Contents

1 Kinetic theory

1.1 Vlasov-Poisson equation

Consider a Cauchy problem of the *Valsov-Poisson system*:

$$\begin{cases} f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, \qquad (t,x,v) \in \mathbb{R}_t^+ \times \mathbb{R}_x^3 \times \mathbb{R}_v^3, \\ E(t,x) = -\nabla_x \Delta_x^{-1} \rho, \\ \rho(t,x) = \int f \, dv, \\ f(0,x,v) = f_0(x,v), \end{cases}$$

where $\gamma = \pm 1$ denotes the charge of particles we are concerned with. For example, $\gamma = -1$ for electrons in plasma and $\gamma = +1$ for galaxies. For the boundaryless problem in which the conservative potential function vanishes at infinity, we have

$$E = \nabla_x \left(\tfrac{1}{4\pi|x|} * \gamma\rho\right) = -\frac{x}{4\pi|x|^3} * \rho$$

for $\gamma = -1$. (ρ denotes mass density.)

Lemma 1.1.

$$\|\rho(t)\|_{L_x^{5/3}} \lesssim 1.$$

Proof.

$$\begin{aligned} \rho(t,x) = \int f(t,x,v) \, dv &\le \int_{|v|<R} f \, dv + \frac{1}{R^2} \int_{|v|\ge R} |v|^2 f \, dv \\ &\lesssim R^3 + R^{-2} \int |v|^2 f \, dv. \end{aligned}$$

Set $R^3 = R^{-2} \int |v|^2 f \, dv$ to get

$$\rho(t,x)^{5/3} \lesssim \int |v|^2 f \, dv.$$

Take $N = 3$, $p = 2$, and $\lambda = 2$. Then,

$$0 < \frac{1}{p} = \frac{1}{q} + \frac{\lambda}{N} - 1$$

implies $q = 6/5$, so we can bound $\|E(t)\|_2$ by interpolation of $\|\rho(t)\|_{6/5}$ and $\|\rho(t)\|_1$:

$$\|E(t)\|_{L_x^2} \lesssim \|\frac{1}{|x|^2} *_x \rho(t,x)\|_{L_x^2} \lesssim \|\rho(t)\|_{6/5} \le \|\rho\|_1^{7/12} \|\rho\|_{5/3}^{5/12}.$$

Thus

$$\|E(t)\|_2 \lesssim \|\rho(t)\|_{5/3}^{5/12} \lesssim \left(\iint |v|^2 f \, dv \, dx\right)^{1/4}.$$

It means $(\iint |v|^2 f \, dv \, dx)^{1/2}$ bounds $(\iint |v|^2 f \, dv \, dx)$, hence the total kinetic energy of the system remains bounded in any time even if $\gamma = +1$. As a corollary, $\|\rho\|_{5/3}$ is also bounded. $\square$

Lemma 1.2. *For* $1 \le q < \frac{N}{N-2} = 3 < p \le \infty$,

$$\|E(t,x)\|_{L^\infty_x} \lesssim \|\rho(t,x)\|_{L^p_x}^{\frac{\frac{2}{N}-1+\frac{1}{q}}{\frac{1}{q}-\frac{1}{p}}} \|\rho(t,x)\|_{L^q_x}^{\frac{1-\frac{1}{p}-\frac{2}{N}}{\frac{1}{q}-\frac{1}{p}}}.$$

Proof. Fix time t. For $2p < N < 2q$,

$$\begin{aligned}
4\pi|E(t,x)| &= |\tfrac{1}{|x|^2} *_x \rho(t,x)| \\
&\le \int_{|x-y|<R} \frac{\rho(t,y)}{|x-y|^2}\,dy + \int_{|x-y|\ge R} \frac{\rho(t,y)}{|x-y|^2}\,dy \\
&\le \|\rho\|_{p'}\Big(\int_{|y|<R} \frac{dy}{|y|^{2p}}\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_{|y|\ge R} \frac{dy}{|y|^{2q}}\Big)^{1/q} \\
&\simeq \|\rho\|_{p'}\Big(\int_0^R r^{N-1-2p}\,dr\Big)^{1/p} + \|\rho\|_{q'}\Big(\int_R^\infty r^{N-1-2q}\,dr\Big)^{1/q} \\
&\simeq \|\rho\|_{p'} R^{\frac{N}{p}-2} + \|\rho\|_{q'} R^{\frac{N}{q}-2}.
\end{aligned}$$

By choosing R such that $\|\rho\|_{p'} R^{\frac{N}{p}-2} = \|\rho\|_{q'} R^{\frac{N}{q}-2}$, we get

$$\|E(t,x)\|_{L^\infty_x} \lesssim \|\rho(t,x)\|_{L^{p'}_x}^{\frac{\frac{2}{N}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{q}}} \|\rho(t,x)\|_{L^{q'}_x}^{\frac{\frac{1}{p}-\frac{2}{N}}{\frac{1}{p}-\frac{1}{q}}},$$

hence the inequality by interchaning p and q with their conjugates. $\square$

1.2 Schaeffer's global existence proof

Theorem (Schaeffer, 1991)**.** *Let $f_0 \in C^1_{c,x,v}$ and $f_0 \geq 0$. Then, the Cauchy problem for the VP system has a unique C^1 global solution.*

Definition 1.1. For a local solution f,

$$Q(t) := 1 + \sup\{|v| : f(s,x,v) \neq 0 \text{ for some } s \in [0,t],\ x \in \mathbb{R}^3_x\}.$$

Lemma 1.3.

$$\|E(t)\|_{L^\infty_x} \lesssim Q(t)^{4/3}.$$

Proof. If $p = \infty$ and $N = 3$, then $1 \leq q < 3$ on the previous lemma implies

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{1-q/3} \|\rho\|_q^{q/3}.$$

For example, if we let $(p,q) = (\infty, 1)$ or $(\infty, 5/3)$, then

$$\|E\|_\infty \lesssim \|\rho\|_\infty^{2/3} \|\rho\|_1^{1/3}, \qquad \|E\|_\infty \lesssim \|\rho\|_\infty^{4/9} \|\rho\|_{5/3}^{5/9}.$$

These are important since $\|\rho\|_1$ and $\|\rho\|_{5/3}$ remain bounded uniformly on time.

