Analysis 2 : General Topology

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Preface

One way to state the definition for general topology is the abstract study of topologies and topological spaces. The word topology is used in two different contexts: analytic sense and geometric sense. When we are talking the stories of doughnuts and coffee mugs, they are in fact involved in topology of geometric sense, which is also referred to as a branch of mathematics that studies constinuous structures of spaces such as manifolds or CW complexes. In analysis, the topology is mentioned greatly unrelatedly to the doughnuts, but it refers to the minimal structure that is required in order to define concepts of limit and continuity. More precisely, once a structure called "topology" is settled on a set, then we can expand basic analytic theories about limit and continuity. Normed spaces are the first examples which possess a particularly nice topology. With the topologies, we can describe formally whether a sequence converges or a function is continuous. This book is interested in the latter issues as noted in the title of the book.

According to the usage of topologies, similarly as mentioned, there are two large branches of general topology; both contribute to build nice frameworks for the wide regions of mathematics. One is for algebraic topology and studies the category of convenient spaces in which well-known constructions and computational tools are available, and the other is for abstract analysis. In general topology focused on analysis, we are more concerned with the implications among individual topologies and special properties of them, rather than the global shapes of topological spaces. For real analysis or functional analysis, general topology provides with extremely important viewpoints for recognizing the various convergence modes of functional sequences. An interesting feature of general topology is that the basic topology in analysis is a preliminary of the abstract study of the spaces used in algebraic topology, hence everyone starts to learn it from analysis.

The purpose of this book is to grasp a big picture and learn basic languages in order to establish frameworks for the next study of modern analysis such as harmonic analysis or functional analysis following after calculus topics, in a quite abstract viewpoints. In particular, we mainly focus on finding admissible answers for the following questions:

- Why are topologies defined in that way? Is that a suitably optimized definition?
- What can metric spaces or normed spaces do more than topological spaces?
- What properties are needed to take and use sequences for describing topologies instead of general nets without any anxiety?
- What does the definition of compactness mean? What roles do they do in practical analysis?

- What are the purposes of introduction of the compactness related concepts such as sequentially compact, σ-compact, or relatively compact spaces?
- Why do locally compact Hausdorff spaces so frequently appear?
- Why is the uniform convergence natural in a continuous function space?
- What is the hidden meaning of complicated theorems of like Arzela-Ascoli or Stone-Weierstrass?

For the first in this book, the basic topological structures including metrics, topologies, and uniformities are introduced in Chapter 1. Although many texts do not cover uniform spaces, they are greatly useful in studying nonmetrizable topologies. In Chapter 2, we learn about continuity of functions and maps. Continuous maps functionally connects two different topological spaces and allow us to compare them. Homeomorphisms and some connectivity will be also covered. Chapter 3 is dedicated to the deeper study of convergence of sequences or nets. In Chapter 4, 5, and 6, we learn compact spaces, separability axioms, and continuous function spaces.

In this book, we are going to assume the reader is already familiar to the theory of normed spaces and elementary foundations of calculus including the epsilon-delta definitions. For instance, we can require the reader to know what the uniform convergence is and that it can be regarded as just a convergence in the properly defined norm on a space of functions.

This book would not be a good choice for a standard course text relative to the other existing great books, because it is written to be helpful in self-teaching. It has been tried to put convincing explanations at every newly defined concept and to cram supplementary stories that are not necessary, but they might not be really satisfied. Nevertheless, I will be very satisfied only if just one of readers could enjoy math with this book.

CHAPTER 1

Topological structures

Firstly we discuss how far the definitions of analytic notions such as limit and continuity can be extended. One of the main interests in general topology is to make extended version of mathematical calculus, on the sets on which algebraic operations are not allowed. We should note that, however, some properties must be compromised when we try to generalize something.

Recall that we have measured the closeness of two points in a normed space by taking the norm at the algebraic subtraction of their position vectors. As the first trial, we can consider dismissing the algebraic operations. This trial has succeeded to find a structure for measuring the nearness between points and to generalize limit of sequences and continuity of functions. Nevertheless, we compromised the theory of differentiation and integration in the lack of algebraic structures. Topology is the term for this successful solution. In other words, for the most part, wonderful statements that are purely related to limits and continuity were possible to be extended without big flaws, even if we forget the vector space operations by introducing the concept of topology.

More precisely, topological structure on a set may refer to either an additional function on the set or a more complicated mathematical device which solves the problem by being put on the set. The norms are typical examples of topological structures, and so is “topology”.

1. Metric

Metric is a generalization of norm and induces a special example of "topology". For example, every subset of a normed vector space is equipped with a natural metric. Metrics which are in between norms and topologies will be helpful to catch the intuition.

The propositions below are not needed to be memorized by force. Later, by applying the results on topologies to metrics as examples, we will naturally find that metric provides with a surprisingly appropriate and widely-applicable tool to understand the nature of mathematical analysis. To give a short answer for the essentiality of metric is "a countable uniform topology" in a sense; understanding what it means would be one of primary goals of this chapter.

1.1. Metric structure. Metric can be viewed as the first successful trial to find an abstract framework for studying limits. In this subsection, we discuss the definitions and examples of metric spaces and its basic functions.

1.1.1. *Metric spaces.* A metric on a set is defined as a function which assigns a nonnegative real number to an unordered pair of two points. The assigned real number has the meaning of distance between the points. A metric space is just a set endowed with a metric.

Definition 1.1. Let X be a set. A *metric* is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ such that

(1) $d(x, y) = 0$ iff $x = y$, (nondegeneracy)
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$, (symmetry)
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (triangle inequality)

A pair (X, d) of a set X and a metric on X is called a *metric space*. We often write it simply X.

The most familiar metric comes from the standard norm on Euclidean space $\mathbb{R}^d$. Notice that the third axiom, the triangle inequality, is named after the one for norms. In this context, we can see metrics as a generalization of norms for spaces that does not admit a vector space structure. In particular, when we study analysis on Euclidean spaces or more generally normed spaces, the metric given in the following example is considered as the standard. Therefore, if not particularly mentioned, then we will implicitly assume this induced metric out of the norm for any subset of a Euclidean space. Moreover, every subset of normed space is also an example of metric space because the metric function can be always inherited to every subset from a metric space.

Example 1.1. A normed space X is a metric space. Precisely, the norm structure naturally defines a real-valued function d on $X \times X$ defined by $d(x, y) := \|x - y\|$ and it satisfies the axioms of metric.

Proof. It is quite easy. Just recall the axioms of norm and deduce the conclusion for each axiom of metric. □

Example 1.2. Let (X, d) be a metric space. Every subset of X has a natural induced metric, just the restriction of original metric d.

Proof. Obvious. □

In fact, the converse holds; every metric space can be viewed as a subset of a normed space. This deeper result on the relation between normed spaces and metric spaces is discovered by Kuratowski []. Since it does not play any important role in the rest of the book, we may jump to Example 1.3. To state the theorem, we introduce an isometry, a map preserving metrics.

DEFINITION 1.2. Let X and Y be metric spaces. A map $\phi : X \to Y$ is called an *isometry* if $d(x, y) = d(\phi(x), \phi(y))$ for all $x, y \in X$. If there is a bijective isometry between X and Y, then we say the spaces are *isometric*.

Every isometry is clearly injective so that it is bijective if and only if it is surjective. Also, the inverse of bijective isometry is an isometry, so the bijective isometries define an equivalent relation on the set of metric spaces. If two metric spaces are isometric, we can view them as virtually same, in the "category" of metric spaces. The following theorem tells another characterization of metric spaces.

THEOREM 1.1 (Kuratowski embedding). *Every metric space is isometric to a subset of a normed space. In other words, for every metric space (X, d), there is an isometry ϕ from X to a normed space.*

PROOF. Choose any point $p \in X$. Let Y be the space of bounded real-valued functions on X. It is a normed space with uniform norm. Define $\phi : X \to Y$ by $\phi(x)(t) = d(x, t) - d(p, t)$. Note that $\phi(x)$ is bounded with $\|\phi(x)\| = \sup_{t\in X} |d(x, t) - d(p, t)| = d(x, p)$. Then,

$$\|\phi(x) - \phi(y)\| = \sup_{t\in X} |\phi(x)(t) - \phi(y)(t)| = \sup_{t\in X} |d(x, t) - d(y, t)| = d(x, y).$$

This proves ϕ is a isometry. $\square$

REMARK. The space Y is somtimes denoted by $\ell^\infty(X)$, and it is in fact a Banach space. In addition, the image of the isometry ϕ is in a closed subspace $C_b(X) \subset \ell^\infty(X)$, the space of bounded real-valued continuous functions.

