Numerical methods in mathematical finance

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Preface

These notes are the basis of my lecture *Numerical methods in mathematical finance* given at Karlsruhe Institute of Technology in the winter term 2014/15 and 2016/17. The purpose of this notes is to help students who have missed parts of the course to fill these gaps, and to provide a service for those students who can concentrate better if they do not have to copy what I write on the blackboard.

It is *not* the purpose of these notes, however, to replace the lecture itself, or to write a text which could compete with the excellent books about the subject. This is why the style of presentation is rather sketchy. As a rule of thumb, one could say that these notes only cover what I *write* during the lecture, but not everything I *say*.

There are still many typos and possibly also other mistakes. Of course, I will try to correct any mistake I find as soon as possible, but please be aware of the fact that you cannot rely on these notes.

Karlsruhe, winter term 2016/17,
Tobias Jahnke

Contents

Chapter 1

Options and arbitrage

References: [BK04, Sey09]

1.1 European options

Financial markets trade investments into stocks of a company, commodities (e.g. oil, gold), etc.
Stocks and commodities are risky assets, because their future value cannot be predicted. Bonds are considered as riskless assets in this lecture. If $B(t_0)$ is invested at time t_0 into a bond with a risk-free interest rate $r > 0$, then the value of the bond at time $t \geq t_0$ is simply

$$B(t) = e^{r(t-t_0)} B(t_0). \tag{1.1}$$

Simplifying assumption: continuous payment of interest

Remark. Discrete payment of interest: obtain $r \cdot \Delta t \cdot B(0)$ after time Δt. Value at $t = n\Delta t$:

$$\tilde{B}(t) = (1 + r \cdot \Delta t)^n B(0) = (1 + rt/n)^n B(0)$$

For $n \longrightarrow \infty$ and $\Delta t \longrightarrow 0$:

$$\lim_{n\to\infty} \tilde{B}(t) = \lim_{n\to\infty} (1 + rt/n)^n B(0) = e^{rt} B(0) = B(t)$$

(continuous payment of interest)

Spot contract: buy or sell an asset (e.g. a stock, a commodity etc.) with immediate delivery
Financial derivatives: contracts about future payments or deliveries with certain conditions

1. Forwards and futures: agreement between two parties to buy or sell an asset at a certain time in the future for a certain delivery price
2. Swaps: contracts regulating an exchange of cash flows at different future times (e.g. currency swap, interest rate swaps, credit default swaps)
3. Options

Definition 1.1.1 (European option)

- *A **European call option** is a contract which gives the holder (=buyer) of the option the right to buy an underlying risky asset at a future maturity date (expiration time) T at a fixed exercise price (strike) K from the writer (=seller) of the option.*
  Typical assets: stocks, parcels of stocks, stock indices, currencies, commodities, ...
  Difference to forwards and futures: At maturity the holder can choose if he wants to buy the asset or not.
- ***European put option:** Similar to call option, but vice versa, i.e. the holder can sell the underlying to the writer.*

Example: At time $t = 0$ Mr. J. buys 5 European call options. Each of these options gives him the right to buy 10 shares of the company KIT at maturity $T > 0$ at the exercise price of $K = 120$€ per share.

- **Case 1:** At time $t = T$, the market price of KIT is 150€ per share.
  Mr. J. exercises his options, i.e. he buys $5 \cdot 10 = 50$ KIT shares at the price of $K = 120$€ per share and sells the shares on the market for 150€ per share. Hence, he wins $50 \cdot 30 = 1500$€.
- **Case 2:** At time $t = T$, the market price of KIT is 100€ per share.
  Hence, Mr. J. does not exercise his options.

What are options good for?

- Speculation
- Hedging ("insurance" against changing market values)

Since an option gives an advantage to the holder, the option has a certain value.
For given T and K the value $V(t, S)$ of the option must depend on the time t and the current price S of the underlying.
For an European option we know that the value at the maturity T is

$$V(T,S) = \begin{cases} (S-K)^+ := \max\{S-K, 0\} & \text{(European call)} \\ (K-S)^+ := \max\{K-S, 0\} & \text{(European put).} \end{cases}$$

The functions $S \mapsto (S-K)^+$ and $S \mapsto (K-S)^+$ are called the **payoff functions** of a call or put, respectively.

The **goal of this course** is to answer the following question:

What is the fair price $V(t, S)$ of an option for $t < T$?

Why is this question important? In order to sell/buy an option, we need to know the fair price.
Why is this question non-trivial? Because the value of the risky asset is random. In particular, the price $S(T)$ at the future expiration time T is not yet known when we buy/sell the option at time $t = 0$.

1.2 More types of options

Variations of the basic principle:

- **European options** can be exercised only at the maturity date.
- **American options** can be exercised at any time before and including the maturity date.
- **Bermuda options** can be exercised at a set of times.

The names "European", "American", "Bermuda" etc. have no geographical meaning. American options can be traded in Europe, European options can be traded in the USA, etc.

- **Vanilla options** = standard options, i.e. European, American or Bermuda calls/puts
- **Exotic options** = non-standard options

Examples for exotic options:

- **Path-dependent option:** The payoff function does not only depend on the price $S(T)$ of the underlying at time T, but on the entire path $t \mapsto S(t)$ for $t \in [0, T]$.
  - **Asian options:** The payoff function depends on the average price, e.g.
  $$\left(\frac{1}{T} \int_0^T S(t)\, dt - K \right)^+$$
  (payoff of an average price call).
  - **Barrier options:** The payoff depends on the question if the price of the underlying has crossed a certain (upper or lower) barrier.
  - **Lookback options:** The payoff depends on $\max_{t\in[0,T]} S(t)$ or $\min_{t\in[0,T]} S(t)$.
- **Options on several assets:**
  - **Basket options:** The payoff depends on the weighted sum of the prices S_i of several assets, e.g.
  $$\left(\sum_{i=1}^d c_i S_i - K \right)^+, \qquad c_i > 0$$

(payoff of a basket call)

- **Rainbow options:** The payoff depends on the relation between the assets, e.g. $\max\{S_1, \dots, S_d\}$.

- **Binary options:** The payoff function has only two possible values
- **Compound options:** Options on options

Remark: There are even more types of options.

1.3 Arbitrage and modelling assumptions

Example. Consider

- a stock with price $S(t)$
- a European call option with maturity $T = 1$, strike $K = 100$, and value $V(t, S(t))$
- a bond with price $B(t)$

Initial data: $S(0) = 100, B(0) = 100, V(0) = 10$.
Assumption: At time $t = 1$, we either have

$$\begin{array}{ll} \text{“up”:} & B(1) = 110,\ S(1) = 120 \\ \text{“down”:} & B(1) = 110,\ S(1) = 80 \end{array}$$

or

At $t = 0$, Mrs. C. buys 0.4 bonds, one call option and sells 0.5 stock (“short selling”).
Value of the portfolio at $t = 0$:

$$0.4 \cdot B(0) + 1 \cdot V(0) - 0.5 \cdot S(0) = 0.4 \cdot 100 + 1 \cdot 10 - 0.5 \cdot 100 = 0$$

Value of the portfolio at $t = 1$ is

$$0.4 \cdot B(1) + 1 \cdot \underbrace{V(1, S(1))}_{=(S(1)-K)^+} - 0.5 \cdot S(1)$$

Two cases:

$$\begin{array}{rl} \text{“up”:} & 0.4 \cdot 110 + 1 \cdot (120 - 100)^+ - 0.5 \cdot 120 = 44 + 20 - 60 = 4 \\ \text{“down”:} & 0.4 \cdot 110 + 1 \cdot (80 - 100)^+ - 0.5 \cdot 80 \quad = 44 + 0 - 40 \ = 4 \end{array}$$

In both cases, Mrs. C. wins 4€ without any risk or investment!
Why is this possible? Because the price $V(0) = 10$ of the option is too low!

Definition 1.3.1 (Arbitrage) ***Arbitrage*** *is the existence of a portfolio, which*

- *requires no initial investment, and*
- *which cannot cause any loss, but very likely a gain.*

Remark. A bond will always yield a risk-less gain, but it requires an investment.

