

# $C^*$ -algebras

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**Definition.** In this note, an *algebra* refers to a vector space over  $\mathbb{C}$  that has a pseudo-ring structure; always associative but possibly nonunital.

**Definition.** A normed  $*$ -algebra  $\mathcal{A}$  is called  $C^*$ -algebra if

- (1)  $\mathcal{A}$  is Banach,
- (2)  $\mathcal{A}$  satisfies the  $C^*$ -identity:  $\|x^*x\| = \|x\|^2$ .

**Theorem.** Every nonunital  $C^*$ -algebra is a maximal ideal of a unital  $C^*$ -algebra.

*Proof.* Let  $\mathcal{A}$  be a nonunital  $C^*$ -algebra. It is enough to show the existence of unital  $C^*$ -algebra  $\tilde{\mathcal{A}}$  such that  $\mathcal{A}$  is a normed  $*$ -subalgebra of  $\tilde{\mathcal{A}}$  with codimension one. It is because a subalgebra is a maximal ideal if and only if the quotient can have a natural ring structure that makes a field.

*Step 1: Construct a unital normed  $*$ -algebra.* Since  $\mathcal{A}$  is a Banach space, the space of bounded operators  $B(\mathcal{A})$  is a Banach algebra. We can recognize  $\mathcal{A}$  as a normed subalgebra of  $B(\mathcal{A})$  because the left multiplication  $(y \mapsto xy) \in B(\mathcal{A})$  has the norm

$$\|(y \mapsto xy)\| = \sup_{y \in \mathcal{A}} \frac{\|xy\|}{\|y\|}$$

that is shown to be equal to  $\|x\|$  by putting  $y = x^*$  and applying the  $C^*$ -identity. Define an algebra  $\tilde{\mathcal{A}}$  as the subalgebra:

$$\tilde{\mathcal{A}} := \{ (y \mapsto xy + \lambda y) \in B(\mathcal{A}) : x \in \mathcal{A}, \lambda \in \mathbb{C} \}.$$

Since  $\tilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$  as algebras, let us write the map  $y \mapsto xy + \lambda y$  as  $(x, \lambda)$ . Then,  $\tilde{\mathcal{A}}$  is a normed  $*$ -algebra with induced norm and involution

$$\|(x, \lambda)\| = \sup_{y \in \mathcal{A}} \frac{\|xy + \lambda y\|}{\|y\|}, \quad (x, \lambda)^* = (x^*, \bar{\lambda}).$$

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Then,  $\mathcal{A}$  is a normed  $*$ -subalgebra of  $\tilde{\mathcal{A}}$  because the norm and involution of  $\mathcal{A}$  agree with  $\tilde{\mathcal{A}}$ .

*Step 2:  $\tilde{\mathcal{A}}$  is Banach.* Suppose  $(x_n, \lambda_n)$  is Cauchy in  $\tilde{\mathcal{A}}$ . Since  $\mathcal{A}$  is complete so that it is closed in  $\tilde{\mathcal{A}}$ , we can induce a norm on the quotient  $\tilde{\mathcal{A}}/\mathcal{A}$  so that the canonical projection is (uniformly) continuous so that  $\lambda_n$  is Cauchy. Also, the inequality  $\|(x, \lambda)\| \leq \|(x, \lambda)\| + |\lambda|$  shows that  $x_n$  is Cauchy in  $\mathcal{A}$ .

Since a finite dimensional normed space is always Banach and  $\mathcal{A}$  is Banach,  $\lambda_n$  and  $x_n$  converge. Finally, the inequality  $\|(x, \lambda)\| \leq \|x\| + |\lambda|$  implies that  $(x_n, \lambda_n)$  converges.