We have seen metrics can be seen as the generalization of norms. However, there are also many examples of metrics that are not involved directly in the norms. Even if they are far from subsets of a normed space, we can apply our intuition of norms. The examples below are given without proofs.

EXAMPLE 1.3. Let X be a set. Then, a function $d : X \times X \to \mathbb{R}_{\geq 0}$ defined by

$$d(x, y) := \begin{cases} 0 & , x = y \\ 1 & , x \neq y \end{cases}$$

is a metric on X. This metric is sometimes called *discrete metric* because balls can separate all single points out.

EXAMPLE 1.4. Let d be a metric on a set X. Let $f : [0, \infty) \to [0, \infty)$ be a function such that $f^{-1}(0) = \{0\}$. If f is monotonically increasing and subadditive, then $f \circ d$ satisfies the triangle inequality, hence is another metric on X. Note that a function f is called subadditive if

$$f(x + y) \leq f(x) + f(y)$$

for all x, y in the domain.

Example 1.5. Let $G = (V, E)$ be a connected graph. Define $d : V \times V \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the distance of two vertices; the length of shortest path connecting two vertices. Then, (V, d) is a metric space.

Example 1.6. Let $\mathcal{P}(X)$ be the power set of a finite set X. Define $d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{Z}_{\geq 0} \subset \mathbb{R}_{\geq 0}$ as the cardinality of the symmetric difference; $d(A, B) := |(A - B) \cup (B - A)|$. Then $(\mathcal{P}(X), d)$ is a metric space.

Example 1.7. Let C be the set of all compact subsets of $\mathbb{R}^d$. Recall that a subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded. Then, $d : C \times C \to \mathbb{R}_{\geq 0}$ defined by

$$d(A, B) := \max\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \ \sup_{b \in B} \inf_{a \in A} \|a - b\| \}$$

is a metric on C. It is a little special case of *Hausdorff metric*.

1.1.2. *Limits and continuity.* The name "metric" confuses the main role of metrics. Metrics are often recognized as something measures a distance and belonging to the study of geoemtry. We cannot say that it is false, but it should be mentioned that a metric is far from geometric structures. It is rather an analytic structure.

Metrics are not interested in measuring a distance between two points; the main function of metrics is to make balls. A ball centered at a point is defined as a set of points such that the distance from the center point is less than a fixed number. The balls centered at each point provide a concrete images of "system of neighborhoods at a point" in a more intuitive sense. In this viewpoint, a metric can be considered as a structure that lets someone accept the notion of neighborhoods more friendly, which is vital for analysis of limits and continuity. The neighborhoods will be defined precisely in the next section. To sum up, metrics allow to define limits and continuity and to think in an intuitive way.

Definition 1.3. Let X be a metric space. A set of the form

$$\{y \in X : d(x, y) < \varepsilon\}$$

for $\varepsilon > 0$ is called a *ball centered at x with radius ε* and denoted by $B(x, \varepsilon)$ or $B_\varepsilon(x)$.

The balls defined as above are also called open balls in order to distinguish from the closed balls; $\overline{B(x, \varepsilon)} = \{y \in X : d(x, y) \leq \varepsilon\}$. The terms "open" and "closed" will be discussed again in the next section. Now let us reformulate the definitions of limits and continuity with balls, which we used in the usual calculus on Euclidean spaces or normed spaces. Compare the following definitions to what we know.

Definition 1.4. Let $\{x_n\}_n$ be a sequence of points on a metric space (X, d). We say that a point x is a *limit* of the sequence or the sequence *converges to x* if for arbitrarily small ball $B(x, \varepsilon)$, we can find n_0 such that $x_n \in B(x, \varepsilon)$ for all $n > n_0$. If it is satisfied, then we write

$$\lim_{n \to \infty} x_n = x,$$

or simply

$$x_n \to x \qquad \text{as} \qquad n \to \infty.$$

We say a sequence is *convergent* if it converges to a point. If it does not converge to any points, then we say the sequence *diverges*.

Definition 1.5. A function $f : X \to Y$ between metric spaces is called *continuous at* $x \in X$ if for any ball $B(f(x), \varepsilon) \subset Y$, there is a ball $B(x, \delta) \subset X$ such that

$$f(B(x, \delta)) \subset B(f(x), \varepsilon).$$

The function f is called *continuous* if it is continuous at every point on X.

The convergence can be also represented by limit of real numbers:

Proposition 1.2. *Let x_n be a sequence in a metric space X and $x \in X$. Then,*

$$\lim_{n\to\infty} x_n = x \iff \lim_{n\to\infty} d(x_n, x) = 0.$$

There are a lot of deeper propositions and results for limit and continuity, but we postpone to mention them to later because they are generalized to topological spaces.

Note that taking either ε or δ really means taking a ball of the very radius. For continuity of a function, we can describe it intuitively that no matter how small ball is taken in the codomain, we can take much smaller ball in the domain. What we sould know is that in what shape the balls centered at each point are distributed. It is because the shape of the set of balls determines the continuity or convergence. To make a vivid illustration, let us give an example.

Example 1.8. Let X be the discrete metric space in Example 1.3. Every ball centered at a point x with respect to the discrete metric is either a singleton $B(x, \varepsilon) = \{x\}$ when $\varepsilon \leq 1$, or the entire space $B(x, \varepsilon) = X$ when $\varepsilon > 1$. In particular, a sequence $\{x_n\}_n$ converges to x if and only if it is eventually x; there is a positive integer n_0 such that $x_n = x$ for all $n > n_0$.

Example 1.9. Let X and Y be metric spaces. If X is equipped with the discrete metric in Example 1.3, then every function $f : X \to Y$ is continuous on the discrete metric.

Example 1.10. An isometry is always continuous.

The set of balls at each point determines properties about limits and continuity. Intuitively, the balls indicate the varying degrees of neighborhoods and relative nearness from a point. Refer to Example 2.1.

1.2. Topological equivalence. A metric can be viewed as a function that takes a sequence as input and returns whether the sequence converges or diverges. That is, metric acts like a criterion which decides convergence of sequences. Take note on the fact that the sequence of real numbers defined by $x_n = \frac{1}{n}$ converges in standard metric but diverges in discrete metric. Like this example, even for the same sequence on a same set, the convergence depends on the attached metrics. What we are interested in is comparison of metrics and to find a proper relation structure.

1.2.1. *Refinement relation.* If a sequence converges in a metric d_2 but diverges in another metric d_1, we would say d_1 has stronger rules to decide the convergence. Refinement relation formalizes the idea.

Definition 1.6. Let d_1 and d_2 are metric on a set X. We say d_1 is *stronger than* d_2 (equivalently, d_2 is *weaker than* d_1) or d_1 *refines* d_2, if for any $x \in X$ and for arbitrary $\varepsilon > 0$ we can find $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon).$$

The notations B_1 and B_2 refer to balls defined with the metrics d_1 and d_2 respectively.

PROPOSITION 1.3. *The refinement relation is a preorder.*

PROOF. It is enough to show the transitivity. Suppose there are three metric d_1, d_2, and d_3 on a set X such that d_1 is stronger than d_2 and d_2 is stronger than d_3. For $i = 1, 2, 3$, let B_i be a notation for the balls defined with the metric d_i.

Take $x \in X$ and $\varepsilon > 0$ arbitrarily. Then, we can find $\varepsilon' > 0$ such that

$$B_2(x, \varepsilon') \subset B_3(x, \varepsilon).$$

Also, we can find $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon').$$

Therefore, we have $B_1(x, \delta) \subset B_3(x, \varepsilon)$ which implies that d_1 refines d_3. □

This definition looks quite strange, but is directly related to the way how a metric gives rise to a topology, which we have not defined yet. Intuitively, the neighborhood systems of balls from a metric "refine" the others. The relation can be described in several ways.

PROPOSITION 1.4. *Let d_1 and d_2 be metrics on a set X. Then, the followings are equivalent:*

(1) *the metric d_1 is stronger than d_2,*
(2) *every sequence that converges to $x \in X$ in d_1 converges to x in d_2,*
(3) *the identity function $\mathrm{id} : (X, d_1) \to (X, d_2)$ is continuous.*

PROOF. (1)⇒(2) Let $\{x_n\}_n$ be a sequence in X that converges to x in d_1. By the assumption, for an arbitrary ball $B_2(x, \varepsilon) = \{y : d_2(x, y) < \varepsilon\}$, there is $\delta > 0$ such that

$$B_1(x, \delta) \subset B_2(x, \varepsilon),$$

where $B_1(x, \delta) = \{y : d_1(x, y) < \delta\}$. Since $\{x_n\}_n$ converges to x in d_1, there is an integer n_0 such that

$$n > n_0 \quad \Longrightarrow \quad x_n \in B_1(x, \delta).$$

Combining them, we obtain an integer n_0 such that

$$n > n_0 \quad \Longrightarrow \quad x_n \in B_2(x, \varepsilon).$$

It means $\{x_n\}$ converges to x in the metric d_2.