Assumptions for modelling an idealized market:

(A1) Arbitrage is impossible (no-arbitrage principle).

(A2) There is a risk-free interest rate $r > 0$ which applies for all credits. Continuous payment of interest according to (1.1).

(A3) No transaction costs, taxes, etc. Trading is possible at any time. Any fraction of an asset can be sold. Liquid market, i.e. selling an asset does not change its value significantly.

(A4) A seller can sell assets he/she does not own yet ("short selling", cf. Mrs. C. above)

(A5) No dividends on the underlying asset are paid.

1.4 Arbitrage bounds

Consider European options with strike $K > 0$ and maturity T on an underlying with price $S(t)$. Let $V_P(t,S)$ and $V_C(t,S)$ be the values of a put option and call option, respectively.

Lemma 1.4.1 (Put-call parity) *Under the assumptions (A1)-(A5) we have*

$$S(t) + V_P(t,S(t)) - V_C(t,S(t)) = e^{-r(T-t)}K$$

for all $t \in [0,T]$.

Proof. Buy one stock, buy a put, write (sell) a call. Then, the value of this portfolio is

$$\pi(t) = S(t) + V_P(t,S(t)) - V_C(t,S(t))$$

and at maturity

$$\pi(T) = S(T) + V_P(T,S(T)) - V_C(T,S(T)) = S(T) + (K - S(T))^+ - (S(T) - K)^+ = K.$$

Hence, the portfolio is risk-less. No arbitrage: The profit of the portfolio must be the same as the profit for investing $\pi(t)$ into a bond at time t:

$$\pi(T) = K \overset{!}{=} e^{r(T-t)}\pi(t) \quad \Longrightarrow \quad e^{-r(T-t)}K = \pi(t) = S(t) + V_P(t,S(t)) - V_C(t,S(t)). \quad \blacksquare$$

Lemma 1.4.2 (Bounds for European calls and puts) *Under the assumptions (A1)-(A5), the following inequalities hold for all* $t \in [0,T]$ *and all* $S = S(t) \geq 0$*:*

$$\left(S - e^{-r(T-t)}K\right)^+ \leq V_C(t,S) \leq S \tag{1.2}$$

$$\left(e^{-r(T-t)}K - S\right)^+ \leq V_P(t,S) \leq e^{-r(T-t)}K \tag{1.3}$$

Proof.

- It is obvious that $V_C(t,S) \geq 0$ and $V_P(t,S) \geq 0$ for all $t \in [0,T]$ and $S \geq 0$.

- Assume that $V_C(t, S(t)) > S(t)$ for some $S(t) \geq 0$.
  Write (sell) a call, buy the stock and put the difference $\delta := V_C(t, S(t)) - S(t) > 0$ in your pocket.
  At $t = T$, there are two scenarios:

  If $S(T) > K$: Must sell stock at the price K to the owner of the call.
  Gain: $K + \delta > 0$
  If $S(T) \leq K$: Gain $S(T) + \delta > 0$

  $\Longrightarrow$ Arbitrage! Contradiction!
- Put-call parity:

$$S - e^{-r(T-t)}K = V_C(t, S) - \underbrace{V_P(t, S)}_{\geq 0} \leq V_C(t, S)$$

  This proves (1.2). The proof of (1.3) is left as an exercise. ∎

Remark. Similar inequalities can be shown for American options (exercise).

1.5 A simple discrete model

Consider

- a stock with price $S(t)$
- a European option with maturity T, strike K, and value $V(t, S(t))$
- a bond with price $B(t) = e^{rt}B(0)$

Suppose that the initial data $S(0) = S_0$ and $B(0) = 1$ are known, and that (A1)-(A5) hold. Goal: Find $V(0, S_0)$.
Simplifying assumption: At time $t = T$, there are only two scenarios

or

$$\begin{aligned} &\text{“up”:} \quad S(T) = u \cdot S_0 \quad \text{with probability } p \\ &\text{“down”:} \quad S(T) = d \cdot S_0 \quad \text{with probability } 1 - p \end{aligned}$$

Assumption: $0 < d \leq e^{rT} \leq u$ and $p \in (0, 1)$
In both cases, we have $B(T) = e^{rT}B(0) = e^{rT}$.

Replication strategy: Construct portfolio with c_1 bonds and c_2 stocks such that

$$c_1 B(t) + c_2 S(t) \stackrel{!}{=} V(t, S(t))$$

for $t \in \{0, T\}$. For $t = T$, this means

$$\begin{aligned} \text{case “up”:} \quad & c_1 e^{rT} + c_2 u S_0 \stackrel{!}{=} V(T, uS_0) =: V_u \\ \text{case “down”:} \quad & c_1 e^{rT} + c_2 d S_0 \stackrel{!}{=} V(T, dS_0) =: V_d \end{aligned}$$

V_u and V_d are known if u and d are known. The unique solution is (check!)

$$c_1 = \frac{uV_d - dV_u}{(u-d)e^{rT}} \qquad\qquad c_2 = \frac{V_u - V_d}{(u-d)S_0}.$$

Hence, the fair price of the option is

$$V(0,S_0) = c_1 \underbrace{B(0)}_{=1} + c_2 S_0 = \frac{uV_d - dV_u}{(u-d)e^{rT}} + \frac{V_u - V_d}{(u-d)}$$

which yields (check!)

$$V(0,S_0) = e^{-rT}\Big(qV_u + (1-q)V_d\Big) \qquad \text{with} \qquad q := \frac{e^{rT}-d}{u-d}. \tag{1.4}$$

Remark: The value of the option does **not** depend on p.

Since $0 < d \leq e^{rT} \leq u$ by assumption, $q \in [0,1]$ can be seen as a probability. Now, define a new probability distribution $\mathbb{P}_q$ by

$$\mathbb{P}_q\Big(S(T) = uS_0\Big) = q, \qquad\qquad \mathbb{P}_q\Big(S(T) = dS_0\Big) = 1-q$$

(q instead of p). Then, we have

$$\mathbb{P}_q\Big(V(T,S(T)) = V_u\Big) = q, \qquad\qquad \mathbb{P}_q\Big(V(T,S(T)) = V_d\Big) = 1-q$$

and hence

$$qV_u + (1-q)V_d = \mathbb{E}_q\Big(V(T,S(T))\Big)$$

can be regarded as the **expectation** of the payoff $V(T,S(T))$ with respect to $\mathbb{P}_q$. In (1.4), this expectation is multiplied by an **discounting** factor e^{-rT}.
Interpretation: In order to have an amount of $B(t)$ at time t, we have to invest $B(0) = e^{-rt}B(t)$ into a bond at time $t = 0$.
The probability q has the property that

$$\mathbb{E}_q(S(T)) = quS_0 + (1-q)dS_0 = \frac{e^{rT}-d}{u-d}uS_0 + \frac{u-e^{rT}}{u-d}dS_0 = e^{rT}S_0.$$

Hence, the expected (with respect to $\mathbb{P}_q$) value of $S(T)$ is exactly the amount we obtain when we invest S_0 into a bond. Therefore, $\mathbb{P}_q$ is called the **risk-neutral probability**.

Moral of the story so far:

> **Under the risk-neutral probability, the price of a European option is the discounted expectation of the payoff.**

Chapter 2

Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space[1]: $\Omega \neq \emptyset$ is a set, $\mathcal{F}$ is a σ-algebra (or σ-field) on Ω, and $\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ is a probability measure.
A probability space is complete if $\mathcal{F}$ contains all subsets G of Ω with $\mathbb{P}$-outer measure zero, i.e. with

$$\mathbb{P}^*(G) := \inf \left\{ \mathbb{P}(F) : F \in \mathcal{F} \text{ and } G \subset F \right\} = 0.$$

Any probability space can be completed. Hence, we can assume that every probability space in this lecture is complete.

