Analysis II : General Topology

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Preface

The word topology is used in two different contexts: analytic sense and geometric sense. When we are talking the stories of doughnuts and coffee mugs, they are in fact involved in topology of geometric sense, which is also referred to as a branch of mathematics that studies spaces such as manifolds or CW complexes. In analysis, the topology is mentioned greatly unrelatedly to the doughnuts, but it refers to the minimal structure that is required in order to define concepts of limit and continuity. More precisely, once a structure called topology is settled on a set, then we can expand some basic analytic theories using limit and continuity. Normed spaces are the first examples which possess a certain topology. With the topology, we could describe formally why a sequence converges to a point or whether a function is continuous. Simply, we say a norm induces a natural topology. This book is of course interested in the latter issues, as we have seen the title of the book.

Then, why should we study topologies for analysis? In Euclidean spaces or the space of continuout functions on a compact set that we have usually studied, we have a fixed standard norm to determine if a sequence converges. For different topologies in analysis provides a sort of different criteria, the convergence changes even for a same sequence when we judge it in different topologies so that we can argue in what sense a sequence converges. Especially in real analysis or functional analysis it is an extremely important way of recognizing the various convergence modes of functional sequences. A nonnegligible problem that occurs here is that some convergences cannot be formulated with neither norms nor metrics anymore, like pointwise convergence of functional sequences defined with uncountable domain. Thereby, the study of topologies necessarily made sophisticated connections with mathematical analysis.

General topology is the abstract study of topologies and topological spaces. Similarly as mentioned above, there are two large branches of general topology; both are contributed to build nice frameworks of other mathematics. One is for algebraic topology and studies the category of convinient spaces in which well-known constructions and computational tools are available, and the other is for doing abstract analysis. One interesting feature of general topology is that the basic topology in analysis is a preliminary of the abstract study of the spaces used in algebraic topology, so everyone starts to learn it from analysis.

The purpose of this book is to catch a big picture and learn basic languages in order for preparing the next study of modern analysis such as harmonic analysis or functional analysis following after calculus topics, in the quite abstract viewpoints. In particular, we mainly focus on finding admissible answers for notoriously hazy questions:

- Why are topologies defined in that way?
- What can metric spaces do more than topological spaces?

- When can we use sequences instead of nets without any anxiety?
- What does compactness mean?
- Why do locally compact Hausdorff spaces so frequently appears?
- What is the importance of complicated theorems of like Arzela-Ascoli or Stone-Weierstrass?

In this book, we are going to assume the reader is already familiar to the theory of normed spaces and elementary foundations of calculus including the epsilon-delta definitions. For instance, we can require the reader to know what the uniform convergence is because it can be regarded as just a convergence in a properly defined norm on a space of functions. For the first, the basic topological structures including metrics, topologies, and uniformities are introduced in Chapter 1. Although many texts do not cover uniform spaces, they are greatly useful in studying nonmetrizable topologies. In Chapter 2, we learn about continuity of functions and maps. Continuous maps functionally connects two different topological spaces and allow us to compare them. Homeomorphisms and some connectivity will be also covered. Chapter 3 is dedicated to the deeper study of convergence of sequences or nets, which can be said as the In Chapter 4, 5, and 6, we learn compact spaces, separability axioms, and continuous function spaces.

This book is written in order to be used in my imaginary lectures for students who lost their chances to learn professional educations for various reasons. I tried to put convincing explanations at every newly defined concept as with clear logics as possible and cram supplementary stories that are not necessary, which might not be really satisfied. Therefore, I think this book would never be a good choice for a standard course text relative to the other existing great books. I will be very satisfied only if one of you just could enjoy math with this book.

CHAPTER 1

Topological structures

Firstly we discuss to what extent the definition of analytic notions such as limit and continuity can be extended. One of main interest in general topology is to make extended version of mathematical calculus on a set without algebraic operations. However, lay it up in the heart that several properties must be compromised when we try to make generalizations.

Recall that we measured how near the two points are by taking absolute value of algebraic subtraction of two position vectors in normed spaces. How can we dismiss the subtraction? Saying only the results, mathematicians succeeded to generalize limit of sequences and continuity of functions, but compromising the theory of differentiation and integration. Topology is the term for this successful solution. In other words, for the most part, wonderful statements purely related to limit and continuity were possible to be extended without big flaws even if we forget the vector space operations by introducing the concept of topology, but differentiation and integration could not on the other hand.

Topological structure refers to an additional function on a given set or a more complicated mathematical device which solves the problem by being put on a set. Norm is a typical example of topological structure, and so is “topology”.

1. Metric

Metric is a generalization of norm and a special example of "topology". For example, every subset of a normed vector space is equipped with a natural metric. Since general topology might be too abstract for novices to grasp, we will make a bridge from norms to topologies.

Metric was the first successful trial to find an abstract framework for studying limits. Later, we will find that metric provides a surprisingly appropriate and widely-applicable tool to understand the nature of mathematical analysis.

1.1. Metric spaces. A metric is a function which assigns a nonnegative real number, which has a meaning of distance, to a pair of two points. A metric space is just a set endowed with a metric.

DEFINITION 1.1. Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x, y) = 0$ iff $x = y$, (nondegenracy)
(2) $d(x, y) = d(y, x)$, (symmetry)
(3) $d(x, z) \leq d(x, y) + d(y, z)$. (triangle inequality)

A pair (X, d) of a set X and a metric on X is called a *metric space*. We often write it simply X.

The most familiar metric comes from the standard norm on Euclidean space $\mathbb{R}^d$. Notice that the third axiom, the triangle inequality, is named after the one for norms. In this context, we can see metrics as a generalization of norms for spaces not admitting the vector space structure. When we use an analytic theory on Euclidean spaces or more generally normed spaces, the metric given in the following example is considered as the standard. Therefore, if not particularly mentioned, then we will implicitly assume this induced metric out of the norm for any subset of a Euclidean space. Moreover, every subset of normed space is also an example of metric space because the metric function can be always inherited to every subset of a given metric space.

