Interchanging limits, derivatives, and integrals

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1. Limit and derivative

f_n pointwisely, Df_n uniformly

2. Limit and integral

We want to find a criterion for This question asks the convergence

$$f_n \to f \quad \text{in} \quad L^1.$$

Theorem 2.1 (Lebesgue dominated convergence theorem). *Let $\{f_\alpha\}_\alpha$ be a net of measurable functions $(X, \mu) \to \mathbb{R}$. Define a maximal function*

$$Mf(x) = \sup_\alpha |f_\alpha(x)|.$$

If $\|Mf\|_1 < \infty$, then

$$\lim_\alpha |f_\alpha(x) - f(x)| = 0 \quad a.e. \quad \implies \quad \lim_\alpha \|f_\alpha - f\|_{L^1} = 0.$$

continuity application

3. Derivative and integral

Theorem 3.1 (Scheffe). *Let $\{f_n\}_n$ be a sequence of nonnegative functions in L^1. Suppose it converges to f pointwisely. Then,*

$$\lim_{n\to\infty} \|f_n\|_1 = \|f\|_1 \quad \iff \quad \lim_{n\to\infty} \|f_n - f\|_1 = 0.$$

This question asks under what conditions the following convergence holds: for fixed t_0,

$$\lim_{t\to t_0} \|N_t f(x, t_0) - \partial_t f(x, t_0)\|_{L^1_x} = 0,$$

where the Newton quotient is defined as

$$N_t f(x, t_0) := \frac{f(x,t) - f(x, t_0)}{t - t_0}$$

for $t \neq t_0$

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Theorem 3.2 (Leibniz rule)**.** *Let I be an interval in $\mathbb{R}$ that containing t_0. Let $f : X \times I \to \mathbb{R}$ be a measurable function such that for a.e. x the function $t \mapsto f(x,t)$ is continuous on I and differentiable on I°.*

If $\|\partial_t f\|_{L^1_x(L^\infty_t)} < \infty$, then

$$\frac{d}{dt}\int f(x,t)\,dx = \int \frac{\partial}{\partial t} f(x,t)\,dx.$$

Proof. Define a maximal function

$$Mf(x,t_0) = \sup_{t \in I \setminus \{t_0\}} |N_t f(x,t_0)|.$$

By the mean value theorem, we get

$$\|\partial_t f\|_{L^1_x(L^\infty_t)} < \infty \quad \Longrightarrow \quad \|Mf(x,t_0)\|_{L^1_x} < \infty.$$

Apply the LDCT. □

F is absolutely continuous,

$$\partial_t F = f \quad \Longleftrightarrow \quad F(x,t) = \int_c^t f(x,s)\,dx.$$

Then, for

$$T_h f(x,0) := \frac{1}{h}\int_0^h f(x,s)\,ds,$$

For $\|f\|_{L^1_x L^\infty_t} = \|\sup_t |f(x,t)|\|_{L^1_x} < \infty$ we have

$$\begin{aligned} |T_h f(x,0)| &\le \frac{1}{h}\int_0^h |f(x,s)|\,ds \\ &\le \left[\frac{1}{h}\int_0^h ds\right] \cdot \sup_t |f(x,t)| \\ &= \sup_t |f(x,t)|. \end{aligned}$$

Thus,

$$Mf(x,0) = \sup_h |T_h f(x,0)| \le \sup_t |f(x,t)| \in L^1_x.$$

Since $f(x,0) \in L^1_x$, by the Lebesgue differentiation theorem, we get

$$\lim_{h \to 0} T_h f(x,0) = f(x,0)$$

for a.e. x.