Since the velocity support of f is bounded by finite $Q(t)$, we have

$$\rho(t,x) = \int_{|v|<Q(t)} f(t,x,v)\, dv \lesssim Q(t)^3 \|f_0(x,v)\|_{L^\infty_v} \lesssim Q(t)^3,$$

so letting $q = \frac{5}{3}$,

$$\|E(t)\|_{L^\infty_x} \lesssim \|\rho(t)\|_{L^\infty_x}^{4/9} \lesssim Q(t)^{4/3}. \qquad \square$$

Decompose $[t-\Delta, t] \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ as

$$\begin{aligned}
U &= \Big\{ (s,x,v) :\ |v| \geq P, \quad |v - \widehat{V}(t)| \geq P, \quad |y - \widehat{X}(s)| \geq r \Big\}, \\
B &= \Big\{ (s,x,v) :\ |v| \geq P, \quad |v - \widehat{V}(t)| \geq P \Big\} \setminus U, \\
G &= \{ (s,x,v) :\ |v| < P \quad \text{or} \quad |v - \widehat{V}(t)| < P \} = (B \cup U)^c.
\end{aligned}$$

Later we choose $P = Q^{4/11}$, $r = R \max\{|v|^{-3}, |v - \widehat{V}(t)|^{-3}\}$, and $R = Q^{16/33} \log^{1/2} Q$. Also, later we choose $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$. The condition $|v| \geq P$ for U is not redundant.

1.2.1 Ugly set

The following observation suggests a lower bound of relative velocity.

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$|y - \widehat{X}(s)| \geq \frac{1}{4}|v - \widehat{V}(t)||s - s_0|$$

for some $s_0 \in [t - \Delta, t]$, where $\Delta \cdot \sup_{s \leq t} \|E(s)\|_\infty < \frac{P}{4}$.

Proof. Since $\Delta \|E(s)\|_\infty < \frac{P}{4}$, we have

$$|v - w| < \frac{P}{4} \quad \text{and} \quad |\widehat{V}(t) - \widehat{V}(s)| < \frac{P}{4}.$$

The condition $|v - \widehat{V}(t)| \geq P$ implies

$$\frac{1}{2}|v - \widehat{V}(t)| \leq |v - \widehat{V}(t)| - \frac{P}{4} - \frac{P}{4} < |w - \widehat{V}(s)|.$$

Let $Z(s) := y - \widehat{X}(s)$. Then,

$$\begin{aligned} Z'(s) &= w - \widehat{V}(s), \\ Z''(s) &= \gamma[E(s, y, w) - E(s, \widehat{X}(s), \widehat{V}(s))]. \end{aligned}$$

Let $s_0 \in [t - \Delta, t]$ minimize $s \mapsto |Z(s)|$ and expand Z as

$$Z(s) = Z(s_0) + Z'(s_0)(s - s_0) + \frac{Z''(\sigma)}{2}(s - s_0)^2$$

for some σ between s and s_0. Then,

$$|Z(s_0) + Z'(s_0)(s - s_0)| \geq |Z'(s_0)(s - s_0)| \geq \frac{1}{2}|v - \widehat{V}(t)||s - s_0|$$

and

$$\begin{aligned} |\frac{Z''(\sigma)}{2}(s - s_0)^2| \leq \|E(t)\|_\infty (s - s_0)^2 &\leq \|E(t)\|_\infty \Delta |s - s_0| \\ &\leq \frac{P}{4}|s - s_0| \leq \frac{1}{4}|v - \widehat{V}(t)||s - s_0| \end{aligned}$$

proves

$$|y - \widehat{X}(s)| = |Z(s)| \geq \frac{1}{4}|v - \widehat{V}(t)||s - s_0|.$$

□

Claim. *Fix t, x, v. If $|v - \widehat{V}(t)| \geq P$, then*

$$\int_{t-\Delta}^{t} \frac{1}{|y - \widehat{X}(s)|^2} \chi_A(s) \, ds \lesssim \frac{r^{-1}}{|v - \widehat{V}(t)|},$$

where $A = \{s : |y - \widehat{X}(s)| \geq r\}$.

Proof. Since $|y-\widehat{X}(s)| \geq \frac{1}{4}|v-\widehat{V}(t)||s-s_0|$,

$$\begin{aligned}\int_{t-\Delta}^{t} \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds &\leq 16\int_{t-\Delta}^{t} \frac{1}{|v-\widehat{V}(t)|^2|s-s_0|^2}\chi_A(s)\,ds \\ &\leq 32\int_{r}^{\infty} \frac{1}{|v-\widehat{V}(t)|^3|s-s_0|^2}\,d(|v-\widehat{V}(t)||s-s_0|) \\ &= 32\,\frac{r^{-1}}{|v-\widehat{V}(t)|}.\end{aligned}$$

□

Therefore, if we let $r^{-1} \simeq \min\{|v|^3, |v-\widehat{V}(t)|^3\}$, then

$$\int_{t-\Delta}^{t} \frac{1}{|y-\widehat{X}(s)|^2}\chi_A(s)\,ds \lesssim |v|^2$$

so that we have

$$\iiint_U \frac{f(s,y,w)}{|y-\widehat{X}(s)|^2}\,dw\,dy\,ds \lesssim R^{-1}\int |v|^2 f(t,x,v)\,dv\,dx \lesssim R^{-1}$$

when

$$U \subset \{\,(s,x,v):\ |v-\widehat{V}(t)| \geq P, \quad |y-\widehat{X}(s)| \geq R\max\{|v|^{-3}, |v-\widehat{V}(t)|^{-3}\}\,\}.$$

1.2.2 Bad set

Consider U^c. We need to control the union of two regions

$$|y-\widehat{X}(s)| < R|v|^{-3} \quad \text{and} \quad |y-\widehat{X}(s)| < R|v-\widehat{V}(t)|^{-3}.$$

Without any conditions, the integration of fundamental solution with respect to y gives

$$\int_{|y-\widehat{X}(s)|<r} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \simeq r.$$

Claim. *If $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$, then*

$$\int_{U^c} \frac{1}{|y-\widehat{X}(s)|^2}\,dy \lesssim \max\{|w|^{-3}, |w-\widehat{V}(s)|^{-3}\}$$

for $s \in [t-\Delta, t]$.

Proof. It follows from

$$|w| \simeq |v|, \quad |w-\widehat{V}(s)| \simeq |v-\widehat{V}(t)|$$

for $|v| \geq P$ and $|v-\widehat{V}(t)| \geq P$. □

1.3 Velocity averaging lemmas

The velocity averaging lemma is used to get regularity of averaged quantity when boundary condition is not given.