(2)⇒(1) We prove it by contradiction. Assume that for some point $x \in X$ we can find $\varepsilon_0 > 0$ such that there is no $\delta > 0$ satisfying $B_1(x, \delta) \subset B_2(x, \varepsilon_0)$. In other words, at the point x, the difference set $B_1(x, \delta) \setminus B_2(x, \varepsilon_0)$ is not empty for every $\delta > 0$. Thus, we can choose x_n to be a point such that

$$x_n \in B_1\left(x, \tfrac{1}{n}\right) \setminus B_2(x, \varepsilon_0)$$

for each positive integer n by putting $\delta = \frac{1}{n}$.

We claim $\{x_n\}_n$ converges to x in d_1 but not in d_2. For $\varepsilon > 0$, if we let $n_0 = \left\lceil \frac{1}{\varepsilon} \right\rceil$ so that we have $\frac{1}{n_0} \leq \varepsilon$, then

$$n > n_0 \quad \Longrightarrow \quad x_n \in B_1\left(x, \tfrac{1}{n}\right) \subset B_1(x, \varepsilon).$$

So $\{x_n\}_n$ converges to x in d_1. However in d_2, for $\varepsilon = \varepsilon_0$, we can find such n_0 like d_1 since

$$x_n \notin B_2(x, \varepsilon_0)$$

for every n. Therefore, $\{x_n\}$ does not converges to x in d_2.

(1)⇔(3) Obvious by definition. □

Proposition 1.5. *Let d_1 and d_2 be metrics on a set X. Suppose for each point x there exists a constant C which may depend on x such that*

$$d_2(x, y) \leq C d_1(x, y)$$

for all Y. Then, d_1 is stronger than d_2.

Proof 1. Take a ball $B_2(x, \varepsilon)$ measured with d_2. Define $\delta = \varepsilon/C$. If $y \in B_1(x, \delta)$, then $d_1(x, y) < \varepsilon/C$ implies $d_2(x, y) < \varepsilon$ so that we have $y \in B_2(x, \varepsilon)$. Therefore, d_1 is stronger than d_2. □

Proof 2. Let x_n be a sequence converges to x in d_1. Since $d_1(x_n, x) \to 0$, by taking limit on the inequality, we obtain $d_2(x_n, x) \to 0$, hence x_n converges to x in d_2. By the previous proposition, d_1 is stronger than d_2. □

Example 1.11. There is always no stronger metric than the discrete metric. In other words, discrete metric is the strongest metric.

1.2.2. *Topological equivalence of metrics.* There exist two different metrics which give exactly same answers about convergence for all sequences. The same issue also occurs on checking continuity of functions. We consider an equivalence relation on the set of metrics, that is, two equivalent metrics give a common criterion for convergence and continuity. This equivalence relation is obtained from the refinement relation; two metrics are equivalent if they refines each other. In this situation, the two metrics are also said to induce exactly the same topology. Some results on the classification of the metrics by their topologies will be discussed.

Definition 1.7. Two metrics on a set are called *topologically equivalent* if the sets of open balls centered at each point are mutually nested; in other words, they refines each other.

The word "topologically" is frequently omitted. There are various characterizations of equivalence among metrics. Especially the following proposition states that we can recover an equivalence class of metrics when it is known that which sequence converges or diverges.

Proposition 1.6. *Let d_1 and d_2 are metrics on a set X. They are equivalent if and only if they share the same sequential convergence data; a sequence converges to $x \in X$ in d_1 if and only if it converges to x in d_2.*

Proof. It is easily deduced by applying Proposition 1.4 twice. □

Remark. Unlike metrics, there exist two different topologies that have same sequential convergence data. For example, a sequence in an uncountable set with cocountable topology converges to a point if and only if it is eventually at the point, which is same with discrete topology. This means the informations of sequence convergence are not sufficient to uniquely characterize a topology. Instead, convergence data of generalized sequences also called nets, recover the whole topology. For topologies having a property called the first countability, it is enough to consider only usual sequences in spite of nets. What we did in this subsection is not useless because topology induced from metric is a typical example of first countable topologies. These kinds of problems will be profoundly treated in Chapter 3.

Remark. One can ask some results for the equivalence of metrics characterized by a same set of continuous functions. However, they are generally difficult problems: is it possible to recover the base space from a continuous function space or a path space?

The following two theorems give sufficient conditions for equivalence. The first theorem is well used to compare norms on a vector space in particular, and the second theorem is going to be used in the next subsection.

Theorem 1.7. *Let d_1 and d_2 be metrics on a set X. If for each point x there exist two constants C_1 and C_2 which may depend on x such that*

$$d_2(x,y) \leq C_1 d_1(x,y) \quad \text{and} \quad d_1(x,y) \leq C_2 d_2(x,y)$$

for all y in X, then d_1 and d_2 are equivalent.

Proof. It is a corollary of Proposition 1.5. □

Theorem 1.8. *Let d be a metric on a set X and let f be a monotonically increasing subadditive real function on $\mathbb{R}_{\geq 0}$ such that $f^{-1}(0) = \{0\}$ so that $f \circ d$ is a metric. If f is continuous at 0 in addition, then $f \circ d$ is equivalent to d.*

Proof. We have seen that $f \circ d$ is a metric in Example 1.4. Firstly, for any ball $B(x,\varepsilon) = \{y : d(x,y) < \varepsilon\}$, we have a smaller ball

$$B_f(x, f(\varepsilon)) \subset B(x,\varepsilon),$$

where $B_f(x, f(\varepsilon)) = \{y : f(d(x,y)) < f(\varepsilon)\}$, since $f(d(x,y)) < f(\varepsilon)$ implies $d(x,y) < \varepsilon$.

The opposite inclusion only requires the continuity of f. Take an arbitrary ball $B_f(x,\varepsilon)$. Since f is continuous at 0, we can find $\delta > 0$ such that

$$d(x,y) < \delta \quad \Longrightarrow \quad f(d(x,y)) < \varepsilon,$$

which implies $B(x,\delta) \subset B_f(x,\varepsilon)$. □

1.2.3. *Topological equivalence of norms.* A typical example of equivalent metrics occurs when we consider norms. It is natural to extend the concept of topological equivalence to norms. The idea is same; if two norms on a vector space give rise to a same topology, or equivalently, topologically equivalent metrics, then we call them equivalent.

The checking procedure is rather simple; the converse of Theorem 1.7 holds for norms. It is because metrics induced from norms have the property called "translation invariance". The following thereom is often taken as the definition of norm equivalence.

Theorem 1.9. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on a vector space V. They induce the equivalent metrics if and only if there are constants C_1 and C_2 such that*

$$\|x\|_2 \leq C_1 \|x\|_1 \quad \text{and} \quad \|x\|_1 \leq C_2 \|x\|_2$$

for all $x \in V$.

Proof. ($\Leftarrow$) Substitute $x \mapsto x - y$. Then, it is just a corollary of Theorem 1.7.

($\Rightarrow$) For any ball $B_2(0, \varepsilon)$, there is a smaller ball $B_1(0, \delta)$ such that $B_1(0, \delta) \subset B_2(0, \varepsilon)$ by the definition of equivalence of metrics. It means we have

$$\|x\|_1 < \delta \quad \Longrightarrow \quad \|x\|_2 < \varepsilon$$

for all $x \in V$. If we let $C_1 := \varepsilon/\delta$, then it is equivalent to

$$C_1\|x\|_1 < \varepsilon \quad \Longrightarrow \quad \|x\|_2 < \varepsilon.$$

If there is a vector $x \in V$ such that $\|x\|_2 > C_1\|x\|_1$, then we can lead a contradiction by taking ε such that $\|x\|_2 > \varepsilon > C_1\|x\|_1$. Therefore, $\|x\|_2 \leq C_1\|x\|_1$ for all $x \in V$. The other inequality is also shown in the same way. $\square$

Especially, when we work on a vector space with finite dimension such as a Euclidean space $\mathbb{R}^d$, the situation gets better dramatically.

Theorem 1.10. *On a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, all norms are equivalent.*

Proof. Let $\mathbb{F}$ be the field $\mathbb{R}$ or $\mathbb{C}$. Both have the absolute value function that makes the vector space complete. Then, a finite dimensional vector space is isomorphic to $\mathbb{F}^d$ for some d. Fix a basis $\{e_i\}_{i=1}^d$ on $\mathbb{F}^d$ and let $x = \sum_{i=1}^d x_i e_i$ denote an arbitrary element of $\mathbb{F}_d$. We will prove all norms are equivalent to the standard Euclidean norm defined for the fixed basis:

$$\|x\|_2 = \|\sum_{i=1}^d x_i e_i\|_2 := \left(\sum_{i=1}^d |x_i|^2\right)^{\frac{1}{2}}.$$

With this standard norm, we can use any theorems we learned in elementary analysis. For example, we are allowed to use the Bolzano-Weierstrass theorem. We used the subscript 2 for the standard Euclidean norm since the norm is frequently called ℓ^2 norm.

