Functional Analysis I : Topological Vector Space

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CHAPTER 1

Topological vector spaces

1. Elementary properties

definition - how to use the continuity of vector space operations effectively homeomorphism by translation and dialation: local base at 0 uniformity pseudometrics, basic classification translation invariant metric completely regular (up to 3.5) boundedness and continuity

2. Classification

$$
\begin{tikzcd}
 & \text{inner product} \arrow[d] & \\
 & \text{finite seminorms (= normable)} \arrow[d] & \\
 & \text{countable seminorms} \arrow[dl] \arrow[dr] & \\
\text{countable pseudometrics (= metrizable)} \arrow[dr] & & \text{uncountable seminorms (= locally convex)} \arrow[dl] \\
 & \text{uncountable pseudometrics} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \text{Hilbert space} \arrow[d] & \\
 & \text{Banach space} \arrow[d] & \\
 & \text{Fréchet space} \arrow[dl] \arrow[dr] & \\
\text{F-space} \arrow[dr] & & \text{(no name)} \arrow[dl] \\
 & \text{(no name)} &
\end{tikzcd}
$$

PROPOSITION 2.1. *Let ρ be a pseudometric. Then,*

$$B(0,1) \subset \frac{B(0,1) + B(0,1)}{2} \subset \frac{1}{2}B(0,2).$$

If ρ is a seminorm, then the equalities hold.

I say this as $\frac{1}{2}B(0,2)$ is “fatter” than $B(0,1)$.

3. Barreled spaces

3.1. The Baire category theorem.

3.2. Uniform boundedness principle.

THEOREM 3.1 (Uniform boundedness principle). *Let X be a barreled space and Y be a topological vector space. Let $\mathcal{F} \subset B(X,Y)$. If $\mathcal{F}$ is pointwise bounded, then $\mathcal{F}$ is equicontinuous.*

3.3. Open mapping theorem.

THEOREM 3.2 (Open mapping theorem). *Let X be a topological vector space and Y be a metrizable barreled space. Let $T\colon X \to Y$ be linear. If T is surjective and continuous, then T is open.*

PROOF. If we let U be an open neighborhood in X, then we want to show TU is a neighborhood. Because T is surjective so that $\overline{TU}$ is absorbent, $\overline{TU}$ is a neighborhood. Note that an open set intersects $\overline{TU}$ also intersects TU.

If there exist two sequences of balanced open neighborhoods $U_n \subset X$ and $V_n \subset Y$ with

(1) $U_1 + \cdots + U_n \subset U$,
(2) $V_n \subset \overline{TU_n}$,
(3) $\bigcap_{n\in\mathbb{N}} V_n = \{0\}$,

then we can show $V_1 \subset TU$. Here is the proof: Suppose $y \in V_1$. Then,

$$\begin{aligned}
&y \cap V_1 \neq \varnothing \longrightarrow y \cap \overline{TU_1} \neq \varnothing \longrightarrow (y + V_2) \cap TU_1 \neq \varnothing \\
\to\ &(y + TU_1) \cap V_2 \neq \varnothing \longrightarrow (y + TU_1) \cap \overline{TU_2} \neq \varnothing \to ((y + TU_1) + V_3) \cap TU_2 \neq \varnothing \\
\to\ &(y + TU_1 + TU_2) \cap V_3 \neq \varnothing \longrightarrow \quad \cdots .
\end{aligned}$$

From the first columns, and by the conditions (1) and (3), we obtain

$$(y + TU) \cap \bigcap_{n\in\mathbb{N}} V_n \neq \varnothing.$$

Therefore, the set $y + TU$ contains 0, hence $y \in TU$.

Let us show the existence of such sequences. At first, take $U_n = 2^{-n}U$ for (1). Then we can take $\{V_n\}_n$ with (2) as we mentioned above. Simultaneously we can have it satisfy (3) because Y is metrizable. $\square$

COROLLARY 3.3. *Let X be a metric space and Y be a barreled space. Then, the open mapping theorem holds.*

PROOF. The isomorphism $Y \cong X/\ker T$ forces Y to be also metrizable. $\square$

COROLLARY 3.4 (The Banach Isomorphy). *A continuous linear bijection onto a metrizable barreled space is a homeomorphism.*

COROLLARY 3.5 (The first isomorphism theorem). *Let $T : X \to Y$ be a bounded linear operator between Banach spaces. Then, the induced map $X/\ker T \to \operatorname{im} T$ is a topological isomorphism.*

CHAPTER 2

Locally convex spaces

1. Seminorms

minkowski functional locally boundedness polar

2. The Hahn-Banach theorem

3. Weak topology

CHAPTER 3

Operators on Banach space

The range of an operator $T : X \to Y$ is closed if and only if

$$\operatorname{im} T \to X / \ker T$$

is bounded.(maybe)