*Step 3:  $\tilde{\mathcal{A}}$  is  $C^*$ .* The  $C^*$ -identity easily follows from the following inequality:

$$\begin{aligned} \|(x, \lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\ &= \sup_{\|y\|=1} \|(xy + \lambda y)^*(xy + \lambda y)\| \\ &= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y)\| \\ &\leq \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \bar{\lambda}x)y + |\lambda|^2 y\| \\ &= \|(x, \lambda)^*(x, \lambda)\|. \end{aligned} \quad \square$$

## 1. BASICS

### 1.1. Continuous functional calculus.

**Theorem 1.1** (Gelfand-Naimark). *For commutative unital  $C^*$ -algebra  $\mathcal{A}$ , the Gelfand transform gives an isometric  $*$ -isomorphism  $\Gamma : \mathcal{A} \rightarrow C(\sigma(\mathcal{A}))$ .*

*Proof. Step 1:  $\Gamma$  is a  $*$ -homomorphism.* We will show  $h(x^*) = \overline{h(x)}$  for linear characters  $h \in \sigma(\mathcal{A})$ . First assume that  $x \in \mathcal{A}$  is self-adjoint.

By the holomorphic functional calculus,

$$e^{itx} = \sum_{n=1}^{\infty} \frac{(itx)^n}{n!}.$$

Since the involution is continuous,

$$(e^{itx})^* = \sum_{n=1}^{\infty} \frac{(-itx)^n}{n!} = e^{-itx},$$

so we have  $\|e^{itx}\|^2 = \|e^{itx}e^{-itx}\| = 1$ . Then, the inequality

$$1 = \|e^{itx}\| \geq |h(e^{itx})| = |e^{ith(x)}| = e^{-t \operatorname{Im} h(x)}$$

proves  $h(x) \in \mathbb{R}$ .

For arbitrary  $x \in \mathcal{A}$ , if we define self-adjoints

$$\operatorname{Re} x := \frac{x + x^*}{2}, \quad \operatorname{Im} x := \frac{x - x^*}{2i},$$

then

$$h(x^*) = h(\operatorname{Re} x) - ih(\operatorname{Im} x) = \overline{h(\operatorname{Re} x)} - i\overline{h(\operatorname{Im} x)} = \overline{h(\operatorname{Re} x) + ih(\operatorname{Im} x)} = \overline{h(x)}$$

for all  $h \in \sigma(\mathcal{A})$ .

*Step 2:  $\Gamma$  is isometric.* Note that we have

$$\|\widehat{x}\| = \sup_{h \in \sigma(\mathcal{A})} |\widehat{x}(h)| = \sup_{h \in \sigma(\mathcal{A})} |h(x)| = r(x).$$

For self adjoint  $x \in \mathcal{A}$ , since we have  $\|x\|^2 = \|x^*x\| = \|x^2\|$ , the spectral radius coincides with the norm by the Gelfand formula for spectral radius in Banach algebras:

$$r(x) = \lim_{n \rightarrow \infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

Hence

$$\|x\|^2 = \|x^*x\| = \|\widehat{x^*x}\| = \|\widehat{x}^*\widehat{x}\| = \|\widehat{x}\|^2$$

for arbitrary  $x \in \mathcal{A}$ .

*Step 3:  $\Gamma$  is surjective.* The step 1 shows that  $\Gamma(\mathcal{A})$  is a unital  $*$ -subalgebra of  $C(\sigma(\mathcal{A}))$ , and it separates points by definition. By the Stone-Weierstrass theorem,  $\Gamma(\mathcal{A})$  is dense in  $C(\sigma(\mathcal{A}))$ . The step 2 shows that  $\Gamma(\mathcal{A})$  is complete and hence closed so that  $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A}))$ .  $\square$

**Theorem 1.2** (Gelfand-Naimark). *For commutative  $C^*$ -algebra  $\mathcal{A}$ , the Gelfand transform gives an isometric  $*$ -isomorphism  $\Gamma : \mathcal{A} \rightarrow C_0(\sigma(\mathcal{A}))$ .*

## 1.2. Positive elements.

## 1.3. Operator topologies.

**Theorem 1.3.** *Let  $f$  be a linear functional on  $B(H)$  for a Hilbert space  $H$ . Then, TFAE:*

- (1)  $f$  is WOT-continuous,
- (2)  $f$  is SOT-continuous,
- (3)  $f(T) = \sum_{i=1}^n \langle Tx_i, y_i \rangle$  for some  $x_i, y_i$ .

*Proof.* (2)  $\Rightarrow$  (3) is the only nontrivial implication. Suppose  $\square$