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大连大学学士学位论文

对变质量系统动力学公式验证数据的分析

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摘　要

目前，关于变质量系统动力学公式，存在多种不同的表达形式。主流的变质量系统动力学公式含有质量变化率的项，是纯粹的数学形式的外推和猜测或者数据的拟合，但是缺乏科学的论证和有效的实验证明。由于变质量动力学方程的正确与否关系到许多以此为前提的工作的可靠性，因此用实验验证现有的变质量系统动力学方程是必要的。本文就此问题，设计了相关实验，并对选取的两种动力学公式进行验证。通过实验，本文测得了沙漏经过气垫导轨上定长距离的始末速度，时间间隔，以及该过程中的漏沙量，并用三种不变质量的情况估算出变质量时的空气阻力及误差范围。在预测前，本文还就测得的数据的合理性作了一定的分析，并定义了另一种间接测得的时间间隔和均速度。在此基础上，本文又用已选取的两种动力学公式对末速度和时间间隔分别预测，并与实际数据作比较。本文发现在实验误差允许范围内，实际测得的时间间隔和末速度与第二种动力学方程给出的预测基本一致，但与第一种动力学方程给出的预测相差甚远．因此本文对两种动力学方程的正确性作出明确判断：$f_r = dmv/dt$是错误的,$f_r = m(t)dv/dt$是正确的．因此，任何以$f_r = dmv/dt$ 为前提的工作都是不可靠的．

关键词：变质量系统；动力学公式；实验验证

Abstract

Currently,there are many different forms of expression on the dynamic equation of variable mass systems. The mainstream of the dynamic equation of variable mass systems which contains the rate of mass change is just the extrapolation and speculation of the pure mathematical form or data fitting, but all of these theories lack for scientific demonstration and effective experimental proof. Since whether the dynamic equation of variable mass is right or not relates to the reliability of much work which are based on the equation, for this reason, it is necessary that using the experimental results to prove the dynamic equation of variable mass systems. In view of this problem, this paper designs some relative experiments, and verifies the two selected dynamic formula. This paper measures the speed of the start and end, the time interval and the amount of the sand in this process of the sandglass while it passes through the fixed distance of the air track by the experiment. Meanwhile, this paper uses three invariant mass situations to estimate the air resistance and its error range when the mass varies. Before the forecast, this article also analyzes the reasonableness of the measured data, and defines the time interval and the mean velocity which are measured by the other indirect method. On this basis, this paper predicts the final speed and the time interval respectively by the two dynamic formulas as mentioned earlier, and compares the results of the prediction with the actual data. This article finds that in the range of experimental allowable error the actual measured time interval and the final speed are consistent with the predictions which are given by the second dynamic equation, but quite different from the predictions which the first dynamic equation gives. Therefore, this paper makes specific judgments about the validity of the two dynamic equations that the $f_r = dmv/dt$ is wrong, while the $f_r = m(t)dv/dt$ is right. Therefore, any puts the $f_r = dmv/dt$ to use as the prerequisite for the work is not reliable.

Key words: Variable mass system；Dynamic equation；Experimental proof

目 录

1 绪论

尽管对变质量系统的研究已有超过两百年的历史,但在这个邻域的研究却比较零星. 目前，有记载的最早研究质量改变的物体问题的是18世纪的瑞士物理学家柏努利[1], 他研究推力对有喷水装置的船的作用效果，并得出了这个特例下的一个运动方程. 捷克的科学家冯布阔伊[2]是第一个系统研究一般变质量系统的人. 在1812年，他得到他的关于变质量系统的运动方程，并开始着手用他的方程解决大量的实例. 冯布阔伊的工作可以说是变质量系统理论延生的标志. 1812年，威廉穆尔[3]在英格兰用数学方法推导出他的火箭运动方程. 1819年，泊松[4]基于拉格朗日方程，得到了变质量系统一般公式. 在1856出版的一书中，塔特和斯蒂尔[5]加进了一个质量变化率的项，并假定质量的改变会对物体有一个连续的沿运动方向的推动力. 自此以后，不停的产生各种版本的变质量系统的动力学方程. 直到现在还存在着好多种版本. 其中一种较为主流的版本为：

$$-f_r = \frac{\mathrm{d}mv}{\mathrm{d}t}$$

还有一种猜想：

$$-f_r = m(t)\frac{\mathrm{d}v}{\mathrm{d}t}$$

1.1 选题意义

虽然前人对于变质量系统动力学公式的研究有各种不同的版本，但就其研究方法却基本只有以下两种：

- 寻找一种或者说自己创造一种看起来似乎合理的表达式.主要表现在已有的方程中加入各种修补项，当然包括上文提到的质量变化项. 填增修补项的主要目的是对实验数据更好的接近和拟合，使得其表达式看似更合理，而不关心真正的物理意义.

- 把变质量系统看作不变质量系统来处理并推导出表达式. 主要表现为把物体分成单元，一份一份的沿平行于速度方向抛出去(不具一般性)，再用

牛顿第二定律来解．大多教科书给出的变质量系统的动力学方程就是用这种方法推导出来的.

前人研究变质量系统动力学公式的方法，必定导致其研究成果缺乏科学的论证和有效的实验证明．对于变质量系统中,现有的主流的动力学方程的多工作只停在数据的拟合和理论推导,存在以下不足:

- 对于变质量系统中，现有的主流的多种形式动力学方程中几乎都包含了质量变化率的项，是纯粹的数学形式的外推和猜测或者数据的拟合，不关心其真正的物理意义.
- 并没有可靠的物理实验证据证明其正确性。变质量动力学方程的正确与否，关系到许多以此假说为前提的工作的可靠性，因此用实验证现有的变质量系统动力学方程是必要的。

1.2 主要研究方法及步骤

本文将理论推导,实验验证以及数据处理三种方法相结合. 对变质量系统的两种动力学方程的准确性做出判断．具体做法如下：

- 从以下两种动力学方程理论推导出便于实验验证的形式.

$$-f_r = \frac{\mathrm{d}mv}{\mathrm{d}t}$$

$$-f_r = m(t)\frac{\mathrm{d}v}{\mathrm{d}t}$$

- 通过实验计设和测量，测得验证动力学方程所需的相关数据.
- 用计算机对实验数据进行处理，将两种动力学方程得出的末速度和末了时间的预测与实验值分别作对比,以对变质量系统的两种动力学方程的准确性做出判断。

1.3 待解决的问题

通过对研究变质量动力学方程的方法及步骤的确定，本文待解决的问题主要有下面三方面：

- 根据两种基本假设的动力学方程,给出两种动力学方程各自的末速度和时间间隔的预测.
- 实验装置的合理性和可行性分析．实验数据独立的核实.
- 通过对实验数据的分析，给出真实末速度和时间间隔及其误差.将两种动力学方程得出的末速度和末了时间的预测与实验值分别作对比, 以便对变质量系统的两种动力学方程的准确性做出判断。在数据的分析及计算时,计算平均阻力是关键.

2 两种公式

我们选取两种动力学公式作为研究对象，分别为：

$$-f_r = \frac{dmv}{dt} \tag{1}$$

$$-f_r = m(t)\frac{dv}{dt} \tag{2}$$

下面我们将结合实验，并由方程1和方程2推导出两种便于实验验证的形式.

2.1 第一种预测公式

对方程1两边同时积分

$$-\int_{t_0}^{t'} f_r dt = \int_{t_0}^{t'} dmv$$

得

$$\bar{f}_r \cdot (t' - t_0) = m_0 v_0 - m'v'$$

因此时间时隔和末速度分别为

$$\tau = t' - t_0 = \frac{m_0 v_0 - m'v'}{\bar{f}_r} = \frac{m_0 v_0 - (m_0 - \Delta m)v'}{\bar{f}_r} \tag{3}$$

$$v' = \frac{m_0 v_0 - \bar{f}_r \cdot (t' - t_0)}{m'} = \frac{m_0 v_0 - \bar{f}_r \tau}{m'} = v_0 + \frac{1}{m_0 - \Delta m}(\Delta m v_0 - \bar{f}_r \tau) \tag{4}$$

2.2 第二种预测公式

对方程2两边同时积分

$$-\int_{t_0}^{t'} f_r dt = \int_{t_0}^{t'} m(t)\frac{dv}{dt} dt = mv\Big|_{t_0}^{t'} - \int_{t_0}^{t'} v\frac{dm}{dt} dt$$

即

$$-\bar{f}_r \tau = m'v' - m_0 v_0 - \overline{v\frac{dm}{dt}}\tau \tag{5}$$

因此时间时隔和末速度分别为

$$\tau = \frac{m_0 v_0 - m'v'}{\bar{f}_r - \overline{v\frac{dm}{dt}}} = \frac{m_0 v_0 - (m_0 - \Delta m)v'}{\bar{f}_r - \overline{v\frac{dm}{dt}}} \tag{6}$$

$$\begin{aligned} v' &= \frac{-\bar{f}_r \tau + m_0 v_0 + \overline{v\frac{dm}{dt}}\tau}{m'} \\ &= v_0 + \left(\frac{m_0}{m'} - 1\right)v_0 - \left(\bar{f}_r - \overline{v\frac{dm}{dt}}\right)\frac{\tau}{m'} \\ &= v_0 + \frac{1}{m_0 - \Delta m}\left[\Delta m v_0 - \left(\bar{f}_r - \overline{v\frac{dm}{dt}}\right)\tau\right] \end{aligned} \tag{7}$$

我们通过对实验数据分析，我们发现速度和质量的变化率几乎是时间的一次函数,并由此，我们可以证明$\overline{v\frac{dm}{dt}} \lessapprox \bar{v}\overline{\frac{dm}{dt}}$. 证明过程见附录A.

2.3 两种预测公式的比较

上文我们分别得出了两种预测公式，为了对比方便，我们把两种预测公式列在一起：

- 两种时间间隔的预测

$$\tau = t' - t_0 = \frac{m_0 v_0 - (m_0 - \Delta m)v'}{\boxed{\bar{f}_r}} \tag{8}$$

$$\tau = \frac{m_0 v_0 - (m_0 - \Delta m)v'}{\boxed{\bar{f}_r - \overline{v\frac{dm}{dt}}}} \tag{9}$$

- 两种末速度的预测

$$v' = v_0 + \frac{1}{m_0 - \Delta m}[\Delta m v_0 - \boxed{\bar{f}_r}\tau] \tag{10}$$

$$v' = v_0 + \frac{1}{m_0 - \Delta m}\left[\Delta m v_0 - \boxed{\left(\bar{f}_r - \overline{v\frac{dm}{dt}}\right)}\tau\right] \tag{11}$$

很显然，两二种预测公式差别仅在$\bar{f}_r$ 和$\bar{f}_r - \overline{v\frac{dm}{dt}}$上，而$-\overline{v\frac{dm}{dt}}$远远大于0, 因此这个差别也必将导致两种预测结果的明显差别:第一种公式在时间间隔和末速度的预测都将比第二种公式的预测大.

3 实验的设计及测量

3.1 实验设计

对于上面的推出的两种变质量的末速度和通过定长的时间间隔的预测，需要对变质量情况下的平均阻力$\bar{f}_r$进行估算或测量. 变质量使我们想到了最简单的沙漏，我们可以让沙漏一边漏沙，一边通过一定距离.

对于恒定质量下的平均阻力，根据下式我们可以测算得到不变质量下的平均阻力.

$$\bar{f}_r = m\bar{a} = m\frac{v_0 - v'}{\tau}$$

要测不变质量的平均阻力，我们得测一个不变质量在一段运动的初末速度以及时间间隔. 这个使我们想到在一根滑轨上，安置两个光电门，让滑块以一初速依次通过两光电门，通过光电门可测算通过其的速度，以及经过两光电门间的时间间隔.

而对于变质量情况下的阻力却无法直接或间接地测量. 基于此，我们不得不设法从不变质量的平均阻力引出变质量情况下的平均阻力. 考虑到变质量过程中，质量和速度都在不断变化，因此，我们想到用一系列具有特定恒定质量，特定的平均速度系统的平均阻力组合出变质量情况下的阻力，并给出误差范围，以保证真实阻力必定落在我们给出的误差范围内.

综合上述考虑，我们设计了如图1的实验装置，考虑变质量时，在滑轨上让沙漏一边漏沙，一边以一初速度依次通过两光电门，以测得真实初末速度和时间间隔. 再用一些定质量来组合出变质量这种情况下的阻力，以得到预测所需的所有量，以及实际值. 这样我们即可完成预测，而且有实际数据与预测对比，最终判断出预测的准确性.

3.2 实验装置

我们的实验装置如图1(实际的实验装置与该图略有区别，但本质上没有区别)，实验中所用到的仪器有：

1. 气垫导轨及附件
2. 电脑通用计数器
3. 两个计速光电门
4. 微音气泵(型号:DC-III)
5. 双沙漏滑块(自制:由一个滑块和两沙漏组成)

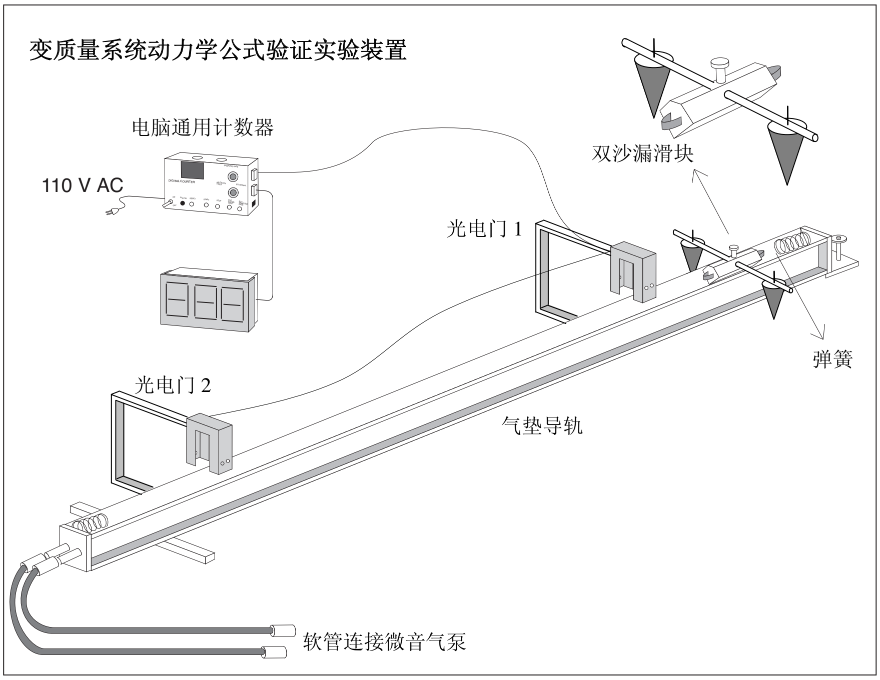

图 1: 实验装置图.

3.3 实验步骤

实验中，我们让沙漏带着一定量的沙子以一定的初速度依次通过两个光电门．具体的实验步骤和注意事项如下：

1. 调整气轨,使气轨处于水平状态(认为当滑块在气轨上能静止或低速的作往复运动时为水平状态)．并将两个光电门拉开一定距离，固定在气轨底座上．

2. 测量多组不变质量以计算平均阻力时．给两沙漏中装入等量的一定量沙子，并调平沙漏．然后以一定的初速度让沙漏依次通过两个相距1m的光电门．注意此过程中沙漏不漏沙．

3. 测量多组变质量的情况．与步骤2类似，但此过程中沙漏处开自由漏沙状态，即运动过程中沙漏的质量在不断减小．

3.4 实验测得的数据

实验中我们要测得或简单计算出的不变质量和变质量情况下的物理量(数据见附录B和C)及其记号如下：

- 初速度v_0：经过光电门1的速度.非直接测得量，实际上是滑块通过光电门1的平均速度(遮光宽度Δs/遮光时间Δt).

- 末速度v'：经过光电门2的速度．(类似光电门1)

- 时间间隔τ：经过两光电门的时间差(为便于计算，记经过光电门1的时刻为$t_0 = -\tau/2$,经过光电门2的时刻为$t' = \tau/2$,).

- 距离s：两光电门间的距离.

- 初质量m_0：经过光电门1时的质量．非直接测得量，实际上是用实验前装入沙子后的总质量减去经过光电门1前漏掉的质量.

- 漏沙质量Δm：双沙漏滑块在经过两光电门之间漏掉沙子的质量．经过光电门2时的质量为m',则$\Delta m = m_0 - m'$. 当然漏掉沙子的质量仅对于变质量的情况而言的.

另外，为得到变质量过程中沙漏漏沙的规律，本文将对沙漏漏沙过程进行单独地研究，设计以下四种方法测沙漏漏沙的速度．具体测量数据见附录D.

- **静态法:** 将两漏斗装入等量给定质量的砂子，然后测量砂子全部漏完的时间，以计算漏砂的速度.

- **动态法一:** 此方法是将漏斗(单个)装好给定质量砂子后，当松开漏斗口开始漏砂时，按下秒表．到了一定时间用一硬薄木板截砂，并按下秒表．当砂漏完时，按下秒表．并称量木板上的砂子质量.

- **动态法二:** 该方法是在单只漏斗漏砂过程中的某一时刻，用一只硬薄木板截砂，同是按下秒表，当砂子漏完时计下时间，并称量砂子的质量.

- **动态法三:** 单只漏斗漏砂过程中，用一圆形器具以一定速度经过漏砂，在器具两边沿与砂子接触时计下这两个时刻，并衡量器具所接到的砂子质量.

4 实验数据的处理

我们根据实验数据可以得到不变质量时的平均速度与平均阻力的关系．但是在变质量过程中的阻力我们无直接测量或得到．而两种两种公式的预测都是变质量情况下的预测，并且都用到了阻力，因此想要顺利进行两种公式的预测，就得设法由不变质量时阻力得到变质量时阻力．本文具体做法是：

1. 对实测的数据的合理性做一定的分析，为预测时采用什么样的数据提供参考.
2. 在不变质量情况下，得到三种质量的平均速度与平均阻力的关系.
3. 对三种质量的平均速度与平均阻力数据进行适当分组，组合出变质量过程中的阻力及误差区间.
4. 将第3步得到阻力及误差区间同其它数据一同代入两种预测公式进行预测.

4.1 实测数据的合理性分析

4.1.1 沙漏漏沙的速率的测量

沙漏漏沙的速率的测量实验过程中由于仪器和实验方法上的粗略，可能造成数据有较大误差，动态法一，二由于用一硬薄木板截沙，在截沙时

1. 由于空中还有少量沙的下落时间没有记录，所以可能会使实验结果偏大.
2. 在沙子下落完时，由于不能使漏斗口饱和下落沙子可能使实验结果偏小.

在此基础上，为了减小实验1,2的误差，并设计了实验动态三，当然，误差是避免不了的．由实验结果初步可以得出以下几个结论：

1. 在静态时，同一漏斗下，漏斗中沙子越多，沙子漏速越大．
2. 在动态时，同一漏斗下，漏斗沙子多，漏速大．
3. 同一质量的沙子，在动态漏速比静态漏速大．
4. 沙子从开始漏到结束，漏沙速率变化不大．

4.1.2 不变质量的测量

不变质量的测量数据将用于预测变质量过程中的阻力及其范围，因此在进行阻力预测前对不变质的数据合理性的检验是必要．对于不变质量的情况，由于滑块在运动过程中始终是减速的，实际数据需满足以下两个约束：

$$\frac{s}{v_0} < \tau < \frac{s}{v'} \tag{12}$$

$$v' < \frac{s}{\tau} < v_0 \tag{13}$$

但由于实验中，实验装置的限制，以及人为的一些操作原因，难免造成测得的数据误差较大，使得测得的数据并不一定满足上述约束．基于此，我们需要检测实测的数据是否满足以上约束．图2是恒定质量(392g)的s/v'及s/v_0与τ的比较，图3是恒定质量(392g) 的v_0,v',s/τ,$(v_0+v')/2$的比较.从图2和图3中，可以看出对于恒定质量(392g)的数据有三个异常点(图中已用红圈圈出)．

我们用同样的方法检测322.8g的实测的数据是否满足约束，图4和图5中，可以看出对于恒定质量(322g)的数据有一个异常点(图中已用红圈圈出)．

我们用同样的方法检测223.0g的实测的数据是否满足约束，图6和图7中，可以看出对于恒定质量(223g)的数据有一个异常点(图中已用红圈圈出)．

我们可以看到，对于不变质量的测量数据总体上还是比较合理的，只存在极个别数据不是很合理，但这些数据偏离理想位置也并不是很大．

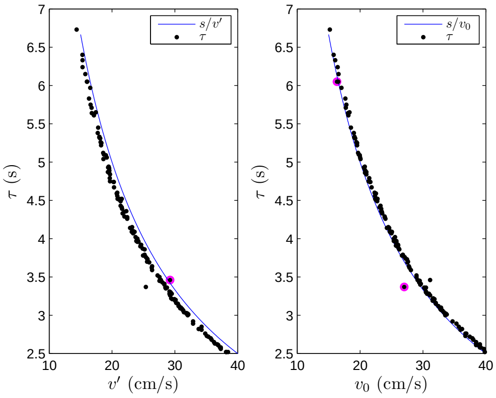

图 2: 392.6g 的s/v'及s/v_0与τ的比较

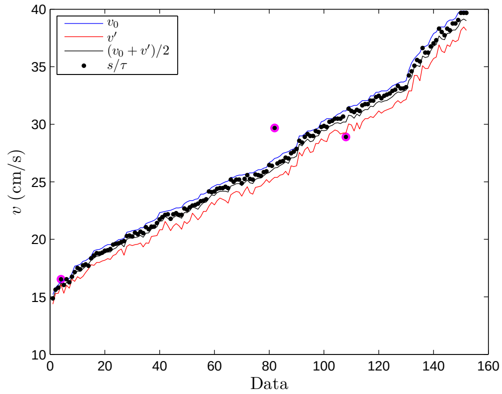

图 3: 392.6g 的v_0,v',s/τ,$(v_0+v')/2$的比较

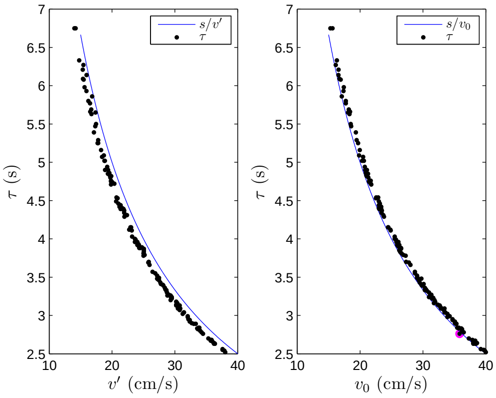

图 4: 322.8g 的s/v'及s/v_0与τ的比较

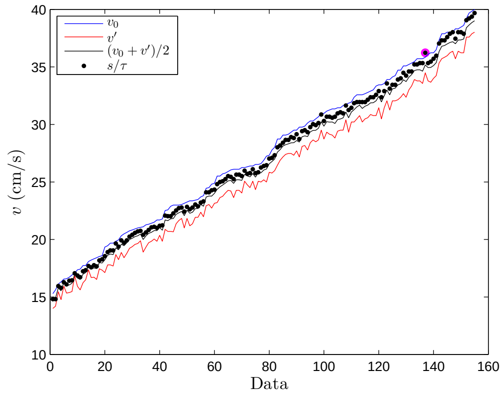

图 5: 322.8g 的v_0,v',s/τ,$(v_0+v')/2$的比较

4.1.3 变质量的测量

变质量的数据将用于与预测得出的结果对比，以判断两种理论的正确性．所以变质量的数据的合理性，对预测与实测数据的对比工作将产生影响．下面我们对变质量测量的数据进行检验，同样，变质量的数据也应满足式12和式13.图8是变质量情况的s/v'及s/v_0与τ的比较，图9是变质量情况的v_0,v',s/τ,$(v_0+v')/2$的比较.

