

Contents

1	Kinetic theory	2
1.1	Vlasov-Poisson equation	2
1.1.1	Schaeffer's global existence proof	3
1.2	Velocity averaging lemmas	4
2	Sturm-Liouville theory	7
2.1	Self-adjointness	7
2.2	Regular Sturm-Liouville problem	7
2.3	Legendre's equation	8
2.4	Bessel's equation	9
3	Peetre's theorem	10
4	Characteristic curve	12
4.1	Wave equation	13
4.2	Burgers' equation	15
5	Weak convergences	16
6	Fixed point methods	17
6.1	Picard-Lindelöf theorem	17
7	Statements in functional analysis and general topology	18
8	Ultrafilter	19
9	Selected analysis problems	20
10	Physics problem	25
10.1	Resonance	25

1 Kinetic theory

1.1 Vlasov-Poisson equation

Consider a Cauchy problem of the *Vlasov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, & (t, x, v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t, x) = \nabla_x \Delta_x^{-1} \rho, \\ \rho(t, x) = \int f dv, \\ f(0, x, v) = f_0(x, v). \end{cases}$$

For boundaryless problem, we have

$$E = \nabla_x \left(-\frac{1}{4\pi|x|} * \rho \right) = \frac{x}{4\pi|x|^3} * \rho.$$

Theorem 1.1 (Schaeffer, 1991). *Let $f_0 \in C_{0,x,v}^1$ and $f_0 \geq 0$. Then, the Cauchy problem for the VP system has a unique C^1 global solution.*

Definition 1.1. For a local solution f ,

$$Q(t) := 1 + \sup\{|v| : f(s, x, v) \neq 0 \text{ for some } s \in [0, t], x \in \mathbb{R}_x^3\}.$$

Electric field can be controlled by density ρ , henceforth by Q .

Lemma 1.2. *For $1 \leq q < \frac{N}{N-2} = 3 < p \leq \infty$,*

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^p}^{\frac{\frac{2}{N}-1+\frac{1}{q}}{\frac{1}{q}-\frac{1}{p}}} \|\rho(t, x)\|_{L_x^q}^{\frac{1-\frac{1}{p}-\frac{2}{N}}{\frac{1}{q}-\frac{1}{p}}}.$$

Furthermore,

$$\|E(t, x)\|_{L_x^\infty} \lesssim Q(t)^\varepsilon$$

for any $0 < \varepsilon \leq 2$.

Proof. Fix time t . For $2p < N < 2q$,

$$\begin{aligned} 4\pi|E(x)| &= \left| \frac{1}{|x|^2} * \rho(x) \right| \\ &\leq \int_{|x-y|<R} \frac{\rho(y)}{|x-y|^2} dy + \int_{|x-y|\geq R} \frac{\rho(y)}{|x-y|^2} dy \\ &\leq \|\rho\|_{p'} \left(\int_{|y|<R} \frac{dy}{|y|^{2p}} \right)^{1/p} + \|\rho\|_{q'} \left(\int_{|y|\geq R} \frac{dy}{|y|^{2q}} \right)^{1/q} \\ &\simeq \|\rho\|_{p'} \left(\int_0^R r^{N-1-2p} dr \right)^{1/p} + \|\rho\|_{q'} \left(\int_R^\infty r^{N-1-2q} dr \right)^{1/q} \\ &\simeq \|\rho\|_{p'} R^{\frac{N}{p}-2} + \|\rho\|_{q'} R^{\frac{N}{q}-2}. \end{aligned}$$

By choosing R such that $\|\rho\|_{p'} R^{\frac{N}{p}-2} = \|\rho\|_{q'} R^{\frac{N}{q}-2}$, we get

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^{p'}}^{\frac{\frac{2}{p}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho(t, x)\|_{L_x^{q'}}^{\frac{\frac{1}{p}-\frac{2}{q}}{\frac{1}{p}-\frac{1}{q}}},$$

hence the inequality.

