1.1 Feed-forward and Feedback control

Feedforward control has no feedback from the output. To solve this, the controller is set using the *inversion control strategy*.

$$C(s) = \frac{1}{P(s)}$$

If the resulting $C(s)$ is improper $C(s)$ can be modified to the following:

$$C(s) = \frac{1}{P(s) \cdot (\tau s + 1)}$$

For good tracking up to 2π, set $\frac{1}{\tau} \geq 10 * 2\pi \Rightarrow \tau \leq 0.016$

Feedback control feeds the output back to the reference. High-gain control in particular frequencies with a unity-feedback control loop is a good control strategy. At those frequencies, we achieve good tracking, disturbance rejection, and sensitivity.

1.1 Intro terms

Open-loop poles refer to the poles of $P(s)C(s)$.

Closed-loop poles are the roots of $N_C N_P + D_C D_P$.

Open left-half plane (OLHP) refer to $Re(s) < 0$.

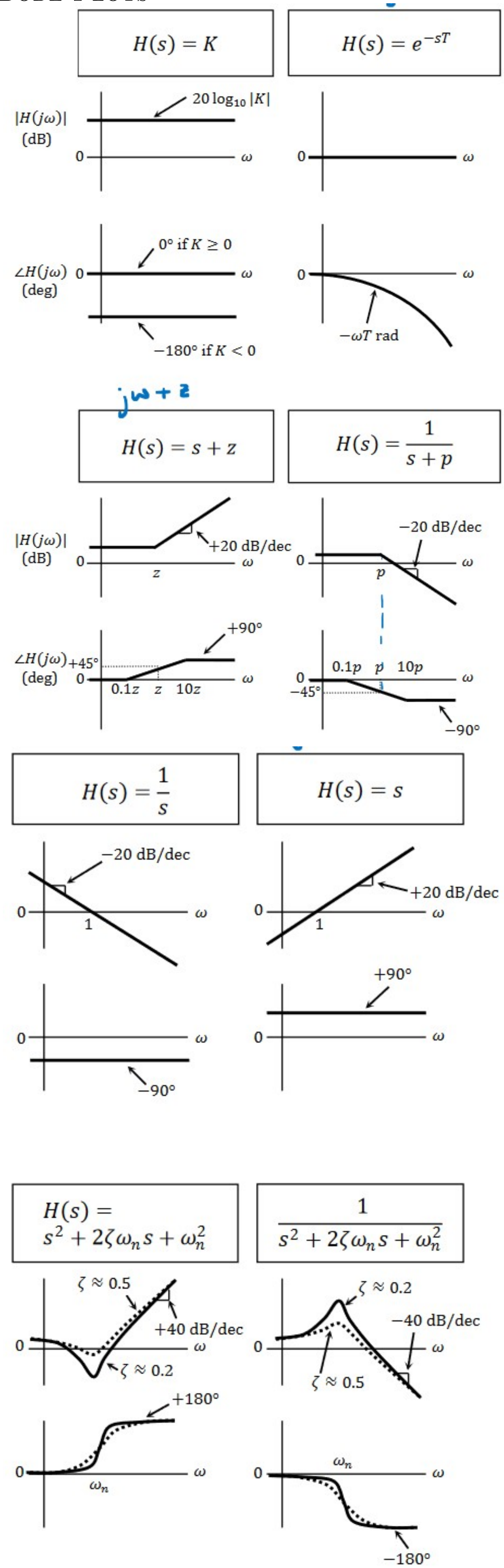
Close left-half plane (CLHP) refer to $Re(s) \leq 0$.

Open right-half plane (ORHP) and Closed right-half plane (CRHP) are similarly defined.

2.1 Laplace Transforms

$$u_{step}(t) \Rightarrow \frac{1}{s}$$
$$u_{ramp}(t) \Rightarrow \frac{1}{s^2}$$
$$e^{at} u_{step}(t) \Rightarrow \frac{1}{s-a}$$
$$sin(at) u_{step}(t) \Rightarrow \frac{a}{s^2 + a^2}$$
$$cos(at) u_{step}(t) \Rightarrow \frac{s}{s^2 + a^2}$$
$$\delta(t) \Rightarrow 1$$

2.1 Bode Plots



2.2 Routh-Hurwitz

BIBO stable iff proper and all poles lie in OLHP.

