1.1 Feed-forward and Feedback control

Feedforward control has no feedback from the output. To solve this, the controller is set using the *inversion control strategy.*

$$C(s) = \frac{1}{P(s)}$$

If the resulting $C(s)$ is improper $C(s)$ can be modified to the following:

$$C(s) = \frac{1}{P(s) \cdot (\tau s + 1)}$$

For good tracking up to 2π, set $\frac{1}{\tau} \geq 10 * 2\pi \Rightarrow \tau \leq 0.016$

Feedback control feeds the output back to the reference. High-gain control in particular frequencies with a unity-feedback control loop is a good control strategy. At those frequencies, we achieve good tracking, disturbance rejection, and sensitivity.

1.1 Intro terms

Open-loop poles refer to the poles of $P(s)C(s)$.

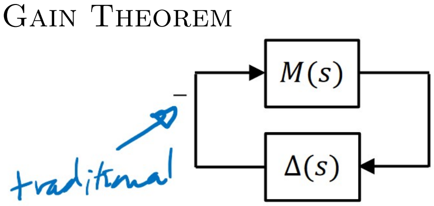
Closed-loop poles are the roots of $N_C N_P + D_C D_P$.

Open left-half plane (OLHP) refer to $Re(s) < 0$.

Close left-half plane (CLHP) refer to $Re(s) \leq 0$.

Open right-half plane (ORHP) and Closed right-half plane (CRHP) are similarly defined.

2.8 Small Gain Theorem



$M(s)$ is known and stable

$\Delta(s)$ is uncertain but stable with $|\Delta(j\omega)| < 1$ for all ω

3.1 Youla Parameterization

For a stable plant. The set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{Q(s)}{1 - P(s)Q(s)} \right\}$$

where $Q(s)$ is <u>proper</u> and <u>stable</u>

Coprime Factorization

Let $P(s)$ be any proper TF. Then there exists <u>stable proper</u> TFs $N_P(s)$, $X_P(s)$, $M_P(s)$, and $Y_P(s)$ such that the following relationships hold for all s:

$$P(s) = \frac{N_P(s)}{M_P(s)}$$

$$N_P(s)X_P(s) + M_P(s)Y_P(s) = 1$$

Youla Parameterization for General $P(s)$

For an unstable plant. Assume there are no CRHP pole-zero cancellations in $P(s)$. Perform a coprime factorization on $P(s)$. Then the set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{X_P(s) + M_P(s)Q(s)}{Y_P(s) - N_P(s)Q(s)} \right\}$$

where $Q(s)$ is <u>proper</u> and <u>stable</u>

3.2 Performance Limitations

At frequencies where $|S(j\omega)| \ll 1$, good tracking, good disturbance rejection, and good sensitivity. Exact opposite when $|S(j\omega)| \gg 1$.

$$S(s) = \frac{1}{1 + L(s)}$$

<u>Ideally</u>:

- $|S(j\omega)|$ is small when $|L(j\omega)|$ is large
- $|S(j\omega)|$ is near one (0 dB) when $|L(j\omega)|$ is small

The complementary sensitivity, $T(s)$, tells us that robust stability is achieved iff there is nominal stability and:

$$|W(j\omega)T_0(j\omega)| < 1 \text{ for all } \omega$$
$$\Leftrightarrow |T_0(j\omega)| < \frac{1}{|W(j\omega)|} \text{ for all } \omega$$

It also tells us how the closed-loop system responds to sensor noise.

$$T(s) = \frac{L(s)}{1 + L(s)}$$

<u>Ideally</u>:

- $|T(j\omega)|$ is small (in fact, $|T(j\omega)| \approx |L(j\omega)|$) at high frequencies when sensor noise is significant, and little feedback effort is used
- $|S(j\omega)|$ is near one (0 dB) at low frequencies, where lots of feedback effort is used

The relationship between the two equations:

$$S(j\omega) + T(j\omega) = \frac{1}{1+L(j\omega)} + \frac{L(j\omega)}{1+L(j\omega)} = 1 \quad (1)$$

$$|S(j\omega)| + |T(j\omega)| \geq 1 \text{ for each } \omega \quad (2)$$

Notes:

- tradeoff between the reasons for wanting $|S(j\omega)|$ to be small and the reason for wanting $|T(j\omega)|$
- both cannot both be "good" (i.e., very small) at the same frequency, but it is possible for both to be "bad" (i.e., very large) at the same frequency
- We never want the magnitude bode plots of S(s) or T(s) to have large peaks

3.3 Interpolation Constraints

Assume that the closed-loop system is stable.