Theorem 1.4 (Velocity averaging)**.** *Let L be a free transport operator $\partial_t + v \cdot \nabla_x$ on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. Then,*

$$\|\int u\varphi\, dv\|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|^{1/2}_{L^2_{t,x,v}} \|Lu\|^{1/2}_{L^2_{t,x,v}}$$

for $\varphi \in C^\infty_c(\mathbb{R}^n_v)$,

Proof. Let $m(t,x) = \int u\varphi\, dv$. By Fourier transform with respect to t and x, we have

$$\widehat{u}(\tau, \xi, v) = \frac{1}{i} \frac{\widehat{Lu}(\tau, \xi, v)}{\tau + v \cdot \xi}$$

and

$$\widehat{m}(\tau, \xi) = \int \widehat{u}(\tau, \xi, v)\varphi(v)\, dv.$$

Fixing τ, ξ, decompose the integral and use Hölder's inequality to get

$$\begin{aligned}
|\widehat{m}(\tau, \xi)| &\le \int_{|\tau + v\cdot\xi| < \alpha} |\widehat{u}\varphi|\, dv + \int_{|\tau + v\cdot\xi| \ge \alpha} \frac{|\widehat{Lu}\varphi|}{|\tau + v \cdot \xi|}\, dv \\
&\le \|\widehat{u}\|^{1/2}_{L^2_v} (\int_{|\tau + v\cdot\xi| < \alpha} |\varphi|^2\, dv)^{1/2} + \|\widehat{Lu}\|^{1/2}_{L^2_v} (\int_{|\tau + v\cdot\xi| \ge \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|^2}\, dv)^{1/2},
\end{aligned}$$

where $\alpha > 0$ is an arbitrary constant that will be determined later. Let

$$I_s(\tau, \xi, \alpha) := \int_{|\tau + v\cdot\xi| < \alpha} |\varphi|^2\, dv, \qquad I_n(\tau, \xi, \alpha) := \int_{|\tau + v\cdot\xi| \ge \alpha} \frac{|\varphi|^2}{|\tau + v \cdot \xi|}\, dv.$$

We are going to estimate the integrals as

$$I_s \lesssim \frac{\alpha}{\sqrt{\tau^2 + |\xi|^2}}, \qquad I_n \lesssim \frac{1}{\alpha\sqrt{\tau^2 + |\xi|^2}}.$$

Define coordinates (v_1, v_2) on $\mathbb{R}_v$ as follows:

$$v_1 := \frac{\tau + v \cdot \xi}{|\xi|} \in \mathbb{R}\ , \qquad v_2 := v - \frac{v \cdot \xi}{|\xi|^2}\xi \ \in \ker(\xi^T) \cong \mathbb{R}^{n-1}.$$

Note that

$$|v|^2 = (v_1 - \frac{\tau}{|\xi|})^2 + |v_2|^2 \qquad \text{and} \qquad \int dv = \iint dv_2\, dv_1.$$

For the first integral, suppose that φ is supported on a ball $|v| \leq R$. If $\frac{|\tau|-\alpha}{|\xi|} > R$, then the region of integration vanishes so that $I_s = 0$. If $|\tau| \leq \alpha + R|\xi|$, then

$$
\begin{aligned}
I_s &\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|}} \int_{|v_2|^2 \leq R^2-(v_1-\frac{\tau}{|\xi|})^2} dv_2\, dv_1 \\
&\lesssim \int_{|v_1|<\frac{\alpha}{|\xi|},\ |v_1|\leq R} \int_{|v_2|\leq R} dv_2\, dv_1 \\
&\lesssim \min\{\frac{2\alpha}{|\xi|}, R\} \cdot R^{n-1} \\
&\lesssim \frac{1}{\sqrt{1+(\frac{|\xi|}{\alpha})^2}} \\
&\lesssim \frac{\alpha}{\sqrt{\tau^2+|\xi|^2}}.
\end{aligned}
$$

For the second integral, suppose that φ is supported on $|v| < C$ so that $|v_1 - \frac{\tau}{|\xi|}|, |v_2| < C$. Then,

$$
\begin{aligned}
I_n &\lesssim \int_{|v_1|\geq \frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \int_{|v_2|<C} \frac{1}{v_1^2|\xi|^2} dv_2\, dv_1 \\
&\simeq \int_{|v_1|\geq \frac{\alpha}{|\xi|},\ |v_1-\frac{\tau}{|\xi|}|<C} \frac{dv_1}{v_1^2|\xi|^2}.
\end{aligned}
$$

If $|\xi| \gtrsim |\tau|$, then

$$
I_n \lesssim \int_{|v_1|\geq \frac{\alpha}{|\xi|}} \frac{dv_1}{v_1^2|\xi|^2} \simeq \frac{1}{\alpha|\xi|} \lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}.
$$

If $|\xi| \ll |\tau|$ such that at least $|\tau| > C|\xi|$, then

$$
\begin{aligned}
I_n &\lesssim \int_{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-C\} \leq v_1 < \frac{|\tau|}{|\xi|}+C} \frac{dv_1}{v_1^2|\xi|^2} \\
&\simeq \frac{1}{|\xi|^2}\left(\frac{1}{\max\{\frac{\alpha}{|\xi|}, \frac{|\tau|}{|\xi|}-C\}} - \frac{1}{\frac{|\tau|}{|\xi|}+C}\right) \\
&\lesssim \frac{1}{\alpha\sqrt{\tau^2+|\xi|^2}}. \quad \text{(This is not easy..!)}
\end{aligned}
$$

To sum up, we have

$$
|\widehat{m}(\tau,\xi)| \lesssim \frac{1}{(\tau^2+|\xi|^2)^{1/4}} (\sqrt{\alpha} \cdot \|\widehat{u}\|_{L_v^2}^{1/2} + \frac{1}{\sqrt{\alpha}} \cdot \|\widehat{Lu}\|_{L_v^2}^{1/2}).
$$

Letting $\alpha = \sqrt{\|\widehat{Lu}\|_{L^2_v} / \|\widehat{u}\|_{L^2_v}}$ and squaring,

$$(\tau^2 + |\xi|^2)^{1/2} |\widehat{m}(\tau, \xi)|^2 \lesssim \|\widehat{u}\|_{L^2_v}^{1/2} \|\widehat{Lu}\|_{L^2_v}^{1/2}.$$

Therefore, the integration on $\mathbb{R}_\tau \times \mathbb{R}^n_\xi$ and Plancheral's theorem gives

$$\|m\|_{H^{1/2}_{t,x}} \lesssim_\varphi \|u\|_{L^2_{t,x,v}}^{1/2} \|Lu\|_{L^2_{t,x,v}}^{1/2}.$$

$\square$

Remark. We can obtain L^p result by applying

$$\min\{\frac{1}{x}, \frac{1}{y}\} \leq \frac{2}{(x^p + y^p)^{1/p}} \leq \max\{\frac{1}{x}, \frac{1}{y}\}$$

on the first integral, and

$$x \gg 1 \quad \Longrightarrow \quad \frac{1}{\max\{a, x - c\}} - \frac{1}{x + c} \lesssim \frac{1}{a(x^p + 1)^{1/p}}$$

on the second integral.

Corollary 1.5. *Let $\mathcal{F}$ be a family of functions on $\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ and $L\mathcal{F}$ are bounded in $L^2_{t,x,v}$, then $\int \mathcal{F}\varphi \, dv$ is bounded in $H^{1/2}_{t,x}$.*

Theorem 1.6. *Let $\mathcal{F}$ be a family of functions on $I_t \times \mathbb{R}^n_x \times \mathbb{R}^n_v$. If $\mathcal{F}$ is weakly relatively compact and $L\mathcal{F}$ is bounded in $L^1_{t,x,v}$, then $\int \mathcal{F}\varphi \, dv$ is relatively compact in $L^1_{t,x}$.*

2 Representation formulas

Theorem 2.1. *Define* $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *by*

$$\Phi(x) = \begin{cases} -\dfrac{1}{2\pi}\log|x| & , \ d=2, \\ \dfrac{\Gamma(\frac{d}{2}+1)}{d(d-2)\pi^{d/2}}\dfrac{1}{|x|^{d-2}} & , \ d \geq 3. \end{cases}$$

1. $u = \Phi$ *solves*
$$-\Delta u = \delta.$$
2. $u = \Phi * f$ *solves*
$$-\Delta u = f.$$

Proof.

