# The Arzela-Ascoli Theorem for Uniform Spaces

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## 1. Introduction

Compactness gives us a notion of a certian "finiteness" of a topological space. When studying analysis, we often encounter situations where we need to check compactness of subsets. There are a lot of tools that help us to check them: the Heine-Borel theorem, the Tychonoff theorem, and the Banach-Alaoglu theorem are the examples. In the space of continuous functions, it is not easy to draw an eidetic image of compact sets and . How can we describe and check the compactness? The Arzela-Ascoli theorem is the great solution of this question.

The Arzela-Ascoli theorem states that compactness in continuous function space is characterized by a property called "equicontinuity". The purpose of this article is to provide varied viewpoints for the compactness and space of continuous functions via the Arzela-Ascoli theorem, which is one of my favorite theorems. Even though the theorem is mostly applied in the space of real-valued functions on a metric space, we are going to discuss in a more general setting, uniform spaces. By doing so, we can learn deeper and more generalized understanding of the nature of topologies on continuous function spaces.

For preliminaries, we will rediscover the relative compactness for uniform spaces at first. Next, the definitions and properties of topologies on continuous function spaces are going to be introduced. After that, various statements with various proofs and some applications of the theorem will follow.

## 2. Relative compactness

**Definition 2.1.** A uniform space is called *relatively compact* or *precompact* if its completion is compact.

The relative compactness is in particular useful in uniform spaces. We might be able to say the familiar definition of relative compactness has come from the following proposition:

**Proposition 2.1.** *A subset of a complete uniform space, like Banach or Fréchet space, is relatively compact if and only if its closure is compact.*

*Proof.* In a complete uniform space, the completion can be charaterized as closure. □

The following two definitions are helpful when we need to check relative compactness.

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**Definition 2.2.** A Cauchy space is called *relatively Cauchy compact* if every net has a Cauchy subnet. [1]

**Definition 2.3.** A uniform space is called *totally bounded* if for every entourage $E$ there is a finite cover $\{U_i\}_i$ with $U_i \times U_i \subset E$ for each index $i$.

**Theorem 2.2.** *Let $X$ be a uniform space. The followings are all equivalent:*

(1) *$X$ is relatively compact;*
(2) *$X$ is relatively Cauchy compact;*
(3) *$X$ is totally bounded.*

*Proof.* Let $\widetilde{X}$ be the completion of $X$ and $\mathcal{U}$ be the uniquely extended uniformity of $\widetilde{X}$ from $X$.

(1) $\Rightarrow$ (2). Every net in $X$ has a subnet that converges in $\widetilde{X}$. The subnet is Cauchy.

(1) $\Rightarrow$ (3). Let $E \in \mathcal{U}$. Since $\{U : U \text{ is open in } \widetilde{X},\ U \times U \subset E\}$ is an open cover of $\widetilde{X}$, there is a finite subcover $\{U_i\}_i$ of $\widetilde{X}$. Then, $\{U_i \cap X\}_i$ is a finite cover of $X$ satisfying $(U_i \cap X) \times (U_i \cap X) \subset E$.

(2) $\Rightarrow$ (1). If we show $\widetilde{X}$ is also relatively Cauchy compact, we are done. That is because of the fact that compactness and net compactness are always equivalent in all topological spaces.

Let $\widetilde{x} : \mathfrak{A} \to \widetilde{X}$ be a net in $\widetilde{X}$. Take an approximating net $x : \mathfrak{A} \times \mathcal{U} \to X$ in $X$ such that

$$(x_{(\alpha,E)}, \widetilde{x}_\alpha) \in E$$

for all $\alpha \in \mathfrak{A}$ and all entourages $E \in \mathcal{U}$. Recall that $(\alpha, E) \succ (\alpha', E')$ if and only if $\alpha \succ \alpha'$ and $E \subset E'$. By the assumption that $X$ is relatively Cauchy compact, there is a Cauchy subnet

$$xh : \mathfrak{B} \xrightarrow{h} \mathfrak{A} \times \mathcal{U} \to X$$

of $x$. Then,

$$\widetilde{x}\alpha h : \mathfrak{B} \xrightarrow{h} \mathfrak{A} \times \mathcal{U} \xrightarrow{\alpha} \mathfrak{A} \to \widetilde{X}$$

is a subnet of $\widetilde{x}$ since both $h$ and the projection $\alpha$ are monotone and cofinal so that so is their composition. We claim that $\widetilde{x}\alpha h$ is Cauchy.

