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## 1 Kinetic theory

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

**Theorem 1.1** (Velocity averaging). *Let  $T$  be a free transport operator  $\partial_t + v \cdot \nabla_x$  on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . Then,*

$$\left\| \int u \varphi dv \right\|_{H_{t,x}^{1/2}} \lesssim_{\varphi} \|u\|_{L_{t,x,v}^2}^{1/2} \|Tu\|_{L_{t,x,v}^2}^{1/2}$$

for  $\varphi \in C_c^\infty(\mathbb{R}_v^n)$ ,

*Proof.* Let  $m(t, x) = \int u \varphi dv$ . By Fourier transform with respect to  $t$  and  $x$ , we have

$$\widehat{u}(\tau, \xi, v) = i \frac{\widehat{Tu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v) \varphi(v) dv.$$

Fixing  $\tau, \xi$ , decompose the integral and use Hölder's inequality to get

$$\begin{aligned} |\widehat{m}(\tau, \xi)| &\leq \int_{|\tau+v \cdot \xi| < \alpha} |\widehat{u} \varphi| dv + \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\widehat{Tu} \varphi|}{|\tau + v \cdot \xi|} dv \\ &\leq \|\widehat{u}\|_{L_v^2}^{1/2} \left( \int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv \right)^{1/2} + \|\widehat{Tu}\|_{L_v^2}^{1/2} \left( \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2} dv \right)^{1/2} \end{aligned}$$

We are going to estimate the integrals as

$$\int_{|\tau+v \cdot \xi| < \alpha} |\varphi|^2 dv \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \quad \int_{|\tau+v \cdot \xi| \geq \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|} dv \lesssim \frac{1}{\alpha \sqrt{\tau^2 + |\xi|^2}}.$$

We may assume that  $\sqrt{\tau^2 + |\xi|^2} \gg 1$ , that is, it is enough to show them for  $\sqrt{\tau^2 + |\xi|^2} \geq C$  with arbitrarily taken constant  $C$ , because the case that  $\sqrt{\tau^2 + |\xi|^2} \lesssim 1$  easily proves the inequality.

Define coordinates  $(v_1, v_2)$  on  $\mathbb{R}_v$  as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}, \quad v_2 := v - \frac{v \cdot \xi}{|\xi|^2} \xi \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = \left(v - \frac{\tau}{|\xi|}\right)^2 + |v_2|^2 \quad \text{and} \quad \int dv = \iint dv_2 dv_1.$$

For the first integral, suppose that  $\varphi$  is supported on a ball  $|v| < R$ . Then,

$$\begin{aligned} \int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2 dv &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2 - (v_1 - \frac{\tau}{|\xi|})^2} dv_2 dv_1 \\ &\lesssim (R^2 - \frac{\tau^2}{|\xi|^2})^{\frac{n-1}{2}} \cdot \frac{2\alpha}{|\xi|}, \end{aligned}$$

where we value the term  $(R^2 - \frac{\tau^2}{|\xi|^2})$  as 0 when  $R^2 < \frac{\tau^2}{|\xi|^2}$ . Since

$$(R^2 - \frac{\tau^2}{|\xi|^2})^{\frac{n-1}{2}} \frac{2\alpha}{|\xi|} \cdot \sqrt{\tau^2 + |\xi|^2} \lesssim \begin{cases} 0, & |\tau| \gg 1 \\ C, & |\xi| \gg 1 \end{cases},$$

we have

$$\int_{|\tau+v\cdot\xi|<\alpha} |\varphi|^2 dv \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}.$$

For the second integral, suppose that  $\varphi$  is supported on  $|v| < C$  so that  $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < C$ . Then,

$$\begin{aligned} \int_{|\tau+v\cdot\xi|\geq\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|} dv &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}, |v_1-\frac{\tau}{|\xi|}<C} \int_{|v_2|<C} \frac{1}{v_1^2|\xi|^2} dv_2 dv_1 \\ &\simeq \int_{|v_1|<\frac{\alpha}{|\xi|}, |v_1-\frac{\tau}{|\xi|}<C} \frac{dv_1}{v_1^2|\xi|^2}. \end{aligned}$$

If  $|\xi| \gtrsim |\tau|$ , then

$$\begin{aligned} \int_{|v_1|<\frac{\alpha}{|\xi|}, |v_1-\frac{\tau}{|\xi|}<C} \frac{dv_1}{v_1^2|\xi|^2} &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \frac{dv_1}{v_1^2|\xi|^2} \\ &\simeq \frac{1}{\alpha|\xi|} \lesssim \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}. \end{aligned}$$

If  $|\xi| \ll |\tau|$  such that at least  $|\tau| > C|\xi|$ , then

$$\begin{aligned} \int_{|v_1|<\frac{\alpha}{|\xi|}, |v_1-\frac{\tau}{|\xi|}<C} \frac{dv_1}{v_1^2|\xi|^2} &\lesssim \int_{|v_1-\frac{\tau}{|\xi|}<C} \frac{dv_1}{v_1^2|\xi|^2} \\ &\simeq \frac{1}{|\xi|^2} \left( \frac{1}{\frac{\tau}{|\xi|} - C} - \frac{1}{\frac{\tau}{|\xi|} + C} \right) \\ &= \frac{2C}{\tau^2 - C^2|\xi|^2} \ll \frac{1}{\sqrt{\tau^2 + |\xi|^2}}, \end{aligned}$$

hence

$$\int_{|\tau+v\cdot\xi|\geq\alpha} \frac{|\varphi|^2}{|\tau+v\cdot\xi|} dv \lesssim \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

To sum up, we have

$$|\widehat{m}(\tau, \xi)| \lesssim \frac{1}{(\tau^2 + |\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Tu}\|_{L_v^2}^{1/2}).$$

Letting  $\alpha = \sqrt{\|\widehat{Tu}\|_{L_v^2} / \|\widehat{u}\|_{L_v^2}}$  and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L_v^2}^{1/2} \|\widehat{Tu}\|_{L_v^2}^{1/2}.$$

Therefore, the integration on  $\mathbb{R}_\tau \times \mathbb{R}_\xi^n$  and Plancheral's theorem gives

$$\|m\|_{H_{t,x}^{1/2}} \lesssim_\varphi \|u\|_{L_{t,x,v}^2}^{1/2} \|Tu\|_{L_{t,x,v}^2}^{1/2}.$$

□

**Corollary 1.2.** *Let  $\mathcal{F}$  be a family of functions on  $\mathbb{R}_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  and  $T\mathcal{F}$  are bounded in  $L_{t,x,v}^2$ , then  $\int \mathcal{F} \varphi dv$  is bounded in  $H_{t,x}^{1/2}$ .*

**Theorem 1.3.** *Let  $\mathcal{F}$  be a family of functions on  $I_t \times \mathbb{R}_x^n \times \mathbb{R}_v^n$ . If  $\mathcal{F}$  is weakly relatively compact and  $T\mathcal{F}$  is bounded in  $L_{t,x,v}^1$ , then  $\int \mathcal{F} \varphi dv$  is relatively compact in  $L_{t,x}^1$ .*

## 2 Peetre's theorem

**Lemma 2.1.** *Suppose a linear operator  $L : C_c^\infty(M) \rightarrow C_c^\infty(M)$  satisfies*

$$\text{supp}(Lu) \subset \text{supp}(u) \quad \text{for } u \in C_c^\infty(X).$$

*For each point  $x \in M$ , there is a bounded neighborhood  $U$  together with a nonnegative integer  $m$  such that*

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

*for  $u \in C_c^\infty(U \setminus \{x\})$ .*

*Proof.* Suppose not. There is a point  $x$  at which the inequality fails; for every bounded neighborhood  $U$  and for every nonnegative  $m$ , we can find  $u \in C_c^\infty(U \setminus \{x\})$  such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large  $C$ . We want to construct a function  $u \in C_c^\infty(U)$  such that  $Lu$  has a singularity at  $x$ .

(Induction step) Take a bounded neighborhood  $U_m$  of  $x$  such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is  $u_m \in C_c^\infty(U_m \setminus \{x\})$  such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset \quad \text{for } i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that  $u \in C_c^\infty(U)$  since the series converges in the inductive topology of the LF space  $C_c^\infty(U)$ : it converges absolutely with respect to the seminorms  $\|\cdot\|_{C^m}$  for all  $m$ :

$$\begin{aligned} \sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\ &\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\ &< \infty. \end{aligned}$$

Also, since the supports of each term are disjoint and  $L$  is locally defined, we have

$$Lu = \sum_{i \geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i \geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i \geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

□

### 3 Elliptic curves

#### 3.1 Reduction of Weierstrass equations

In this subsection, we want to investigate the important constants of elliptic curves such as  $c_4$ ,  $c_6$ ,  $\Delta$ ,  $j$  by calculating equations with hands.