Take a norm $\|\cdot\|$ on $\mathbb{F}^d$. One direction is easy: if we let $C_2 := \sqrt{d} \cdot \max_i \|e_i\|$, then

$$\begin{aligned}\|x\| = \|\sum_{i=1}^d x_i e_i\| &\leq \sum_{i=1}^d |x_i|\|e_i\| \\ &\leq \max_i \|e_i\| \sum_{i=1}^d |x_i| \\ &\leq \max_i \|e_i\| \left(\sum_{i=1}^d 1^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^d |x_i|^2\right)^{\frac{1}{2}} = C_2\|x\|_2.\end{aligned}$$

In addition, this inequality show that the function $(\mathbb{F}^d, \|\cdot\|_2) \to \mathbb{R} : x \mapsto \|x\|$ is continuous.

There are many proofs for the other direction. We give a proof using sequences. Suppose there is no constant C such that the inequality $\|x\|_2 \leq C\|x\|$ holds. In other words, for every positive integer n, we can find $x_n \in \mathbb{F}^d \setminus \{0\}$ such that

$$\|x_n\|_2 > n\|x_n\|.$$

Normalize x_n with respect to $\|\cdot\|_2$ to define a new sequence $y_n := \frac{x_n}{\|x_n\|_2}$. Then, we have

$$\|y_n\|_2 = 1 \quad \text{and} \quad \|y_n\| < \tfrac{1}{n}.$$

Since the set $\{x : \|x\|_2 = 1\}$ is bounded in the standard norm, the Bolzano-Weierstrass theorem implies the existence of a convergent subsequence $\{y_{n_k}\}_k$ of $\{y_n\}$ in the standard norm $\|\cdot\|_2$. For the limit point y of $\{y_{n_k}\}_k$, take limits $k \to \infty$ on the inequalities

$$\big|\|y_{n_k}\|_2 - \|y\|_2\big| \le \|y_{n_k} - y\|_2$$

and

$$\big|\|y_{n_k}\| - \|y\|\big| \le \|y_{n_k} - y\| \le C_2\|y_{n_k} - y\|_2.$$

Then since $\|y_{n_k}\|_2 \to 1$ and $\|y_{n_k}\| \to 0$ as $k \to \infty$, we get $\|y\|_2 = 1$ and $\|y\| = 0$, which is a contradiction to the axiom of norm. This proves that $\|\cdot\|$ is equivalent to the standard norm. □

Remark. Even if we choose $\|\cdot\|_p$ instead of the standard Euclidean norm, the proof still works. It suffices to manipulate the Schwarz inequality in proper way. The reason why we chose the Euclidean norm is that we assumed the Bolzano-Weierstrass theorem was proved in the Euclidean norm.

Remark. More generally, the equivalence of norms is due to the locally compactness and the completeness. In fact, locally compactness is a way to characterize finite dimensional Banach spaces. Hence we may also apply the Heine-Borel theorem or the extreme value theorem instead of the Bolzano-Weierstrass theorem, which are exactly equivalent statements for compactness. Notice a closed ball is compact in such spaces, following the relation diagram:

$$\text{Bounded} \xrightarrow{\text{fin. dim.}} \text{Totally bounded} \xrightarrow{\text{complete}} \text{Compact}.$$

This result is important in functional analysis.

Example 1.12. Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on $\mathbb{R}^2$ defined as

$$\|(x,y)\|_1 := |x| + |y|, \quad \|(x,y)\|_2 := \sqrt{|x|^2 + |y|^2}.$$

Then, since we have inequalities

$$\|(x,y)\|_2 \le \|(x,y)\|_1 \le \sqrt{2}\|(x,y)\|_2$$

for all $(x,y) \in \mathbb{R}^2$, the two norms are equivalent.

1.3. Family of pseudometrics. Our goal in this subsection is to describe a topology generated by several metrics, and, in general, by several "pseudometrics". This idea provides a useful method to construct a metric or topology, which can be applied to a quite wide range of applications.

1.3.1. *Finite family of metrics.* At first, let us look into combination of metrics. Specifically, in a conventional way, metrics are summed to make another metric out of olds since sum of two metrics also satisfies the all axioms of metric. They have been usefully summed because convergence of a sequence in the resulted metric is equivalent to convergence in the summands. See Proposition 1.13. However, here we give a slightly more general construction using norms restricted onto the closed orthant $(\mathbb{R}_{\ge 0})^d$, of which summation becomes just a special case.

Proposition 1.11. *Let $\{d_i\}_{i=1}^d$ be a finite family of metrics on a set X. Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$. Then, $d(x,y) := \|(d_1(x,y),\cdots,d_d(x,y))\|$ is another metric on X.*

Proof. All of the three axioms of metric can be checked without difficulty. $\square$

Remark. Although it is possible to figure out conditions for $f:[0,\infty)^2\to[0,\infty)$ to have $f(d_1,d_2)$ be a metric, we just compromised it for simplicity and usefulness. Also, if we use norms, then the same method can be extended to the case of norms.

Furthermore, the newly defined metric is unique up to equivalence. We prove only for a pair of two metrics, but it is easy to check by mathematical induction that any finite family of metrics can be combined to make new metrics, which are essentially equivalent. In fact, the following proposition can be checked trivially without long proof if we introduce a concept called bases of topology, which will be seen later.

Proposition 1.12. *Let d_1, d_2, d_1', and d_2' be metrics on a set. Let $\|\cdot\|$ and $\|\cdot\|'$ be norms on $\mathbb{R}^2$. If d_1, d_2, and $\|\cdot\|$ are equivalent to d_1', d_2', and $\|\cdot\|'$ repsectively, then $\|(d_1,d_2)\|$ and $\|(d_1',d_2')\|'$ are equivalent metrics.*

Proof. Let $\{y:\|(d_1'(x,y),d_2'(x,y))\|'<\varepsilon\}$ be an arbitrary ball centered at a point x taken by the metric $\|(d_1',d_2')\|'$. By Theorem 1.10, there is a constant $C,C'>0$ such that $C'\|\cdot\|'\le\|\cdot\|_\infty\le C\|\cdot\|$, where we denote $\|(x,y)\|_\infty=\max\{|x|,|y|\}$.

By the equivalence of each pair of metrics, there are $\delta_1>0$ and $\delta_2>0$ such that

$$B_1(x,\delta_1)\subset B_{1'}(x,C'\varepsilon)\quad\text{and}\quad B_2(x,\delta_2)\subset B_{2'}(x,C'\varepsilon),$$

where $B_1,B_{1'},B_2$, and $B_{2'}$ denotes the ball with respect to metrics d_1,d_1',d_2, and d_2' respectively. With δ_1 and δ_2, define $\delta:=\min\{\delta_1,\delta_2\}/C$.

Then, we can manually check the ball of radius δ made in the metric $\|(d_1,d_2)\|$ is contained in the ball of radius ε made in the metric $\|(d_1',d_2')\|'$:

$$\{y:\|(d_1(x,y),d_2(x,y))\|<\delta\}\subset\{y:\|(d_1'(x,y),d_2'(x,y))\|'<\varepsilon\}.$$

The opposite part is shown in the same way symmetrically. $\square$

Example 1.13. Let d_1 and d_2 be metrics. Then

$$d_1(x,y)+d_2(x,y)\quad\text{and}\quad\max\{d_1(x,y),d_2(x,y)\}$$

are equivalent metrics.

Example 1.14. If d_1 and d_2 are equivalent metrics, then d_1+d_2 is also equivalent to d_1 and d_2.

From now, whenever we need to write a combined metric of a family of metrics, we will just adopt the sum d_1+d_2. Then, another characterization of the summed metric is given as follows; see Proposition 1.6 and ponder about the meaning of the following proposition.

Proposition 1.13. *Let d_1 and d_2 be metrics on a set X. A sequecne $\{x_n\}_n$ converges to x in d_1+d_2 if and only if it converges to x in both d_1 and d_2.*

Proof. ($\Rightarrow$) Let $\{x_n\}_n$ be a sequence that converges to x in d_1+d_2. For $\varepsilon>0$, we have an positive integet n_0 such that

$$n>n_0\quad\Longrightarrow\quad d_1(x_n,x)+d_2(x_n,x)<\varepsilon.$$

With this n_0, by the nonnegativity of metric functions, we get $d_1(x_n, x) < \varepsilon$ and $d_2(x_n, x) < \varepsilon$ for $n > n_0$.