Given:

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$					
s^2	k_1	k_2	k_3		
s^1	l_1	l_2			
s^0	m_1				

$$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

All roots of Q(s) are in the OLHP iff all elements in the first column do not have any sign changes.

2.3 Steady-state Calculations

FVT states:

$$f_\infty = \lim_{s \to 0} sF(s)$$

For a standard controller plant combination, the following equations can apply

$$y_\infty = T_{ry}(0) = \lim_{s \to 0} \frac{PC}{1+PC} e_\infty = T_{re}(0) = \lim_{s \to 0} \frac{1}{1+PC}$$

2.4 Transient Response Calculations

Dominant Dynamics: the closer the poles and zeros are to the origin, the more important they are.

Poles and zeros close together in the OLHP essentially cancel each other out

First order systems equations:

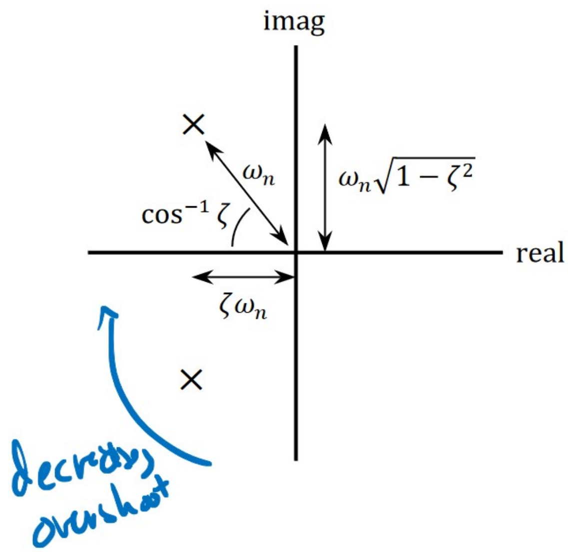
$$H(s) = \frac{1}{\tau s + 1}$$

The 2% settling time is about $T_s \approx 4\tau$.

Second-order systems:

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

System is underdamped when $0 < \zeta < 1$.



2% settling time	$T_s \approx \frac{4}{\zeta\omega_n}$
Oscillation frequency	$\omega_n\sqrt{1-\zeta^2}$
Overshoot	$OS = 100 \exp \frac{-\zeta\pi}{\sqrt{1-\zeta^2}}$
Peak time	$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

OLHP poles make the reponse more sluggish. OLHP zeros make the response more perky

Bandwidth relates to the speed of response. Higher bandwidth means a faster speed of response. Bandwidth is often defined to be the frequency where $G(j\omega)$ is 3 dB less than $|G(0)|$ Bandwidth is also defined as the gain-crossover frequency, which is the frequency where $|P(j\omega)C(j\omega)| = 0dB$

2.5 Root Locus Methods

$$Q(s) = D(s) + K \cdot N(s)P(s)C(s) = \frac{N(s)}{D(s)}$$

m is the order of N(s) n is the order of D(s)

Rules:

- RL is symmetric about the real axis.
- RL plot consists of n branches.
- RL is a continuous function of K.
- The root locus starts at the roots of D(s)
- m branches terminate at the m roots of N(s). The remaining n-m branches go off to infinity. It is centered around σ.