**Step 1.** The Riemann-Roch theorem proves that every curve of genus 1 with a specified base point can be described by the first kind of Weierstrass equation. Explicitly, the first form of Weierstrass equation is

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \quad (1)$$

**Step 2.** *Elimination of  $xy$  and  $y$ .* Factorize the left hand side

$$y(y + a_1x + a_3) = x^3 + a_2x^2 + a_4x + a_6.$$

By translation

$$\boxed{x \mapsto x, \quad y \mapsto y - \frac{1}{2}(a_1x + a_3)}$$

we have

$$\begin{aligned} y^2 - \left(\frac{1}{2}(a_1x + a_3)\right)^2 &= x^3 + a_2x^2 + a_4x + a_6, \\ y^2 &= x^3 + \left(\frac{1}{4}a_1^2 + a_2\right)x^2 + \left(\frac{1}{2}a_1a_3 + a_4\right)x + \left(\frac{1}{4}a_3^2 + a_6\right), \\ y^2 &= x^3 + \frac{1}{4}(a_1^2 + 4a_2)x^2 + \frac{1}{2}(a_1a_3 + 2a_4)x + \frac{1}{4}(a_3^2 + 4a_6). \end{aligned}$$

Introduce new coefficients  $b$  to write it as

$$y^2 = x^3 + \frac{1}{4}b_2x^2 + \frac{1}{2}b_4x + \frac{1}{4}b_6.$$

By scaling

$$\boxed{x \mapsto x, \quad y \mapsto \frac{1}{2}y}$$

we get

$$y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6. \quad (2)$$

**Step 3.** *Elimination of  $x^2$ .* By translation

$$\boxed{x \mapsto x - \frac{1}{12}b_2}$$



we have

$$\begin{aligned} y^2 = 4 \left( x^3 - 3 \cdot \frac{1}{12} b_2 x^2 + 3 \cdot \frac{1}{12^2} b_2^2 x - \frac{1}{12^3} b_2^3 \right) \\ + b_2 \left( x^2 - 2 \cdot \frac{1}{12} b_2 x + \frac{1}{12^2} b_2^2 \right) \\ + 2b_4 \left( x - \frac{1}{12} b_2 \right) \\ + b_6, \end{aligned}$$

so

$$\begin{aligned} y^2 &= 4x^3 + \left( 4 \cdot 3 \cdot \frac{1}{12^2} b_2^2 - 2 \cdot \frac{1}{12} b_2^2 + 2b_4 \right) x + \left( -4 \cdot \frac{1}{12^3} b_2^3 + \frac{1}{12^2} b_2^3 - 2 \cdot \frac{1}{12} b_2 b_4 + b_6 \right) \\ &= 4x^3 + \frac{1}{12} (-b_2^2 + 24b_4) x + \frac{1}{216} (b_2^3 - 36b_2 b_4 + 216b_6). \end{aligned}$$

Write it as

$$y^2 = 4x^3 - \frac{1}{12} c_4 x - \frac{1}{216} c_6.$$

We want to match the coefficients of  $y^2$  and  $x^3$  but also want the coefficients of  $c_4 x$  and  $c_6$  to be integers. Iterative scaling implies

$$\begin{aligned} x \mapsto \frac{1}{6}x : \quad & 216y^2 = 4x^3 - 3c_4x - c_6 \\ y \mapsto \frac{1}{36}y : \quad & y^2 = 24x^3 - 18c_4x - 6c_6 \\ x \mapsto \frac{1}{6}x : \quad & 9y^2 = x^3 - 27c_4x - 54c_6 \\ y \mapsto \frac{1}{3}y : \quad & y^2 = x^3 - 27c_4x - 54c_6. \end{aligned}$$

Thus, we get the famous third form of Weierstrass equation:

$$y^2 = x^3 - 27c_4x - 54c_6. \tag{3}$$

**Theorem 3.1.** *Let*

$$E : y^2 = x^3 - Ax - B.$$

*TFAE:*

- (1) *A point  $(x, y)$  is a singular point of  $E$ .*
- (2)  *$y = 0$  and  $x$  is a double root of  $x^3 - Ax - B$ .*
- (3)  *$\Delta = 0$ .*

*Proof.* (1) $\Rightarrow$ (2)  $\partial_y f = 0$  implies  $y = 0$ .  $f = \partial_x f = 0$  implies  $x$  is a double root of  $x^3 - Ax - B$ .  $A$  determines whether  $x$  is either cusp or node.  $\square$

## 4 Characteristic curve

Algorithm:

- (1) Establish the associated vector field by substituting  $u \mapsto y$ .
- (2) Find the integral curve.
- (3) Eliminate the auxiliary variables to get an algebraic equation.
- (4) Verify the computed solution is in fact the real solution.

**Proposition 4.1.** *Suppose that there exists a smooth solution  $u : \Omega \rightarrow \mathbb{R}_y$  of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, & (t, x) \in \Omega \subset \mathbb{R}_{t \geq 0} \times \mathbb{R}_x, \\ u(0, x) = x, & \text{at } x \in \mathbb{R}, \end{cases}$$

and let  $M$  be the embedded surface defined by  $y = u(t, x)$ .

Let  $\gamma : I \rightarrow \Omega \times \mathbb{R}_y$  be an integral curve of the vector field

$$X = \frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that  $\gamma(0) \in M$ . Then,  $\gamma(\theta) \in M$  for all  $\theta \in I$ .

*Proof.* We may assume  $\gamma$  is maximal. Define  $\tilde{\gamma} : \tilde{I} \rightarrow M$  as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that  $\tilde{\gamma}(0) = \gamma(0)$ . Since  $X$  and  $\tilde{X}$  coincide on  $M$ , the curve  $\tilde{\gamma}$  is also an integral curve of  $X$  with  $\tilde{\gamma}(0) = \gamma(0)$ . By the uniqueness of the integral curve, we get  $\tilde{I} \subset I$  and  $\gamma(\theta) = \tilde{\gamma}(\theta)$  for all  $\theta \in \tilde{I}$ .

Since  $M$  is closed in  $E$ , the open interval  $\tilde{I} = \gamma^{-1}(M)$  is closed in  $I$ , hence  $\tilde{I} = I$  by the connectedness of  $I$ .  $\square$

**Definition 4.1.** The projection of the integral curve  $\gamma$  onto  $\Omega$  is called a *characteristic*.

This proposition implies that we might be able to describe the points on the surface  $M$  explicitly by finding the integral curves of the vector field  $X$ . Once we find a necessary condition of the form of algebraic equation, we can demonstrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since  $X$  does not depend on  $u$ , we can solve the ODE: let  $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$  be the integral curve of  $X$  such that  $\gamma(0) = (0, \xi, \xi)$ . Then, the system of ODEs

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad y(\theta) = \xi, \quad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t, x) = \frac{-1 + \sqrt{1 + 4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain  $\Omega$  as

$$\Omega = \{ (t, x) : tx > -\frac{1}{4} \}.$$

#### 4.1 Wave equation

$$\begin{aligned}u_{tt} - c^2 u_{xx} &= 0 \quad \text{for } t, x > 0, \\ u(0, x) &= g(x), \quad u_t(0, x) = h(x), \quad u_x(t, 0) = \alpha(t).\end{aligned}$$

Define  $v := u_t - cu_x$ . Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ v(0, x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t, x) = h(x - ct) - cg'(x - ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ u(0, x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ u_x(t, 0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter  $\xi > 0$ ,

$$\begin{aligned}\frac{dt}{d\theta} &= 1, & \frac{dx}{d\theta} &= -c, & \frac{dy}{d\theta} &= -v(t, x), \\ t(0) &= 0, & x(0) &= \xi, & y(0) &= g(\xi)\end{aligned}$$

is solved as

$$t(\theta) = \theta, \quad x(\theta) = -c\theta + \xi, \quad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta') d\theta',$$

hence for  $x > ct > 0$ ,

$$\begin{aligned}u(t, x) &= g(\xi) - \int_0^\theta v(s, \xi - cs) ds \\ &= g(x + ct) \\ &= \frac{3g(x + ct) - g(x - ct)}{2} - \int_0^t h(x + c(t - 2s)) ds\end{aligned}$$

## 4.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + uu_x = 0.$$

- (1) Suppose  $u(0, x) = \tanh(x)$ . For what values of  $t > 0$  does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (2) Suppose  $u(0, x) = -\tanh(x)$ . For what values of  $t > 0$  does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the  $tx$ -plane.
- (3) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases}.$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and “paste” the solution together.

## 5 Fundamental differential geometry

### 5.1 Manifold and Atlas

**Definition 5.1.** A *locally Euclidean space*  $M$  of dimension  $m$  is a Hausdorff topological space  $M$  for which each point  $x \in M$  has a neighborhood  $U$  homeomorphic to an open subset of  $\mathbb{R}^d$ .

**Definition 5.2.** A *manifold* is a locally Euclidean space satisfying the one of following equivalent conditions: second countability, blabla

**Definition 5.3.** A *chart* or a *coordinate system* for a locally Euclidean space is a map  $\varphi$  is a homeomorphism from an open set  $U \subset M$  to an open subset of  $\mathbb{R}^d$ . A chart is often written by a pair  $(U, \varphi)$ .