($\Leftarrow$) Suppose a sequence $\{x_n\}_n$ is converges to x in both d_1 and d_2. For $\varepsilon > 0$, we may find positive integers n_1 and n_2 such that $n > n_1$ and $n > n_2$ imply $d_1(x_n, x) < \frac{\varepsilon}{2}$ and $d_2(x_n, x) < \frac{\varepsilon}{2}$ respectively. If we define $n_0 := \max\{n_1, n_2\}$, then

$$n > n_0 \quad \Longrightarrow \quad d_1(x_n, x) + d_2(x_n, x) < \varepsilon. \qquad \square$$

1.3.2. *Countable family of metrics.* There is also a method for combining not only finite family of metrics, but also countable family of metrics. Since the sum of countably many positive numbers may diverges to infinity, we cannot sum the metrics directly. The strategy used here is to "bound" the metrics. We call a metric bounded when the range of metric function is bounded.

PROPOSITION 1.14. *Every metric possesses an equivalent bounded metric.*

PROOF. Let d be a metric on a set. Let f be a bounded, monotonically increasing, and subadditive function on $\mathbb{R}_{\geq 0}$ that is continuous at 0 and satisfies $f^{-1}(0) = \{0\}$. The mostly used examples are

$$f(x) = \frac{x}{1+x} \quad \text{and} \quad f(x) = \min\{x, 1\}.$$

Then, $f \circ d$ is a bounded metric equivalent to d by Theorem 1.8. $\square$

DEFINITION 1.8. Let d be a metric on a set X. A *standard bounded metric* means either metric

$$\min\{d, 1\} \quad \text{or} \quad \frac{d}{d+1},$$

and we will denote it by $\hat{d}$.

The supremum of the standard bounded metric is 1. Every metric can be bounded above by not only 1 but also an arbitrary constant, keeping the topological equivalence, just by giving the constant as a coefficient to $\hat{d}$. Then, following propositions give methods for combining countable metrics.

PROPOSITION 1.15. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a countable family of metrics on a set X. Then a function $d : X^2 \to \mathbb{R}_{\geq 0}$ defined by*

$$d(x, y) := \sum_{i\in\mathbb{N}} 2^{-i}\hat{d}_i(x, y)$$

is a metric. Furthermore, a sequence $\{x_n\}_n$ converges in d if and only if it converges in every d_i.

PROOF. The function is well-defined by the monotone convergence theorem. The only nontrivial axiom is the triangle inequality. Consider the triangle inequality of truncated sum of metrics

$$\sum_{i=1}^{k} 2^{-i}\hat{d}_i(x, z) \leq \sum_{i=1}^{k} 2^{-i}\hat{d}_i(x, y) + \sum_{i=1}^{k} 2^{-i}\hat{d}_i(y, z).$$

By taking limit $k \to \infty$, we obtain the triangle inequality for d, hence a metric. Let us show the rest part.

($\Rightarrow$) We have an ineuqlaity $d_i \leq 2^i \hat{d}$ for each i, so convergence in d implies the convergence in each $\hat{d}_i$. See Theorem 1.7. The equivalence of $\hat{d}_i$ and d_i implies the desired result.

($\Leftarrow$) Suppose a sequence $\{x_n\}_n$ converges to a point x in d_i for every index i. Take an arbitrary small ball $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad \Longrightarrow \quad \hat{d}_i(x_n, x) < \frac{\varepsilon}{2}.$$

Define $k := \lceil 1 - \log_2 \varepsilon \rceil$ so that we have $2^{-k} \leq \frac{\varepsilon}{2}$. With this k, define

$$n_0 := \max_{1 \leq i \leq k} n_i.$$

If $n > n_0$, then

$$\begin{aligned} d(x_n, x) &= \sum_{i=1}^{k} 2^{-i} \hat{d}_i(x_n, x) + \sum_{i=k+1}^{\infty} 2^{-i} \hat{d}_i(x_n, x) \\ &< \sum_{i=1}^{k} 2^{-i} \frac{\varepsilon}{2} + \sum_{i=k+1}^{\infty} 2^{-i} \\ &< \frac{\varepsilon}{2} + 2^{-k} \leq \varepsilon, \end{aligned}$$

so x_n converges to x in the metric d. □

Proposition 1.16. *Let $\{d_i\}_{i \in \mathbb{N}}$ be a countable family of metrics on a set X. Then a function $d : X^2 \to \mathbb{R}_{\geq 0}$ defined by*

$$d(x, y) := \sup_{i \in \mathbb{N}} i^{-1} \hat{d}_i(x, y)$$

is a metric. Furthermore, a sequence $\{x_n\}_n$ converges in d if and only if it converges in every d_i.

Proof. The function is well-defined by the least upper bound property of real numbers. The triangle inequality and the direction ($\Rightarrow$) have the same proof with the previous one.

($\Leftarrow$) Suppose a sequence $\{x_n\}_n$ converges to a point x in each d_i, and take an arbitrary small ball $B(x, \varepsilon) = \{y : d(x, y) < \varepsilon\}$ with metric d. By the assumption, we can find n_i for each i satisfying

$$n > n_i \quad \Longrightarrow \quad \hat{d}_i(x_n, x) < \varepsilon.$$

Define $k := \lceil \frac{1}{\varepsilon} \rceil$ so that we have $k^{-1} \leq \varepsilon$. With this k, define

$$n_0 := \max_{1 \leq i \leq k} n_i.$$

If $n > n_0$, then

$$i^{-1} \hat{d}_i(x, y) \leq \hat{d}_i(x, y) < \varepsilon \quad \text{for} \quad i \leq k$$

and

$$i^{-1} \hat{d}_i(x, y) \leq i^{-1} < k^{-1} \leq \varepsilon \quad \text{for} \quad i > k$$

imply $d(x_n, x) < \varepsilon$, which means that x_n converges to x in the metric d. □

From the two propositions and Proposition 1.6, we get an additional corollary:

COROLLARY 1.17. *Let $\{d_i\}_{i\in\mathbb{N}}$ be a countable family of metrics on a set X. Then, two metrics*

$$d(x,y) := \sum_{i\in\mathbb{N}} 2^{-i} d_i(x,y) \quad \textit{and} \quad d(x,y) := \sup_{i\in\mathbb{N}} i^{-1} d_i(x,y)$$

are equivalent.

The sequences $\{2^{-i}\}_i$ and $\{i^{-1}\}_i$ in the above propositions can be replaced into any positive real sequences $\{a_i\}_{i=1}^{\infty}$ such that

$$\sum_{i\in\mathbb{N}} a_i < \infty \quad \text{and} \quad \lim_{i\to\infty} a_i = 0,$$

respectively.

REMARK. A metric

$$d'(x,y) = \sup_{i\in\mathbb{N}} d_i(x,y)$$

is not used because the convergence in this metric is a stronger condition than the convergence with respect to each metric d_i. In other words, this metric generates a finer(stronger) topology than the topology generated by subbase of balls. For example, the topology on $\mathbb{R}^{\mathbb{N}}$ generated by this metric defined with the projection pseudometrics is exactly what we often call the box topology.

1.3.3. *Pseudometrics.* It is often required to consider combining an uncountable family of generalized metrics, called pseudometrics. To motivate and introduce pseudometrics, we give an example problem. Let $X \times Y$ be a cartesian product of two metric spaces (X, d_X) and (Y, d_Y). We can ask which metric should be chosen in the most natural way, and a possible answer is as follows:

$$(x_n, y_n) \to (x,y) \quad \iff \quad x_n \to x \quad \text{and} \quad y_n \to y$$

as $n \to \infty$. We wish to recognize this as the sum of two different convergences: one is $x_n \to x$, and the other is $y_n \to y$. So then try to define two metric functions ρ_X and ρ_Y on $X \times Y$ such that

$$\rho_X((x,y),(x',y')) = d_X(x,x') \quad \text{and} \quad \rho_Y((x,y),(x',y')) = d_Y(y,y').$$

We expect these functions to satisfy axioms of metrics, but they fail on the identity of indiscernibles: $\rho_X((x,y),(x',y')) = 0$ does not imply $(x,y) = (x',y')$. However, the thing is, we can still define convergence or continuity with them; they also give rise to topologies, which lack some good features including the uniqueness of the limit point of a convergent sequence. We define pseudometrics by missing the nondegeneracy condition from the original definition of metric. Compare it with the definition of metric.