2.1 Stochastic proceses and filtrations

Definition 2.1.1 (Stochastic process) *Let $\mathcal{T}$ be an ordered set (e.g. $\mathcal{T} = [0, \infty), \mathcal{T} = \mathbb{N}$). A **stochastic process** is a family $X = \{X_t : t \in \mathcal{T}\}$ of random variables*

$$X_t : \Omega \longrightarrow \mathbb{R}^d.$$

Below, we will often simply write X_t instead of $\{X_t : t \in \mathcal{T}\}$.
Equivalent notations: $X(t, \omega)$, $X(t)$, $X_t(\omega)$, X_t, ...
For a fixed $\omega \in \Omega$, the function $t \mapsto X_t(\omega)$ is called a realization (or path or trajectory) of X.

The path of a stochastic process is associated to some $\omega \in \Omega$. As time evolves, more information about ω becomes available.

Example (cf. chapter 2 in [Shr04]). Toss a coin three times. Possible results are:

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
HHH	HHT	HTH	HTT	THH	THT	TTH	TTT

(H = heads, T = tails).

[1] See "2.2.2 What is $(\Omega, \mathcal{F}, \mathbb{P})$ anyway?" in the book [CT04] for a nice discussion of this concept.

- Before the first toss, we only know that $\omega \in \Omega$.
- After the first toss, we know if the final result will belong to

$$\{HHH, HHT, HTH, HTT\} \text{ or to } \{THH, THT, TTH, TTT\}.$$

  These sets are "resolved by the information". Hence, we know in which of the sets

$$\{w_1, w_2, w_3, w_4\}, \{w_5, w_6, w_7, w_8\}$$

  ω is.
- After the second toss, the sets

$$\{HHH, HHT\}, \{HTH, HTT\}, \{THH, THT\}, \{TTH, TTT\}$$

  are resolved, and we know in which of the sets

$$\{w_1, w_2\}, \{\omega_3, w_4\}, \{w_5, w_6\}, \{w_7, w_8\}$$

  ω is.

This motivates the following definition.

Definition 2.1.2 (Filtration)

- *A **filtration** is a family $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $t \geq s \geq 0$.*
  A filtration models the fact that more and more information about a process is known as time evolves.
- *If $\{X_t : t \geq 0\}$ is a family of random variables and X_t is $\mathcal{F}_t$-measurable, then $\{X_t : t \geq 0\}$ is **adapted** to (or **nonanticipating** with respect to) $\{\mathcal{F}_t : t \geq 0\}$.*
  Interpretation: At time t we know for each set $S \in \mathcal{F}_t$ if $\omega \in S$ or not. The value of X_t is revealed at time t.
- *For every $s \in [0, t]$ let $\sigma\{X_s\}$ be the σ-algebra generated by X_s, i.e. the smallest σ-algebra on Ω containing the sets*

$$X_s^{-1}(B) \text{ for all } B \in \mathcal{B}$$

  where $\mathcal{B}$ denotes the Borel σ-algebra. By definition $\sigma\{X_s\}$ is the smallest σ-algebra where X_s is measurable.

2.2 The Wiener process

Robert Brown 1827, Louis Bachelier 1900, Albert Einstein 1905, Norbert Wiener 1923

Definition 2.2.1 (Normal distribution) *A random variable $X : \Omega \longrightarrow \mathbb{R}^d$ with $d \in \mathbb{N}$ is **normal** if it has a multivariate **normal (Gaussian) distribution** with mean $\mu \in \mathbb{R}^d$ and a symmetric, positive definite covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$, i.e.*

$$\mathbb{P}(X \in B) = \int_B \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\Big(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\Big)\, dx$$

for all Borel sets $B \subset \mathbb{R}^d$. Notation: $X \sim \mathcal{N}(\mu, \Sigma)$

Remarks:

1. If $X \sim \mathcal{N}(\mu, \Sigma)$, then $\mathbb{E}(X) = \mu$ and $\Sigma = (\sigma_{ij})$ with $\sigma_{ij} = \mathbb{E}[(X_i - \mu_i)(X_j - \mu_j)]$.
2. Standard normal distribution $\Leftrightarrow \mu = 0$, $\Sigma = I$ (identity matrix).
3. If $X \sim \mathcal{N}(\mu, \Sigma)$ and $Y = v + TX$ for some $v \in \mathbb{R}^d$ and a regular matrix $T \in \mathbb{R}^{d\times d}$, then
$$Y \sim \mathcal{N}(v + T\mu, T\Sigma T^T). \tag{2.1}$$
4. Warning: In one dimension, the covariance matrix is simply a number, namely the variance. Unfortunately, the variance is usually denoted by σ^2 instead of σ in the literature, which is somewhat confusing.

Definition 2.2.2 (Wiener process, Brownian motion)
*(a) A continuous-time stochastic process $\{W_t : t \in [0, T)\}$ is called a **standard Brownian motion** or **standard Wiener process** if it has the following properties:*

1. *$W_0 = 0$ (with probability one)*
2. *Independent increments: For all $0 \le t_1 < t_2 < \ldots < t_n < T$ the random variables*
$$W_{t_2} - W_{t_1}, \quad W_{t_3} - W_{t_2}, \quad \ldots \quad , \quad W_{t_n} - W_{t_{n-1}}$$
are independent.
3. *$W_t - W_s \sim \mathcal{N}(0, t-s)$ for any $0 \le s < t < T$.*
4. *There is a $\tilde{\Omega} \subset \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ such that $t \mapsto W_t(\omega)$ is continuous for all $\omega \in \tilde{\Omega}$.*

(b) If $W_t^{(1)}, \ldots, W_t^{(d)}$ are independent one-dimensional Wiener processes, then $W_t = \Big(W_t^{(1)}, \ldots, W_t^{(d)}\Big)$ is called a d-dimensional Wiener process, and

$$W_t - W_s \sim \mathcal{N}(0, (t-s)I).$$

Existence of Brownian motion was irst proved in a mathematically rigorous way by Norbert Wiener in 1923.
The Wiener process will serve as the "source of randomness" in our model of the financial market.

Notation: $W_t = W_t(\omega) = W(t,\omega) = W(t)$

Numerical simulation of a Wiener process (d=1). Choose step-size $\tau > 0$, put $t_n = n\tau$ and $\tilde{W}_0 = 0$.

for $n = 0, 1, 2, 3, \ldots$
  Generate random number $Z_n \sim \mathcal{N}(0,1)$
  $\tilde{W}_{n+1} = \tilde{W}_n + \sqrt{\tau} Z_n$
end for

For $\tau \longrightarrow 0$ the interpolation of $\tilde{W}_0, \tilde{W}_1, \tilde{W}_2, \ldots$ approximates a path of the Wiener process.

How smooth is a path of a Wiener process? Consider only $d = 1$.

Hölder continuity and non-differentiability

A function $f : (a,b) \longrightarrow \mathbb{R}$ is **Hölder continuous of order α** for some $\alpha \in [0,1]$ if there is a constant C such that

$$|f(t) - f(s)| \leq C|t-s|^\alpha \qquad \text{for all } s,t \in (a,b).$$

If $\alpha = 0$, then f is bounded.
If $\alpha > 0$, then f is uniformly continuous.
If $\alpha = 1$, then f is Lipschitz continuous.

A path of the Wiener process on a bounded interval is

- Hölder continuous of order $\alpha \in [0, \frac{1}{2})$ with probability one, but
- not Hölder continuous of order $\alpha \geq \frac{1}{2}$ with probability one.

A path of the Wiener process is nowhere differentiable with probability one.

Proofs: [Ste01], chapter 5

Unbounded total variation

Let $[a,b]$ be an interval and let

$$P_N = (t_n)_{n=0}^N, \qquad a = t_0 < t_1 < \ldots < t_N = b$$

be a partition of $[a,b]$ with $|P_N| = \max_n |t_n - t_{n-1}|$.
Example: equidistant partition, $\tau = (b-a)/N$, $t_n = a + n \cdot \tau$, $|P_N| = \tau$.
The **total variation** of a function $f : (a,b) \longrightarrow \mathbb{R}$ is

$$TV_{a,b}(f) = \lim_{\substack{N \longrightarrow \infty \\ |P_N| \longrightarrow 0}} \sum_{n=1}^{N} |f(t_n) - f(t_{n-1})|. \tag{2.2}$$

The limit is to be understood in the sense that partitions are nested, i.e. more and more points are added to the points from previous partitions.
Remark: An equivalent definition is $TV_{a,b}(f) = \sup_{P_N} \sum_{n=1}^{N} |f(t_n) - f(t_{n-1})|$.