In order to show this, take an arbitrary entourage $E$. We may assume $E$ is symmetric. By the Cauchyness of $xh$, we can find $\beta_0 \in \mathfrak{B}$ such that

$$\beta, \beta' \succ \beta_0 \quad \Longrightarrow \quad (x_{h(\beta)}, x_{h(\beta')}) \in E.$$

Moreover, we may assume $h(\beta_0) = (\alpha_0, E_0)$ satisfies $(\alpha_0, E_0) \succ (\alpha_0, E)$ so that

$$(x_{h(\beta_0)}, \widetilde{x}_{\alpha h(\beta_0)}) = (x_{(\alpha_0,E_0)}, \widetilde{x}_{\alpha_0}) \in E_0 \subset E.$$

Then, for $\beta, \beta' \succ \beta_0$, the following three relations

$$(x_{h(\beta)}, \widetilde{x}_{\alpha h(\beta)}) \in E_0 \subset E, \quad (x_{h(\beta)}, x_{h(\beta')}) \in E, \quad (x_{h(\beta')}, \widetilde{x}_{\alpha h(\beta')}) \in E_0 \subset E$$

implies $(\widetilde{x}_{\alpha h(\beta)}, \widetilde{x}_{\alpha h(\beta')}) \in E^3$. Therefore, $\widetilde{x}\alpha h : \mathfrak{B} \to \widetilde{X}$ is Cauchy.

[1] There are three well-known different definitions: Willard subnet, Kelly subnet, and Aarnes-Andenæs subnet. However, since the existence of each subnet associated to a common eventuality filter is equivalent, there will be no conflicts among them. In this note, we define a subnet as the monotone cofinal function between the index sets. See Eric schechter's book [].

(3) $\Rightarrow$ (1). Clearly, $\widetilde{X}$ is totally bounded. A complete totally bounded space is compact. $\square$

In the last of the proof above, we used the famous result of complete totally bounded spaces. This can be easily proved with diagonal subsequence extracting argument under the metric space condition, but we need the subtle application of the axiom of choice when we require only uniformness of the space.

**Theorem 2.3.** *A complete totally bounded uniform space is compact.*

*Proof 1.* Let $X$ be a totally bounded complete uniform space. Let $\mathcal{U}$ and $\mathcal{T}$ be the uniformity and topology of $X$ respectively. Let $x : \mathfrak{A} \to X$ be a net in $X$. We are going to show $x$ has a Cauchy subnet.

*Step 1: Applying Zorn's lemma.*

Define a subset $\mathfrak{A}' \subset \mathfrak{A} \times \mathcal{T} \times \mathcal{U}$ by

$$(\alpha, U, E) \in \mathfrak{A}' \quad \Longleftrightarrow \quad x_\alpha \in U,\ U^2 \subset E,\ \text{and } x^{-1}(U) \text{ is cofinal in } \mathfrak{A}.$$

The third condition is a necessary condition for $U$ to contain a limit of a subnet of $x$. Define a subset $Z \subset \mathcal{P}(\mathfrak{A}')$ by

$$\mathfrak{B} \in Z \quad \Longleftrightarrow \quad \pi_{\mathcal{T}}(\mathfrak{B}) \subset \mathcal{T} \text{ is directed.}$$

The image of $\pi_{\mathcal{T}}$ will play a similar role like a "filter". We apply Zorn's lemma on $Z$ to make an "ultrafilter".

