Compact sets

1. Let $X \subset \mathbb{R}^d$. Show that if X is bounded then every sequence in X has a convergent subsequence. (Bolzano-Weierstrass)

2. Let $X \subset \mathbb{R}^d$. Show that if every sequence in X has a convergent subsequence, then X is closed and bounded.

3. Let $X \subset \mathbb{R}^d$ be compact. Suppose an infinite set $\mathcal{C} \subset \mathcal{P}(X)$ only contains closed subsets of X. Show that if $\bigcap_{C \in A} C$ is nonempty for all finite subset $A \subset \mathcal{C}$, then $\bigcap_{C \in \mathcal{C}} C$ is nonempty.

Continuous functions

1. Let X be a set. Let $f_n : X \to \mathbb{R}$ be a sequence of functions. Show that f_n converges to $f : X \to \mathbb{R}$ uniformly if and only if $\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$.

2. Let $X \subset \mathbb{R}^d$. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. Show that if f_n converges to $f : X \to \mathbb{R}$ uniformly, then f is also continuous. (In other words, the set of real-valued continuous functions $C(X)$ is always closed under the topology of uniform convergence.)

3. Let $X \subset \mathbb{R}^d$ be compact. Show that is $f : X \to \mathbb{R}$ is continuous then it is uniformly continuous.

4. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions. Show that if $f_n \to f$ pointwisely and $f_n' \to g$ uniformly, then $g = f'$.

Measures

Let X be a set and $\mathcal{F}$ be a σ-algebra on X. A *measure* on $\mathcal{F}$ is a function $\mu : \mathcal{F} \to [0, \infty]$ such that

- $\mu(\varnothing) = 0$,
- $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for a sequence of disjoint sets $E_i \in \mathcal{F}$. (countable-additivity)

We call an element in $\mathcal{F}$ *measurable* (when we are known $\mathcal{F}$).

1. Show that if E_i is a monotonically increasing sequence of measurable subsets, then $\mu(\bigcup_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$. (Continuity from below)
2. Show that if E_i is a monotonically decreasing sequence of measurable subsets, then $\mu(\bigcap_{i=1}^{\infty} E_i) = \lim_{i\to\infty} \mu(E_i)$ when given $\mu(E_1) < \infty$. (Continuity from above)
3. Show that there is no measure μ defined on the entire power set $\mathcal{P}(\mathbb{R})$ such that $\mu([a, b]) = b - a$ and $\mu(x + E) = \mu(E)$ for $x \in \mathbb{R}$, $E \subset \mathbb{R}$. (Hint: Define an equivalence relation on $\mathbb{R}$ such that $x \sim y$ iff $x - y \in \mathbb{Q}$. Take $N \subset [0, 1)$ such that N contains precisely one member of each equivalence class. Show $1 \leq \sum_{r\in\mathbb{Q}\cap[0,1)} \mu(N) \leq 3$ to lead a contradiction.)

Measurable functions

Let X be a set. A σ-algebra $\mathcal{F}$ on X is also called a *measurable structure* and X with $\mathcal{F}$ is called a *measurable space*. A function $f : X \to Y$ between measurable spaces is called *measurable* if the measurability of $E \subset Y$ implies the measurability of $f^{-1}(E)$.

On $\mathbb{R}$, the smallest σ-algebra containing open sets is called *Borel σ-algebra* and its elements are called *Borel sets*. We will denote it by $\mathcal{B}(\mathbb{R})$. For a function $f : X \to \mathbb{R}$ where X is a measurable space, we call f just measurable if $f^{-1}(E)$ is measurable for all Borel sets E.

1. Let X be a measurable space. Show that if $f, g : X \to \mathbb{R}$ is measurable, then $f + g$, $|f|$, f^2, and fg are all measurable.
2. Let X be a measurable space and f_n be a sequence of bounded measurable functions. Show that $g = \sup_n f_n$ and $h = \limsup_n f_n$ are measurable.
3. Let X be a measurable space. Show that if $f : X \to \mathbb{R}$ is measurable, then $|f|$ is measurable.