1. Fix $\varphi \in C_c^\infty$. We want to show
$$-\int \Phi\Delta\varphi = \varphi(0).$$
Divide and apply Stokes' theorem twice to get
$$\begin{aligned} \int \Phi\Delta\varphi &= \int_{|x|<\varepsilon} \Phi\Delta\varphi + \int_{|x|\geq\varepsilon} \Phi\Delta\varphi \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi - \int_{|x|\geq\varepsilon} \nabla\Phi\cdot\nabla\varphi + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma. \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi + \int_{|x|\geq\varepsilon} \varphi\Delta\Phi - \int_{|x|=\varepsilon} \varphi\nabla\Phi\cdot d\sigma + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma \\ &= \int_{|x|<\varepsilon} \Phi\Delta\varphi - \int_{|x|=\varepsilon} \varphi\nabla\Phi\cdot d\sigma + \int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma. \end{aligned}$$
The first integral is bounded as
$$|\int_{|x|<\varepsilon} \Phi\Delta\varphi| \lesssim_\varphi |\int_{|x|<\varepsilon} \Phi| \lesssim \begin{cases} \varepsilon^2|\log\varepsilon| & , \ d=2, \\ \varepsilon^2 & , \ d\geq 3. \end{cases}$$
The third integral is bounded as
$$|\int_{|x|=\varepsilon} \Phi\nabla\varphi\cdot d\sigma| \lesssim_\varphi |\int_{|x|=\varepsilon} \Phi\, d\sigma| \lesssim \begin{cases} \varepsilon|\log\varepsilon| & , \ d=2, \\ \varepsilon & , \ d\geq 3. \end{cases}$$
For the second integral, since
$$\nabla\Phi = -\frac{1}{d\,\alpha(d)}\frac{x}{|x|^d},$$
we have

□

3 Sturm-Liouville theory

3.1 Self-adjointness

Let $I = [a, b]$ and

$$L = -\frac{1}{w(x)}\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)\right],$$
$$0 \le p(x) \in C^\infty(I), \quad q(x) \in C^\infty(I), \quad 0 < w(x) \in C^\infty(I).$$

We expect L to be self-adjoint. In this regard, our interest is ellimination of the difference term

$$\langle f, Lg\rangle - \langle Lf, g\rangle = p(f'g - fg')|_a^b.$$

Name	Operator	Domain	B.C.
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Periodic
Helmholtz	$L = -\frac{d^2}{dx^2}$	$[a, b]$	Separated Robin
Legendre	$L = -\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right)$	$[-1, 1]$	None
A. Legendre	$L = -\left[\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right) - \frac{m^2}{1-x^2}\right]$	$[-1, 1]$	Dirichlet
Hermite	$L = -e^{x^2}\left[\frac{d}{dx}\left(e^{-x^2}\frac{d}{dx}\right)\right]$	$(-\infty, \infty)$	Polynomial growth
Laguerre			

3.2 Regular Sturm-Liouville problem

We mean *regular Sturm-Liouville problems* by the case that p does not vanish on the boundary of I that we should cancel $f'g - fg'|_a^b$. View the Sturm-Liouville operator L as a non-densely defined operator on the space $C^\infty(I)$ with inner product $\langle f, g\rangle = \int_I fgw$ with domain

$$V = \{\, u \in C^\infty(I) : \alpha_0 u(a) + \alpha_1 u'(a) = 0,\ \beta_0 u(b) + \beta_1 u'(b) = 0 \,\},$$

the subspace for the *separated* Robin boundary condition.

Proposition 3.1. *The operator $L : V \to C^\infty(I)$ is self-adjoint when $C^\infty(I)$ has the inner product $\langle f, g\rangle = \int_I fgw$.*

We are interested in the eigenvalue problem of $L : V \to C^\infty(I)$ on V. Fortunately, if we choose a constant $z \in \mathbb{C} \setminus \mathbb{R}$, then $(L-z)^{-1} : C^\infty(I) \to V$ is well-defined.

Proposition 3.2. *If z is not an eigenvalue of L, then $L - z : V \to C^\infty(I)$ is bijective.*

Proof. The injectivity follows from the definition of eigenvalues. We may assume that L is injective by translation $q \mapsto q - \lambda$.

Suppose $f \in C^\infty(I)$. The surjectivity is equivalent to the existence of a second order inhomogeneous boundary problem:

$$-pu'' - p'u' - qu = fw,$$
$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Let u_a, u_b be the unique solutions of the corresponding homogeneous equation with initial conditions

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0, \quad \beta_0 u(b) + \beta_1 u'(b) = 0.$$

Then we can define $L^{-1} : C^\infty([0,1]) \to D(L)$ by

$$L^{-1}f(x) := u_a(x) \int_x^1 \frac{u_b}{W[u_a,u_b]} \frac{f}{(-p)} w \;+\; u_b(x) \int_0^x \frac{u_a}{W[u_a,u_b]} \frac{f}{(-p)} w,$$

where $W[u_a,u_b] := u_a u_b' - u_b u_a'$ denotes the Wronskian. This formula is derived from variation of parameters: we can compute c_a and c_b from the fact that

$$\begin{pmatrix} 0 \\ \frac{f}{(-p)} w \end{pmatrix} = \begin{pmatrix} u_a & u_b \\ u_a' & u_b' \end{pmatrix} \begin{pmatrix} c_a' \\ c_b' \end{pmatrix} \quad \Longrightarrow \quad L(c_a u_a + c_b u_b) = f.$$

Then, we can easily check that

$$L^{-1} L u = u$$

for $u \in D(L)$, which implies L is surjective. □

3.3 Legendre's equation

The Legendre equation is

$$(1-x^2)u'' - 2xu' + l(l+1)u = 0, \quad \text{on } [-1,1].$$

The Sturm-Liouville operator is

$$L = -\frac{d}{dx}\left((1-x^2)\frac{d}{dx}\right).$$

Since $p(\pm 1) = 0$, the operator $L : C^\infty([-1,1]) \to C^\infty([-1,1])$ is self-adjoint on the whole domain.