First, we claim $Z \neq \varnothing$. For an entourage $E \in \mathcal{U}$, using totally boundedness, we can find a finite open cover $\{U_i\}_i$ of $X$ such that $U_i^2 \subset E$ for all $i$. Since $\bigcup_i x^{-1}(U_i) = x^{-1}(X) = \mathfrak{A}$, there is at least one $i$ such that $x^{-1}(U_i)$ is a cofinal subset of $\mathfrak{A}$. If we choose any $\alpha \in x^{-1}(U_i)$, then the singleton $\{(\alpha, U_i, E)\}$ is an element of $Z$, because singleton can be always said to be directed. (We cannot choose $U = X$ since no entourages may contain $X^2$ without totally boundedness.)

The upper bound of each chain is obtained by union. Therefore, there is a maximal element in $Z$. Let it denoted by $\mathfrak{M}$. Here are several facts about $\mathfrak{M}$:

(1) $\mathfrak{M} \subset \mathfrak{A}'$ inherits the order relation from $\mathfrak{A} \times \mathcal{T} \times \mathcal{U}$,
(2) if $U \in \pi_{\mathcal{T}}(\mathfrak{M})$, then $\alpha \in x^{-1}(U)$ and $U^2 \subset E$ imply $(\alpha, U, E) \in \mathfrak{M}$ by the maximality,
(3) $\mathfrak{M} \in Z$, i.e. $\pi_{\mathcal{T}}(\mathfrak{M})$ is directed,
(4) $\mathfrak{M} \subset \mathfrak{A}'$, i.e. $x^{-1}(U)$ is cofinal for $U \in \pi_{\mathcal{T}}(\mathfrak{M})$.

*Step 2: Verification of Cauchy subnet.*

The goal is to show $x \circ \pi_{\mathfrak{A}} : \mathfrak{M} \to X$ is a subnet that is Cauchy. So, we need to show the three conditions: directedness of $\mathfrak{M}$, monotone cofinality of $\pi_{\mathfrak{A}}|_{\mathfrak{M}}$, and finally Cauchyness.

(directedness) For $(\alpha, U, E), (\alpha', U', E') \in \mathfrak{M}$ we can find $V \in \pi_{\mathcal{T}}(\mathfrak{M})$ such that $U \cup U' \supset V$ by the directedness $\pi_{\mathcal{T}}(\mathfrak{M})$. Since $x^{-1}(V)$ is cofinal, there is $\beta \in x^{-1}(V)$ satisfies $\alpha, \alpha' \succ \beta$. Then, $(\beta, V, E \cap E') \in \mathfrak{M}$ gives an upper bound.

(monotone cofinality) Monotonicity is trivial. Take any $(\alpha, U, E) \in \mathfrak{M}$. For any $\alpha' \in \mathfrak{A}$ there is $(\beta, U, E) \in \mathfrak{M}$ such that $\alpha, \alpha' \succ \beta$ since $x^{-1}(U)$ is cofinal.

(Cauchyness) We claim $\pi_{\mathcal{U}}(\mathfrak{M}) = \mathcal{U}$. Assume $E \notin \pi_{\mathcal{U}}(\mathfrak{M})$. Let $\{V_i\}_i$ be a finite open cover of $X$ such that $V_i^2 \subset E$ for all $i$. Suppose for every $i$ there exists $U_i \in \pi_{\mathcal{T}}(\mathfrak{M})$

such that $x^{-1}(U_i \cap V_i)$ is bounded above, i.e. not cofinal. If we let $U \in \pi_{\mathcal{T}}(\mathfrak{M})$ be an upper bound of $\{U_i\}_i$, then $x^{-1}(U \cap V_i) \subset x^{-1}(U_i \cap V_i)$ is clearly bounded above for all $i$, so

$$\bigcup_i x^{-1}(U \cap V_i) = x^{-1}(U \cap X) = x^{-1}(U)$$

is also bounded above, which gives a contradiction to the cofinality of $x^{-1}(U)$. This implies the existence of an open set $V$ such that $V^2 \subset E$ and $x^{-1}(U \cap V)$ is cofinal for all $U \in \pi_{\mathcal{T}}(\mathfrak{M})$. With this $V$, we can deduce that the cofinality of $x^{-1}(U \cap V)$ lets the following collection

