

## Contents

<b>1</b>	<b>Kinetic theory</b>	<b>2</b>
1.1	Vlasov-Poisson equation . . . . .	2
1.1.1	Schaeffer's global existence proof . . . . .	3
1.2	Velocity averaging lemmas . . . . .	4
<b>2</b>	<b>Sturm-Liouville theory</b>	<b>7</b>
2.1	Self-adjointness . . . . .	7
2.2	Regular Sturm-Liouville problem . . . . .	7
2.3	Legendre's equation . . . . .	8
2.4	Bessel's equation . . . . .	9
<b>3</b>	<b>Peetre's theorem</b>	<b>10</b>
<b>4</b>	<b>Characteristic curve</b>	<b>12</b>
4.1	Wave equation . . . . .	13
4.2	Burgers' equation . . . . .	15
<b>5</b>	<b>Weak convergences</b>	<b>16</b>
<b>6</b>	<b>Fixed point methods</b>	<b>17</b>
6.1	Picard-Lindelöf theorem . . . . .	17
<b>7</b>	<b>Statements in functional analysis and general topology</b>	<b>18</b>
<b>8</b>	<b>Ultrafilter</b>	<b>19</b>
<b>9</b>	<b>Selected analysis problems</b>	<b>20</b>
<b>10</b>	<b>Physics problem</b>	<b>25</b>
10.1	Resonance . . . . .	25

# 1 Kinetic theory

## 1.1 Vlasov-Poisson equation

Consider a Cauchy problem of the *Vlasov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t, x) = \nabla_x \Delta_x^{-1} \rho, \\ \rho(t, x) = \int f dv, \\ f(0, x, v) = f_0(x, v). \end{cases}$$

For boundaryless problem, we have

$$E = \nabla_x \left( -\frac{1}{4\pi|x|} * \rho \right) = \frac{x}{4\pi|x|^3} * \rho.$$

**Theorem 1.1** (Schaeffer, 1991). *Let  $f_0 \in C_{0,x,v}^1$  and  $f_0 \geq 0$ . Then, the Cauchy problem for the VP system has a unique  $C^1$  global solution.*

**Definition 1.1.** For a local solution  $f$ ,

$$Q(t) := 1 + \sup\{|v| : f(s, x, v) \neq 0 \text{ for some } s \in [0, t], x \in \mathbb{R}_x^3\}.$$

Electric field can be controlled by density  $\rho$ , henceforth by  $Q$ .

**Lemma 1.2.** *For  $1 \leq q < \frac{N}{N-2} = 3 < p \leq \infty$ ,*

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^p}^{\frac{\frac{2}{N}-1+\frac{1}{q}}{\frac{1}{q}-\frac{1}{p}}} \|\rho(t, x)\|_{L_x^q}^{\frac{1-\frac{1}{p}-\frac{2}{N}}{\frac{1}{q}-\frac{1}{p}}}.$$

Furthermore,

$$\|E(t, x)\|_{L_x^\infty} \lesssim Q(t)^\varepsilon$$

for any  $0 < \varepsilon \leq 2$ .

Note that  $|E| \lesssim Q(t)^\varepsilon$  does not prove global boundedness of  $Q(t)$ .

*Proof.* Fix time  $t$ . For  $2p < N < 2q$ ,

$$\begin{aligned} 4\pi|E(x)| &= \left| \frac{1}{|x|^2} * \rho(x) \right| \\ &\leq \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^2} dy + \int_{|x-y|\geq R} \frac{\rho(y)}{|x-y|^2} dy \\ &\leq \|\rho\|_{p'} \left( \int_{|y|<R} \frac{dy}{|y|^{2p}} \right)^{1/p} + \|\rho\|_{q'} \left( \int_{|y|\geq R} \frac{dy}{|y|^{2q}} \right)^{1/q} \\ &\simeq \|\rho\|_{p'} \left( \int_0^R r^{N-1-2p} dr \right)^{1/p} + \|\rho\|_{q'} \left( \int_R^\infty r^{N-1-2q} dr \right)^{1/q} \\ &\simeq \|\rho\|_{p'} R^{\frac{N}{p}-2} + \|\rho\|_{q'} R^{\frac{N}{q}-2}. \end{aligned}$$

By choosing  $R$  such that  $\|\rho\|_{p'} R^{\frac{N}{p}-2} = \|\rho\|_{q'} R^{\frac{N}{q}-2}$ , we get

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^{p'}}^{\frac{\frac{2}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho(t, x)\|_{L_x^{q'}}^{\frac{\frac{1}{p}-\frac{2}{q}}{\frac{1}{p}-\frac{1}{q}}},$$

hence the inequality.

