C^*-algebras

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Contents

1. Basics	2
1.1. Continuous functional calculus	2
1.2. Positive elements	3
1.3. Operator topologies	3
2. von Neumann algebras	3

Definition. In this note, an *algbera* refers to a vector space over $\mathbb{C}$ that has a pseudo-ring structure; always associative but possibly nonunital.

Definition. An normed $*$-algebra $\mathcal{A}$ is called *C^*-algebra* if

(1) $\mathcal{A}$ is Banach,
(2) $\mathcal{A}$ satisfies the C^*-identity: $\|x^*x\| = \|x\|^2$.

Theorem. *Every nonunital C^*-algebra is a maximal ideal of a unital C^*-algebra.*

Proof. Let $\mathcal{A}$ be a nonunital C^*-algebra. It is enough to show the existence of unital C^*-algebra $\widetilde{\mathcal{A}}$ such that $\mathcal{A}$ is a normed $*$-subalgebra of $\widetilde{\mathcal{A}}$ with codimension one. It is because a subalgebra is a maximal ideal if and only if the quotient can have a natural ring structure that makes a field.

Step 1: Construct a unital normed $*$-algebra. Since $\mathcal{A}$ is a Banach space, the space of bounded operators $B(\mathcal{A})$ is a Banach algebra. We can recognize $\mathcal{A}$ as a normed subalgebra of $B(\mathcal{A})$ because the left multiplication $(y \mapsto xy) \in B(\mathcal{A})$ has the norm

$$\|(y \mapsto xy)\| = \sup_{y \in \mathcal{A}} \frac{\|xy\|}{\|y\|}$$

that is shown to be equal to $\|x\|$ by putting $y = x^*$ and applying the *C^*-identity*. Define an algebra $\widetilde{\mathcal{A}}$ as the subalgebra:

$$\widetilde{\mathcal{A}} := \{\, (y \mapsto xy + \lambda y) \in B(\mathcal{A}) : x \in \mathcal{A},\ \lambda \in \mathbb{C} \,\}.$$

Since $\widetilde{\mathcal{A}} \cong \mathcal{A} \oplus \mathbb{C}$ as algebras, let us write the map $y \mapsto xy + \lambda y$ as (x, λ). Then, $\widetilde{\mathcal{A}}$ is a normed $*$-algebra with induced norm and involution

$$\|(x, \lambda)\| = \sup_{y \in \mathcal{A}} \frac{\|xy + \lambda y\|}{\|y\|}, \qquad (x, \lambda)^* = (x^*, \overline{\lambda}).$$

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Then, $\mathcal{A}$ is a normed $*$-subalgebra of $\widetilde{\mathcal{A}}$ because the norm and involution of $\mathcal{A}$ agree with $\widetilde{\mathcal{A}}$.

Step 2: $\widetilde{\mathcal{A}}$ is Banach. Suppose (x_n, λ_n) is Cauchy in $\widetilde{\mathcal{A}}$. Since $\mathcal{A}$ is complete so that it is closed in $\widetilde{\mathcal{A}}$, we can induce a norm on the quotient $\widetilde{\mathcal{A}}/\mathcal{A}$ so that the canonical projection is (uniformly) continuous so that λ_n is Cauchy. Also, the inequality $\|x\| \leq \|(x,\lambda)\| + |\lambda|$ shows that x_n is Cauchy in $\mathcal{A}$.

Since a finite dimensional normed space is always Banach and $\mathcal{A}$ is Banach, λ_n and x_n converge. Finally, the inequality $\|(x,\lambda)\| \leq \|x\| + |\lambda|$ implies that (x_n, λ_n) converges.

Step 3: $\widetilde{\mathcal{A}}$ is C^*. The C^*-identity easily follows from the following inequality:

$$\begin{aligned}
\|(x,\lambda)\|^2 &= \sup_{\|y\|=1} \|xy + \lambda y\|^2 \\
&= \sup_{\|y\|=1} \|(xy+\lambda y)^*(xy+\lambda y)\| \\
&= \sup_{\|y\|=1} \|y^*((x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y)\| \\
&\leq \sup_{\|y\|=1} \|(x^*x + \lambda x^* + \overline{\lambda} x)y + |\lambda|^2 y\| \\
&= \|(x,\lambda)^*(x,\lambda)\|.
\end{aligned}$$

□

1. Basics

1.1. **Continuous functional calculus.**

Theorem 1.1 (Gelfand-Naimark)**.** *For commutative unital C^*-algebra $\mathcal{A}$, the Gelfand transform gives an isometric *-isomorphism $\Gamma : \mathcal{A} \to C(\sigma(\mathcal{A}))$.*

Proof. Step 1: Γ is a $*$-homomorphism. We will show $h(x^*) = \overline{h(x)}$ for linear characters $h \in \sigma(\mathcal{A})$. First assume that $x \in \mathcal{A}$ is self-adjoint.