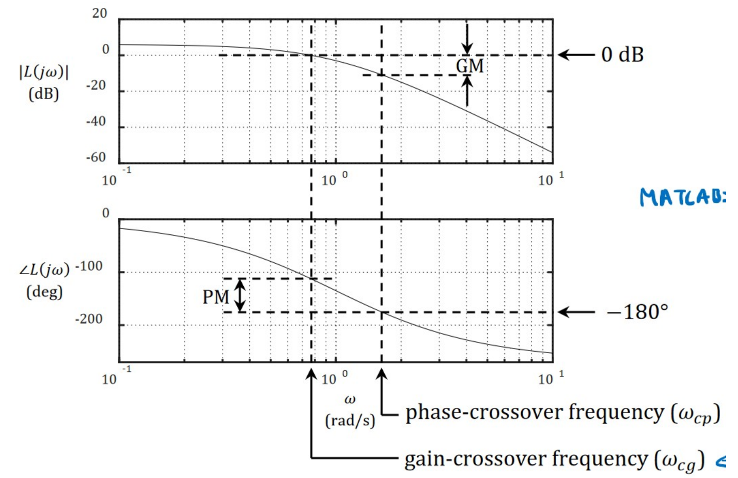
$$\sigma = \frac{\Sigma(\text{roots of D(s)}) - \Sigma(\text{roots of N(s)})}{n-m}$$

- no-yes-no rule. A test point s located on the real axis is part of the root locus if and only if the total number of (finite) real poles and zeros of P(s)C(s) to the right of s is odd.

2.6 Nyquist Plots

If the number of CCW encirclements of the critical point $s = -1$ by the Nyquist plot of L(s) equals the number of OLS poles in the ORHP.

2.6 Phase Margin and Gain Margin



MATLAB:

$$PM \approx 100\zeta \text{ [deg]}$$

2.7 Lead, Lag, PID

Lead compensation increases PM.

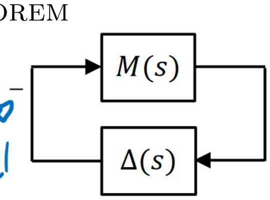
Lag compensation boosts low-frequency gain, or can increase PM.

PID control is simple controller and is commonly used.

$$C(s) = K_P + \frac{K_I}{s} + \frac{K_D s}{\tau s + 1}$$

$$C(s) = \frac{K_D s^2 + K_P s + K_I}{s(\tau s + 1)}$$

2.8 Small Gain Theorem



traditional

$M(s)$ is known and stable

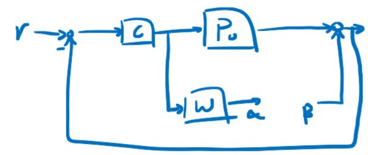
$\Delta(s)$ is uncertain but stable with $|\Delta(j\omega)| < 1$ for all ω

$$|\Delta(j\omega)M(j\omega) < 1| \text{ for all } \omega \text{ and all admissible } \Delta(s)$$

or, equivalently,

$$|M(j\omega) \leq 1| \text{ for all } \omega$$

Extract $\Delta(s)$ to get the effective $M(s)$:



$T_{\beta\alpha} = \frac{-CW}{1+P_oC} \Rightarrow M = \frac{+CW}{1+P_oC} = \frac{3s^2}{(s+1)^2(s+3)}$

Analysis:

- $M(s)$ is stable ✓
- $|M(j\omega)| \leq 1$ for all ω

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$\therefore$ control system is robustly stable

2.9 Loopshaping

$$L(s) = P(s)C(s)$$

At low frequencies, we want:

$$|L(j\omega)| \gg 1$$

At high frequencies, we want:

$$|L(j\omega)| \ll 1$$

3.1 Youla Parameterization

For a stable plant. The set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{Q(s)}{1 - P(s)Q(s)} \right\}$$

where $Q(s)$ is proper and stable

Coprime Factorization

Let $P(s)$ be any proper TF. Then there exists stable proper TFs $N_P(s)$, $X_P(s)$, $M_P(s)$, and $Y_P(s)$ such that the following relationships hold for all s:

$$P(s) = \frac{N_P(s)}{M_P(s)}$$

$$N_P(s)X_P(s) + M_P(s)Y_P(s) = 1$$

Youla Parameterization for General $P(s)$

For an unstable plant. Assume there are no CRHP pole-zero cancellations in $P(s)$. Perform a coprime factorization on $P(s)$. Then the set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{X_P(s) + M_P(s)Q(s)}{Y_P(s) - N_P(s)Q(s)} \right\}$$

where $Q(s)$ is proper and stable

3.2 Performance Limitations

At frequencies where $|S(j\omega)| \ll 1$, good tracking, good disturbance rejection, and good sensitivity. Exact opposite when $|S(j\omega)| \gg 1$.