If $p = \infty$ and $N = 3$, then $1 \leq q < 3$ implies

$$\|E\|_\infty^3 \lesssim \|\rho\|_\infty^{3-q} \|\rho\|_q^q.$$

For example, if we let $(p, q) = (\infty, 1)$ or $(\infty, 5/3)$, then

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{2/3} \|\rho\|_1^{1/3}, \quad \|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

To control E with $Q(t)$,

$$\rho(t, x) = \int_{|v| < Q(t)} f(t, x, v) dv \lesssim Q(t)^3 \|f_0(x, v)\|_{L_v^\infty},$$

so by letting $\varepsilon = 3 - q$,

$$\|E(t, x)\|_{L_x^\infty} \lesssim \|\rho(t, x)\|_{L_x^\infty}^{(3-q)/3} \lesssim Q(t)^\varepsilon$$

for any $0 < \varepsilon \leq 2$. (L^p -norm of ρ for $p < \infty$ well propagates by the measure preserving property.) \square

For a characteristic curve $s \mapsto (X(s; t, x, v), V(s; t, x, v))$, we have

$$f(t, x, v) = f(s, X(s; t, x, v), V(s; t, x, v))$$

and

$$\begin{aligned} & \iint f(s, x, v) \varphi(x, v) dv dx \\ &= \iint f(s, X(s; t, x, v), V(s; t, x, v)) \varphi(X(s; t, x, v), V(s; t, x, v)) dv dx \\ &= \iint f(t, x, v) \varphi(X(s; t, x, v), V(s; t, x, v)) dv dx. \end{aligned}$$

1.1.1 Schaeffer's global existence proof

Fix a characteristic $(X^*(s), V^*(s)) = (X(s; t^*, x^*, v^*), V(s; t^*, x^*, v^*))$. For $t \in [0, t^*]$,

$$\begin{aligned} \int_t^{t^*} |E(s, X^*(s))| ds &\lesssim \int_t^{t^*} \iint \frac{f(s, x, v)}{|x - X^*(s)|^2} dv dx ds \\ &= \int_t^{t^*} \iint \frac{f(t, x, v)}{|X(s; t, x, v) - X^*(s)|^2} dv dx ds. \end{aligned}$$

$$\begin{aligned}
|v - V(t; t^*, x, v)| &= |V(t^*; t^*, x, v) - V(t; t^*, x, v)| \\
&\leq \int_t^{t^*} |E(s, X(s, x, v))| ds \lesssim (t^* - t)Q^{4/3}.
\end{aligned}$$

1.2 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 1.3 (Velocity averaging). *Let L be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. Then,*

$$\left\| \int u \varphi dv \right\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}$$

for $\varphi \in C_c^\infty(\mathbb{R}_v^n)$,

Proof. Let $m(t, x) = \int u \varphi dv$. By Fourier transform with respect to t and x , we have

$$\widehat{u}(\tau, \xi, v) = \frac{1}{i} \frac{\widehat{Lu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v) \varphi(v) dv.$$

Fixing τ, ξ , decompose the integral and use Hölder's inequality to get

$$\begin{aligned}
|\widehat{m}(\tau, \xi)| &\leq \int_{|\tau+v \cdot \xi| < \alpha} |\widehat{u} \varphi| dv + \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\widehat{Lu} \varphi|}{|\tau + v \cdot \xi|} dv \\
&\leq \|\widehat{u}\|_{L_v^2}^{1/2} \left(\int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv \right)^{1/2} + \|\widehat{Lu}\|_{L_v^2}^{1/2} \left(\int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2} dv \right)^{1/2},
\end{aligned}$$

where $\alpha > 0$ is an arbitrary constant that will be determined later. Let

$$I_s(\tau, \xi, \alpha) := \int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv, \quad I_n(\tau, \xi, \alpha) := \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|} dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \quad I_n \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

Define coordinates (v_1, v_2) on \mathbb{R}_v as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}, \quad v_2 := v - \frac{v \cdot \xi}{|\xi|^2} \xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v_1 - \frac{\tau}{|\xi|})^2 + |v_2|^2 \quad \text{and} \quad \int dv = \iint dv_2 dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| \leq R$. If $\frac{|\tau| - \alpha}{|\xi|} > R$, then the region of integration vanishes so that $I_s = 0$. If $|\tau| \leq \alpha + R|\xi|$, then

$$\begin{aligned} I_s &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 dv_1 \\ &\lesssim \int_{|v_1| < \frac{\alpha}{|\xi|}, |v_1| \leq R} \int_{|v_2| \leq R} dv_2 dv_1 \\ &\lesssim \min\left\{\frac{2\alpha}{|\xi|}, R\right\} \cdot R^{n-1} \\ &\lesssim \frac{1}{\sqrt{1 + (\frac{|\xi|}{\alpha})^2}} \\ &\lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}. \end{aligned}$$