从图8和图9中，可以看出对于变质量数据有11个异常点(图中已用红圈圈出).

相对于不变质量的测量数据，变质量的测量数据误差要大得多，异常数据

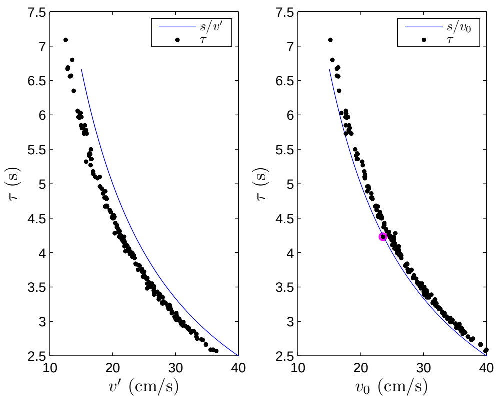

图 6: 223.0g 的s/v'及s/v_0与τ的比较

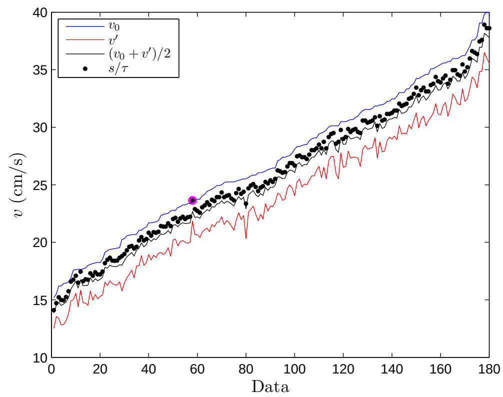

图 7: 223.0g 的v_0,v',s/τ,$(v_0+v')/2$的比较

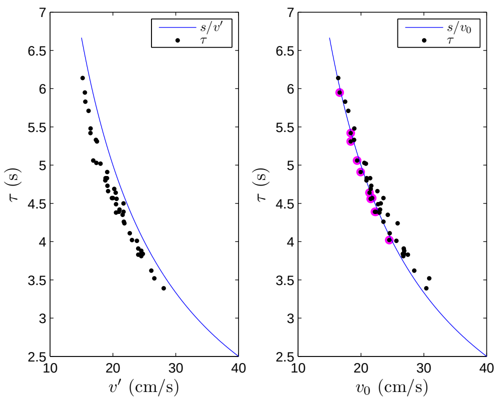

图 8: 变质量情况的s/v'及s/v_0与τ的比较

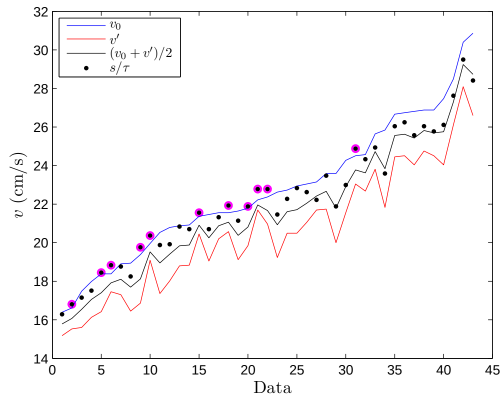

图 9: 变质量情况的v_0,v',s/τ,$(v_0+v')/2$的比较

点的个数达到了总数据量的1/4,这可能是变质量情况测量中，系统的稳定性不是很好导致的,不过这些数据离理想位置距离并不大，我们猜测，这些异常数据将对预测产生一定的影响，但这种影响一定是较小的，并且将不会对两种变质量公式的正确性判断起决定性作用．．

4.2 不变质量时的平均速度与平均阻力的关系

对于质量不变情况，我们很容易得到沙漏在两光电门间的平均速度为

$$\bar{v} = \frac{s}{\tau} \tag{14}$$

在4.4.1(实测数据的合理性分析)中,我们发现，我们测得的数据误差较大，并存在一些不合理的数据，为消除这些不合理数据影响，我们对平均速度$\bar{v}$ 和时间间隔τ作了另一种定义：

$$\bar{v}' = \frac{v_0 + v'}{2} \tag{15}$$

$$\tau' = \frac{s}{\bar{v}'} \tag{16}$$

本文下面的所有数据处理和预测，都是基于以上两种平均速度的定义(所有图的左边的图对应式14的定义，右图对应式15 和式16的定义,在下面的分析中，我们主要针对第一种定义).

平均阻力为

$$-\bar{f}_r = \overline{ma} = m\bar{a} = m\frac{v' - v_0}{\tau} \tag{17}$$

实验中，我们分别对223.0g, 322.8g, 392.6g 这三个质量水平做了多组测量，直接测得或简单计算可得的每组数据包括：τ,v',v_0. 再由式14和式17，我们得到3种质量水平下的多组平均速度与平均阻力，图10,图11和图12 分别为三种质量水平下平均速度与平均阻力的关系.

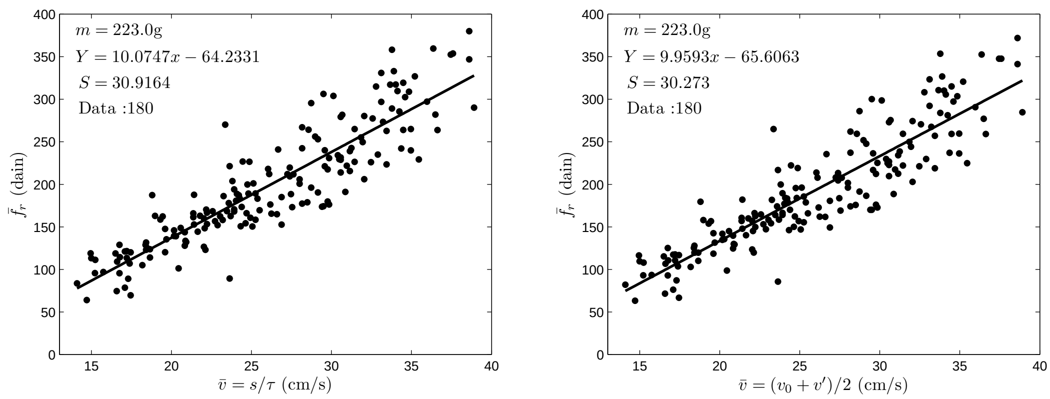

图 10: 223.0g质量水平下的平均速度与平均阻力的关系

分别比较图10，图11和图12中左右两图，我们发现两种不同平均速度定义下的定质量的平均速度与平均阻力的关系差异很小. 即不合理数据对定质量的平均速度与平均阻力的关系预测有一定的影响，但这种影响很少.

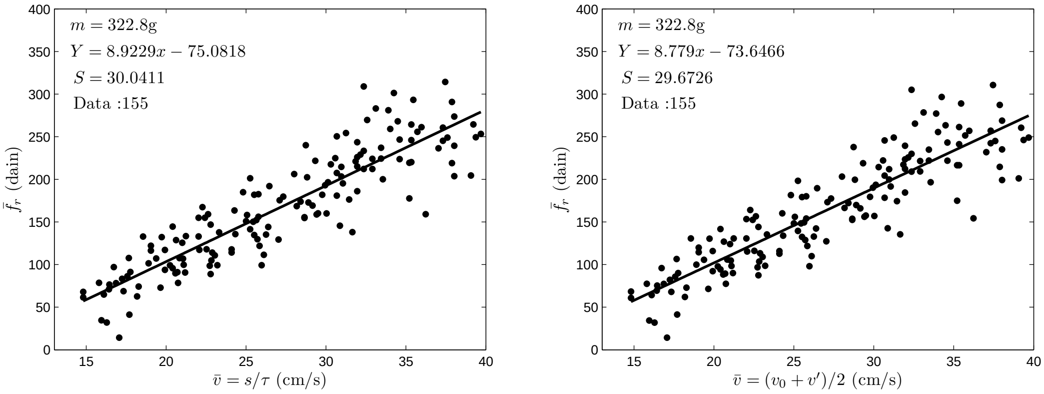

图 11: 322.8g质量水平下的平均速度与平均阻力的关系

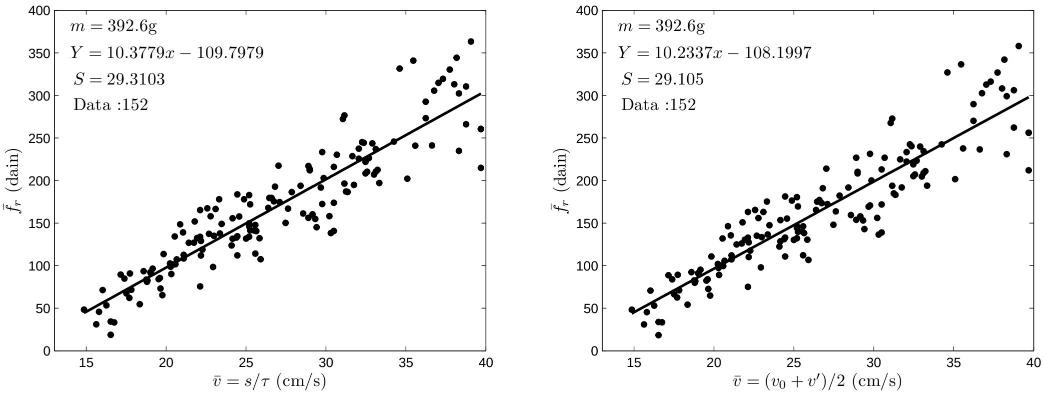

图 12: 392.6g质量水平下的平均速度与平均阻力的关系

4.3 不同速度,不同质量组合出阻力

变质量过程中的阻力我们无法直接测得，但我们可以利用不变质量的情况组合出变质量过程中的阻力．本文实验中，变质量系统在运动过程中，质量和速度都在不断减小．在所有变质量实验中，我们固定实验前装入沙子的质量为412.6g, 经过光电门1和光电门2的质量分别为 m_0, m'；经过光电门1和光电门2的速度分别为v_0, v'．所选三种质量(m_{s1}, m_{s2}, m_{s3})和速度(v_{s1}, v_{s2}, v_{s3})必需是递减的，并且在范围上要覆盖变质量情况的质量和速度变化范围，即满足以下关系

$$m_{s1} > m_0 > m_{s2} > m' > m_{s3}$$

$$\bar{v}_{s1} > v_0 > \bar{v}_{s2} > v' > \bar{v}_{s3}$$

本文用已测得三种固定质量(分别作为m_{s1}, m_{s2}, m_{s3})水平下多组实验数据，我们从定质量的数据中直接挑出符合条件的数据组．这样我们可得到平均阻力$\bar{f}_r = (\bar{f}_{rs1} + \bar{f}_{rs2} + \bar{f}_{rs3})/3$与平均速度$\bar{v} = (\bar{v}_{s1} + \bar{v}_{s2} + \bar{v}_{s3})/3$的关系如图13

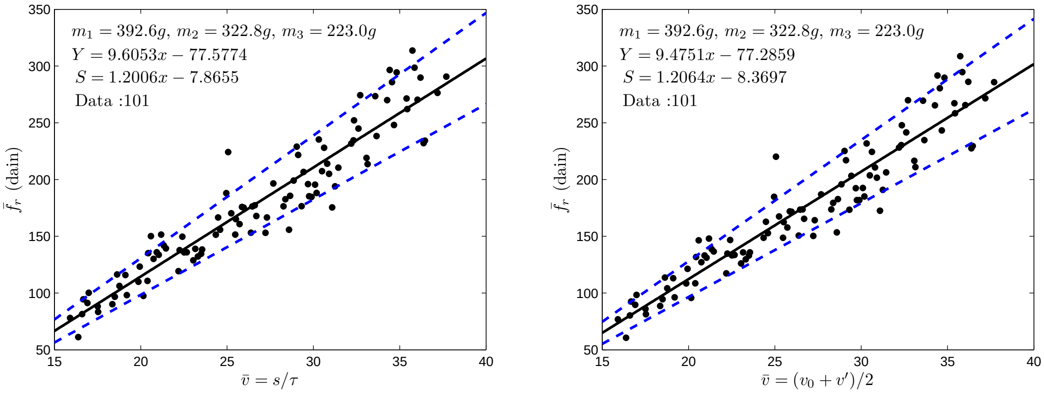

图 13: 不同质量不同速度组合出的平均速度与平均阻力的关系

其中，实线为均速度与平均阻力的一次拟合曲线，其方程如图13中Y表达式．另外，随着平均速度的不断增大，均方差也在不断增，因此我们还对均方差做了一次拟合，其方程如图13中S表达式. 图13中的虚线方程为$Y \pm S$.

另外，比较图13中左右两图，我们发现两种不同平均速度定义下的定质量的平均速度与平均阻力的关系差异很小．即不合理数据对变质量的平均速度与平均阻力的关系预测有一定的影响，但这种影响很少．原因也是很明显的，变质量的平均速度与平均阻力的关系是由三种不变质量的情况组合出来的，而不合理数据对三种不变质量的情况的影响就较小．

4.4 两种预测

上文中我们得到的两种公式的预测，两种预测公式中均有$\bar{f}_r$, 而本文是用组合出的阻力代替真实的平均阻力，由于组合出的阻力有一定的误差区间，必然使得本文的预测也有一定的误差区间．下面我们分别对末速度和时间间隔进行两种预测．

由于两种预测公式中均要求输入$\bar{f}_r$， 本文用图13 中三条直线分别作为$\bar{f}_r$的上限，均值，下限代入预测公式．同时还需代入以下的实测数据或由实测数据间接得到的数据：

- 预测τ: m_0, v_0, Δm, v'; 第二种预测还需$\overline{v\frac{\mathrm{d}m}{\mathrm{d}t}}$
- 预测v': m_0, v_0, Δm, τ; 第二种预测还需$\overline{v\frac{\mathrm{d}m}{\mathrm{d}t}}$

代入数据后，我们分别得到两种预测的末速度和时间间隔的上线，均值和下限，即图上的3种点, 再对43组实验数据做出了43组预测点,并做了线性拟合,如图14和图15所示.

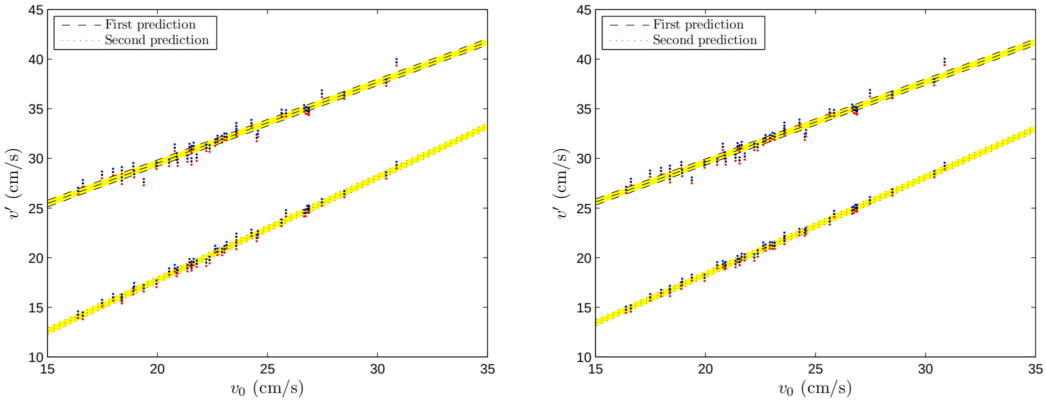

图 14: 两种末速度预测及其误差

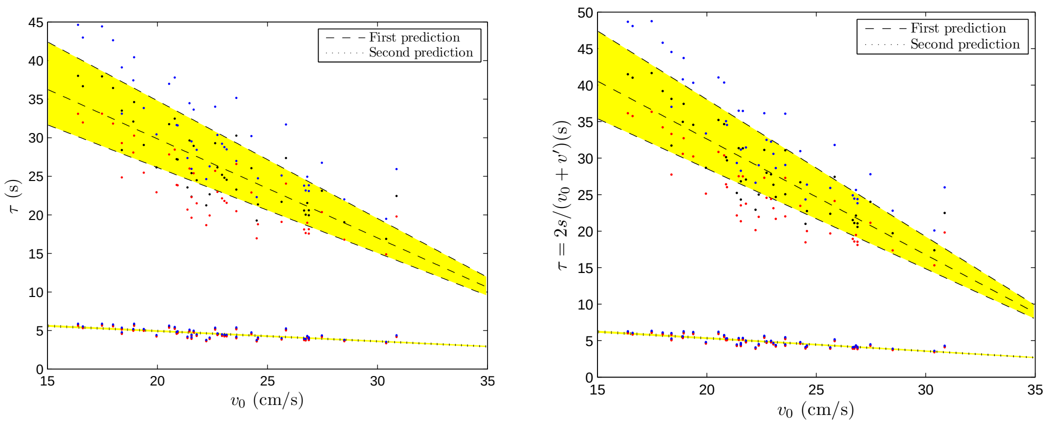

图 15: 两种时间间隔预测及其误差

从图14和图15中我们也可以看出，每种预测都由两条虚线构成了一条预测带，实际上这是我们保守的预测，如果两种预测中有一种是正确的，则应保证：

- 真实的值应落于该种预测带中.
- 由于实验测量的误差，允许有少量点落于带外附近.

另外，我们发现图14的末速度预测带越来越宽，而图15中的时间间隔预测带越来越窄. 就其原因：

- 在两个末速度预测公式中，阻力项均位置分子中. 阻力变化范围越大，也直接使得末速度的预测区间越来越大，反应到图中也就是预测带越来越宽.
- 在两个时间间隔预测公式上，阻力项位于分母中. 阻力变化范围越大，却使得时间间隔的预测区间越来越小，反应到图中也就是预测带越来越窄.

再分别比较图14和图15中左右两图，我们发现两种不同平均速度定义下的预测带差异也并不是很明显，在x轴上的截矩略有差异.

为直接验证两公式是否正确，本文将实验测得末速度和时间间隔实际值与两种预测的末速度和时间间隔的上线，均值和下限三条拟合曲线作比较，分别如图16和图17.

从图16中，可以看出真实的末速度大多落在或非常接近第二个预测带；而对于第一个预测带，不仅没有任何点落其间，而且所有点都远离它. 同时，我们还在同一个坐标系下作出了$v' = v_0$这个曲线，很明显，第一种预测末速度位于该曲线上方($v_0 > v'$)，而第二条位于该曲线下方($v_0 < v'$)，实验数据全部位于该直线下方，这足以说明，实验过程中，滑块是作减速运动. 第二种预测则呈现加速，显然是错误的.

从图17中，可以看出真实的时间间隔大多落在或非常接近第二个预测带；而对于第一个预测带，不仅没有任何点落其间，而且所有点都远离它. 同时，

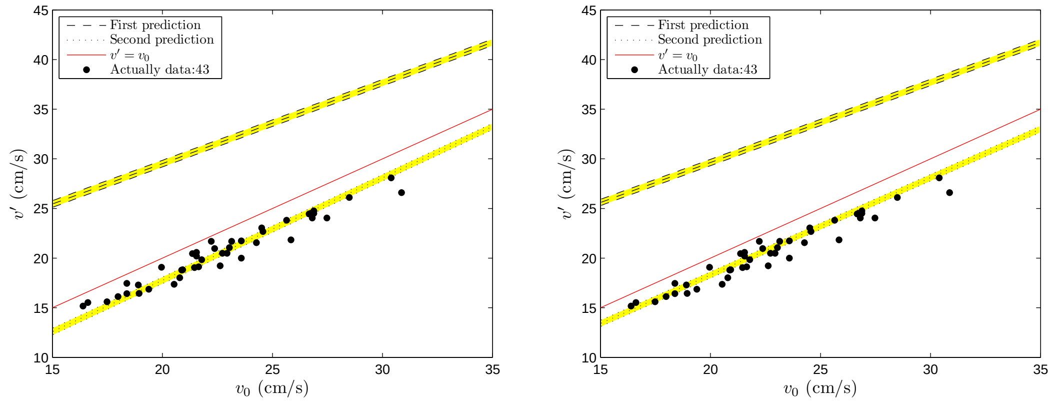

图 16: 两种末速度预测及实际末速度对比图

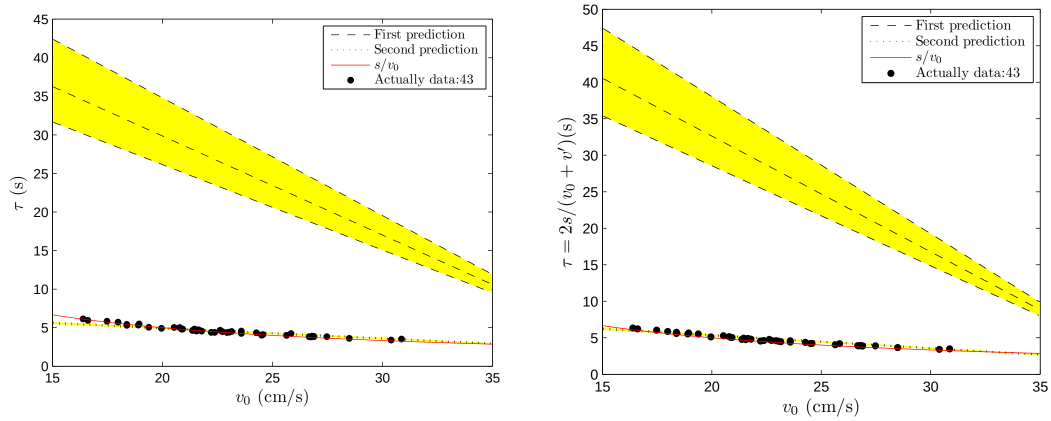

图 17: 两种时间间隔预测及实际末速度对比图

我们还在同一坐标系下作出了s/v_0的曲线，我们发现所有实测数据点都离这条曲线很近，这说明，整个运动过程中速度变化较小($s/v_0 \approx s/\bar{v}$).

