623 Computational Finance, Fall 2014.
Lecturer: Sergey Nadtochiy.

Chapter I. Lattice (tree) methods.
Partial information can be found in Chapter 10 in "The Mathematics of Financial Derivatives" by Wilmott, Howison and Dewynne 1998.

1 One-period binomial model

A simplest imaginable financial model consists of two time periods ("now= 0" and "future= T"), one tradable asset S, and a bank account (an instrument for short-term borrowing/lending) B with constant interest rate r, such that:

- At time zero, the price of a unit of tradable risky asset (stock) S is $S_0 = s_0 = \text{const}$.
- Its time-T price is
$$S_T = \begin{cases} s_u & \text{with probability } p \in (0,1), \\ s_d & \text{with probability } 1-p, \end{cases}$$
with deterministic $s_d \leq s_u$.
- The riskless asset (bank account) B has initial price $B_0 = 1$ and the terminal value $B_T = 1 + rT$.

See Figure 1.

Note that we need to ensure that our model does not contradict the No-Arbitrage (NA) principle.

Def 1. *In the above model,* **arbitrage** *is a trading strategy, given by a number π_0, such that*

- $\mathbb{P}(\pi_0 S_T - \pi_0 s_0(1+rT) \geq 0) = 1$*;*
- $\mathbb{P}(\pi_0 S_T - \pi_0 s_0(1+rT) > 0) > 0$*.*

Thm 1. *The above model is free of arbitrage (satisfies the NA principle) if and only if*

$$s_d < (1+rT)s_0 < s_u \tag{1}$$

Proof:
"Only if". Provided there is no arbitrage in the model, let's prove that (1) is satisfied. Assume $s_d \geq (1+rT)s_0$. Then borrow s_0 from your bank account (short a bond with principal s_0) and buy one unit of the risky asset ($\pi_0 = 1$). A time T your payoff is at least $s_d - (1+rT)s_0 \geq 0$, and with positive probability it is $s_u - (1+rT)s_0 > 0$. Hence arbitrage. Assuming $s_u \leq (1+rT)s_0$, we obtain arbitrage by doing the opposite to the above.

Exercise 1. *Prove the other implication: if $s_d < (1+rT)s_0 < s_u$ then there is no arbitrage. Hint: make use of the arguments that follow.*

∎

See Figure 2.

1.1 Pricing and hedging of derivatives

Suppose we have an option paying off at T. Denote its value (price) at time $t = 0$ by $V_0 = \text{const}$. At time T, its payoff V_T is a function of S_T: it takes a value v_u if $S_T = s_u$, and v_d if $S_T = s_d$.

Such option can be a call with strike K, then, $V_T = (S_T - K)^+$. See Figure 4.

Q 1. *What is the current price of this option V_0? Or, more precisely, which values of V_0 are possible (i.e. do not violate the NA assumption)?*

It turns out (and this is, in fact, what boosted the development of financial mathematics in the last few decades) that the option price V_0 in such a model is determined **uniquely** by the **NA principle**!

Rem 1. *Note that, as will be clear from what follows, V_0 **is not** given by the "average payoff" $pv_u + (1-p)v_d$. In fact, there is no reason why the price should be given by the mathematical expectation.*

Before we show how the price V_0 can be computed in the above model, we address another problem of great importance in finance – the problem of *hedging*. In fact, the two problems turn out to have the same solution.

Main idea: *hedge* (eliminate) the *risk* (randomness) of the short position in the option by trading the underlying asset.

Rem 2. *What makes us think it is ever possible at all? Note that the values of S and V have a particular correlation: when one goes up, so does the other one. So there is a chance to "control" the changes in option price by trading the underlying.*

To construct the hedging strategy, note that,

- if we sell an option at $t = 0$ for V_0 and put the money on the bank account,
- and buy π_0 shares of S at $t = 0$,
- then the amount of money we hold on the bank account at $t = 0$ is $\gamma_0 = V_0 - \pi_0 s_0$.

At time T we have

- the payoff from the bank account $\gamma_0 B_T = \gamma_0(1 + rT) = (V_0 - \pi_0 s_0)(1 + rT)$
- and the cumulative payoff from the option and the asset is
  - either $\pi_0 s_u - v_u$
  - or $\pi_0 s_d - v_d$

Since we want to make the *total value* of our positions be deterministic, we need to choose π_0 (the only free variable above) so that

$$-v_u + \pi_0 s_u = -v_d + \pi_0 s_d$$

Thus

$$\pi_0 = \frac{v_u - v_d}{s_u - s_d},$$

which is known as the **delta-hedging** formula.

Recall that our initial investment is zero. Since the total payoff at time T is *deterministic*, due to *NA principle*, it has to be zero as well. Thus, we have

$$(V_0 - \pi_0 s_0)(1 + rT) - v_u + \pi_0 s_u = 0 \tag{2}$$

$$V_0(1 + rT) = v_u - \pi_0(s_u - s_0(1 + rT)) = \frac{(s_0(1 + rT) - s_d)v_u + v_d(s_u - s_0(1 + rT))}{s_u - s_d}$$

$$V_0 = \frac{1}{1 + rT}\left(\frac{s_0(1 + rT) - s_d}{s_u - s_d}v_u + \frac{s_u - s_0(1 + rT)}{s_u - s_d}v_d\right)$$

From the above example we can make few very important (and very deep, despite what it might seem) observations:

- The following representation holds

  $$V_0 = \frac{1}{1 + rT}\left(qv_u + (1 - q)v_d\right) = \mathbb{E}^{\mathbb{Q}}\left(\frac{V_T}{B_T}\right),$$

  i.e. the price of the option can be computed as a **discounted expectation, but using the "probability"** q **instead of** p, with

  $$q = \frac{s_0(1 + rT) - s_d}{s_u - s_d}.$$

  This q is called the *risk-neutral* probability. It has no interpretation as an actual probability of an event happening in the real-world (or, our model for the real world), and it is only used as a tool for representing the option price.