**Definition 5.4.** An *atlas*  $\mathcal{F}$  is a collection  $\mathcal{F} = \{(U_\alpha, \varphi_\alpha) \mid \alpha \in A\}$  of charts on  $M$  such that  $\bigcup_{\alpha \in A} U_\alpha = M$ .

**Definition 5.5.** A *differentiable manifold* is a manifold on which a differentiable structure is equipped.

The definition of differentiable structure will be given in the next subsection. Actually, a differentiable structure can be defined for a locally Euclidean space.

### 5.2 Definition of Differentiable Structure

**Definition 5.6.** An atlas  $\mathcal{F}$  is called *differentiable* if any two charts  $\varphi_\alpha, \varphi_\beta \in \mathcal{F}$  is *compatible*: each *transition function*  $\tau_{\alpha\beta}: \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta)$  which is defined by  $\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1}$  is differentiable.

It is called a *gluing condition*.

**Definition 5.7.** For two differentiable atlases  $\mathcal{F}, \mathcal{F}'$ , the two atlases are *equivalent* if  $\mathcal{F} \cup \mathcal{F}'$  is also differentiable.

**Definition 5.8.** An differentiable atlas  $\mathcal{F}$  is called *maximal* if the following holds: if a chart  $(U, \varphi)$  is compatible to all charts in  $\mathcal{F}$ , then  $(U, \varphi) \in \mathcal{F}$ .

**Definition 5.9.** A *differentiable structure* on  $M$  is a maximal differentiable atlas.

To differentiate a function on a flexible manifold, first we should define the differentiability of a function. A differentiable structure, which is usually defined by a maximal differentiable atlas, is roughly a collection of differentiable functions on  $M$ . When the charts is already equipped on  $M$ , it is natural to define a function  $f: M \rightarrow \mathbb{R}$  differentiable if the functions  $f \circ \varphi^{-1}: \mathbb{R}^d \rightarrow \mathbb{R}$  is differentiable.

The gluing condition makes the differentiable function for a chart is also differentiable for any charts because  $f \circ \varphi_\alpha^{-1} = (f \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\alpha^{-1}) = (f \circ \varphi_\beta^{-1}) \circ \tau_{\alpha\beta}$ . If a

function  $f$  is differentiable on an atlas  $\mathcal{F}$ , then  $f$  is also differentiable on any atlases which is equivalent to  $\mathcal{F}$  by the definition of the equivalence relation for differential atlases. We can construct the equivalence classes respected to this equivalence relation.

Therefore, we want to define a differentiable structure as a one of the equivalence classes. However the differentiable structure is frequently defined as a maximal atlas for the convenience since each equivalence class is determined by a unique maximal atlas.

**Example 5.1.** While the circle  $S^1$  has a unique smooth structure,  $S^7$  has 28 smooth structures. The number of smooth structures on  $S^4$  is still unknown.

**Definition 5.10.** A continuous function  $f: M \rightarrow N$  is differentiable if  $\psi \circ f \circ \varphi^{-1}$  is differentiable for charts  $\varphi, \psi$  on  $M, N$  respectively.

### 5.3 Curves

**Definition 5.11.** For  $f: M \rightarrow \mathbb{R}$  and  $(U, \phi)$  a chart,

$$df \left( \frac{\partial}{\partial x^\mu} \right) := \frac{\partial f \circ \phi^{-1}}{\partial x^\mu}.$$

**Definition 5.12.** Let  $\gamma: I \rightarrow M$  be a smooth curve. Then,  $\dot{\gamma}(t)$  is defined by a tangent vector at  $\gamma(t)$  such that

$$\dot{\gamma}(t) := d\gamma \left( \frac{\partial}{\partial t} \right).$$

Let  $\phi: M \rightarrow N$  be a smooth map. Then,  $\phi(t)$  can refer to a curve on  $N$  such that

$$\phi(t) := \phi(\gamma(t)).$$

Let  $f: M \rightarrow \mathbb{R}$  be a smooth function. Then,  $\dot{f}(t)$  is defined by a function  $\mathbb{R} \rightarrow \mathbb{R}$  such that

$$\dot{f}(t) := \frac{d}{dt} f \circ \gamma.$$

**Proposition 5.1.** Let  $\gamma: I \rightarrow M$  be a smooth curve on a manifold  $M$ . The notation  $\dot{\gamma}^\mu$  is not confusing thanks to

$$(\dot{\gamma})^\mu = (\dot{\gamma}^\mu).$$

In other words,

$$dx^\mu(\dot{\gamma}) = \frac{d}{dt} x^\mu \circ \gamma.$$

## 5.4 Connection computation

$$\begin{aligned}
\nabla_X Y &= X^\mu \nabla_\mu (Y^\nu \partial_\nu) \\
&= X^\mu (\nabla_\mu Y^\nu) \partial_\nu + X^\mu Y^\nu (\nabla_\mu \partial_\nu) \\
&= X^\mu \left( \frac{\partial Y^\nu}{\partial x^\mu} \right) \partial_\nu + X^\mu Y^\nu (\Gamma_{\mu\nu}^\lambda \partial_\lambda) \\
&= X^\mu \left( \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda \right) \partial_\nu.
\end{aligned}$$

The covariant derivative  $\nabla_X Y$  does not depend on derivatives of  $X^\mu$ .

$$Y_{;\mu}^\nu = \nabla_\mu Y^\nu = \frac{\partial Y^\nu}{\partial x^\mu}, \quad Y_{;\mu}^\nu = (\nabla_\mu Y)^\nu = \frac{\partial Y^\nu}{\partial x^\mu} + \Gamma_{\mu\lambda}^\nu Y^\lambda.$$

**Theorem 5.2.** *For Levi-civita connection for  $g$ ,*

$$\Gamma_{ij}^l = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

*Proof.*

$$\begin{aligned}
(\nabla_i g)_{jk} &= \partial_i g_{jk} - \Gamma_{ij}^l g_{lk} - \Gamma_{ik}^l g_{jl} \\
(\nabla_j g)_{kl} &= \partial_j g_{kl} - \Gamma_{jk}^l g_{li} - \Gamma_{jl}^l g_{kl} \\
(\nabla_k g)_{ij} &= \partial_k g_{ij} - \Gamma_{ki}^l g_{lj} - \Gamma_{kj}^l g_{il}
\end{aligned}$$

If  $\nabla$  is a Levi-civita connection, then  $\nabla g = 0$  and  $\Gamma_{ij}^k = \Gamma_{ji}^k$ . Thus,

$$\Gamma_{ij}^l g_{kl} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

$$\Gamma_{ij}^l = \frac{1}{2}g^{kl}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}).$$

□

## 5.5 Geodesic equation

**Theorem 5.3.** *If  $c$  is a geodesic curve, then components of  $c$  satisfies a second-order differential equation*

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0.$$

*Proof.* Note

$$0 = \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{\gamma}^\mu \nabla_\mu (\dot{\gamma}^\lambda \partial_\lambda) = (\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu + \dot{\gamma}^\nu \dot{\gamma}^\lambda \Gamma_{\nu\lambda}^\mu) \partial_\mu.$$



Since

$$\dot{\gamma}^\nu \partial_\nu \dot{\gamma}^\mu = \dot{\gamma}(\dot{\gamma}^\mu) = d\dot{\gamma}^\mu(\dot{\gamma}) = d\dot{\gamma}^\mu \circ d\gamma \left( \frac{\partial}{\partial t} \right) = d\dot{\gamma}^\mu \left( \frac{\partial}{\partial t} \right) = \ddot{\gamma}^\mu,$$

we get a second-order differential equation

$$\frac{d^2 \gamma^\mu}{dt^2} + \Gamma_{\nu\lambda}^\mu \frac{d\gamma^\nu}{dt} \frac{d\gamma^\lambda}{dt} = 0$$

for each  $\mu$ .