为定量的描述实测数据与两种预测的相似程度,本文参照性相关系数的定义[6], 给出实测点与直线的相似程度的定义:

- 对于点$(x_i, y_i)(i = 1, \cdots, n)$ 与直线y=ax+b=f(x)的相似程度r定义为

$$r_x = \frac{\sum_{i=1}^{n}\left(x_i - \overline{x_i}\right)\left(f(x_i) - \overline{y_i}\right)}{\sqrt{\sum_{i=1}^{n}\left(x_i - \overline{x_i}\right)^2 \sum_{i=1}^{n}\left(f(x_i) - \overline{y_i}\right)^2}}$$

$$r_y = \frac{\sum_{i=1}^{n}\left(y_i - \overline{y_i}\right)\left(f^{-1}(y_i) - \overline{x_i}\right)}{\sqrt{\sum_{i=1}^{n}\left(y_i - \overline{y_i}\right)^2 \sum_{i=1}^{n}\left(f^{-1}(y_i) - \overline{x_i}\right)^2}}$$

$$r = \sqrt{r_x \cdot r_y}$$

相似程度r的取值范围在0到1之间,当r越接近1,$(x_i, y_i)(i = 1, \cdots, n)$与直线越相似.

按照上面的定义,我们得到两种预测的相似程度,结果如表1所示.

表 1: 两种动力学方程预测与实测数据的相似程度

预测的种类	τ预测	v'预测
第一种	0.0783	0.2617
第二种	0.9996	1.0000

另外，我们比较图17中的两个子图，发现，左图中多数点在该曲线以上，但也有不少点在其下方，这是不合逻辑的．右图中，所有的数据点都落在该曲线上方;右图的预测带也相对于右图的预测带有所上移，几乎全部位于s/v_0上方，我们将在4.4.1中详细讨论产生这一差异的原因．

除此之外，我们还用二次曲线对不变质量时的平均速度与平均阻力的关系，不同质量不同速度组合出的平均速度与平均阻力的关系，末速度和时间间隔的预测进行了拟合，该部分见附录F，除了曲线上略有差别外，二次拟合得出的结论与一次拟合相同：实测数据符合第二种动力学公式的预测，而与每一种预测几乎没什么联系．

4.4.1 预测误差分析

在(实测数据的合理性分析)中，我们指出了实测数据存在不合理的地方：实测的$\tau < s/v_0$．在本节中，我们来讨论这些不合理的数据对预测产生的影响．

为了更好的看出有问题的数据对预测的影响，我们将图9的横坐标换为v_0，并与两种预测作比较，如图19和图18.从图19的左图中可明显的看出：

- $s/\tau > v_0$或离v_0较近的s/τ对应的实测数据v'主要分布在预测带上方，并离预测带相对较远.
- $s/\tau < (v_0 + v')/2$ 对应的实测数据v'主要分布在预测带下方，并离预测带相对较远.
- $(v_0 + v')/2 < s/\tau < v_0$ 对应的实测数据v'则主要分布于预测带上或离预测带较近.

右图是我们另一种平均速度和时间隔定义下的预测结果. 显然，右图中，预测的效果比左图的效果稍好，真实的末速度都落于曲线s/v_0上方，但与左图差异不大.

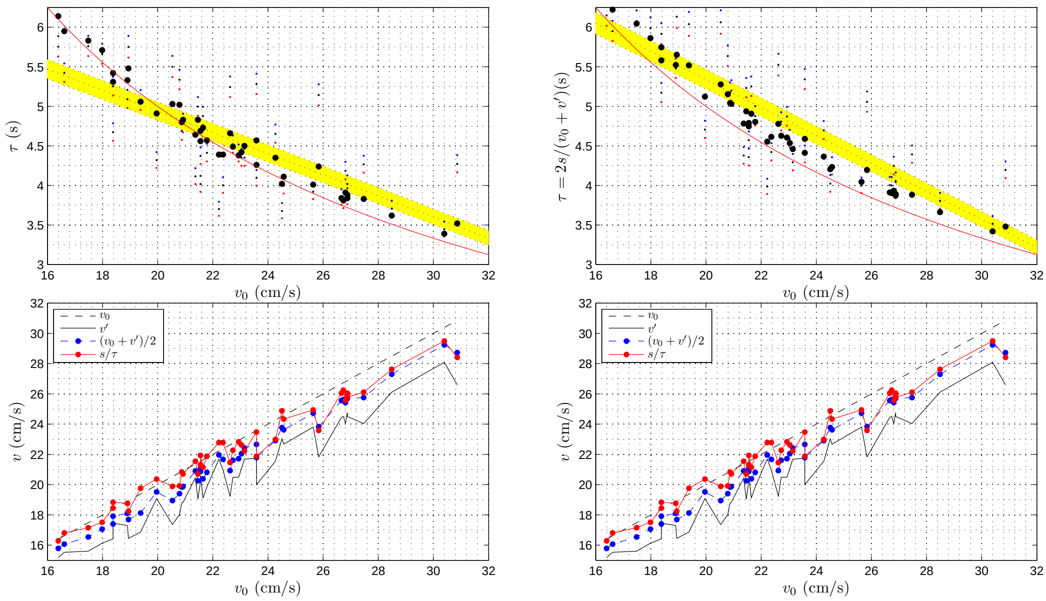

图 18: 时间间隔预测数据与不合理数据比照图

再观察图18,我们可以从图中找出4.4中不合逻辑(存在一些点$\tau < s/v_0$)现象的原因. 比较图18的两子，可以得到以下结论：

- 凡是$s/\tau > v_0$对应的实测τ就落在曲线s/v_0下方. 其实也必然如此

$$\frac{s}{\tau} > v_0 \Longrightarrow \frac{1}{\tau} > \frac{v_0}{s} \Longrightarrow \tau < \frac{s}{v_0}$$

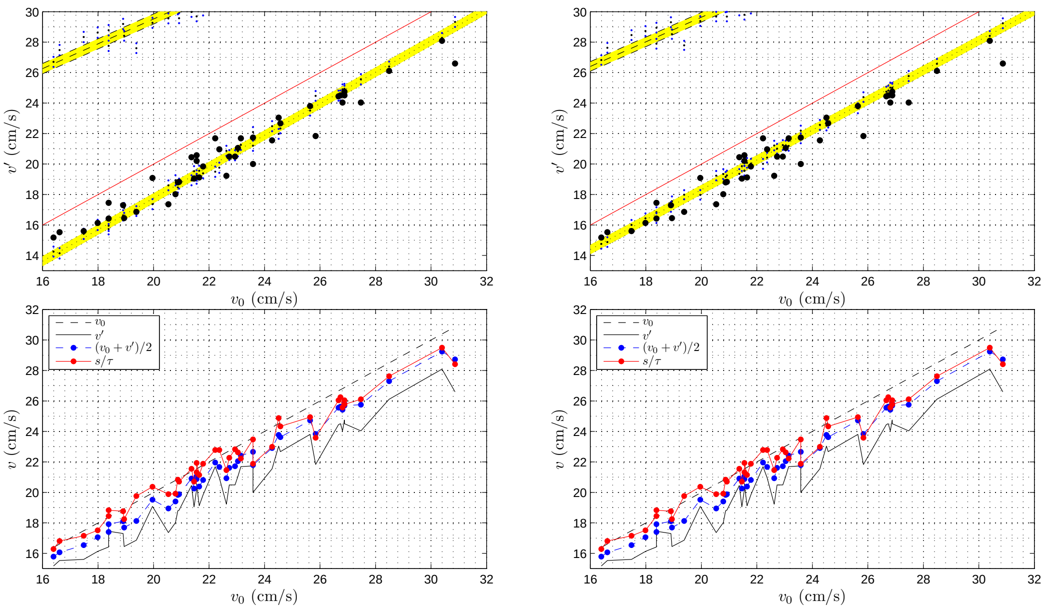

图 19: 末速度预测数据与不合理数据比照图

- τ的第二种预测带与实测数据存在的误差主要是由于τ的预测带是线性拟合出来的．但这不影响我们对两种公式直接判断．另外，我们用了方程$\tau = \frac{a}{v_0}$拟合了初速度与时间间隔的关系，其中a是拟合出的参数，这一部分过程和结果见附录G．

右图是我们另一种平均速度和时间隔定义下的预测结果．显然，右图中，末速度的预测的效果比左图的效果稍好，真实的末速度更接近于预测带(前两个点较为明显)，但与左图总体差异不大．

5 讨论及结论

5.1 讨论

本文在实验的基础上，对实验的数据进行处理，将变质量情况实测时间间隔和末速度与两种动力学公式的预测结果进行对比，以判断两种动力学公式的正确性．

大多课本中的牛顿第二定律的形式为：$F = dp/dt$, 而$p = mv$ ，因此对于不变质量，公式退化为$F = ma$. 显然，不管是$F = dp/dt$, 还是$F = ma$,对不变质

量都是成立的．对于变质系统，是不是就只有$F = dp/dt$成立，$F = ma$就不适用了？

研究变质量问题需要对变质量有一个清醒的认识．前人对变质量问题的研究，总是把变质量问题与动量守恒相混淆．在经典力学中常遇到一些所谓的“质量变化”的问题，如火箭在飞行中质量不断减少，雨滴问题等。这些问题的共同特点把物体分成单元，一份一份的沿平行于速度方向抛出去(不具一般性)。实质是运用牛顿定三定律，作用力与反作用力大小相等，方向相反，得出”变质量”系统的动力学的运动方程．

如果$F = 0 = dp/dt$ 正确，那么$p = mv$将是一个常数，因此当m 减小，v就会增大，因此一个减质量物体就会加速运动？实际上我们在漏沙实验并没有发现沙漏因漏沙而加速．如果$F = 0 = ma = mdv/dt$ 正确，那么v将保持不变．这与我们实际情况是一至的，我们把让一个沙漏静止在滑轨上漏沙，并没有发现沙漏因此而运动．

因此，无论是变质量还是不变质量系统，$F = ma$都是成立的．

5.2 结论

本文所用的判别性实验是合理可行的，虽然实验中由于实验条件的限制，不可避免存在误差和少量错误的数据．但本文实验结果表明很清楚的显示：

- 实测的时间间隔和末速度与第二种动力学方程给出的预测基本一至．
- 测的时间间隔和末速度与第二种动力学方程给出的预测相差甚远．

这使得我们很有信心对两种动力学方程的正确性作出明确判断：

☹ 第一种动力学方程是错误的.即以下动力学方程是错误的.

$$-f_r = \frac{\mathrm{d}mv}{\mathrm{d}t}$$

☺ 第二种动力学方程是正确的.即以下动力学方程是正确的.

$$-f_r = m(t)\frac{\mathrm{d}v}{\mathrm{d}t}$$

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附录

A 公式证明

我们通过对实验数据分析，发现速度和质量的变化率几乎是时间的一次函数, 因此有：

$$v(t) = -k_v t + \bar{v} = -\bar{a}t + \bar{v} = -\frac{v_0 - v'}{\tau}t + \frac{s}{\tau} \tag{18}$$

$$\frac{dm}{dt} = -\overline{\frac{d^2m}{dt^2}}t + \overline{\frac{dm}{dt}} = -\frac{\frac{dm}{dt}|_{t_0} - \frac{dm}{dt}|_{t'}}{\tau}t + \frac{\Delta m}{\tau} = -k_m t + \frac{\Delta m}{\tau} \tag{19}$$

为简化计算,令$dm/dt = u(t)$,则上式可化为$u(t) = -\bar{\dot{u}}t + \bar{u}$. 将式18和式19代入式5中最后一项得：

$$\begin{aligned}
\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} v\frac{dm}{dt}dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} (\bar{v} - \bar{a}t)(\bar{u} - \bar{\dot{u}}t)dt = \overline{v\frac{dm}{dt}}\tau \\
&= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} [\bar{v}\bar{u} - (\bar{u}\bar{a} + \bar{v}\bar{\dot{u}})t + \bar{a}\bar{\dot{u}}t^2]dt \\
&= \bar{v}\bar{u}\tau - \frac{1}{2}(\bar{u}\bar{a} + \bar{v}\bar{\dot{u}})t^2\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} + \frac{1}{3}\bar{a}\bar{\dot{u}}t^3\Big|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \\
&= \bar{v}\bar{u}\tau + \frac{1}{12}\bar{a}\bar{\dot{u}}\tau^3 \\
&= \bar{v}\bar{u}\tau\Big(1 + \frac{\bar{a}\bar{\dot{u}}\tau^2}{12\bar{v}\bar{u}}\Big)
\end{aligned}$$

因此有

$$\begin{aligned}
\overline{v\frac{dm}{dt}} &= \bar{v}\bar{u}\Big(1 + \frac{\bar{a}\bar{\dot{u}}\tau^2}{12\bar{v}\bar{u}}\Big) \\
&= \frac{S}{\tau}\frac{\Delta m}{\tau}\Big[1 + \frac{(v_0 - v')(\frac{dm}{dt}|_{t_0} - \frac{dm}{dt}|_{t'})}{12\frac{S}{\tau}\frac{\Delta m}{\tau}}\Big] \\
&= \frac{S}{\tau}\frac{\Delta m}{\tau}\Big[1 - \frac{(v_0 - v')(\frac{dm}{dt}|_{t'} - \frac{dm}{dt}|_{t_0})}{12\frac{S}{\tau}\frac{\Delta m}{\tau}}\Big] \lessapprox \bar{v}\overline{\frac{dm}{dt}}
\end{aligned} \tag{20}$$

B 不变质量数据

表 2: 不变质量数据

m=392.6g									单位：Δt/ms, τ/s		
Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ
65.8	69.6	6.73	45.8	49.2	4.67	38.4	40.6	3.92	31.2	32.8	3.15
63.3	65.4	6.40	44.8	48.0	4.60	38.2	40.2	3.87	31.2	32.8	3.15
62.3	65.3	6.33	44.7	48.0	4.56	38.1	39.7	3.86	30.8	32.6	3.12
61.2	62.3	6.05	44.6	46.4	4.52	37.5	40.1	3.78	30.8	32.7	3.12
60.8	65.3	6.24	44.5	47.2	4.51	37.3	39.7	3.79	30.5	32.4	3.09
60.5	62.5	6.05	44.4	48.2	4.59	37.0	39.4	3.37	30.5	32.1	3.08
60.3	63.5	6.15	44.2	47.3	4.51	36.9	39.4	3.76	30.4	32.3	3.10
58.7	60.5	5.97	44.0	46.8	4.49	36.7	39.1	3.74	30.4	32.1	3.08
56.6	61.2	5.83	44.0	47.2	4.52	36.4	39.0	3.73	30.4	32.0	3.07
56.4	59.7	5.71	43.9	47.9	4.52	36.3	38.6	3.69	30.2	31.9	3.06
56.1	60.3	5.75	43.3	46.4	4.41	36.2	39.1	3.70	30.0	31.8	3.04
55.3	59.6	5.64	43.0	46.8	4.43	36.0	37.9	3.64	29.9	31.4	3.03
55.1	58.4	5.61	42.8	46.3	4.39	35.9	38.0	3.62	29.8	31.2	3.00
54.6	57.4	5.65	42.8	44.9	4.36	35.6	37.9	3.59	29.7	31.4	3.02
53.9	56.2	5.45	42.6	45.5	4.35	34.5	36.3	3.50	29.7	31.2	3.02
52.6	56.4	5.38	42.5	46.1	4.33	34.5	36.7	3.52	29.6	31.1	3.01
52.4	55.6	5.32	41.8	45.5	4.29	34.3	36.0	3.46	28.8	30.4	2.92
52.3	55.6	5.33	41.8	44.6	4.28	34.0	35.7	3.43	28.3	30.4	2.89
52.0	55.2	5.31	41.6	44.6	4.26	34.0	36.3	3.45	28.0	29.2	2.85
51.5	55.0	5.26	41.2	43.7	4.14	33.9	36.2	3.45	27.8	29.2	2.81
51.2	54.6	5.25	40.8	43.1	4.15	33.8	35.3	3.40	27.6	29.6	2.82
51.2	54.8	5.22	40.6	43.5	4.14	33.7	35.3	3.41	27.2	28.5	2.73
50.8	53.8	5.12	40.4	42.8	4.10	33.1	34.8	3.36	27.2	28.7	2.76
50.5	53.5	5.10	40.4	42.4	4.09	33.0	35.0	3.35	27.1	28.7	2.76
50.1	52.6	5.09	40.2	42.6	4.09	32.8	35.1	3.36	26.7	28.3	2.72
50.0	52.2	5.06	40.0	42.8	4.07	32.8	34.3	3.31	26.4	28.0	2.70
49.8	53.7	5.04	39.9	43.2	4.09	32.8	34.1	3.30	26.3	27.9	2.68
48.5	51.6	4.93	39.4	41.9	3.97	32.6	33.9	3.28	26.0	27.1	2.61
48.4	51.2	4.92	39.4	41.6	4.00	32.4	34.0	3.28	26.0	27.5	2.63
48.2	51.4	4.94	39.2	41.4	3.97	32.3	34.3	3.28	26.0	27.6	2.65
48.1	51.2	4.86	39.1	41.5	3.97	32.1	34.2	3.26	25.7	27.1	2.61
48.0	51.1	4.89	39.1	42.1	4.02	32.1	34.2	3.46	25.6	27.2	2.62
47.6	50.8	4.84	39.0	41.3	3.96	31.7	33.3	3.19	25.6	26.8	2.58
47.6	51.7	4.87	38.9	40.7	3.91	31.7	33.4	3.21	25.5	26.9	2.58
46.8	50.8	4.75	38.8	41.6	3.96	31.6	34.0	3.22	25.2	26.8	2.56
46.6	50.9	4.79	38.8	41.8	3.97	31.6	33.2	3.20	25.1	26.2	2.52
46.4	49.4	4.74	38.6	40.8	3.90	31.4	33.8	3.21	25.1	26.0	2.52
46.2	49.3	4.74	38.4	40.7	3.91	31.2	33.1	3.16	25.1	26.2	2.52

表 3: 不变质量数据

						m=322.8g			单位：Δt/ms, τ/s		
Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ
65.4	71.4	6.75	45.9	50.4	4.71	36.0	38.8	3.69	29.9	32.8	3.09
64.0	70.4	6.75	44.4	47.9	4.53	35.7	38.5	3.66	29.8	31.8	3.04
62.0	64.7	6.27	44.3	48.3	4.54	34.8	37.8	3.57	29.5	31.2	2.98
61.3	67.7	6.33	43.7	48.3	4.54	34.8	37.2	3.55	29.5	32.0	3.02
60.4	62.7	6.14	43.5	48.4	4.49	34.4	36.8	3.52	29.3	31.2	2.99
60.3	65.2	6.21	43.5	46.8	4.44	34.4	36.5	3.49	29.2	31.2	2.99
59.8	65.0	6.09	43.5	46.2	4.40	34.3	36.4	3.49	28.8	31.1	2.95
59.1	64.6	6.08	43.4	45.8	4.39	34.1	36.4	3.46	28.8	30.9	2.94
58.5	59.4	5.86	43.2	47.6	4.46	33.9	36.6	3.47	28.5	30.6	2.90
57.6	62.8	5.93	43.0	45.8	4.38	33.9	36.1	3.43	28.5	30.9	2.92
57.4	64.0	5.98	42.8	47.2	4.42	33.6	36.8	3.48	28.4	30.3	2.89
56.4	61.6	5.80	42.8	46.8	4.39	33.6	35.6	3.40	28.3	30.0	2.89
56.4	60.6	5.77	42.7	45.6	4.34	33.5	35.5	3.39	28.2	29.5	2.84
55.3	57.6	5.65	42.6	45.6	4.37	33.2	36.0	3.41	28.0	29.6	2.84
54.9	59.9	5.69	42.2	44.7	4.31	33.0	35.2	3.36	28.0	29.6	2.83
54.6	59.8	5.63	42.0	45.5	4.29	33.0	34.9	3.33	28.0	29.8	2.83
54.3	60.5	5.66	40.9	43.6	4.15	32.8	35.1	3.34	27.9	29.0	2.76
54.0	57.3	5.50	40.8	43.4	4.15	32.6	34.9	3.32	27.7	29.6	2.83
53.8	57.7	5.47	40.3	44.0	4.12	32.3	33.9	3.24	27.6	29.7	2.82
51.7	58.4	5.39	40.2	43.2	4.11	32.3	34.8	3.30	27.6	29.4	2.80
51.4	56.2	5.29	39.2	43.1	4.03	32.1	34.1	3.26	27.4	29.2	2.78
50.8	56.2	5.25	39.2	42.3	4.00	32.0	34.3	3.26	26.8	28.3	2.70
50.8	56.5	5.25	38.9	42.1	3.99	31.9	34.4	3.27	26.4	28.0	2.68
50.4	53.5	5.09	38.8	41.6	3.96	31.9	34.7	3.26	26.4	27.9	2.68
50.1	54.8	5.16	38.6	41.2	3.92	31.8	34.0	3.23	26.3	27.8	2.66
49.2	53.0	5.02	38.4	41.3	3.93	31.8	33.9	3.22	26.3	27.6	2.64
48.8	54.3	5.07	38.3	42.3	3.96	31.6	33.9	3.23	26.3	27.5	2.63
48.5	53.2	5.02	38.3	41.2	3.90	31.5	32.9	3.16	26.1	28.0	2.67
48.2	52.0	4.94	38.3	41.2	3.90	31.4	34.1	3.20	26.1	27.5	2.63
48.0	51.6	4.9	38.1	41.6	3.92	31.3	33.1	3.18	26.0	27.6	2.63
47.8	51.1	4.86	38.1	39.9	3.85	30.9	33.1	3.14	25.9	27.6	2.64
47.6	50.9	4.83	38.0	40.4	3.89	30.7	32.8	3.13	25.5	26.6	2.56
47.5	50.3	4.82	37.9	40.8	3.88	30.7	32.5	3.13	25.2	26.6	2.55
47.4	52.9	4.90	37.9	39.9	3.83	30.7	33.1	3.13	25.1	26.4	2.54
47.0	51.7	4.85	37.7	41.1	3.88	30.6	32.8	3.13	25.0	26.3	2.52
46.9	50.7	4.80	37.7	39.9	3.87	30.5	32.7	3.11			
46.7	50.4	4.75	37.6	40.0	3.81	30.4	32.4	3.09			
46.5	49.9	4.74	37.2	39.7	3.79	30.3	32.5	3.09			
46.1	50.4	4.76	36.7	40.0	3.78	30.0	32.5	3.07			
46.1	49.1	4.72	36.6	38.7	3.70	29.9	31.8	3.04			