For the second integral, suppose that φ is supported on $|v| < C$ so that $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < C$. Then,

$$\begin{aligned} I_n &\lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|} < C} \int_{|v_2| < C} \frac{1}{v_1^2 |\xi|^2} dv_2 dv_1 \\ &\simeq \int_{|v_1| \geq \frac{\alpha}{|\xi|}, |v_1 - \frac{\tau}{|\xi|} < C} \frac{dv_1}{v_1^2 |\xi|^2}. \end{aligned}$$

If $|\xi| \gtrsim |\tau|$, then

$$I_n \lesssim \int_{|v_1| \geq \frac{\alpha}{|\xi|}} \frac{dv_1}{v_1^2 |\xi|^2} \simeq \frac{1}{\alpha |\xi|} \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

If $|\xi| \ll |\tau|$ such that at least $|\tau| > C|\xi|$, then

$$\begin{aligned} I_n &\lesssim \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - C\} \leq v_1 < \frac{|\tau|}{|\xi|} + C} \frac{dv_1}{v_1^2 |\xi|^2} \\ &\simeq \frac{1}{|\xi|^2} \left(\frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|} - C\}} - \frac{1}{\frac{|\tau|}{|\xi|} + C} \right) \\ &\lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}. \quad (\text{This is not easy..!}) \end{aligned}$$

To sum up, we have

$$|\widehat{m}(\tau, \xi)| \lesssim \frac{1}{(\tau^2 + |\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Lu}\|_{L_v^2}^{1/2}).$$

Letting $\alpha = \sqrt{\|\widehat{Lu}\|_{L_v^2} / \|\widehat{u}\|_{L_v^2}}$ and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2} \|\widehat{Lu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$ and Plancheral's theorem gives

$$\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Lu\|_{L_{t,x,v}^2}^{1/2}.$$

□

Remark. We can obtain L^p result by applying

$$\min\left\{\frac{1}{x}, \frac{1}{y}\right\} \leq \frac{2}{(x^p + y^p)^{1/p}} \leq \max\left\{\frac{1}{x}, \frac{1}{y}\right\}$$

on the first integral, and

$$x \gg 1 \implies \frac{1}{\max\{a, x - c\}} - \frac{1}{x + c} \lesssim \frac{1}{a(x^p + 1)^{1/p}}$$

on the second integral.

Corollary 1.4. *Let \mathcal{F} be a family of functions on $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. If \mathcal{F} and $L\mathcal{F}$ are bounded in $L_{t,x,v}^2$, then $\int \mathcal{F} \varphi dv$ is bounded in $H_{t,x}^{1/2}$.*

Theorem 1.5. *Let \mathcal{F} be a family of functions on $I_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$. If \mathcal{F} is weakly relatively compact and $L\mathcal{F}$ is bounded in $L_{t,x,v}^1$, then $\int \mathcal{F} \varphi dv$ is relatively compact in $L_{t,x}^1$.*

2 Sturm-Liouville theory

2.1 Self-adjointness

Let $I = [a, b]$ and

$$L = -\frac{1}{w(x)} \left[\frac{d}{dx} \left(p(x) \frac{d}{dx} \right) + q(x) \right],$$

$$0 \leq p(x) \in C^\infty(I), \quad q(x) \in C^\infty(I), \quad 0 < w(x) \in C^\infty(I).$$

We expect L to be self-adjoint. In this regard, our interest is elimination of the difference term

$$\langle f, Lg \rangle - \langle Lf, g \rangle = p(f'g - fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[\frac{d}{dx} \left((1-x^2) \frac{d}{dx} \right) - \frac{m^2}{1-x^2} \right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2} \left[\frac{d}{dx} \left(e^{-x^2} \frac{d}{dx} \right) \right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

2.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that p does not vanish on the boundary of I that we should cancel $f'g - fg'|_a^b$. View the Sturm-Liouville operator L as a non-densely defined operator on the space $C^\infty(I)$ with inner product $\langle f, g \rangle = \int_I fgw$ with domain

$$V = \{ u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0, \beta_0 u(b) + \beta_1 u'(b) = 0 \},$$

the subspace for the *separated* Robin boundary condition.

