Differential Geometry

Lecture by Ikhan Choi
Notes by Ikhan Choi

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CHAPTER 1

Manifolds

1. Vector bundle

2. Differentiable manifold

2.1. Manifold and Atlas.

Definition 2.1. A *locally Euclidean space* M of dimension m is a Hausdorff topological space M for which each point $x \in M$ has a neighborhood U homeomorphic to an open subset of $\mathbb{R}^d$.

Definition 2.2. A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

Definition 2.3. A *chart* or a *coordinate system* for a locally Euclidean space is a map φ is a homeomorphism from an open set $U \subset M$ to an open subset of $\mathbb{R}^d$. A chart is often written by a pair (U, φ).

Definition 2.4. An *atlas* $\mathcal{F}$ is a collection $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$ of charts on M such that $\bigcup_{\alpha \in A} U_\alpha = M$.

Definition 2.5. A *differentiable maifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

2.2. Definition of Differentiable Structure.

Definition 2.6. An atlas $\mathcal{F}$ is called *differentiable* if any two charts $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$ is *compatible*: each *transition function* $\tau_{\alpha\beta} \colon \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ which is defined by $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$ is differentiable.

It is called a *gluing condition*.

Definition 2.7. For two differentiable atlases $\mathcal{F}, \mathcal{F}'$, the two atlases are *equivalent* if $\mathcal{F} \cup \mathcal{F}'$ is also differentiable.

Definition 2.8. An differentiable atlas $\mathcal{F}$ is called *maximal* if the following holds: if a chart (U, φ) is compatible to all charts in $\mathcal{F}$, then $(U, \varphi) \in \mathcal{F}$.

Definition 2.9. A *differentiable structure* on M is a maximal differentiable atlas.

To differentiate a function on a flexible manofold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on M. When the charts is already equipped on M, it is natural to define a function $f \colon M \to \mathbb{R}$ differentiable if the functions $f \circ \varphi^{-1} \colon \mathbb{R}^d \to \mathbb{R}$ is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$. If a function f is differentiable on an atlas $\mathcal{F}$, then f is also differentiable on any atlases which is equivalent to $\mathcal{F}$ by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

Example 2.1. While the circle S^1 has a unique smooth structure, S^7 has 28 smooth structures. The number of smooth structures on S^4 is still unknown.

Definition 2.10. A continuous function $f\colon M \to N$ is differentiable if $\psi \circ f \circ \varphi^{-1}$ is differentiable for charts φ, ψ on M, N respectively.

Definition 2.11 (Partition of unity).

3.

Definition 3.1. For $f\colon M \to \mathbb{R}$ and (U, ϕ) a chart,

$$df\left(\frac{\partial}{\partial x^\mu}\right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

Definition 3.2. Let $\gamma\colon I \to M$ be a smooth curve. Then, $\dot{\gamma}(t)$ is defined by a tangent vector at $\gamma(t)$ such that

$$\dot{\gamma}(t) := d\gamma\left(\frac{\partial}{\partial t}\right).$$

Let $\phi\colon M \to N$ be a smoth map. Then, $\phi(t)$ can refer to a curve on N such that

$$\phi(t) := \phi(\gamma(t)).$$

Let $f\colon M \to \mathbb{R}$ be a smooth function. Then, $\dot{f}(t)$ is defined by a function $\mathbb{R} \to \mathbb{R}$ such that

$$\dot{f}(t) := \frac{d}{dt} f \circ \gamma.$$

Proposition 3.1. *Let $\gamma\colon I \to M$ be a smooth curve on a manifold M. The notation $\dot{\gamma}^\mu$ is not confusing thanks to*

$$(\dot{\gamma})^\mu = (\dot{\gamma^\mu}).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt} x^\mu \circ \gamma.$$

CHAPTER 2

Riemannian geometry

1. Connection

1.1. Connection.

$$
\begin{aligned}
\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\
&= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma^\lambda_{\mu\nu} \partial_\lambda) \\
&= X^\mu \left(\frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda \right) \partial_\nu .
\end{aligned}
$$

The covariant derivative $\nabla_X Y$ does not depend on derivatives of X^μ.

$$
Y^\nu_{,\mu} = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \qquad Y^\nu_{;\mu} = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma^\nu_{\mu\lambda} Y^\lambda .
$$

Theorem 1.1. *For Levi-civita connection for g,*

$$
\Gamma^l_{ij} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).
$$

Proof.

$$
\begin{aligned}
(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma^l_{ij} g_{lk} - \Gamma^l_{ik} g_{jl} \\
(\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma^l_{jk} g_{li} - \Gamma^l_{ji} g_{kl} \\
(\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma^l_{ki} g_{lj} - \Gamma^l_{kj} g_{il}
\end{aligned}
$$

If ∇ is a Levi-civita connection, then $\nabla g = 0$ and $\Gamma^k_{ij} = \Gamma^k_{ji}$. Thus,

$$
\begin{aligned}
\Gamma^l_{ij} g_{kl} &= \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}). \\
\Gamma^l_{ij} &= \frac{1}{2} g^{kl} (\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).
\end{aligned}
$$

$\square$

2.

THEOREM 2.1. *If c is a geodesic curve, then components of c satisfies a second-order differential equation*

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0.$$

PROOF. Note

$$0 = \nabla_{\dot\gamma}\dot\gamma = \dot\gamma^\mu\nabla_\mu(\dot\gamma^\lambda\partial_\lambda) = (\dot\gamma^\nu\partial_\nu\dot\gamma^\mu + \dot\gamma^\nu\dot\gamma^\lambda\Gamma^\mu_{\nu\lambda})\partial_\mu.$$

Since

$$\dot\gamma^\nu\partial_\nu\dot\gamma^\mu = \dot\gamma(\dot\gamma^\mu) = d\dot\gamma^\mu(\dot\gamma) = d\dot\gamma^\mu \circ d\gamma\left(\frac{\partial}{\partial t}\right) = d\dot\gamma^\mu\left(\frac{\partial}{\partial t}\right) = \ddot\gamma^\mu,$$

we get a second-order differential equation

$$\frac{d^2\gamma^\mu}{dt^2} + \Gamma^\mu_{\nu\lambda}\frac{d\gamma^\nu}{dt}\frac{d\gamma^\lambda}{dt} = 0$$

for each μ. □