表 4: 不变质量数据

					m=223.0g				单位：Δt/ms, τ/s		
Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ	Δt_1	Δt_2	τ
65.9	79.9	7.09	44.6	52.8	4.68	37.8	43.3	3.96	31.3	35.9	3.27
64.5	73.8	6.80	44.5	52.4	4.68	37.8	42.6	3.92	31.2	35.8	3.26
61.7	74.7	6.57	44.4	51.2	4.62	36.9	41.2	3.81	31.0	34.7	3.21
61.7	78.0	6.67	44.5	53.2	4.67	36.8	41.4	3.82	31.0	34.3	3.21
60.9	77.8	6.69	43.9	49.5	4.54	36.5	42.3	3.84	30.8	34.5	3.20
60.8	75.9	6.56	43.9	49.3	4.52	36.5	42.1	3.83	30.8	34.2	3.18
60.4	72.5	6.35	43.4	50.8	4.59	36.4	40.5	3.76	30.6	34.6	3.18
59.1	67.1	6.03	43.3	49.8	4.53	36.3	40.0	3.72	30.3	33.2	3.12
56.9	66.6	5.97	43.0	49.6	4.50	36.0	40.5	3.72	30.3	34.0	3.14
56.7	64.2	5.85	43.0	50.0	4.54	35.6	41.6	3.75	30.3	33.9	3.13
56.7	69.4	6.06	42.8	50.2	4.51	35.3	39.6	3.64	30.0	34.0	3.12
56.7	63.1	5.73	42.8	50.0	4.50	35.3	39.2	3.63	30.0	33.1	3.08
56.5	67.8	6.02	42.5	45.8	4.23	35.2	40.3	3.65	29.7	33.5	3.07
56.3	68.0	5.96	42.3	48.4	4.37	35.1	39.9	3.65	29.6	32.7	3.04
55.6	68.8	5.97	42.1	48.4	4.40	35.1	40.0	3.67	29.2	32.0	2.99
55.3	63.4	5.78	42.1	49.0	4.43	34.7	39.4	3.62	29.2	33.4	3.05
55.1	66.8	5.85	41.6	47.9	4.34	34.5	38.8	3.57	29.1	32.6	3.01
54.9	64.7	5.75	41.4	47.4	4.31	34.4	38.8	3.57	29.0	32.3	2.99
54.8	66.3	5.81	40.9	47.1	4.26	34.4	38.1	3.55	28.9	33.2	3.02
54.8	65.4	5.81	40.8	47.8	4.30	34.0	37.6	3.51	28.9	32.7	3.02
54.0	64.8	5.73	40.6	46.4	4.22	33.9	39.0	3.55	28.5	32.4	2.97
52.3	60.5	5.50	40.4	46.8	4.24	33.8	37.7	3.48	28.5	32.0	2.96
51.9	61.6	5.41	40.1	46.0	4.18	33.6	39.2	3.55	28.4	31.2	2.91
51.6	60.1	5.36	40.1	45.9	4.18	33.3	36.9	3.43	28.3	32.1	2.94
51.5	60.9	5.43	40.0	45.0	4.11	33.2	36.4	3.40	28.2	32.2	2.95
51.4	61.3	5.44	39.7	46.4	4.18	33.2	36.4	3.39	28.1	31.4	2.92
51.3	61.4	5.43	39.6	45.7	4.16	33.2	38.5	3.50	28.1	31.1	2.90
51.2	60.4	5.36	39.6	46.0	4.15	33.2	39.2	3.48	28.0	32.3	2.96
49.4	63.4	5.32	39.6	46.7	4.20	32.8	36.0	3.36	28.0	31.6	2.93
49.2	60.7	5.27	39.6	47.5	4.23	32.8	37.7	3.45	27.8	30.4	2.86
48.6	59.2	5.18	39.5	45.6	4.12	32.8	37.6	3.43	27.8	30.7	2.86
48.5	58.0	5.10	39.4	44.3	4.06	32.7	35.8	3.35	27.8	31.2	2.89
48.4	56.9	5.08	39.3	45.5	4.13	32.6	36.6	3.38	27.7	31.3	2.90
48.4	59.1	5.14	39.2	44.8	4.10	32.6	36.5	3.36	27.6	30.0	2.82
48.3	55.7	5.10	39.2	49.2	4.28	32.4	36.5	3.35	27.6	31.0	2.87
47.5	55.7	4.96	39.0	44.2	4.05	32.3	36.6	3.38	27.3	30.8	2.84
47.5	53.1	4.89	38.8	43.7	4.01	32.0	37.6	3.39	27.0	30.0	2.78
47.1	55.6	4.96	38.7	43.2	3.99	31.8	35.6	3.27	26.6	29.1	2.73
46.9	54.8	4.93	38.7	44.6	4.03	31.7	35.2	3.27	26.6	29.3	2.74
46.1	52.8	4.80	38.4	45.7	4.09	31.7	35.6	3.29	26.4	29.9	2.75
46.1	54.2	4.86	38.4	44.6	4.04	31.7	35.5	3.28	25.6	28.7	2.67
46.0	53.0	4.79	38.3	45.4	4.02	31.6	35.4	3.27	25.6	28.7	2.66
45.9	53.5	4.80	38.2	42.8	3.96	31.4	34.4	3.24	25.1	27.4	2.57
45.7	52.5	4.78	38.0	44.0	3.98	31.4	36.6	3.32	25.0	27.8	2.59
44.8	52.3	4.67	37.9	43.2	3.94	31.3	34.8	3.23	25.0	28.1	2.59

C 变质量数据

表 5: 变质量数据

				m=412.6g			单位：Δt/ms, τ/s, 漏沙量/g		
Δt_1	Δt_2	τ	起始漏沙	漏沙量Δm	Δt_1	Δt_2	τ	起始漏沙	漏沙量Δm
61.0	65.9	6.14	27.9	166.7	35.1	38.3	3.62	27.1	101.8
60.2	64.4	5.95	25.9	171.1	32.9	35.6	3.39	29.4	92.2
57.2	64.1	5.83	27.1	165.6	32.4	37.6	3.52	27.1	103.0
55.6	62.0	5.71	28.9	160.7	26.4	29.1	2.78	23.2	79.4
54.4	60.9	5.42	28.7	157.5					
54.4	57.3	5.31	33.2	147.9					
52.9	57.8	5.33	26.9	154.7					
52.8	60.8	5.48	27.5	147.3					
51.6	59.3	5.06	25.9	135.8					
50.1	52.4	4.91	22.4	143.9					
48.7	57.6	5.03	27.0	137.0					
48.1	55.5	5.02	27.6	145.8					
47.9	53.2	4.80	35.4	135.7					
47.8	53.1	4.83	27.8	132.4					
46.8	48.9	4.64	24.0	131.7					
46.6	52.5	4.83	24.0	139.6					
46.4	49.5	4.69	32.0	134.6					
46.4	48.6	4.56	26.6	126.3					
46.2	52.3	4.73	23.4	138.6					
45.9	50.4	4.57	28.6	126.0					
45.0	46.1	4.39	29.9	128.9					
44.7	47.7	4.39	28.6	126.9					
44.2	52.0	4.66	28.4	128.0					
44.0	48.8	4.49	23.2	131.6					
43.6	48.8	4.38	26.9	128.2					
43.4	47.5	4.42	28.2	128.4					
43.2	46.1	4.50	24.5	127.7					
42.4	46.0	4.26	28.4	125.8					
42.4	50.0	4.57	23.9	131.1					
41.2	46.4	4.35	28.6	124.7					
40.8	43.4	4.02	29.7	110.5					
40.7	44.1	4.11	23.0	114.5					
39.0	42.0	4.01	28.7	118.0					
38.7	45.8	4.24	25.7	115.7					
37.5	40.9	3.84	29.7	110.4					
37.4	40.8	3.81	30.6	107.9					
37.3	41.6	3.91	30.2	106.6					
37.2	40.4	3.84	28.9	106.1					
37.2	40.8	3.88	27.6	105.6					
36.4	41.6	3.83	32.4	112.3					

D 漏砂速率的测量

表 6: 漏砂速率测量数据

静态法测量漏砂速率的测量数据

砂子质量(g)	漏沙时间(s)	速率(g/s)
187.0	14.495	12.901
	14.544	12.858
	14.447	12.944
	14.434	12.956
	14.654	12.761
93.7	7.195	13.023
	7.300	12.836
	7.365	12.722
	7.244	12.935
	7.346	12.755
62.0	4.733	13.100
	4.804	12.906
	4.828	12.842
	4.878	12.710
	4.969	12.477

动态法一测量漏砂速率的测量数据

砂子质量(g)	漏沙时间(s)	速率(g/s)
57.0	4.526	12.594
167.3	13.204	12.670
181.6	14.196	12.792
88.4	6.910	12.793
158.0	12.286	12.860
161.7	12.305	13.141
176.4	13.197	13.336
57.6	4.311	13.361
81.2	6.025	13.477
77.7	5.435	14.296

动态法二测量漏砂速率的测量数据

砂子质量(g)	漏沙时间(s)	速率(g/s)
24.1	2.004	12.026
24.7	2.025	12.196
16.0	1.284	12.461
24.3	1.939	12.532
21.6	1.713	12.609
22.1	1.720	12.674
90.2	7.082	12.737
94.5	7.395	12.779
88.1	6.893	12.781
16.6	1.297	12.799
90.0	7.059	12.877
88.6	6.879	12.884
88.8	6.764	13.128
90.5	6.875	13.160

动态法三测量漏砂速率的测量数据

砂子质量(g)	漏沙时间(s)	速率(g/s)
15.8	1.433	11.026
18.5	1.604	11.534
19.9	1.711	11.631
92.0	7.816	11.771
21.2	1.790	11.844
91.8	7.721	11.890
23.6	1.964	12.016
92.5	7.693	12.024
19.3	1.600	12.063
19.1	1.582	12.073
19.1	1.565	12.204
88.8	7.264	12.225
86.4	7.038	12.276
87.2	7.077	12.322
86.0	6.937	12.397
93.2	7.454	12.503

E 预测公式的另一种形式和预测结果

我们仍然选取以下两种动力学公式作为研究对象，分别为：

$$-f_r = \frac{dmv}{dt} \tag{21}$$

$$-f_r = m(t)\frac{dv}{dt} \tag{22}$$

由方程21和方程22推导出另一种预测形式.

E.1 第一种预测公式

对方程21两边同时积分

$$-\int_{t_0}^{t'} f_r dt = \int_{t_0}^{t'} dmv$$

得

$$\bar{f}_r \cdot (t' - t_0) = m_0 v_0 - m' v'$$

因此时间时隔和末速度分别为

$$\tau = t' - t_0 = \frac{m_0 v_0 - m' v'}{\bar{f}_r} \tag{23}$$

$$v' = \frac{m_0 v_0 - \bar{f}_r \cdot (t' - t_0)}{m'} = \frac{m_0 v_0 - \bar{f}_r \tau}{m'} \tag{24}$$

E.2 第二种预测公式

由

$$-f_r = m(t)\frac{dv}{dt}$$

两边同时积分

$$
\begin{aligned}
-\int_{t_0}^{t'} f_r dt &= \int_{t_0}^{t'} m\frac{dv}{dt}dt \\
&= mv\Big|_{t_0}^{t'} - \int_{t_0}^{t'} v\frac{dm}{dt}dt \\
&= m'v' - m_0v_0 - v_\xi \int_{t_0}^{t'} \frac{dm}{dt}dt
\end{aligned}
$$

其中$0 < v_{\min} < v_\xi < v_{\max}$. 因为在每次实验中，速度变化不大，即区间$[v_{\min}, v_{\max}]$长度较小．所以可用$\bar{v} = \frac{v_0+v'}{2}$代替$v_\xi$, 由此产生的速度$\bar{v}$误差范围为$[v_{\min}, v_{\max}]$．所以上式可化为

$$
-\overline{f_r}\tau = m'v' - m_0v_0 - \bar{v}(m' - m_0)
$$

因此时间时隔和末速度分别为

$$
\tau = \frac{m_0v_0 - m'v' - \bar{v}(m_0 - m')}{\bar{f}_r} = \frac{m_0v_0 - m'v' - \bar{v}\Delta m}{\bar{f}_r} \tag{25}
$$

$$
v' = \frac{m_0v_0 - \bar{f}_r\tau - \bar{v}\Delta m}{m'} \tag{26}
$$

E.3 两种预测公式的比较

上文我们分别得出了两种预测公式，为了对比方便，我们把两种预测公式列在一起：

- 两种时间间隔的预测

$$
\tau = \frac{m_0v_0 - m'v'}{\bar{f}_r} \tag{27}
$$

$$
\tau = \frac{m_0v_0 - m'v' - \boxed{\bar{v}\Delta m}}{\bar{f}_r} \tag{28}
$$

- 两种末速度的预测

$$v' = \frac{m_0 v_0 - \bar{f}_r \tau}{m'} \tag{29}$$

$$v' = \frac{m_0 v_0 - \bar{f}_r \tau - \boxed{\bar{v}\Delta m}}{m'} \tag{30}$$

很显然，两二种预测公式差别仅每二个公式分子上比每一个多一项$-\bar{v}\Delta m < 0$, 因此这个差别也必将导致两种预测结果的明显差别:第一种公式在时间间隔和末速度的预测都将比第二种公式的预测大.

E.4 两种预测结果

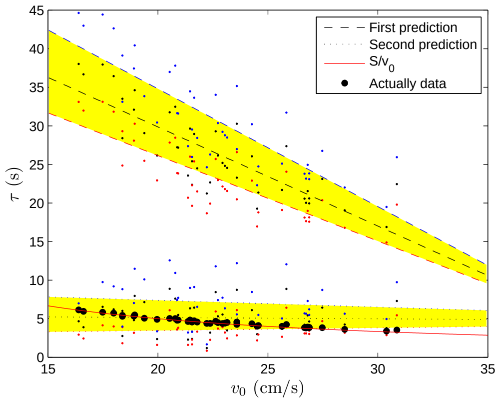


图 21: 两种时间间隔预测和实验数据的对比

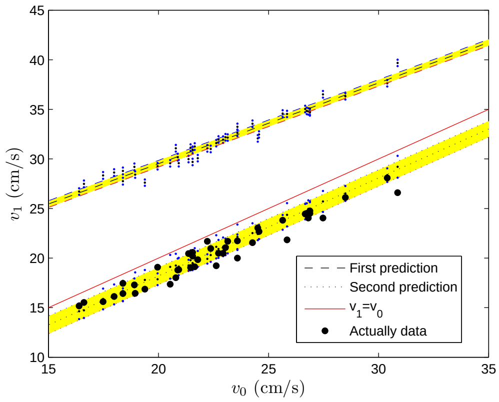


图 22: 两种末速度预测和实验数据的对比

F 二次拟合的结果

正文中，我们处理数据时，全部都用一次曲线拟合，这可能会使我们的结果存在局限性．基于此，我们在此用二次曲线拟合来重新处理所有数据．

F.1 不变质量时的平均速度与平均阻力的关系

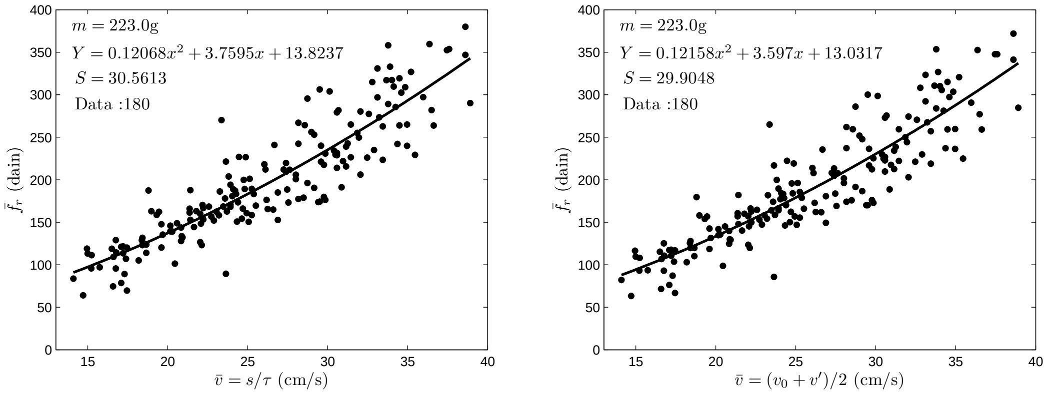


图 23: 223.0g质量水平下的平均速度与平均阻力的关系

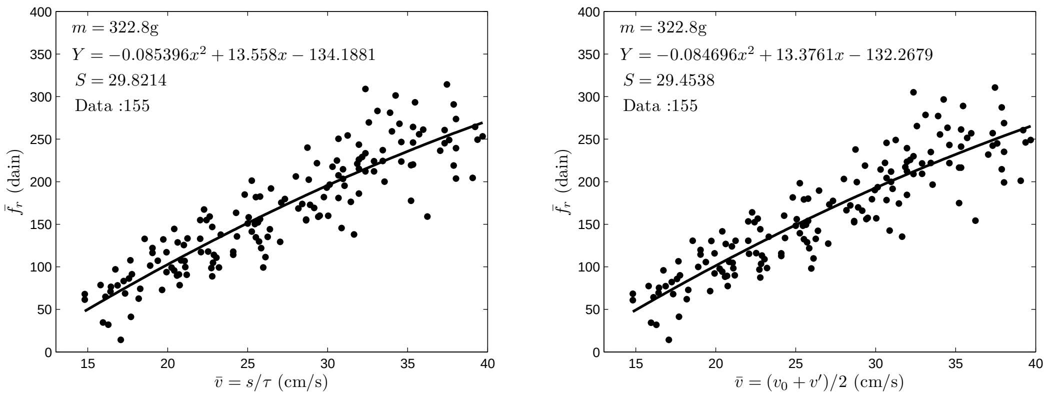


图 24: 322.8g质量水平下的平均速度与平均阻力的关系

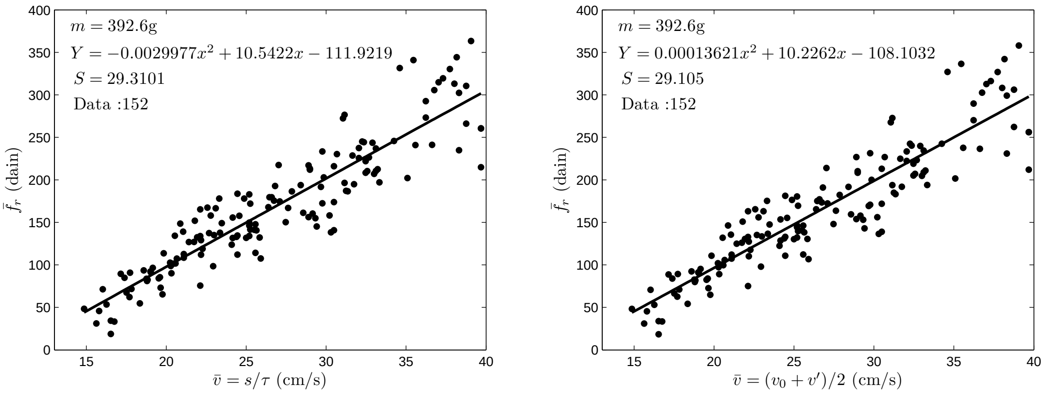


图 25: 392.6g质量水平下的平均速度与平均阻力的关系

F.2 不同速度,不同质量组合出阻力

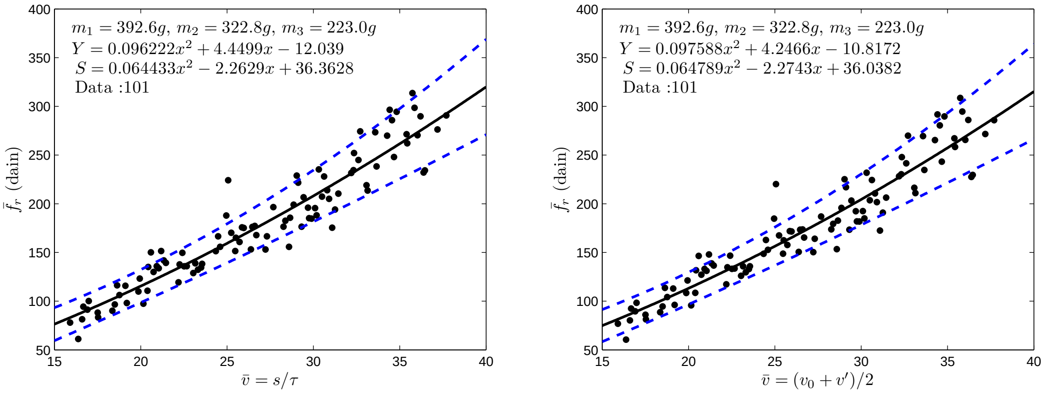


图 26: 不同质量不同速度组合出的平均速度与平均阻力的关系

F.3 两种预测

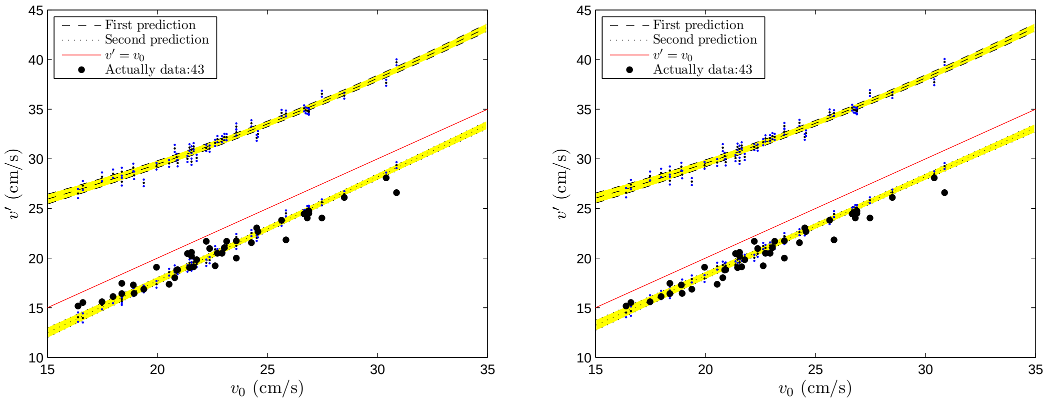


图 27: 两种末速度预测及其误差

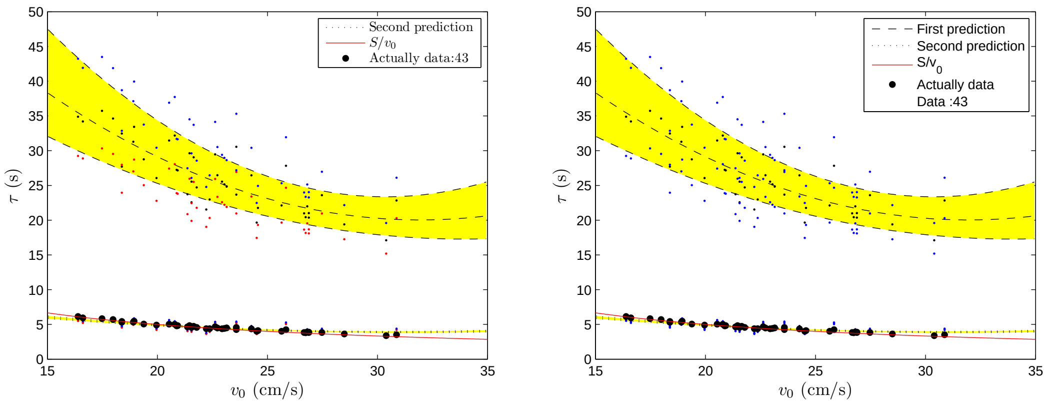


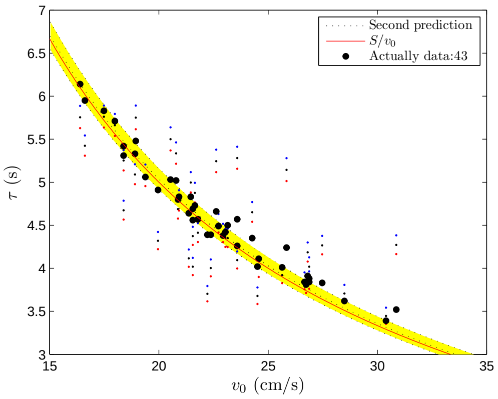
图 28: 两种时间间隔预测及其误差

G 另一种τ拟合

除了对τ作线性和二次拟合，由于我们知道，τ随是v_0递减函数，并且τ恒大于0.并且有关系:$s/v_0 < \tau < s/v_1$.因此我们还按下式对τ作了拟合：

$$\tau = \frac{a}{v_0}\text{或}\tau = \frac{s}{bv_0}$$

其中a(或b)是拟合参数．图29是两种不同均速度定义的τ按v_0的反比例函数拟合图.