$$S(s) = \frac{1}{1 + L(s)}$$

<u>Ideally</u>:

- $|S(j\omega)|$ is small when $|L(j\omega)|$ is large
- $|S(j\omega)|$ is near one (0 dB) when $|L(j\omega)|$ is small

The complementary sensitivity, $T(s)$, tells us that robust stability is achieved iff there is nominal stability and:

$$|W(j\omega)T_0(j\omega)| < 1 \text{ for all } \omega$$
$$\Leftrightarrow |T_0(j\omega)| < \frac{1}{|W(j\omega)|} \text{ for all } \omega$$

It also tells us how the closed-loop system responds to sensor noise.

$$T(s) = \frac{L(s)}{1 + L(s)}$$

<u>Ideally</u>:

- $|T(j\omega)|$ is small (in fact, $|T(j\omega)| \approx |L(j\omega)|$) at high frequencies when sensor noise is significant, and little feedback effort is used
- $|S(j\omega)|$ is near one (0 dB) at low frequencies, where lots of feedback effort is used

The relationship between the two equations:

$$S(j\omega) + T(j\omega) = \frac{1}{1 + L(j\omega)} + \frac{L(j\omega)}{1 + L(j\omega)} = 1 \qquad (1)$$
$$|S(j\omega)| + |T(j\omega)| \geq 1 \text{ for each } \omega \qquad (2)$$

Notes:

- tradeoff between the reasons for wanting $|S(j\omega)|$ to be small and the reason for wanting $|T(j\omega)|$
- both cannot both be "good" (i.e., very small) at the same frequency, but it is possible for both to be "bad" (i.e., very large) at the same frequency
- We never want the magnitude bode plots of S(s) or T(s) to have large peaks

3.3 Interpolation Constraints

Assume that the closed-loop system is stable.

- If $P(s)C(s)$ has a CRHP zero at $s = z$, then $S(z) = 1$ and $T(z) = 0$
- If $P(s)C(s)$ has a CRHP pole at $s = p$, then $S(p) = 0$ and $T(p) = 1$

3.4 Performance Limitations Due to ORHP

Bandwidth rule of thumb is

$$\text{bandwidth } > 2p$$

Lemma

If:

- The closed-loop system is stable
- The plan has an open-loop unstable real pole at $s = p > 0$
- The reference signal is a unit step

Then the tracking error, $e(t) = r(t) - y(t)$ necessarily satisfies:

$$\int_0^\infty e(t)e^{-pt}\, dt = 0$$

An ORHP pole in the plant leads to overshoot in the closed-loop step response

$$y_{OS} \geq (1 - 0.9y_\infty)(e^{pt_r} - 1) > 0$$
$$t_r \leq \frac{1}{p} ln\left(1 + \frac{y_{OS}}{1 - 0.9y_\infty}\right)$$

Bode Sensitivy Integral

Assume that:

- The controller stabilizes the closed-loop system
- The loop gain, $L(s) = P(s)C(s)$, has a ***relative degree*** of at least two (i.e., it has at least two more poles than zeros).

Let $N_p \geq 0$ denote the number of ORHP poles of $L(s)$ and denote the ORHP poles by $p_1, \ldots, p_{N_p}$. Then the sensitivity function must satisify

$$\int_0^\infty ln|S(j\omega)|\, d\omega = \sum_{i=1}^{N_p} Re(p_i) \geq 0$$

If there are no unstable poles, then the equation reduces to

$$\int_0^\infty ln|S(j\omega)|\, d\omega = 0$$

Where the negative area equals the positive area through a *waterbed effect.*

If there are an unstable poles, then the area of sensitivity increase must be *greater than* the area of sensitivity decrease.