If f is differentiable and f' is integrable, then it can be shown that

$$TV_{a,b}(f) = \int_a^b |f'(t)|\, dt$$

Conversely: If a function f has bounded total variation, then its derivative exists for almost all $x \in [a, b]$.
Consequence: A path of the Wiener process has unbounded total variation with probability one.

Filtration of the Wiener process

The natural filtration of the Wiener process on $[0, T]$ is given by

$$\{\mathcal{F}_t : t \in [0, T]\}, \qquad \mathcal{F}_t = \sigma\{W_s,\ s \in [0, t]\}$$

(cf. Definition 2.1.2). For technical reasons, however, it is more advantageous to use an **augmented** filtration called the **standard Brownian filtration**. See pp. 50-51 in [Ste01] for details.

2.3 Construction of the Itô integral (step 1 and 2)

References: [KP99, Øks03, Shr04, Ste01]

The model considered in 1.5 is clearly too simple: only two discrete times, only two possible prices of $S(T)$.

Goal: Construct a more realistic model for the dynamics of $S(t)$.

Naïve Ansatz:

$$\underbrace{\frac{dX}{dt} = f(t, X)}_{\substack{\text{ordinary differential} \\ \text{equation}}} + \underbrace{g(t, X)Z(t)}_{\text{random noise}}, \qquad Z(t) = ?$$

Apply explicit Euler method: Choose $t \geq 0$ and $N \in \mathbb{N}$, let $\tau = t/N$, $t_n = n \cdot \tau$ and define approximations $X_n \approx X(t_n)$ by

$$X_{n+1} = X_n + \tau f(t_n, X_n) + \tau g(t_n, X_n) Z(t_n) \qquad (n = 0, 1, 2, \ldots).$$

In the special case $f(t,X) = 0$ and $g(t,X) = 1$, we want that $X_n = W(t_n)$ is the Wiener process, i.e. we postulate that

$$W(t_{n+1}) \stackrel{!}{=} W(t_n) + \tau Z(t_n).$$

This yields

$$X_{n+1} = X_n + \tau f(t_n, X_n) + g(t_n, X_n)\Big(W(t_{n+1}) - W(t_n)\Big)$$

and after N steps

$$X_N = X_0 + \tau \sum_{n=0}^{N-1} f(t_n, X_n) + \sum_{n=0}^{N-1} g(t_n, X_n)\Big(W(t_{n+1}) - W(t_n)\Big). \tag{2.3}$$

Keep t fixed, let $N \longrightarrow \infty$, $\tau = t/N \longrightarrow 0$. Then, (2.3) should somehow converge to

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \underbrace{\int_0^t g(s, X(s)) dW(s)}_{(\star)}. \tag{2.4}$$

Problem: We cannot define $(\star)$ as a pathwise Riemann-Stieltjes integral (cf. appendix B). When $N \longrightarrow \infty$, the sum

$$\sum_{n=0}^{N-1} g(t_n, X_n(\omega))\Big(W(t_{n+1}, \omega) - W(t_n, \omega)\Big)$$

diverges with probability one, because a path of the Wiener process has unbounded total variation with probability one.

New goal: Define the integral

$$\mathcal{I}_t[u](\omega) = \int_0^t u(s, \omega)\, dW_s(\omega)$$

in a "reasonable" way for the following class of functions.

Definition 2.3.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\{\mathcal{F}_t : t \in [0,T]\}$ be the standard Brownian filtration. Then, we define $\mathcal{H}^2[0,T]$ to be the class of functions*

$$u = u(t,\omega), \qquad u : [0,T] \times \Omega \longrightarrow \mathbb{R}$$

with the following properties:

- *$(t,\omega) \mapsto u(t,\omega)$ is $(\mathcal{B} \times \mathcal{F})$-measurable.*
- *u is adapted to $\{\mathcal{F}_t : t \in [0,T]\}$, i.e. $u(t,\cdot)$ is $\mathcal{F}_t$-measurable.*
- $\mathbb{E}\left(\int_0^T u^2(t,\omega)\, dt\right) < \infty$

Step 1: Itô integral for elementary functions

Definition 2.3.2 (Elementary functions) *A function $\phi \in \mathcal{H}^2[0,T]$ is called **elementary** if it is a stochastic step function of the form*

$$\begin{aligned}\phi(t,\omega) &= a_0(\omega)\mathbf{1}_{[0,0]}(t) + \sum_{n=0}^{N-1} a_n(\omega)\mathbf{1}_{(t_n,t_{n+1}]}(t) \\ &= a_0(\omega)\mathbf{1}_{[0,t_1]}(t) + \sum_{n=1}^{N-1} a_n(\omega)\mathbf{1}_{(t_n,t_{n+1}]}(t)\end{aligned}$$

with a partition $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$. The random variables a_n must be $\mathcal{F}_{t_n}$-measurable with $\mathbb{E}(a_n^2) < \infty$. Here and below,

$$\mathbf{1}_{[c,d]}(t) = \begin{cases} 1 & \text{if } t \in [c,d] \\ 0 & \text{else} \end{cases} \tag{2.5}$$

is the indicator function of an interval $[c,d]$.

Skizze

For $0 \le c < d \le T$, the only reasonable way to define the Itô integral of an indicator function $\mathbf{1}_{(c,d]}$ is

$$\mathcal{I}_T[\mathbf{1}_{(c,d]}](\omega) = \int_0^T \mathbf{1}_{(c,d]}(s)\, dW(s,\omega) = \int_c^d dW(s,\omega) = W(d,\omega) - W(c,\omega).$$

Hence, by linearity, we define the Itô integral of an elementary function by

$$\mathcal{I}_T[\phi](\omega) = \sum_{n=0}^{N-1} a_n(\omega)\big(W(t_{n+1},\omega) - W(t_n,\omega)\big).$$

Lemma 2.3.3 (Itô isometry for elementary functions) *For all elementary functions we have*

$$\mathbb{E}\left(\mathcal{I}_T[\phi]^2\right) = \mathbb{E}\left(\int_0^T \phi^2(t,\omega)\, dt\right)$$

or equivalently

$$\|\mathcal{I}_T[\phi]\|_{L^2(d\mathbb{P})} = \|\phi\|_{L^2(dt\times d\mathbb{P})}$$

with

$$\|\phi\|_{L^2(dt\times d\mathbb{P})} = \left(\int_\Omega \int_0^T \phi^2(t,\omega)\, dt\, d\mathbb{P}\right)^{\frac{1}{2}} = \left(\mathbb{E}\left(\int_0^T \phi^2(t,\omega)\, dt\right)\right)^{\frac{1}{2}}.$$

Proof. Since

$$\phi^2(t,\omega) = a_0^2(\omega)\mathbf{1}_{[0,0]}(t) + \sum_{n=0}^{N-1} a_n^2(\omega)\mathbf{1}_{(t_n,t_{n+1}]}(t)$$

we obtain

$$\mathbb{E}\left(\int_0^T \phi^2(t,\omega)\,dt\right) = \sum_{n=0}^{N-1} \mathbb{E}\left(a_n^2\right)(t_{n+1}-t_n) \tag{2.6}$$

for the right-hand side. If we let $\Delta W_n = W(t_{n+1}) - W(t_n)$, then

$$\mathcal{I}_T[\phi]^2 = \left(\sum_{n=0}^{N-1} a_n \Delta W_n\right)^2 = \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} a_n a_m \Delta W_n \Delta W_m. \tag{2.7}$$