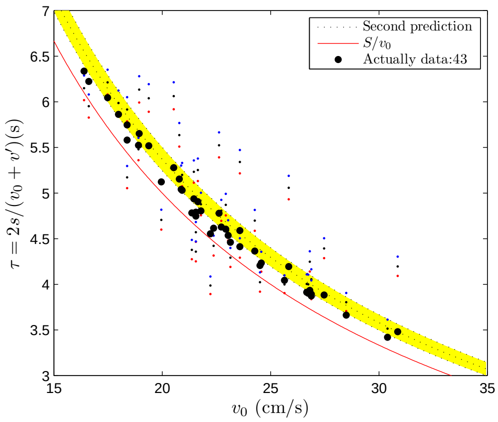


图 29: τ按v_0的反比例函数拟合

H 外文文献及其翻译

10

Conservation of Momentum

10–1 Newton's Third Law

On the basis of Newton's second law of motion, which gives the relation between the acceleration of any body and the force acting on it, any problem in mechanics can be solved in principle. For example, to determine the motion of a few particles, one can use the numerical method developed in the preceding chapter. But there are good reasons to make a further study of Newton's laws. First, there are quite simple cases of motion which can be analyzed not only by numerical methods, but also by direct mathematical analysis. For example, although we know that the acceleration of a falling body is 32 ft/sec^2, and from this fact could calculate the motion by numerical methods, it is much easier and more satisfactory to analyze the motion and find the general solution, $s = s_0 + v_0t + 16t^2$. In the same way, although we can work out the positions of a harmonic oscillator by numerical methods, it is also possible to show analytically that the general solution is a simple cosine function of t, and so it is unnecessary to go to all that arithmetical trouble when there is a simple and more accurate way to get the result. In the same manner, although the motion of one body around the sun, determined by gravitation, can be calculated point by point by the numerical methods of Chapter 9, which show the general shape of the orbit, it is nice also to get the exact shape, which analysis reveals as a perfect ellipse.

Unfortunately, there are really very few problems which can be solved exactly by analysis. In the case of the harmonic oscillator, for example, if the spring force is not proportional to the displacement, but is something more complicated, one must fall back on the numerical method. Or if there are two bodies going around the sun, so that the total number of bodies is three, then analysis cannot produce a simple formula for the motion, and in practice the problem must be done numerically. That is the famous three-body problem, which so long challenged human powers of analysis; it is very interesting how long it took people to appreciate the fact that perhaps the powers of mathematical analysis were limited and it might be necessary to use the numerical methods. Today an enormous number of problems that cannot be done analytically are solved by numerical methods, and the old three-body problem, which was supposed to be so difficult, is solved as a matter of routine in exactly the same manner that was described in the preceding chapter, namely, by doing enough arithmetic. However, there are also situations where both methods fail: the simple problems we can do by analysis, and the moderately difficult problems by numerical, arithmetical methods, but the very complicated problems we cannot do by either method. A complicated problem is, for example, the collision of two automobiles, or even the motion of the molecules of a gas. There are countless particles in a cubic millimeter of gas, and it would be ridiculous to try to make calculations with so many variables (about 10^{17}—a hundred million billion). Anything like the motion of the molecules or atoms of a gas or a block or iron, or the motion of the stars in a globular cluster, instead of just two or three planets going around the sun—such problems we cannot do directly, so we have to seek other means.

In the situations in which we cannot follow details, we need to know some general properties, that is, general theorems or principles which are consequences of Newton's laws. One of these is the principle of conservation of energy, which was discussed in Chapter 4. Another is the principle of conservation of momentum, the subject of this chapter. Another reason for studying mechanics further is that there are certain patterns of motion that are repeated in many different circumstances, so it is good to study these patterns in one particular circumstance. For example, we shall study collisions; different kinds of collisions have much in common. In the flow of fluids, it does not make much difference what the fluid is, the laws of the flow are similar. Other problems that we shall study are vibrations and oscillations and, in particular, the peculiar phenomena of mechanical waves—sound, vibrations of rods, and so on.

In our discussion of Newton's laws it was explained that these laws are a kind of program that says "Pay attention to the forces," and that Newton told us only two things about the nature of forces. In the case of gravitation, he gave us the complete law of the force. In the case of the very complicated forces between atoms, he was not aware of the right laws for the forces; however, he discovered one rule, one general property of forces, which is expressed in his Third Law, and that is the total knowledge that Newton had about the nature of forces—the law of gravitation and this principle, but no other details.

This principle is that *action equals reaction.*

What is meant is something of this kind: Suppose we have two small bodies, say particles, and suppose that the first one exerts a force on the second one, pushing it with a certain force. Then, simultaneously, according to Newton's Third Law, the second particle will push on the first with an equal force, in the opposite direction; furthermore, these forces effectively act in the same line. This is the hypothesis, or law, that Newton proposed, and it seems to be quite accurate, though not exact (we shall discuss the errors later). For the moment we shall take it to be true that action equals reaction. Of course, if there is a third particle, not on the same line as the other two, the law does *not* mean that the total force on the first one is equal to the total force on the second, since the third particle, for instance, exerts its own push on each of the other two. The result is that the total effect on the first two is in some other direction, and the forces on the first two particles are, in general, neither equal nor opposite. However, the forces on each particle can be resolved into parts, there being one contribution or part due to each other interacting particle. Then each *pair* of particles has corresponding components of mutual interaction that are equal in magnitude and opposite in direction.

10–2 Conservation of momentum

Now what are the interesting consequences of the above relationship? Suppose, for simplicity, that we have just two interacting particles, possibly of different mass, and numbered 1 and 2. The forces between them are equal and opposite; what are the consequences? According to Newton's Second Law, force is the time rate of change of the momentum, so we conclude that the rate of change of momentum p_1 of particle 1 is equal to minus the rate of change of momentum p_2 of particle 2, or

$$dp_1/dt = -dp_2/dt. \tag{10.1}$$

Now if the *rate of change* is always equal and opposite, it follows that the *total change* in the momentum of particle 1 is equal and opposite to the *total change* in the momentum of particle 2; this means that if we *add* the momentum of particle 1 to the momentum of particle 2, the rate of change of the sum of these, due to the mutual forces (called internal forces) between particles, is zero; that is

$$d(p_1 + p_2)/dt = 0. \tag{10.2}$$

There is assumed to be no other force in the problem. If the rate of change of this sum is always zero, that is just another way of saying that the quantity $(p_1 + p_2)$ does not change. (This quantity is also written $m_1v_1 + m_2v_2$, and is called the *total momentum* of the two particles.) We have now obtained the result that the total momentum of the two particles does not change because of any mutual interactions between them. This statement expresses the law of conservation of momentum in that particular example. We conclude that if there is any kind of force, no matter how complicated, between two particles, and we measure or calculate $m_1v_1 + m_2v_2$, that is, the sum of the two momenta, both before and after the forces act, the results should be equal, i.e., the total momentum is a constant.

If we extend the argument to three or more interacting particles in more complicated circumstances, it is evident that so far as internal forces are concerned, the total momentum of all the particles stays constant, since an increase in momentum of one, due to another, is exactly compensated by the decrease of the second, due to the first. That is, all the internal forces will balance out, and therefore cannot change the total momentum of the particles. Then if there are no forces from the outside (external forces), there are no forces that can change the total momentum; hence the total momentum is a constant.

It is worth describing what happens if there are forces that do *not* come from the mutual actions of the particles in question: suppose we isolate the interacting particles. If there are only mutual forces, then, as before, the total momentum of the particles does not change, no matter how complicated the forces. On the other hand, suppose there are also forces coming from the particles outside the isolated group. Any force exerted by outside bodies on inside bodies, we call an *external* force. We shall later demonstrate that the sum of all external forces equals the rate of change of the total momentum of all the particles inside, a very useful theorem.

The conservation of the total momentum of a number of interacting particles can be expressed as

$$m_1v_1 + m_2v_2 + m_3v_3 + \cdots = \text{a constant}, \tag{10.3}$$

if there are no net external forces. Here the masses and corresponding velocities of the particles are numbered 1, 2, 3, 4, . . . The general statement of Newton's Second Law for each particle,

$$f = \frac{d}{dt}(mv), \tag{10.4}$$

is true specifically for the *components* of force and momentum in any given direction; thus the x-component of the force on a particle is equal to the x-component of the rate of change of momentum of that particle, or

$$f_x = \frac{d}{dt}(mv_x), \tag{10.5}$$

and similarly for the y- and z-directions. Therefore Eq. (10.3) is really three equations, one for each direction.

In addition to the law of conservation of momentum, there is another interesting consequence of Newton's Second Law, to be proved later, but merely stated now. This principle is that the laws of physics will look the same whether we are standing still or moving with a uniform speed in a straight line. For example, a child bouncing a ball in an airplane finds that the ball bounces the same as though he were bouncing it on the ground. Even though the airplane is moving with a very high velocity, unless it changes its velocity, the laws look the same to the child as they do when the airplane is standing still. This is the so-called *relativity principle*. As we use it here we shall call it "Galilean relativity" to distinguish it from the more careful analysis made by Einstein, which we shall study later.

We have just derived the law of conservation of momentum from Newton's laws, and we could go on from here to find the special laws that describe impacts and collisions. But for the sake of variety, and also as an illustration of a kind of reasoning that can be used in physics in other circumstances where, for example, one might not know Newton's laws and might take a different approach, we shall discuss the laws of impacts and collisions from a completely different point of view. We shall base our discussion on the principle of Galilean relativity, stated above, and shall end up with the law of conservation of momentum.

We shall start by assuming that nature would look the same if we run along at a certain speed and watch it as it would if we were standing still. Before dis-

10–3

cussing collisions in which two bodies collide and stick together, or come together and bounce apart, we shall first consider two bodies that are held together by a spring or something else, and are then suddenly released and pushed by the spring or perhaps by a little explosion. Further, we shall consider motion in only one direction. First, let us suppose that the two objects are exactly the same, are nice symmetrical objects, and then we have a little explosion between them. After the explosion, one of the bodies will be moving, let us say toward the right, with a velocity v. Then it appears reasonable that the other body is moving toward the left with a velocity v, because if the objects are alike there is no reason for right or left to be preferred and so the bodies would do something that is symmetrical. This is an illustration of a kind of thinking that is very useful in many problems but would not be brought out if we just started with the formulas.

The first result from our experiment is that equal objects will have equal speed, but now suppose that we have two objects made of different materials, say copper and aluminum, and we make the two *masses* equal. We shall now suppose that if we do the experiment with two masses that are equal, even though the objects are not identical, the velocities will be equal. Someone might object: "But you know, you could do it backwards, you did not have to *suppose* that. You could *define* equal masses to mean two masses that acquire equal velocities in this experiment." We follow that suggestion and make a little explosion between the copper and a very large piece of aluminum, so heavy that the copper flies out and the aluminum hardly budges. That is too much aluminum, so we reduce the amount until there is just a very tiny piece, then when we make the explosion the aluminum goes flying away, and the *copper* hardly budges. That is not enough aluminum. Evidently there is some right amount in between; so we keep adjusting the amount until the velocities come out equal. Very well then—let us turn it around, and say that when the velocities are equal, the masses are equal. This appears to be just a definition, and it seems remarkable that we can transform physical laws into mere definitions. Nevertheless, there *are* some physical laws involved, and if we accept this definition of equal masses, we immediately find one of the laws, as follows.

Suppose we know from the foregoing experiment that two pieces of matter, A and B (of copper and aluminum), have equal masses, and we compare a third body, say a piece of gold, with the copper in the same manner as above, making sure that its mass is equal to the mass of the copper. If we now make the experiment between the aluminum and the gold, there is nothing in logic that says *these* masses must be equal; however, the *experiment* shows that they actually are. So now, by experiment, we have found a new law. A statement of this law might be: If two masses are each equal to a third mass (as determined by equal velocities in this experiment), then they are equal to each other. (This statement does *not* follow at all from a similar statement used as a postulate regarding *mathematical* quantities.) From this example we can see how quickly we start to infer things if we are careless. It is *not* just a definition to say the masses are equal when the velocities are equal, because to say the masses are equal is to imply the mathematical laws of equality, which in turn makes a prediction about an experiment.

As a second example, suppose that A and B are found to be equal by doing the experiment with one strength of explosion, which gives a certain velocity; if we then use a stronger explosion, will it be true or not true that the velocities now obtained are equal? Again, in logic there is nothing that can decide this question, but experiment shows that it *is* true. So, here is another law, which might be stated: If two bodies have equal masses, as measured by equal velocities at one velocity, they will have equal masses when measured at another velocity. From these examples we see that what appeared to be only a definition really involved some laws of physics.

In the development that follows we shall assume it is true that equal masses have equal and opposite velocities when an explosion occurs between them. We shall make another assumption in the inverse case: If two identical objects, moving in opposite directions with equal velocities, collide and stick together by some kind of glue, then which way will they be moving after the collision? This is again a

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symmetrical situation, with no preference between right and left, so we assume that they stand still. We shall also suppose that any two objects of equal mass, even if the objects are made of different materials, which collide and stick together, when moving with the same velocity in opposite directions will come to rest after the collision.

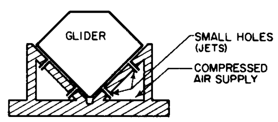

Fig. 10–1. End view of linear air trough.

10–3 Momentum *is* conserved!

We can verify the above assumptions experimentally: first, that if two stationary objects of equal mass are separated by an explosion they will move apart with the same speed, and second, if two objects of equal mass, coming together with the same speed, collide and stick together they will stop. This we can do by means of a marvelous invention called an air trough,* which gets rid of friction, the thing which continually bothered Galileo (Fig. 10–1). He could not do experiments by sliding things because they do not slide freely, but, by adding a magic touch, we can today get rid of friction. Our objects will slide without difficulty, on and on at a constant velocity, as advertised by Galileo. This is done by supporting the objects on air. Because air has very low friction, an object glides along with practically constant velocity when there is no applied force. First, we use two glide blocks which have been made carefully to have the same weight, or mass (their weight was measured really, but we know that this weight is proportional to the mass), and we place a small explosive cap in a closed cylinder between the two blocks (Fig. 10–2). We shall start the blocks from rest at the center point of the track and force them apart by exploding the cap with an electric spark. What should happen? If the speeds are equal when they fly apart, they should arrive at the ends of the trough at the same time. On reaching the ends they will both bounce back with practically opposite velocity, and will come together and stop at the center where they started. It is a good test; when it is actually done the result is just as we have described (Fig. 10–3).

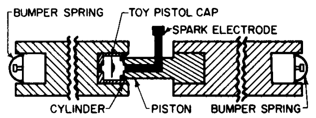

Fig. 10–2. Sectional view of gliders with explosive interaction cylinder attachment.

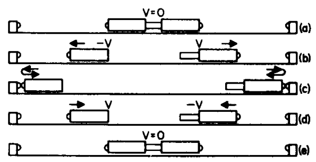

Fig. 10–3. Schematic view of action-reaction experiment with equal masses.

Now the next thing we would like to figure out is what happens in a less simple situation. Suppose we have two equal masses, one moving with velocity v and the other standing still, and they collide and stick; what is going to happen? There is a mass $2m$ altogether when we are finished, drifting with an unknown velocity. What velocity? That is the problem. To find the answer, we make the assumption that if we ride along in a car, physics will look the same as if we are standing still. We start with the knowledge that two equal masses, moving in opposite directions with equal speeds v, will stop dead when they collide. Now suppose that while this happens, we are riding by in an automobile, at a velocity $-v$. Then what does it look like? Since we are riding along with one of the two masses which are coming together, that one appears to us to have zero velocity. The other mass, however, going the other way with velocity v, will appear to be coming toward us at a velocity $2v$ (Fig. 10–4). Finally, the combined masses after collision will seem to be passing by with velocity v. We therefore conclude that an object with velocity $2v$, hitting an equal one at rest, will end up with velocity v, or what is mathematically exactly the same, an object with velocity v hitting and sticking to one at rest will produce an object moving with velocity $v/2$. Note that if we multiply the mass and the velocity beforehand and add them together, $mv + 0$, we get the same answer as when we multiply the mass and the velocity of everything afterwards, $2m$ times $v/2$. So that tells us what happens when a mass of velocity v hits one standing still.

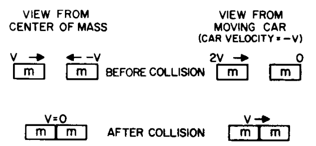

Fig. 10–4. Two views of an inelastic collision between equal masses.

In exactly the same manner we can deduce what happens when equal objects having *any* two velocities hit each other.

Suppose we have two equal bodies with velocities v_1 and v_2, respectively, which collide and stick together. What is their velocity v after the collision? Again we ride by in an automobile, say at velocity v_2, so that one body appears to be at rest. The other then appears to have a velocity $v_1 - v_2$, and we have the same case that we had before. When it is all finished they will be moving at $\frac{1}{2}(v_1 - v_2)$ with respect to the car. What then is the actual speed on the ground?

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* H. V. Neher and R. B. Leighton, *Amer. Jour. of Phys.* **31**, 255 (1963).

when it moves as a whole. Therefore momentum, as a mechanical quantity, is difficult to hide. Nevertheless, momentum *can* be hidden—in the electromagnetic field, for example. This case is another effect of relativity.

One of the propositions of Newton was that interactions at a distance are instantaneous. It turns out that such is not the case; in situations involving electrical forces, for instance, if an electrical charge at one location is suddenly moved, the effects on another charge, at another place, do not appear instantaneously—there is a little delay. In those circumstances, even if the forces are equal the momentum will not check out; there will be a short time during which there will be trouble, because for a while the first charge will feel a certain reaction force, say, and will pick up some momentum, but the second charge has felt nothing and has not yet changed its momentum. It takes time for the influence to cross the intervening distance, which it does at 186,000 miles a second. In that tiny time the momentum of the particles is not conserved. Of course after the second charge has felt the effect of the first one and all is quieted down, the momentum equation will check out all right, but during that small interval momentum is not conserved. We represent this by saying that during this interval there is another kind of momentum besides that of the particle, mv, and that is momentum in the electromagnetic field. If we add the field momentum to the momentum of the particles, then momentum is conserved at any moment all the time. The fact that the electromagnetic field can possess momentum and energy makes that field very real, and so, for better understanding, the original idea that there are just the forces between particles has to be modified to the idea that a particle makes a field, and a field acts on another particle, and the field itself has such familiar properties as energy content and momentum, just as particles can have. To take another example: an electromagnetic field has waves, which we call light; it turns out that light also carries momentum with it, so when light impinges on an object it carries in a certain amount of momentum per second; this is equivalent to a force, because if the illuminated object is picking up a certain amount of momentum per second, its momentum is changing and the situation is exactly the same as if there were a force on it. Light can exert pressure by bombarding an object; this pressure is very small, but with sufficiently delicate apparatus it is measurable.

Now in quantum mechanics it turns out that momentum is a different thing—it is no longer mv. It is hard to define exactly what is meant by the velocity of a particle, but momentum still exists. In quantum mechanics the difference is that when the particles are represented as particles, the momentum is still mv, but when the particles are represented as waves, the momentum is measured by the number of waves per centimeter: the greater this number of waves, the greater the momentum. In spite of the differences, the law of conservation of momentum holds also in quantum mechanics. Even though the law $f = ma$ is false, and all the derivations of Newton were wrong for the conservation of momentum, in quantum mechanics, nevertheless, in the end, that particular law maintains itself!

momentum. Starting from simple, symmetrical cases, we have demonstrated the law for more complex cases. We could, in fact, do it for any rational mass ratio, and since every ratio is exceedingly close to a rational ratio, we can handle every ratio as precisely as we wish.

10–4 Momentum and energy

All the foregoing examples are simple cases where the bodies collide and stick together, or were initially stuck together and later separated by an explosion. However, there are situations in which the bodies do *not* cohere, as, for example, two bodies of equal mass which collide with equal speeds and then rebound. For a brief moment they are in contact and both are compressed. At the instant of maximum compression they both have zero velocity and energy is stored in the elastic bodies, as in a compressed spring. This energy is derived from the kinetic energy the bodies had before the collision, which becomes zero at the instant their velocity is zero. The loss of kinetic energy is only momentary, however. The compressed condition is analogous to the cap that releases energy in an explosion. The bodies are immediately decompressed in a kind of explosion, and fly apart again; but we already know that case—the bodies fly apart with equal speeds. However, this speed of rebound is less, in general, than the initial speed, because not all the energy is available for the explosion, depending on the material. If the material is putty no kinetic energy is recovered, but if it is something more rigid, some kinetic energy is usually regained. In the collision the rest of the kinetic energy is transformed into heat and vibrational energy—the bodies are hot and vibrating. The vibrational energy also is soon transformed into heat. It is possible to make the colliding bodies from highly elastic materials, such as steel, with carefully designed spring bumpers, so that the collision generates very little heat and vibration. In these circumstances the velocities of rebound are practically equal to the initial velocities; such a collision is called *elastic*.