If  $p = \infty$  and  $N = 3$ , then  $1 \leq q < 3$  implies

$$\|E\|_\infty^3 \lesssim \|\rho\|_\infty^{3-q} \|\rho\|_q^q.$$

For example, if we let  $(p, q) = (\infty, 1)$  or  $(\infty, 5/3)$ , then

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{2/3} \|\rho\|_1^{1/3}, \quad \|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

Since the velocity support of  $f$  is bounded by finite  $Q(t)$ , we have

$$\rho(t, x) = \int_{|v| < Q(t)} f(t, x, v) dv \lesssim Q(t)^3 \|f_0(x, v)\|_{L_v^\infty} \lesssim Q(t)^3,$$

so by letting  $\varepsilon = 3 - q$ ,

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^\infty}^{(3-q)/3} \lesssim Q(t)^\varepsilon$$

for any  $0 < \varepsilon \leq 2$ . ( $L^p$ -norm of  $\rho$  for  $p < \infty$  well propagates by the measure preserving property.)  $\square$

For a characteristic curve  $s \mapsto (X(s; t, x, v), V(s; t, x, v))$ , we have

$$f(t, x, v) = f(s, X(s; t, x, v), V(s; t, x, v))$$

and

$$\begin{aligned} & \iint f(s, x, v) \varphi(x, v) dv dx \\ &= \iint f(s, X(s; t, x, v), V(s; t, x, v)) \varphi(X(s; t, x, v), V(s; t, x, v)) dv dx \\ &= \iint f(t, x, v) \varphi(X(s; t, x, v), V(s; t, x, v)) dv dx. \end{aligned}$$

### 1.1.1 Schaeffer's global existence proof

Fix a characteristic  $(X^*(s), V^*(s)) = (X(s; t^*, x^*, v^*), V(s; t^*, x^*, v^*))$ . For  $t \in [0, t^*]$ ,

$$\begin{aligned} \int_t^{t^*} |E(s, X^*(s))| ds &\lesssim \int_t^{t^*} \iint \frac{f(s, x, v)}{|x - X^*(s)|^2} dv dx ds \\ &= \int_t^{t^*} \iint \frac{f(t, x, v)}{|X(s; t, x, v) - X^*(s)|^2} dv dx ds. \end{aligned}$$

$$\begin{aligned}
|v - V(t; t^*, x, v)| &= |V(t^*; t^*, x, v) - V(t; t^*, x, v)| \\
&\leq \int_t^{t^*} |E(s, X(s, x, v))| ds \lesssim (t^* - t)Q^{4/3}.
\end{aligned}$$

## 1.2 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

**Theorem 1.3** (Velocity averaging). *Let  $L$  be a free transport operator  $\partial_t + v \cdot \nabla_x$  on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . Then,*

$$\left\| \int u \varphi dv \right\|_{H_{t,x}^{1/2}} \lesssim_{\varphi} \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}$$

for  $\varphi \in C_c^\infty(\mathbb{R}_v^n)$ ,

*Proof.* Let  $m(t, x) = \int u \varphi dv$ . By Fourier transform with respect to  $t$  and  $x$ , we have

$$\widehat{u}(\tau, \xi, v) = \frac{1}{i} \frac{\widehat{Lu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v) \varphi(v) dv.$$

Fixing  $\tau, \xi$ , decompose the integral and use Hölder's inequality to get

$$\begin{aligned}
|\widehat{m}(\tau, \xi)| &\leq \int_{|\tau+v \cdot \xi| < \alpha} |\widehat{u} \varphi| dv + \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\widehat{Lu} \varphi|}{|\tau + v \cdot \xi|} dv \\
&\leq \|\widehat{u}\|_{L_v^2}^{1/2} \left( \int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv \right)^{1/2} + \|\widehat{Lu}\|_{L_v^2}^{1/2} \left( \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2} dv \right)^{1/2},
\end{aligned}$$

where  $\alpha > 0$  is an arbitrary constant that will be determined later. Let

$$I_s(\tau, \xi, \alpha) := \int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv, \quad I_n(\tau, \xi, \alpha) := \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|} dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \quad I_n \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