□

## 6 Vector calculus on spherical coordinates

$$\begin{aligned}
V &= (V_r, V_\theta, V_\phi) \\
&= V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi} \quad (\text{normalized coords}) \\
&= V_r \frac{\partial}{\partial r} + \frac{1}{r} V_\theta \frac{\partial}{\partial \theta} + \frac{1}{r \sin \theta} V_\phi \frac{\partial}{\partial \phi} \quad (\Gamma(TM)) \\
&= V_r dr + r V_\theta d\theta + r \sin \theta V_\phi d\phi \quad (\Omega^1(M)) \\
&= r^2 \sin \theta V_r d\theta \wedge d\phi + r \sin \theta V_\theta d\phi \wedge dr + r V_\phi dr \wedge d\theta \quad (\Omega^2(M)). \\
\nabla \cdot V &= \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} (r^2 \sin \theta V_r) + \frac{\partial}{\partial \theta} (r \sin \theta V_\theta) + \frac{\partial}{\partial \phi} (r V_\phi) \right] \\
\Delta u &= \frac{1}{r^2 \sin \theta} \left[ \frac{\partial}{\partial r} \left( r^2 \sin \theta \frac{\partial}{\partial r} u \right) + \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} u \right) + \frac{\partial}{\partial \phi} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} u \right) \right]
\end{aligned}$$

Let  $(\xi, \eta, \zeta)$  be an orthogonal coordinate that is *not* normalized. Then,

$$\begin{aligned}
\sharp &= g = \text{diag}(\|\partial_\xi\|^2, \|\partial_\eta\|^2, \|\partial_\zeta\|^2) \\
\hat{x} &= \|\partial_x\|^{-1} \partial_x = \|\partial_x\| dx = \|\partial_y\| \|\partial_z\| dy \wedge dz
\end{aligned}$$

In other words, we get the normalized differential forms in sphereical coordinates as follows:

$$dr, \quad r d\theta, \quad r \sin \theta d\phi, \quad (r d\theta) \wedge (r \sin \theta d\phi), \quad (r \sin \theta d\phi) \wedge (dr), \quad (dr) \wedge (r d\theta).$$

$$\begin{aligned}
\text{grad} : \nabla &= \left[ \frac{1}{\|\partial_x\|} \frac{\partial}{\partial x} \cdot -, \frac{1}{\|\partial_y\|} \frac{\partial}{\partial y} \cdot -, \frac{1}{\|\partial_z\|} \frac{\partial}{\partial z} \cdot - \right] \\
\text{curl} : \nabla &= \left[ \frac{1}{\|\partial_y\| \|\partial_z\|} \left( \frac{\partial}{\partial y} (\|\partial_z\| \cdot -) - \frac{\partial}{\partial z} (\|\partial_y\| \cdot -) \right), \right. \\
&\quad \frac{1}{\|\partial_z\| \|\partial_x\|} \left( \frac{\partial}{\partial z} (\|\partial_x\| \cdot -) - \frac{\partial}{\partial x} (\|\partial_z\| \cdot -) \right), \\
&\quad \left. \frac{1}{\|\partial_x\| \|\partial_y\|} \left( \frac{\partial}{\partial x} (\|\partial_y\| \cdot -) - \frac{\partial}{\partial y} (\|\partial_x\| \cdot -) \right) \right] \\
\text{div} : \nabla &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[ \frac{\partial}{\partial x} (\|\partial_y\| \|\partial_z\| \cdot -), \frac{\partial}{\partial y} (\|\partial_z\| \|\partial_x\| \cdot -), \frac{\partial}{\partial z} (\|\partial_x\| \|\partial_y\| \cdot -) \right] \\
\Delta &= \frac{1}{\|\partial_x\| \|\partial_y\| \|\partial_z\|} \left[ \frac{\partial}{\partial x} \left( \frac{\|\partial_y\| \|\partial_z\|}{\|\partial_x\|} \frac{\partial}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\|\partial_z\| \|\partial_x\|}{\|\partial_y\|} \frac{\partial}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\|\partial_x\| \|\partial_y\|}{\|\partial_z\|} \frac{\partial}{\partial z} \right) \right]
\end{aligned}$$

$$\text{grad} = \frac{1}{\|\cdot\|^1} (\nabla) \cdot \|\cdot\|^0$$

$$\text{curl} = \frac{1}{\|\cdot\|^2} (\nabla \times) \|\cdot\|^1$$

$$\text{div} = \frac{1}{\|\cdot\|^3} (\nabla \cdot) \|\cdot\|^2$$

## 7 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator  $T$  induces a Hilbert space structure on its domain. If the inclusion is bounded, then  $T$  has the bounded inverse. If the inclusion is compact, then  $T$  has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting  $L^2$  inner product on  $C([0, 1])$ , define  $\phi(f) = \int_0^{\frac{1}{2}} f$ .
- Every separable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem  $\rightarrow$  continuous embedding is really an embedding.
- $D(\Omega)$  is defined by a *countable strict* inductive limit of  $D_K(\Omega)$ ,  $K \subset \Omega$  compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever  $\alpha < \beta$  the induced topology by  $\mathcal{T}_\beta$  coincides with  $\mathcal{T}_\alpha$ )
- A net  $(\phi_d)_d$  in  $D(\Omega)$  converges if and only if there is a compact  $K$  such that  $\phi_d \in D_K(\Omega)$  for all  $d$  and  $\phi_d$  converges uniformly.
- The integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as  $\delta$  is called *singular*.
- $D'$  is equipped with the weak\* topology.
- $\frac{\partial}{\partial x} : D' \rightarrow D'$  is continuous. They commute (Schwarz theorem holds).
- $D \rightarrow S \rightarrow L^p$  are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$  and  $H \subset \widehat{\mathbb{C}}$  have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

## 8 Algebraic integer

### 8.1 Quadratic integer

**Theorem 8.1.** *Every quadratic field is of the form  $\mathbb{Q}(\sqrt{d})$  for a square-free  $d$ .*

**Theorem 8.2.** *Let  $d$  be a square-free.*

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z} + \sqrt{d}\mathbb{Z} & , d \equiv 2, 3 \pmod{4} \\ \mathbb{Z} + \frac{1 + \sqrt{d}}{2}\mathbb{Z} & , d \equiv 1 \pmod{4} \end{cases}$$

$$\Delta_{\mathbb{Q}(\sqrt{d})} = \begin{cases} 4d & , d \equiv 2, 3 \pmod{4} \\ d & , d \equiv 1 \pmod{4} \end{cases}$$

**Example 8.1.**

$$\Delta_{\mathbb{Q}(i)} = -4, \quad \Delta_{\mathbb{Q}(\sqrt{2})} = 8, \quad \Delta_{\mathbb{Q}(\gamma)} = 5, \quad \Delta_{\mathbb{Q}(\omega)} = -3$$

where  $\gamma := \frac{1+\sqrt{5}}{2}$  and  $\omega = \zeta_3$ .

**Theorem 8.3.** *Let  $\theta^3 = hk^2$  for  $h, k$  square-free's.*

$$\mathcal{O}_{\mathbb{Q}(\theta)} = \begin{cases} \mathbb{Z} + \theta\mathbb{Z} + \frac{\theta^2}{k}\mathbb{Z} & , m \not\equiv \pm 1 \pmod{9} \\ \mathbb{Z} + \theta\mathbb{Z} + \frac{\theta^2 \pm \theta k + k^2}{3k}\mathbb{Z} & , m \equiv \pm 1 \pmod{9} \end{cases}$$

**Corollary 8.4.** *If  $\theta^3$  is a square free integer, then*

$$\mathcal{O}_{\mathbb{Q}(\theta)} = \mathbb{Z}[\theta].$$

### 8.2 Integral basis

**Theorem 8.5.** *Let  $\alpha \in K$ .  $\text{Tr}_K(\alpha) \in \mathbb{Z}$  if  $\alpha \in \mathcal{O}_K$ .  $N_K(\alpha) \in \mathbb{Z}$  if and only if  $\alpha \in \mathcal{O}_K$ .*

**Theorem 8.6.** *Let  $\{\omega_1, \dots, \omega_n\}$  be a basis of  $K$  over  $\mathbb{Q}$ . If  $\Delta(\omega_1, \dots, \omega_n)$  is square-free, then  $\{\omega_1, \dots, \omega_n\}$  is an integral basis.*

**Theorem 8.7.** *Let  $\{\omega_1, \dots, \omega_n\}$  be a basis of  $K$  over  $\mathbb{Q}$  consisting of algebraic integers. If  $p^2 \mid \Delta$  and it is not an integral basis, then there is a nonzero algebraic integer of the form*

$$\frac{1}{p} \sum_{i=1}^n \lambda_i \omega_i.$$

### 8.3 Fractional ideals

**Theorem 8.8.** *Every fractional ideal of  $K$  is a free  $\mathbb{Z}$ -module with rank  $[K : \mathbb{Q}]$ .*

*Proof.* This theorem holds because  $K/\mathbb{Q}$  is separable and  $\mathbb{Z}$  is a PID.

□

## 8.4 Frobenius element

Consider an abelian extension  $L/K$ . Let  $\mathfrak{p}$  be a prime in  $\mathcal{O}_K$ . Since  $L/K$  is Galois, the followings do not depend on the choice of  $\mathfrak{P}$  over  $\mathfrak{p}$ .

**Lemma 8.9.** *The following sequence of abelian groups is exact:*

$$0 \longrightarrow I(\mathfrak{P}|\mathfrak{p}) \longrightarrow D(\mathfrak{P}|\mathfrak{p}) \longrightarrow \text{Gal}(k(\mathfrak{P})/k(\mathfrak{p})) \longrightarrow 0,$$

where  $k(\mathfrak{P}) := \mathcal{O}_L/\mathfrak{P}$  and  $k(\mathfrak{p}) := \mathcal{O}_K/\mathfrak{p}$  are residue fields.

The Frobenius element is defined as an element of  $D(\mathfrak{P}|\mathfrak{p})/I(\mathfrak{P}|\mathfrak{p}) \cong \text{Gal}(k(\mathfrak{P})/k(\mathfrak{p}))$ , which is a cyclic group.

