Morse Theory

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1. Morse functions

Definition 1.1. Let M be a manifold. A *Morse function* is a smooth function $f : M \to \mathbb{R}$ such that all critical points are nondegenerate.

Proposition 1.1. *Let M be an embedded submanifold of $\mathbb{R}^n$. For almost every point $p \in \mathbb{R}^n$, the function $f : M \to \mathbb{R} : x \mapsto \|x - p\|^2$ is Morse.*

Proof. Suppose that $p \in \mathbb{R}^n$ makes f be not Morse so that it possesses a degenerate critical point. Note that the notation x can denote not only a point variable on M but also the embedding map $M \hookrightarrow \mathbb{R}^n$. Let $N \subset M \times \mathbb{R}^n$ be the normal bundle of the tangent bundle TM and define a map $\varphi : N \to \mathbb{R}^n$ such that $\varphi(x, y) = x + y$. We claim that the point $(x, p - x)$ is contained in N and φ is critical at this point if f is degenerate at x.

The differential of f is

$$df_x(v) = 2(x - p) \cdot dx(v) = 2(x - p) \cdot v,$$

so x is critical point if and only if $x - p$ is proportional to $T_x M$.

Let $\{x^i\}_{i=1}^m$ be orthonormal coordinates for M and let $\{e_j\}_{j=1}^{n-m}$ be an orthonormal frame field of N. Define coordinate functions $\{x^i, y^j\}$ on the manifold N by

$$x^i(x, y) := x^i(x), \quad \text{and} \quad y^j(x, y) := y \cdot e_j(x).$$

Then,

$$\left\{ \frac{\partial x}{\partial x^1}, \cdots, \frac{\partial x}{\partial x^m}, \frac{\partial y}{\partial y^1}, \cdots, \frac{\partial y}{\partial y^{n-m}} \right\}$$

always form an orthonormal basis on $\mathbb{R}^n$ and

Since

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial x}{\partial x^i} + \frac{\partial y}{\partial x^i} \quad \text{and} \quad \frac{\partial \varphi}{\partial y^j} = \frac{\partial y}{\partial y^j},$$

we have

$$\begin{aligned} \frac{\partial \varphi}{\partial x^i} \cdot \frac{\partial x}{\partial x^k} &= \delta_{ik} - y \cdot \frac{\partial^2 x}{\partial x^i \, \partial x^k}, & \qquad \frac{\partial \varphi}{\partial x^i} \cdot \frac{\partial y}{\partial y^l} &= -y \cdot \frac{\partial^2 y}{\partial x^i \, \partial y^l}, \\ \frac{\partial \varphi}{\partial y^j} \cdot \frac{\partial x}{\partial x^k} &= 0, & \qquad \frac{\partial \varphi}{\partial y^j} \cdot \frac{\partial y}{\partial y^l} &= \delta_{jl}. \end{aligned}$$

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To represent $d\varphi(\partial_{x^1}, \cdots, \partial_{y^{n-m}})$ with matrix, we can write

$$\begin{pmatrix} \frac{\partial \varphi}{\partial x^i} \\ \frac{\partial \varphi}{\partial y^j} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x^k} & \frac{\partial y}{\partial y^l} \end{pmatrix} = \begin{pmatrix} \mathrm{Id} - y \cdot \frac{\partial^2 x}{\partial x^i \, \partial x^k} & -y \cdot \frac{\partial^2 y}{\partial x^i \, \partial y^l} \\ 0 & \mathrm{Id} \end{pmatrix}.$$

Then,

$$\frac{\partial^2 f}{\partial x^i \, \partial x^j} = 2 \left(\mathrm{Id} + (x-p) \cdot \frac{\partial^2 x}{\partial x^i \, \partial x^j} \right)$$

deduces that $d\varphi$ is not surjective at $(x, p-x)$. Therefore, by the Sard theorem, set of such p has measure zero. $\square$

Proposition 1.2. *Let M be a manifold. The set of Morse functions is dense in $C^\infty(M)$.*

Proof. Let f be a smooth function on M. Embed M in $\mathbb{R}^{d-1}$ such that $x \mapsto (x_2, \cdots, x_d)$. Then, $x \mapsto (f(x), x_2, \cdots, x_d)$ gives an embedding into R^d. Define a sequence $\{\varepsilon_n\}_n \subset \mathbb{R}^n$ such that $\varepsilon_n \to 0$ and the sequence of functions

$$f_n(x) := \frac{\|x + ne_1 + \varepsilon_n\|^2 - n^2}{2n}$$

is Morse, where $\{e_i\}$ denotes the standard basis of $\mathbb{R}^d$. This can be done by the previous proposition. Then,

$$\begin{aligned} f_n(x) &= \frac{(f(x) + n + \varepsilon_n \cdot e_1)^2 + \cdots + (x_n + \varepsilon_n \cdot e_d)^2 - n^2}{2n} \\ &= f(x) + \frac{\|x + \varepsilon_n\|}{2n} + \varepsilon_n \cdot e_1 \end{aligned}$$

proves that $\|f_n - f\|_{C^k(K)} \to 0$ on every compact $K \subset M$. $\square$

Theorem 1.3 (Morse lemma)**.** *Let p be a nondegenerate critical point of a Morse function f on a manifold M. Then, there exists a local chart (U, φ) of p such that*

$$f \circ \varphi^{-1}(x_1, \cdots, x_m) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$$

for some k. This chart is called Morse chart.

Proof. $\square$

Corollary 1.4. *The critical points of a Morse function are isolated. In particular, on a compact manifold are finitely many critical points of a Morse function.*

2. Pseudo-gradients

Definition 2.1. Let f be a Morse function on a manifold M. A *pseudo-gradient* adapted to f is a vector field X such that

(1) $df(X) < 0$ at all noncritical points,
(2) there is a Morse chart at critical points in which $X = \operatorname{grad} f$, where the metric is induced from the chart.

Proposition 2.1. *A pseudo-gradient always exists for any Morse functions.*

Proof. Cover the manifold with charts such that every critical point is contained in a unique chart, which is Morse. For each chart (U, φ), we can define a vector field on U by

$$X := -d\varphi^{-1}(\mathrm{grad}(f \circ \varphi^{-1})),$$

using the standard metric on $\varphi(U)$. Then, we have

$$df(X) = -\langle \mathrm{grad}(f \circ \varphi^{-1}), \mathrm{grad}(f \circ \varphi^{-1}) \rangle \leq 0,$$

where the equality holds only at critical points. With a partition of unity, the vector fields are combined and easily checked to be pseudogradient. □

Definition 2.2. Let p be a critical point of a Morse function f on a manifold M. Denote $\varphi^s : M \to M$ by the flow of a pseudo-gradient. A *stable manifold* is defined as

$$W^s(p) := \{\, x \in M : \lim_{s \to \infty} \varphi^s(x) = p \,\},$$

and an *unstable manifold* is defined as

$$W^u(p) := \{\, x \in M : \lim_{s \to -\infty} \varphi^s(x) = p \,\}.$$

Proposition 2.2. *The stable manifolds and unstable manifolds are manifolds. Further, they are diffeomorphic open disks. Moreover, the index of p is equal to*

$$\dim W^u(p) = \operatorname{codim} W^s(p)$$

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