EXAMPLE 1.1. A normed space X is a metric space. Precisely, the norm structure naturally defines a real-valued function d on $X \times X$ defined by $d(x, y) := \|x - y\|$ and it satisfies the axioms of metric.

PROOF. It is quite easy. Just recall the axioms of norm and deduce the conclusion for each axiom of metric. □

EXAMPLE 1.2. Let (X, d) be a metric space. Every subset of X has a natural induced metric, just the restriction of original metric d.

PROOF. Obvious. □

In fact, the converse holds; every metric space can be viewed as a subset of a normed space. This deeper result on the relation between normed spaces and metric spaces is discovered by Kuratowski[]. Since the theorem does not play any important role in the whole book, readers who want to read fast may skip. To state the theorem, we introduce isometry, a map preserving metrics.

Definition 1.2. Let X and Y be metric spaces. A map $\phi : X \to Y$ is called an *isometry* if $d(x, y) = d(\phi(x), \phi(y))$ for all $x, y \in X$. If there is a bijective isometry between X and Y, then we say the spaces are *isometric*.

Every isometry is clearly injective so that it is bijective if and only if it is surjective. Also, the inverse of bijective isometry is also an isometry. If two metric spaces are isometric, we can view them as virtually same, in the "category" of metric spaces. The following theorem tells another characterization of metric spaces.

Proposition 1.3 (Kuratowski embedding). *Every metric space is isometric to a subset of a normed space. In other words, for every metric space (X, d), there is an isometry ϕ from X to a normed space.*

Proof. Choose any point $p \in X$. Let Y be the space of bounded real-valued functions on X. It is a normed space with uniform norm. Define $\phi : X \to Y$ by $\phi(x)(t) = d(x, t) - d(p, t)$. Note that $\phi(x)$ is bounded with $\|\phi(x)\| = \sup_{t\in X} |d(x, t) - d(p, t)| = d(x, p)$. Then,

$$\|\phi(x) - \phi(y)\| = \sup_{t\in X} |\phi(x)(t) - \phi(y)(t)| = \sup_{t\in X} |d(x, t) - d(y, t)| = d(x, y).$$

This proves ϕ is a isometry. □

Remark. The space Y is somtimes denoted by $\ell^\infty(X)$, and it is in fact a Banach space. In addition, the image of the isometry ϕ is in a closed subspace $C_b(X) \subset \ell^\infty(X)$, the space of bounded real-valued continuous functions.

We have seen metrics can be seen as the generalization of norms. However, there are also many examples of metrics that are not involved directly in the norms. Even if they are far from subsets of a normed space, we can apply our intuition of balls. The examples below are given without proofs.

Example 1.4. Let X be a set. Then, a function $d : X \times X \to \mathbb{R}_{\geq 0}$ defined by

$$d(x, y) := \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases}$$

is a metric on X. This metric is sometimes called *discrete metric* because balls can separate all single points out.

Example 1.5. Let d be a metric on a set X. Let $f : [0, \infty) \to [0, \infty)$ be a monotonically increasing function such that $f^{-1}(0) = \{0\}$. If f is subadditive, in other words f satisfies

$$f(x + y) \leq f(x) + f(y)$$

for all $x, y \in \mathbb{R}_{\geq 0}$, then $f \circ d$ is another metric on X.

Example 1.6. Let $G = (V, E)$ be a connected graph. Define $d : V \times V \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the distance of two vertices; the length of shortest path connecting two vertices. Then, (V, d) is a metric space.

Example 1.7. Let $\mathcal{P}(X)$ be the power set of a finite set X. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the cardinality of the symmetric difference; $d(A, B) := |(A - B) \cup (B - A)|$. Then $(\mathcal{P}(X), d)$ is a metric space.

Example 1.8. Let C be the set of all compact subsets of $\mathbb{R}^n$. Recall that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded. Then, $d : C \times C \to \mathbb{R}_{\geq 0}$ defined by

$$d(A,B) := \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\}$$

is a metric on C. It is a little special case of *Hausdorff metric.*

1.2. Limits and continuity. Many freshmen misunderstand the main role of metric for its name. They recognize metric as something measures a distance and belonging to the study of geoemtry. We cannot strongly affirm it is false, but I hope to mention that a metric is quite far from geometric structures, and is rather an analytic structure. Meaning, metric is in fact not interested in measuring a distance between two points; the main function of metric is to make balls. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuistic sense. Metric can be considered as a device to let someone accept the notion of neighborhoods more friendly, which is vital for analysis of limits and continuity.

Definition 1.3. Let X be a metric space. A set of the form

$$\{y \in X : d(x,y) < \varepsilon\}$$

for $\varepsilon > 0$ is called a *ball centered at* x and denoted by $B(x,\varepsilon)$ or $B_\varepsilon(x)$.

The balls are also called open balls in order to distinguish from the closed balls; $\overline{B(x,\varepsilon)} = \{y \in X : d(x,y) \leq \varepsilon\}$. The terms openness and closedness will be discussed again in the next section. Now let us reformulate the definitions of limits and continuity with balls, which we actually use in the usual calculus on Euclidean spaces or generally on normed spaces. Compare the following definitions to what we remember.

Definition 1.4. Let $\{x_n\}_n$ be a sequence of points on a metric space (X,d). We say that a point x is a *limit* of the sequence or the sequence *converges to* x if for arbitrarily small ball of size ε cenetered at x, $B(x,\varepsilon)$, we can find n_0 such that $x_n \in B(x,\varepsilon)$ for all $n > n_0$. If it is satisfied, then we write

$$\lim_{n\to\infty} x_n = x,$$

or simply

$$x_n \to x$$

as $n \to \infty$. If there is no such limit x, then we say the sequence *diverges.*

Definition 1.5. A function $f : X \to Y$ between metric spaces is called *continuous at* $x \in X$ if for any ball $B(f(x),\varepsilon) \subset Y$ centered at $f(x)$ there is a ball $B(x,\delta) \subset X$ centered at x such that

$$f(B(x,\delta)) \subset B(f(x),\varepsilon).$$

The function f is called *continuous* if it is continuous at every point on X.

