Diachrony of Spectra

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Postech - Unist - Kaist Joint Seminar

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Introduction

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Question

Why is it defined like this?

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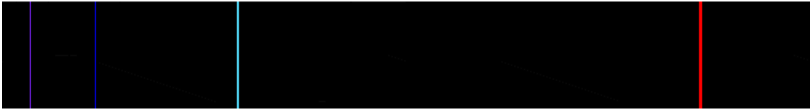
Hydrogen spectral series



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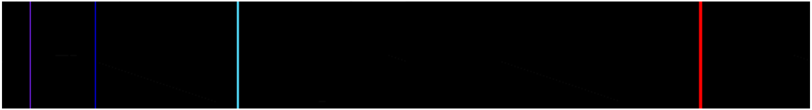

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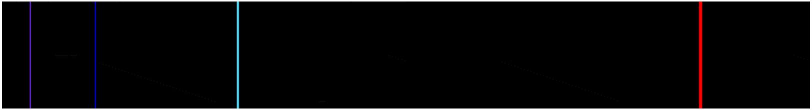

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Question

How can we explain and compute this phenomenon?

Rydberg's formula : Bohr model

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The constant h is called the Planck constant and $\hbar := \frac{h}{2\pi}$.

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Proposition (Rydberg formula)

The wavelengths λ of absorbed or emitted photons from a hydrogen atom is estimated by the following formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \textit{for} \quad n_1, n_2 \in \mathbb{N},$$

where $R := \frac{k^2e^4m}{4\pi\hbar^3c}$ is the Rydberg constant.

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Let's solve.

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- Since $\phi_E(t) \propto e^{-iEt}$ is easily solved, the main difficulty is ψ_E.

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Don't be so pedantic in doing physics.

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- Eigenvalues embody the possible energies of an electron, so we can give the Rydberg formula a reasonable explanation.
- This result explains not only the discretized energy spectrum but also the number of each orbitals!

Conclusion of Section 1

$$\begin{array}{c} \text{Partial Differential Equations with Time Evolution} \\ \Big\downarrow \text{ Separation of variables} \\ \text{Simultaneous Eigenvalue Problems} \\ \Big\downarrow \\ \text{Study of Eigenvalues} = \text{Study of Hydrogen Spectrum} \end{array}$$

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From now, we basically assume $\mathbb{C}$ as the scalar field for vector spaces.

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Theorem (Spectral theorem for normal matrices)

A complex square matrix A is normal if unitarily normalizable.

Spectral theorem of normal matrices

Hilbert space

Bounded operators

Spectral theorem of compact normal operators

Spectral theorem of elliptic operators (Bonus)

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$\downarrow$

The eigenvalues are distributed like

$$0 < \lambda_1 < \lambda_2 < \cdots \to \infty.$$

Contents

C^*-algebras

First we give definitions:

Psudo-definition

An algebra is a vector space with multiplication.

Definition

A C^*-*algebra* is a complex associative algebra with norm $\|\cdot\|$ such that:

1. $\|x^*x\| = \|x\|^2$

Example 1 : Bounded operators

Example 2 : Continuous functions

Spectra, multiplicative homomorphisms, maximal ideals

Gelfand-Naimark theorem

Algebraic variety

Coordinate ring

Maximal ideal is a point

Problem of unified codomains

Functoriality