Define coordinates  $(v_1, v_2)$  on  $\mathbb{R}_v$  as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}, \quad v_2 := v - \frac{v \cdot \xi}{|\xi|^2} \xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v_1 - \frac{\tau}{|\xi|})^2 + |v_2|^2 \quad \text{and} \quad \int dv = \iint dv_2 dv_1.$$

For the first integral, suppose that  $\varphi$  is supported on a ball  $|v| \leq R$ . If  $\frac{|\tau| - \alpha}{|\xi|} > R$ , then the region of integration vanishes so that  $I_s = 0$ . If  $|\tau| \leq \alpha + R|\xi|$ , then

$$\begin{aligned} I_s &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 dv_1 \\ &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}, |v_1| \leq R} \int_{|v_2| \leq R} dv_2 dv_1 \\ &\lesssim \min\left\{\frac{2\alpha}{|\xi|}, R\right\} \cdot R^{n-1} \\ &\lesssim \frac{1}{\sqrt{1 + (\frac{|\xi|}{\alpha})^2}} \\ &\lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}. \end{aligned}$$

For the second integral, suppose that  $\varphi$  is supported on  $|v| < C$  so that  $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < C$ . Then,

$$\begin{aligned} I_n &\lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|} < C} \int_{|v_2| < C} \frac{1}{v_1^2 |\xi|^2} dv_2 dv_1 \\ &\simeq \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|} < C} \frac{dv_1}{v_1^2 |\xi|^2}. \end{aligned}$$

If  $|\xi| \gtrsim |\tau|$ , then

$$I_n \lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}} \frac{dv_1}{v_1^2 |\xi|^2} \simeq \frac{1}{\alpha |\xi|} \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

If  $|\xi| \ll |\tau|$  such that at least  $|\tau| > C|\xi|$ , then

$$\begin{aligned} I_n &\lesssim \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - C\} \leq v_1 < \frac{|\tau|}{|\xi|} + C} \frac{dv_1}{v_1^2 |\xi|^2} \\ &\simeq \frac{1}{|\xi|^2} \left( \frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - C\}} - \frac{1}{\frac{|\tau|}{|\xi|} + C} \right) \\ &\lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}. \quad (\text{This is not easy..!}) \end{aligned}$$

To sum up, we have

$$|\widehat{m}(\tau, \xi)| \lesssim \frac{1}{(\tau^2 + |\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Lu}\|_{L_v^2}^{1/2}).$$

Letting  $\alpha = \sqrt{\|\widehat{Lu}\|_{L_v^2} / \|\widehat{u}\|_{L_v^2}}$  and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2} \|\widehat{Lu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on  $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$  and Plancheral's theorem gives

$$\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}.$$

□

*Remark.* We can obtain  $L^p$  result by applying

$$\min\left\{\frac{1}{x}, \frac{1}{y}\right\} \leq \frac{2}{(x^p + y^p)^{1/p}} \leq \max\left\{\frac{1}{x}, \frac{1}{y}\right\}$$

on the first integral, and

$$x \gg 1 \implies \frac{1}{\max\{a, x - c\}} - \frac{1}{x + c} \lesssim \frac{1}{a(x^p + 1)^{1/p}}$$

on the second integral.

**Corollary 1.4.** *Let  $\mathcal{F}$  be a family of functions on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  and  $L\mathcal{F}$  are bounded in  $L_{t,x,v}^2$ , then  $\int \mathcal{F} \varphi dv$  is bounded in  $H_{t,x}^{1/2}$ .*

**Theorem 1.5.** *Let  $\mathcal{F}$  be a family of functions on  $I_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  is weakly relatively compact and  $L\mathcal{F}$  is bounded in  $L_{t,x,v}^1$ , then  $\int \mathcal{F} \varphi dv$  is relatively compact in  $L_{t,x}^1$ .*

## 2 Sturm-Liouville theory

### 2.1 Self-adjointness

Let  $I = [a, b]$  and

$$L = -\frac{1}{w(x)} \left[ \frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) \right],$$

$$0 \leq p(x) \in C^\infty(I), \quad q(x) \in C^\infty(I), \quad 0 < w(x) \in C^\infty(I).$$

We expect  $L$  to be self-adjoint. In this regard, our interest is elimination of the difference term

$$\langle f, Lg \rangle - \langle Lf, g \rangle = p(f'g - fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[ \frac{d}{dx} \left( (1-x^2) \frac{d}{dx} \right) - \frac{m^2}{1-x^2} \right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2} \left[ \frac{d}{dx} \left( e^{-x^2} \frac{d}{dx} \right) \right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

### 2.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that  $p$  does not vanish on the boundary of  $I$  that we should cancel  $f'g - fg'|_a^b$ . View the Sturm-Liouville operator  $L$  as a non-densely defined operator on the space  $C^\infty(I)$  with inner product  $\langle f, g \rangle = \int_I fgw$  with domain

$$V = \{ u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0, \beta_0 u(b) + \beta_1 u'(b) = 0 \},$$

the subspace for the *separated* Robin boundary condition.