Its eigenvalues and corresponding eigenspaces are

l	Eigenvalue $l(l+1)$	Eigenbasis
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$
3	12	$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$
4	20	$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$

If we admit

$$Q_0(x) = \frac{1}{2}\log\frac{1+x}{1-x}, \quad Q_1(x) = 1 - \frac{1}{2}x\log\frac{1+x}{1-x}, \ \cdots \ \in L^2(-1,1) \setminus C^\infty([-1,1])$$

as eigenvectors of L, then the self-adjointness fails on the extended domain. For example,

$$\begin{aligned} \langle Q_0, Lf \rangle - \langle LQ_0, f \rangle &= p(x)\big(Q_0'(x)f(x) - Q_0(x)f'(x)\big)\big|_{-1}^{1} \\ &= f(1) - f(-1) \end{aligned}$$

does not vanish in general even for $f \in C^\infty([-1,1])$.

3.4 Bessel's equation

The Bessel equation is

$$x^2u'' + xu' + (k^2x^2 - \nu^2)u = 0, \qquad \text{on } (0,\infty).$$

The Sturm-Liouville operator is

$$-\frac{1}{x}\left[\frac{d}{dx}\left(x\frac{d}{dx}\right) - \nu^2\frac{1}{x}\right].$$

4 Peetre's theorem

Lemma 4.1. *Suppose a linear operator $L : C_c^\infty(M) \to C_c^\infty(M)$ satisfies*

$$\operatorname{supp}(Lu) \subset \operatorname{supp}(u) \quad \textit{for} \quad u \in C_c^\infty(X).$$

For each point $x \in M$, there is a bounded neighborhood U together with a nonnegative integer m such that

$$\|Lu\|_{C^0} \lesssim \|u\|_{C^m}$$

for $u \in C_c^\infty(U \setminus \{x\})$.

Proof. Suppose not. There is a point x at which the inequality fails; for every bounded neighborhood U and for every nonnegative m, we can find $u \in C_c^\infty(U \setminus \{x\})$ such that

$$\|Lu\|_{C^0} \geq C\|u\|_{C^m},$$

for arbitrarily large C. We want to construct a function $u \in C_c^\infty(U)$ such that Lu has a singularity at x.

(Induction step) Take a bounded neighborhood U_m of x such that

$$U_m \subset U \setminus \bigcup_{i=0}^{m-1} \overline{U}_i.$$

There is $u_m \in C_c^\infty(U_m \setminus \{x\})$ such that

$$\|Lu_m\|_{C^0} > 4^m \|u_m\|_{C^m}.$$

Note that

$$\operatorname{supp}(u_i) \cap \operatorname{supp}(u_j) = \varnothing \quad \text{for} \quad i \neq j.$$

Define

$$u := \sum_{i \geq 0} 2^{-i} \frac{u_i}{\|u_i\|_{C^i}}.$$

We have that $u \in C_c^\infty(U)$ since the series converges in the inductive topology of the LF space $C_c^\infty(U)$: it converges absolutely with respect to the seminorms $\|\cdot\|_{C^m}$ for all m:

$$\begin{aligned}
\sum_{i \geq 0} \|2^{-i} \frac{u_i}{\|u_i\|_{C^i}}\|_{C^m} &= \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} \\
&\leq \sum_{0 \leq i < m} 2^{-i} \frac{\|u_i\|_{C^m}}{\|u_i\|_{C^i}} + \sum_{i \geq m} 2^{-i} \\
&< \infty.
\end{aligned}$$

Also, since the supports of each term are disjoint and L is locally defined, we have

$$Lu = \sum_{i\geq 0} 2^{-i} \frac{Lu_i}{\|u_i\|_{C^i}}.$$

Thus,

$$\|Lu\|_{C^0} = \sup_{i\geq 0} 2^{-i} \frac{\|Lu_i\|_{C^0}}{\|u_i\|_{C^i}} > \sup_{i\geq 0} 2^{-i} \cdot 4^i = \infty,$$

which leads a contradiction.

$\square$

5 Characteristic curve

Algorithm:

(1) Establish the associated vector field by substituting $u \mapsto y$.

(2) Find the integral curve.

(3) Eliminate the auxiliary variables to get an algebraic equation.

(4) Verify the computed solution is in fact the real solution.

Proposition 5.1. *Suppose that there exists a smooth solution $u : \Omega \to \mathbb{R}_y$ of an initial value problem*

$$\begin{cases} u_t + u^2 u_x = 0, & (t,x) \in \Omega \subset \mathbb{R}_{t\geq 0} \times \mathbb{R}_x, \\ u(0,x) = x, & \text{at } x \in \mathbb{R}, \end{cases}$$

and let M be the embedded surface defined by $y = u(t,x)$.

Let $\gamma : I \to \Omega \times \mathbb{R}_y$ be an integral curve of the vector field

$$\frac{\partial}{\partial t} + y^2 \frac{\partial}{\partial x}$$

such that $\gamma(0) \in M$. Then, $\gamma(\theta) \in M$ for all $\theta \in I$.

Proof. We may assume γ is maximal. Define $\tilde{\gamma} : \tilde{I} \to M$ as the maximal integral curve of the vector field

$$\tilde{X} = \frac{\partial}{\partial t} + u^2 \frac{\partial}{\partial x} \in \Gamma(TM)$$

such that $\tilde{\gamma}(0) = \gamma(0)$. Since X and $\tilde{X}$ coincide on M, the curve $\tilde{\gamma}$ is also an integral curve of X with $\tilde{\gamma}(0) = \gamma(0)$. By the uniqueness of the integral curve, we get $\tilde{I} \subset I$ and $\gamma(\theta) = \tilde{\gamma}(\theta)$ for all $\theta \in \tilde{I}$.

Since M is closed in E, the open interval $\tilde{I} = \gamma^{-1}(M)$ is closed in I, hence $\tilde{I} = I$ by the connectedness of I. □

Definition 5.1. The projection of the integral curve γ onto Ω is called a *characteristic.*

This proposition implies that we might be able to describe the points on the surface M explicitly by finding the integral curves of the vector field X. Once we find a necessary condition of the form of algebraic equation, we can demostrate the computed hypothetical solution by explicitly checking if it satisfies the original PDE.