Proposition 2.1. *The operator $L : V \rightarrow C^\infty(I)$ is self-adjoint when $C^\infty(I)$ has the inner product $\langle f, g \rangle = \int_I fgw$.*

We are interested in the eigenvalue problem of $L : V \rightarrow C^\infty(I)$ on V . Fortunately, if we choose a constant $z \in \mathbb{C} \setminus \mathbb{R}$, then $(L - z)^{-1} : C^\infty(I) \rightarrow V$ is well-defined.

Proposition 2.2. *If z is not an eigenvalue of L , then $L - z : V \rightarrow C^\infty(I)$ is bijective.*

Proof. The injectivity follows from the definition of eigenvalues. We may assume that L is injective by translation $q \mapsto q - \lambda$.

Suppose $f \in C^\infty(I)$. The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$\begin{aligned} -pu'' - p'u' - qu &= fw, \\ \alpha_0 u(a) + \alpha_1 u'(a) &= 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0. \end{aligned}$$

Let u_a, u_b be the unique solutions of the corresponding homogeneous equation with initial conditions

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Then we can define $L^{-1} : C^\infty([0, 1]) \rightarrow D(L)$ by

$$L^{-1}f(x) := u_a(x) \int_x^1 \frac{u_b}{W[u_a, u_b]} \frac{f}{(-p)} w + u_b(x) \int_0^x \frac{u_a}{W[u_a, u_b]} \frac{f}{(-p)} w,$$

where $W[u_a, u_b] := u_a u'_b - u_b u'_a$ denotes the Wronskian. This formula is derived from variation of parameters: we can compute c_a and c_b from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u'_a & u'_b \end{pmatrix} \begin{pmatrix} c'_a \\ c'_b \end{pmatrix} \implies L(c_a u_a + c_b u_b) = f.$$

Then, we can easily check that

$$L^{-1}Lu = u$$

for $u \in D(L)$, which implies L is surjective. □

2.3 Legendre's equation

The Legendre equation is

$$(1 - x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1, 1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx} \left((1 - x^2) \frac{d}{dx} \right).$$

Since $p(\pm 1) = 0$, the operator $L : C^\infty([-1, 1]) \rightarrow C^\infty([-1, 1])$ is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

l	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2} \log \frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x \log \frac{1+x}{1-x}, \quad \dots \in L^2(-1, 1) \setminus C^\infty([-1, 1])$$

as eigenvectors of L , then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x)(Q'_0(x)f(x) - Q_0(x)f'(x)) \Big|_{-1}^1 \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for $f \in C^\infty([-1, 1])$.

2.4 Bessel's equation

The Bessel equation is

$$x^2 u'' + xu' + (k^2 x^2 - \nu^2)u = 0, \quad \text{on } (0, \infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x} \left[\frac{d}{dx} \left(x \frac{d}{dx} \right) - \nu^2 \frac{1}{x} \right].$$

3 Peetre's theorem

Lemma 3.1. *Suppose a linear operator $L : C_c^\infty(M) \rightarrow C_c^\infty(M)$ satisfies*

$$\text{supp}(Lu) \subset \text{supp}(u) \quad \text{for } u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m , we can find $u \in C_c^\infty(U \setminus \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C . We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x .

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m :

$$\begin{aligned} \sum_{i \geq 0} \left\| 2^{-i} \frac{u_i}{\|u_i\|_{C^i}} \right\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\ &\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\ &< \infty. \end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

□

4 Characteristic curve

Algorithm:

- (1) Establish the associated vector field by substituting $u \mapsto y$.
- (2) Find the integral curve.
- (3) Eliminate the auxiliary variables to get an algebraic equation.
- (4) Verify the computed solution is in fact the real solution.

Proposition 4.1. *Suppose that there exists a smooth solution $u : \Omega \rightarrow \mathbb{R}_y$ of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, & (t, x) \in \Omega \subset \mathbb{R}_{t \geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, & \text{at } x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t, x)$.

Let $\gamma : I \rightarrow \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\tilde{\gamma} : \tilde{I} \rightarrow M$ as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\tilde{\gamma}(0) = \gamma(0)$. Since X and \tilde{X} coincide on M , the curve $\tilde{\gamma}$ is also an integral curve of X with $\tilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\tilde{I} \subset I$ and $\gamma(\theta) = \tilde{\gamma}(\theta)$ for all $\theta \in \tilde{I}$.