By definition, the Wiener process has independent increments with $\mathbb{E}(\Delta W_n) = 0$ and $\mathbb{E}(\Delta W_n^2) = \mathbb{V}(\Delta W_n) = t_{n+1} - t_n$. If $n > m$, then $a_m a_n \Delta W_m$ is $\mathcal{F}_{t_n}$-measurable, and since ΔW_n is independent of $\mathcal{F}_{t_n}$, it follows that

$$\mathbb{E}\left(a_n a_m \Delta W_n \Delta W_m\right) = \begin{cases} \mathbb{E}\left(a_n a_m \Delta W_m\right)\mathbb{E}\left(\Delta W_n\right) = 0 & \text{if } n \neq m \\ \mathbb{E}\left(a_n^2\right)(t_{n+1}-t_n) & \text{if } n = m. \end{cases}$$

Hence, taking the expectation of (2.7) gives

$$\mathbb{E}\left(\mathcal{I}_T[\phi]^2\right) = \sum_{n=0}^{N-1} \mathbb{E}\left(a_n^2\right)(t_{n+1}-t_n). \tag{2.8}$$

Comparing (2.6) and (2.8) yields the assertion. ∎

Step 2: Itô integral on $\mathcal{H}^2[0,T]$

Lemma 2.3.4 *For any $u \in \mathcal{H}^2[0,T]$ there is a sequence $(\phi_k)_{k\in\mathbb{N}}$ of elementary functions $\phi_k \in \mathcal{H}^2[0,T]$ such that*

$$\lim_{k\to\infty} \|u - \phi_k\|_{L^2(dt\times d\mathbb{P})} = 0$$

Proof: Section 6.6 in [Ste01].

Let $u \in \mathcal{H}^2[0,T]$ and let $(\phi_k)_{k\in\mathbb{N}}$ be elementary functions such that

$$u = \lim_{k\to\infty} \phi_k \qquad \text{in } L^2(dt \times d\mathbb{P})$$

as in Lemma 2.3.4. The linearity of $\mathcal{I}_T[\cdot]$ and Lemma 2.3.3 yield

$$\|\mathcal{I}_T[\phi_j] - \mathcal{I}_T[\phi_k]\|_{L^2(d\mathbb{P})} = \|\mathcal{I}_T[\phi_j - \phi_k]\|_{L^2(d\mathbb{P})} = \|\phi_j - \phi_k\|_{L^2(dt\times d\mathbb{P})} \longrightarrow 0$$

for $j, k \longrightarrow \infty$. Hence, $(\mathcal{I}_T[\phi_k])_k$ is a Cauchy sequence in the Hilbert space $L^2(d\mathbb{P})$. Thus, $(\mathcal{I}_T[\phi_k])_k$ converges in $L^2(d\mathbb{P})$, and we can define

$$\mathcal{I}_T[u] = \lim_{k\to\infty} \mathcal{I}_T[\phi_k].$$

The choice of the sequence does not matter: If $(\psi_k)_{k\in\mathbb{N}}$ is another sequance of elementary functions with $u = \lim_{k\to\infty} \psi_k$ in $L^2(dt \times d\mathbb{P})$, then by Lemma 2.3.3 we obtain for $k \longrightarrow \infty$

$$\begin{aligned}\|\mathcal{I}_T[\phi_k] - \mathcal{I}_T[\psi_k]\|_{L^2(d\mathbb{P})} &= \|\mathcal{I}_T[\phi_k - \psi_k]\|_{L^2(d\mathbb{P})} \\ &= \|\phi_k - \psi_k\|_{L^2(dt\times d\mathbb{P})} \\ &\leq \|\phi_k - u\|_{L^2(dt\times d\mathbb{P})} + \|u - \psi_k\|_{L^2(dt\times d\mathbb{P})} \longrightarrow 0.\end{aligned}$$

Theorem 2.3.5 (Itô isometry) *For all $u \in \mathcal{H}^2[0,T]$ we have*

$$\|\mathcal{I}_T[u]\|_{L^2(d\mathbb{P})} = \|u\|_{L^2(dt\times d\mathbb{P})}.$$

Proof: Let $(\phi_k)_{k\in\mathbb{N}}$ again be elementary functions such that $u = \lim_{k\to\infty} \phi_k$ in $L^2(dt \times d\mathbb{P})$; cf. Lemma 2.3.4. Then

$$\lim_{k\to\infty} \|\phi_k\|_{L^2(dt\times d\mathbb{P})} = \|u\|_{L^2(dt\times d\mathbb{P})},$$

because the reverse triangle inequality yields

$$\Big|\|\phi_k\|_{L^2(dt\times d\mathbb{P})} - \|u\|_{L^2(dt\times d\mathbb{P})}\Big| \leq \|\phi_k - u\|_{L^2(dt\times d\mathbb{P})} \to 0.$$

By the same argument, we obtain

$$\lim_{k\to\infty} \|\mathcal{I}_T[\phi_k]\|_{L^2(d\mathbb{P})} = \|\mathcal{I}_T[u]\|_{L^2(d\mathbb{P})}$$

Now the assertion follows from Lemma 2.3.3 by taking the limit. ∎

2.4 Martingales

Definition 2.4.1 (conditional expectation) *Let X be an integrable random variable, and let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then, Y is a **conditional expectation** of X with respect to $\mathcal{G}$ if Y is $\mathcal{G}$-measurable and if*

$$\begin{aligned}&\mathbb{E}(X\mathbf{1}_A) = \mathbb{E}(Y\mathbf{1}_A) && \text{for all } A \in \mathcal{G}, \\ \Leftrightarrow\quad &\int_A X(\omega)\, d\mathbb{P}(\omega) = \int_A Y(\omega)\, d\mathbb{P}(\omega) && \text{for all } A \in \mathcal{G}.\end{aligned}$$

In this case, we write $Y = \mathbb{E}(X \mid \mathcal{G})$.

"This definition is not easy to love. Fortunately, love is not required."
J.M. Steele in [Ste01], p. 45.

Interpretation. $\mathbb{E}(X \mid \mathcal{G})$ is a random variable on $(\Omega, \mathcal{G}, \mathbb{P})$ and hence on $(\Omega, \mathcal{F}, \mathbb{P})$, too. Roughly speaking, $\mathbb{E}(X \mid \mathcal{G})$ is the best approximation of X detectable by the events in $\mathcal{G}$. The more $\mathcal{G}$ is refined, the better $\mathbb{E}(X \mid \mathcal{G})$ approximates X.

Examples.

1. If $\mathcal{G} = \{\Omega, \emptyset\}$, then $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$.
2. If $\mathcal{G} = \mathcal{F}$, then $\mathbb{E}(X \mid \mathcal{G}) = X$.
3. If $F \in \mathcal{F}$ with $\mathbb{P}(F) > 0$ and

$$\mathcal{G} = \{\emptyset, F, \Omega \setminus F, \Omega\}$$

then it can be shown that

$$\mathbb{E}(X \mid \mathcal{G})(\omega) = \begin{cases} \dfrac{1}{\mathbb{P}(F)} \displaystyle\int_F X d\mathbb{P} & \text{if } \omega \in F \\ \dfrac{1}{\mathbb{P}(\Omega \setminus F)} \displaystyle\int_{\Omega \setminus F} X d\mathbb{P} & \text{if } \omega \in \Omega \setminus F. \end{cases}$$

4. If X is independent of $\mathcal{G}$, then $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X)$.

Lemma 2.4.2 (Properties of the conditional expectation) *For all integrable random variables X and Y and all sub-σ-algebras $\mathcal{G} \subset \mathcal{F}$, the conditional expectation has the following properties:*

- *Linearity:* $\mathbb{E}(X + Y \mid \mathcal{G}) = \mathbb{E}(X \mid \mathcal{G}) + \mathbb{E}(Y \mid \mathcal{G})$
- *Positivity: If $X \geq 0$, then $\mathbb{E}(X \mid \mathcal{G}) \geq 0$.*
- *Tower property: If $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$ are sub-σ-algebras, then*

$$\mathbb{E}\Big(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}\Big) = \mathbb{E}(X \mid \mathcal{H})$$

- $\mathbb{E}\Big(\mathbb{E}(X \mid \mathcal{G})\Big) = \mathbb{E}(X)$
- *Factorization property: If Y is $\mathcal{G}$-measurable and $|XY|$ and $|Y|$ are integrable, then*

$$\mathbb{E}(XY \mid \mathcal{G}) = Y\mathbb{E}(X \mid \mathcal{G})$$

Proof: Exercise.