Since X does not depend on u, we can solve the ODE: let $\gamma(\theta) = (t(\theta), x(\theta), y(\theta))$ be the integral curve of X such that $\gamma(0) = (0, \xi, \xi)$. Then, the system of ODEs

$$\begin{aligned} \frac{dt}{d\theta} &= 1, & t(0) &= 0, \\ \frac{dx}{d\theta} &= y(\theta)^2, & x(0) &= \xi, \\ \frac{dy}{d\theta} &= 0, & y(0) &= \xi \end{aligned}$$

is solved as

$$t(\theta) = \theta, \qquad y(\theta) = \xi, \qquad x(\theta) = \xi^2\theta + \xi.$$

Therefore,

$$u(t,x) = \frac{-1 + \sqrt{1+4tx}}{2t}.$$

From this formula, we would be able to determine the suitable domain Ω as

$$\Omega = \{ (t,x) : tx > -\tfrac{1}{4} \}.$$

5.1 Wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 \quad \text{for} \quad t, x > 0, \\ &u(0,x) = g(x), \qquad u(0,x) = h(x), \qquad u_x(t,0) = \alpha(t). \end{aligned}$$

Define $v := u_t - cu_x$. Then we have

$$\begin{cases} v_t + cv_x = 0 & t, x > 0, \\ \quad v(0,x) = h(x) - cg'(x). \end{cases}$$

By method of characteristic,

$$v(t,x) = h(x-ct) - cg'(x-ct).$$

Then, we can solve two system

$$\begin{cases} u_t - cu_x = v, & x > ct > 0, \\ \quad u(0,x) = g(x), \end{cases}$$

and

$$\begin{cases} u_t - cu_x = v, & ct > x > 0, \\ \quad u_x(t,0) = \alpha(t), \end{cases}$$

For the first system, introducing parameter $\xi > 0$,

$$\frac{dt}{d\theta} = 1, \qquad \frac{dx}{d\theta} = -c, \qquad \frac{dy}{d\theta} = -v(t,x),$$
$$t(0) = 0, \qquad x(0) = \xi, \qquad y(0) = g(\xi)$$

is solved as

$$t(\theta) = \theta, \qquad x(\theta) = -c\theta + \xi, \qquad y(\theta) = g(\xi) + \int_0^\theta -v(\theta', \xi - c\theta')\, d\theta',$$

hence for $x > ct > 0$,

$$\begin{aligned} u(t,x) &= g(\xi) - \int_0^\theta v(s, \xi - cs)\, ds \\ &= g(x+ct) \\ &= \frac{3g(x+ct) - g(x-ct)}{2} - \int_0^t h(x + c(t-2s))\, ds \end{aligned}$$

5.2 Burgers' equation

Consider the inviscid Burgers' equation

$$u_t + u u_x = 0.$$

(1) Suppose $u(0, x) = \tanh(x)$. For what values of $t > 0$ does the solution of the quasi-linear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(2) Suppose $u(0, x) = -\tanh(x)$. For what values of $t > 0$ does the solution of the quasilinear PDE remain smooth and single valued? Given an approximation sketch of the characteristics in the tx-plane.

(3) Suppose

$$u(0, x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x < 1, \\ 1, & 1 \leq x \end{cases} .$$

Sketch the characteristics. Solve the Cauchy problem. Hint: solve the problem in each region separately and "paste" the solution together.

6 Weak convergences

7 Existence theorems for ODE

7.1 Picard-Lindelöf theorem

Let $I = [0, T] \subset \mathbb{R}_t$ and $\Omega = \overline{B_r(a)} \subset \mathbb{R}^d_x$. Consider the following initial value problem:

$$x' = f(t, x), \qquad x(0) = a.$$

Theorem 7.1 (Global existence, $\Omega = \mathbb{R}^d$)**.** *If f is $C_t \operatorname{Lip}_x$ on $I \times \mathbb{R}^d$, the equation has a unique C^1 global solution on I.*

Proof. Step 1: Construction of an approximation. Define a sequence of functions $\{x_n\}$ as

$$x'_{n+1} = f(t, x_n(t)), \quad x_{n+1}(0) = a; \qquad x_0 \equiv a.$$

The explicit formula is given by

$$x_{n+1}(t) = a + \int_0^t f(s, x_n(s))\, dx.$$

The sequence belongs to $C(I)$.

Step 2: Convergence of the approximation. Let

$$\sup_{t \in I} |f(t,x) - f(t,y)| \le K|x-y| \qquad \text{and} \qquad \sup_{t \in I} |f(t,a)| \le M.$$

First we have

$$|x_1(t) - x_0(t)| \le \int_0^t |f(s,a)|\, ds \le Mt.$$

By induction, we have

$$\begin{aligned}
|x_{n+1}(t) - x_n(t)| &\le \int_0^t |f(s, x_n(s)) - f(s, x_{n-1}(s))|\, ds \\
&\le K \int_0^t |x_n(s) - x_{n-1}(s)|\, dx \\
&\le MK^n \int_0^t \frac{s^n}{n!}\, ds \\
&= MK^n \frac{t^{n+1}}{(n+1)!}.
\end{aligned}$$

This proves the absolute convergence

$$\sum_{n=0}^{n} \|x_{n+1} - x_n\|_{C_t} \lesssim e^{KT} - 1,$$

hence x_n converges uniformly.

Step 3: Verification of the approximation. Let x^* be the limit of x_n. Then, by limiting

$$x_{n+1}(t) = a + \int_0^t f(s, x_n(s))\, ds,$$

we get

$$x^*(t) = a + \int_0^t f(s, x^*(s))\, ds.$$

Thus, x^* is a solution and it is easy to check x^* is C^1. $\square$

Theorem 7.2 (Local existence)**.** *If f is $C_t^0 \operatorname{Lip}_x$ on $I \times \Omega$, then the equation has a unique C^1 local solution.*

The interval of existence may be arbitrarily chosen such that

$$T \leq R \cdot \|f\|^{-1}_{C_{t,x}(I\times\Omega)}.$$

Proof. Define $\varphi : C([0,T], \overline{B(x_0,R)}) \to C([0,T], \overline{B(x_0,R)})$ as:

$$\varphi(x)(t) := x_0 + \int_0^t f(s, x(s))\, ds.$$

It is well-defined since

$$\begin{aligned} |\varphi(x)(t) - x_0| &\leq \int_0^t |f(s, x(s))|\, ds \\ &\leq TM \leq R. \end{aligned}$$

It is a contraction since we have

$$\begin{aligned} |\varphi(x)(t) - \varphi(y)(t)| &\leq \int_0^t |f(s, x(s)) - f(s, y(s))|\, ds \\ &\leq \int_0^t K|x(s) - y(s)|\, ds \\ &\leq TK\|x(s) - y(s)\| \end{aligned}$$

so that

$$\|\varphi(x) - \varphi(y)\| \leq TK\|x - y\|$$

$\square$

The above one looses the Lipschitz condition to local condition.

8 Statements in functional analysis and general topology

Function analysis:

- Suppose a densely defined operator T induces a Hilbert space structure on its domain. If the inclusion is bounded, then T has the bounded inverse. If the inclusion is compact, then T has the compact inverse.
- A closed subspace of an incomplete inner product space may not have orthogonal complement: setting L^2 inner product on $C([0,1])$, define $\phi(f) = \int_0^{\frac{1}{2}} f$.
- Every seperable Banach space is linearly isomorphic and homeomorphic. But there are two non-isomorphic Banach spaces.
- open mapping theorem -> continuous embedding is really an embedding.
- $D(\Omega)$ is defined by a *countable stict* inductive limit of $D_K(\Omega)$, $K \subset \Omega$ compact. Hence it is not metrizable by the Baire category theorem. (Here strict means that whenever $\alpha < \beta$ the induced topology by $\mathcal{T}_\beta$ coincides with $\mathcal{T}_\alpha$)
- A net $(\phi_d)_d$ in $D(\Omega)$ converges if and only if there is a compact K such that $\phi_d \in D_K(\Omega)$ for all d and ϕ_d converges uniformly.
- Th integration with a locally integrable function is a distribution. This kind of distribution is called *regular*. The nonregular distribution such as δ is called *singular*.
- D' is equipped with the weak* topology.
- $\frac{\partial}{\partial x} \colon D' \to D'$ is continuous. They commute (Schwarz theorem holds).
- $D \to S \to L^p$ are continuous (immersion) but not imply closed subspaces (embedding).