  Probability p is the *objective* or *physical* probability.

  The associated measures are usually denoted by $\mathbb{P}$ and $\mathbb{Q}$.

- The option's value V_0 is independent of p! Recall what we have done:

  - built a model for the underlying (s_0, s_u, s_d, p),
  - constructed a hedging (replicating) strategy for the option,
  - and used the NA argument to compute V_0, the value of the option in this model.

  Note that V_0 depends on the possible paths of the underlying $\{s_0, s_u\}$ and $\{s_0, s_d\}$ but not on the probability of taking one or the other. Instead, we assign the artificial probabilities q and $1 - q$ to these paths to compute V_0.

  Exercise 2. *Compare two models, having the same parameters* s_0*,* s_d*,* s_u *and* r*: one with* $p = 0.5$ *and the other with* $p = 0.99$*. The price of the option is the same in both models, however, it is clear that in the first one the option will pay off a positive amount with much higher probability. Can you explain what is wrong?*

- Under $\mathbb{P}$

  $$\mathbb{E}^{\mathbb{P}}\left(S_T | S_0\right) = ps_u + (1 - p)s_d,$$

  but under $\mathbb{Q}$

  $$\mathbb{E}^{\mathbb{Q}}\left(S_T | S_0\right) = qs_u + (1 - q)s_d = \frac{(1 + rT)s_0 - s_d}{s_u - s_d}s_u + \frac{s_u - (1 + rT)s_0}{s_u - s_d}s_d = (1 + rT)s_0$$

  Thus, the **discounted asset price** (S/B) **is a martingale** under the measure $\mathbb{Q}$, which is also called the **risk-neutral measure**.

Def 2. *In a (B, S) market, a* **risk-neutral probability** *is a probability measure under which the discounted risky asset S/B is a* **martingale***.*

Thm 2. *(Fundamental Theorem of Asset Pricing (FTAP) 1)* **NA** *is equivalent to the existence of a* **risk-neutral measure***.*

Def 3. *A (B, S) market is* **complete** *if any contingent claim V can be hedged perfectly by trading in B and S.*

Thm 3. *(FTAP 2) Assume NA holds. Then, the market is* **complete** *if and only if the risk-neutral measure is* **unique***.*

Q 2. *How do we choose the parameters of the model: s_u, s_d, p?*

This choice can be based on the (presumably known) probabilistic characteristics of the return of the underlying asset S_T/S_0. More precisely, we can choose the above triplet so that it matches the *expectation* and *variance* of the return:

$$\mathbb{E}\left(S_T/S_0\right) = \mu, \qquad \mathtt{Var}\left(S_T/S_0\right) = \mathbb{E}\left(S_T/S_0 - \mathbb{E}\left(S_T/S_0\right)\right)^2 = \sigma^2, \tag{3}$$

where μ and σ are some known numbers.

Notice that we have two equations and three unknowns. In order to get a well-posed problem we need to make additional specifications. Typically, the considerations of symmetry lead to

$$s_u = us_0, \qquad s_d = ds_0, \qquad u = \frac{1}{d}. \tag{4}$$

Q 3. *How does the pricing and hedging change if the risky asset pays a* **dividend** *of rate D?*

In this case, the effective payoff of the risky asset is

$$S_T(1+DT) = \begin{cases} s_u(1+DT) \\ s_d(1+DT) \end{cases}$$

The discounted payoff is

$$S_T(1+DT)/B_T = \begin{cases} s_u \frac{1+DT}{1+rT} \\ \\ s_d \frac{1+DT}{1+rT} \end{cases}$$

Thus, in the case of dividends, the **risk neutral probability can be computed just like in the case of zero dividend, with a different interest rate** $\tilde{r}$, such that:

$$\frac{1}{1+\tilde{r}T} = \frac{1+DT}{1+rT},$$
$$\tilde{r} = \frac{r-D}{1+DT} \tag{5}$$

Rem 3. *When T is small, we have*

$$\tilde{r} \approx r - D$$

However, it is important to remember that the **discounting is still done with the original interest rate** r**!**