DEFINITION 1.9. A function $\rho : X \times X \to \mathbb{R}_{\geq 0}$ is called a *pseudometric* if

(1) $\rho(x,x) = 0$ for all $x \in X$,
(2) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$, (symmetry)
(3) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$. (triangle inequality)

Since pseudometrics also give rise to a topology, we can define

(1) convergence of a sequence;
(2) continuity between a set endowed with a pseudometric;
(3) refinement and equivalence relations;
(4) countable sum of bounded pseudometrics to make a new pseudometric.

Furthermore, every statement for metrics can be generalized to pseudometrics. Check that we have not used the condition that $d(x, y) = 0$ implies $x = y$. However, there is one big problem: the limit of a convergent sequence is not unique with a pseudometric.

Example 1.15. Let $\rho(x, y) = \rho((x_1, x_2), (y_1, y_2)) = |x_1 - y_1|$ be a pseudometric on $\mathbb{R}^2$. Consider a sequence $\{(\frac{1}{n}, 0)\}_n$. Since $(0, c)$ satisfies

$$\rho((\tfrac{1}{n}, 0), (0, c)) = \tfrac{1}{n} \to 0 \quad \text{as} \quad n \to \infty$$

for any real number c, the sequence converges to (x_1, x_2) if and only if $x_1 = 0$.

Although a sequence may have several limits in a pseudometric, a sum of a family of pseudometrics can have the sequences have at most one limit. In this case, the sum would satisfy the axioms of a metric.

Definition 1.10. A family of pseudometrics $\{\rho_\alpha\}_\alpha$ on a set X is said to *separates points* if the condition

$$\rho_\alpha(x, y) = 0 \quad \text{for all } \alpha$$

implies $x = y$.

Proposition 1.18. *A finite family of pseudometrics $\{\rho_i\}_{i=1}^N$ separates points if and only if the pseudometric $\rho := \sum_{i=1}^N \rho_i$ is a metric.*

Proposition 1.19. *A countable family of pseudometrics $\{\rho_i\}_{i\in\mathbb{N}}$ separates points if and only if the pseudometric defined by*

$$\rho(x, y) := \sum_{i\in\mathbb{N}} 2^{-i}\tilde{\rho}_i(x, y) \quad \textit{or} \quad \sup_{i\in\mathbb{N}} \tilde{\rho}_i(x, y),$$

where $\tilde{\rho}_i$ is either $\min\{\rho_i, 1\}$ or $\rho_i/(\rho_i + 1)$, is a metric.

In other words, we can say that a topology generated by countable family of metrics is metrizable.

We give the first example of a topology which cannot be given by a metric.

Example 1.16. sequence space pointwise convergence.

Exercises. Determine true or false and give a reason briefly:

(1) Every nonempty set can be endowed with a metric.
(2) The squared sum of two metrics is a metric.
(3) Every convergent sequence in a metric space is bounded.

Problems.

PROBLEM 1.1. Find a metric d on $\mathbb{R}$ such that a sequence $x_n = \frac{1}{n}$ converges to 1 and a sequence $y_n = 1 - \frac{1}{n}$ converges to 0.

PROBLEM 1.2. Find a metric d on $\mathbb{R}$ such that a sequence $\{x_n\}_n$ defined by $x_n = x + \frac{1}{n}$ is convergent with respect to d if and only if $x \neq 0$.

PROBLEM 1.3. Let d a metric on a set X and $p > 0$. Show that a function d_p defined by $d_p(x, y) := d(x, y)^p$ is a metric and it is equivalent to d.

PROBLEM 1.4. Compute the range of the function $f(x) = \|x\|_p / \|x\|_q$, where $\|x\|_p^p := \sum_{i=1}^d |x_i|^p$ for $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$.

2. Topology

We define topology and introduce some supplementary notions.

2.1. Filters. Suppose we want to find a proper way to define limit and convergence. Recall how we define convergence of a sequence of real numbers: we say a sequence $(x_n)_{n\in\mathbb{N}}$ converges to a number x if for each $\varepsilon > 0$ there is $n_0(\varepsilon) \in \mathbb{N}$ such that $|x - x_n| < \varepsilon$ whenever $n > n_0$. Simply, x_n is close to x if n is close to the infinity. Observe the two necessary structures to make this possible; the "system of neighborhoods" at each point x, and the total order on the index set $\mathbb{N}$ the set of natural numbers. Without the order structure, we would not be able to formulate the intuition of the direction toward which a sequence is converging. Even though the order on $\mathbb{N}$ is totally defined so that we can compare every pair of two elements, but it can be generalized to the case of partial orders.

Definition 2.1. A subset $\mathcal{D}$ of a poset is called *(upward) directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $a \leq c$ and $b \leq c$. Similarly, $\mathcal{D}$ is called *downward directed* if for every $a, b \in \mathcal{D}$ there is $c \in \mathcal{D}$ such that $c \leq a$ and $c \leq b$.

The directedness of a partially ordered set is an essential notion to define limit.

Let X be a set and $x \in X$. Then, the power set $\mathcal{P}(X)$ is a poset with inclusion relation. The filter bases are defined abstractly:

Definition 2.2. A *filter base* is a nonempty and downward directed subset of a poset.

And concretely:

Definition 2.3. A collection $\mathcal{B}_x$ of subsets of X is called a *filter base at x* if every element of $\mathcal{B}_x$ contains x and it forms a nonempty downward directed subset; every $U \in \mathcal{B}_x$ contains x, and for all $U_1, U_2 \in \mathcal{B}_x$ there is $U \in \mathcal{B}_x$ such that $U \subset U_1 \cap U_2$.

Among filters, we can give a relation structure as follows.

Definition 2.4. Let $\mathcal{B}_x, \mathcal{B}'_x$ be filter bases at x. We say $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$, or a *refinement* of $\mathcal{B}_x$ if for every $U \in \mathcal{B}_x$ there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$.

As synonyms, all the following expressions tell the same situation.

(1) $\mathcal{B}'_x$ is *finer than* $\mathcal{B}_x$,
(2) $\mathcal{B}'_x$ is *stronger than* $\mathcal{B}_x$,
(3) $\mathcal{B}_x$ is *coarser than* $\mathcal{B}'_x$,
(4) $\mathcal{B}_x$ is *weaker than* $\mathcal{B}'_x$.

The relation is a preorder so that we can consider the equivalence classes on which the natural partial order can be defined.

Proposition 2.1. *The refinement relation between filter bases is a preorder, and each equivalence class contains a unique maximal element.*

Proof. To show a relation is a preorder, we need to check transitivity. Suppose $\mathcal{B}''_x$ is finer than $\mathcal{B}'_x$ and $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$. For any $U \in \mathcal{B}_x$, there is $U' \in \mathcal{B}'_x$ such that $U' \subset U$, and there is also $U'' \in \mathcal{B}''_x$ such that $U'' \subset U'$. Since $U'' \subset U$, we can conclude $\mathcal{B}''_x$ is finer than $\mathcal{B}_x$.

We can say two filter bases are equivalent if they are both finer than each other. Consider an equivalence class of filter bases and just denote it by A. Then, $\bigcup_{\mathcal{B}_x \in A} \mathcal{B}_x$ is also contained in A since it is equivalent to an arbitrary filter base $\mathcal{B}_x$ in A. It is also easy to check that this is maximal. □

Now we define filters.

DEFINITION 2.5. A *filter at x* is the maximal element of an equivalence class of filter bases at x.

In other words, filters have one-to-one correspondence to the equivalence classes of filter bases. A filter is identified to an equivalence class of filter bases. They can be also characterized by three axioms.

THEOREM 2.2. *A collection $\mathcal{F}_x$ of subsets of X is a filter at x if and only if every element contains x and it is closed under supersets and finite intersections;*

(1) *$x \in U$ for $U \in \mathcal{F}_x$,*
(2) *if $U \subset V$ and $U \in \mathcal{F}_x$, then $V \in \mathcal{F}_x$,*
(3) *if $U, V \in \mathcal{F}_x$, then $U \cap V \in \mathcal{F}_x$.*

PROOF. □

Many references use the above theorem as the definition of filter because it is useful for someone who wants to check whether a given family is a filter.

THEOREM 2.3. *A filter $\mathcal{F}'_x$ is finer than another filter $\mathcal{F}_x$ if and only if $\mathcal{F}'_x \supset \mathcal{F}_x$.*

PROOF. □

The following examples will be helpful to catch the intuition.

EXAMPLE 2.1. Let x be a point in a metric space. The set of all open balls cenetered at x is a filter base at x. The set of all open balls containing x is aslo a filter base and they are equivalent. A filter equivalent to these filter bases are called *neighborhood filter at x* or *neighborgood system of x*.

EXAMPLE 2.2. Let S be a subset of a set. The set of all subsets containing S is a filter at x for every $x \in S$. If $S = \{x\}$, then it is called a *principal filter* at x.