That the velocities *before* and *after* an elastic collision are equal is not a matter of conservation of momentum, but a matter of conservation of *kinetic energy*. That the speeds of the bodies rebounding after a symmetrical collision are equal to *each other*, however, is a matter of conservation of momentum.

We might similarly analyze collisions between bodies of different masses, different initial velocities, and various degrees of elasticity, and determine the final velocities and the loss of kinetic energy, but we shall not go into the details of these processes.

Elastic collisions are especially interesting for systems that have no internal "gears, wheels, or parts." Then when there is a collision there is nowhere for the energy to be impounded, because the objects that move apart are in the same condition as when they collided. Therefore, between very elementary objects, the collisions are always elastic or very nearly elastic. For instance, the collisions between atoms or molecules in a gas are said to be perfectly elastic. Although this is an excellent approximation, even such collisions are not *perfectly* elastic; otherwise one could not understand how energy in the form of light or heat radiation could come out of a gas. Once in a while, in a gas collision, a low-energy infrared ray is emitted, but this occurrence is very rare and the energy emitted is very small. So, for most purposes, collisions of molecules in gases are considered to be perfectly elastic.

As an interesting example, let us consider an *elastic* collision between two objects of *equal mass*. If they come together with the same speed, they would come apart at that same speed, by symmetry. But now look at this in another circumstance, in which one of them is moving with velocity v and the other one is at rest. What happens? We have been through this before. We watch the symmetrical collision from a car moving along with one of the objects, and we find that if a stationary body is struck elastically by another body of exactly the same mass, the moving body stops, and the one that was standing still now moves away with the same speed that the other one had; the bodies simply exchange velocities. This behavior can easily be demonstrated with a suitable impact apparatus. More generally, if both bodies are moving, with different velocities, they simply exchange velocity at impact.

Another example of an almost elastic interaction is magnetism. If we arrange a pair of U-shaped magnets in our glide blocks, so that they repel each other, when one drifts quietly up to the other, it pushes it away and stands perfectly still, and now the other goes along, frictionlessly.

The principle of conservation of momentum is very useful, because it enables us to solve many problems without knowing the details. We did not know the details of the gas motions in the cap explosion, yet we could predict the velocities with which the bodies came apart, for example. Another interesting example is rocket propulsion. A rocket of large mass, M, ejects a small piece, of mass m, with a terrific velocity V relative to the rocket. After this the rocket, if it were originally standing still, will be moving with a small velocity, v. Using the principle of conservation of momentum, we can calculate this velocity to be

$$v = \frac{m}{M} \cdot V.$$

So long as material is being ejected, the rocket continues to pick up speed. Rocket propulsion is essentially the same as the recoil of a gun: there is no need for any air to push against.

10–5 Relativistic momentum

In modern times the law of conservation of momentum has undergone certain modifications. However, the law is still true today, the modifications being mainly in the definitions of things. In the theory of relativity it turns out that we do have conservation of momentum; the particles have mass and the momentum is still given by mv, the mass times the velocity, *but the mass changes with the velocity*, hence the momentum also changes. The mass varies with velocity according to the law

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}, \tag{10.7}$$

where m_0 is the mass of the body at rest and c is the speed of light. It is easy to see from the formula that there is negligible difference between m and m_0 unless v is very large, and that for ordinary velocities the expression for momentum reduces to the old formula.

The components of momentum for a single particle are written as

$$p_x = \frac{m_0 v_x}{\sqrt{1 - v^2/c^2}}, \quad p_y = \frac{m_0 v_y}{\sqrt{1 - v^2/c^2}}, \quad p_z = \frac{m_0 v_z}{\sqrt{1 - v^2/c^2}}, \tag{10.8}$$

where $v^2 = v_x^2 + v_y^2 + v_z^2$. If the x-components are summed over all the interacting particles, both before and after a collision, the sums are equal; that is, momentum is conserved in the x-direction. The same holds true in any direction.

In Chapter 4 we saw that the law of conservation of energy is not valid unless we recognize that energy appears in different forms, electrical energy, mechanical energy, radiant energy, heat energy, and so on. In some of these cases, heat energy for example, the energy might be said to be "hidden." This example might suggest the question, "Are there also hidden forms of momentum—perhaps heat momentum?" The answer is that it is very hard to hide momentum for the following reasons.

The random motions of the atoms of a body furnish a measure of heat energy, if the *squares* of the velocities are summed. This sum will be a positive result, having no directional character. The heat is there, whether or not the body moves as a whole, and conservation of energy in the form of heat is not very obvious. On the other hand, if one sums the *velocities*, which have direction, and finds a result that is not zero, that means that there is a drift of the entire body in some particular direction, and such a gross momentum is readily observed. Thus there is no random internal lost momentum, because the body has net momentum only when it moves as a whole. Therefore momentum, as a mechanical quantity, is difficult to hide. Nevertheless, momentum *can* be hidden—in the electromagnetic field, for example. This case is another effect of relativity.

One of the propositions of Newton was that interactions at a distance are instantaneous. It turns out that such is not the case; in situations involving electrical forces, for instance, if an electrical charge at one location is suddenly moved, the effects on another charge, at another place, do not appear instantaneously—there is a little delay. In those circumstances, even if the forces are equal the momentum will not check out; there will be a short time during which there will be trouble, because for a while the first charge will feel a certain reaction force, say, and will pick up some momentum, but the second charge has felt nothing and has not yet changed its momentum. It takes time for the influence to cross the intervening distance, which it does at 186,000 miles a second. In that tiny time the momentum of the particles is not conserved. Of course after the second charge has felt the effect of the first one and all is quieted down, the momentum equation will check out all right, but during that small interval momentum is not conserved. We represent this by saying that during this interval there is another kind of momentum besides that of the particle, mv, and that is momentum in the electromagnetic field. If we add the field momentum to the momentum of the particles, then momentum is conserved at any moment all the time. The fact that the electromagnetic field can possess momentum and energy makes that field very real, and so, for better understanding, the original idea that there are just the forces between particles has to be modified to the idea that a particle makes a field, and a field acts on another particle, and the field itself has such familiar properties as energy content and momentum, just as particles can have. To take another example: an electromagnetic field has waves, which we call light; it turns out that light also carries momentum with it, so when light impinges on an object it carries in a certain amount of momentum per second; this is equivalent to a force, because if the illuminated object is picking up a certain amount of momentum per second, its momentum is changing and the situation is exactly the same as if there were a force on it. Light can exert pressure by bombarding an object; this pressure is very small, but with sufficiently delicate apparatus it is measurable.

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10-9

翻译

动量守恒

10-1 牛顿第三定律

牛顿第=定律给出了任何物体的加速度与作用在它上面的力之间的关系，在这个基础上，原则上可以解决任何力学问题．例如，为了确定几个粒子的运动。人们可以利用前面一章中所展开的数值方法．但是我们有充分的理由来进一步研究牛顿定律。首先，有一些十分简单的运动不仅可以用数值方法分析，也可以直接进行数学分析．比如，我们知道落体的加速是32英尺 / 秒后，由这个事实虽然可以用数值方法计算出运动，但是分析这个运动并找到一般解$ss_0+v_0t+16t^2$。，则更为容易也更令人满意．同样，虽然我们可以按数值方法计算简谐振子的位置，但我们也能用分析方法表明一般解是简单的t的余弦函数，因此，当存在一种简单而又更为精确的方法以得出结果时，再去用一系列麻烦的算术运算能毫无必要了．同理一个行星由引力决定的绕太阳的运行固然可以用第九章的数值解法逐点地加以计算，从而找到轨道的一般形状，但能够得到准确的形状–分析表明这是一个完整的椭圆－就更好了．

遗憾的是，只有很少问题能够以分析方法精确求解．例如就简谐振子来说，如果弹簧力不是正比于位移，而是更为复杂的话，人们就只得又回到数值解法上来．或者：假如有两个天体绕太阳运行，使天体的总数是三个，那么分析法就无法得出一个简单的运动公式，实际上这个问题只能作数值解．这就是著名的三体问题，它曾经长时间地向人们的分析能力挑战；十分有趣的是，人们曾经花了那么长时间才领悟到也许数学分析的能力是有限的，因而使用数值解法是必要的这个事实．今天，大量无法以分析方法解决的问题已由数值方法解出，那个曾被认为是如此困难的古老的三体问题，已作为常规计算准确地按上一章所描述的方式进行充分的演算后，加以解决了．然而，也有一些两种方法都失效的情况：对简单的问题我们可以用分析方法，对适当困难的问题可以用数值和算术方法；但是对非常困难的问题则这两种方法都不能用了．侧如：两辆汽车的碰撞，或者甚至气体中分子的运动，就是一种复杂的问题．在一立方毫米的气体中有数不清的粒子，而试图用这么许多变量（约10^{17}个－即十亿

亿个)来作计算将是荒谬的．任何问题，如果不是只有二、三个行星绕太阳运行，而是诸如象气体、术块、铁块中的分子或原子的运动，或在球状星团中许多恒星的运动之类这样的问题，我们就不能直接去解，因此只好借助于其他手段．

在那种无法了解细节的情况下，我们需要知道某些一般性质，亦即需要知道作为牛顿定律结果的一般性定理或原则．在第四章讨论过的能量守恒定律就是其中之一．另一个是动量守恒定律，这是本章的课题．进一步研究力学的另一个理由是：有某些运动模式在许多不同的状况下一再重复地出觋，因而，在一个特定情况下研究这些模式是有益的．例如，我们将研究碰撞，不阿类型的碰撞有许多共同之处．又如在流体的流动中，到底是哪一种流体这个问题并没有多大关系，这是因为流动的定律是类似的．我们将研究的其他一些问题是振动及振荡，特别是，机械波的特殊现象–声，杆的振动，等等．

在我们对牛顿定律的讨论中已经解释过：过些定律是一种处理问题的方案，它告诉我们："要注意力！"而在有关力的性质方面牛顿只向我们讲了两件事．在引力情况中，他留给我们一条完整的力的定律．关于原子间的非常复杂的作用力，他并不知道力的正确的规律：然而，他发现了一条有关力的一般性质的规则，并在第三定律中对此作了阐明，这就是牛顿在有关力的性质上：所具有的全部知识–引力定律和第三定律，再没有其他细节了．

牛顿第三定律是：作用等于反作用．

它的含义如下：假设我们有两个小物体，比如说两个粒子，第一个粒子对第二个粒子施加一个力，即用一个一定的力推它．那么，按照牛顿第三定律，第二个粒子同时以大小柑等、方向相反的力推第一个粒子；而且，这力实际上沿同一根线超作用．这就是牛顿提出的假设，或者说定律，它看来是相当准确的，尽管并不严格正确(以后我们将讨论它的误差)．暂时我们将认为作用等于反作用是正确的．当然，假如有第三个粒予，它不与其他两个粒子在同一条直线上，则这个定律并不意味着作用在第一个粒于上的总的力等于作用在第二个粒子上的总的力，因为，比方说，第三个粒子对这两个粒子中的每一个都要施加推力．结果作用在前两个粘子上的总效应是在某个别的方面上，从而一般说来，作用在前两个粒子上的力大小既不相等，方向也不相反．然而，作用在每

个粒子上的力总可以分解为若干部分，每一个与之相互作用的粒子都有一份贡献或一个部分．因而，每一对粒子都有相应的彼此相反作用的分量，它们大小相等、方向相反．

10－2动量守恒

现在来看一下，上述联系有什么有趣的结果?为了简单起见，我们假设只有两个互相作用的粒子，质量可能不同，并分别编为1号及2号．它们之间的力相等而方向相反；这会有什么结果呢？ 按照牛顿第二定律，力是动量对时间的变化率，于是我们得出粒子l的动量p_1的变化率等于粒子2的动量p_2变化率的负值．即

$$dp_1/dt = -dp_2/dt$$

现在，如果变化率总是数值相等、方向相反，就可知道粒子1动量的总变化与粒子2动量的总变化数值相等、方向相反；这意味着，如果我们把粒子Ｉ的动量与粒于2的动量相加，那么由于粒子之间相互作用力(称为内力)引起的两个粒子动量之和的变化率为零；即

$$d(p_1 + p_2)/dt = 0$$

在这个问题中假定没有其他作用力．如果这个和的变化率总是零，这正是量$(p_1 + p_2)$不发生变化的另一种说法(这个量也可写成$m_1v_1 + m_2v_2$。并称为这两个粒子的总动量)．现在我们得出两个粒子的总动量不因它们之间的任何相互作用而改变的结论．这个说法表示了在这个特例下的动量守恒定律．我们断言：如果两个粒子间存在着任何类型的力(不管这个力怎样复杂），我们在力作用之前及力作用之后去测量或计算(m_1v_1)，即两个动量之和，则结果总是相等的，也就是说，总动量是一个常数．

假如我们把论证引伸到更复杂的三十或多个相互作用粒子的情况，那么很明显，当只考虑内力时，所有粒子的总动量保持不变，因为其中一个粒子由另一个粒子引起的动量的增加，恰好严格地被前者引起的后音动量的减少所补偿．也就是说，所有的内力将互相抵消，因此不可能改变粒子的总动量．于是，如果没有来自外界的力(外力)，那么就投有什么力可以改变总动量，因此

总动量是一个常数.

值得一淡的是，如果存在一些并非来自所说的粒子间的相互作用的力：假定我们把柑互作用的粒子隔离开来，这时会出现什么情况?如果只有相互作用力，那么同以前一样，无论这些力多么复杂，粒子的总动量不变．另一方面，假定还有来自隔离开来的那一群以外的粒子的作用力．我们称任何外部物体施加于内部物体的力为外力．必后我们将证明所有外力之和等于所有内部粒子动量总和的变化率．这是一个非常有用的定理。

如果没有净的外力，一群相互作用粒子的总动量守恒可以表示为

$$m_1v_1 + m_2v_2 + m_3v_3 + \cdots = 常数$$

这里将粒子的质量和相应的速度顺序编为．1，2，3，4，$\cdots$等．对每个粒子，牛顿第二定律的一般表述是：

$$f = \frac{d}{dt}(mv)$$

特别是对力和动量在任何给定方向上的分量也同样成立；这样作用在一个粒子上的力的x分量就等于该粒子动量变化率的x分量，即

$$f_x = \frac{d}{dt}(mv_x)$$

对y和z方向也如此．所以方程(10.3)实际上是三个方程，每个方向一个．

除动量守恒定律外，牛顿第二定律还有另一个有趣的结果，现在先提一下，以后再证明．这个原理就是：无论我们保持静止状态，还是沿一条直线作匀速运动，物理定律将都是相同的．例如，一个在飞机上拍皮球的孩于，会发现皮球跳得和他过去在地面上拍时一样高．即使飞机”极高速度飞行，只要它不改变飞行速度，物理定律在孩子看来总是和飞机静止时完全一样．过称为相对性原理．当我们在这里使用这个原理时，将称它为＂彻利略相对性＂，以与爱因斯坦所作的更仔细的分析相区别，后者我们将在以后研究.

我们刚从牛顿定律推导出了动量守恒定律，由此出发，我们可以接下去找出一些描写碰撞的定律．但是为多样化起见，同时也为了阐明一种在物理学上可用于其他情况(比方说，人们也许并不知道牛顿定律，也许另辟其他途径)的

推理方式，我们将从一个完全不同的观点讨论碰撞定律．我们的讨论将从上述伽利略相对性原理出发，而”得出动量守恒定律告终．

我们将从下列假定出发：我们”一定速度运动并观察自然界时，自然界在我们看来和我们静止不动时完全相同．在讨论那种两个物体碰撞后粘在一起，或者来到一起再弹开的情况之前，我们将首先考虑用弹簧或其他东西联结在一起的两个物体，突然放开它们，使它们受到弹簧的推力或者某种轻微爆炸的情形．此外，我们将只考虑一个方向上的运础．我们先假定，两个物体完全相同、十分对称，接着两者之间发生了轻微爆炸．爆炸后，其中一个物体将以速度v向右运动，另一个物体将以速度v向左运动．由于选两个物体是类同的: 因而没有什么理由认为它们对左或右会有所偏爱，故两个物体的行为应该是对称的、因此，认定另一物体以速度v向左运动看来是合理的．这里阐明了一种在许多问题中都十分有用的思维方式，如果我们只从公式入手，那就显不出来了．

我们这个实验的第一个结论是相同的物体将有相等的速率，现在假设两个物体由不同材料比如说铜和铝制成，并令它们的质量相等．我们将假定，如果用两个质量相等的物体做实验，即使它们不是全同的，它们的速度也将是相等的．有人可能反驳说：”但是你知道，你可以反过来，不必击作假设，你可以定义相同的质量为在这个实验中获得相等速度的两个物体的质量”．我们按照这个建议，并在铜块与体积很大的铝块之间作一次轻微爆炸，铝块是如此之重，以致铜块飞出去后，铝块几乎不动．由于铝太多，因而我们把铝块减少到只剩下很薄一片，于是当我们再作一次爆炸时，铝块飞走了，而铜块却几乎不动．这说明铝又太少了．很明显，在两种铝的数量之间有某个正确的数值；于是我们继续调整铝的数量直至速度相等为止．好，现在我们反过来，并认为当速度相等时，质量也相等．这似乎只是一个定义，看来很奇怪我们可以把一些物理定律变成仅仅是一些定义。然而，这里已经包含了某些物理定律，假如我们采纳这个质量相等的定义，我们立即就可得到如下的一条定律．

假设我们从上面的实验知道，两块材料A与B(铜和铝)具有相等的质最．我们用上述的同样方式将铜块和第三块材料，比如金块，进行比较，并确认它的质量等于铜块的质量．如果我们现在用铝和金做实验，在逻辑上并不能说明这

些质量必须相等；然而实验表明它们实际上是相等的．所以通过实验，我们发现了一条新的定律这条定律的一种说法可能是这样的：如果两个质量分别等于第三个质量(由在这个实验中速度相等来确定)，那么它们彼此相等．(这个表述完全不能从用于有关数学量的公设的相似的陈述中推得.)从这个倒子我们可以看到，假如我们不小心的话，我们会多么轻易地推出结论！说速度相等时质量相等，这决不仅仅是一个定义，因为说质量相等就含有数学上有关相等的定律的意思，而这个相等的定律又可反过来对有关实验作出预言．

作为第二个侧子，假设实验时用某一强度的爆炸使A,B占两个物体获得一定的速度，从而发现它们相等；那么如果我们再使用更强烈的爆炸，这时所获得的速度是不是还相等呢?同样在逻辑上根本不能确定这个问题，但实验证明确实如此．这样，我们又有了一条定律，它可以表述为：如果在某一速度时按照速度相等方法来测定两个物体具有相等质量．则在另一个速度下测量，它们也将有相同的质量．从这些例子中我们看出，表面上看来只是一个定义的东西实际上包含了某些物理定律．

在下面的论证中，我们将假设：当在两个物体间发生爆炸时，相等的质量将具有数值相等，方向相反的速度这个命题成立．在相反的情况下，我们将作另一个假设：如果两个以相等的速度在相反的方向上运动的全同物体碰撞后被某种粘胶粘在一起，那么碰撞后它们将以什么方式运动呢?这又是一个对左和右没有特别偏重的对称的情况，所以我们假定它们将保持静止．我们还要假定，任何两个质量相同的物体，即使由不同材料制成，在以相等的速度沿相反的方向运动并碰撞和粘在一起时，它们碰撞后将保持静止．

10-3动量是守恒的

我们可以用实验来验证上述假设：即，第一，如果两个相等质量的静止物体发生爆炸后分开时，它们将以同样的速率分开运动；第二，两个相等质量的物体以同样的速率相向运动，碰撞并粘合后，它们将停止运动．我们可以利用一个称为气垫的惊人的发明来做实验，它能摆脱不断使伽利略深感麻烦的摩擦力(图10-1)．伽利略不能用光滑的东西来做实验，因为那些物体不能自由地滑动，但是在今天，加上一个不可思议的接触后，我们就能摆脱掉摩擦力．我们的物体正如伽利略所宣称的那样，可以毫无困难地以不变的速度滑动．这是

通过“空气来托起物体而实现的．因为空气只有极其微小的摩擦力，当不加力时，物体实际上就以不变的速度滑行．首先，我们使用两个经过精心制作具有同样的重量或质量的滑块(实际上是测出了它们的重量，但是我们知道重量是正比于质量的).在两个滑块问的一个封闭气缸中放进一个小的雷管(图10-2)．开始时，将两个滑块静止置放在槽的中心，然后利用电火花引爆雷管，迫使它们分开．这时会出现什么呢?如果在它们飞开时速率相等，就应当同时到达气垫的两端．到达两端后，它们实际上又将以相反的速度弹回，然后又跑到一起，并停在开始运动时的起点–中心处．这是一个很好的试验;经过实践以后，结果正如上述(图10-3)．

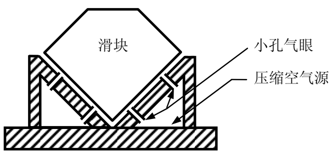

图 1: 线性气垫侧视图

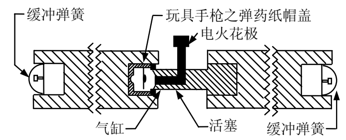

图 2: 带有爆破作用气缸附件的尚志截面图

接下来，我们要解决的是在稍为复杂一些的情况下会发生什么．假设我们有两个质量相等的物体，一个以速度v运动，另一个静止不动，它们碰撞后结合在一起；那时又将发生什么情况?结果是一个质量为$2m$的物体以一个未知速度移动．速度多大呢?问题就在于此．为了找到答案，假定当我们驱车前进时，物理规律在我们看来和静止时完全一样．我们从两个质量相等以相同的速率v沿相反的方向运动的物体发生碰撞后，将静止不动出发．现在假设在发生这种情况时，我们乘在一辆以速度$-v$开行的汽车上．那么它看上去象什么呢?由于我们随着两个相向运动的物体中的一个一起前进，因而这一个物体在我们看来速度为0．而另一个以速度v向相反方向运动的物体．在我们看来就以速度$2v$向我们走来(图10-4)．最后，在碰撞后结合起来的物体看来以速度v经过．因此我们得出结论，一个速度为$2v$的物体碰到另一个静止的质量相等的物体时，结果将以速度v运动，或者用数学上完全等价的方式来说是：一个速度v的物体撞在另一个静止物体上非结台在一起时，将产生一个以度$v/2$运动的物体．注意，如果我们将事前的质量与速度分别相乘再相．加得到mvO，与我们将事后的每一个物体的质量与速度相乘，即2m乘以$v/2$所得的答案相同．这

