Functional Analysis II : Operator Algebra

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Theorem 2.1. *If x, y commutes, then $\sigma(xy) \subset \sigma(x)\sigma(y)$.*

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1. Basics

2. The Gelfand-Naimark theorems

2.1. Commutative Banach algebras. The Gelfand representation $A \to C_0(\widehat{A})$ can be defined for commutative Banach algebras.

DEFINITION 2.1. A *symmetric* Banach algebra is an involutive Banach algebra for which the Gelfand representation preserves the involution. We will not consider non-symmetric involutive Banach algebras in this section.

Notice the following implication:

$$\begin{tikzcd}
 & \text{symmetirc Banach algebra} \arrow[dr] & \\
C^*\text{-algebra} \arrow[ur] \arrow[dr] & & \text{Banach algebra.} \\
 & \text{semisimple Banach algebra} \arrow[ur] &
\end{tikzcd}$$

Let A be a commutative Banach algebra.

THEOREM 2.1. *If A is semisimple, then the Gelfand representation is a monomorphism; it is injective.*

PROOF. It is because the kernel is given by the Jacobson radical. □

THEOREM 2.2. *If A is symmetric, then the Gelfand representation is an epimorphism; it has a dense range.*

PROOF. The image is closed under all operations except involution, separates points, and vanishes nowhere. If A is symmetric, then the image is closed under involution. Thus, by the Stone-Weierstrass theorem, we get the result. □

C^*-algebras are semisimple and symmetric (even if it is noncommutative).

THEOREM 2.3. *A C^*-algebra is semisimple.*

THEOREM 2.4. *A C^*-algebra is symmetric.*

PROOF 1. It is by Arens. □

PROOF 2. It is by Fukamiya. □

Furethermore,

THEOREM 2.5. *If A is a commutative C^*-algebra, then the Gelfand representation is isometric.*

Since an isometry is injective and has a closed range, therefore, it should be isometric *-isomorphism.

CHAPTER 4

Von Neumann algebras