3.5 Performance Limitations Due to ORHP Zeros

If the plant has an ORHP real zero at s = z, then good performance can be achieved if the closed-loop bandwidth is significantly smaller than z. ORHP Zeroes:

- Zeroes near the origin are worse than ORHP zeroes far from the origin
- ORHP zeroes lead to *undershoot* in the step responses
- You cannot get rid of CRHP zeroes, since $P(s)C(s)$ is the numerator of $T_{ry}(s)$

Lemma

$$\int_0^\infty y(t)e^{-zt}\,dt = 0$$
$$y_{US} \geq \frac{0.98y_\infty}{e^{zt_s}-1} > 0$$
$$t_r \leq \frac{1}{z}ln\left(1+\frac{0.98y_\infty}{y_{US}}\right)$$

Bandwidth rule of thumb given a ORHP zero:

$$\text{bandwidth } < z/2$$

Poisson Integral

$$\int_0^\infty ln|S(j\omega)|W(\omega)\,d\omega = \pi\sum_{i=1}^{N_p} ln\left|\frac{p_i+z}{p_i^*-z}\right| \geq 0$$
$$W(\omega) = \frac{2z}{z^2+\omega^2}$$

4.1 Basic MIMO Concepts

$$y = (I+PCH)^{-1}PCr$$
$$L_o = PC$$
$$S_o = (I+L_o)^{-1}$$
$$T_o = L_o(I+L_o)^{-1}$$
$$S_o(s)+T_o(s) = I$$

The *minor* of a matrix is the determinant of any square submatrix obtained by deleting certain row's and/or columns of the matrix

The *rank* of a matrix is the max number of linearly independent column/row vectors.

To find $\phi(s)$, find all the minors. 1st order is just the element, while second order is as described above. Find the lowest common denominator of all non-zero minors.

A MIMO system is stable iff all the individual elements of the system are stable, and if all MIMO system poles are in the OLHP

The zero polynomial $z(s)$ is the greatest common divisor of all the r-th order minors. Steps:

1. Find the minors for the MIMO system
2. Determine $\phi(s)$
3. Make each minor as a fraction over $\phi(s)$
4. Find the GCD of the r-th order minors.

4.2 Decentralized Control

- Assume plant is square
- Ignore cross-channel interactions in the plant, resulting in a MIMO controller that is diagonal
- If there is too much coupling, then the resulting system is unstable

4.3 Decoupling Control

- A decoupling block is added: $P_{aug} = PW$
- Design a decentralized controller for the augmented plant: $C = W\bar{C}$

Steps:

1. Decouple the plant at dc. i.e. Solve $P(0)$
2. Solve $W(s) = P_1^{-1}(0)$, to get $P_{aug} = P * W$
3. Solve for the controllers, ignoring the cross-channel interactions
4. $C(s) = W\bar{C}(s)$

4.4 Sequential Loop Closure

- Accounts for cross-channel interactions
- Feedback loops are closed sequentially instead of in parallel
- **This approach only applies if the plant has a triangular structure.**
- Ex:
  - Given Plant:
    $$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}$$
    $y_1 = P_{11}u_1 + P_{12}u_2$ and $y_2 = P_{22}u_2$
  - first design C_{22} to stabilize P_22 since it is independent of P_{11} and P_{12}
  - Then design C_{22}

5.1 Basic State-Space Framework and Manipulations

State-space models are not unique. Infinite number of state-space realizations that correspond to the same transfer functions They follow the form:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Manipulation 1: Convert from SS to TF

$$Y(s) = [D + C(sI - A)^{-1}B] \cdot U(s)$$

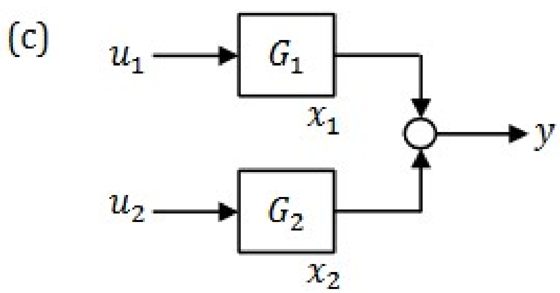
Manipulation 2: Poles and zeros of a SS system