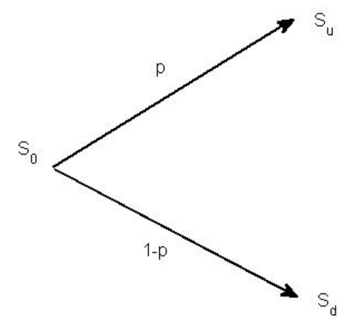

Figure 1: One-period binomial model

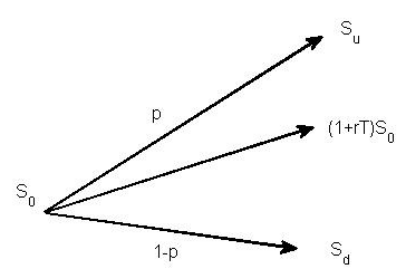

Figure 2: One-period binomial model

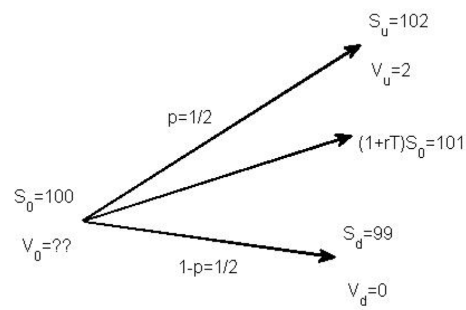

Figure 3: Option in a one-period binomial model

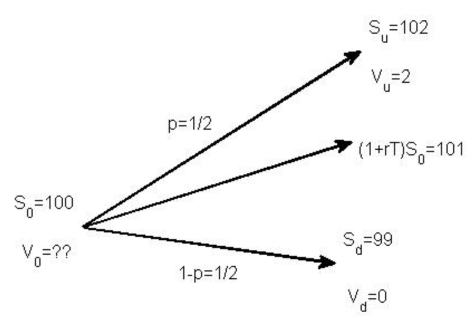

Figure 4: Option in a one-period binomial model

2 Multi-period tree models

Notation: risk-neutral measure = pricing measure = equivalent martingale measure (EMM).

One-period binomial model is very limited:

- It only allows for "buy and hold" strategies for contingent claims, without additional trading in between. In such models, one can only bet on changes in the underlying asset (say, "it will go down" or "up").

  However, the prices of derivatives are also affected by other market indicators which reflect the "market point of view" about the future (such as volatility). Hence, derivatives prices fluctuate as the values of the indicators change.

  Trading derivatives, one may (and typically does) want to exploit these fluctuations. The one-step model does not allow for such trading strategies.

- We may need to price options with several different maturities (expiries) in the same model. Hence, we need to allow for payoffs at several moments of time.

- In addition, there exist so-called *path-dependant derivatives*, whose payoffs, and, consequently, prices, are different from those of the European-type claims. A model needs more than one time step in order to reproduce these differences.

- Common sense: a model should reproduce the *important* observed phenomena.

A natural extension of the previous model is to increase the number of time steps, adding new possible values for the underlying asset. Then, we end up with a tree in Figure 5.

However, a tree with N time steps will have $2^{N+1}-1$ nodes! This makes any numerical computations in this model, essentially, impossible.

A way out is to construct the tree so that it is *recombining*: see Figure 6. Then the number of nodes grows with the number of steps N as $\frac{1}{2}(N+1)(N+2)$.

A simple way to construct such a tree is to take all up-steps to be of the form

$$S_n \mapsto \begin{cases} uS_n, & p, \\ dS_n, & 1-p, \end{cases}$$

for some fixed constants $u > d > 0$ and all $n = 0, \dots, N$. All jumps are assumed to be independent.

Then, it is easy to see that, for all $n = 0, \dots, N$, we have:

$$S_n \in \left\{s_n^0, s_n^1, \dots, s_n^n\right\},$$

where $s_n^m = d^{n-m}u^m s_0$. Then, from the node (n, m), we can only jump to $(n+1, m+1)$ or $(n+1, m)$:

$$s_n^m \mapsto \begin{cases} s_{n+1}^{m+1}, \\ s_{n+1}^{m}, \end{cases}$$

We fix T and assume that each time step is of length $\Delta t := \frac{T}{N}$. Then, the riskless asset B starts from $B_0 = 1$ and grows at rate r:

$$B_n \mapsto B_n(1 + rT)$$

It is easy to see that this model satisfies NA if and only if each single-period sub-model satisfies NA. In other words, if

$$d < 1 + r\Delta t < u$$

2.1 Pricing and hedging of European options in a multi-period binomial model

Assume that we have constructed a tree model, as shown in the previous subsection and in Figure 7. Assume that the market filtration is generated by the two assets (the risky asset and the bank account).

Consider a European option which pays $V_N = F(S_N)$ at the time of expiry N, where F is a given function (e.g. $F(S_N) = (S_N - K)^+$).

Q 4. *What is the arbitrage-free price of this option, V_0?*

From FTAP 1, we know that the price is given by

$$V_0 = \mathbb{E}^{\mathbb{Q}}(V_T/B_T) = \mathbb{E}^{\mathbb{Q}}(F(S_T)/B_T),$$

under some EMM $\mathbb{Q}$. From the homogeneity of the original model, it makes sense to search for $\mathbb{Q}$ under which the risky asset, at each step, jumps up or down with probabilities q and $1-q$, respectively, independently across steps.

We can obtain the probability q (or, the associated probability measure $\mathbb{Q}$) from the martingale property of S/B. Notice that $((1+r\Delta t)^{-n} S_n)_{n=0}^{N}$ must be a $\mathbb{Q}$-martingale:

$$\mathbb{E}\left((1+r\Delta t)^{-(n+1)} S_{n+1} \middle| S_n\right) = (1+r\Delta t)^{-n} S_n,$$

$$(1+r\Delta t)^{-(n+1)} q u S_n + (1+r\Delta t)^{-(n+1)} (1-q) d S_n = (1+r\Delta t)^{-n} S_n,$$

Solving this equation, we obtain the desired expression for q, which coincides with the one we had in the one-period case (setting $s_u = us_0$ and $s_d = ds_0$):

$$q = \frac{1 + r\Delta t - d}{u - d} \tag{6}$$

However to price the option, we would need to compute the probability of each path, and sum over all paths. The problem is that the number of paths grows exponentially! Hence, numerically, this is not feasible.

This is why we will use the **backward recursion** instead.