Definition 2.4.3 (martingale) *Let X_t be a stochastic process which is adapted to a filtration $\{\mathcal{F}_t : t \geq 0\}$ of $\mathcal{F}$. If*

1. *$\mathbb{E}(|X_t|) < \infty$ for all $0 \leq t < \infty$, and*
2. *$\mathbb{E}(X_t | \mathcal{F}_s) = X_s$ for all $0 \leq s \leq t < \infty$,*

then X_t is called a ***martingale****. A martingale X_t is called continuous if there is a set $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that the path $t \mapsto X_t(\omega)$ is continuous for all $\omega \in \Omega_0$.*

Interpretation: A martingale models a fair game. Observing the game up to time s does not give any advantage for future times.

Examples. It can be shown that each of the following processes is a continuous martingale with respect to the standard Brownian filtration:

$$W_t, \qquad W_t^2 - t, \qquad \exp\left(\alpha W_t - \frac{\alpha^2}{2}t\right)$$

Proof: Exercise.

2.5 Construction of the Itô integral (step 3 and 4)

Step 3: The Itô integral as a process

So far we have defined the Itô integral $\mathcal{I}_T[u](\omega)$ over the interval $[0,T]$ for **fixed** T. For applications in mathematical finance, however, we want to consider $\{\mathcal{I}_t[u](\omega) : t \in [0,T]\}$ as a stochastic **process**.

If $u(s,\omega) \in \mathcal{H}^2[0,T]$, then $\mathbf{1}_{[0,t]}(s)u(s,\omega) \in \mathcal{H}^2[0,T]$. Can we define $\mathcal{I}_t[u](\omega)$ by $\mathcal{I}_T[\mathbf{1}_{[0,t]}u](\omega)$?

Problem: The integral $\mathcal{I}_T[\mathbf{1}_{[0,t]}u](\omega)$ is only defined in $L^2(d\mathbb{P})$. Hence, the value $\mathcal{I}_T[\mathbf{1}_{[0,t]}u](\omega)$ is arbitrary on sets Z_t of $\mathbb{P}$-measure zero. This is the case for every $t \in [0,T]$, and since the set $[0,T]$ is uncountable[2], the union

$$\bigcup_{t\in[0,T]} Z_t$$

(i.e. the set where the process is not well-defined) could be "very large"! Fortunately, this can be fixed:

Theorem 2.5.1 *For any $u \in \mathcal{H}^2[0,T]$ there is a process $\{X_t : t \in [0,T]\}$ that is a continuous martingale with respect to the standard Brownian filtration $\mathcal{F}_t$ such that the event*

$$\{\omega \in \Omega : X_t(\omega) = \mathcal{I}_T[\mathbf{1}_{[0,t]}u](\omega)\}$$

has probability one for each $t \in [0,T]$.

A proof can be found in [Ste01], Theorem 6.2, pages 83-84.

[2]We only know that countable unions of null sets have measure zero, but this is not true for uncountabel unions.

Step 4: The Itô integral on $\mathcal{L}^2_{loc}[0,T]$

So far we have defined the Itô integral for functions $u \in \mathcal{H}^2[0,T]$; cf. Definition 2.3.1. Such functions must satisfy

$$\mathbb{E}\left(\int_0^T u^2(t,\omega)\,dt\right) < \infty, \tag{2.9}$$

and this condition is sometimes too restrictive. With some more work, the Itô integral can be extended to the class $\mathcal{L}^2_{loc}[0,T]$, i.e. to all functions

$$u = u(t,\omega), \qquad u : [0,T] \times \Omega \longrightarrow \mathbb{R}$$

with the following properties:

- $(t,\omega) \mapsto u(t,\omega)$ is $(\mathcal{B} \times \mathcal{F})$-measurable.
- u is adapted to $\{\mathcal{F}_t : t \in [0,T]\}$.
- $\mathbb{P}\left(\int_0^T u^2(t,\omega)\,dt < \infty\right) = 1.$

The first two conditions are the same as for $\mathcal{H}^2[0,T]$, but the third condition is weaker than (2.9). If $y : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then $u(t,\omega) = y\big(W(t,\omega)\big) \in \mathcal{L}^2_{loc}[0,T]$, because $t \mapsto y\big(W(t,\omega)\big)$ is continuous with probability one and hence bounded on $[0,T]$ with probability one.

Details: Chapter 7 in [Ste01].

Notation

The process X constructed above is called the **Itô integral** (Itô Kiyoshi 1944) of $u \in \mathcal{L}^2_{loc}[0,T]$ and is denoted by

$$X(t,\omega) = \int_0^t u(s,\omega)\,dW(s,\omega).$$

The Itô integral over an arbitrary interval $[a,b] \subset [0,T]$ is defined by

$$\int_a^b u(s,\omega)\,dW(s,\omega) = \int_0^b u(s,\omega)\,dW(s,\omega) - \int_0^a u(s,\omega)\,dW(s,\omega).$$

Alternative notations:

$$\int_a^b u(s,\omega)\,dW(s,\omega) = \int_a^b u(s,\omega)\,dW_s(\omega) = \int_a^b u_s(\omega)\,dW_s(\omega) = \int_a^b u_s\,dW_s$$

Properties of the Itô integral

Lemma 2.5.2 *Let $c \in \mathbb{R}$ and $u, v \in \mathcal{L}^2_{loc}[0,T]$. The Itô integral on $[a,b] \subset [0,T]$ has the following properties:*

1. *Linearity:*
$$\int_a^b \Big(cu(s,\omega) + v(s,\omega)\Big)\, dW_s(\omega) = c\int_a^b u(s,\omega)\, dW_s(\omega) + \int_a^b v(s,\omega)\, dW_s(\omega)$$
with probability one.
2. $\mathbb{E}\left(\int_a^b u(s,\omega)\, dW_s(\omega)\right) = 0$
3. $\int_a^t u(s,\omega)\, dW_s(\omega)$ *is $\mathcal{F}_t$-measurable for $t \geq a$.*
4. *Itô isometry on $[a,b]$:*
$$\mathbb{E}\left(\left(\int_a^b u(s,\omega)\, dW_s(\omega)\right)^2\right) = \mathbb{E}\left(\int_a^b u^2(s,\omega)\, ds\right)$$
(cf. Theorem 2.3.5).
5. *Martingale property: The Itô integral*
$$X(t,\omega) = \int_0^t u(s,\omega)\, dW(s,\omega).$$
of a function $u \in \mathcal{H}^2[0,T]$ is a continuous martingale with respect to the standard Brownian filtration; cf. Theorem 2.5.1. If $u \in \mathcal{L}^2_{loc}[0,T]$, then the Itô integral is only a local martingale; cf. Proposition 7.7 in [Ste01].

The first four properties can be shown by considering elementary functions and passing to the limit.

2.6 Stochastic differential equations and the Itô formula

Definition 2.6.1 (SDE) *A **stochastic differential equation (SDE)** is an equation of the form*
$$X(t) = X(0) + \int_0^t f\big(s, X(s)\big)\, ds + \int_0^t g\big(s, X(s)\big)\, dW(s). \tag{2.10}$$

The solution $X(t)$ of (2.10) is called an ***Itô process****.*

The last term is an Itô integral, with $W(t)$ denoting the Wiener process. The functions $f : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ are called drift and diffusion coefficients, respectively. These functions are typically given while $X(t) = X(t,\omega)$ is unknown.
This equation is actually not a **differential** equation, but an **integral** equation! Often people write

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$

as a shorthand notation for (2.10). Some people even "divide by dt" in order to make the equation look like a differential equation, but this is more than audacious since "dW_t/dt" does not make sense.

