Diachrony of Spectra

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Introduction

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$$\mathrm{Spec}(\mathbb{Z}) = \{ 2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, 7\mathbb{Z}, 11\mathbb{Z}, \cdots \}.$$

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Question

Why is it defined like this?

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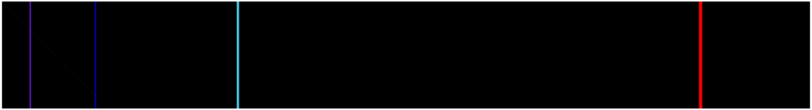
Hydrogen spectral series



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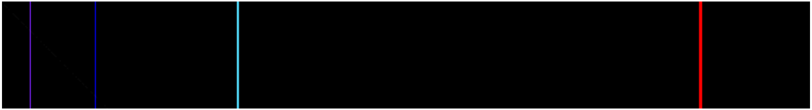

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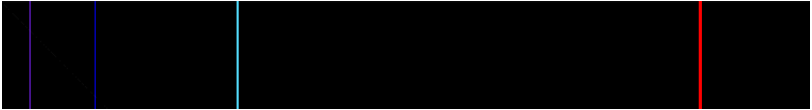

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Question

How can we explain and compute this phenomenon?

Rydberg's formula : Bohr model

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The constant h is called the Planck constant and $\hbar := \frac{h}{2\pi}$.

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From the three relations

$$mvr = n\hbar, \quad \frac{mv^2}{r} = -k\frac{(+e)(-e)}{r^2}, \quad E = K + V = \frac{1}{2}mv^2 - k\frac{e^2}{r},$$

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Proposition (Rydberg formula)

The wavelengths λ of absorbed or emitted photons from a hydrogen atom is estimated by the following formula:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right), \quad \textit{for} \quad n_1, n_2 \in \mathbb{N},$$

where $R := \frac{k^2e^4m}{4\pi\hbar^3c}$ is the Rydberg constant.

Rydberg's formula : Schrödinger equation

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In quantum mechanics, an electron around a hydrogen atom is described by the Schrödinger equation: for $(t, x) \in \mathbb{R}^{1+3}$

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Solving the euqation, we obtain the probability distribution function $|\Psi(t, x)|^2$ of the electron at time t!

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Remark

The first one is not mathematically correct statement because we should resolve some technical issues on convergence.

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$$\mathcal{H} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

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Also, how can we compute them?

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The Beginning of Spectral Theory

Rydberg's formula : Schrödinger equation

By long long calculations, we can obtain the following heuristically:

Proposition

The eigenvalues of $\mathcal{H} = -\Delta - |x|^{-1}$ *is*

Separation of variables

Spectral theorem of normal matrices

Spectral theorem of compact operators

Spectral theorem of elliptic operators

Banach algebras and C*-algebras

Example 1 : Bounded operators

Example 2 : Continuous functions

Spectra, multiplicative homomorphisms, maximal ideals

Gelfand-Naimark theorem

Algebraic variety

Coordinate ring

Maximal ideal is a point

Problem of unified codomains

Functoriality