- Denote by $V(n,m)$ the **value function** of the option at node (n,m): the price of the option after the nth time step, given that the underlying value at that time is s_n^m ($S_n = s_n^m$). In other words,

$$V(n,m) = \frac{1}{1+r\Delta t} \mathbb{E}^{\mathbb{Q}}\left(F(S_N) \,|\, S_n = s_n^m\right)$$

- We compute the value function of the option recursively, backwards, starting from time N, since at that time the price is known – it coincides with the payoff: $V(N,m) = F(s_N^m)$.

- Applying the results from the previous section to the right-most subgraphs (sub-models) of the tree (see Figure 8), we obtain

$$V(N-1,m) = (1+r\Delta t)^{-1} \left(qV(N,m+1) + (1-q)V(N,m) \right),$$

where q is given by (6).

- Working backwards, we see that the above method works for any moment of time n: one can simply treat the value of the option at time $n+1$ as an intermediate payoff, and apply the results established for the one-period model. Thus, we obtain the following recursive formula, for $n = 1, \dots, N$:

$$V(n-1,m) = (1+r\Delta t)^{-1} \left(qV(n,m+1) + (1-q)V(n,m) \right), \quad m = 0, \dots, n,$$

where

$$q = \frac{1 + r\Delta t - d}{u - d}$$

Together with the terminal condition

$$V(N,m) = F(s_N^m), \quad m = 0, \dots, N,$$

it allows us to compute the initial price $V_0 = V(0,0)$.

Hedging.

Recall that in a single-period model we valued options by *hedging*. In the present multi-period model, we can price the option without computing the hedging strategy – simply using the risk-neutral measure.

However, in the multi-period tree model, it is not too hard to find a hedging strategy for a short position in the option: i.e. a strategy of trading in B and S, such that, together with the short position in the option, the total value of the portfolio remains zero at all times.

Rem 4. *Note that, in this case, hedging strategy is given by a sequence* $(\pi_1, \pi_2, \dots, \pi_N)$*, where* π_n *denotes the number shares of* S *in the hedging portfolio at the end of the time interval* $[n-1, n]$*, before rebalancing at time* n*.*

- We have seen that the risk-neutral measure is unique. Hence, the market is complete and there exists a unique hedge for the option V

- Due to the Markov nature of the model, we expect the hedging strategy of a short position in V to be given by the delta-hedging formula:

$$\pi_n = \pi(n,m) = \frac{V(n,m+1) - V(n,m)}{s_n^{m+1} - s_n^m}$$

- Let us check that it does, indeed, work.
  - The change in the value of the hedging strategy between the times $n-1$ and n is given by:

$$\Delta_n = \pi_n(S_n - S_{n-1}) + (V_{n-1} - \pi_n S_{n-1}) r\Delta t - V_n + V_{n-1}$$

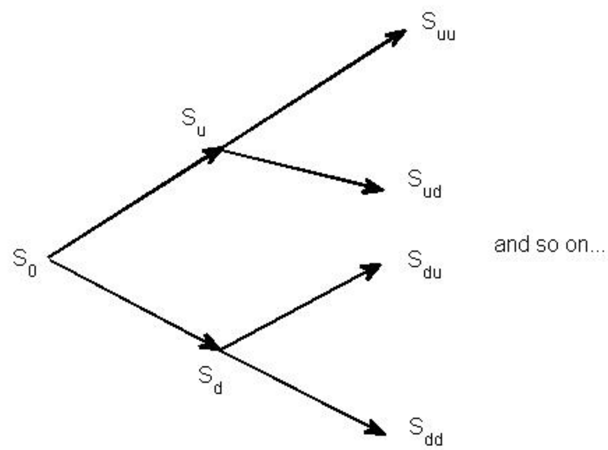

Figure 5: Multi-period binomial model as a tree

– Assume that we are at node $(n-1,m)$ at time $n-1$, and that the next jump is upward. Then, the above becomes:

$$\begin{aligned}\Delta_n = &\frac{V(n,m+1)-V(n,m)}{s_n^{m+1}-s_n^m}\left(s_n^{m+1}-s_{n-1}^m-s_{n-1}^m r\Delta t\right)\\ &-V(n,m+1)+(qV(n,m+1)+(1-q)V(n,m))\\ &=(V(n,m+1)-V(n,m))\left(\frac{u-1}{u-d}-\frac{r\Delta t}{u-d}\right)\\ &-V(n,m+1)+(qV(n,m+1)+(1-q)V(n,m))\\ &=(V(n,m+1)-V(n,m))\,(1-q)\\ &-V(n,m+1)+(qV(n,m+1)+(1-q)V(n,m))=0\end{aligned}$$

Similarly, we obtain zero change if the next jump is downward.

2.2 Pricing exotic options on trees

Asian options

The payoff of an Asian option depends on the terminal value of the underlying as well as on its average value. For example, the floating-strike Asian call has payoff