**Definition 8.1.** For an unramified prime  $\mathfrak{p} \subset \mathcal{O}_K$  so that  $I(\mathfrak{P}|\mathfrak{p})$  is trivial, the Frobenius element  $\phi(\mathfrak{P}|\mathfrak{p}) \in \text{Gal}(L/K)$  is defined by

$$\phi_{\mathfrak{P}|\mathfrak{p}}(\mathfrak{P}) = \mathfrak{P}, \quad \text{and} \quad \phi_{\mathfrak{P}|\mathfrak{p}}(x) \equiv x^{|\mathcal{O}_K/\mathfrak{p}|} \pmod{\mathfrak{P}} \quad \text{for } x \in \mathcal{O}_L.$$

The first condition is equivalent to  $\phi_{\mathfrak{P}|\mathfrak{p}} \in D(\mathfrak{P}|\mathfrak{p})$ . In fact, the Frobenius element is in fact a generator of the cyclic group  $D(\mathfrak{P}|\mathfrak{p}) \cong \text{Gal}(k(\mathfrak{P})/k(\mathfrak{p}))$  by the Galois theory of finite fields.

*Remark.* Fermat's little theorem states

$$\phi_{\mathfrak{P}|\mathfrak{p}}(x) \equiv x \pmod{\mathfrak{p}}, \text{ for } x \in \mathcal{O}_K,$$

which means  $\phi_{\mathfrak{P}|\mathfrak{p}}$  fixes the field  $\mathcal{O}_K/\mathfrak{p}$  so that  $\phi_{\mathfrak{P}|\mathfrak{p}} \in \text{Gal}(k(\mathfrak{P})/k(\mathfrak{p}))$ .

## 8.5 Quadratic Dirichlet character

Let  $K = \mathbb{Q}(\sqrt{D})$  be a quadratic field with discriminant  $D$  and  $L = \mathbb{Q}(\zeta_D)$  be the cyclotomic field for  $\zeta_D = e^{\frac{2\pi i}{D}}$ .

$$\begin{array}{ccc} D(\mathfrak{P}/p) \subset \text{Gal}(L/\mathbb{Q}) & \cong & (\mathbb{Z}/D\mathbb{Z})^\times & L = \mathbb{Q}(\zeta_D) \\ & \downarrow q & \downarrow \chi_K = (\frac{D}{\cdot}) & \\ D(\mathfrak{p}/p) \subset \text{Gal}(K/\mathbb{Q}) & \cong & \{\pm 1\} & K = \mathbb{Q}(\sqrt{D}). \end{array}$$

For  $p \nmid D$  so that  $p$  is unramified, let  $\sigma_p := (\zeta_D \mapsto \zeta_D^p) \in \text{Gal}(L/\mathbb{Q})$ . Then, what is  $\sigma_p|_K$  in  $\text{Gal}(K/\mathbb{Q})$ . In other words, for  $\sigma_p(\zeta_D) = \zeta_D^p$  which is true:  $\sigma_p(\sqrt{D}) = \pm\sqrt{D}$ ?

Note that  $\sigma$  satisfies the condition to be the Frobenius element:  $\sigma_p = \phi_{\mathfrak{P}|p}$ . Therefore,  $q(\phi_{\mathfrak{P}|p}) = \phi_{\mathfrak{p}|p} = \sigma_p|_K$  is also a Frobenius element. There are only two cases:

(1) If  $f = |D(\mathfrak{p}/p)| = 1$ , then  $\sigma|_K$  is the identity, so  $\chi_K(p) = 1$

(2) If  $f = |D(\mathfrak{p}/p)| = 2$ , then  $\sigma|_K$  is not trivial, so  $\chi_K(p) = -1$

Artin reciprocity:  $(\mathbb{Z}/D\mathbb{Z})^\times$  is extended to  $I_K^S$ .

## 9 Diophantine equations

### 9.1 Quadratic equation on a plane

Ellipse is reduced by finitely many computations.

Especially for hyperbola, here is a strategy to use infinite descent.

- (1) Let midpoint to be origin.
- (2) Find the subgroup of  $\text{SL}_2(\mathbb{Z})$  preserving the image of hyperbola (which would be isomorphic to  $\mathbb{Z}$ ).
- (3) Find an impossible region.
- (4) Assume a solution and reduce it to the either impossible region or the ground solution.

**Example 9.1** (Pell's equation). Consider

$$x^2 - 2y^2 = 1.$$

A generator of hyperbola generating group is  $g = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$ . It has a ground solution  $(1, 0)$  and impossible region  $1 < x < 3$ . If  $(a, b)$  is a solution with  $a > 0$ , then we can find  $n$  such that  $g^n(a, b)$  is in the region  $[1, 3]$ . The possible case is  $g^n(a, b) = (1, 0)$ .

**Example 9.2** (IMO 1988, the last problem). Consider a family of equations

$$x^2 + y^2 - kxy - k = 0.$$

By the vieta jumping, a generator is  $g : (a, b) \mapsto (b, kb - a)$ . It has an impossible region  $xy < 0 : x^2 + y^2 - kxy - k \geq x^2 + y^2 > 0$ . If  $(a, b)$  is a solution with  $a > b$ , then we can find  $n$  such that  $g^n(a, b)$  is in the region  $xy \leq 0$ . Only possible case is  $g^n(a, b) = (\sqrt{k}, 0)$  or  $g^n(a, b) = (0, -\sqrt{k})$ . In other words, the equation has a solution iff  $k$  is a perfect square.



## 9.2 The Mordell equations

(The reciprocity laws let us learn not only which prime splits, but also which prime factors a given polynomial has.)

$$y^2 = x^3 + k$$

There are two strategies for the Mordell equations:

- $x^2 - 2x + 4$  has a prime factor of the form  $4k + 3$
- $x^3 = N(y - a)$  for some  $a$ .

First case:  $k = 7, -5, -6, 45, 6, 46, -24, -3, -9, -12$ .

**Example 9.3.** Solve  $y^2 = x^3 + 7$ .

*Proof.* Taking mod 8,  $x$  is odd and  $y$  is even. Consider

$$y^2 + 1 = (x + 2)(x^2 - 2x + 4).$$

Since

$$x^2 - 2x + 4 = (x - 1)^2 + 3,$$

there is a prime  $p \equiv 3 \pmod{4}$  that divides the right hand side. Taking mod  $p$ , we have

$$y^2 \equiv -1 \pmod{p},$$

which is impossible. Therefore, the equation has no solutions.  $\square$

**Example 9.4.** Solve  $y^2 = x^3 - 2$ .

*Proof.* Taking mod 8,  $x$  and  $y$  are odd. Consider a ring of algebraic integers  $\mathbb{Z}[\sqrt{-2}]$ . We have

$$N(y - \sqrt{-2}) = (y - \sqrt{-2})(y + \sqrt{-2}) = x^3.$$

For a common divisor  $\delta$  of  $y \pm \sqrt{-2}$ , we have

$$N(\delta) \mid N((y - \sqrt{-2}) - (y + \sqrt{-2})) = N(2\sqrt{-2}) = |(2\sqrt{-2})(-2\sqrt{-2})| = 8.$$

On the other hand,

$$N(\delta) \mid x^3 \equiv 1 \pmod{2},$$

so  $N(\delta) = 1$  and  $\delta$  is a unit. Thus,  $y \pm \sqrt{-2}$  are relatively prime. Since the units in  $\mathbb{Z}[\sqrt{-2}]$  are  $\pm 1$ , which are cubes,  $y \pm \sqrt{-2}$  are cubics in  $\mathbb{Z}[\sqrt{-2}]$ .

Let

$$y + \sqrt{-2} = (a + b\sqrt{-2})^3 = a(a^2 - 6b^2) + b(3a^2 - 2b^2)\sqrt{-2},$$

so that  $1 = b(3a^2 - 2b^2)$ . We can conclude  $b = \pm 1$ . The possible solutions are  $(x, y) = (3, \pm 5)$ , which are in fact solutions.  $\square$

## 10 The local-global principle

### 10.1 The local fields

Let  $f \in \mathbb{Z}[x]$ .

*Does  $f = 0$  have a solution in  $\mathbb{Z}$ ?*

*Does  $f = 0$  have a solution in  $\mathbb{Z}/(p^n)$  for all  $n$ ?*

*Does  $f = 0$  have a solution in  $\mathbb{Z}_p$ ?*

In the first place, here is the algebraic definition.

**Definition 10.1.** Let  $p \in \mathbb{Z}$  be a prime. The ring of the  $p$ -adic integers  $\mathbb{Z}_p$  is defined by the inverse limit:

$$\mathbb{Z}_p := \varprojlim_{n \in \mathbb{N}} \mathbb{F}_{p^n} \longrightarrow \cdots \longrightarrow \mathbb{Z}/(p^3) \longrightarrow \mathbb{Z}/(p^2) \longrightarrow \mathbb{F}_p.$$

**Definition 10.2.**  $\mathbb{Q}_p = \text{Frac } \mathbb{Z}_p$ .

Secondly, here is the analytic definition.

