Strategies for classical differential geometry

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1. Introduction

- Formulate what conditions are given and what is the objective.
- Find coefficients of a particular vector (α or ν) over an appropriate basis($\{\mathbf{T}\}$, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, $\{$.
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2. Curves on a plane

2.1. Arc-length parameterization.

$$s(t) := \int_0^t \|\dot{\alpha}\|$$

3. Curves in a space

$\kappa \neq 0$ at every point

Definition. Let α be a curve.

$$\mathbf{T} := \frac{\dot{\alpha}}{\|\dot{\alpha}\|}, \quad \mathbf{N} := \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|}, \quad \mathbf{B} := \mathbf{T} \times \mathbf{N}.$$

Definition.

$$\kappa := \mathbf{T}' \cdot \mathbf{N}, \quad \tau := -\mathbf{B}' \cdot \mathbf{N}.$$

Theorem 3.1 (Frenet-Serret formula)**.** *Let α be a unit speed curve.*

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}.$$

Proof. The vectors $\mathbf{T}', \mathbf{B}', \mathbf{N}$ are collinear. □

Theorem 3.2. *Let α be a unit speed curve.*

$$\begin{aligned} \alpha' &= \mathbf{T} \\ \alpha'' &= \kappa \mathbf{N} \\ \alpha''' &= -\kappa^2 \mathbf{T} + \kappa' \mathbf{N} + \kappa\tau \mathbf{B} \end{aligned}$$

Skew-symmetricity is due to the fact the differential of an orthogonal matrix forms a skew symmetric matrix.

Example 3.3. Let α be a curve in $\mathbb{R}^3$. If the normal line always passes through a point, then α is contained in a circle.

Proof. Let α be a unit speed curve.

$$\langle \alpha - c, \mathbf{T} \rangle = \langle \alpha - c, \mathbf{T} \rangle = 0.$$

$$\langle \alpha - c, \mathbf{N} \rangle = -\frac{1}{\kappa}.$$

$\square$

4. Surfaces in a space

5. Curves on a surface