$$V_N = (S_N - \frac{1}{N+1}\sum_{n=0}^{N} S_n)^+$$

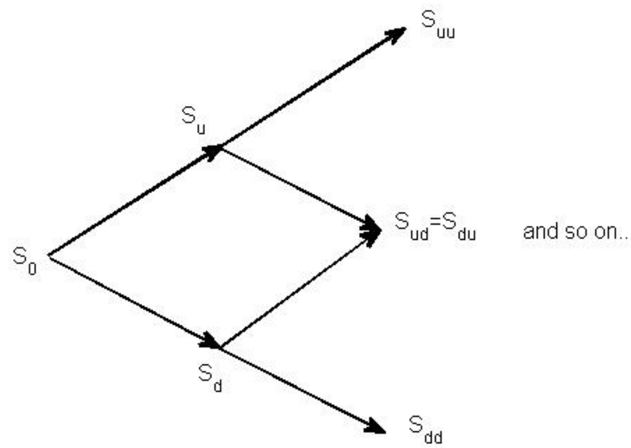

Figure 6: Recombining tree

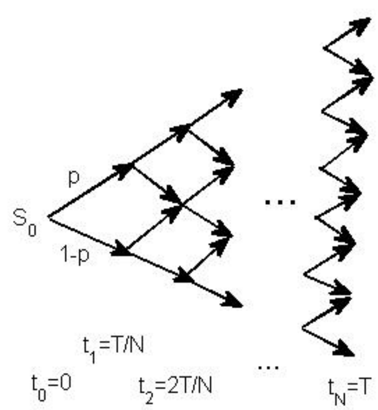

Figure 7: N-step recombining tree

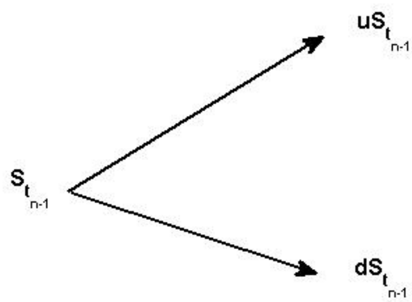

Figure 8: Single-period sub-tree (or, sub-model)

FTAP 1 implies that

$$V_0 = \mathbb{E}^{\mathbb{Q}}(V_N/B_N)$$

As we discussed before, pricing by iterating over the paths is not feasible.

Q 5. *Can we price Asian options by backward recursion?*

NO! The reason is that the value function $V(n,m)$ is not well defined: the price of an option depends on the past values of S, not only on the current node (n,m)!

Barrier options

Barrier options are also path-dependent. Nevertheless, we can price them by backward recursion.

Barrier option has a terminal payoff function $F(S_N)$, and, in addition, it can be

- knock-in or knock-out;
- with upper or lower barrier.

For example, the payoff of up-and-out call is given by

$$V_N = (S_N - K)^+ \mathbf{1}_{\max_{0\le t\le N} S_t < U}$$

Note that a knock-in option can always be represented as a combination of a knock-out and a European:

$$F(S_N)\mathbf{1}_{\max_{0\le t\le N} S_t \ge U} = F(S_N) - F(S_N)\mathbf{1}_{\max_{0\le t\le N} S_t < U}$$

Let us price a knock-out option with terminal payoff function F and the upper barrier U. As usual, we recall

$$V_0 = \mathbb{E}^{\mathbb{Q}}(V_N/B_N)$$

Note that the price of the option, again, depends on the entire path of S – hence it is not a priori clear how to define the value function. However, in this case, the dependence is particularly simple. Let us introduce, for any $n = 0, \dots, N-1$ and $m = 0, \dots, n$, the price of the option conditional on the fact that the barrier has not been hit in the past:

$$V(n,m) = \frac{1}{(1+r\Delta t)^{N-n}} \mathbb{E}^{\mathbb{Q}} \left(F(S_N) \mathbf{1}_{\max_{0\leq t\leq N} S_t < U} \,|\, S_n = s_n^m, \max_{0\leq t\leq n-1} S_t < U \right)$$

Notice that, if $s_n^m \geq U$ we have $V(n,m) = 0$. If $s_n^m < U$, we condition on the next value of S, to obtain:

$$V(n,m) = \frac{1}{(1+r\Delta t)^{N-n}} \cdot$$

$$\sum_{j=m}^{m+1} \mathbb{E}^{\mathbb{Q}} \left(V_N \,|\, S_{n+1} = s_{n+1}^j, S_n = s_n^m, \max_{0\leq t\leq n-1} S_t < U \right) \mathbb{Q}(S_{n+1} = s_{n+1}^j \,|\, S_n = s_n^m, \max_{0\leq t\leq n-1} S_t < U)$$

$$= \frac{1}{(1+r\Delta t)^{N-n}} \sum_{j=m}^{m+1} \mathbb{E}^{\mathbb{Q}} \left(V_N \,|\, S_{n+1} = s_{n+1}^j, \max_{0\leq t\leq n} S_t < U \right) \mathbb{Q}(S_{n+1} = s_{n+1}^j \,|\, S_n = s_n^m)$$

$$= \frac{1}{1+r\Delta t} \left(V(n+1,m)(1-q) + V(n+1,m+1)q \right)$$

Thus, for all $n = 0, \dots, N-1$ and $m = 0, \dots, n$, we obtain:

$$V(n,m) = \begin{cases} 0, & s_n^m \geq U, \\ \frac{1}{1+r\Delta t} \left(qV(n+1,m+1) + (1-q)V(n+1,m) \right), & s_n^m < U, \end{cases}$$

which produces the desired recursive relation. Along with the terminal condition, for $m = 0, \dots, N$,

$$V(N,m) = \begin{cases} 0, & s_N^m \geq U, \\ F(s_N^m), & s_N^m < U, \end{cases}$$

it allows to compute the option's price $V(0,0)$ numerically.

American options

American option pays off $F(S_\tau)$ at the time τ chosen by the option's buyer.