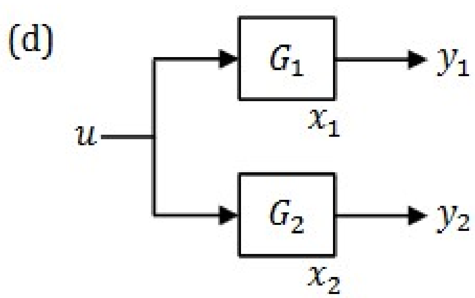
- Poles of the SS system are $eig(A)$ == roots of $det(sI - A)$
- Zeroes are the values of s where the rank of the following matrix falls below its normal rank:

$$R(s) = \begin{bmatrix} (sI - A) & -B \\ C & D \end{bmatrix}$$

  Which can be computed by finding the the roots of $det(R(s))$

Manipulation 3: Interconnecting state-space models





(c)

$\dot{x}_1 = A_1x_1 + B_1u_1$
$\dot{x}_2 = A_2x_2 + B_2u_2$
$y = (C_1x_1 + D_1u_1) + (C_2x_2 + D_2u_2)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 & D_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

(d)

$\dot{x}_1 = A_1x_1 + B_1u$
$\dot{x}_2 = A_2x_2 + B_2u$
$y_1 = (C_1x_1 + D_1u)$
$y_2 = (C_2x_2 + D_2u)$

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}u$$
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}u$$

Manipulation 4: Converting from TF to SS

- SISO case, creates a SS with controllable canonical form (CCF):

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{bmatrix} x + \begin{bmatrix} d \end{bmatrix} u$$

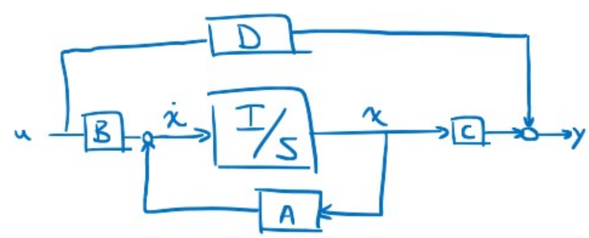
- SIMO case:

$$G(s) = \begin{bmatrix} A_1 & 0 & \cdots & B_1 \\ 0 & A_2 & & B_2 \\ \vdots & & \ddots & \vdots \\ C_1 & 0 & \cdots & D_1 \\ 0 & C_2 & & D_2 \\ \vdots & & \ddots & \vdots \end{bmatrix}$$

- MIMO case:

$$G(s) = \begin{bmatrix} A_1 & 0 & \cdots & B_1 & 0 & \cdots \\ 0 & A_2 & & 0 & B_2 & \\ \vdots & & \ddots & \vdots & & \ddots \\ C_1 & C_2 & \cdots & D_1 & D_2 & \cdots \end{bmatrix}$$

Manipulation 5: Block Diagram of SS Model



5.2 Controllability, Observability, Stabilizability, and Detectability

Modes that show up on the output y(t) are said to be *Observable.* If all modes are observable, the state-space realization is said to *observable.* If not, it's *unobservable*

Modes that are affected by the input are said to *controllable* If all modes are controllable, the state-space realization is said to *controllable.* If not, it's *uncontrollable*

A state-space realization where all unstable modes are controllable is said to be *stabilizable.*

$$\text{controllability} \Rightarrow \text{stabilizability}$$

A state-space realization where all unstable modes are observable is said to be *detectable.*

$$\text{observability} \Rightarrow \text{detectability}$$

A system is controllable and observable iff the corresponding TF matrix has no P-Z cancellations

A system is stabilizable and detectable iff the corresponding TF matrix has no <u>unstable</u> P-Z cancellations

5.2.2 Controllability and Stabilizability

A, B is controllable iff the n x (nm) *controllability matrix* has rank n.

$$M_c = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Using the PBH controllability test, A, B is controllable iff

$$\text{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n \text{ for all } \lambda \in \text{eig}(A)$$

Recall: eig(A) is found as the solutions to λ, for $\det(A - \lambda I)$

5.2.3 Observability and Detectability

A, C is observable iff the (pn) x n *observability matrix* has rank n.