**Definition 10.3.** Let  $p \in \mathbb{Z}$  be a prime. Define a absolute value  $|\cdot|_p$  on  $\mathbb{Q}$  by  $|p^m a|_p = \frac{1}{p^m}$ . The local field  $\mathbb{Q}_p$  is defined by the completion of  $\mathbb{Q}$  with respect to  $|\cdot|_p$ .

**Definition 10.4.**  $\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ .

**Example 10.1.** Observe

$$\begin{aligned} 3^{-1} &\equiv 2_5 \pmod{5} \\ &\equiv 32_5 \pmod{5^2} \\ &\equiv 132_5 \pmod{5^3} \\ &\equiv 1313132_5 \pmod{5^7} \cdots \end{aligned}$$

Therefore, we can write

$$3^{-1} = \overline{132}_5 = 2 + 3p + p^2 + 3p^3 + p^4 + \cdots$$

for  $p = 5$ . Since there is no negative power of 5,  $3^{-1}$  is a  $p$ -adic integer for  $p = 5$ .

**Example 10.2.**

$$\begin{aligned} 7 &\equiv 1_3^2 \pmod{3} \\ &\equiv 111_3^2 \pmod{3^3} \\ &\equiv 20111_3^2 \pmod{3^5} \\ &\equiv 120020111_3^2 \pmod{3^9} \cdots \end{aligned}$$

Therefore, we can write

$$\sqrt{7} = \cdots 120020111_3 = 1 + p + p^2 + 2p^4 + 2p^7 + p^8 + \cdots$$

for  $p = 3$ . Since there is no negative power of 3,  $\sqrt{7}$  is a  $p$ -adic integer for  $p = 3$ .

There are some pathological and interesting phenomena in local fields. Actually note that the values of  $|\cdot|_p$  are totally disconnected.

**Theorem 10.1.** *The absolute value  $|\cdot|_p$  is nonarchimedean: it satisfies  $|x + y|_p \leq \max\{|x|_p, |y|_p\}$ .*

*Proof.* Trivial. □

**Theorem 10.2.** *Every triangle in  $\mathbb{Q}_p$  is isosceles.*

**Theorem 10.3.**  *$\mathbb{Z}_p$  is a discrete valuation ring: it is local PID.*

*Proof.* asdf □

**Theorem 10.4.**  *$\mathbb{Z}_p$  is open and compact. Hence  $\mathbb{Q}_p$  is locally compact Hausdorff.*

*Proof.*  $\mathbb{Z}_p$  is open clearly. Let us show limit point compactness. Let  $A \subset \mathbb{Z}_p$  be infinite. Since  $\mathbb{Z}_p$  is a finite union of cosets  $p\mathbb{Z}_p$ , there is  $\alpha_0$  such that  $A \cap (\alpha_0 + p\mathbb{Z}_p)$  is infinite. Inductively, since

$$\alpha_n + p^{n+1}\mathbb{Z}_p = \bigcup_{1 \leq x < p} (\alpha_n + xp^{n+1} + p^{n+2}\mathbb{Z}_p),$$

we can choose  $\alpha_{n+1}$  such that  $\alpha_n \equiv \alpha_{n+1} \pmod{p^{n+1}}$  and  $A \cap (\alpha_{n+1} + p^{n+2}\mathbb{Z}_p)$  is infinite. The sequence  $\{\alpha_n\}$  is Cauchy, and the limit is clearly in  $\mathbb{Z}_p$ . □

## 10.2 Hensel's lemma

**Theorem 10.5** (Hensel's lemma). *Let  $f \in \mathbb{Z}_p[x]$ . If  $f$  has a simple solution in  $\mathbb{F}_p$ , then  $f$  has a solution in  $\mathbb{Z}_p$ .*

*Proof.* asdf □

*Remark.* Hensel's lemma says: for  $X$  a scheme over  $\mathbb{Z}_p$ ,  $X$  is smooth iff  $X(\mathbb{Z}_p) \twoheadrightarrow X(\mathbb{F}_p) \dots ???$

**Example 10.3.**  $f(x) = x^p - x$  is factorized linearly in  $\mathbb{Z}_p[x]$ .

## 10.3 Sums of two squares

**Theorem 10.6** (Euler). *A positive integer  $m$  can be written as a sum of two squares if and only if  $v_p(m)$  is even for all primes  $p \equiv 3 \pmod{4}$ .*

**Lemma 10.7.** *Let  $p$  be a prime with  $p \equiv 1 \pmod{4}$ . Every  $p$ -adic integer is a sum of two squares of  $p$ -adic integers.*

## 11 Ultrafilter

**Theorem 11.1.** *Let  $\mathcal{U}$  be an ultrafilter on a set  $S$  and  $X$  be a compact space. For  $f: S \rightarrow X$ , the limit  $\mathcal{U}\text{-}\lim f$  always exists.*

**Theorem 11.2.** *Let  $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$  be a product space of compact spaces  $X_\alpha$ . A net  $\{f_d\}_{d \in \mathcal{D}}$  on  $X$  has a convergent subnet.*

*Proof 1.* Use Tychonoff. Compactness and net compactness are equivalent.  $\square$

*Proof 2.* It is a proof without Tychonoff. Let  $\mathcal{U}$  be an ultrafilter on a set  $\mathcal{D}$  containing all  $\uparrow d$ . Define a directed set  $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$  as  $(d, U) \prec (d', U')$  for  $U \supset U'$ . Let  $f: \mathcal{E} \rightarrow X$  be a net defined by  $f_{(d, U)} = f_d$ .

By the previous theorem,  $\mathcal{U}\text{-}\lim \pi_\alpha f_d$  exists for each  $\alpha$ . Define  $f \in X$  such that  $\pi_\alpha f = \mathcal{U}\text{-}\lim \pi_\alpha f_d$ . Let  $G = \prod_\alpha G_\alpha \subset X$  be any open neighborhood of  $f$  where  $G_\alpha = X_\alpha$  except finite. Then  $G_\alpha$  is an open neighborhood of  $\pi_\alpha f$  so that we have  $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$  by definition of convergence with ultrafilter.<sup>9</sup> Since  $U_\alpha = \mathcal{D}$  except finites, we can take an upper bound  $U_0 \in \mathcal{U}$ . Then, by taking any  $d_0 \in U_0$ , we have  $f_{(d, U)} \in G$  for every  $(d, U) \succ (d_0, U_0)$ . This means  $f = \lim_{\mathcal{E}} f_{(d, U)}$ , so we can say  $\lim_{\mathcal{E}} f_{(d, U)}$  exists.  $\square$

## 12 Universal coefficient theorem

**Lemma 12.1.** *Suppose we have a flat resolution*

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

*Then, we have a exact sequence*

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Tor}_1^R(A, B) \longrightarrow P_1 \otimes B \longrightarrow P_0 \otimes B \longrightarrow A \otimes B \longrightarrow 0.$$

**Theorem 12.2.** *Let  $R$  be a PID. Let  $C_\bullet$  be a chain complex of flat  $R$ -modules and  $G$  be a  $R$ -module. Then, we have a short exact sequence*

$$0 \longrightarrow H_n(C) \otimes G \longrightarrow H_n(C; G) \longrightarrow \mathrm{Tor}(H_{n-1}(C), G) \longrightarrow 0,$$

*which splits, but not naturally.*

*Proof 1.* We have a short exact sequence of chain complexes

$$0 \longrightarrow Z_\bullet \longrightarrow C_\bullet \longrightarrow B_{\bullet-1} \longrightarrow 0$$

where every morphism in  $Z_\bullet$  and  $B_\bullet$  are zero. Since modules in  $B_{\bullet-1}$  are flat, we have a short exact sequence

$$0 \longrightarrow Z_\bullet \otimes G \longrightarrow C_\bullet \otimes G \longrightarrow B_{\bullet-1} \otimes G \longrightarrow 0$$

and the associated long exact sequence

$$\cdots \longrightarrow H_n(B; G) \longrightarrow H_n(Z; G) \longrightarrow H_n(C; G) \longrightarrow H_{n-1}(B; G) \longrightarrow H_{n-1}(Z; G) \longrightarrow \cdots$$

where the connecting homomorphisms are of the form  $(i_n: B_n \rightarrow Z_n) \otimes 1_G$  (It is better to think diagram chasing than a natural construction). Since morphisms in  $B$  and  $Z$  are zero (if it is not, then the short exact sequence of chain complexes are not exact, we have

$$\cdots \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n(C; G) \longrightarrow B_{n-1} \otimes G \longrightarrow Z_{n-1} \otimes G \longrightarrow \cdots$$

Since

$$0 \longrightarrow \mathrm{Tor}_1^R(H_n, G) \longrightarrow B_n \otimes G \longrightarrow Z_n \otimes G \longrightarrow H_n \otimes G \longrightarrow 0$$

for all  $n$ , the exact sequence splits into short exact sequence by images

$$0 \longrightarrow H_n \otimes G \longrightarrow H_n(C; G) \longrightarrow \mathrm{Tor}_1^R(H_{n-1}, G) \longrightarrow 0.$$