Rem 5. *American-style options do not quite fit within the standard framework, and, in particular, FTAP can not be applied in this case. The reason is that such an option can not be described as a single contingent claim (i.e. random variable), whose payoff is determined by the sources of randomness specified in the model (in this case, the path of S). Instead, the payoff of an American option depends both on the model sources of randomness and on the choice of the option buyer. Thus, American option has a whole range of different payoffs, depending on the choice of the exercise strategy τ, and the buyer can choose the payoff that she prefers.*

A rational agent will choose the best possible exercise time, which is based only on the information available up to that time. Then, the seller should try to sell the option at the highest possible price: the price that corresponds to the best choice of τ. Using this idea, one can develop a precise mathematical argument to show that the price of the American option is given by:

$$\sup_\tau \mathbb{E}^{\mathbb{Q}} \left(F(S_\tau)/B_\tau \right),$$

where the supremum is taken over all stopping times τ with values in $0, \dots, N$.

Let us develop a backward recursion for pricing American options. Similar to the barrier options, in addition to the time and the value of S, here, we need to keep track of another quantity – whether the option has been exercised before or not. Thus, for $n = 0, \ldots, N-1$ and $m = 0, \ldots, n$, we define:

$$V(n,m) = \sup_{\tau} \mathbb{E}^{\mathbb{Q}} \left(\frac{1}{(1+r\Delta t)^{\tau-n}} F(S_\tau) \,|\, S_n = s_n^m, \tau \geq n \right),$$

which is the price of the option at the node (n,m) conditional on the fact that it has not been exercised before.

Repeating the arguments used for barrier options, we obtain the following recursive relation:

$$V(n,m) = \max \left(\frac{1}{1+r\Delta t} \left(qV(n+1,m+1) + (1-q)V(n+1,m) \right), F(s_n^m) \right),$$

for all $n = 0, \ldots, N-1$ and $m = 0, \ldots, n$. Along withe the terminal condition $V(N,m) = F(s_N^m)$, the above formula provides an efficient algorithm for computing the price of an American option.

Exercise region is the set of nodes (n,m), such that $V(n,m) = F(s_n^m)$. Its compliment is called the **continuation region**.

Rem 6. *If the dividend rate is zero, American call is the same as European call. If the interest rate is zero, then, American put is the same as European put.*

Proof:

Under $\mathbb{Q}$, S/B is a mtg. Notice also that the function $x \mapsto (x-K)^+$ convex and increasing. Then, Jensen's inequality implies:

$$\mathbb{E}^{\mathbb{Q}} \frac{(S_T - K)^+}{B_T} = \mathbb{E}^{\mathbb{Q}} \left(\mathbb{E}^{\mathbb{Q}} \left((\frac{S_T}{B_T} - \frac{K}{B_T})^+ \,|\, \mathcal{F}_\tau \right) \right) \geq \mathbb{E}^{\mathbb{Q}} (\mathbb{E}^{\mathbb{Q}} (\frac{S_T}{B_T} \,|\, \mathcal{F}_\tau) - \frac{K}{B_T})^+ \geq \mathbb{E}^{\mathbb{Q}} \frac{(S_\tau - K)^+}{B_\tau},$$

for any stopping time $\tau \leq T$. Similarly, one can analyze put options, when the interest rate is zero.

Remarks about implementation of the pricing algorithms on binomial trees

- One only needs to create an array of size $(N+1) \times (N+1)$, to store the value function V, and a vector of terminal values of the risky asset $(s_N^0, \ldots, s_N^N)$. (In the case of American options, we actually need the whole array $(s_n^m)_{n=0,\ldots,N, m=0,\ldots,n}$.)

- To fill in the values of V, we start from the "leaves" of the tree, $V(N, \cdot)$, which we fill in according to the terminal payoff values. Then, we proceed backwards according to the recursion formulas.

- Recall that we defined $s_n^m = d^{n-m} u^m s_0$, to obtain a recombining tree. In fact, this particular construction is not necessary: one can define the possible values of the risky asset at time n by specifying an arbitrary vector $(s_n^0, \ldots, s_n^n)$, for each $n = 1, \ldots, N$. Then, all the pricing algorithms remain the same, except that the associated risk-neutral probabilities q may be different at each node: $q = (q(n,m))$.

2.3 Implied trees

Here, we discuss the **calibration** of tree models to the prices of **European call (or put) options**.

Recall that, in a binomial model, options prices are determined uniquely by $r, \Delta t, d, u$. However, we are often forced to fix these parameters due to other considerations. More precisely, the sample paths of the underlying processes are usually observed, so we cannot change them, but their probabilities are not observed directly – hence we prefer to manipulate the probability measures to fit the observed options' prices.

It turns out, there is another class of tree models, where the above parameters $r, \Delta t, d, u$ can be fixed, and there is still enough flexibility to fit option prices by choosing an appropriate $\mathbb{Q}$. In addition, the resulting calibration algorithm is relatively simple.

Trinomial trees

Key feature: trinomial tree model is **incomplete**! Hence, there is a flexibility in choosing the risk-neutral measure $\mathbb{Q}$.

To construct a trinomial tree, we assume that, for many node (n, m) the risky asset S can move to $(n+1, m+1)$, $(n+1, m)$ or $(n+1, m-1)$. More precisely, from s_n^m, at time $n = 0, \dots, N-1$, the process S can have the following jumps:

$$s_n^m \mapsto \begin{cases} s_{n+1}^{m+1} = u s_n^m, & p_u(n,m), \\ s_{n+1}^{m} = s_n^m, & 1 - p_u(n,m) - p_d(n,m), \\ s_{n+1}^{m-1} = \frac{1}{u} s_n^m, & p_d(n,m), \end{cases}$$

where $s_n^m = s^m = u^m s_0$, with a fixed constant $u > 1$; and $p_u(n,m)$ and $p_d(n,m)$ are the probabilities of moving up and down, respectively, with $p_u, p_d \geq 0$ and $p_u + p_d \in [0, 1]$. All jumps are assumed to be independent.