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Using the PBH observability test, A, C is controllable iff

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n \text{ for all } \lambda \in \text{eig}(A)$$

Recall: eig(A) is found as the solutions to λ, for $\det(A - \lambda I)$

5.3 Plain State Feedback Control (Pole Placement)

The plain state-feedback controller $u = -Kx + r$ can arbitrarily place poles. The plant **must be controllable**. If it is uncontrollable but stabilizable, we cannot place poles arbitrarily but we can find a K to stabilizie $A - BK$.

K is a row vector.

Steps:

$$A_{CLS} = A - BK$$
$$\pi(s) = det(sI - A_{CLS})$$
$$\pi_{des}(s) = \pi(s)$$

CLS poles are at $\text{eig}(A_{CLS}) = \text{eig}(A - BK)$

5.4 Observer

Consider the following Plant:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and Observer

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y})$$
$$\hat{y} = C\hat{x} + Du$$

The estimation error is given by $e(t) = x(t) - \hat{x}(t)$. Then:

- $\dot{e} = (A - HC)e$, and $e(t = \infty) = 0$ for any IC $e(0)$ iff $\text{eig}(A - HC) \in$ OLHP.
- The poles of $\text{eig}(A - HC)$ can be placed arbitrarily iff (A, C) is <u>observable</u>.
- The poles of $\text{eig}(A - HC)$ can be placed in OLHP iff (A, C) is only <u>detectable</u>.

H is a column vector.

Steps:

$$\pi(s) = det(sI - (A - HC))$$
$$\pi_{des}(s) = \pi(s)$$

Resulting state-space realization:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ HC & A - HC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix} u$$
$$\bar{y} = I \cdot \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

5.4 Observer-based state-feedback control

Mixture of solving for K and H, using the above steps. The CLS for the observer-based state-feedback controller:

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

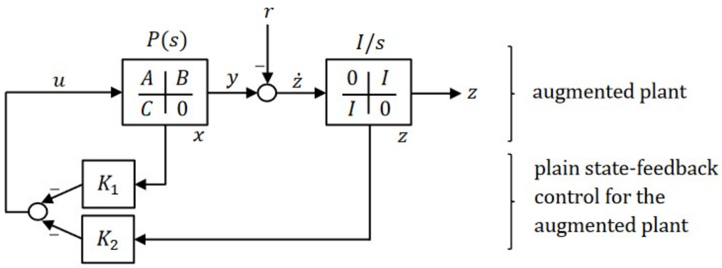
The closed loop poles are found at $\text{eig}(A - HC) \cup \text{eig}(A - BK)$

General rule-of-thumb is to place $\text{eig}(A - HC)$ about twice as fast (i.e., real parts twice as large) as $\text{eig}(A - BK)$.

K: Arbitrary pole placement is possible iff (A, B) is controllable, and stabilization is possible iff (A, B) is stabilizable.

H: Arbitrary placement of poles of the observer error-dynamics is possible iff (A, C) is observable, and they can be placed (not necessarily arbitrarily) in the OLHP iff (A, C) is detectable.

5.5 Plain state-feedback controller with integral action



$$\begin{bmatrix}\dot{x}\\ \dot{z}\end{bmatrix} = \begin{bmatrix}A & 0\\ C & 0\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}B\\ 0\end{bmatrix}u + \begin{bmatrix}0\\ -I\end{bmatrix}r$$
$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u + \tilde{B}_r r$$

The CLS poles ($\text{eig}(\tilde{A} - \tilde{B}K)$) can be placed arbitrarily iff $(\tilde{A}, \tilde{B})$ is controllable Essentially, the same steps as in 5.3, however A is replaced with $\tilde{A}$. Still need to calculate the characteristic polynomial using $\det(sI - (\tilde{A} - \tilde{B}K))$

5.5 Observer-based state feedback controller with integral action

Essentially, the same steps as in 5.4, however A is replaced with $\tilde{A}$.

CLS poles: $\text{eig}(\tilde{A} - \tilde{B}K) \cup \text{eig}(A - HC)$

K: Arbitrary pole placement is possible iff $(\tilde{A}, \tilde{B})$ is controllable. Stabililization is possible iff $\tilde{A}, \tilde{B}$ is stabilizable.