For splitting,

□

*Proof 2.* Since  $R$  is PID, we can construct a flat resolution of  $G$

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow G \longrightarrow 0.$$

Since modules in  $C_\bullet$  are flat so that the tensor product functors are exact and  $P_1 \rightarrow P_0$  and  $P_0 \rightarrow G$  induce the chain maps, we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet \otimes P_1 \longrightarrow C_\bullet \otimes P_0 \longrightarrow C_\bullet \otimes G \longrightarrow 0.$$

Then, we have the associated long exact sequence

$$\cdots \longrightarrow H_n(C; P_1) \longrightarrow H_n(C; P_0) \longrightarrow H_n(C; G) \longrightarrow H_{n-1}(C; P_1) \longrightarrow H_{n-1}(C; P_0) \longrightarrow \cdots$$

Since flat tensor product functor commutes with homology functor from chain complexes, we have

$$\cdots \longrightarrow H_n \otimes P_1 \longrightarrow H_n \otimes P_0 \longrightarrow H_n(C; G) \longrightarrow H_{n-1} \otimes P_1 \longrightarrow H_{n-1} \otimes P_0 \longrightarrow \cdots$$

Since

$$0 \longrightarrow \text{Tor}_1^R(G, H_n) \longrightarrow H_n \otimes P_1 \longrightarrow H_n \otimes P_0 \longrightarrow H_n \otimes G \longrightarrow 0$$

for all  $n$ , the exact sequence splits into short exact sequence by images

$$0 \longrightarrow H_n \otimes G \longrightarrow H_n(C; G) \longrightarrow \text{Tor}_1^R(G, H_{n-1}) \longrightarrow 0.$$

□

*Proof 3.* (??) By tensoring  $G$ , we get the following diagram.

$$\begin{array}{ccccc}
H_n \otimes G & & & & H_{n-1} \otimes G \\
& \searrow & & & \nearrow \\
& \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G & \\
& \nearrow & & \nwarrow & \\
C_n \otimes G & & & & C_{n-1} \otimes G \\
& \searrow & & \nearrow & \\
& \text{im } \partial_n \otimes G & & & \\
& \nearrow & & & \\
& \text{Tor}_1(H_{n-1}, G) & & & 
\end{array}$$

Every aligned set of consecutive arrows indicates an exact sequence. Notice that epimorphisms and cokernels are preserved, but monomorphisms and kernels are not. Especially,  $\text{coker } \partial_{n+1} \otimes G = \text{coker}(\partial_{n+1} \otimes 1_G)$  is important.

Consider the following diagram.

$$\begin{array}{ccccc}
 H_n(C; G) & & H_n \otimes G & & \\
 \searrow & & \downarrow & & \\
 & & \text{coker } \partial_{n+1} \otimes G & & \text{ker } \partial_{n-1} \otimes G \\
 & & \downarrow & \nearrow & \uparrow \text{monic!} \\
 & & \text{im } \partial_n \otimes G & & C_{n-1} \otimes G \\
 & \nearrow & \searrow & \nearrow & \\
 \text{Tor}_1(H_{n-1}, G) & & \text{im}(\partial_n \otimes 1_G) & & 
 \end{array}$$

Since  $\ker \partial_{n-1}$  is free,

If we show  $\text{im}(\partial_n \otimes 1_G) \rightarrow \ker \partial_{n-1} \otimes G$  is monic, then we can get

$$\begin{aligned}
 H_n(C; G) &= \ker(\text{coker } \partial_{n+1} \otimes G \rightarrow \text{im}(\partial_n \otimes 1_G)) \\
 &= \ker(\text{coker } \partial_{n+1} \otimes G \rightarrow \ker \partial_{n-1} \otimes G).
 \end{aligned}$$

□

### 13 Analysis problems

**Problem 13.1.** The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

*Solution.* Let  $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$ . Divide the unit circle  $\mathbb{R}/2\pi\mathbb{Z}$  by  $7k$  uniform arcs. There are at least  $2^k/7k$  integers that are not exceed  $2^k$  and are in a same arc. Let  $S$  be the integers and  $x_0$  be the smallest element. Since,  $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$  for  $x \in S$ ,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also,  $1 \leq x - x_0 \leq x \leq 2^k$ ,  $x - x_0 \in A_k$ .

$$|A_k| \geq \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\geq \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\ &\geq \sum_{k=1}^N (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\ &= \sum_{k=1}^N \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\ &> \sum_{k=1}^N \frac{2^k}{2^{k+2}} \frac{1}{7k} \\ &= \frac{1}{28} \sum_{k=1}^N \frac{1}{k} \\ &\rightarrow \infty. \end{aligned}$$

□



**Problem 13.2.** If  $|xf'(x)| \leq M$  and  $\frac{1}{x} \int_0^x f(y) dy \rightarrow L$ , then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .

*Solution.* Since

$$\begin{aligned} \left| f(x) - \frac{F(x) - F(a)}{x - a} \right| &\leq \frac{1}{x - a} \int_a^x |f(x) - f(y)| dy \\ &= \frac{1}{x - a} \int_a^x (x - y) |f'(c)| dy \\ &\leq \frac{M}{x - a} \int_a^x \frac{x - y}{c} dy \\ &\leq M \frac{x - a}{a} \end{aligned}$$

by the mean value theorem and

$$f(x) - L = \left[ f(x) - \frac{F(x) - F(a)}{x - a} \right] + \frac{x}{x - a} \left[ \frac{F(x)}{x} - L \right] + \frac{a}{x - a} \left[ \frac{F(a)}{a} - L \right],$$

we have for any  $\varepsilon > 0$

$$\limsup_{x \rightarrow \infty} |f(x) - L| \leq \varepsilon$$

where  $a$  is defined by  $\frac{x-a}{a} = \frac{\varepsilon}{M}$ . □

**Problem 13.3.** Let  $f_n : I \rightarrow I$  be a sequence of real functions that satisfies  $|f_n(x) - f_n(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$ , where  $I = [0, 1]$ . Then, it has a uniformly convergent subsequence.

*Solution.* By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that  $f_n$  converges to a function  $f : \mathbb{Q} \cap I \rightarrow I$  pointwisely.

*Step [.1]* For  $n \geq 4$ , we claim

$$|x - y| \leq \frac{1}{n} \implies |f_n(x) - f_n(y)| \leq \frac{5}{n}. \quad (1)$$

Fix  $x \in I$  and take  $z \in I$  such that  $|x - z| = \frac{2}{n}$  so that

$$|f_n(x) - f_n(z)| \leq |x - z| = \frac{2}{n}.$$

If  $y$  satisfies  $|x - y| \leq \frac{1}{n}$ , then we have  $|y - z| \geq |x - z| - |x - y| \geq \frac{1}{n}$ , so we get

$$|f_n(y) - f_n(z)| \leq |y - z| \leq |y - x| + |x - z| \leq \frac{3}{n}.$$

Combining these two inequalities proves what we want.

*Step [.2]* For  $\varepsilon > 0$  and  $N := \lceil \frac{15}{\varepsilon} \rceil$  we claim

$$|x - y| \leq \frac{1}{N} \quad \text{and} \quad n > N \implies |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{3} \quad (2)$$

when  $N \geq 4$ . It is allowed for  $|x - y|$  to have the following two cases:

$$|x - y| \leq \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x - y| \leq \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \leq \frac{5}{n} < \frac{5}{N} \leq \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \leq |x - y| \leq \frac{1}{N} \leq \frac{\varepsilon}{15}.$$

Hence the claim is proved.

*Step [.3]* We will prove  $f$  is uniformly continuous. For  $\varepsilon > 0$ , take  $\delta := \frac{1}{N}$ , where  $N := \lceil \frac{15}{\varepsilon} \rceil$ . We will show

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

for  $x, y \in \mathbb{Q} \cap I$  and  $N \geq 4$ . Fix rational numbers  $x$  and  $y$  in  $I$  which satisfy  $|x - y| < \delta$ . Since  $f_n(x)$  and  $f_n(y)$  converges to  $f(x)$  and  $f(y)$  respectively, we may take an integer  $n_x$  and  $n_y$ , such that

$$n > n_x \implies |f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad (3)$$

and

$$n > n_y \implies |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \quad (4)$$

Choose an integer  $n$  such that  $n > \max\{n_x, n_y, N\}$ . Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since  $f$  is continuous on a dense subset  $\mathbb{Q} \cap I$ , it has a unique continuous extension on the whole  $I$ . Let it denoted by the same notation  $f$ .