$$\mathfrak{M} \cup \{\, (\alpha, U \cap V, E) : U \in \pi_{\mathcal{T}}(\mathfrak{M}),\ \alpha \in U \cap V,\ (U \cap V)^2 \in E \,\}$$

be a subset of $\mathfrak{A}'$. Furthermore, it is contained in $Z$ as an element because $\{U \cap V\}_{U \in \pi_{\mathcal{T}}(\mathfrak{M})}$ is directed. It is a contradiction to the maximality of $\mathfrak{M}$, therefore, $\pi_{\mathcal{U}}(\mathfrak{M}) = \mathcal{U}$. The Cauchyness follows easily from this. □

*Proof 2.* This proof is by DL Frank, Columbia university, 1965 [ ].

□

Empirically, compactness checking problems seem to be fallen into two cases: one is to show both completeness and relative compactness, and the other is to apply Tychonoff's theorem. Followings are some examples.

**Example 2.4** (The Heine-Borel theorem)**.** A subset of a Euclidean space is compact if and only if it is closed and bounded.

**Example 2.5** (The Blaschke selection theorem)**.** Let $H(X)$ be the metric space of nonempty compact subsets of a metric space $X$ with Hausdorff metric. If $X$ is compact, then $H(X)$ is compact.

**Example 2.6** (The Banach-Alaoglu theorem)**.** Let $X$ be a locally convex space and $X^*$ the continuous dual of $X$ with weak* topology. If $B \subset X^*$ satisfies $\sup_{f \in B} |f(x)| < \infty$ for each $x \in X$, then $B$ is compact.

*Proof.* We can embed $X^*$ into $\mathbb{F}^X$. □

## 3. Topologies on continuous function spaces

## 4. Variations on the Arzela-Ascoli theorem

**Theorem 4.1** (Arzela-Ascoli, conventional version)**.** *Let $X$ be a compact space. For $\{f_n\}_{n \in \mathbb{N}} \subset C(X)$, if it is equicontinuous and pointwisely bounded, then there is a subsequence that uniformly converges.*

Let $\mathcal{T}_p$ be the topology of pointwise convergence and $\mathcal{T}_c$ be the topology of compact convergence.

**Theorem 4.2** (Arzela-Ascoli, metrized version)**.** *Let $X$ be a hemicompact space and $Y$ be a metric space. If $\mathcal{F} \subset C(X, Y)$ is equicontinuous and relatively compact in $\mathcal{T}_p$, then it is relatively compact in $\mathcal{T}_c$.*

*Proof.* Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ and $K \subset X$ be a compact.

By equicontinuity, for each $k \in \mathbb{N}$ a finite open cover $\{U_s\}_{s\in S_k}$ with a finite set $S_k \subset K$ can be taken such that $x \in U_s \Rightarrow d(f(x), f(s)) < \frac{1}{k}$ for all $f \in \mathcal{F}$. By the pointwise relative compactness, we can extract a subsequence $\{f_m\}_{m\in\mathbb{N}}$ of $\{f_n\}_n$ such that $\{f_m(s)\}_m$ is Cauchy for each $s \in \bigcup_{k\in\mathbb{N}} S_k$ by the diagonal argument.

For every $\varepsilon > 0$, let $k = \lceil \varepsilon^{-1} \rceil$ so that $\frac{1}{k} \leq \varepsilon$. Let $m_{0,s}$ be an index such that $m, m' > m_{0,s} \Rightarrow d(f_m(s), f_{m'}(s)) < \varepsilon$, and define $m_0 = \max\{m_{0,s} : s \in S_k\}$. Then, for arbitrary $x \in K$, we obtain $m, m' > m_0 \Rightarrow$

$$d(f_m(x), f_{m'}(x)) \leq d(f_m(x), f_m(s)) + d(f_m(s), f_{m'}(s)) + d(f_{m'}(s), f_{m'}(x)) < 3\varepsilon$$

by taking $s \in S_k$ such that $x \in U_s$. Thus, $\{f_m\}_m$ is a subsequence of $\{f_n\}_n$ that is uniformly Cauchy on $K$. □

If $C(X, Y)$ is not metrizable, or if $Y$ is uniform but not metrizable, the subsequence extracting procedure is no more available.