- If $P(s)C(s)$ has a CRHP zero at $s = z$, then $S(z) = 1$ and $T(z) = 0$
- If $P(s)C(s)$ has a CRHP pole at $s = p$, then $S(p) = 0$ and $T(p) = 1$

3.4 Performance Limitations Due to ORHP

Bandwidth rule of thumb is

$$\text{bandwidth } > 2p$$

Lemma

If:

- The closed-loop system is stable
- The plan has an open-loop unstable real pole at $s = p > 0$
- The reference signal is a unit step

Then the tracking error, $e(t) = r(t) - y(t)$ necessarily satisfies:

$$\int_0^\infty e(t)e^{-pt}\,dt = 0$$

An ORHP pole in the plant leads to overshoot in the closed-loop step response

$$y_{OS} \geq (1 - 0.9y_\infty)(e^{pt_r} - 1) > 0$$

$$t_r \leq \frac{1}{p} ln\left(1 + \frac{y_{OS}}{1 - 0.9y_\infty}\right)$$

Bode Sensitivy Integral

Assume that:

- The controller stabilizes the closed-loop system
- The loop gain, $L(s) = P(s)C(s)$, has a ***relative degree*** of at least two (i.e., it has at least two more poles than zeros).

Let $N_p \geq 0$ denote the number of ORHP poles of $L(s)$ and denote the ORHP poles by $p_1, \ldots, p_{N_p}$. Then the sensitivity function must satisify

$$\int_0^\infty ln|S(j\omega)|\,d\omega = \sum_{i=1}^{N_p} Re(p_i) \geq 0$$

If there are no unstable poles, then the equation reduces to

$$\int_0^\infty ln|S(j\omega)|\,d\omega = 0$$

Where the negative area equals the positive area through a *waterbed effect.*

If there are an unstable poles, then the area of sensitivity increase must be *greater than* the area of sensitivity decrease.

3.5 Performance Limitations Due to ORHP Zeros

If the plant has an ORHP real zero at s = z, then good performance can be achieved if the closed-loop bandwidth is significantly smaller than z. ORHP Zeroes:

- Zeroes near the origin are worse than ORHP zeroes far from the origin
- ORHP zeroes lead to *undershoot* in the step responses
- You cannot get rid of CRHP zeroes, since $P(s)C(s)$ is the numerator of $T_{ry}(s)$

Lemma

$$\int_0^\infty y(t)e^{-zt}\,dt = 0$$

$$y_{US} \geq \frac{0.98y_\infty}{e^{zt_s} - 1} > 0$$

$$t_r \leq \frac{1}{z} ln\left(1 + \frac{0.98y_\infty}{y_{US}}\right)$$

Bandwidth rule of thumb given a ORHP zero:

$$\text{bandwidth } < z/2$$

Poisson Integral

$$\int_0^{\infty} ln|S(j\omega)|W(\omega)\,d\omega = \pi \sum_{i=1}^{N_p} ln\left|\frac{p_i + z}{p_i^* - z}\right| \geq 0$$
$$W(\omega) = \frac{2z}{z^2 + \omega^2}$$

4.1 Basic MIMO Concepts

$$\begin{aligned} y &= (I + PCH)^{-1}PCr \\ L_o &= PC \\ S_o &= (I + L_o)^{-1} \\ T_o &= L_o(I + L_o)^{-1} \\ S_o(s) + T_o(s) &= I \end{aligned}$$

The *minor* of a matrix is the determinant of any square submatrix obtained by deleting certain row's and/or columns of the matrix

The *rank* of a matrix is the max number of linearly independent column/row vectors.