Since M is closed in E , the open interval $\tilde{I} = \gamma^{-1}(M)$ is closed in I , hence $\tilde{I} = I$ by the connectedness of I . \square

Definition 4.1. The projection of the integral curve γ onto Ω is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X . Once we find a necessary condition of the form of algebraic equation, we can demonstrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u , we can solve the ODE: let $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$ be the integral curve of X such that $\gamma(0) = (0, \xi, \xi)$. Then, the system of ODEs

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad y(\theta) = \xi, \quad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t, x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega = \{ (t, x) : tx > -\frac{1}{4} \}.$$

4.1 Wave equation

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } t, x > 0, \\ u(0, x) &= g(x), \quad u_t(0, x) = h(x), \quad u_x(t, 0) = \alpha(t).\end{aligned}$$

Define $v := u_t - cu_x$. Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0, x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t, x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0, x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t, 0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter $\xi > 0$,

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t, x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi)\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad x(\theta) = -c\theta + \xi, \quad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta') d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned}u(t, x) &= g(\xi) - \int_0^\theta v(s, \xi - cs) ds \\ &= g(x + ct) \\ &= \frac{3g(x + ct) - g(x - ct)}{2} - \int_0^t h(x + c(t - 2s)) ds\end{aligned}$$

4.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

- (1) Suppose $u(0, x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx -plane.
- (2) Suppose $u(0, x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx -plane.
- (3) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and “paste” the solution together.

5 Weak convergences

6 Fixed point methods

6.1 Picard-Lindelöf theorem

Theorem 6.1. *Consider the following initial value problem:*

$$x' = f(t, x), \quad x(0) = x_0,$$

where

$$f(-, x) \in C([0, T]), \quad f(t, -) \in \text{Lip}(\overline{B(x_0, R)}).$$

If T and R satisfy

- (1) $TK < 1$,
- (2) $TM \leq R$,
- (3) $|f(t, x)| \leq M$ on $[0, T] \times \overline{B(x_0, R)}$,

then the equation has a unique solution on $[-T, T]$.

Proof. Define $\varphi : C([0, T], \overline{B(x_0, R)}) \rightarrow C([0, T], \overline{B(x_0, R)})$ as:

$$\varphi(x)(t) := x_0 + \int_0^t f(s, x(s)) ds.$$

It is well-defined since

$$\begin{aligned} |\varphi(x)(t) - x_0| &\leq \int_0^t |f(s, x(s))| ds \\ &\leq TM \leq R. \end{aligned}$$

It is a contraction since we have

$$\begin{aligned} |\varphi(x)(t) - \varphi(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))| ds \\ &\leq \int_0^t K|x(s) - y(s)| ds \\ &\leq TK\|x - y\| \end{aligned}$$

so that

$$\|\varphi(x) - \varphi(y)\| \leq TK\|x - y\|$$

□

The above one looses the Lipschitz condition to local condition.

7 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0, 1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every separable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem \rightarrow continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable strict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by \mathcal{T}_β coincides with \mathcal{T}_α)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- The integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x} : D' \rightarrow D'$ is continuous. They commute (Schwarz theorem holds).
- $D \rightarrow S \rightarrow L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

8 Ultrafilter

Definition 8.1. An *ultrafilter* is a synonym for maximal filter. If we sat \mathcal{U} is an *ultrafilter on a set A* , then it means \mathcal{U} is a maximal filter as a directed subset of $\mathcal{P}(A)$.

existence of ultrafilter.

Theorem 8.1. *Let \mathcal{U} be an ultrafilter on a set A and X be a compact space. For a function $f : A \rightarrow X$, the limit $\mathcal{U}\text{-lim } f$ always exists.*

Theorem 8.2. *Let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ be a product space of compact spaces X_α . A net $f : \mathcal{D} \rightarrow X$ has a convergent subnet.*

Proof 1. Use Tychonoff. Compactness and net compactness are equivalent. \square

Proof 2. It is a proof without Tychonoff. Let \mathcal{U} be a ultrafilter on a set \mathcal{D} containing all $\uparrow d$. Define a directed set $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$ as $(d, U) \succ (d', U')$ for $U \subset U'$. Let $f : \mathcal{E} \rightarrow X$ be a subnet of $f : \mathcal{D} \rightarrow X$ defined by $f_{(d, U)} = f_d$.