There are a lot of deeper and valuable propositions and results for limit and continuity, but we postpone to mention them to later because they are generalized to topological spaces. However, for a while, let us just check that taking either ε or δ really means taking a ball of the very radius. For continuity of a function, we can intuitively describe it by saying that no matter how small ball is taken in the codomain, we can take much smaller ball in the domain. Then, what we want to know would be how the balls centered at each point are distributed because what the set of balls looks like may determine the continuity or convergence. To make a vivid illustration, let us give an example.

Example 1.9. Let X be the discrete metric space in Example 1.4. Every ball centered at a point x with respect to the discrete metric is either a singleton $B(x,\varepsilon) = \{x\}$ when $\varepsilon \leq 1$, or the entire space $B(x,\varepsilon) = X$ when $\varepsilon > 1$. In particular, a sequence $\{x_n\}_n$ converges to x if and only if it is eventually x; there is a positive integer n_0 such that $x_n = x$ for all $n > n_0$.

Example 1.10. Let X and Y be metric spaces. If X is equipped with the discrete metric in Example 1.4, then every function $f : X \to Y$ is continuous.

Example 1.11. An isometry is always continuous.

The set of balls at each point plays an important role in determining properties for limits and continuity. Intuitively, the balls indicate the varying degrees of neighborhoods and relative nearness from a point. Refer to Example 2.4.

1.3. Topological equivalence. Take note on the fact that the sequence of real numbers defined by $x_n = \frac{1}{n}$ diverges in discrete metric. Like this example, even for the same sequence on a same set, the convergence depends on the attached metric. However, we cannot conversely say that different metrics always provide different criteria for convergence. In other words, when we consider a metric as a function that takes a sequence as input and outputs whether a sequence converges or diverges, there may be two different metrics which gives exactly same answer about convergence. Of course, the continuity of functions has the same issue. This allows us to think an equivalence relation on the set of metrics, that is, two equivalent metrics give a common criterion for convergence and continuity. In this situation, it can be paraphrased into that the two metrics induce exactly same topology. Some definitions and theorems for the equivalence checking will be given as follows.

Definition 1.6. Let d_1 and d_2 are metric on a set X. The two metrics are called *topologically equivalent* if the sets of open balls at each point are mutually nested; for any $x \in X$ and for arbitrary $\varepsilon > 0$, we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$B_1(x,\delta_1) \subset B_2(x,\varepsilon) \quad \text{and} \quad B_2(x,\delta_2) \subset B_1(x,\varepsilon),$$

where the notations B_1 and B_2 refer to balls defined with the metrics d_1 and d_2 respectively.

The word "topologically" is frequently omitted. This definition looks quite strange, but is directly related to the way how a metric gives rise to a topology. There are various characterizations of equivalence among metrics.

Proposition 1.12. *Let d_1 and d_2 are metric on a set X. They are equivalent if and only if they share the same sequential convergence data; a sequence converges in d_1 if and only if it converges in d_2.*

Proof. ($\Rightarrow$) Suppose d_1 and d_2 are equivalent. It is enough to show that the convergence of a sequence in d_1 implies the convergence in d_2 because the converse is shown in the same way.

Let $\{x_n\}_n$ be a sequence in X that converges to x in d_1. Take an arbitrary ball $B_2(x, \varepsilon) = \{y : d_2(x, y) < \varepsilon\}$. By the definition of equivalence, there is $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon),$$

where $B_1(x, \delta) = \{y : d_1(x, y) < \delta\}$. Since $\{x_n\}_n$ converges to x in d_1, there is an integer n_0 such that

$$n > n_0 \implies x_n \in B_1(x, \delta).$$

Combining them, we obtain an integer n_0 such that

$$n > n_0 \implies x_n \in B_2(x, \varepsilon).$$

It means $\{x_n\}$ converges to x in the metric d_2.

($\Leftarrow$) We prove it by contradiction. Assuming d_1 and d_2 are not equivalent, we will construct a sequence that converges in one metric but not in the other metric.

Since d_1 and d_2 are not equivalent, without loss of generality, for some point $x \in X$, we can find $\varepsilon_0 > 0$ such that there is no $\delta > 0$ satisfying $B_1(x, \delta) \subset B_2(x, \varepsilon_0)$. In other words, at the point x, the difference set $B_1(x, \delta) \setminus B_2(x, \varepsilon_0)$ is not empty for every $\delta > 0$. Thus, we can choose x_n to be a point such that

$$x_n \in B_1\left(x, \tfrac{1}{n}\right) \setminus B_2(x, \varepsilon_0)$$

for each positive integer n.

We claim $\{x_n\}_n$ converges to x in d_1 but not in d_2. For $\varepsilon > 0$, if we let $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$ so that we have $\frac{1}{n_0} \leq \varepsilon$, then

$$n > n_0 \implies x_n \in B_1\left(x, \tfrac{1}{n}\right) \subset B_1(x, \varepsilon).$$

So $\{x_n\}_n$ converges to x in d_1. However in d_2, for $\varepsilon = \varepsilon_0$, we can find such n_0 like d_1 since

$$x_n \notin B_2(x, \varepsilon_0)$$

for every n. Therefore, $\{x_n\}$ does not converges to x in d_2. $\square$

Proposition 1.13. *Let d_1 and d_2 are metric on a set X. They are equivalent if and only if the two identity functions $I : (X, d_1) \to (X, d_2)$ and $I : (X, d_2) \to (X, d_1)$ are continuous.*

Proof. The continuity of $I : (X, d_1) \to (X, d_2)$ is equivalent to the existence of δ such that $B_1(x, \delta) \subset B_2(x, \varepsilon)$. The opposite part is also true vice versa. $\square$

Remark. Generally, there exist two different topologies that have same sequential convergence data. For example, a sequence in an uncountable set with cocountable topology converges to a point if and only if it is eventually at the point, which is same with discrete topology. This means the informations of sequence convergence are not sufficient to uniquely characterize a topology. Instead, convergence data of generalized sequences also called nets, recover the whole topology. For topologies having a property called the first countability, it is enough to consider only usual sequences in spite of nets. What we did in this subsection is not useless because topology induced from metric is a typical example of first countable topologies. These kinds of problems will be profoundly treated in Chapter 3.