就告诉我们一个速度为v的物体撞在一个静止的物体上时会出现什么情况．

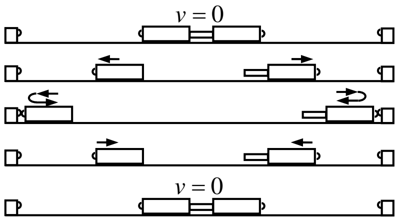

图 3: 两个质量相等的物体的作用—反作用实验示意图

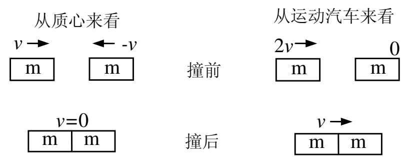

图 4: 质量相等的物体进行的非弹性碰撞的两种情况

我们可以用完全同样的方式推导出当两个质量相等的物体以任意两种速度相碰撞时会出现什么情况．

假设我们有两个质量相等的物体，分别具有速度v_1及v_2。．它们碰撞并结合在一起．试问碰撞后，它们的速度v是多少?我们再从乘上一辆速度为$v2$的汽车来看，则一个物体就象是静止的，而另一个物体就象具有：$(v_1 - v_2)$的速度，于是我们就得到了同以前一样的情况．当所有过一切都完成后，它们相对于汽车将以$\frac{1}{2}(v_1 - v_2)$的速度运动.那么它们相对于地而的实际速度是多少呢?答案是$v = \frac{1}{2}(v_1 - v_2) + v_2 = \frac{1}{2}(v_1 + v_2)$ (图10-5)．我们再次注意到

$$mv_1 + mv_2 = 2m \cdot \frac{(v_1 + v_2)}{2}$$

于是利用这个原理，对于任何质量相等的物体碰撞后结合在一起的情况，我们都能加以分析．事实上，我们虽然只是计算了一维的情况，但是假如我们坐在一辆沿某个倾斜方向运动的汽车上，我们就可以对更为复杂的碰撞找出更多的东西，这个原型是相同的，只是细节上更加复杂而已．

为了从实验上检验当一个以速度v运动的物体与另一个速度为0的质量相等的物体碰撞在一起后，是否会组成一个以速度$v/2$运动的物体，我们可以用气垫装器进行如下的实验．在气垫中放入三个质量相等的物体，其中两个物体开始时由爆发汽缸装置连接在一起，第三个物体非常靠近，但和它们稍为隔开一点点，它还带有二一个粘性缓冲器以致在另一个物体碰上它时，就会和它粘在一起．现在，在爆炸后一刹那，我们有两个质量为m，分别以相等面相反的速度v运动的物体．过一会儿其中一个物体将碰撞在第三个物体上，构成

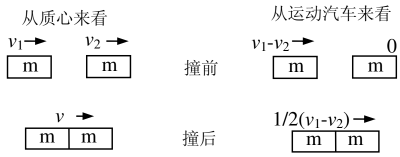

图 5: 质量相等的物体进行的另一种非弹性碰撞的两种情况

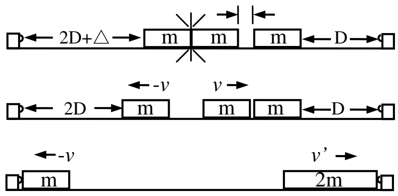

图 6: 验证以速度v运动的质量为m的物体与一个质量相同的静止物体碰撞结合在一起以质量$2m$,速度$\frac{1}{2}v$运动的实验

一个质量为$2m$的物体，我们相信，它将以速度$v/2$运动．我们怎样测出它确实是$v/2$呢?把物体在气垫上的初始位置作这样安排，使得两端的距离不同，而是按2：1的比例．这样继续以速度v运动的第一个物体，在一给定时间内所通过的距离将是那两个连在一起的物体通过的距离的2倍(假定第二个物体在与第三个物体碰撞前只通过一段很小的距离)．质量为m的物体与质量为$2m$的物体应当同时到达终点，我们去试一下时，就会发现确实如此(图10- 6).

我们要解决的下一个问题是，如果有两个不同质量的物体,情况又会怎样．让我们取一个质量为m的物体和一个质量为$2m$的物体，并利用我们的爆炸作用．这时将会发生什么呢?如果爆炸后m以速度v运动，那么$2m$又以什么速度运动呢?今第二个和第三个质量之间的距离为零，重复我们刚才作过的实验，当我们试一下后，会得出同样的结果，也就是说，超作用的质量m和$2m$各达到速度$-v$及$v/2$．这样，m与$2m$之间的直接的反作用与先是在m和m之间对称地反作用，随后m又与第三个m发生碰撞并结合在一起所得出的结果完全相同．而且，我们还发现，从气垫两端弹回的质量为m和$2m$的物体的速度与原来(几乎)完全相反，如果它们粘在一起，就会停止不动．

现在我们要问的另一个问题是：如果具有速度v的质量沿m的物体与另一个静止的质量为$2m$的物作碰撞并结合在一起，会发生什什么情况呢？这个问题利用伽利略相对性原理很容易回答．因为我们只要坐在一辆以速度$-v/2$ 运动曲的汽车里观察刚才描写的碰撞就行了(图10－7）．从汽车上看，速度是

$$V_1' = v - v(\text{汽车})v + v/2 = \frac{3}{2}v\text{及}V_2' = -\frac{v}{2} - v(\text{汽车}) = -\frac{v}{2} + \frac{v}{2} = 0$$

在碰撞后，质量$3m$在我们看来以速度$v/2$运动.于是我们就得到了碰撞前后的速度比是3:1的答案：如果一个质量为m的物体与一个质量为$2m$的静止的物体相碰撞，并结合在一起，则整个物体就以原先m的速度的$\frac{1}{3}$运动.一般的规则又是；各个物休的质量与速度乘积之和保持不变，即$mv+0-3m\times v/3$,这样，我们就一步一步逐渐建证起动量守恒定理.

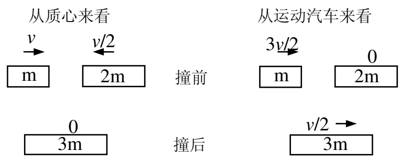

图 7: m和$2m$之间的非弹性碰撞的两种情况

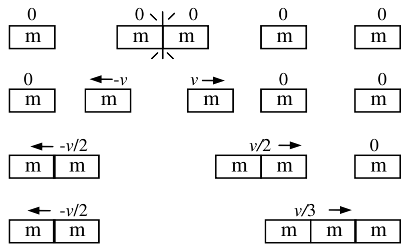

图 8: $2m$与$3m$之间的作用与反作用

现在的情况是1对2．利用同样的论证，我们可以预言1对3，2对3等等的结果，从静止开始的2对3的情况如图10－8所示.

在每一种情况中，我们发现，第一个物休的质量乘它的速度，加上第二个物体的质量乘它的速度，等于最后物体的总质量乘它的速度．因此，这都是一些动量守恒的例证．从简单的、对称的情况出发，我们用实验说明了在较复杂情况下的守恒定律．事实上，对于任何质量比是有理数的情况，我们都能这样做，并且由于任何一个比值都可以充分接近于一个有理数的比值，因此，我们能够以任何精确度处理任何比值的情形.

10-4动量和能量

上述的所有例子都是物体发生碰撞结台在一起，或者是先结合在一起，以后又由于爆炸而被分开的简单的情况．然而也有一些物体不粘台在一起的情况，例如，两十质量相等的物体以相同速率发生碰撞后弹开．在很短的时间内，它们发生接触，彼此都受到压缩．在压缩最大的那一瞬间，它们的速度都是0，而能量则贮存在弹性物体内，就象压缩弹簧的情形一样．这个能量是由物体碰撞之前所具有的动能转化而来的，而在速度为O的那一瞬间，它们的动能就变为0．然而，动能只是暂时失去．压缩状况类似于爆炸时释放能量的雷管．在某种爆炸的状况下，这些物体立即膨胀并又相互飞开；但是我们已经知道在这种情况下，物体是以相同速率飞开的．然而，一般说来，弹开的速率要

比原来的速率小，因为并非所有的能量都对爆炸有用，这与材料性质有关．如果材料是油灰，动能就不会恢复，如果材料是比较硬的，通常会再获得一定的动能．在碰撞中，其余的动能转化为热和振动能－物体变热并作振动．振动能量也很快转变为热能．用钢这样的高弹性材料制成一些碰撞物体，再用精心设计的弹簧缓冲器，有可能使得在碰撞中产生的热和振动很小．在这些情形中，弹回来的速度实际上等于初始速度；这种碰撞称为弹性碰撞．

弹性碰撞的前、后速度相等这件事与动量守恒无关，而与动能的守恒有关。然而，在对称的碰撞后，物体弹开的速率彼此相等却与动量守恒有关．

我们可以类似地分析不同质量、不同初始速度和不同弹性程度的物体之间的碰撞，确定最终速度和动能的损失，但是我们将不去详细探讨这些过程．

对于没有内部的"齿轮、转轴或部件"的系统来说，弹性碰撞是特别有趣的．这样在发生碰撞时，没有地方可以消耗能量，因为那些弹开的物体与它们在碰撞时的状态相同．因此，在非常基本的物体之间的碰撞总是高弹性的，或者非常接近于弹性的．例如，气体中分子或原子间的碰撞就被认为是完全弹性的．虽然这是一个非常好的近似，但即使这样的碰撞也不是完全弹性的；不然人们就会无法理解能量怎么会以光或热辐射的形式从气体中释放出来．在气体分子的碰撞中，偶而会有低能红外线发射出来; 但这种情况是非常罕见的，所发射的能量也是非常微小的．所以，对于大多数场合，气体中的分子碰撞被认为是完全弹性的．

作为一个有趣的例子，让我们考虑两个质量相等的物体之间的弹性碰撞．如果它们以同样速率相碰撞，那么，根据对称性原理，它们应当以相同的速率弹开．但是现在我们来看一下另一种情况下的这种碰撞，即其中的一个物体以速度v运动，而另一个物体保持静止．当两者碰撞时会出现什么情况?其实我们在前面已碰到过这种情况．从跟着物体中的一个一起运动的汽车中来观察对称的碰撞，我们发现，如果静止的物体与另一个质量恰好相同的物体发生弹性碰撞，则运动着的物体停了下来。而曾经是静止的物体现在以另一个物体曾经具有的同样的速度运动；两个物体只不过变换一下速度而已．用适当的碰撞装置很容易演示这个现象，更一般地说，假如两个物体以不同的速度运动，那么在碰撞时它们仅仅简单地交换一下速度．

另一个几乎是完全弹性的相互作用的例子为磁性。假如在我们的滑执上放置一对U形磁性，使它们彼此推斥，那么当一块磁铁静静地移向另一块磁铁时，这块磁铁会把它推走，而自己则完全保持静止，被推走的一块磁铁则无摩擦地向前滑动．动量守恒原理是非常有用的，因为它使我们在毋需了解细节的情况下也能解决许多题，例如，我们并不知道在雷管引爆时气体的运动情况，然而却能预知物体分离的速度．另一个有趣的例子是火箭的推进．一枚具有很大质量M的火箭用极大的速度V(相对于火箭来说)排出质量为m的小块后，如果火箭原来静止的话，它将以很小的速度v运动．利用动量守恒原理，我们可以计算出这个速度为

$$v = \frac{m}{M} \cdot V$$

只要不断地排出物质，火箭就一直加速．火箭的推进本质上与枪的反冲是一回事：不需要任何作反推的空气．

10-5 相对论性动量

现在已对动量守恒定律作了一些修正．然而，今天这条定律仍是正确的，修正主要是在事物的定义上．在相对论中，我们的确也有动量守恒定律；粒子具有质量，而动量仍由mv，即质量乘以速度给出，但是重量随速度改变，因此动量也发生改变．质量随速度的变化遵从以下规律

$$m = m_0/\sqrt{1 - v^2/c^2}$$

这里m_0是物体的静止质量，c是光速．从这个公式很容易看出, 除非v非常在，否则m与m_0的差别就可忽略，面对通常的速度，动量的表示式就还原为原来的公式．

单个粒子的动量分量可以写为

$$P_x = m_0 v_x/\sqrt{1 - v^2/c^2}$$

$$P_y = m_0 v_y/\sqrt{1 - v^2/c^2}$$

$$P_z = m_0 v_z / \sqrt{1 - v^2/c^2}$$

这里$v^2 = v_x^2 + v_y^2 + v_z^2$. 如果对所有相互作用粒子在碰撞前后的$x$动量分量分别求和，则两个和相等，也就是说，在$x$方向上的动量守恒. 同样的情况对任何方向都成立.

在第四章中，我们看到，只有承认能量能表现为电能、机械能、辐射能、热能等等不同形式，能量守恒定律才确实成立. 在某些这类情况中，例如热能，能量可以说成是"隐藏"的. 这个例子可能使我们联想到这样一个问题："是不是也存在着动量的隐藏形式－或许是某种热动量呢?"答案是由于下述理出由隐藏动量是很困难的.

如果把各个原子的速度的平方相加，一个物体内原子的无规则运动就提供了热能的一种量度. 速度平方和将是正的，不具有方向上的特征. 物体内热的存在与物体是否作整体运动无关，并且以热这种形式的能量守恒不是很明显的. 另一方面，如果我们把速度相加，由于速度是有方向的，若发现其结果不为零，这就意味着整个物体在某个特定方向上有移动，而这样显著的动量是很容易观察到的. 因为只有物体作整体运动时，它才有净动量，所以就不存在内部无规则动量损耗. 因此动量作为一个力学量是难以隐藏起来的. 然而，例如在电磁场内动量也可以被隐藏起来. 这种情况是另一种相对论效应.

牛顿的前提之一是认为在一段距离内的相互作用是瞬时的. 结果发现情况并非如此；比如，在包含着电力的情况下，如果在某一个位置上的一个电荷突然移动，其对在另一个位置上的另一个电荷的影响并不是瞬时的－稍有一点推迟. 在那种状况下，即使彼此作用的力是相等的，动量仍与之不符；这样，在一段短时间内将出现麻烦，因为有一段时间，第一个电荷将感受一定的反作用力，即获得了某些动量，但第二个电荷却丝毫也不受影响，也不改变它的动量. 影响跨过它们之间的距离所需要的时间，即以每秒. 186,000英里的速度跨过这段距离的时间. 在这段很短的时间内，粒子的动量是不守恒的. 当然，在第二个电荷感受到第一个电荷的作用并且一切都稳定下来之后，动量的方程就完全成立，但在那段小小的时间间隔中动量是不守恒的. 为了表明这一点，我们说在这段时间内除粒子的动量mv外还有另一类动量存在，这就是电磁场的动

量．如果我们将电磁场的动量加在粒子的动量上，则在所有时间内动量每一时刻都守恒．电磁场具有动量和能量这个事实使场的存宅在更为真实．因此，更好的理解是，原来那种认为只有粒子之间存在力的概念必须修正为：粒子具有场，场作用在另一个粒子上，而场本身具有我们所熟悉的性质，比如正象粒子那样带有能量和动量．再举另外一个例子：电磁场中存在着我们称之为光的电磁波，结果光也具有动量．所以，光撞击一个物体时，它在每秒钟内传递了一定大小的动量；这相当于一个力，因为，如果被照射物体每秒钟获得一定的动量，它的动量就发生变化，这种情况与有一个力作用在它上面完全相同．光撞击在物体上时会施加一个压力;这个压力很小，但用足够灵敏的仪器可以测量出来

在量子力学中，动量是另一回事－它不再是mv了．物体的速度的含义已难于确切定义，但是动量仍然存在．在量子力学中，差别在于当粒子表现为粒子时，动量仍是mv，但是当粒子表现为波时，动量就用每厘米的波数来量度：波数越大，动量就越大．尽管存在这些差别: 动量守恒定律在量子力学中仍然成立．虽然$f = ma$不成立，所有从牛顿定律出发的有关动量守恒的推导也都不成立，然而，在量子力学中，这条特殊定律却最后仍然有效！

I MatLab Script

I.1 AvgVvsFrConMass.m

```
% AvgVvsFrConMass.m
% ***********************************************
% This fuctions is for the constant mass system to plot the figure of
% average velocity vs average resistance(fr) and output the v1,v2,average
% v,average fr and the error of the curve fitting
% Zhou Lvwen:  zhou.lv.wen@gmail.com

function [v1,v2,avgv,avgfr,error] = AvgVvsFrConMass(xls,sheet,m,ifplot,n)
d = 1;%cm
s = 100;%cm
global Vmmode

data = xlsread(xls,sheet,'a3:c200');
t1 = data(:,1)/1000;%s
t2 = data(:,2)/1000;%s
tau= data(:,3);%s
v1 = d./t1;%cm/s
v2 = d./t2;%cm/s
[avgv,index] = sort(s./tau);%cm/s
v1 = v1(index);
v2 = v2(index);
tau = tau(index);
[tau,vbar] = ChangeTauAndBarV(tau,v1,v2);

avgfr = m*(v1-v2)./tau;%dain
a = polyfit(avgv,avgfr,n);
avgvfit = avgv;
avgfrfit = polyval(a,avgvfit);
error = avgfrfit-avgfr;
Standard_deviation = sqrt(sum((error).^2)./(length(avgfr)-1));

if ifplot
    plot(avgv,avgfr,'.k',avgvfit,avgfrfit,'k-','markersize',15,...
                                            'linewidth',2);
    ylim([0,400]);xlim([13,40]);

    if Vmmode == 1
        xlabelstr = strcat('$\bar v = ','s/\tau$ (cm/s)');
    else
        xlabelstr = strcat('$\bar v = ','(v_0+v^\prime)/2$ (cm/s)');
    end

    xlabel(xlabelstr,'Interpreter','latex','fontsize',13)
```

```
    ylabel('$\bar{f_r}$ (dain)','Interpreter','latex','fontsize',13)
    text(14,380,strcat('$m={ }$ ',sheet,'g'),...
                'Interpreter','latex','fontsize',13)
    text(14,350,strcat('$Y = ',poly2str(a,'x'),'$'),...
                'Interpreter','latex','fontsize',13)
    text(14.2,320,strcat('$S={ }$ ',num2str(Standard_deviation)),...
                'Interpreter','latex','fontsize',13)
    text(14.2,290,strcat('Data :',num2str(length(tau))),...
                'Interpreter','latex','fontsize',13)
end
```

I.2 AptitudeGrouping.m

```
% AptitudeGrouping.m
% ******************************************
% This function choose and combine proper average v and average fr from
% three constant mass system to approach the resistance of variable mass
% system.
% Zhou Lvwen:  zhou.lv.wen@gmail.com

function [a,aerror] = AptitudeGrouping(ifplot,n)
if nargin==0
   ifplot = 1;
   n = 2;% polyfit degree
end
global Vmmode
data=xlsread('DataOfVariableMass','412.6g','A3:E46');
M = [392.6,322.8,223.0];
mlim=[384.99  283.87];
d = 1;%cm
s = 100;%cm
E=0.16;

t1 = data(:,1)/1000;%s
t2 = data(:,2)/1000;%s
tau= data(:,3);%s
v1 = d./t1;%cm/s
v2 = d./t2;%cm/s

v1 = mean(v1-v2)./(mlim(1)-mlim(2))*M(1)+v1-...
     mean(v1-v2)./(mlim(1)-mlim(2))*mlim(1);
v2 = mean(v1-v2)./(mlim(1)-mlim(2))*M(3)+v1-...
     mean(v1-v2)./(mlim(1)-mlim(2))*mlim(1);

[v1,I]=sort(v1);
v2=v2(I);
tau=tau(I);
```

```
[tau,vbar] = ChangeTauAndBarV(tau,v1,v2);

ratio=(M(1)-M(2))/(M(1)-M(3));
vInsert = v2+(v1-v2).*ratio;
a1=polyfit(v1,vInsert,1);
a2=polyfit(v1,v2,1);

v1i=14:0.15:42;
vIi=polyval(a1,v1i);
v2i=polyval(a2,v1i);

xls='DataOfConstantMass';
sheet={'392.6','322.8','223.0'};

[v1m1,v2m1,avgvm1,avgfrm1,errorm1]=AvgVvsFrConMass(xls,sheet{1},M(1),0,n);
[v1m2,v2m2,avgvm2,avgfrm2,errorm2]=AvgVvsFrConMass(xls,sheet{2},M(2),0,n);
[v1m3,v2m3,avgvm3,avgfrm3,errorm3]=AvgVvsFrConMass(xls,sheet{3},M(3),0,n);

Index=[];
I1=[];
I2=[];
I3=[];
for i=1:length(v1i)
    e=E;
    for j=1:length(avgvm1)
        if abs(v1i(i)-avgvm1(j))<e
           e=abs(v1i(i)-avgvm1(j));
           I1=j;
        end
    end
    e=E;
    for j=1:length(avgvm2)
        if abs(vIi(i)-avgvm2(j))<e
           e=abs(vIi(i)-avgvm2(j));
           I2=j;
        end
    end
    e=E;
    for j=1:length(avgvm3)
        if abs(v2i(i)-avgvm3(j))<e
           e=abs(v2i(i)-avgvm3(j));
           I3=j;
        end
    end
    if (isempty(I1)+isempty(I2)+isempty(I3))==0
        Index=[Index;I1 I2 I3];
    end
    I1=[];I2=[];I3=[];
end
```

```
vGroup=[avgvm1(Index(:,1)),avgvm2(Index(:,2)),avgvm3(Index(:,3))];
frGroup=[avgfrm1(Index(:,1)),avgfrm2(Index(:,2)),avgfrm3(Index(:,3))];
errormGroup=[errorm1(Index(:,1)),errorm2(Index(:,2)),errorm3(Index(:,3))];

avgvg=mean(vGroup,2);
avgfrg=mean(frGroup,2);
avgerror=mean(abs(errormGroup),2);

a=polyfit(avgvg,avgfrg,n);
avgfrgfit1=polyval(a,avgvg);
S=sqrt(sum((avgfrgfit1-avgfrg).^2)./(length(avgfrgfit1)-1));
avgvgfit=15:0.1:40;
avgfrgfit=polyval(a,avgvgfit);

aerror=polyfit(avgvg,avgerror,n);
errori=polyval(aerror,avgvgfit);

if ifplot == 1
    plot(avgvg,avgfrg,'.k',avgvgfit,avgfrgfit,'k','markersize',15,...
                                                 'linewidth',2);
    hold on;
    plot(avgvgfit,[avgfrgfit-errori;avgfrgfit+errori],'b--','linewidth',2)
    if Vmmode == 1
        xlabelstr = strcat('$\bar v = ','s/\tau$');
    else
        xlabelstr = strcat('$\bar v = ','(v_0+v^\prime)/2$');
    end
    xlabel(xlabelstr,'Interpreter','latex','fontsize',13);
    ylabel('$\bar{f_r}$ (dain)','Interpreter','latex','fontsize',13)
    text(16,380,'$m_1 = 392.6g$, $m_2 = 322.8g$, $m_3 = 223.0g$ ',...
                'Interpreter','latex','fontsize',13)
    text(16,360,strcat('$Y = ',poly2str(a,'x'),'$'),...
                'Interpreter','latex','fontsize',13)
    text(16.2,340,strcat('$S = ',poly2str(aerror,'x'),'$'),...
                'Interpreter','latex','fontsize',13)
    text(16.2,320,strcat('Data :',num2str(length(Index))),...
                'Interpreter','latex','fontsize',13)
end
```