To find $\phi(s)$, find all the minors. 1st order is just the element, while second order is as described above. Find the lowest common denominator of all non-zero minors.

A MIMO system is stable iff all the individual elements of the system are stable, and if all MIMO system poles are in the OLHP

The zero polynomial $z(s)$ is the greatest common divisor of all the r-th order minors. Steps:

1. Find the minors for the MIMO system
2. Determine $\phi(s)$
3. Make each minor as a fraction over $\phi(s)$
4. Find the GCD of the r-th order minors.

4.2 Decentralized Control

- Assume plant is square
- Ignore cross-channel interactions in the plant, resulting in a MIMO controller that is diagonal
- If there is too much coupling, then the resulting system is unstable

4.3 Decoupling Control

- A decoupling block is added: $P_{aug} = PW$
- Design a decentralized controller for the augmented plant: $C = W\bar{C}$

Steps:

1. Decouple the plant at dc. i.e. Solve $P(0)$
2. Solve $W(s) = P_1^{-1}(0)$, to get $P_{aug} = P * W$
3. Solve for the controllers, ignoring the cross-channel interactions
4. $C(s) = W\bar{C}(s)$

4.4 Sequential Loop Closure

- Accounts for cross-channel interactions
- Feedback loops are closed sequentially instead of in parallel
- **This approach only applies if the plant has a triangular structure.**
- Ex:
  - Given Plant:

    $$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$$

    $y_1 = P_{11}u_1 + P_{12}u_2$ and $y_2 = P_{22}u_2$
  - first design C_{22} to stabilize P_22 since it is independent of P_{11} and P_{12}
  - Then design C_{22}

5.1 Basic State-Space Framework and Manipulations

State-space models are not unique. Infinite number of state-space realizations that correspond to the same transfer functions They follow the form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

Manipulation 1: Convert from SS to TF

$$Y(s) = [D + C(sI - A)^{-1}B] \cdot U(s)$$

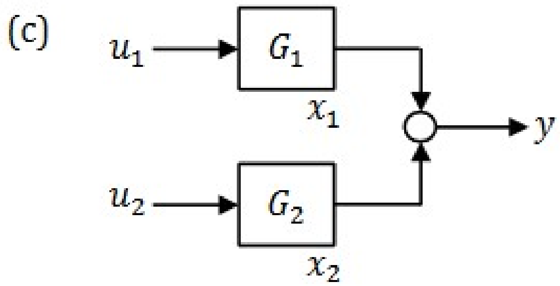
Manipulation 2: Poles and zeros of a SS system

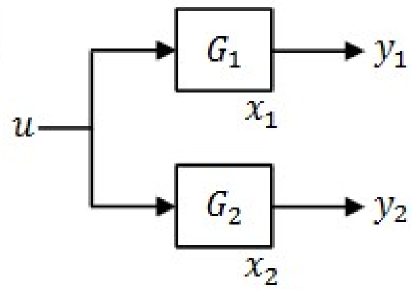
- Poles of the SS system are $eig(A)$ == roots of $det(sI - A)$
- Zeroes are the values of s where the rank of the following matrix falls below its normal rank:

  $$R(s) = \begin{bmatrix} (sI - A) & -B \\ C & D \end{bmatrix}$$

  Which can be computed by finding the the roots of $det(R(s))$

Manipulation 3: Interconnecting state-space models





(c)

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$
$$\dot{x}_2 = A_2 x_2 + B_2 u_2$$
$$y = (C_1 x_1 + D_1 u_1) + (C_2 x_2 + D_2 u_2)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 & D_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

(d)

$$\dot{x}_1 = A_1 x_1 + B_1 u$$
$$\dot{x}_2 = A_2 x_2 + B_2 u$$
$$y_1 = (C_1 x_1 + D_1 u)$$
$$y_2 = (C_2 x_2 + D_2 u)$$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} u$$

Manipulation 4: Converting from TF to SS

- SISO case, creates a SS with controllable canonical form (CCF):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix} x + \begin{bmatrix} d \end{bmatrix} u$$

