3.1 Youla Parameterization

For a stable plant. The set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{Q(s)}{1 - P(s)Q(s)} \right\}$$

where $Q(s)$ is proper and stable

Coprime Factorization

Let $P(s)$ be any proper TF. Then there exists stable proper TFs $N_P(s)$, $X_P(s)$, $M_P(s)$, and $Y_P(s)$ such that the following relationships hold for all s:

$$P(s) = \frac{N_P(s)}{M_P(s)}$$
$$N_P(s)X_P(s) + M_P(s)Y_P(s) = 1$$

Youla Parameterization for General $P(s)$

For an unstable plant. Assume there are no CRHP pole-zero cancellations in $P(s)$. Perform a coprime factorization on $P(s)$. Then the set of all stabilizing controllers equals the following set:

$$\left\{ C(s) : C(s) = \frac{X_P(s) + M_P(s)Q(s)}{Y_P(s) - N_P(s)Q(s)} \right\}$$

where $Q(s)$ is proper and stable

3.2 Performance Limitations

At frequencies where $|S(j\omega)| \ll 1$, good tracking, good disturbance rejection, and good sensitivity. Exact opposite when $|S(j\omega)| \gg 1$.

$$S(s) = \frac{1}{1 + L(s)}$$

Ideally:

- $|S(j\omega)|$ is small when $|L(j\omega)|$ is large
- $|S(j\omega)|$ is near one (0 dB) when $|L(j\omega)|$ is small

The complementary sensitivity, $T(s)$, tells us that robust stability is achieved iff there is nominal stability and:

$$|W(j\omega)T_0(j\omega)| < 1 \text{ for all } \omega$$
$$\Leftrightarrow |T_0(j\omega)| < \frac{1}{|W(j\omega)|} \text{ for all } \omega$$

It also tells us how the closed-loop system responds to sensor noise.

$$T(s) = \frac{L(s)}{1 + L(s)}$$

Ideally:

- $|T(j\omega)|$ is small (in fact, $|T(j\omega)| \approx |L(j\omega)|$) at high frequencies when sensor noise is significant, and little feedback effort is used
- $|S(j\omega)|$ is near one (0 dB) at low frequencies, where lots of feedback effort is used

The relationship between the two equations:

$$S(j\omega) + T(j\omega) = \frac{1}{1 + L(j\omega)} + \frac{L(j\omega)}{1 + L(j\omega)} = 1 \quad (1)$$
$$|S(j\omega)| + |T(j\omega)| \geq 1 \text{ for each } \omega \quad (2)$$

Notes:

- tradeoff between the reasons for wanting $|S(j\omega)|$ to be small and the reason for wanting $|T(j\omega)|$
- both cannot both be "good" (i.e., very small) at the same frequency, but it is possible for both to be "bad" (i.e., very large) at the same frequency
- We never want the magnitude bode plots of S(s) or T(s) to have large peaks

3.3 Interpolation Constraints Assume that the closed-loop system is stable.

- If $P(s)C(s)$ has a CRHP zero at $s = z$, then $S(z) = 1$ and $T(z) = 0$
- If $P(s)C(s)$ has a CRHP pole at $s = p$, then $S(p) = 0$ and $T(p) = 1$

3.4 Performance Limitations Due to ORHP

Bandwidth rule of thumb is

$$\text{bandwidth } > 2p$$

Lemma

If:

- The closed-loop system is stable
- The plan has an open-loop unstable real pole at $s = p > 0$
- The reference signal is a unit step

Then the tracking error, $e(t) = r(t) - y(t)$ necessarily satisfies:

$$\int_0^\infty e(t)e^{-pt}\, dt = 0$$

An ORHP pole in the plant leads to overshoot in the closed-loop step response

$$y_{OS} \geq (1 - 0.9y_\infty)(e^{pt_r} - 1) > 0$$
$$t_r \leq \frac{1}{p} ln \left(1 + \frac{y_{OS}}{1 - 0.9y_\infty}\right)$$

Bode Sensitivy Integral

Assume that:

- The controller stabilizes the closed-loop system
- The loop gain, $L(s) = P(s)C(s)$, has a ***relative degree*** of at least two (i.e., it has at least two more poles than zeros).

Let $N_p \geq 0$ denote the number of ORHP poles of $L(s)$ and denote the ORHP poles by $p_1, \ldots, p_{N_p}$. Then the sensitivity function must satisify

$$\int_0^\infty ln|S(j\omega)|\, d\omega = \sum_{i=1}^{N_p} Re(p_i) \geq 0$$

If there are no unstable poles, then the equation reduces to

$$\int_0^\infty ln|S(j\omega)|\, d\omega = 0$$

Where the negative area equals the positive area through a *waterbed effect.*

If there are an unstable poles, then the area of sensitivity increase must be *greater than* the area of sensitivity decrease.

3.5 Performance Limitations Due to ORHP Zeroes

If the plant has an ORHP real zero at s = z, then good performance can be achieved if the closed-loop bandwidth is significantly smaller than z. ORHP Zeroes:

- Zeroes near the origin are worse than ORHP zeroes far from the origin
- ORHP zeroes lead to *undershoot* in the step responses
- You cannot get rid of CRHP zeroes, since $P(s)C(s)$ is the numerator of $T_{ry}(s)$

Lemma

$$\int_0^\infty y(t)e^{-zt}\, dt = 0$$
$$y_{US} \geq \frac{0.98 y_\infty}{e^{zt_s} - 1} > 0$$
$$t_r \leq \frac{1}{z} ln\left(1 + \frac{0.98 y_\infty}{y_{US}}\right)$$

Bandwidth rule of thumb given a ORHP zero:

$$\text{bandwidth } < z/2$$

Poisson Integral

$$\int_0^\infty ln|S(j\omega)| W(\omega)\, d\omega = \pi \sum_{i=1}^{N_p} ln\left|\frac{p_i + z}{p_i^* - z}\right| \geq 0$$
$$W(\omega) = \frac{2z}{z^2 + \omega^2}$$

4.1 Basic MIMO Concepts

$$y = (I + PCH)^{-1} PCr$$
$$L_o = PC$$
$$S_o = (I + L_o)^{-1}$$
$$T_o = L_o(I + L_o)^{-1}$$
$$S_o(s) + T_o(s) = I$$

The *minor* of a matrix is the determinant of any square submatrix obtained by deleting certain row's and/or columns of the matrix

The *rank* of a matrix is the max number of linearly independent column/row vectors.

To find $\phi(s)$, find all the minors. 1st order is just the element, while second order is as described above. Find the lowest common denominator of all non-zero minors.

A MIMO system is stable iff all the individual elements of the system are stable, and if all MIMO system poles are in the OLHP

The zero polynomial $z(s)$ is the greatest common divisor of all the r-th order minors. Steps:

1. Find the minors for the MIMO system
2. Determine $\phi(s)$
3. Make each minor as a fraction over $\phi(s)$
4. Find the GCD of the r-th order minors.