EXAMPLE 2.3. The set of all subsets of $\mathbb{N}$ whose complement is finite is a filter, but it is not a filter at a point. However, it is intuitively a filter at infinity.

2.2. Topologies. Before defining topology, recall that it plays the most important role in the definition of continuous functions to deal with neighborhoods of a point. We want a structure to give a notion of neighborhoods of a point such as metrics, in other words, we want to generalize metric in a suitable way. There is a conventional definition of topology: topology is defined as a subset of the power set of underlying space satisfying some axioms, and it is said to consist of open sets so that a topology indicates that which subsets are open or not. However, this definition is so abstract that it might allow first-readers to lose its intuitions. Thereby, we attempt to take another way. Before introducing topology, we shall define a topological basis. Topological bases are often used to describe a particular topology as bases of vector spaces do. The main definition of topology will follow.

Let X be a set.

Definition 2.6. A collection $\mathcal{B}$ of subsets of X is called a *topological base* or simply a *base on* X if

$$\{U : x \in U \in \mathcal{B}\}$$

is a filter base at x for every $x \in X$.

A topological base is a kind of global version of filter base.

Definition 2.7. Let $\mathcal{B}$ be a topological base on X and $x \in X$. A filter base at x is called a *local base* at x if it is equivalent to the filter base $\{U : x \in U \in \mathcal{B}\}$. If a local base is a filter at x, then it is called *neighborhood filter* of x.

All the followings are synonyns:

(1) local base
(2) neighborhood system
(3) fundamental system of neighborhoods
(4) complete system of neighborhoods
(5) filter base of neighborhood filter

As we have done in the previous section, we can settle the refinement order on the set of topological bases.

Definition 2.8. Let $\mathcal{B}, \mathcal{B}'$ be topological bases on X. We say $\mathcal{B}$ is *coarser* or *weaker* than $\mathcal{B}'$, and $\mathcal{B}'$ is *finer*, *stronger* than $\mathcal{B}$, or a *refinement* of $\mathcal{B}$ if every local base $\mathcal{B}'_x$ is finer than $\mathcal{B}_x$ at every point $x \in X$.

Proposition 2.4. *The refinement relation between topological bases is a preorder, and each equivalence class contains a unique maximal element.*

Proof. □

A topology is defined to be the maximal element, which means in fact an equivalence class of topological bases.

Definition 2.9. A *topology on* X is the maximal element of an equivalence class of topological bases on X.

There is also a criterion for topology.

Theorem 2.5. *A collection $\mathcal{T}$ of subsets of X is a topology on X if and only if*

(1) $\varnothing, X \in \mathcal{T}$,
(2) *if* $\{U_\alpha\}_{\alpha \in \mathcal{A}} \subset \mathcal{T}$, *then* $\bigcup_{\alpha \in \mathcal{A}} U_\alpha \in \mathcal{T}$,
(3) *if* $U_1, U_2 \in \mathcal{T}$, *then* $U_1 \cap U_2 \in \mathcal{T}$.

Proof. □

Theorem 2.5 is usually used as a definition of topology because it allows us to check without difficulty whether a collection of subsets is a topology.

Since all topological structures are made to generalize the standard metric of Euclidean space, so drawing balls for representing base elements is always helpful in the whole story of general topology.

2.3. Bases and subbases.

Definition 2.10. Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. If $\mathcal{T}$ is the coarsest topology containing $\mathcal{B}$, then we say the topology $\mathcal{T}$ is *generated by* $\mathcal{B}$.

Theorem 2.6. *Let $\mathcal{B}$ and $\mathcal{T}$ be a base and a topology on a set X. The followings are equivalent:*

(1) *$\mathcal{B}$ generates $\mathcal{T}$,*
(2) *$\mathcal{B}$ and $\mathcal{T}$ are equivaent bases,*
(3) *$\mathcal{T}$ is the set of all arbitrary unions of elements of $\mathcal{B}$.*

Definition 2.11. Let $\mathcal{S} \subset \mathcal{P}(X)$. If a topology $\mathcal{T}$ is the coarsest topology containing $\mathcal{S}$, then we say $\mathcal{S}$ is called a *subbase* of $\mathcal{T}$.

Proposition 2.7. *Let $\mathcal{S} \subset \mathcal{P}(X)$. The set of finite intersections of elements of $\mathcal{S}$ is a basis.*

Here is the metric space example.

Example 2.4. Let X be a metric space. A set of all balls $\mathcal{B} = \{B(x, \varepsilon) : x \in X,\, \varepsilon > 0\}$ is a base on X because for every point $x \in B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$, we have $x \in B(x, \varepsilon) \subset B(x_1, \varepsilon_1) \cap B(x_2, \varepsilon_2)$ where $\varepsilon = \min\{\varepsilon_1 - d(x, x_1),\, \varepsilon_2 - d(x, x_2)\}$.

In metric spaces, of course, there can exist infinitely many bases, but they are hardly considered except $\mathcal{B}$. Sometimes in the context of metric spaces, the term neighborhood or basis are used to say $\mathcal{B}$. As we have seen, balls in metric spaces are the main concept to state ε-δ argument. This example would show a basis is fundamental language to describe the nature of limits in metric spaces.

2.4. Open sets and neighborhoods. def:nbhd and neighborhood filter convergence and limit

2.5. Closed sets and limit points. closure dense set,

2.6. Interior and closure.

3. Uniformity

3.1. Uniform spaces. uniformness of metric

3.2. Entourages. Uniform spaces are generalization of metric spaces. The uniform structure is required to define uniform continuity, uniform convergence, completeness, etc. Although the definition of uniform structure is not so easy at first, they have enormous advantage to learn. For example, they are extremely useful in functional analysis since every compatible topology on algebraic structures such as topological group and topological vector space must admit a natural uniform structure. Hence, we can use completeness or something uniform without unnecessary concerns.

DEFINITION 3.1 (Uniform space). A *uniform space* is a set X equipped with a filter of binary relations $\mathcal{U} \subset \mathcal{P}(X^2)$ such that for every $E \in \mathcal{U}$,

(1) reflexivity: $(x, x) \in E$ for all $x \in X$,
(2) triangle inequality: $\exists E' \in \mathcal{U} : E' \circ E' \subset E$,
(3) symmetry: $E^{-1} \in \mathcal{U}$,

where $\Delta_X = \{(x, x) : x \in X\}$ and

$$E \circ F = \{(x, z) : (x, y) \in E, (y, z) \in F\}, \quad E^{-1} = \{(y, x) : (x, y) \in E\}.$$

The collection $\mathcal{U}$ is called a *uniformity*, and a relation $E \in \mathcal{U}$ is called an *entourage*. If $(x, y) \in E$, then we say x and y are E-close.

DEFINITION 3.2. Let $(X, \mathcal{U})$ be a uniform space. Let τ be a set containing all $U \subset X$ such that for every $x \in U$ there is an entourage E with $E_x \subset U$. Then τ defines a topology on X, which is called *uniform topology*, or *induced topology*.

DEFINITION 3.3. A uniform space is called *Hausdorff* if there is an entourage E such that $x \in E$ and $y \notin E$ for every pair of distinct points $x, y \in X$. This is equivalent for the induced topology to be Hausdorff.

Note that the axioms for the definition of uniform spaces bear a similarity with the one of metric spaces. For one exception, the Hausdorffness implies the nondegeneracy. A uniform space is defined by the collection of relations that embody the concept of nearness. Unlike neighborhoods in general topological space, an entourage measures the nearness not pointwisely(locally) but uniformly(globally). We have the following hierarchy:

$$\text{topological space} \supset \text{uniform space} \supset \text{metric space}.$$

EXAMPLE 3.1. Let G be a topological group. Let U be an open neighborhood of the identity e. Define

$$E_U := \{(g, h) : gh^{-1} \in U\}.$$

Then, the set of E_U forms a uniformity. The difficult part is the triangle inequality, which can be shown from the continuity of group operation.

3.3. Pseudometrics. Metric can be regarded as the "countably" uniform structure in some sense. In other texts, for this reason, one frequently introduces metric instead of uniformity in order to avoid superfluously complicated and less intuitive notions of uniform structures, when only doing elementary analysis not requiring uncountable local bases.

One of the mostly used way of characterizing uniformity is to induce the fundamental system of entourages from a family of pseudometrics. The manner is simple: just take all pseudoballs as the fundamental system of entourages.

Definition 3.4. Let

The proof of the following theorem is based on Bourbaki's text (General topology part2, chapter 9).

Theorem 3.1. *Every uniformity is induced by a family of pesudometrics.*

Proof. □

CHAPTER 2

Continuity

1. Continuous functions

1.1. Various continuity. continuity, Cauchy continuity, uniform continuity, Lipschitz continuity

Example 1.1. An isometry between metric spaces is Lipschitz continuous with costant 1.