I.3 formulae.m

```
% formulae.m
% *********************************************
% This function to present disparate predicton from two equations of
% dynamics:
%           -fr=d(mv)/dt    and  -fr=m(t)dm/dt
% The prediction deduce by lou: deduce==1, by zhou: deduce==2;
```

```
% Zhou Lvwen:  zhou.lv.wen@gmail.com

function [taup,v1p] = formulae(m0,Dm,v0,v1,v,tau,fr,kind)
global deduce

if deduce==1
    F = fr+(kind==2)*100./tau.*Dm./tau;
    taup = (m0.*v0-(m0-Dm).*v1)./F;
    v1p = v0 + (Dm.*v0-F.*tau)./(m0-Dm);
else
    vdm = (kind==2)*v.*Dm;
    taup = (m0.*v0-(m0-Dm).*v1-vdm)./fr;
    v1p = (m0.*v0-fr.*tau-vdm)./(m0-Dm);
end
```

I.4 MainConMass.m

```
% MainConMass.m
% ********************************************
% This is main script to plot the figure of average velocity vs average
% resistance(fr)(For three constant mass(392.6g, 322.8g and 223.0g) systems)
% Zhou Lvwen:  zhou.lv.wen@gmail.com

global deduce Vmmode
n = 2; % polyfit degree
Vmmode = 2;
deduce = 1;

xls='DataOfConstantMass';
M = [392.6,322.8,223.0];
sheet={'392.6','322.8','223.0'};
ifplot = 1;% plot or not
for i=1:3
    figure
    [v1,v2,avgv,avgfr,error]=AvgVvsFrConMass(xls,sheet{i},M(i),ifplot,n);
end
```

I.5 PredictTau.m

```
% PredictTau.m
% ********************************************
% This is main script to predict time interval.
% Zhou Lvwen:  zhou.lv.wen@gmail.com

global deduce Vmmode
n = 1;% polyfit degree
vtaufit = 1; % vtaufit==1 for 'tau = a*v0+b'; vtaufit == 2 for 'tau = a/v0'
Vmmode = 2;  % Vmmode = 1 for vbar = s/tau;
```

```
                % Vmmode = 2 for vbar = (v0+v1)/2, tau = s/vbar;
deduce = 1;     % The prediction deduce by lou: deduce==1, by zhou: deduce==2;

data = xlsread('DataOfVariableMass','412.6g','A3:E46');
m0 = 412.6-data(:,4);%g
Dm = data(:,5);%g
d = 1;%cm
s = 100;%cm
t1 = data(:,1)./1000;%s
t2 = data(:,2)./1000;%s
tau= data(:,3);%s
v0 = d./t1;%cm/s
v1 = d./t2;%cm/s

[v0,I]=sort(v0);
v1=v1(I);
tau=tau(I);
Dm=Dm(I);
m0=m0(I);
[tau,vbar] = ChangeTauAndBarV(tau,v0,v1);

[a,aerror] = AptitudeGrouping(0,n);
frmax = polyval(a,vbar)+polyval(aerror,vbar);
fr=polyval(a,vbar);
frmin = polyval(a,vbar)-polyval(aerror,vbar);
str={'--k',':k'};
H=[];
for i=1:2
    for kind=1:2
        [taupmax,v1pmax] = formulae(m0,Dm,v0,v1,v0,tau,frmax,kind);
        [taupmin,v1pmin] = formulae(m0,Dm,v0,v1,v1,tau,frmin,kind);
        [taup,v1p] = formulae(m0,Dm,v0,v1,vbar,tau,fr,kind);

        v1fit=[15.04:0.1:35]';
        if vtaufit == 1
            amean=polyfit(v0,taup,n);
            taupfit=polyval(amean,v1fit);

            amax=polyfit(v0,taupmax,n);
            taupmaxfit=polyval(amax,v1fit);

            amin=polyfit(v0,taupmin,n);
            taupminfit=polyval(amin,v1fit);
        else
            a0=100;
            tauf=inline('a./x','a','x');

            [amean,resid2]=lsqcurvefit(tauf,a0,v0,taup);
            taupfit=tauf(amean,v1fit);
```

```
            [amax,resid2]=lsqcurvefit(tauf,a0,v0,taupmax);
            taupmaxfit=tauf(amax,v1fit);

            [amin,resid2]=lsqcurvefit(tauf,a0,v0,taupmin);
            taupminfit=tauf(amin,v1fit);
        end
        figure(i);
        hold on
        hid=fill([v1fit;flipud(v1fit)],[taupmaxfit;flipud(taupminfit)],'y');
        set(hid,'EdgeColor','none');

        h=plot(v1fit,taupfit,   str{kind},...
               v1fit,taupmaxfit,str{kind},...
               v1fit,taupminfit,str{kind});
        if i ==1
            plot(v0,taup,'.k',v0,taupmax,'.r',v0,taupmin,'.b','markersize',5)
        end
        H=[H h];
    end

    if i ==1
        legend(H(1,:),{'First prediction','Second prediction'},...
                       'Interpreter','latex');
    else
        h1=plot(v0,tau,'.k','markersize',15);
        h2=plot([15:0.05:35],s./[15:0.05:35],'r');
        legend([H(1,3:4),h2,h1],...
               {'First prediction','Second prediction','$s/v_0$',...
                strcat('Actually data: ',num2str(length(v1)))},...
                'Interpreter','latex');
    end
    xlim([15,35]);
    box on
    ylabelstr = '$ \tau $ (s)';
    if Vmmode==1
        ylabelstr = '$ \tau $ (s)';
    else
        ylabelstr = strcat('$ \tau = 2s/(v_0+v^\prime)$(s)');
    end
    xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
    ylabel(ylabelstr,'Interpreter','latex','fontsize',13)
end
```

I.6 PredictVelocity.m

```
% PredictVelocity.m
% ***********************************************
```

```
% This is main script to predict finally velocity.
% Zhou Lvwen:  zhou.lv.wen@gmail.com

global deduce Vmmode
n = 1;% polyfit degree
vtaufit = 1; % vtaufit==1 for 'tau = a*v0+b'; vtaufit == 2 for 'tau = a/v0'
Vmmode = 2;  % Vmmode = 1 for vbar = s/tau;
             % Vmmode = 2 for vbar = (v0+v1)/2, tau = s/vbar;
deduce = 1;  % The prediction deduce by lou: deduce==1, by zhou: deduce==2;

data = xlsread('DataOfVariableMass','412.6g','A3:E46');
m0 = 412.6-data(:,4);%g
Dm = data(:,5);%g
d = 1;%cm
s = 100;%cm
t1 = data(:,1)./1000;%s
t2 = data(:,2)./1000;%s
tau= data(:,3);%s
v0 = d./t1;%cm/s
v1 = d./t2;%cm/s

[v0,I] = sort(v0);
v1 = v1(I);
tau = tau(I);
Dm = Dm(I);
m0 = m0(I);

[tau,vbar] = ChangeTauAndBarV(tau,v0,v1);

[a,aerror] = AptitudeGrouping(0,n);
frmax = polyval(a,vbar)+polyval(aerror,vbar);
fr = polyval(a,vbar);
frmin = polyval(a,vbar)-polyval(aerror,vbar);
hold on
str={'--k',':k'};
H=[];
for i = 1:2
    for kind=1:2
        [taupmax,v1pmax] = formulae(m0,Dm,v0,v1,v0,tau,frmax,kind);
        [taupmin,v1pmin] = formulae(m0,Dm,v0,v1,v1,tau,frmin,kind);
        [taup,v1p] = formulae(m0,Dm,v0,v1,s./tau,tau,fr,kind);

        v0fit = [15.04:0.1:35]';

        amean = polyfit(v0,v1p,n);
        v1pfit = polyval(amean,v0fit);

        amax = polyfit(v0,v1pmax,n);
        v1pmaxfit = polyval(amax,v0fit);
```

```
        amin = polyfit(v0,v1pmin,n);
        v1pminfit = polyval(amin,v0fit);

        figure(i);
        hold on
        hid = fill([v0fit;flipud(v0fit)],[v1pmaxfit;flipud(v1pminfit)],'y');
        set(hid,'EdgeColor','none');
        if i == 1
            plot(v0,v1p,'.k',v0,v1pmax,'.r',v0,v1pmin,'.b','markersize',5);
        end
        h = plot(v0fit,v1pfit,   str{kind},...
                 v0fit,v1pmaxfit,str{kind},...
                 v0fit,v1pminfit,str{kind});
        H = [H h];
    end

    if i ==1
        legend(H(1,:),{'First prediction','Second prediction'},...
                      'Interpreter','latex',2);
    else
        h1 = plot(v0,v1,'.k','markersize',15);
        h2 = plot([15 35],[15,35],'r');% plot the line "v1=v0"
        legend([H(1,3:4),h2,h1],
              {'First prediction','Second prediction','$v^\prime=v_0$',...
               strcat('Actually data: ',num2str(length(v1)))},...
               'Interpreter','latex',2);
    end
    xlim([15,35])
    box on
    xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
    ylabel('$ v^\prime $ (cm/s)','Interpreter','latex','fontsize',13);
end
```

I.7 MainPrediction.m

```
% MainPrediction.m
% *********************************************
% This is main script to predict time interval(tau) and finally velocity(v1).
% Zhou Lvwen:  zhou.lv.wen@gmail.com

global deduce Vmmode
n = 2;% polyfit degree
vtaufit = 1; % vtaufit==1 for 'tau = a*v0+b'; vtaufit == 2 for 'tau = a/v0'
Vmmode = 2;  % Vmmode = 1 for vbar = s/tau;
             % Vmmode = 2 for vbar = (v0+v1)/2, tau = s/vbar;
deduce = 1;  % The prediction deduce by lou: deduce==1, by zhou: deduce==2;
```

```
data = xlsread('DataOfVariableMass','412.6g','A3:E46');
m0 = 412.6-data(:,4);%g
Dm = data(:,5);%g
d = 1;%cm
s = 100;%cm
t1 = data(:,1)./1000;%s
t2 = data(:,2)./1000;%s
tau= data(:,3);%s
v0 = d./t1;%cm/s
v1 = d./t2;%cm/s

[v0,I]=sort(v0);
v1=v1(I);
tau=tau(I);
Dm=Dm(I);
m0=m0(I);
[tau,vbar] = ChangeTauAndBarV(tau,v0,v1);

[a,aerror] = AptitudeGrouping(0,n);
frmax = polyval(a,vbar)+polyval(aerror,vbar);
fr=polyval(a,vbar);
frmin = polyval(a,vbar)-polyval(aerror,vbar);
str={'--k',':k'};
Ht=[];
Hv=[];

figure('name','predict time interval(tau)');
figure('name','predict finally velocity(v1)');
for kind=1:2
    [taupmax,v1pmax] = formulae(m0,Dm,v0,v1,v0,tau,frmax,kind);
    [taupmin,v1pmin] = formulae(m0,Dm,v0,v1,v1,tau,frmin,kind);
    [taup,v1p] = formulae(m0,Dm,v0,v1,vbar,tau,fr,kind);

    v0fit=[15.04:0.1:35]';
    % ---------------------------------predict tau-------------------------
    if vtaufit == 1
         atmean=polyfit(v0,taup,n);
         taupfit=polyval(atmean,v0fit);

        atmax=polyfit(v0,taupmax,n);
        taupmaxfit=polyval(atmax,v0fit);

        atmin=polyfit(v0,taupmin,n);
        taupminfit=polyval(atmin,v0fit);
    else
        a0=100;
        tauf=inline('a./x','a','x');

        [atmean,resid2]=lsqcurvefit(tauf,a0,v0,taup);
```

```
        taupfit=tauf(atmean,v0fit);

        [atmax,resid2]=lsqcurvefit(tauf,a0,v0,taupmax);
        taupmaxfit=tauf(atmax,v0fit);

        [atmin,resid2]=lsqcurvefit(tauf,a0,v0,taupmin);
        taupminfit=tauf(atmin,v0fit);
    end

    figure(1)
    hold on
    hidt=fill([v0fit;flipud(v0fit)],[taupmaxfit;flipud(taupminfit)],'y');
    set(hidt,'EdgeColor','none');
    plot(v0,taup,'.k',v0,taupmax,'.r',v0,taupmin,'.b','markersize',5)

    ht=plot(v0fit,taupfit,  str{kind},...
           v0fit,taupmaxfit,str{kind},...
           v0fit,taupminfit,str{kind});
    Ht=[Ht ht];

    % ----------------------------------predict v1--------------------------
    avmean = polyfit(v0,v1p,n);
    v1pfit = polyval(avmean,v0fit);

    avmax = polyfit(v0,v1pmax,n);
    v1pmaxfit = polyval(avmax,v0fit);

    avmin = polyfit(v0,v1pmin,n);
    v1pminfit = polyval(avmin,v0fit);

    figure(2)
    hold on
    hidv = fill([v0fit;flipud(v0fit)],[v1pmaxfit;flipud(v1pminfit)],'y');
    set(hidv,'EdgeColor','none');
    plot(v0,v1p,'.k',v0,v1pmax,'.b',v0,v1pmin,'.b','markersize',5)

    hv = plot(v0fit,v1pfit,   str{kind},...
             v0fit,v1pmaxfit,str{kind},...
             v0fit,v1pminfit,str{kind});
    Hv = [Hv hv];
end

figure(1)
h1t=plot(v0,tau,'.k','markersize',15);
h2t=plot([15:0.05:35],s./[15:0.05:35],'r');
legend([Ht(1,2),h2t,h1t],{'Second prediction','$S/v_0$',...
strcat('Actually data: ',num2str(length(v1)))},'Interpreter','latex')
xlim([15,35])
box on
```

```
ylabelstr = '$ \tau $ (s)';
if Vmmode==1
    ylabelstr = '$ \tau $ (s)';
else
    ylabelstr = strcat('$ \tau = 2s/(v_0+v^\prime)$(s)');
end
xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
ylabel(ylabelstr,'Interpreter','latex','fontsize',13)
gca1=gca;

%------------------------------------------------------------------------
figure(2)
h1v = plot(v0,v1,'.k','markersize',15);
h2v = plot([15 35],[15,35],'r');% plot the line "v1=v0"
legend([Hv(1,:),h2v,h1v],{'First prediction','Second prediction',...
      '$v^\prime=v_0$',strcat('Actually data:',num2str(length(v1)))},...
      'Interpreter','latex',2)

xlim([15,35])
box on
xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
ylabel('$ v^\prime $ (cm/s)','Interpreter','latex','fontsize',13)

gca2=gca;
```

I.8 Comparev.m

```
% Comparev.m
% *********************************************
% This is main script to find some exceptional date by compare v0, v1,
% (v0+v1)/2 and s/tau. Reasonable date should subject to:
%                       v1<(v0+v1)/2 approximate s/tau <v0
% We find the unreasonable date and analyze the prediction, and find the
% relationship between them.

% Zhou Lvwen:  zhou.lv.wen@gmail.com

MainPrediction

fig1=figure('name','predict time interval(tau)',...
            'position',[20,40,600,700]);
s1=subplot('position',[0.1,0.56,0.8,0.4]);
pos1=get(s1,'Position');
delete(s1);
hax1=copyobj(gca1,fig1);
set(hax1,'Position', pos1);
xlim([16,32])
ylim([3,6.25])
```

```
grid minor

fig2=figure('name','predict finally velocity(v1)',...
            'position',[620,40,600,700]);
s2=subplot('position',[0.1,0.56,0.8,0.4]);
pos2=get(s2,'Position');
delete(s2);
hax2=copyobj(gca2,fig2);
set(hax2,'Position', pos2);
xlim([16,32])
ylim([13,30])
grid minor

close(figure(1),figure(2))

data=xlsread('DataOfVariableMass','412.6g','A3:E46');
m0 = 412.6-data(:,4);%g
Dm = data(:,5);%g
d = 1;%cm
s = 100;%cm
t1 = data(:,1)./1000;%s
t2 = data(:,2)./1000;%s
tau= data(:,3);%s
v0 = d./t1;%cm/s
v1 = d./t2;%cm/s

[v0,I]=sort(v0);
v1=v1(I);
tau=tau(I);

for i=3:4
    figure(i)
    v01=mean([v0 v1],2);
    vbar=s./tau;
    x=v0;%1:length(tau);
    subplot('position',[0.1,0.1,0.8,0.4])
    hold on
    plot(x,v0,'k--',x,v1,'k-');
    plot(x,v01,'.--b',x,vbar,'.-r','markersize',14);
    legend({'$v_0$','$v^\prime$','$(v_0+v^\prime)/2$','$s/\tau$'},...
            'Interpreter','latex',2)
    xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
    ylabel('$ v $ (cm/s)','Interpreter','latex','fontsize',13);
    grid minor
    box on
    xlim([16,32])
    ylim([15,32])
end
```

I.9 ChangeTauAndBarV.m

```
% ChangeTauAndBarV.m
% ********************************************
% This function to redefine average velocity(vbar) and time interval(tau).
% Zhou Lvwen:  zhou.lv.wen@gmail.com

function [Tau,vbar] = ChangeTauAndBarV(tau,v0,v1)
global deduce Vmmode
s = 100;
if Vmmode==1&deduce==1
    vbar = s./tau;
    Tau = tau;
elseif Vmmode==2&deduce==1
    vbar = (v0+v1)./2;
    Tau = s./vbar;
elseif Vmmode==1&deduce==2
    vbar = s./tau;
    Tau = tau;
elseif Vmmode==2&deduce==2
    vbar = (v0+v1)./2;
    Tau = tau;
end
```

I.10 DataAnalysis.m

```
% DataAnalysis.m
% ********************************************
% This is main script to find some exceptional date by compare v0, v1,
% (v0+v1)/2 and s/tau. Reasonable date should subject to:
%           s/v0<tau <s/v1 &  v1<(v0+v1)/2 approximate s/tau <v0
% Figure 1 give the relationship between tau(measured data) and s/v1, and
% tau and s/v0.
% Figure 2 compare v0, v1,(v0+v1)/2 and s/tau.

% Zhou Lvwen:  zhou.lv.wen@gmail.com

nx = 1;
ns = 3;

xls={'DataOfConstantMass','DataOfVariableMass'};
sheet={'392.6','322.8','223.0','412.6g'};
if nx==2&ns~=4;
   error('DataOfVariableMass.xls only have sheet sheet{4}');
elseif nx==1&ns==4
    error('DataOfConstantMass.xls dont have sheet{4}');
end
```

```
data = xlsread(xls{nx},sheet{ns},'a3:c200');
d = 1;%cm
s = 100;%cm
t1 = data(:,1)/1000;%s
t2 = data(:,2)/1000;%s
tau= data(:,3);%s
v0 = d./t1;%cm/s
v1 = d./t2;%cm/s
[v0,I]=sort(v0);
v1 = v1(I);
tau = tau(I);
x=15:0.02:40;

figure('name','Compare s/v1,s/v0 and tau by Finally velocity')
subplot(1,2,1)
hold on
I = find(s./v1<tau); % Find unreasonable data.
for i = 1:length(I)
   plot(v1(I(i)),tau(I(i)),'mo','markersize',5,'linewidth',1.5)
end
h1=plot(x,s./x,v1,tau,'.k','markersize',10);
box on
legend(h1,{'$s/v^\prime$','$\tau$'},'Interpreter','latex')
xlabel('$ v^\prime $ (cm/s)','Interpreter','latex','fontsize',13);
ylabel('$ \tau $ (s)','Interpreter','latex','fontsize',13)

subplot(1,2,2)
hold on
I = find(s./v0>tau); % Find unreasonable data.
for i = 1:length(I)
   plot(v0(I(i)),tau(I(i)),'mo','markersize',5,'linewidth',1.5)
end
h2=plot(x,s./x,v0,tau,'.k','markersize',10);
box on
legend(h2,{'$s/v_0$','$\tau$'},'Interpreter','latex')
xlabel('$ v_0 $ (cm/s)','Interpreter','latex','fontsize',13);
ylabel('$ \tau $ (s)','Interpreter','latex','fontsize',13)


figure('name','Compare initial, finally, average and median velocity')
Vaverage = s./tau;
Vmedian = (v1+v0)./2;
x = 1:length(v0);
hold on
I = find(v0<Vaverage|Vaverage<v1); % Find unreasonable data.
for i = 1:length(I)
   plot(x(I(i)),Vaverage(I(i)),'mo','markersize',5,'linewidth',1.5)
end
h3 =plot(x,v0,'b-',x,v1,'r-',x,Vmedian,'-k',...
```

```
                              x,Vaverage,'.k','markersize',10);
box on
legend(h3,{'$v_0$','$v^\prime$','$(v_0+v^\prime)/2$','$s/\tau$'},...
           'Interpreter','latex',2)
xlabel('Data','Interpreter','latex','fontsize',13)
ylabel('$ v $ (cm/s)','Interpreter','latex','fontsize',13);
```

I.11 MatchDegree.m

```
% MatchDegree.m
% *********************************************
% This function to compute the match degree between (xi,yi) and carve y =
% f(x) .  f(x)=poly(a);
% Zhou Lvwen:  zhou.lv.wen@gmail.com

function r = MatchDegree(xi,yi,a)
x = mean(xi);
y = mean(yi);

fx = polyval(a,xi);
fy = subs(finverse(poly2sym(a)),yi);

rx = sum((yi-y).*(fy-x))/sqrt(sum((yi-y).^2)*sum((fy-x).^2));
ry = sum((fx-y).*(xi-x))/sqrt(sum((fx-y).^2)*sum((xi-x).^2));
r = sqrt(rx*ry);

%r = abs(sum((fx-y).*(xi-x))/sqrt(sum((fx-y).^2)*sum((xi-x).^2)));
```