**Theorem 4.3** (Arzela-Ascoli, generalized version)**.** *Let $X$ be a topological space and $Y$ be a uniform space. If $\mathcal{F} \subset C(X, Y)$ is equicontinuous and relatively compact in $\mathcal{T}_p$, then it is relatively compact in $\mathcal{T}_c$.*

*Proof 1.* Every net in $\mathcal{F}$ has a subnet that is pointwisely Cauchy. Then we are done if we prove every pointwisely Cauchy net in $\mathcal{F}$ is in fact uniformly Cauchy on each compact set $K \subset X$.

Let $f : \mathfrak{A} \to \mathcal{F}$ be a pointwisely Cauchy net. Let $E$ be an arbitrary entourage in $Y$. By equicontinuity, every point $s \in X$ has a neighborhood $U_s$ such that

$$x \in U_s \quad \Longrightarrow \quad (f(x), f(s)) \in E$$

for all $f \in \mathcal{F}$. We can find a finite set $S \subset K$ such that $\{U_s\}_{s\in S}$ is a cover of $X$. (In here, we do not need to extract a subnet because the net is already pointwisely Cauchy.) Let $\alpha_{0,s}$ be an index such that

$$\alpha, \alpha' \succ \alpha_{0,s} \quad \Longrightarrow \quad (f_\alpha(s), f_{\alpha'}(s)) \in E,$$

and define $\alpha_0 = \max\{\alpha_{0,s}\}_{s\in S}$. Then, for arbitrary $x \in K$ there is $s \in S$ such that $x \in U_s$, so we get

$$\alpha, \alpha' \succ \alpha_0 \quad \Longrightarrow \quad (f_\alpha(x), f_{\alpha'}(x)) = (f_\alpha(x), f_\alpha(s)) \circ (f_\alpha(s), f_{\alpha'}(s)) \circ (f_{\alpha'}(s), f_{\alpha'}(x)) \in E^3.$$

Thus, $f$ is uniformly Cauchy on $K$. □

*Proof 2.* □

*Proof 3.* We are going to prove the two topologies coincide in $\mathcal{F}$. Note that subspace topologies $\mathcal{T}_p|_\mathcal{F}$ and $\mathcal{T}_c|_\mathcal{F}$ are also generated by pseudometrics $\rho_{\{x\}}(f, g) = d(f(x), g(x))$ and $\rho_K(f, g) = \sup\{d(f(x), g(x)) : x \in K\}$ defined on $\mathcal{F}$ respectively. Since $\mathcal{T}_p|_\mathcal{F} \subset \mathcal{T}_c|_\mathcal{F}$ clearly, it is enough to show the converse.

Take a subbasis element $B_K(f_0, \varepsilon) \cap \mathcal{F}$ of $\mathcal{T}_c|_\mathcal{F}$ with $f_0 \in \mathcal{F}$. By equicontinuity of $\mathcal{F}$, there is a finite cover $\{U_s\}_{s\in S}$ of a compact subset $K$ such that $x \in U_s \Rightarrow d(f(x), f(s)) < \varepsilon$ for all $f \in \mathcal{F}$.

If $f \in \mathcal{F}$ satisfies $\rho_{\{s\}}(f, f_0) < \varepsilon$ for all $s \in S$, then we get

$$\rho_K(f, f_0) \leq \sup_{x \in K} d(f(x), f(s)) + d(f(s), f_0(s)) + \sup_{x \in K} d(f_0(s), f_0(x)) < \varepsilon$$

by taking $s \in S$ such that $x \in U_s$. This means

$$\bigcap_{s \in S} B_{\{s\}}(f_0, \varepsilon) \subset B_K(f_0, 3\varepsilon).$$

It implies that the subbasis element contains an open set in $\mathcal{T}_p|_{\mathcal{F}}$. Therefore, $\mathcal{T}_c|_{\mathcal{F}} \subset \mathcal{T}_p|_{\mathcal{F}}$. $\square$

## 5. Applications