Remark. One can ask some results for the equivalence of metrics characterized by a same set of continuous functions. However, they are generally difficult problems: is it possible to recover the base space from a continuous function space or a path space?

The following two theorems give sufficient conditions for equivalence. The first theorem is well used to compare norms on a vector space in particular, and the second theorem is going to be used in the next subsection.

Theorem 1.14. *Let d_1 and d_2 are metric on a set X. If for each point x there exist two constants C_1 and C_2 which may depend on x such that*

$$d_2(x,y) \le C_1 d_1(x,y) \quad \text{and} \quad d_1(x,y) \le C_2 d_2(x,y)$$

for all y in X, then d_1 and d_2 are equivalent.

Proof. Since $d_1(x,y) < \varepsilon/C_1$ implies $d_2(x,y) < \varepsilon$ and $d_2(x,y) < \varepsilon/C_2$ implies $d_1(x,y) < \varepsilon$, we have

$$B_1\left(x, \frac{\varepsilon}{C_1}\right) \subset B_2(x,\varepsilon), \quad B_2\left(x, \frac{\varepsilon}{C_2}\right) \subset B_1(x,\varepsilon).$$

By letting $\delta_1 = \varepsilon/C_1$ and $\delta_2 = \varepsilon/C_2$, we can see the two metrics are equivalent. □

Theorem 1.15. *Let d be a metric on a set X and let f be a monotonically increasing subadditive real function on $\mathbb{R}_{\ge 0}$ such that $f^{-1}(0) = \{0\}$ so that $f \circ d$ is a metric. If f is continuous at 0 in addition, then $f \circ d$ is equivalent to d.*

Proof. We have seen that $f \circ d$ is a metric in Example 1.5. Firstly, for any ball $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$, we have a smaller ball

$$B_f(x, f(\varepsilon)) \subset B(x,\varepsilon),$$

where $B_f(x, f(\varepsilon)) = \{y : f(d(x,y)) < f(\varepsilon)\}$, since $f(d(x,y)) < f(\varepsilon)$ implies $d(x,y) < \varepsilon$.

The second inclusion requires the continuity of f. Take an arbitrary ball $B_f(x,\varepsilon)$. Since f is continuous at 0, we can find $\delta > 0$ such that

$$d(x,y) < \delta \quad \Longrightarrow \quad f(d(x,y)) < \varepsilon,$$

which implies $B(x,\delta) \subset B_f(x,\varepsilon)$. □

It is also natural to apply the concept of topological equivalence to norms. The idea is same; if two norms gives rise to a same topology, or equivalently, topologically equivalent metrics, then we call them equivalent. However, the checking procedure becomes rather simple; the converse of Theorem 1.14 holds for norms. It is because metrics on a vector space induced from norms has the property called translation invariance. The following thereom is often taken as the definition of norm equivalence.

Theorem 1.16. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. They induce the equivalent metrics if and only if there are constants C_1 and C_2 such that*

$$\|x\|_2 \le C_1 \|x\|_1 \quad \text{and} \quad \|x\|_1 \le C_2 \|x\|_2$$

for all $x \in V$.

PROOF. ($\Leftarrow$) It is a corollary of Theorem 1.14.

($\Rightarrow$) For any ball $B_2(0,\varepsilon)$, there is a smaller ball $B_1(0,\delta)$ such that $B_1(0,\delta) \subset B_2(0,\varepsilon)$ by the definition of equivalence of metrics. It means we have

$$\|x\|_1 < \delta \quad \Longrightarrow \quad \|x\|_2 < \varepsilon$$

for all $x \in V$. If we let $C_1 := \varepsilon/\delta$, then it is equivalent to

$$C_1\|x\|_1 < \varepsilon \quad \Longrightarrow \quad \|x\|_2 < \varepsilon.$$

If there is a vector $x \in V$ such that $\|x\|_2 > C_1\|x\|_1$, then we can lead a contradiction by taking ε between $\|x\|_2$ and $C_1\|x\|_1$. Therefore, $\|x\|_2 \leq C_1\|x\|_1$ for all $x \in V$. The other inequality is also shown by the same way. □

Especially, when we work on a vector space with finite dimension such as a Euclidean space $\mathbb{R}^n$, the situation gets better dramatically.

THEOREM 1.17. *On a finite dimensional vector space over a complete field such as $\mathbb{R}$ and $\mathbb{C}$, all norms are equivalent.*

PROOF. □

REMARK. Equivalence of norms on finite dimensional space is due to the locally compactness and the completeness. A closed ball follows the relation diagram:

$$\text{Bounded} \xrightarrow{\text{fin. dim.}} \text{Totally bounded} \xrightarrow{\text{complete}} \text{Compact}.$$

It is also an important result in functional analysis.

1.4. Bounding and combining metrics. In this subsection, we want to describe a topology generated by several metrics. Precisely, we can sum the metrics to make another metric out of old two metrics. Fortunately, the new metric is unique up to equivalence. By repeating the summation, it is easy to check any finite family of metrics can be combined to make a new metric.