By the previous theorem, $\mathcal{U}\text{-lim } \pi_\alpha f_d \in X_\alpha$ exists for each α . Define $f \in X$ such that $\pi_\alpha f = \mathcal{U}\text{-lim } \pi_\alpha f_d$. Let $G = \prod_\alpha G_\alpha \subset X$ be any open neighborhood of f . Then, $\pi_\alpha f \in G_\alpha$ and we have $G_\alpha = X_\alpha$ except finite. For α , we can take $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$ by definition of convergence with ultrafilter. Since $U_\alpha = \mathcal{D}$ except finites, we can take an upper bound $U_0 \in \mathcal{U}$ of $\{U_\alpha\}_\alpha$. Then, by taking any $d_0 \in U_0$, we have $f_{(d, U)} \in G$ for every $(d, U) \succ (d_0, U_0)$. This means $f = \lim_{\mathcal{E}} f_{(d, U)}$, so we can say $\lim_{\mathcal{E}} f_{(d, U)}$ exists. \square

9 Selected analysis problems

Problem 9.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \leq x - x_0 \leq x \leq 2^k$, $x - x_0 \in A_k$.

$$|A_k| \geq \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\ &\geq \sum_{k=1}^N (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^N \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &> \sum_{k=1}^N \frac{2^k}{2^{k+2}} \frac{1}{7k} \\ &= \frac{1}{28} \sum_{k=1}^N \frac{1}{k} \\ &\rightarrow \infty. \end{aligned}$$

□

Problem 9.2. If $|xf'(x)| \leq M$ and $\frac{1}{x} \int_0^x f(y) dy \rightarrow L$, then $f(x) \rightarrow L$ as $x \rightarrow \infty$.

Solution. Since

$$\begin{aligned} \left| f(x) - \frac{F(x) - F(a)}{x - a} \right| &\leq \frac{1}{x - a} \int_a^x |f(x) - f(y)| dy \\ &= \frac{1}{x - a} \int_a^x (x - y) |f'(c)| dy \\ &\leq \frac{M}{x - a} \int_a^x \frac{x - y}{c} dy \\ &\leq M \frac{x - a}{a} \end{aligned}$$

by the mean value theorem and

$$f(x) - L = \left[f(x) - \frac{F(x) - F(a)}{x - a} \right] + \frac{x}{x - a} \left[\frac{F(x)}{x} - L \right] + \frac{a}{x - a} \left[\frac{F(a)}{a} - L \right],$$

we have for any $\varepsilon > 0$

$$\limsup_{x \rightarrow \infty} |f(x) - L| \leq \varepsilon$$

where a is defined by $\frac{x-a}{a} = \frac{\varepsilon}{M}$. □

Problem 9.3. Let $f_n : I \rightarrow I$ be a sequence of real functions that satisfies $|f_n(x) - f_n(y)| \leq |x - y|$ whenever $|x - y| \geq \frac{1}{n}$, where $I = [0, 1]$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap I \rightarrow I$ pointwisely.

Step [.1] For $n \geq 4$, we claim

$$|x - y| \leq \frac{1}{n} \implies |f_n(x) - f_n(y)| \leq \frac{5}{n}. \quad (1)$$

Fix $x \in I$ and take $z \in I$ such that $|x - z| = \frac{2}{n}$ so that

$$|f_n(x) - f_n(z)| \leq |x - z| = \frac{2}{n}.$$

If y satisfies $|x - y| \leq \frac{1}{n}$, then we have $|y - z| \geq |x - z| - |x - y| \geq \frac{1}{n}$, so we get

$$|f_n(y) - f_n(z)| \leq |y - z| \leq |y - x| + |x - z| \leq \frac{3}{n}.$$

Combining these two inequalities proves what we want.

Step [.2] For $\varepsilon > 0$ and $N := \lceil \frac{15}{\varepsilon} \rceil$ we claim

$$|x - y| \leq \frac{1}{N} \quad \text{and} \quad n > N \implies |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{3} \quad (2)$$

when $N \geq 4$. It is allowed for $|x - y|$ to have the following two cases:

$$|x - y| \leq \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x - y| \leq \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \leq \frac{5}{n} < \frac{5}{N} \leq \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \leq |x - y| \leq \frac{1}{N} \leq \frac{\varepsilon}{15}.$$

Hence the claim is proved.

