Finite Group Theory

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Definition 2.1 (Sylow p-subgroup)**.** Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. A *Sylow p-subgroup* is a subgroup of order p^a. We are going to denote the set of Sylow p-subgroups by $\mathrm{Syl}_p(G)$ and the number of Sylow p-subgroups by $n_p(G)$.

Theorem 2.1 (The Sylow theorem)**.** *Let G be a finite group of order $n = p^a m$ for a prime $p \nmid m$. Then,*

$$p \mid n_p - 1, \qquad n_p \mid m$$

for some $k \in \mathbb{N}$.

Proof. Step 1: Sylow p-subgroups exist. We apply mathematical induction. The base step is trivial. Suppose every finite group of order less than n possesses a Sylow p-subgroup.

By applying the orbit-stabilizer theorem for the action $G \curvearrowright G$ by conjugation, build the class equation

$$|G| = |Z(G)| + \sum_i |G : C_G(g_i)|.$$

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There are two cases: $p \mid |Z(G)|$ or $p \nmid |Z(G)|$.

Case 1: $p \mid |Z(G)|$. The group G has a normal subgroup of order p by applying Cauchy's theorem for abelian groups on the center. Then, the inverse image of a Sylow p-subgroup of the quotient group is also a Sylow p-subgroup of G.

Case 2: $p \nmid |Z(G)|$. Since $p \mid n$, we have $p \nmid |G : C_G(g)|$ for some $g \in G$. Then, a Sylow p-subgroup of the centralizer is also a Sylow p-subgroup of G.

Therefore, we are done for Step 1.

Step 2: Sylow p-subgroup that is normal is unique. Note that p does not divide the order of the quotient group. Every p-subgroup should be contained in the Sylow p-subgroup, the kernel of the quotient map. The Sylow p-subgroup is clearly unique.

Step 3: Sylow p-subgroups get action by conjugation. Let P be a Sylow p-subgroup of G. We construct class equations via the orbit-stabilizer theorm for various actions to extract information on n_p. Note that stabilizers in any setwise conjugation action is exactly normalizers.

(1) The action $P \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = 1 + \sum_i |P : N_P(P_i)|$$
since $P = N_P(P_i)$ implies $P \trianglelefteq N_G(P_i)$ and $P = P_i$.

(2) Suppose the action $G \curvearrowright \mathrm{Syl}_p(G)$ is not transitive. Take another Sylow p-subgroup P' is not conjugate with P in G. The two actions $P \curvearrowright \mathrm{Orb}_G(P)$ and $P' \curvearrowright \mathrm{Orb}_G(P)$ gives
$$|\operatorname{Orb}_G(P)| = 1 + \sum_i |P : N_P(P_i)| = \sum_i |P' : N_{P'}(P_i)|.$$
It deduces $|\operatorname{Orb}_G(P)| \equiv 0, 1 \pmod p$ simultaneously, which leas a contradiction.

(3) The action $G \curvearrowright \mathrm{Syl}_p(G)$ gives
$$n_p = |G : N_G(P_i)|$$
for all $P_i \in \mathrm{Syl}_p(G)$ because the action is transitive.

Then, (1) proves $p \mid n_p - 1$, and (3) proves $n_p \mid m$. □

Corollary 2.2. *Let G be a finite group. Then,*

(1) *every pair of two Sylow p-subgroup is conjugate.*
(2) *every p-subgroup is contained in a Sylow p-subgroup.*
(3) *a Sylow p-subgroup is normal if and only if $n_p = 1$.*

Theorem 2.3. *Alternative proof for existence of p-groups.*

Proof. Let $|G| = p^{a+b}m$. Let $\mathcal{P}_{p^a}$ be the set of all p^a-sets in G. Give $G \curvearrowright \mathcal{P}_{p^a}$ by left multiplication. Since $v_p(|\mathcal{P}_{p^a}|) = v_p(\binom{p^a(p^b m)}{p^a}) = b$, there is an orbit $\mathcal{O}$ such that $v_p(|\mathcal{O}|) \le b$. We have transitive action $G \curvearrowright \mathcal{O}$ and the stabilizer H satisfies $p^a \mid |G|/|\mathcal{O}| = |H|$. Since $H \curvearrowright \mathcal{O}$ trivially, $H \curvearrowright A$ for $A \in \mathcal{O} \subset \mathcal{P}_{p^a}$. It is only possible when $H \subset A$, hence $|H| = p^a$. □

What we want to find is subgroup lattices. A subgroup lattice particularly contains data about orders and conjugacy classes of subgroups.

In order to show the existence of subgroups of paricular orders:

(1) p-group theory, (including Cuachy and Sylow)
(2) extension theory, (what can subgroups of subgroups do?)
(3) normalizers,
(4) kernel of permutation representation

In order to find the size of conjugacy classes:

(1) measure the order of normalizers, (find some groups normalize a subgroup)
(2) count elements,

3. Extensions

Proposition 3.1. *Let N and H be groups. Then, the following objects have one-to-one correspondences among each other.*

(1) *isomorphic types of groups G such that a sequence*
$$0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0$$
is exact and right split,
(2) *isomorphic types of groups G such that $N \trianglelefteq G \geq H$ with $G = NH$ and $N \cap H = 1$,*
(3) *group actions $H \curvearrowright N$ preserving the group structure of N.*

Definition 3.1. The group G in the previous proposition is called the *semidirect product* of N and H.

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0.$$

Four data $G, F, \varphi : G \to \mathrm{Aut}(F), c : G \times G \to F$ completely determine the extension E.

Suppose we have an extension $F \to E \to G$. There is a *set-theoretic section* $s : G \to E$. The number of s is $|G||F|$.

Definition of *action* φ: For two sections s and s', $s(g)$ and $s'(g)$ acts on F equivalently. Thus, we can define a *group homomorphism* $\varphi : G \to \mathrm{Aut}(F)$ independently on sections.

Definition of *2-cocycle* c: It is a *set-theoretic function* $c : G \times G \to F$ defined by $c(g, g') = s(g)s(g')s(gg')^{-1}$ for a section s. Actually, c depends on the section s, and c measures how much s fails to be a group homomorphism. It requires the cocycle condition for the associativity of group operation, i.e.
$$c(g, h)c(gh, k) = \varphi_g(c(h, k))c(g, hk)$$
should be satisfied. Conversely, a map $G \times G \to F$ satisfying the condition the cocycle condition gives a associative group operation on G.

If F is abelian, then the set of cocycles forms an abelian group, and is denoted by $Z^2(G, F)$. The boundaries are also defined in abelian F case.

(1) φ, c is trivial $\Leftrightarrow$ direct product,
(2) c is trivial $\Leftrightarrow$ s is a homomorphism $\Leftrightarrow$ semidirect product,
(3) φ is trivial $\Leftrightarrow$ central extension.

Group cohomology is defined for a group G and G-module A (three data: G, A, φ. What is important is that the cohomology depends on the action of G on A.

If φ is trivial so that A is just an abelian group, then the universal coefficient theorem can be applied.