PROPOSITION 1.18. *Sum of two metrics are a metric.*

PROOF. Obvious. □

PROPOSITION 1.19. *Let d_1, d_2, d_1', and d_2' be metrics on a set. If d_1 and d_2 are equivalent to d_1' and d_2' repsectively, then $d_1 + d_2$ and $d_1' + d_2'$ are equivalent.*

PROOF. Let $\{y : d_1'(x,y) + d_2'(x,y) < \varepsilon\}$ be an arbitrary ball centered at a point x taken by the metric $d_1' + d_2'$. By the equivalence, we can find $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$B_1(x,\delta_1) \subset B_{1'}(x,\tfrac{\varepsilon}{2}) \quad \text{and} \quad B_2(x,\delta_2) \subset B_{2'}(x,\tfrac{\varepsilon}{2}).$$

Let $\delta := \min\{\delta_1,\delta_2\}$. Then, the ball of radius δ in the metric $d_1 + d_2$ is contained in the ball of radius ε in the metric $d_1' + d_2'$:

$$\{y : d_1(x,y) + d_2(x,y) < \delta\} \subset \{y : d_1'(x,y) + d_2'(x,y) < \varepsilon\}.$$

The opposite part is shown in the same way. □

In fact, there is also a method for combining not only finite family of metrics, but also countable family of metrics. Since the sum of countably many numbers may diverges, we cannot sum the metrics directly. The strategy used here is to "bound" the metrics. We call a metric bounded when the image of metric is bounded.

Proposition 1.20. *Every metric possesses an equivalent bounded metric.*

Proof. Let d be a metric on a set. Let f be a bounded, monotonically increasing, and subadditive function on $\mathbb{R}_{\geq 0}$ that is continuous at 0 and satisfies $f^{-1}(0) = \{0\}$. The mostly used examples are

$$f(x) = \frac{x}{1+x} \quad \text{or} \quad f(x) = \min\{x, 1\}.$$

Then, $f \circ d$ is a bounded metric equivalent to d by Theorem 1.15. $\square$

Problems.

Problem 1.1. Show that there is a metric d on $\mathbb{R}$ such that a sequence $\{x_n\}_n$ defined by $x_n = x + \frac{1}{n}$ is convergent with respect to d if and only if $x \neq 0$.

Problem 1.2. Let $\{x_n\}_n$ be a convergent sequence in a metric space X. Show that for each point $p \in X$ there is $M > 0$ such that $d(x_n, p) < M$ for all n.

Problem 1.3. Let d and d' be metrics on a set X. Suppose that a sequence $\{x_n\}_n$ in X converges to x in d and converges to x' in d'. Show that $x = x'$.

Problem 1.4. Let d a metric on a set X. Show that a function d_p defined by $d_p(x, y) = d(x, y)^p$ is a metric equivalent to d for every $p > 0$.

2. Topology

We define topology and introduce some supplementary notions.

2.1. Filters. Suppose we want to find a proper way to define limit and convergence. Recall how we define convergence of a sequence of real numbers: we say a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a number x if for each $\varepsilon > 0$ there is $n_0(\varepsilon) \in \mathbb{N}$ such that $|x - x_n| < \varepsilon$ whenever $n > n_0$. Simply, x_n is close to x if n is close to the infinity. Observe the two necessary structures to make this possible; the "system of neighborhoods" at each point x, and the total order on the index set $\mathbb{N}$ the set of natural numbers. Without the order structure, we would not be able to formulate the intuition of the direction toward which a sequence is converging. Even though the order on $\mathbb{N}$ is totally defined so that we can compare every pair of two elements, but it can be generalized to the case of partial orders.

Definition 2.1. A subset $\mathcal{D}$ of a poset is called *(upward) directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$. Similarly, $\mathcal{D}$ is called *downward directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $c \leq a$ and $c \leq b$.

The directedness of a partially ordered set is an essential notion to define limit.

Let X be a set and $x \in X$. Then, the power set $\mathcal{P}(X)$ is a poset with inclusion relation. The filter bases are defined abstractly:

Definition 2.2. A *filter base* is a nonempty and downward directed subset of a poset.

And concretely:

Definition 2.3. A collection $\mathcal{B}_x$ of subsets of X is called a *filter base at x* if every element of $\mathcal{B}_x$ contains x and it forms a nonempty downward directed subset; every $U \in \mathcal{B}_x$ contains x, and for all $U_1, U_2 \in \mathcal{B}_x$ there is $U \in \mathcal{B}_x$ such that $U \subset U_1 \cap U_2$.

Among filters, we can give a relation structure as follows.

Definition 2.4. Let $\mathcal{B}_x, \mathcal{B}'_x$ be filter bases at x. We say $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$, or a *refinement* of $\mathcal{B}_x$ if for every $U \in \mathcal{B}_x$ there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$.

As synonyms, all the following expressions tell the same situation.

(1) $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$,
(2) $\mathcal{B}'_x$ is *stronger than* $\mathcal{B}_x$,
(3) $\mathcal{B}_x$ is *coarser than* $\mathcal{B}'_x$,
(4) $\mathcal{B}_x$ is *weaker than* $\mathcal{B}'_x$.

The relation is a preorder so that we can consider the equivalence classes on which the natural partial order can be defined.

Proposition 2.1. *The refinement relation between filter bases is a preorder, and each equivalence class contains a unique maximal element.*

Proof. To show a relation is a preorder, we need to check transitivity. Suppose $\mathcal{B}''_x$ is finer than $\mathcal{B}'_x$ and $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$. For any $U \in \mathcal{B}_x$, there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$, and there is also $U'' \in \mathcal{B}''_x$ such that $U'' \subset U'$. Since $U'' \subset U$, we can conclude $\mathcal{B}''_x$ is finer than $\mathcal{B}_x$.

We can say two filter bases are equivalent if they are both finer than each other. Consider an equivalence class of filter bases and just denote it by A. Then, $\bigcup_{\mathcal{B}_x \in A} \mathcal{B}_x$ is also contained in A since it is equivalent to an arbitrary filter base $\mathcal{B}_x$ in A. It is also easy to check that this is maximal. □

Now we define filters.

DEFINITION 2.5. A *filter at* x is the maximal element of an equivalence class of filter bases at x.

In other words, filters have one-to-one correspondence to the equivalence classes of filter bases. A filter is identified to an equivalence class of filter bases. They can be also characterized by three axioms.

THEOREM 2.2. *A collection $\mathcal{F}_x$ of subsets of X is a filter at x if and only if every element contains x and it is closed under supersets and finite intersections;*

(1) $x \in U$ *for* $U \in \mathcal{F}_x$,
(2) *if* $U \subset V$ *and* $U \in \mathcal{F}_x$, *then* $V \in \mathcal{F}_x$,
(3) *if* $U, V \in \mathcal{F}_x$, *then* $U \cap V \in \mathcal{F}_x$.