1.2. Sequential continuity.

2. Continuous maps

2.1. Mono and epi.

2.2. Subspaces and quotient spaces.

2.3. Product space.

2.4. Homeomorphisms. continuous bijection open map how to show two spaces are not homeomorphic - topological property: connected, compact

3. Connectedness

3.1. Connected spaces. component

3.2. Path connected spaces.

3.3. Locally connected spaces.

3.4. Homotopy.

CHAPTER 3

Convergence

1. Nets

product of two directed sets projection is monotone final uniformity is itself an upward directed set by reverse inclusion, like $\mathbb{R}_{\geq 0}$. cofinality and subsequence

eventuality filter, three definitions of subnets

2. Sequences

sequential spaces, first countable

3. Completeness

completion

CHAPTER 4

Compactness

Definition 0.1. Let X be a topological space. A *cover* of a subset $A \subset X$ is a collection $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets of X such that $A \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. If U_α are all open, then it is called *open cover*.

Definition 0.2. Let X be a topological space. A subset $K \subset X$ is called *compact* if every open cover of K has a finite subcover.

Proposition 0.1. *Let X be a topological space with a basis $\mathcal{B}$. A subset $K \subset X$ is compact if and only if every cover of the form $\{B_x \in \mathcal{B}\}_{x \in K}$ has a finite subcover.*

Remark. Let $\mathcal{P}$ be a property of a fuction $f\colon X \to Y$, such as continuity If we say f has $\mathcal{P}$ at a point x, then it would implies that x has a neighborhood U such that

0.1. Properties of compactness.

Theorem 0.2. *Let X and Y be topological spaces. For a continuous map $f\colon X \to Y$, the image $f(K)$ is compact for compact $K \subset X$.*

Remark. This is why the term "compact space" is widely used.

Corollary 0.3 (The extreme value theorem). *A continuous function on a closed interval has a global maximum and*

Heine-Cantor,

0.2. Characterizations of compactness.

$$\begin{tikzcd}
\text{net compact} \arrow[d, leftrightarrow] & \text{sequentially compact} \arrow[d] & \\
\text{compact} \arrow[r] & \text{countably compact} \arrow[r] \arrow[u, dashed, bend right, "\text{sequential}"'] \arrow[l, dashed, bend left, "\substack{\text{metrizable} \\ \textit{or } \text{2nd countable}}"] & \text{limit point compact} \arrow[l, dashed, bend left, "T_1"]
\end{tikzcd}$$

1. Relative compactness

PROPOSITION 1.1. *Let X be a locally compact Hausdorff space. For a subset A of X, the followings are all equivalent:*

(1) *a uniformity on X makes A have compact completion,*
(2) *the set A has compact closure in X,*
(3) *every uniformity on X makes A have compact completion.*

CHAPTER 5

Separation axioms

1. Separation axioms

2. Metrization theorems

CHAPTER 6

Function spaces

1. Compact-open topology

1.1. Definition.

Definition 1.1. Let X and Y be topological spaces. The *continuous functions space* $C(X,Y)$ is the set of continuous functions from X to Y. If $Y=\mathbb{R}$ or $\mathbb{C}$, then the continuous function space is denoted by $C(X)$.

1.2. Compact convergence. topology of compact convergence metrizability and hemicompact topology of uniform convergence uniform structure of pointwise convergence In considering the continuous function space, Y will be assumed to be a metric space because of its usefulness in most applications. Then, there are two useful topologies on $C(X,Y)$. Since there is a difficulty to deal with open sets or basis directly in a function space, the convergence will be a reliable alternative to describe the topologies. Before giving definition of the topologies, define pseudometrics ρ_K on $C(X,Y)$ by

$$\rho_K(f,g)=\sup_{x\in K} d(f(x),g(x))$$

for $K\subset X$ compact.

Definition 1.2. Let X and Y be topological spaces. The *topology of pointwise convergence* on $C(X,Y)$ is a subspace topology inherited from the product topology on Y^X.

Proposition 1.1. *Let X be a topological space and Y be a metric space. The topology of pointwise convergence on $C(X,Y)$ is generated by pseudometrics $\rho_{\{x\}}$, namely all $\{g: d(f(x),g(x))<\varepsilon\}$ for $f\in C(X,Y)$, $\varepsilon>0$, and $x\in X$.*

Definition 1.3. Let X be a topological space and Y be a metric space. The *topology of compact convergence* on $C(X,Y)$ is a topology generated by pseudometrics ρ_K, namely all $\{g:\rho_K(f,g)<\varepsilon\}$ for $f\in C(X,Y)$, $\varepsilon>0$, and compact $K\subset X$.

Proposition 1.2. *Let $C(X,Y)$ be a continuous function space for a topological space X and a metric space Y. A functional sequence in $C(X,Y)$ converges in the topology of compact convergence if and only if the functional sequence converges compactly.*

Theorem 1.3. *Let X be a topological space and Y be a metric space. If X is hemicompact, in other words, X has a sequence of compact subsets $\{K_n\}_{n\in\mathbb{N}}$ such that every compact subset of X is contained in K_n for some $n\in\mathbb{N}$, then the topology of compact convergence on $C(X,Y)$ is metrizable.*

Proof. bounding and merging pseudometrics $\square$

1.3. Exponentiability. locally compact Hausdorff spaces exponential space $\frac{\varepsilon}{3}$ argument

2. Rings of continuous functions

2.1. $C(X), C_0(X), C_b(X)$.

3. Important theorems on function space

3.1. The Arzela-Ascoli theorem. The Arzela-Ascoli theorem is a main technique to verify compactness of a subspace of continuous function space. The theorem requires the notion of equicontinuity, which lifts pointwise compactness up onto compactness in topology of compact convergence.

Definition 3.1. Let X be a topological space and Y be a metric space. A subset $\mathcal{F} \subset C(X,Y)$ is called *(pointwise or locally) equicontinuous* if for every $\varepsilon > 0$ and each $x_0 \in X$, there is an open neighborhood U of x_0 such that $x \in U \Rightarrow d(f(x), f(x_0)) < \varepsilon$ for all $f \in \mathcal{F}$.

Compare with the following definition:

Definition 3.2. Let X be a metric space and Y be a metric space. A subset $\mathcal{F} \subset C(X,Y)$ is called *uniformly equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$ for all $f \in \mathcal{F}$.

The uniform equicontinuity is what the Rudin's book says it just equicontinuous.

Theorem 3.1 (Arzela-Ascoli, conventional version). *Let X be a compact space. For $(f_n)_{n\in\mathbb{N}} \subset C(X)$, if it is equicontinuous and pointwisely bounded, then there is a subsequence that uniformly converges.*

Theorem 3.2 (Arzela-Ascoli, metrized version). *Let X be a hemicompact space and Y be a metric space. Let $\mathcal{T}_p$ and $\mathcal{T}_c$ be the topology of pointwise and compact convergence on $C(X,Y)$ relatively. For $\mathcal{F} \subset C(X,Y)$, if $\mathcal{F}$ is equicontinuous and relatively compact in $\mathcal{T}_p$, then $\mathcal{F}$ is relatively compact in $\mathcal{T}_c$.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and $K \subset X$ be a compact. By equicontinuity, for each $k \in \mathbb{N}$ a finite open cover $\{U_s\}_{s\in S_k}$ with a finite set $S_k \subset K$ can be taken such that $x \in U_s \Rightarrow d(f(x), f(s)) < \frac{1}{k}$ for all $f \in \mathcal{F}$. By the pointwise relative compactness, we can extract a subsequence $\{f_m\}_{m\in\mathbb{N}}$ of $\{f_n\}_n$ such that $\{f_m(s)\}_m$ is Cauchy for each $s \in \bigcup_{k\in\mathbb{N}} S_k$ by the diagonal argument.

For every $\varepsilon > 0$, let $k = \lceil (\frac{\varepsilon}{3})^{-1} \rceil$ and $m_0 = \max\{m_{0,s} : s \in S_k\}$ where $m_{0,s}$ satisfies that $m, m' > m_{0,s} \Rightarrow d(f_m(s), f_{m'}(s)) < \frac{\varepsilon}{3}$. By taking $s \in S_k$ such that $x \in U_s$ for arbitrary $x \in K$, we obtain, for $m, m' > m_0$,

$$d(f_m(x), f_{m'}(x)) \leq d(f_m(x), f_m(s)) + d(f_m(s), f_{m'}(s)) + d(f_{m'}(s), f_{m'}(x)) < \varepsilon.$$

Thus, $\{f_m\}_m$ is a subsequence of $\{f_n\}_n$ that is uniformly Cauchy on K. ☐

converse of Arzela-Ascoli

3.2. The Stone-Weierstrass theorem.