Two special cases:

- If $g\big(t, X(t)\big) \equiv 0$, then (2.10) is reduced to

$$X(t) = X(0) + \int_0^t f\big(s, X(s)\big)\, ds.$$

  If $X(t)$ is differentiable, this is equivalent to the initial value problem

$$\frac{dX(t)}{dt} = f\big(t, X(t)\big), \qquad\qquad X(0) = X_0.$$

- For $f\big(t, X(t)\big) \equiv 0$, $g\big(t, X(t)\big) \equiv 1$ and $X(0) = 0$, (2.10) turns into

$$X(t) = \underbrace{X(0)}_{=0} + \underbrace{\int_0^t f\big(s, X(s)\big)\, ds}_{=0} + \int_0^t \underbrace{g\big(s, X(s)\big)}_{=1}\, dW(s) = W(t) - W(0) = W(t).$$

Computing Riemann integrals via the basic definition is usually very tedious. The fundamental theorem of calculus provides an alternative which is more convenient in most cases. For Itô integrals, the situation is similar: The approximation via elementary functions which is used to *define* the Itô integral is rarely used to *compute* the integral. What is the counterpart of the fundamental theorem of calculus for the Itô integral?

Theorem 2.6.2 (Itô formula) *Let X_t be the solution of the SDE*

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t$$

and let $F(t,x)$ be a function with continuous partial derivatives $\partial_t F = \frac{\partial F}{\partial t}$, $\partial_x F = \frac{\partial F}{\partial x}$, and $\partial_x^2 F = \frac{\partial^2 F}{\partial x^2}$. Then, we have for $Y_t := F(t, X_t)$ that

$$\begin{aligned} dY_t &= \partial_t F \, dt + \partial_x F \, dX_t + \frac{1}{2}(\partial_x^2 F) g^2 \, dt \\ &= \left(\partial_t F + (\partial_x F) f + \frac{1}{2}(\partial_x^2 F) g^2 \right) dt + (\partial_x F) g \, dW_t. \end{aligned} \tag{2.11}$$

with $f = f(t, X_t)$, $g = g(t, X_t)$, $\partial_x F = \partial_x F(t, X_t)$, and so on.

Notation. Evaluations of the derivatives of F are to be understood in the sense of, e.g.,

$$\partial_x F(s, X_s) := \partial_x F(t,x)\big|_{(t,x)=(s,X_s)}$$

and so on.

The proof of Theorem 2.6.2 is sketched at the end of this section.

Remarks:

1. If $y(t)$ is a smooth deterministic fuctions, then according to the chain rule the derivative of $t \mapsto F\big(t, y(t)\big)$ is

   $$\frac{d}{dt} F\big(t, y(t)\big) = \partial_t F\big(t, y(t)\big) + \partial_x F\big(t, y(t)\big) \cdot \frac{dy(t)}{dt}$$

   and in shorthand notation

   $$dF = \partial_t F dt + \partial_x F dy.$$

   The Itô formula can be considered as a stochastic version of the chain rule, but the term $\frac{1}{2}(\partial_x^2 F) \cdot g^2 dt$ is surprising since such a term does not appear in the deterministic chain rule.

2. Let $f(t, X_t) = 0$, $g(t, X_t) = 1$, $X_t = W_t$ and suppose that $F(t,x) = F(x)$ does not depend on t. Then, the Itô formula yields for $Y_t := F(W_t)$ that

   $$dY_t = F'(W_t) dW_t + \frac{1}{2} F''(W_t) dt$$

   which is the shorthand notation for

   $$F(W_t) = F(W_0) + \int_0^t F'(W_s) dW_s + \frac{1}{2} \int_0^t F''(W_s) ds.$$

   This can be seen as a counterpart of the fundamental theorem of calculus. Again, the last term is surprising, because for a suitable deterministic function $v(t) = v_t$ we obtain

   $$F(v_t) = F(v_0) + \int_0^t F'(v_s) dv_s.$$

Example 1. Consider the integral

$$\int_0^t W_s \, dW_s.$$

$X_t := W_t$ solves the SDE with $f(t, X_t) \equiv 0$ and $g(t, X_t) \equiv 1$. For

$$F(t,x) = x^2, \qquad\qquad Y_t = F(t, X_t) = X_t^2 = W_t^2$$

the Itô formula

$$dY_t = \left(\partial_t F + (\partial_x F) f + \frac{1}{2} (\partial_x^2 F) g^2 \right) dt + (\partial_x F) g \, dW_t$$

yields

$$\begin{aligned} & d(W_t^2) = 0 + 0 + \frac{1}{2} \cdot 2 \cdot 1^2 \, dt + 2W_t \cdot 1 \, dW_t = dt + 2W_t \, dW_t \\ \implies \qquad & W_t \, dW_t = \frac{1}{2} \left(d(W_t^2) - dt \right) \end{aligned}$$

This means that

$$\int_0^t W_s \, dW_s = \frac{1}{2} \int_0^t 1 \, d(W_s^2) - \frac{1}{2} \int_0^t 1 \, ds = \frac{1}{2} W_t^2 - \frac{1}{2} t.$$

Example 2. The solution of the SDE

$$dY_t = \mu Y_t \, dt + \sigma Y_t \, dW_t$$

with constants $\mu, \sigma \in \mathbb{R}$ and deterministic initial value $Y_0 \in \mathbb{R}$ is given by

$$Y_t = \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right) Y_0.$$

This process is called a **geometric Brownian motion** and is often used in mathematical finance to model stock prices (see below).

Proof. Let $f(t, X_t) \equiv 0$, $g(t, X_t) \equiv 1$, $X_t = W_t$ as before, but now with

$$F(t,x) = \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) t + \sigma x \right) Y_0$$

and derivatives

$$\partial_t F(t,x) = \left(\mu - \frac{\sigma^2}{2} \right) F(t,x), \qquad \partial_x^i F(t,x) = \sigma^i F(t,x), \qquad i \in \{1, 2\}.$$

Hence, the Itô formula applied to $Y_t = F(t, X_t) = F(t, W_t)$ yields

$$\begin{aligned} dY_t &= \left(\left(\mu - \frac{\sigma^2}{2} \right) Y_t + 0 + \frac{1}{2} \sigma^2 Y_t \cdot 1^2 \right) dt + \sigma Y_t \cdot 1 \, dW_t \\ &= \mu Y_t \, dt + \sigma Y_t \, dW_t. \end{aligned}$$

■

Ordinary differential equations can have multiple solutions with the same initial value, and solutions do not necessarily exist for all times. Hence, we cannot expect that every SDE has a unique solution. As in the ODE case, however, existence and uniqueness can be shown under certain assumptions concerning the coefficients f and g:

Theorem 2.6.3 (existence and uniqueness)
Let $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ be functions with the following properties:

- ***Lipschitz condition:*** *There is a constant $L \geq 0$ such that*

$$|f(t,x) - f(t,y)| \leq L|x-y|, \qquad |g(t,x) - g(t,y)| \leq L|x-y| \tag{2.12}$$

  for all $x, y \in \mathbb{R}$ and $t \geq 0$.
- ***Linear growth condition:*** *There is a constant $K \geq 0$ such that*

$$|f(t,x)|^2 \leq K \left(1 + |x|^2\right), \qquad |g(t,x)|^2 \leq K \left(1 + |x|^2\right) \tag{2.13}$$

  for all $x \in \mathbb{R}$ and $t \geq 0$.

Then, the SDE

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t), \qquad t \in [0,T]$$

with deterministic initial value $X(0) = X_0$ has a continuous adapted solution and

$$\sup_{t \in [0,T]} \mathbb{E}\left(X^2(t)\right) < \infty.$$

If both $X(t)$ and $\widetilde{X}(t)$ are such solutions, then

$$\mathbb{P}\Big(X(t) = \widetilde{X}(t) \text{ for all } t \in [0,T]\Big) = 1.$$

Proof: Theorem 9.1 in [Ste01] or Theorem 4.5.3 in [KP99].

Remark: The assumptions can be weakened.

Sketch of the proof of the Itô formula (Theorem 2.6.2).