PROOF. □

Many references use the above theorem as the definition of filter because it is useful for someone who wants to check whether a given family is a filter.

THEOREM 2.3. *A filter $\mathcal{F}'_x$ is finer than another filter $\mathcal{F}_x$ if and only if $\mathcal{F}'_x \supset \mathcal{F}_x$.*

PROOF. □

The following examples will be helpful to catch the intuition.

EXAMPLE 2.4. Let x be a point in a metric space. The set of all open balls cenetered at x is a filter base at x. The set of all open balls containing x is aslo a filter base and they are equivalent. A filter equivalent to these filter bases are called *neighborhood filter at* x.

EXAMPLE 2.5. Let S be a subset of a set. The set of all subsets containing S is a filter at x for every $x \in S$. If $S = \{x\}$, then it is called a *principal filter* at x.

EXAMPLE 2.6. The set of all subsets of $\mathbb{N}$ whose complement is finite is a filter, but it is not a filter at a point. However, it is intuitively a filter at infinity.

2.2. Topologies. Before defining topology, recall that it plays the most important role in the definition of continuous functions to deal with neighborhoods of a point. We want a structure to give a notion of neighborhoods of a point such as metrics, in other words, we want to generalize metric in a suitable way. There is a conventional definition of topology: topology is defined as a subset of the power set of underlying space satisfying some axioms, and it is said to consist of open sets so that a topology indicates that which subsets are open or not. However, this definition is so abstract that it might allow first-readers to lose its intuitions. Thereby, we attempt to take another way. Before introducing topology, we shall define a topological basis. Topological bases are often used to describe a particular topology as bases of vector spaces do. The main definition of topology will follow.

Let X be a set.

DEFINITION 2.6. A collection $\mathcal{B}$ of subsets of X is called a *topological base* or simply a *base on* X if

$$\{U : x \in U \in \mathcal{B}\}$$

is a filter base at x for every $x \in X$.

A topological base is a kind of global version of filter base.

DEFINITION 2.7. Let $\mathcal{B}$ be a topological base on X and $x \in X$. A filter base at x is called a *local base* at x if it is equivalent to the filter base $\{U : x \in U \in \mathcal{B}\}$. If a local base is a filter at x, then it is called *neighborhood filter* of x.

All the followings are synonyns:

(1) local base
(2) neighborhood system
(3) fundamental system of neighborhoods
(4) complete system of neighborhoods
(5) filter base of neighborhood filter

As we have done in the previous section, we can settle the refinement order on the set of topological bases.

DEFINITION 2.8. Let $\mathcal{B}, \mathcal{B}'$ be topological bases on X. We say $\mathcal{B}$ is *coarser* or *weaker* than $\mathcal{B}'$, and $\mathcal{B}'$ is *finer*, *stronger* than $\mathcal{B}$, or a *refinement* of $\mathcal{B}$ if every local base $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$ at every point $x \in X$.

PROPOSITION 2.7. *The refinement relation between topological bases is a preorder, and each equivalence class contains a unique maximal element.*

PROOF. □

A topology is defined to be the maximal element, which means in fact an equivalence class of topological bases.

DEFINITION 2.9. A *topology on* X is the maximal element of an equivalence class of topological bases on X.

There is also a criterion for topology.

THEOREM 2.8. *A collection $\mathcal{T}$ of subsets of X is a topology on X if and only if*

(1) $\varnothing, X \in \mathcal{T}$,
(2) *if* $\{U_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{T}$, *then* $\bigcup_{\alpha\in\mathcal{A}} U_\alpha \in \mathcal{T}$,
(3) *if* $U_1, U_2 \in \mathcal{T}$, *then* $U_1 \cap U_2 \in \mathcal{T}$.

PROOF. □

Theorem 2.8 is usually used as a definition of topology because it allows us to check without difficulty whether a collection of subsets is a topology.

Since all topological structures are made to generalize the standard metric of Euclidean space, so drawing balls for representing base elements is always helpful in the whole story of general topology.

2.3. Bases and subbases.

Definition 2.10. Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. If $\mathcal{T}$ is the coarsest topology containing $\mathcal{B}$, then we say the topology $\mathcal{T}$ is *generated by* $\mathcal{B}$.

Theorem 2.9. *Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. The followings are equivalent:*

(1) *$\mathcal{B}$ generates $\mathcal{T}$,*
(2) *$\mathcal{B}$ and $\mathcal{T}$ are equivaent bases,*
(3) *$\mathcal{T}$ is the set of all arbitrary unions of elements of $\mathcal{B}$.*

Definition 2.11. Let $\mathcal{S} \subset \mathcal{P}(X)$. If a topology $\mathcal{T}$ is the coarsest topology containing $\mathcal{S}$, then we say $\mathcal{S}$ is called a *subbase* of $\mathcal{T}$.

Proposition 2.10. *Let $\mathcal{S} \subset \mathcal{P}(X)$. The set of finite intersections of elements of $\mathcal{S}$ is a basis.*

Here is the metric space example.

Example 2.11. Let X be a metric space. A set of all balls $\mathcal{B} = \{B(x,\varepsilon) : x \in X, \varepsilon > 0\}$ is a base on X because for every point $x \in B(x_1,\varepsilon_1) \cap B(x_2,\varepsilon_2)$, we have $x \in B(x,\varepsilon) \subset B(x_1,\varepsilon_1) \cap B(x_2,\varepsilon_2)$ where $\varepsilon = \min\{\varepsilon_1 - d(x,x_1), \varepsilon_2 - d(x,x_2)\}$.

In metric spaces, of course, there can exist infinitely many bases, but they are hardly considered except $\mathcal{B}$. Sometimes in the context of metric spaces, the term neighborhood or basis are used to say $\mathcal{B}$. As we have seen, balls in metric spaces are the main concept to state ε-δ argument. This example would show a basis is fundamental language to describe the nature of limits in metric spaces.

