# Functional Analysis I : Topological Vector Space

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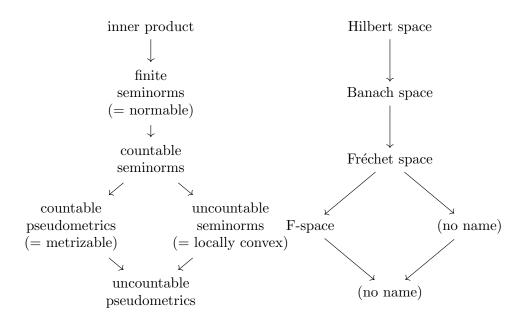
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# Category of topological vector spaces

#### 1. Elementary properties

definition - how to use the continuity of vector space operations effectively homeomorphism by translation and dialation: local base at 0 uniformity pseudometrics, basic classification translation invariant metric completely regular (up to 3.5) boundedness and continuity

#### 2. Classification



Proposition 2.1. Let  $\rho$  be a pseudometric. Then,

$$B(0,1) \subset \frac{B(0,1) + B(0,1)}{2} \subset \frac{1}{2}B(0,2).$$

If  $\rho$  is a seminorm, then the equalities hold.

I say this as  $\frac{1}{2}B(0,2)$  is "fatter" than B(0,1).

### Barreled spaces and bornological spaces

#### 0.1. The Baire category theorem.

#### 0.2. Uniform boundedness principle.

Theorem 0.2 (Uniform boundedness principle). Let X be a barreled space and Y be a topological vector space. Let  $\mathcal{F} \subset B(X,Y)$ . If  $\mathcal{F}$  is pointwise bounded, then  $\mathcal{F}$  is equicontinuous.

#### 0.3. Open mapping theorem.

Theorem 0.3 (Open mapping theorem). Let X be a topological vector space and Y be a metrizable barreled space. Let  $T \colon X \to Y$  be linear. If T is surjective and continuous, then T is open.

PROOF. If we let U be an open neighborhood in X, then we want to show TU is a neighborhood. Because T is surjective so that  $\overline{TU}$  is absorbent,  $\overline{TU}$  is a neighborhood. Note that an open set intersects  $\overline{TU}$  also intersects TU.

If there exist two sequences of balanced open neighborhoods  $U_n \subset X$  and  $V_n \subset Y$  with

- (1)  $U_1 + \cdots + U_n \subset U$ ,
- (2)  $V_n \subset \overline{TU_n}$ ,
- (3)  $\bigcap_{n\in\mathbb{N}} V_n = \{0\},\$

then we can show  $V_1 \subset TU$ . Here is the proof: Suppose  $y \in V_1$ . Then,

$$y \cap V_1 \neq \varnothing \xrightarrow{} y \cap \overline{TU_1} \neq \varnothing \xrightarrow{} (y + V_2) \cap TU_1 \neq \varnothing$$

$$(y + TU_1) \cap V_2 \neq \varnothing \xleftarrow{} (y + TU_1) \cap \overline{TU_2} \neq \varnothing \xrightarrow{} ((y + TU_1) + V_3) \cap TU_2 \neq \varnothing$$

$$(y + TU_1 + TU_2) \cap V_3 \neq \varnothing \xrightarrow{} \cdots$$

From the first columns, and by the conditions (1) and (3), we obtain

$$(y+TU)\cap\bigcap_{n\in\mathbb{N}}V_n\neq\varnothing.$$

Therefore, the set y + TU contains 0, hence  $y \in TU$ .

Let us show the existence of such sequences. At first, take  $U_n = 2^{-n}U$  for (1). Then we can take  $\{V_n\}_n$  with (2) as we mentioned above. Simultaneously we can have it satisfy (3) because Y is metrizable.

Corollary 0.4. Let X be metrizable and Y be barreled. Then, the open mapping theorem holds.

Proof. The quotient of metrizable space is also metrizable, so Y is a metrizable barreled space.  $\Box$ 

COROLLARY 0.5 (The Banach Isomorphy). A continuous linear bijection onto a metrizable barreled space is a homeomorphism, i.e. topological isomorphism.

COROLLARY 0.6 (The first isomorphism theorem). Let  $T: X \to Y$  be a bounded linear operator between Banach spaces. Then, the induced map  $X/\ker T \to \operatorname{im} T$  is a topological isomorphism.

# Locally convex spaces

### 1. Seminorms

minkowski functional locally boundedness polar

### 2. The Hahn-Banach theorem

## 3. Weak topology

### Operators on Banach space

DO NOT contain topics the can be generalized within Banach algebras or any other operator algebras (e.g. polar decomposition, Gelfand theory, functional calculus, spectral resolution)

Theorem 0.1. Let X be complete and Y be complete metrizable. The range of a continuous operator  $T: X \to Y$  is closed if and only if the induced linear isomorphism

$$\frac{X}{\ker T} \to \operatorname{im} T$$

has a continuous inverse so that it becomes a topological isomorphism.

Proof. One direction is easy.

For the other direction, suppose im T is closed in Y. Note that the metrizability condition of Y is set in order to apply the open mapping theorem.

COROLLARY 0.2. Let  $T: X \to Y$  be a bounded operator between Banach spaces. Then, T is bounded below if and only if  $\operatorname{im} T$  is closed and T is injective.

#### 1. Spectral theory

When a Banach algebra is realized as a concrete operator space, then the spectral theory on it changes drastically. For example we can categorize three cases for a linear operator between Banach spaces to fail the invertibility:

- (1) it is not injective; (point spectrum)
- (2) it is injective, its range is not dense; (residual spectrum)
- (3) it is injective, its range is dense, but not closed; (continuous spectrum)

## 2. Compact operators

## 3. Unbounded operators

## 4. Nuclear operators

## 5. Fredholm theory

## 6. Perturbation theory