**Proposition 2.1.** *The operator  $L : V \rightarrow C^\infty(I)$  is self-adjoint when  $C^\infty(I)$  has the inner product  $\langle f, g \rangle = \int_I fgw$ .*

We are interested in the eigenvalue problem of  $L : V \rightarrow C^\infty(I)$  on  $V$ . Fortunately, if we choose a constant  $z \in \mathbb{C} \setminus \mathbb{R}$ , then  $(L - z)^{-1} : C^\infty(I) \rightarrow V$  is well-defined.

**Proposition 2.2.** *If  $z$  is not an eigenvalue of  $L$ , then  $L - z : V \rightarrow C^\infty(I)$  is bijective.*

*Proof.* The injectivity follows from the definition of eigenvalues. We may assume that  $L$  is injective by translation  $q \mapsto q - \lambda$ .

Suppose  $f \in C^\infty(I)$ . The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$\begin{aligned} -pu'' - p'u' - qu &= fw, \\ \alpha_0 u(a) + \alpha_1 u'(a) &= 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0. \end{aligned}$$

Let  $u_a, u_b$  be the unique solutions of the corresponding homogeneous equation with initial conditions

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Then we can define  $L^{-1} : C^\infty([0, 1]) \rightarrow D(L)$  by

$$L^{-1}f(x) := u_a(x) \int_x^1 \frac{u_b}{W[u_a, u_b]} \frac{f}{(-p)} w + u_b(x) \int_0^x \frac{u_a}{W[u_a, u_b]} \frac{f}{(-p)} w,$$

where  $W[u_a, u_b] := u_a u'_b - u_b u'_a$  denotes the Wronskian. This formula is derived from variation of parameters: we can compute  $c_a$  and  $c_b$  from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u'_a & u'_b \end{pmatrix} \begin{pmatrix} c'_a \\ c'_b \end{pmatrix} \implies L(c_a u_a + c_b u_b) = f.$$

Then, we can easily check that

$$L^{-1}Lu = u$$

for  $u \in D(L)$ , which implies  $L$  is surjective. □

### 2.3 Legendre's equation

The Legendre equation is

$$(1 - x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx} \left( (1 - x^2) \frac{d}{dx} \right).$$



Since  $p(\pm 1) = 0$ , the operator  $L : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1])$  is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

$l$	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x \log \frac{1+x}{1-x}, \quad \dots \in L^2(-1, 1) \setminus C^\infty([-1, 1])$$

as eigenvectors of  $L$ , then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x)(Q'_0(x)f(x) - Q_0(x)f'(x)) \Big|_{-1}^1 \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for  $f \in C^\infty([-1, 1])$ .

## 2.4 Bessel's equation

The Bessel equation is

$$x^2 u'' + xu' + (k^2 x^2 - \nu^2)u = 0, \quad \text{on } (0, \infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x} \left[ \frac{d}{dx} \left( x \frac{d}{dx} \right) - \nu^2 \frac{1}{x} \right].$$

### 3 Peetre's theorem

**Lemma 3.1.** *Suppose a linear operator  $L : C_c^\infty(M) \rightarrow C_c^\infty(M)$  satisfies*

$$\text{supp}(Lu) \subset \text{supp}(u) \quad \text{for } u \in C_c^\infty(X).$$

*For each point  $x \in M$ , there is a bounded neighborhood  $U$  together with a nonnegative integer  $m$  such that*

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

*for  $u \in C_c^\infty(U \setminus \{x\})$ .*

*Proof.* Suppose not. There is a point  $x$  at which the inequality fails; for every bounded neighborhood  $U$  and for every nonnegative  $m$ , we can find  $u \in C_c^\infty(U \setminus \{x\})$  such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large  $C$ . We want to construct a function  $u \in C_c^\infty(U)$  such that  $Lu$  has a singularity at  $x$ .