2.4. Open sets and neighborhoods. def:nbhd and neighborhood filter convergence and limit

2.5. Closed sets and limit points. closure dense set,

2.6. Interior and closure.

3. Uniformity

3.1. Uniform spaces. uniformness of metric

3.2. Entourages. Uniform spaces are generalization of metric spaces. The uniform structure is required to define uniform continuity, uniform convergence, completeness, etc. Although the definition of uniform structure is not so easy at first, they have enormous advantage to learn. For example, they are extremely useful in functional analysis since every compatible topology on algebraic structures such as topological group and topological vector space must admit a natural uniform structure. Hence, we can use completeness or something uniform without unnecessary concerns.

Definition 3.1 (Uniform space). A *uniform space* is a set X equipped with a filter of binary relations $\mathcal{U} \subset \mathcal{P}(X^2)$ such that for every $E \in \mathcal{U}$,

(1) reflexivity: $(x, x) \in E$ for all $x \in X$,
(2) triangle inequality: $\exists E' \in \mathcal{U} : E' \circ E' \subset E$,
(3) symmetry: $E^{-1} \in \mathcal{U}$,

where $\Delta_X = \{(x, x) : x \in X\}$ and

$$E \circ F = \{(x, z) : (x, y) \in E, (y, z) \in F\}, \quad E^{-1} = \{(y, x) : (x, y) \in E\}.$$

The collection $\mathcal{U}$ is called a *uniformity*, and a relation $E \in \mathcal{U}$ is called an *entourage*. If $(x, y) \in E$, then we say x and y are E-close.

Definition 3.2. Let $(X,\mathcal{U})$ be a uniform space. Let τ be a set containing all $U \subset X$ such that for every $x \in U$ there is an entourage E with $E_x \subset U$. Then τ defines a topology on X, which is called *uniform topology*, or *induced topology*.

Definition 3.3. A uniform space is called *Hausdorff* if there is an entourage E such that $x \in E$ and $y \notin E$ for every pair of distinct points $x, y \in X$. This is equivalent for the induced topology to be Hausdorff.

Note that the axioms for the definition of uniform spaces bear a similarity with the one of metric spaces. For one exception, the Hausdorffness implies the nondegeneracy. A uniform space is defined by the collection of relations that embody the concept of nearness. Unlike neighborhoods in general topological space, an entourage measures the nearness not pointwisely(locally) but uniformly(globally). We have the following hierarchy:

$$\text{topological space} \supset \text{uniform space} \supset \text{metric space}.$$

Example 3.1. Let G be a topological group. Let U be an open neighborhood of the identity e. Define

$$E_U := \{(g, h) : gh^{-1} \in U\}.$$

Then, the set of E_U forms a uniformity. The difficult part is the triangle inequality, which can be shown from the continuity of group operation.

3.3. Pseudometrics. Metric can be regarded as the "countably" uniform structure in some sense. In other texts, for this reason, one frequently introduces metric instead of uniformity in order to avoid superfluously complicated and less intuitive notions of uniform structures, when only doing elementary analysis not requiring uncountable local bases.

One of the mostly used way of characterizing uniformity is to induce the fundamental system of entourages from a family of pseudometrics. The manner is simple: just take all pseudoballs as the fundamental system of entourages.

Definition 3.4. Let

The proof of the following theorem is based on Bourbaki's text (General topology part2, chapter 9).

Theorem 3.2. *Every uniformity is induced by a family of pesudometrics.*

Proof. □

CHAPTER 2

Continuity

1. Continuous functions

1.1. Various continuity. continuity, Cauchy continuity, uniform continuity, Lipschitz continuity

Example 1.1. An isometry between metric spaces is Lipschitz continuous with costant 1.

1.2. Sequential continuity.

2. Continuous maps

2.1. Mono and epi.

2.2. Subspaces and quotient spaces.

2.3. Product space.

2.4. Homeomorphisms. continuous bijection open map how to show two spaces are not homeomorphic - topological property: connected, compact

3. Connectedness

3.1. Connected spaces. component

3.2. Path connected spaces.

3.3. Locally connected spaces.

3.4. Homotopy.

CHAPTER 3

Convergence

1. Nets

product of two directed sets projection is monotone final uniformity is itself an upward directed set by reverse inclusion, like $\mathbb{R}_{\geq 0}$. cofinality and subsequence

eventuality filter, three definitions of subnets

2. Sequences

sequential spaces, first countable

3. Completeness

completion

CHAPTER 4

Compactness

Definition 0.1. Let X be a topological space. A *cover* of a subset $A \subset X$ is a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X such that $A \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. If U_α are all open, then it is called *open cover*.

Definition 0.2. Let X be a topological space. A subset $K \subset X$ is called *compact* if every open cover of K has a finite subcover.

Proposition 0.1. *Let X be a topological space with a basis $\mathcal{B}$. A subset $K \subset X$ is compact if and only if every cover of the form $\{B_x \in \mathcal{B}\}_{x \in K}$ has a finite subcover.*

Remark. Let $\mathcal{P}$ be a property of a fuction $f \colon X \to Y$, such as continuity If we say f has $\mathcal{P}$ at a point x, then it would implies that x has a neighborhood U such that

0.1. Properties of compactness.

Theorem 0.2. *Let X and Y be topological spaces. For a continuous map $f \colon X \to Y$, the image $f(K)$ is compact for compact $K \subset X$.*

Remark. This is why the term "compact space" is widely used.

Corollary 0.3 (The extreme value theorem). *A continuous function on a closed interval has a global maximum and*

Heine-Cantor,

0.2. Characterizations of compactness.

$$
\begin{tikzcd}
\text{net compact} \arrow[d, leftrightarrow] & \text{sequentially compact} \arrow[d, shift right] & \\
\text{compact} \arrow[r] & \text{countably compact} \arrow[u, dashed, shift right, "\text{sequential}"'] \arrow[l, dashed, bend left, "\substack{\text{metrizable} \\ \textit{or}\ \text{2nd countable}}"] \arrow[r] & \text{limit point compact} \arrow[l, dashed, bend left, "T_1"]
\end{tikzcd}
$$

CHAPTER 5

Separation axioms

1. Separation axioms

2. Metrization theorems

CHAPTER 6

Function spaces

1. Compact-open topology

1.1. Definition.

Definition 1.1. Let X and Y be topological spaces. The *continuous functions space* $C(X, Y)$ is the set of continuous functions from X to Y. If $Y = \mathbb{R}$ or $\mathbb{C}$, then the continuous function space is denoted by $C(X)$.