General topology:

- $H \subset \mathbb{C}$ and $H \subset \widehat{\mathbb{C}}$ have distinct Cauchy structures which give a same topology. In addition, the latter is precompact while the former is not.

9 Ultrafilter

Definition 9.1. An *ultrafilter* is a synonym for maximal filter. If we sat $\mathcal{U}$ is an *ultrafilter on a set* A, then it means $\mathcal{U}$ is a maximal filter as a directed subset of $\mathcal{P}(A)$.

existence of ultrafilter.

Theorem 9.1. *Let $\mathcal{U}$ be an ultrafilter on a set A and X be a compact space. For a function $f : A \to X$, the limit $\mathcal{U}\text{-}\lim f$ always exists.*

Theorem 9.2. *Let $X = \prod_{\alpha \in \mathcal{A}} X_\alpha$ be a product space of compact spaces X_α. A net $f : \mathcal{D} \to X$ has a convergent subnet.*

Proof 1. Use Tychonoff. Compactness and net compactness are equivalent. □

Proof 2. It is a proof without Tychonoff. Let $\mathcal{U}$ be a ultrafilter on a set $\mathcal{D}$ contataining all $\uparrow d$. Define a directed set $\mathcal{E} = \{(d, U) \in \mathcal{D} \times \mathcal{U} : d \in U\}$ as $(d, U) \succ (d', U')$ for $U \subset U'$. Let $f : \mathcal{E} \to X$ be a subnet of $f : \mathcal{D} \to X$ defined by $f_{(d,U)} = f_d$.

By the previous theorem, $\mathcal{U}\text{-}\lim \pi_\alpha f_d \in X_\alpha$ exsits for each α. Define $f \in X$ such that $\pi_\alpha f = \mathcal{U}\text{-}\lim \pi_\alpha f_d$. Let $G = \prod_\alpha G_\alpha \subset X$ be any open neighborhood of f. Then, $\pi_\alpha f \in G_\alpha$ and we have $G_\alpha = X_\alpha$ except finite. For α, we can take $U_\alpha := \{d : \pi_\alpha f_d \in G_\alpha\} \in \mathcal{U}$ by definition of convergence with ultrafilter Since $U_\alpha = \mathcal{D}$ except finites, we can take an upper bound $U_0 \in \mathcal{U}$ of $\{U_\alpha\}_\alpha$. Then, by taking any $d_0 \in U_0$, we have $f_{(d,U)} \in G$ for every $(d, U) \succ (d_0, U_0)$. This means $f = \lim_{\mathcal{E}} f_{(d,U)}$, so we can say $\lim_{\mathcal{E}} f_{(d,U)}$ exists. □

10 Selected analysis problems

Problem 10.1. The following series diverges:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}}.$$

Solution. Let $A_k := [1, 2^k] \cap \{x : |\sin x| < \frac{1}{k}\}$. Divide the unit circle $\mathbb{R}/2\pi\mathbb{Z}$ by $7k$ uniform arcs. There are at least $2^k/7k$ integers that are not exceed 2^k and are in a same arc. Let S be the integers and x_0 be the smallest element. Since, $|x - x_0| \pmod{2\pi} < \frac{2\pi}{7k}$ for $x \in S$,

$$|\sin(x - x_0)| < |x - x_0| \pmod{2\pi} < \frac{2\pi}{7k} < \frac{1}{k}.$$

Also, $1 \le x - x_0 \le x \le 2^k$, $x - x_0 \in A_k$.

$$|A_k| \ge \frac{2^k}{7k}.$$

Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^{1+|\sin n|}} &\ge \sum_{n \in A_N} \frac{1}{n^{1+|\sin n|}} \\
&\ge \sum_{k=1}^{N} (|A_k| - |A_{k-1}|) \frac{1}{2^{k+1}} \\
&= \sum_{k=1}^{N} \frac{|A_k|}{2^{k+1}} - \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&= \frac{|A_N|}{2^{N+1}} + \sum_{k=1}^{N-1} \frac{|A_k|}{2^{k+2}} \\
&> \sum_{k=1}^{N} \frac{2^k}{2^{k+2}} \frac{1}{7k} \\
&= \frac{1}{28} \sum_{k=1}^{N} \frac{1}{k} \\
&\to \infty.
\end{aligned}$$

$\square$

Problem 10.2. If $|xf'(x)| \le M$ and $\frac{1}{x}\int_0^x f(y)\,dy \to L$, then $f(x) \to L$ as $x \to \infty$.

Solution. Since

$$\begin{aligned} \left| f(x) - \frac{F(x)-F(a)}{x-a} \right| &\le \frac{1}{x-a}\int_a^x |f(x)-f(y)|\,dy \\ &= \frac{1}{x-a}\int_a^x (x-y)|f'(c)|\,dy \\ &\le \frac{M}{x-a}\int_a^x \frac{x-y}{c}\,dy \\ &\le M\frac{x-a}{a} \end{aligned}$$

by the mean value theorem and

$$f(x)-L = \left[f(x) - \frac{F(x)-F(a)}{x-a}\right] + \frac{x}{x-a}\left[\frac{F(x)}{x} - L\right] + \frac{a}{x-a}\left[\frac{F(a)}{a} - L\right],$$

we have for any $\varepsilon > 0$

$$\limsup_{x\to\infty} |f(x)-L| \le \varepsilon$$

where a is defined by $\frac{x-a}{a} = \frac{\varepsilon}{M}$. $\square$

Problem 10.3. Let $f_n : I \to I$ be a sequence of real functions that satisfies $|f_n(x) - f_n(y)| \le |x-y|$ whenever $|x-y| \ge \frac{1}{n}$, where $I = [0,1]$. Then, it has a uniformly convergent subsequence.

Solution. By the Bolzano-Weierstrass theorem and the diagonal argument for subsequence extraction, we may assume that f_n converges to a function $f : \mathbb{Q} \cap I \to I$ pointwisely.

*Step [.*1] For $n \ge 4$, we claim

$$|x-y| \le \frac{1}{n} \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{5}{n}. \tag{1}$$

Fix $x \in I$ and take $z \in I$ such that $|x-z| = \frac{2}{n}$ so that

$$|f_n(x) - f_n(z)| \le |x-z| = \frac{2}{n}.$$

If y satisfies $|x-y| \le \frac{1}{n}$, then we have $|y-z| \ge |x-z| - |x-y| \ge \frac{1}{n}$, so we get

$$|f_n(y) - f_n(z)| \le |y-z| \le |y-x| + |x-z| \le \frac{3}{n}.$$

Combining these two inequalities proves what we want.

