

# Functional Analysis II : Operator Algebra

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## CHAPTER 1

# **Banach algebras**

## 1. Spectra

**2. Gelfand theory**

THEOREM 2.1. *If  $x, y$  commutes, then  $\sigma(xy) \subset \sigma(x)\sigma(y)$ .*





## CHAPTER 2

# Operators on Hilbert spaces

**1. Spectral theory**

## CHAPTER 3

### $C^*$ -algebras

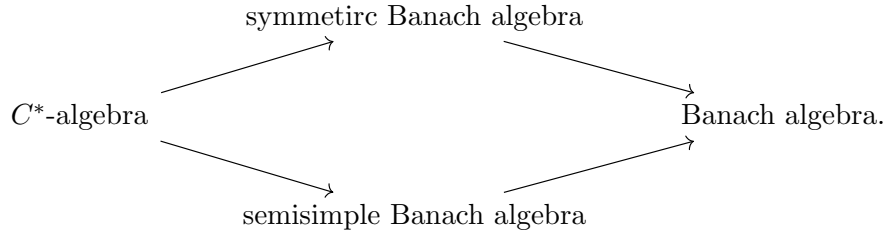
## 1. Basics

## 2. The Gelfand-Naimark theorems

**2.1. Commutative Banach algebras.** The Gelfand representation  $A \rightarrow C_0(\widehat{A})$  can be defined for commutative Banach algebras.

DEFINITION 2.1. A *symmetric* Banach algebra is an involutive Banach algebra for which the Gelfand representation preserves the involution. We will not consider non-symmetric involutive Banach algebras in this section.

Notice the following implication:



Let  $A$  be a commutative Banach algebra.

THEOREM 2.1. *If  $A$  is semisimple, then the Gelfand representation is a monomorphism; it is injective.*

PROOF. It is because the kernel is given by the Jacobson radical.  $\square$

THEOREM 2.2. *If  $A$  is symmetric, then the Gelfand representation is an epimorphism; it has a dense range.*

PROOF. The image is closed under all operations except involution, separates points, and vanishes nowhere. If  $A$  is symmetric, then the image is closed under involution. Thus, by the Stone-Weierstrass theorem, we get the result.  $\square$

$C^*$ -algebras are semisimple and symmetric (even if it is noncommutative).

THEOREM 2.3. *A  $C^*$ -algebra is semisimple.*

THEOREM 2.4. *A  $C^*$ -algebra is symmetric.*

PROOF 1. It is by Arens.  $\square$

PROOF 2. It is by Fukamiya.  $\square$

Furthermore,

THEOREM 2.5. *If  $A$  is a commutative  $C^*$ -algebra, then the Gelfand representation is isometric.*

Since an isometry is injective and has a closed range, therefore, it should be isometric  $^*$ -isomorphism.



## CHAPTER 4

# Von Neumann algebras