1.2. Compact convergence. topology of compact convergence metrizability and hemicompact topology of uniform convergence uniform structure of pointwise convergence In considering the continuous function space, Y will be assumed to be a metric space because of its usefulness in most applications. Then, there are two useful topologies on $C(X, Y)$. Since there is a difficulty to deal with open sets or basis directly in a function space, the convergence will be a reliable alternative to describe the topologies. Before giving definition of the topologies, define pseudometrics ρ_K on $C(X, Y)$ by

$$\rho_K(f, g) = \sup_{x \in K} d(f(x), g(x))$$

for $K \subset X$ compact.

Definition 1.2. Let X and Y be topological spaces. The *topology of pointwise convergence* on $C(X, Y)$ is a subspace topology inherited from the product topology on Y^X.

Proposition 1.1. *Let X be a topological space and Y be a metric space. The topology of pointwise convergence on $C(X, Y)$ is generated by pseudometrics $\rho_{\{x\}}$, namely all $\{g : d(f(x), g(x)) < \varepsilon\}$ for $f \in C(X, Y)$, $\varepsilon > 0$, and $x \in X$.*

Definition 1.3. Let X be a topological space and Y be a metric space. The *topology of compact convergence* on $C(X, Y)$ is a topology generated by pseudometrics ρ_K, namely all $\{g : \rho_K(f, g) < \varepsilon\}$ for $f \in C(X, Y)$, $\varepsilon > 0$, and compact $K \subset X$.

Proposition 1.2. *Let $C(X, Y)$ be a continuous function space for a topological space X and a metric space Y. A functional sequence in $C(X, Y)$ converges in the topology of compact convergence if and only if the functional sequence converges compactly.*

Theorem 1.3. *Let X be a topological space and Y be a metric space. If X is hemicompact, in other words, X has a sequence of compact subsets $\{K_n\}_{n \in \mathbb{N}}$ such that every compact subset of X is contained in K_n for some $n \in \mathbb{N}$, then the topology of compact convergence on $C(X, Y)$ is metrizable.*

Proof. bounding and merging pseudometrics $\square$

1.3. Exponentiability. locally compact Hausdorff spaces exponential space $\frac{\varepsilon}{3}$ argument

2. Rings of continuous functions

2.1. $C(X), C_0(X), C_b(X)$**.**

3. Important theorems on function space

3.1. The Arzela-Ascoli theorem. The Arzela-Ascoli theorem is a main technique to verify compactness of a subspace of continuous function space. The theorem requires the notion of equicontinuity, which lifts pointwise compactness up onto compactness in topology of compact convergence.

Definition 3.1. Let X be a topological space and Y be a metric space. A subset $\mathcal{F} \subset C(X,Y)$ is called *(pointwise or locally) equicontinuous* if for every $\varepsilon > 0$ and each $x_0 \in X$, there is an open neighborhood U of x_0 such that $x \in U \Rightarrow d(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$.

Compare with the following definition:

Definition 3.2. Let X be a metric space and Y be a metric space. A subset $\mathcal{F} \subset C(X,Y)$ is called *uniformly equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$.

The uniform equicontinuity is what the Rudin's book says it just equicontinuous.

Theorem 3.1 (Arzela-Ascoli, conventional version). *Let X be a compact space. For $(f_n)_{n\in\mathbb{N}} \subset C(X)$, if it is equicontinuous and pointwisely bounded, then there is a subsequence that uniformly converges.*

Theorem 3.2 (Arzela-Ascoli, metrized version). *Let X be a hemicompact space and Y be a metric space. Let $\mathcal{T}_p$ and $\mathcal{T}_c$ be the topology of pointwise and compact convergence on $C(X,Y)$ relatively. For $\mathcal{F} \subset C(X,Y)$, if $\mathcal{F}$ is equicontinuous and relatively compact in $\mathcal{T}_p$, then $\mathcal{F}$ is relatively compact in $\mathcal{T}_c$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and $K \subset X$ be a compact. By equicontinuity, for each $k \in \mathbb{N}$ a finite open cover $\{U_s\}_{s\in S_k}$ with a finite set $S_k \subset K$ can be taken such that $x \in U_s \Rightarrow d(f(x), f(s)) < \frac{1}{k}$ for all $f \in \mathcal{F}$. By the pointwise relative compactness, we can extract a subsequence $\{f_m\}_{m\in\mathbb{N}}$ of $\{f_n\}_n$ such that $\{f_m(s)\}_m$ is Cauchy for each $s \in \bigcup_{k\in\mathbb{N}} S_k$ by the diagonal argument.

For every $\varepsilon > 0$, let $k = \lceil (\frac{\varepsilon}{3})^{-1} \rceil$ and $m_0 = \max\{m_{0,s} : s \in S_k\}$ where $m_{0,s}$ satisfies that $m, m' > m_{0,s} \Rightarrow d(f_m(s), f_{m'}(s)) < \frac{\varepsilon}{3}$. By taking $s \in S_k$ such that $x \in U_s$ for arbitrary $x \in K$, we obtain, for $m, m' > m_0$,

$$d(f_m(x), f_{m'}(x)) \le d(f_m(x), f_m(s)) + d(f_m(s), f_{m'}(s)) + d(f_{m'}(s), f_{m'}(x)) < \varepsilon.$$

Thus, $\{f_m\}_m$ is a subsequence of $\{f_n\}_n$ that is uniformly Cauchy on K. $\square$

converse of Arzela-Ascoli

3.2. The Stone-Weierstrass theorem.