Step [.3] We will prove f is uniformly continuous. For $\varepsilon > 0$, take $\delta := \frac{1}{N}$, where $N := \lceil \frac{15}{\varepsilon} \rceil$. We will show

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

for $x, y \in \mathbb{Q} \cap I$ and $N \geq 4$. Fix rational numbers x and y in I which satisfy $|x - y| < \delta$. Since $f_n(x)$ and $f_n(y)$ converges to $f(x)$ and $f(y)$ respectively, we may take an integer n_x and n_y , such that

$$n > n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad (3)$$

and

$$n > n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \quad (4)$$

Choose an integer n such that $n > \max\{n_x, n_y, N\}$. Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since f is continuous on a dense subset $\mathbb{Q} \cap I$, it has a unique continuous extension on the whole I . Let it denoted by the same notation f .

Step [4] Finally, we are going to show $f_n \rightarrow f$ uniformly. For $\varepsilon > 0$, let $N := \lceil \frac{15}{\varepsilon} \rceil$. The uniform continuity of f allows to have $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{2}{3}\varepsilon. \quad (5)$$

Take a rational $r \in I$, depending on $x \in I$, such that $|x - r| < \min\{\frac{1}{N}, \delta\}$. Then, by (2) and (5), given $n > N \geq 4$, we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any $x \in I$. By limiting $n \rightarrow \infty$, we obtain

$$\lim_{n \rightarrow \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since ε and x are arbitrary, we can deduce the uniform convergence of f_n as $n \rightarrow \infty$. \square

Problem 9.4. A measurable subset of \mathbb{R} with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

Solution. Let $E \subset \mathbb{R}$ be measurable with $\mu(E) > 0$. We may assume E is bounded so that we have $E \subset I$ for a closed bounded interval since \mathbb{R} is σ -compact. Let n be a positive integer arbitrarily taken. Then, we can find N such that $\sum_{k=1}^N \frac{1}{k} > (n-1) \frac{\mu(I)}{\mu(E)}$.

Assume that every point x in E is contained in at most $n-1$ sets among

$$E, \frac{1}{2}E, \frac{1}{3}E, \dots, \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k \in A} \frac{1}{k}E = \emptyset$$

for any subset $A \subset \{1, \dots, N\}$ with $|A| \geq n$. Define

$$E_A := \bigcap_{k \in A} \frac{1}{k}E \cap \bigcap_{k' \in A} \left(\frac{1}{k'}E \right)^c$$

for $A \subset \{1, \dots, N\}$. Then, $\mu(E_A) = 0$ for $|A| \geq n$.

Note that we have

$$\mu\left(\frac{1}{k}E\right) = \sum_{k \in A} \mu(E_A) = \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) = \sum_{k=1}^N \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A) = \sum_{|A| < n} |A| \mu(E_A)$$

by double counting, and since E_A are disjoint, we have

$$\sum_{|A| < n} |A| \mu(E_A) = (n-1) \sum_{0 < |A| < n} \mu(E_A) \leq (n-1) \mu(I),$$

hence a contradiction to

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) > (n-1) \mu(I).$$

Therefore, we may find an element x that belongs to $\frac{1}{k}E$ for $k \in A$, where $A \subset \{1, \dots, N\}$ with $|A| = n$. Then, $ax \in E$ for all $a \in A \subset \mathbb{Z}$. \square

10 Physics problem

10.1 Resonance

Let m, b, k, A, ω_d be positive real constants. Consider an underdamped oscillator with sinusoidal driving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0, \quad x'(0) = 0.$$

There are some observations:

- (1) The underdamping condition means $b^2 - 4mk < 0$ so that the roots of characteristic equation are imaginary.
- (2) The positivity of m, b implies the real part of solution that will be denoted by $-\beta = -\frac{b}{2m}$ is negative; it shows exponential decay of solutions.
- (3) Introducing the natural frequency $\omega_n = \sqrt{k/m}$, we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega_d t.$$

- (4) The complementary solution is computed as

$$x_c(t) = x_0 e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \rightarrow \infty} x_c(t) = 0.$$

- (5) The condition $\beta > \omega_n$ is equivalent to that the oscillator is underdamped.
- (6) Let m, k be fixed. Then, the solution x_c decays most fastly when b satisfied $b^2 = 4mk$, equivalently, $\beta = \omega_n$.
- (7) When $\omega_d = \omega_n$ such that the amplitude of particular solution diverges.