*Step [.*2] For $\varepsilon > 0$ and $N := \lceil \frac{15}{\varepsilon} \rceil$ we claim

$$|x-y| \le \frac{1}{N} \quad \text{and} \quad n > N \quad \Longrightarrow \quad |f_n(x) - f_n(y)| \le \frac{\varepsilon}{3} \tag{2}$$

when $N \ge 4$. It is allowed for $|x-y|$ to have the following two cases:

$$|x-y| \le \frac{1}{n} \quad \text{or} \quad \frac{1}{n} < |x-y| \le \frac{1}{N}.$$

For the former case, by the inequality (1) we have

$$|f_n(x) - f_n(y)| \le \frac{5}{n} < \frac{5}{N} \le \frac{\varepsilon}{3}.$$

For the latter case, by the assumption at the beginning of the problem, we have

$$|f_n(x) - f_n(y)| \le |x-y| \le \frac{1}{N} \le \frac{\varepsilon}{15}.$$

Hence the claim is proved.

*Step [.*3] We will prove f is uniformly continuous. For $\varepsilon > 0$, take $\delta := \frac{1}{N}$, where $N := \lceil \frac{15}{\varepsilon} \rceil$. We will show

$$|x-y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon$$

for $x, y \in \mathbb{Q} \cap I$ and $N \geq 4$. Fix rational numbers x and y in I which satisfy $|x - y| < \delta$. Since $f_n(x)$ and $f_n(y)$ converges to $f(x)$ and $f(y)$ respectively, we may take an integer n_x and n_y, such that

$$n > n_x \quad \Longrightarrow \quad |f_n(x) - f(x)| < \frac{\varepsilon}{3} \tag{3}$$

and

$$n > n_y \quad \Longrightarrow \quad |f_n(y) - f(y)| < \frac{\varepsilon}{3}. \tag{4}$$

Choose an integer n such that $n > \max\{n_x, n_y, N\}$. Then, combining (3), (2), and (4), we obtain

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Since f is continuous on a dense subset $\mathbb{Q} \cap I$, it has a unique continuous extension on the whole I. Let it denoted by the same notation f.

Step [.4] Finally, we are going to show $f_n \to f$ uniformly. For $\varepsilon > 0$, let $N := \lceil \frac{15}{\varepsilon} \rceil$. The uniform continuity of f allows to have $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \frac{2}{3}\varepsilon. \tag{5}$$

Take a rational $r \in I$, depending on $x \in I$, such that $|x - r| < \min\{\frac{1}{N}, \delta\}$. Then, by (2) and (5), given $n > N \geq 4$, we have an inequality

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(r)| + |f_n(r) - f(r)| + |f(r) - f(x)| \\ &< \frac{\varepsilon}{3} + |f_n(r) - f(r)| + \frac{2}{3}\varepsilon \end{aligned}$$

for any $x \in I$. By limiting $n \to \infty$, we obtain

$$\lim_{n \to \infty} |f_n(x) - f(x)| < \varepsilon.$$

Since ε and x are arbitrary, we can deduce the uniform convergence of f_n as $n \to \infty$. $\square$

Problem 10.4. A measurable subset of $\mathbb{R}$ with positive measure contains an arbitrarily long subsequence of an arithmetic progression. (made by me!)

Solution. Let $E \subset \mathbb{R}$ be measurable with $\mu(E) > 0$. We may assume E is bounded so that we have $E \subset I$ for a closed bounded interval since $\mathbb{R}$ is σ-compact. Let n be a positive integer arbitrarily taken. Then, we can find N such that $\sum_{k=1}^{N} \frac{1}{k} > (n-1)\frac{\mu(I)}{\mu(E)}$.

Assume that every point x in E is contained in at most $n-1$ sets among

$$E,\ \frac{1}{2}E,\ \frac{1}{3}E,\ \cdots,\ \frac{1}{N}E.$$

In other words, it is equivalent to:

$$\bigcap_{k\in A} \frac{1}{k}E = \varnothing$$

for any subset $A \subset \{1, \cdots, N\}$ with $|A| \geq n$. Define

$$E_A := \bigcap_{k\in A} \frac{1}{k}E \cap \bigcap_{k'\in A} \left(\frac{1}{k'}E\right)^c$$

for $A \subset \{1, \cdots, N\}$. Then, $\mu(E_A) = 0$ for $|A| \geq n$.

Note that we have

$$\mu(\tfrac{1}{k}E) = \sum_{k\in A} \mu(E_A) = \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A).$$

Summing up, we get

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) = \sum_{k=1}^{N} \sum_{\substack{k\in A \\ |A|<n}} \mu(E_A) = \sum_{|A|<n} |A|\mu(E_A)$$

by double counting, and since E_A are dijoint, we have

$$\sum_{|A|<n} |A|\mu(E_A) = (n-1) \sum_{0<|A|<n} \mu(E_A) \leq (n-1)\mu(I),$$

hence a contradiction to

$$\sum_{k=1}^{N} \mu(\tfrac{1}{k}E) > (n-1)\mu(I).$$

Therefore, we may find an element x that belongs to $\frac{1}{k}E$ for $k \in A$, where $A \subset \{1, \cdots, N\}$ with $|A| = n$. Then, $ax \in E$ for all $a \in A \subset \mathbb{Z}$. $\square$

11 Physics problem

11.1 Resonance

Let m, b, k, A, ω_d be positive real constants. Consider an underdamped oscillator with sinusoidal diving force described as

$$mx'' + bx' + kx = A \sin \omega_d t, \quad x(0) = x_0,\ x'(0) = 0.$$

There are some observations:

(1) The underdamping condition means $b^2 - 4mk < 0$ so that the roots of characteristic equation are imaginary.

(2) The positivity of m, b implies the real part of solution that will be denoted by $-\beta = -\frac{b}{2m}$ is negative; it shows exponential decay of solutions.

(3) Introducing the natural frequency $\omega_n = \sqrt{k/m}$, we can rewrite the equation as

$$x'' + 2\zeta\omega_n x' + \omega_n^2 x = A \sin \omega t.$$

(4) The complementary solution is computed as

$$x_c(t) = x_0\ e^{-\beta t} \cos \sqrt{\beta^2 - \omega_n^2} t,$$

and it can be verified that this solution is asymptotically stable, i.e.

$$\lim_{t \to \infty} x_c(t) = 0.$$

(5) The condition $\beta > \omega_n$ is equivalent to that the oscillator is underdamped.

(6) Let m, k be fixed. Then, the solution x_c decays most fastly when b satisfied $b^2 = 4mk$, equivalently, $\beta = \omega_n$.

(7) When $\omega_d = \omega_n$ such that the amplitude of particular solution diverges.