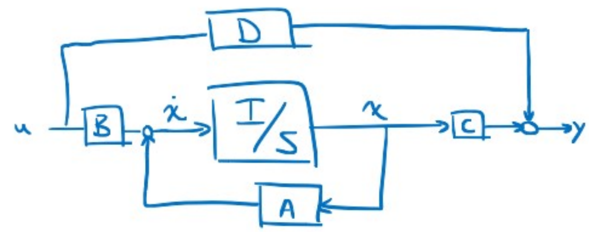
- SIMO case:

$$G(s) = \left[\begin{array}{cc|c} A_1 & 0 & \cdots & B_1 \\ 0 & A_2 & & B_2 \\ \vdots & & \ddots & \vdots \\ C_1 & 0 & \cdots & D_1 \\ 0 & C_2 & & D_2 \\ \vdots & & \ddots & \vdots \end{array}\right]$$

- MIMO case:

$$G(s) = \begin{bmatrix} A_1 & 0 & \cdots & B_1 & 0 & \cdots \\ 0 & A_2 & & 0 & B_2 & \\ \vdots & & \ddots & \vdots & & \ddots \\ C_1 & C_2 & \cdots & D_1 & D_2 & \cdots \end{bmatrix}$$

Manipulation 5: Block Diagram of SS Model



5.2 Controllability, Observability, Stabilizability, and Detectability

Modes that show up on the output y(t) are said to be *Observable*. If all modes are observable, the state-space realization is said to *observable*. If not, it's *unobservable*

Modes that are affected by the input are said to *controllable* If all modes are controllable, the state-space realization is said to *controllable*. If not, it's *uncontrollable*

A state-space realization where <u>all unstable</u> modes are controllable is said to be *stabilizable.*

$$\text{controllability} \Rightarrow \text{stabilizability}$$

A state-space realization where <u>all unstable</u> modes are observable is said to be *detectable.*

$$\text{observability} \Rightarrow \text{detectability}$$

A system is controllable and observable iff the corresponding TF matrix has no P-Z cancellations

A system is stabilizable and detectable iff the corresponding TF matrix has no <u>unstable</u> P-Z cancellations

5.2.2 Controllability and Stabilizability

A, B is controllable iff the n x (nm) *controllability matrix* has rank n.

$$M_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Using the PBH controllability test, A, B is controllable iff

$$\text{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \text{ for all } \lambda \in \text{eig}(A)$$

Recall: eig(A) is found as the solutions to λ, for $\det(A - \lambda I)$

5.2.3 Observability and Detectability

A, C is observable iff the (pn) x n *observability matrix* has rank n.

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Using the PBH observability test, A, C is controllable iff

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \text{ for all } \lambda \in \text{eig}(A)$$

Recall: eig(A) is found as the solutions to λ, for $\det(A - \lambda I)$

5.3 Plain State Feedback Control (Pole Placement)

The plain state-feedback controller $u = -Kx + r$ can arbitrarily place poles. The plant **must be controllable**. If it is uncontrollable but stabilizable, we cannot place poles arbitrarily but we can find a K to stabilizie $A - BK$.

K is a row vector.

Steps:

$$\begin{aligned} A_{CLS} &= A - BK \\ \pi(s) &= det(sI - A_{CLS}) \\ \pi_{des}(s) &= \pi(s) \end{aligned}$$

CLS poles are at $\text{eig}(A_{CLS}) = \text{eig}(A - BK)$

5.4 Observer

Consider the following Plant:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

and Observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + H(y - \hat{y}) \\ \hat{y} &= C\hat{x} + Du \end{aligned}$$

The estimation error is given by $e(t) = x(t) - \hat{x}(t)$. Then:

- $\dot{e} = (A - HC)e$, and $e(t = \infty) = 0$ for any IC $e(0)$ iff $\text{eig}(A - HC) \in$ OLHP.
- The poles of $\text{eig}(A - HC)$ can be placed arbitrarily iff (A, C) is <u>observable</u>.
- The poles of $\text{eig}(A - HC)$ can be placed in OLHP iff (A, C) is only <u>detectable</u>.

