k_{ij}\right)\alpha_k$$
defines a Levi-Civita connection.

4.3. **Parallel transport.**

4.4. **Geodesics.**

4.5. **Curvature.**

5. Global theory