(Induction step) Take a bounded neighborhood  $U_m$  of  $x$  such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is  $u_m \in C_c^\infty(U_m \setminus \{x\})$  such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that  $u \in C_c^\infty(U)$  since the series converges in the inductive topology of the LF space  $C_c^\infty(U)$ : it converges absolutely with respect to the seminorms  $\|\cdot\|_{C^m}$  for all  $m$ :

$$\begin{aligned} \sum_{i \geq 0} \left\| 2^{-i} \frac{u_i}{\|u_i\|_{C^i}} \right\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\ &\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\ &< \infty. \end{aligned}$$

Also, since the supports of each term are disjoint and  $L$  is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

□

## 4 Characteristic curve

Algorithm:

- (1) Establish the associated vector field by substituting  $u \mapsto y$ .
- (2) Find the integral curve.
- (3) Eliminate the auxiliary variables to get an algebraic equation.
- (4) Verify the computed solution is in fact the real solution.

**Proposition 4.1.** *Suppose that there exists a smooth solution  $u : \Omega \rightarrow \mathbb{R}_y$  of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, & (t, x) \in \Omega \subset \mathbb{R}_{t \geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, & \text{at } x \in \mathbb{R}, \end{cases}$$

and let  $M$  be the embedded surface defined by  $y = u(t, x)$ .

Let  $\gamma : I \rightarrow \Omega \times \mathbb{R}_y$  be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that  $\gamma(0) \in M$ . Then,  $\gamma(\theta) \in M$  for all  $\theta \in I$ .

*Proof.* We may assume  $\gamma$  is maximal. Define  $\tilde{\gamma} : \tilde{I} \rightarrow M$  as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that  $\tilde{\gamma}(0) = \gamma(0)$ . Since  $X$  and  $\tilde{X}$  coincide on  $M$ , the curve  $\tilde{\gamma}$  is also an integral curve of  $X$  with  $\tilde{\gamma}(0) = \gamma(0)$ . By the uniqueness of the integral curve, we get  $\tilde{I} \subset I$  and  $\gamma(\theta) = \tilde{\gamma}(\theta)$  for all  $\theta \in \tilde{I}$ .

Since  $M$  is closed in  $E$ , the open interval  $\tilde{I} = \gamma^{-1}(M)$  is closed in  $I$ , hence  $\tilde{I} = I$  by the connectedness of  $I$ .  $\square$

**Definition 4.1.** The projection of the integral curve  $\gamma$  onto  $\Omega$  is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface  $M$  explicitly by finding the integral curves of the vector field  $X$ . Once we find a necessary condition of the form of algebraic equation, we can demonstrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since  $X$  does not depend on  $u$ , we can solve the ODE: let  $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$  be the integral curve of  $X$  such that  $\gamma(0) = (0, \xi, \xi)$ . Then, the system of ODEs

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad y(\theta) = \xi, \quad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t, x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain  $\Omega$  as

$$\Omega = \{ (t, x) : tx > -\frac{1}{4} \}.$$

#### 4.1 Wave equation

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } t, x > 0, \\ u(0, x) &= g(x), \quad u_t(0, x) = h(x), \quad u_x(t, 0) = \alpha(t).\end{aligned}$$

Define  $v := u_t - cu_x$ . Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0, x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t, x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0, x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t, 0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter  $\xi > 0$ ,

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t, x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi)\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad x(\theta) = -c\theta + \xi, \quad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta') d\theta',$$

hence for  $x > ct > 0$ ,

$$\begin{aligned}u(t, x) &= g(\xi) - \int_0^\theta v(s, \xi - cs) ds \\ &= g(x + ct) \\ &= \frac{3g(x + ct) - g(x - ct)}{2} - \int_0^t h(x + c(t - 2s)) ds\end{aligned}$$

## 4.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

- (1) Suppose  $u(0, x) = \tanh(x)$ . For what values of  $t > 0$  does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (2) Suppose  $u(0, x) = -\tanh(x)$ . For what values of  $t > 0$  does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (3) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and “paste” the solution together.

## 5 Weak convergences



## 6 Fixed point methods

### 6.1 Picard-Lindelöf theorem

**Theorem 6.1.** *Consider the following initial value problem:*

$$x' = f(t, x), \quad x(0) = x_0,$$

where

$$f(-, x) \in C([0, T]), \quad f(t, -) \in \text{Lip}(\overline{B(x_0, R)}).$$

If  $T$  and  $R$  satisfy

- (1)  $TK < 1$ ,
- (2)  $TM \leq R$ ,
- (3)  $|f(t, x)| \leq M$  on  $[0, T] \times \overline{B(x_0, R)}$ ,

then the equation has a unique solution on  $[-T, T]$ .