Rem 7. *As before, it is not necessary to choose $s^m = u^m s_0$ – we can choose any other set of possible values $\left\{s^{-N}, \dots, s^N\right\}$. Of course, the subsequent equations would need to be modified accordingly.*

Do not confuse s^m with "s to the power m"!

The riskless asset remains the same: $B_n \mapsto B_n(1 + r\Delta t)$.

Let us characterize the risk-neutral measures:

$$s_n^m = \frac{1}{1 + r\Delta t} \mathbb{E}^{\mathbb{Q}}(S_{n+1} \,|\, S_n = s_n^m)$$

$$(1 + r\Delta t) s^m = q_u(n,m) s^{m+1} + (1 - q_u(n,m) - q_d(n,m)) s^m + q_d(n,m) s^{m-1},$$

$$1 + r\Delta t = q_u(n,m) u + 1 - q_u(n,m) - q_d(n,m) + q_d(n,m)/u$$

One equation and two unknowns.

Exercise 3. *Show that, if the NA condition $\frac{1}{u} < 1 + r\Delta t < u$ is satisfied, then, for any fixed (n,m) there are infinitely many solutions $(q_u(n,m), q_d(n,m))$ to the above equation.*

Pricing with trinomial trees

It is exactly the same as in the binomial case. Let us assume that the risk-neutral probabilities $(q_u(n,m), q_d(n,m))$ are fixed, for all $n = 0, \dots, N-1$ and $m = -n, \dots, n$, and consider, for example, a European option with payoff function $F(S_N)$. Then, the recursive formula for the value function V is given by:

$$V(n,m) = \frac{1}{1 + r\Delta t} \left(q_u(n,m) V(n+1, m+1) + (1 - q_u(n,m) - q_d(n,m)) V(n+1, m) + q_d(n,m) V(n+1, m-1) \right),$$

for $n = 0, \dots, N-1$ and $m = -n, \dots, n$. Similarly, one can adapt the recursive relations for the barrier and american options.

Calibrating to Arrow-Debreu prices

Q 6. *How do we choose $(q_u(n,m), q_d(n,m))$ so that the resulting call prices match the ones observed in the market?*

It turns out that it is very convenient to split the calibration process into two steps by introducing the (artificial) **Arrow-Debreu securities**.

- Introduce $\lambda(n,m)$, the time zero **price of an Arrow-Debreu security**, which **pays** $\$1$ **at time** n**, if the underlying at time** n **is at** s^m, and **pays zero otherwise**.
- Clearly, $\lambda(n,m) = \frac{1}{(1+r\Delta t)^n}\mathbb{Q}\left(S_n = s^m\right)$.
- By **conditioning**, we obtain

$$\mathbb{Q}\left(S_{n+1} = s^m\right) = q_d(n,m+1)\mathbb{Q}(S_n = s^{m+1})$$

$$+q_u(n,m-1)\mathbb{Q}(S_n = s^{m-1}) + (1 - q_d(n,m) - q_u(n,m))\mathbb{Q}(S_n = s^m)$$

- Using the above equation and the **martingale property of** (S/B) **under** $\mathbb{Q}$, we obtain the following system

$$\begin{cases} (1+r\Delta t)\lambda(n+1,m) = q_d(n,m+1)\lambda(n,m+1) \\ +(1 - q_d(n,m) - q_u(n,m))\lambda(n,m) + q_u(n,m-1)\lambda(n,m-1), \\ \\ 1 + r\Delta t = q_u(n,m)u + 1 - q_u(n,m) - q_d(n,m) + q_d(n,m)/u, \end{cases} \tag{7}$$

  which holds for $n = 0, \dots, N-1$ and $m = -n+1, \dots, n+1$, where we assume that $\lambda(n,j) = 0$ whenever $j > n$ or $j < -n$.

- This system can be solved for $\{q_u(n,m), q_d(n,m)\}_{n=0,\dots,N-1,\, m=-n,\dots,n}$, given $\{\lambda(n,m)\}_{n=0,\dots,N-1,\, m=-n,\dots,n}$, as follows.
  - We start with $n = N-1$ and $m = N$. Then, the first equation in (7) becomes:

$$(1+r\Delta t)\lambda(N,N) = q_d(N-1,N+1)\lambda(N-1,N+1)$$

$$+(1 - q_d(N-1,N) - q_u(N-1,N))\lambda(N-1,N) + q_u(N-1,N-1)\lambda(N-1,N-1)$$

$$= q_u(N-1,N-1)\lambda(N-1,N-1),$$

    from which we obtain $q_u(N-1,N-1)$.
  - The second equation gives us $q_d(N-1,N-1)$.
  - Using the first equation with $n = N-1$ and $m = N-1$, we obtain $q_u(N-1,N-2)$.
  - Thus, we proceed backwards in $m = N, N-1, \dots, -N+1$, with fixed $n = N-1$, to obtain

$$\{q_u(N-1,m)\}_{m=-N+1,\dots,N-1}$$

– Then, repeat the same with $n = N-2, \dots, 0$, to obtain the entire family of risk neutral probabilities:

$$\{q_u(n,m), q_d(n,m)\}_{n=0,\dots,N-1,\, m=-n,\dots,n}$$

Calibrating to call prices

Any **European-type contract** can be **replicated** perfectly via the **Arrow-Debreu securities**. Therefore, **having** $\{\lambda(n,m)\}$ **we can compute all the call and put prices**:

$$C(n,K) = \sum_{m=-n}^{n} \frac{1}{(1+r\Delta t)^n} \mathbb{Q}\left(S_n = s^m\right)\left(s^m - K\right)^+ = \sum_{m=-n}^{n} \lambda(n,m)(s^m - K)^+, \tag{8}$$

where $C(n,K)$ is the price of a call option with maturity n and strike K.