*Step [4]* Finally, we are going to show  $f_n \rightarrow f$  uniformly. For  $\varepsilon > 0$ , let  $N := \lceil \frac{15}{\varepsilon} \rceil$ . The uniform continuity of  $f$  allows to have  $\delta > 0$  such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{2}{3}\varepsilon. \quad (5)$$

Take a rational  $r \in I$ , depending on  $x \in I$ , such that  $|x - r| < \min\{\frac{1}{N}, \delta\}$ . Then, by (2) and (5), given  $n > N \geq 4$ , we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any  $x \in I$ . By limiting  $n \rightarrow \infty$ , we obtain

$$\lim_{n \rightarrow \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since  $\varepsilon$  and  $x$  are arbitrary, we can deduce the uniform convergence of  $f_n$  as  $n \rightarrow \infty$ .  $\square$

**Problem 13.4.** A measurable subset of  $\mathbb{R}$  with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

*Solution.* Let  $E \subset \mathbb{R}$  be measurable with  $\mu(E) > 0$ . We may assume  $E$  is bounded so that we have  $E \subset I$  for a closed bounded interval since  $\mathbb{R}$  is  $\sigma$ -compact. Let  $n$  be a positive integer arbitrarily taken. Then, we can find  $N$  such that  $\sum_{k=1}^N \frac{1}{k} > (n-1) \frac{\mu(I)}{\mu(E)}$ .

Assume that every point  $x$  in  $E$  is contained in at most  $n-1$  sets among

$$E, \frac{1}{2}E, \frac{1}{3}E, \dots, \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k \in A} \frac{1}{k}E = \emptyset$$

for any subset  $A \subset \{1, \dots, N\}$  with  $|A| \geq n$ . Define

$$E_A := \bigcap_{k \in A} \frac{1}{k}E \cap \bigcap_{k' \in A} \left( \frac{1}{k'}E \right)^c$$

for  $A \subset \{1, \dots, N\}$ . Then,  $\mu(E_A) = 0$  for  $|A| \geq n$ .

Note that we have

$$\mu\left(\frac{1}{k}E\right) = \sum_{k \in A} \mu(E_A) = \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) = \sum_{k=1}^N \sum_{\substack{k \in A \\ |A| < n}} \mu(E_A) = \sum_{|A| < n} |A| \mu(E_A)$$

by double counting, and since  $E_A$  are disjoint, we have

$$\sum_{|A| < n} |A| \mu(E_A) = (n-1) \sum_{0 < |A| < n} \mu(E_A) \leq (n-1) \mu(I),$$

hence a contradiction to

$$\sum_{k=1}^N \mu\left(\frac{1}{k}E\right) > (n-1) \mu(I).$$

Therefore, we may find an element  $x$  that belongs to  $\frac{1}{k}E$  for  $k \in A$ , where  $A \subset \{1, \dots, N\}$  with  $|A| = n$ . Then,  $ax \in E$  for all  $a \in A \subset \mathbb{Z}$ . □

## 14 Bundles

Show that  $S^n$  has a nonvanishing vector field if and only if  $n$  is odd.

*Solution.* Since  $S^n$  is embedded in  $\mathbb{R}^{n+1}$ , the tangent bundle  $TS^n$  can be considered as an embedded manifold in  $S^n \times \mathbb{R}^{n+1}$  which consists of  $(x, v)$  such that  $\langle x, x \rangle = 1$  and  $\langle x, v \rangle = 0$ , where the inner product is the standard one of  $\mathbb{R}^{n+1}$ .

Suppose  $n$  is odd. We have a vector field  $(x_1, x_2, \dots, x_{n+1}; x_2, -x_1, \dots, -x_n)$  which is nonvanishing.

Conversely, suppose we have a nonvanishing vector field  $X$ . Consider a map

$$\phi : S^n \xrightarrow{X} TS^n \rightarrow S^n \times \mathbb{R}^{n+1} \xrightarrow{\phi} \mathbb{R}^{n+1} \rightarrow S^n.$$

The last map can be defined since  $X$  is nowhere zero. Since this map satisfies  $\langle x, \phi(x) \rangle = 0$  for all  $x \in S^n$ , we can define homotopies from  $\phi$  to the identity map and the antipodal map respectively. Therefore, the antipodal map must have positive degree,  $+1$ , so  $n$  is odd.  $\square$

## 15 Some problems

Problems I made:

1. Let  $f$  be  $C^2$  with  $f''(c) \neq 0$ . Define a function  $\xi$  such that  $f(x) - f(c) = f'(\xi(x))(x - c)$  with  $|\xi - c| \leq |x - c|$ , show that  $\xi'(c) = 1/2$ .
2. Let  $f$  be a  $C^2$  function such that  $f(0) = f(1) = 0$ . Show that  $\|f\| \leq \frac{1}{8}\|f''\|$ .
3. Show that a measurable subset of  $\mathbb{R}$  with positive measure contains an arbitrarily long subsequence of an arithmetic progression.
4. Show that there is no continuous bijection from  $[0, 1]^2 \setminus \{p\}$  to  $[0, 1]^2$ .
5. Show that for a nonnegative sequence  $a_n$  if  $\sum a_n$  diverges then  $\sum \frac{a_n}{1+a_n}$  also diverges.
6. Show that if both limits of a function and its derivative exist at infinity then the former is 0.
7. Show that every real sequence has a monotonic subsequence that converges to the limit superior of the supersequence.
8. Show that if a decreasing nonnegative sequence  $a_n$  converges to 0 and satisfies  $S_n \leq 1 + na_n$  then  $S_n$  is bounded by 1.
9. Show that the set of local minima of a convex function is connected.
10. Show that a smooth function such that for each  $x$  there is  $n$  having the  $n$ th derivative vanish is a polynomial.
11. Show that if a continuously differentiable  $f$  satisfies  $f(x) \neq 0$  for  $f'(x) = 0$ , then in a bounded set there are only finite points at which  $f$  vanishes.
12. Let a function  $f$  be differentiable. For  $a < a' < b < b'$  show that there exist  $a < c < b$  and  $a' < c' < b'$  such that  $f(b) - f(a) = f'(c)(b - a)$  and  $f(b') - f(a') = f'(c')(b' - a')$ .
13. Show that if  $xf'(x)$  is bounded and  $x^{-1} \int_0^x f \rightarrow L$  then  $f(x) \rightarrow L$  as  $x \rightarrow \infty$ .
14. Show that if a sequence of real functions  $f_n: [0, 1] \rightarrow [0, 1]$  satisfies  $|f(x) - f(y)| \leq |x - y|$  whenever  $|x - y| \geq \frac{1}{n}$ , then the sequence has a uniformly convergent subsequence.
15. (Flett)

12. Let  $f$  be a differentiable function with  $f(0) = 0$ . Show that there is  $c \in (0, 1)$  such that  $cf(c) = (1 - c)f'(c)$ .
13. Find the value of  $\lim_{n \rightarrow \infty} \frac{1}{n} \left( \sum_{k=1}^n \frac{1}{n} f\left(\frac{k}{n}\right) - \int_0^1 f(x) dx \right)$ .

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14. Let  $f$  be a continuous function. Show that  $f(x)=c$  cannot have exactly two solutions for every  $c$ .
15. Show that a continuous function that takes on no value more than twice takes on some value exactly once.
16. Let  $f$  be a function that has the intermediate value property. Show that if the preimage of every singleton is closed, then  $f$  is continuous.

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17. Show that if a holomorphic function has positive real parts on the open unit disk then  $|f'(0)| < 2 \operatorname{Re} f(0)$ .
18. Show that if at least one coefficient in the power series of a holomorphic function at each point is 0 then the function is a polynomial.
19. Show that if a holomorphic function on a domain containing the closed unit disk is injective on the unit circle then so is on the disk.
20. Show that for a holomorphic function  $f$  and every  $z_0$  in the domain there are  $z_1 \neq z_2$  such that  $\frac{f(z_1)-f(z_2)}{z_1-z_2} = f'(z_0)$ .
21. For two linearly independent entire functions, show that one cannot dominate the other.
22. Show that uniform limit of injective holomorphic function is either constant or injective.
23. Suppose the set of points in a domain  $U \subset \mathbb{C}$  at which a sequence of bounded holomorphic functions  $(f_n)$  converges has a limit point. Show that  $(f_n)$  compactly converges.

---

24. Show that normal nilpotent matrix equals zero.
25. Show that two matrices  $AB$  and  $BA$  have same nonzero eigenvalues whose both multiplicities are coincide blabla...
26. Show that if  $A$  is a square matrix whose characteristic polynomial is minimal then a matrix commuting  $A$  is a polynomial in  $A$ .
27. Show that if two by two integer matrix is a root of unity then its order divides 12.
28. Show that a finite symmetric group has two generators.

29. Show that a nontrivial normalizer of a  $p$ -group meets its center out of identity.
  30. Show that a proper subgroup of a finite  $p$ -group is a proper subgroup of its normalizer. In particular, every finite  $p$ -group is nilpotent.
  31. Show that the complement of a saturated monoid in a commutative ring is a union of prime ideals.
  32. Show that the Galois group of a quintic over  $\mathbb{Q}$  having exactly three real roots is isomorphic to  $S_5$ .
- 
33. Show that if  $A^\circ \in B$  and  $B$  is closed, then  $(A \cup B)^\circ \subset B$ .
- 
34. Show that the tangent space of the unitary group at the identity is identified with the space of skew Hermitian matrices.
  35. Prove the Jacobi formula for matrix.
  36. Show that  $S^3$  and  $T^2$  are parallelizable.
  37. Show that  $\mathbb{R}P^n = S^n/Z_2$  is orientable if and only if  $n$  is odd.