H: Arbitrary placement of poles of the observer error-dynamics is possible iff (A, C) is observable. They can be placed (not necessarily arbitrarily) in the OLHP iff (A, C) is detectable.

5.6 Optimal State-feedback control

Where $Q \geq 0$ and $R > 0$, minimize:

$$J = \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

Assume that (A, B) is stabilizable and (A, Q) is detectable.

The resulting controller is the stabilizing linear state-feedback controller, $u(t) = -Kx(t)$, where K is computed by finding the unique symmetric PSD solution P to the algebraic Riccati equation (ARE):

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

and setting $K = R^{-1}B^T P$.

Q and R are tuning knobs. Q is commonly a diagonal matrix where $q_{ii} \geq 0$. This results in:

$$x(t)^T Q x(t) = q_{11}x_1^2(t) + q_{22}x_2^2(t) + \cdots + q_{nn}x_n^2(t)$$

Q is commonly chosen as $Q = C^T C$. This leads to:

$$y(t) = Cx(t)$$
$$x(t)^T Q x(t) = ||y(t)||^2$$

R is also commonly a diagonal matrix.

Decreasing R minimizes the control effort, and the poles go off to $-\infty$. Increasing R increases the control effort, and the poles approach the OL poles.

Changing R does the opposite effect.

Increasing Q and R by the same factor does not change K or the pole locations.

Matrix Things

- $\det\begin{bmatrix}a & b\\ c & d\end{bmatrix} = a\det(d) - b\det(c) = ad - bc$

  $\det\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & k\end{bmatrix} = +a\det\begin{bmatrix}e & f\\ h & k\end{bmatrix} - b\det\begin{bmatrix}d & f\\ g & k\end{bmatrix} + c\det$

  $= +a(ek - fh) - b(dk - fg) + c(dh - eg)$

- Inverse of matrix: $M^{-1} = \frac{adj(M)}{det(M)}$

  $$\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

  $$M^{-1} = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & k\end{bmatrix}^{-1} = \frac{1}{det(M)} adj(M)$$

  $$adj(M) = \begin{bmatrix} +det\begin{bmatrix}e & f\\ h & k\end{bmatrix} & -det\begin{bmatrix}b & c\\ h & k\end{bmatrix} & +det\begin{bmatrix}b & c\\ e & f\end{bmatrix}\\ -det\begin{bmatrix}d & f\\ g & k\end{bmatrix} & +det\begin{bmatrix}a & c\\ g & k\end{bmatrix} & -det\begin{bmatrix}a & c\\ d & f\end{bmatrix}\\ +det\begin{bmatrix}d & e\\ g & h\end{bmatrix} & -det\begin{bmatrix}a & b\\ g & h\end{bmatrix} & +det\begin{bmatrix}a & b\\ d & e\end{bmatrix}\end{bmatrix}$$

- $det(M) \neq 0$ for full rank M
- Rank of $n \times m$ matrix M is number of linearly independent columns, or number of linearly independent rows. Both will be the same.
- An $n \times n$ matrix A has n eigenvalues λ associated with n eigenvectors v that satisfy this equation $Av = \lambda v$.

  $$eig(A) = \{\lambda : \det(A - \lambda I) = 0\}$$

  Characteristic polynomial of A: $\pi(s) = \det(A - \lambda I)$

- Let M be a symmetric matrix. The scalar $x^T Mx$ is called a quadratic form. M is positive definite (PD) if $x^T Mx > 0$ for any vector $x \neq 0$ (shortform $M > 0$), it is positive semi-definite (PSD) if $x^T Mx \geq 0$ for any vector x (shortform $M \geq 0$). M is PD iff all eigs of M are positive, PSD iff all eigs are non-negative.

- For block triangular matrices

$$M = \begin{bmatrix} M_1 & 0 \\ M_2 & M_3 \end{bmatrix} \quad or \quad M = \begin{bmatrix} M_1 & M_2 \\ 0 & M_3 \end{bmatrix}$$

$$det(M) = det(M_1) \cdot det(M_3)$$

$$eig(M) = eig(M_1) \cup eig(M_3)$$