*Proof.* Define  $\varphi : C([0, T], \overline{B(x_0, R)}) \rightarrow C([0, T], \overline{B(x_0, R)})$  as:

$$\varphi(x)(t) := x_0 + \int_0^t f(s, x(s)) ds.$$

It is well-defined since

$$\begin{aligned} |\varphi(x)(t) - x_0| &\leq \int_0^t |f(s, x(s))| ds \\ &\leq TM \leq R. \end{aligned}$$

It is a contraction since we have

$$\begin{aligned} |\varphi(x)(t) - \varphi(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| ds \\ &\leq \int_0^t K|x(s) - y(s)| ds \\ &\leq TK\|x - y\| \end{aligned}$$

so that

$$\|\varphi(x) - \varphi(y)\| \leq TK\|x - y\|$$

□

The above one looses the Lipschitz condition to local condition.

## 7 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator  $T$  induces a Hilbert space structure on its domain. If the inclusion is bounded, then  $T$  has the bounded inverse. If the inclusion is compact, then  $T$  has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting  $L^2$  inner product on  $C([0, 1])$ , define  $\phi(f) = \int_0^{\frac{1}{2}} f$ .
- Every separable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem  $\rightarrow$  continuous embedding is really an embedding.
- $D(\Omega)$  is defined by a *countable strict* inductive limit of  $D_K(\Omega)$ ,  $K \subset \Omega$  compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever  $\alpha < \beta$  the induced topology by  $\mathcal{T}_\beta$  coincides with  $\mathcal{T}_\alpha$ )
- A net  $(\phi_d)_d$  in  $D(\Omega)$  converges if and only if there is a compact  $K$  such that  $\phi_d \in D_K(\Omega)$  for all  $d$  and  $\phi_d$  converges uniformly.
- Th integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as  $\delta$  is called *singular*.
- $D'$  is equipped with the weak\* topology.
- $\frac{\partial}{\partial x} : D' \rightarrow D'$  is continuous. They commute (Schwarz theorem holds).
- $D \rightarrow S \rightarrow L^p$  are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$  and  $H \subset \widehat{\mathbb{C}}$  have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

## 8 Ultrafilter

**Definition 8.1.** An *ultrafilter* is a synonym for maximal filter. If we sat  $\mathcal{U}$  is an *ultrafilter on a set  $A$* , then it means  $\mathcal{U}$  is a maximal filter as a directed subset of  $\mathcal{P}(A)$ .

existence of ultrafilter.

**Theorem 8.1.** *Let  $\mathcal{U}$  be an ultrafilter on a set  $A$  and  $X$  be a compact space. For a function  $f : A \rightarrow X$ , the limit  $\mathcal{U}\text{-lim } f$  always exists.*

**Theorem 8.2.** *Let  $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$  be a product space of compact spaces  $X_\alpha$ . A net  $f : \mathcal{D} \rightarrow X$  has a convergent subnet.*

*Proof 1.* Use Tychonoff. Compactness and net compactness are equivalent.  $\square$

*Proof 2.* It is a proof without Tychonoff. Let  $\mathcal{U}$  be a ultrafilter on a set  $\mathcal{D}$  containing all  $\uparrow d$ . Define a directed set  $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$  as  $(d, U) \succ (d', U')$  for  $U \subset U'$ . Let  $f : \mathcal{E} \rightarrow X$  be a subnet of  $f : \mathcal{D} \rightarrow X$  defined by  $f_{(d, U)} = f_d$ .

By the previous theorem,  $\mathcal{U}\text{-lim } \pi_\alpha f_d \in X_\alpha$  exists for each  $\alpha$ . Define  $f \in X$  such that  $\pi_\alpha f = \mathcal{U}\text{-lim } \pi_\alpha f_d$ . Let  $G = \prod_\alpha G_\alpha \subset X$  be any open neighborhood of  $f$ . Then,  $\pi_\alpha f \in G_\alpha$  and we have  $G_\alpha = X_\alpha$  except finite. For  $\alpha$ , we can take  $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$  by definition of convergence with ultrafilter. Since  $U_\alpha = \mathcal{D}$  except finites, we can take an upper bound  $U_0 \in \mathcal{U}$  of  $\{U_\alpha\}_\alpha$ . Then, by taking any  $d_0 \in U_0$ , we have  $f_{(d, U)} \in G$  for every  $(d, U) \succ (d_0, U_0)$ . This means  $f = \lim_{\mathcal{E}} f_{(d, U)}$ , so we can say  $\lim_{\mathcal{E}} f_{(d, U)}$  exists.  $\square$