It turns out that the inverse is also true: **having the call (or put) prices with all maturities** $n = 1, \dots, N$**, and with sufficiently many strikes, one can derive the Arrow-Debreu prices from them**.

The above fact is a consequence of a more general observation, which is known as the **Shimko's method**. Recall that the value of the underlying at time n, S_n, can only take a finite number of values $\{s^{-n}, \dots, s^n\}$. Choose ΔK so that $s^{m-1} \leq s^m - \Delta K_1 < s^m < s^m + \Delta K_2 \leq s^{m+1}$. Notice that the following portfolio

$$\frac{1}{\Delta K_2}\left(C(n, s^m + \Delta K_2) - C(n, s^m)\right) - \frac{1}{\Delta K_1}\left(C(n, s^m) - C(n, s^m - \Delta K_1)\right)$$

pays one dollar at time n if $S_n = s^m$, otherwise it expires worthless. Hence, it is the Arrow-Debreu security, and we have

$$\lambda(n,m) = \frac{1}{\Delta K_2}\left(C(n, s^m + \Delta K_2) - C(n, s^m)\right) - \frac{1}{\Delta K_1}\left(C(n, s^m) - C(n, s^m - \Delta K_1)\right),$$

Rem 8. *Similar trick works in any model – not necessarily on a tree! In general, it is a well-known fact that, given enough strikes, the risk-neutral distribution of the risky asset at maturity can be deduced from the call (or put) prices.*

For simplicity, assume that the market gives us call prices

$$C(n,K),$$

for **all maturities** $n = 1, \dots, N$ **and strikes** $K = s^m$**, with** $m = -(n-1), \dots, n-1$.

Q 7. *What are the risk neutral probabilities* $(q_u(n,m), q_d(n,m))$ *that reproduce these call prices?*

The algorithm

- Choosing $\Delta K_1 = s^m - s^{m-1}$ and $\Delta K_2 = s^{m+1} - s^m$, we obtain, for all $n = 1, \dots, N$ and $m = -n, \dots, n$:

$$\lambda(n,m) = \frac{1}{s^{m+1} - s^m}\left(C(n, s^{m+1}) - C(n, s^m)\right) - \frac{1}{s^m - s^{m-1}}\left(C(n, s^m) - C(n, s^{m-1})\right), \tag{9}$$

  where $s^j = u^j s_0$, for all integers j, $C(n,j) = 0$ if $j \geq n$, and $C(n,j) = s_0 - \frac{s^j}{(1+r\Delta t)^n}$ if $j \leq -n$.

- We can use (9) to find $(\lambda(n,m))$ and, then, obtain the risk-neutral probabilities via (7).

- Alternatively, we can use (7) and (9), to obtain explicit formulas:

$$\begin{cases} q_u(n,m) = \frac{(1+r\Delta t)C(n+1,s^m)-\sum_{j=m+1}^{N}\lambda(n,j)((1+r\Delta t)s^j-s^m)}{\lambda(n,m)(s^{m+1}-s^m)} \\ = \frac{(1+r\Delta t)C(n+1,s^m)/s^m-\sum_{j=m+1}^{N}\lambda(n,j)((1+r\Delta t)u^{j-m}-1)}{\lambda(n,m)(u-1)}, \\ q_d(n,m) = \frac{q_u(n,m)(s^{m+1}-s^m)+s^m-(1+r\Delta t)s^m}{s^m-s^{m-1}} = \frac{q_u(n,m)(u-1)-r\Delta t}{1-\frac{1}{u}}, \end{cases} \tag{10}$$

  for $n = 0, \ldots, N-1$ and $m = -n, \ldots, n$.

- **Thus, if the market provides us with call prices** $C(n,K)$**, for all maturities** $n = 1, \ldots, N$ **and strikes** $K = s^m$**, with** $m = -(n-1), \ldots, n-1$**, then, we can use (9) and (10) to pass from call prices to the risk neutral probabilities** $(q_u(n,m), q_d(n,m))$.

The above algorithm gives an **explicit solution to the problem of calibration**! The resulting trinomial tree is called the **implied tree**.

Shortcomings

- Implied tree models **allow for simple explicit calibration**, but they may require **more strikes and maturities than there are available on the market**.

- However, the main problem of implied trees is the **instability** as the size of the tree grows (e.g. because we want to make the model more realistic, or because we want to match more option prices).

  The problem reveals itself in **negative probabilities** $q_d(n,m)$**,** $q_u(n,m)$ **or** $1 - q_d(n,m) - q_u(n,m)$. Typically, one can make sure (after some work) that **the implied prices of Arrow-Debreu securities,** λ_n^m**, are nonnegative**. But it is **hardly possible to avoid** negative $q_d(n,m)$ and $q_u(n,m)$ using the above formulas directly.

- The above difficulty may be treated, for example, by introducing the **"adaptive" trees**, with more nodes in the areas where better accuracy of the computations is required.

- In general, the abstract problem of **calibration to European options with all possible strikes and maturities** is an **ill-posed inverse problem**!