H is a column vector.

Steps:

$$\begin{aligned} \pi(s) &= det(sI - (A - HC)) \\ \pi_{des}(s) &= \pi(s) \end{aligned}$$

Resulting state-space realization:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & A - HC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u$$

$$\bar{y} = I \cdot \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

5.4 Observer-based state-feedback control

Mixture of solving for K and H, using the above steps. The CLS for the observer-based state-feedback controller:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

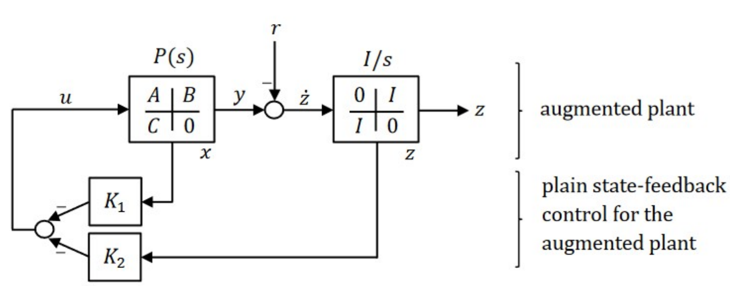
The closed loop poles are found at $\text{eig}(A - HC) \cup \text{eig}(A - BK)$

General rule-of-thumb is to place $\text{eig}(A - HC)$ about twice as fast (i.e., real parts twice as large) as $\text{eig}(A - BK)$.

K: Arbitrary pole placement is possible iff (A, B) is controllable, and stabilization is possible iff (A, B) is stabilizable.

H: Arbitrary placement of poles of the observer error-dynamics is possible iff (A, C) is observable, and they can be placed (not necessarily arbitrarily) in the OLHP iff (A, C) is detectable.

5.5 Plain state-feedback controller with integral action



$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -I \end{bmatrix} r$$

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + \tilde{B}_r r$$

The CLS poles ($\text{eig}(\tilde{A} - \tilde{B}K)$) can be placed arbitrarily iff $(\tilde{A}, \tilde{B})$ is controllable Essentially, the same steps as in 5.3, however A is replaced with $\tilde{A}$. Still need to calculate the characteristic polynomial using $\det(sI - (\tilde{A} - \tilde{B}K))$

5.5 Observer-based state feedback controller with integral action

Essentially, the same steps as in 5.4, however A is replaced with $\tilde{A}$.

CLS poles: $\text{eig}(\tilde{A} - \tilde{B}K) \cup \text{eig}(A - HC)$

K: Arbitrary pole placement is possible iff $(\tilde{A}, \tilde{B})$ is controllable. Stabililization is possible iff $\tilde{A}, \tilde{B}$ is stabilizable.

H: Arbitrary placement of poles of the observer error-dynamics is possible iff (A, C) is observable. They can be placed (not necessarily arbitrarily) in the OLHP iff (A, C) is detectable.

5.6 Optimal State-feedback control

Where $Q \geq 0$ and $R > 0$, minimize:

$$J = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

Assume that (A, B) is stabilizable and (A, Q) is detectable.

The resulting controller is the stabilizing linear state-feedback controller, $u(t) = -Kx(t)$, where K is computed by finding the unique symmetric PSD solution P to the algebraic Riccati equation (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and setting $K = R^{-1}B^T P$.

Q and R are tuning knobs. Q is commonly a diagonal matrix where $q_{ii} \geq 0$. This results in:

$$x(t)^T Q x(t) = q_{11}x_1^2(t) + q_{22}x_2^2(t) + \cdots + q_{nn}x_n^2(t)$$

Q is commonly chosen as $Q = C^T C$. This leads to:

$$y(t) = Cx(t)$$
$$x(t)^T Q x(t) = ||y(t)||^2$$

R is also commonly a diagonal matrix.

Decreasing R minimizes the control effort, and the poles go off to $-\infty$. Increasing R increases the control effort, and the poles approach the OL poles.

Changing R does the opposite effect.

Increasing Q and R by the same factor does not change K or the pole locations.