## 9 Selected analysis problems

**Problem 9.1.** The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

*Solution.* Let  $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$ . Divide the unit circle  $\mathbb{R}/2\pi\mathbb{Z}$  by  $7k$  uniform arcs. There are at least  $2^k/7k$  integers that are not exceed  $2^k$  and are in a same arc. Let  $S$  be the integers and  $x_0$  be the smallest element. Since,  $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$  for  $x \in S$ ,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also,  $1 \leq x - x_0 \leq x \leq 2^k$ ,  $x - x_0 \in A_k$ .

$$|A_k| \geq \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\ &\geq \sum_{k=1}^N (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^N \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &> \sum_{k=1}^N \frac{2^k}{2^{k+2}} \frac{1}{7k} \\ &= \frac{1}{28} \sum_{k=1}^N \frac{1}{k} \\ &\rightarrow \infty. \end{aligned}$$

□

**Problem 9.2.** If  $|xf'(x)| \leq M$  and  $\frac{1}{x} \int_0^x f(y) dy \rightarrow L$ , then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .

*Solution.* Since

$$\begin{aligned} \left| f(x) - \frac{F(x) - F(a)}{x - a} \right| &\leq \frac{1}{x - a} \int_a^x |f(x) - f(y)| dy \\ &= \frac{1}{x - a} \int_a^x (x - y) |f'(c)| dy \\ &\leq \frac{M}{x - a} \int_a^x \frac{x - y}{c} dy \\ &\leq M \frac{x - a}{a} \end{aligned}$$

by the mean value theorem and

$$f(x) - L = \left[ f(x) - \frac{F(x) - F(a)}{x - a} \right] + \frac{x}{x - a} \left[ \frac{F(x)}{x} - L \right] + \frac{a}{x - a} \left[ \frac{F(a)}{a} - L \right],$$

we have for any  $\varepsilon > 0$

$$\limsup_{x \rightarrow \infty} |f(x) - L| \leq \varepsilon$$

where  $a$  is defined by  $\frac{x-a}{a} = \frac{\varepsilon}{M}$ . □

**Problem 9.3.** Let  $f_n : I \rightarrow I$  be a sequence of real functions that satisfies  $|f_n(x) - f_n(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$ , where  $I = [0, 1]$ . Then, it has a uniformly convergent subsequence.

*Solution.* By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that  $f_n$  converges to a function  $f : \mathbb{Q} \cap I \rightarrow I$  pointwisely.

*Step [.1]* For  $n \geq 4$ , we claim

$$|x - y| \leq \frac{1}{n} \implies |f_n(x) - f_n(y)| \leq \frac{5}{n}. \quad (1)$$

Fix  $x \in I$  and take  $z \in I$  such that  $|x - z| = \frac{2}{n}$  so that

$$|f_n(x) - f_n(z)| \leq |x - z| = \frac{2}{n}.$$

If  $y$  satisfies  $|x - y| \leq \frac{1}{n}$ , then we have  $|y - z| \geq |x - z| - |x - y| \geq \frac{1}{n}$ , so we get

$$|f_n(y) - f_n(z)| \leq |y - z| \leq |y - x| + |x - z| \leq \frac{3}{n}.$$

Combining these two inequalities proves what we want.

*Step [.2]* For  $\varepsilon > 0$  and  $N := \lceil \frac{15}{\varepsilon} \rceil$  we claim

$$|x - y| \leq \frac{1}{N} \quad \text{and} \quad n > N \implies |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{3} \quad (2)$$

when  $N \geq 4$ . It is allowed for  $|x - y|$  to have the following two cases:

$$|x - y| \leq \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x - y| \leq \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \leq \frac{5}{n} < \frac{5}{N} \leq \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \leq |x - y| \leq \frac{1}{N} \leq \frac{\varepsilon}{15}.$$

Hence the claim is proved.