By the holomorphic functional calculus,

$$e^{itx} = \sum_{n=1}^{\infty} \frac{(itx)^n}{n!}.$$

Since the involution is continuous,

$$(e^{itx})^* = \sum_{n=1}^{\infty} \frac{(-itx)^n}{n!} = e^{-itx},$$

so we have $\|e^{itx}\|^2 = \|e^{itx}e^{-itx}\| = 1$. Then, the inequality

$$1 = \|e^{itx}\| \geq |h(e^{itx})| = |e^{ith(x)}| = e^{-t\operatorname{Im} h(x)}$$

proves $h(x) \in \mathbb{R}$.

For arbitrary $x \in \mathcal{A}$, if we define self-adjoints

$$\operatorname{Re} x := \frac{x + x^*}{2}, \qquad \operatorname{Im} x := \frac{x - x^*}{2i},$$

then

$$h(x^*) = h(\operatorname{Re} x) - ih(\operatorname{Im} x) = \overline{h(\operatorname{Re} x)} - i\overline{h(\operatorname{Im} x)} = \overline{h(\operatorname{Re} x) + ih(\operatorname{Im} x)} = \overline{h(x)}$$

for all $h \in \sigma(\mathcal{A})$.

Step 2: Γ *is isometric.* Note that we have

$$\|\widehat{x}\| = \sup_{h\in\sigma(\mathcal{A})} |\widehat{x}(h)| = \sup_{h\in\sigma(\mathcal{A})} |h(x)| = r(x).$$

For self adjoint $x \in \mathcal{A}$, since we have $\|x\|^2 = \|x^*x\| = \|x^2\|$, the spectral radius coincides with the norm by the Gelfand formula for spectral radius in Banach algebras:

$$r(x) = \lim_{n\to\infty} \|x^{2^n}\|^{1/2^n} = \|x\|.$$

Hence

$$\|x\|^2 = \|x^*x\| = \|\widehat{x^*x}\| = \|\widehat{x}^*\widehat{x}\| = \|\widehat{x}\|$$

for arbitrary $x \in \mathcal{A}$.

Step 3: Γ *is surjective.* The step 1 shows that $\Gamma(\mathcal{A})$ is a unital $*$-subalgebra of $C(\sigma(\mathcal{A}))$, and it separates points by definition. By the Stone-Weierstrass theorem, $\Gamma(\mathcal{A})$ is dense in $C(\sigma(\mathcal{A}))$. The step 2 shows that $\Gamma(\mathcal{A})$ is complete and hence closed so that $\Gamma(\mathcal{A}) = C(\sigma(\mathcal{A})$. □

Theorem 1.2 (Gelfan-Naimark)**.** *For commutative C^*-algebra $\mathcal{A}$, the Gelfand transform gives an isometric *-isomorphism $\Gamma : \mathcal{A} \to C_0(\sigma(\mathcal{A}))$.*

1.2. **Positive elements.**

1.3. **Operator topologies.**

Theorem 1.3. *Let f be a linear functional on $B(H)$ for a Hilbert space H. Then, TFAE:*

(1) *f is* WOT*-continuous,*
(2) *f is* SOT*-continuous,*
(3) *$f(T) = \sum_{i=1}^n \langle Tx_i, y_i\rangle$ for some x_i, y_i.*

Proof. (2) ⇒ (3) is the only nontrivial implication. By the definition of SOT, there exists $v \in \mathcal{H}^n$ such that

$$|f(T)| \le \|T^{\oplus n}v\|.$$

The functional $f : \mathcal{A} \to \mathbb{C}$ factors through $\mathcal{H}^n$ such that

$$\mathcal{A} \xrightarrow{v} \mathcal{H}^n \to \mathbb{C}.$$

□

2. von Neumann algebras

Theorem 2.1 (Double commutant theorem)**.** *Let $\mathcal{A}$ be a C^*-algebra in $\mathcal{B}(\mathcal{H})$. Then, $\mathcal{A}'' = \overline{\mathcal{A}}^{\text{WOT}}$.*

Proof. Since $\overline{\mathcal{A}}^{\text{WOT}}$ is closed convex, $\overline{\mathcal{A}}^{\text{SOT}} = \overline{\mathcal{A}}^{\text{WOT}}$. Also, $\mathcal{A}''$ is weakly closed, $\overline{\mathcal{A}}^{\text{WOT}} \subset \mathcal{A}''$. Let $T \in \mathcal{A}''$ and $v \in \mathcal{H}^n$. We want to show $Tv \in \overline{\mathcal{A}v}$.

Let P be the orthogonal projection on $\overline{\mathcal{A}v} \subset \mathcal{H}^n$. If $a \in \mathcal{A}$, then $a\overline{\mathcal{A}v} \subset \overline{\mathcal{A}v}$ because the multiplication by a is continuous on $\mathcal{H}^n$. It implies that $\operatorname{ran}(aP) \subset \operatorname{ran}(P)$ and $P(aP) = aP$ for all $a \in \mathcal{A}$. Because $\mathcal{A}$ is closed under the involution, P commutes with a, so $P \in \mathcal{A}'$.

Then, $TP = PT$ implies $Tv = T(Pv) = P(Tv) \in \operatorname{ran}(P) = \overline{\mathcal{A}v}$.

$\square$