*Step [.3]* We will prove  $f$  is uniformly continuous. For  $\varepsilon > 0$ , take  $\delta := \frac{1}{N}$ , where  $N := \lceil \frac{15}{\varepsilon} \rceil$ . We will show

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

for  $x, y \in \mathbb{Q} \cap I$  and  $N \geq 4$ . Fix rational numbers  $x$  and  $y$  in  $I$  which satisfy  $|x - y| < \delta$ . Since  $f_n(x)$  and  $f_n(y)$  converges to  $f(x)$  and  $f(y)$  respectively, we may take an integer  $n_x$  and  $n_y$ , such that

$$n > n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad (3)$$

and

$$n > n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \quad (4)$$

Choose an integer  $n$  such that  $n > \max\{n_x, n_y, N\}$ . Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since  $f$  is continuous on a dense subset  $\mathbb{Q} \cap I$ , it has a unique continuous extension on the whole  $I$ . Let it denoted by the same notation  $f$ .

*Step [4]* Finally, we are going to show  $f_n \rightarrow f$  uniformly. For  $\varepsilon > 0$ , let  $N := \lceil \frac{15}{\varepsilon} \rceil$ . The uniform continuity of  $f$  allows to have  $\delta > 0$  such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{2}{3}\varepsilon. \quad (5)$$

Take a rational  $r \in I$ , depending on  $x \in I$ , such that  $|x - r| < \min\{\frac{1}{N}, \delta\}$ . Then, by (2) and (5), given  $n > N \geq 4$ , we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any  $x \in I$ . By limiting  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since  $\varepsilon$  and  $x$  are arbitrary, we can deduce the uniform convergence of  $f_n$  as  $n \rightarrow \infty$ .  $\square$

**Problem 9.4.** A measurable subset of  $\mathbb{R}$  with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

*Solution.* Let  $E \subset \mathbb{R}$  be measurable with  $\mu(E) > 0$ . We may assume  $E$  is bounded so that we have  $E \subset I$  for a closed bounded interval since  $\mathbb{R}$  is  $\sigma$ -compact. Let  $n$  be a positive integer arbitrarily taken. Then, we can find  $N$  such that  $\sum_{k=1}^N \frac{1}{k} > (n-1) \frac{\mu(I)}{\mu(E)}$ .

Assume that every point  $x$  in  $E$  is contained in at most  $n-1$  sets among

$$E, \frac{1}{2}E, \frac{1}{3}E, \dots, \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k \in A} \frac{1}{k}E = \emptyset$$

for any subset  $A \subset \{1, \dots, N\}$  with  $|A| \geq n$ . Define

$$E_A := \bigcap_{k \in A} \frac{1}{k}E \cap \bigcap_{k' \in A} \left( \frac{1}{k'}E \right)^c$$

for  $A \subset \{1, \dots, N\}$ . Then,  $\mu(E_A) = 0$  for  $|A| \geq n$ .

Note that we have

$$\mu\left(\frac{1}{k}E\right) = \sum_{k \in A} \mu(E_A) = \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) = \sum_{k=1}^N \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A) = \sum_{|A| < n} |A| \mu(E_A)$$

by double counting, and since  $E_A$  are disjoint, we have

$$\sum_{|A| < n} |A| \mu(E_A) = (n-1) \sum_{0 < |A| < n} \mu(E_A) \leq (n-1) \mu(I),$$

hence a contradiction to

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) > (n-1) \mu(I).$$

Therefore, we may find an element  $x$  that belongs to  $\frac{1}{k}E$  for  $k \in A$ , where  $A \subset \{1, \dots, N\}$  with  $|A| = n$ . Then,  $ax \in E$  for all  $a \in A \subset \mathbb{Z}$ .  $\square$



## 10 Physics problem

### 10.1 Resonance

Let  $m, b, k, A, \omega_d$  be positive real constants. Consider an underdamped oscillator with sinusoidal driving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0, \quad x'(0) = 0.$$

There are some observations:

- (1) The underdamping condition means  $b^2 - 4mk < 0$  so that the roots of characteristic equation are imaginary.
- (2) The positivity of  $m, b$  implies the real part of solution that will be denoted by  $-\beta = -\frac{b}{2m}$  is negative; it shows exponential decay of solutions.
- (3) Introducing the natural frequency  $\omega_n = \sqrt{k/m}$ , we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega_d t.$$

- (4) The complementary solution is computed as

$$x_c(t) = x_0 e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \rightarrow \infty} x_c(t) = 0.$$

- (5) The condition  $\beta > \omega_n$  is equivalent to that the oscillator is underdamped.
- (6) Let  $m, k$  be fixed. Then, the solution  $x_c$  decays most fastly when  $b$  satisfied  $b^2 = 4mk$ , equivalently,  $\beta = \omega_n$ .
- (7) When  $\omega_d = \omega_n$  such that the amplitude of particular solution diverges.