(i) Equation (2.11) is the shorthand notation for

$$Y_t = Y_0 + \int_0^t \left(\partial_t F(s, X_s) + \partial_x F(s, X_s) \cdot f(s, X_s) + \frac{1}{2} \partial_x^2 F(s, X_s) \cdot g^2(s, X_s) \right) ds + \int_0^t \partial_x F(s, X_s) \cdot g(s, X_s) dW_s$$

Assume that F is twice continuously differentiable with bounded partial derivatives. (Otherwise F can be approximated by such functions with uniform convergence on compact subsets of $[0, \infty) \times \mathbb{R}$.)
Assume that $(t, \omega) \mapsto f(t, X_t(\omega))$ and $(t, \omega) \mapsto g(t, X_t(\omega))$ are elementary functions. (Otherwise approximate by elementary functions.) Hence, there is a partition $0 = t_0 < t_1 < \ldots < t_N = t$ such that

$$f(t, X_t(\omega)) = f(0, X_0(\omega)) \mathbf{1}_{[0,t_1]}(t) + \sum_{n=1}^{N-1} f(t_n, X_{t_n}(\omega)) \mathbf{1}_{(t_n, t_{n+1}]}(t)$$

and the same equation with f replaced by g.

(ii) Notation: For the rest of the proof, we define

$$f^{(n)} := f(t_n, X_{t_n}), \qquad F^{(n)} := F(t_n, X_{t_n}),$$
$$g^{(n)} := g(t_n, X_{t_n}), \qquad \partial_t F^{(n)} := \partial_t F(t_n, X_{t_n})$$

and so on, and

$$\Delta t_n = t_{n+1} - t_n, \qquad \Delta X_n = X_{t_{n+1}} - X_{t_n}, \qquad \Delta W_n = W_{t_{n+1}} - W_{t_n}.$$

Since f and g are elementary functions, we have

$$\begin{aligned} X_{t_n} &= X_0 + \int_0^{t_n} f(s, X_s)\, ds + \int_0^{t_n} g(s, X_s)\, dW_s \\ &= X_0 + \sum_{k=0}^{n-1} \underbrace{f(t_k, X_{t_k})}_{f^{(k)}} \Delta t_k + \sum_{k=0}^{n-1} \underbrace{g(t_k, X_{t_k})}_{g^{(k)}} \Delta W_k. \end{aligned}$$

and hence

$$\Delta X_n = X_{t_{n+1}} - X_{t_n} = f^{(n)} \Delta t_n + g^{(n)} \Delta W_n.$$

(iii) Telescoping sum:

$$Y_t = Y_{t_N} = Y_0 + \sum_{n=0}^{N-1} (Y_{t_{n+1}} - Y_{t_n}) = Y_0 + \sum_{n=0}^{N-1} \left(F^{(n+1)} - F^{(n)} \right)$$

Apply Taylor's theorem:

$$\begin{aligned} & F^{(n+1)} - F^{(n)} \\ &= \partial_t F^{(n)} \cdot \Delta t_n + \partial_x F^{(n)} \cdot \Delta X_n + \frac{1}{2} \partial_t^2 F^{(n)} \cdot (\Delta t_n)^2 + \partial_t \partial_x F^{(n)} \cdot \Delta t_n \Delta X_n \\ &\quad + \frac{1}{2} \partial_x^2 F^{(n)} \cdot (\Delta X_n)^2 + R_n(\Delta t_n, \Delta X_n) \end{aligned}$$

with a remainder term R_n. Insert this into the telescoping sum.

(iv) Consider the limit $N \longrightarrow \infty$, $\Delta t_n \longrightarrow 0$ with respect to $\| \cdot \|_{L^2(d\mathbb{P})}$. For the first two terms, this yields

$$\lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_t F^{(n)} \cdot \Delta t_n = \lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_t F(t_n, X_{t_n}) \cdot \Delta t_n = \int_0^t \partial_t F(s, X_s) \, ds$$

and

$$\begin{aligned} & \lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_x F^{(n)} \cdot \Delta X_n \\ &= \lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_x F^{(n)} \cdot f^{(n)} \Delta t_n + \lim_{N \to \infty} \sum_{n=0}^{N-1} \partial_x F^{(n)} \cdot g^{(n)} \Delta W_n \\ &= \int_0^t \partial_x F(s, X_s) \cdot f(s, X_s) \, ds + \int_0^t \partial_x F(s, X_s) \cdot g(s, X_s) \, dW_s. \end{aligned}$$

(v) Next, we investigate the "$\partial_x^2 F^{(n)}$ term". Since

$$(\Delta X_n)^2 = \left(f^{(n)} \Delta t_n + g^{(n)} \Delta W_n \right)^2$$

we have

$$\frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot (\Delta X_n)^2 = \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot \left(f^{(n)} \right)^2 (\Delta t_n)^2 \tag{2.14}$$

$$+ \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot f^{(n)} g^{(n)} \Delta t_n \Delta W_n \tag{2.15}$$

$$+ \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot \left(g^{(n)} \right)^2 (\Delta W_n)^2. \tag{2.16}$$

For the right-hand side of (2.14), we obtain

$$\left\| \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot \left(f^{(n)}\right)^2 (\Delta t_n)^2 \right\|_{L^2(d\mathbb{P})}^2 = \mathbb{E}\left[\left(\sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot \left(f^{(n)}\right)^2 (\Delta t_n)^2 \right)^2 \right] \longrightarrow 0.$$

With the abbreviation $\alpha^{(n)} := \partial_x^2 F^{(n)} \cdot f^{(n)} g^{(n)}$ we obtain for the right-hand side of (2.15) that

$$\begin{aligned} \left\| \sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right\|_{L^2(d\mathbb{P})}^2 &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right)^2 \right] \\ &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \mathbb{E}\left(\alpha^{(n)} \alpha^{(m)} \Delta W_n \Delta W_m \right) \Delta t_n \Delta t_m. \end{aligned}$$

Since

$$\mathbb{E}\left(\alpha^{(n)} \alpha^{(m)} \Delta W_n \Delta W_m \right) = \mathbb{E}\left(\alpha^{(n)} \alpha^{(m)} \Delta W_n \right) \underbrace{\mathbb{E}\left(\Delta W_m \right)}_{=0} = 0$$

for $n < m$ and similar for $m < n$, only the terms with $n = m$ have to be considered, which yields

$$\left\| \sum_{n=0}^{N-1} \alpha^{(n)} \Delta t_n \Delta W_n \right\|_{L^2(d\mathbb{P})}^2 = \sum_{n=0}^{N-1} \mathbb{E}\left((\alpha^{(n)})^2 \right) (\Delta t_n)^2 \underbrace{\mathbb{E}\left[(\Delta W_n)^2 \right]}_{=\Delta t_n} \longrightarrow 0.$$

The third term (2.16), however, has a non-zero limit: We show that

$$\lim_{N \to \infty} \frac{1}{2} \sum_{n=0}^{N-1} \partial_x^2 F^{(n)} \cdot \left(g^{(n)}\right)^2 (\Delta W_n)^2 = \frac{1}{2} \int_0^t \partial_x^2 F(s, X_s) \cdot \left(g(s, X_s)\right)^2 \, ds$$

which yields the strange additional term in the Itô formula. With the abbreviation $\beta^{(n)} = \frac{1}{2} \partial_x^2 F^{(n)} \cdot \left(g^{(n)}\right)^2$ we have

$$\begin{aligned} &\left\| \sum_{n=0}^{N-1} \beta^{(n)} \left((\Delta W_n)^2 - \Delta t_n \right) \right\|_{L^2(d\mathbb{P})}^2 \\ &= \mathbb{E}\left[\left(\sum_{n=0}^{N-1} \beta^{(n)} \left((\Delta W_n)^2 - \Delta t_n \right) \right)^2 \right] \\ &= \mathbb{E}\left[\sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \beta^{(n)} \beta^{(m)} \left((\Delta W_n)^2 - \Delta t_n \right) \left((\Delta W_m)^2 - \Delta t_m \right) \right]. \end{aligned}$$

For $n < m$ we have

$$\begin{aligned}&\mathbb{E}\left[\beta^{(n)}\beta^{(m)}\left((\Delta W_n)^2-\Delta t_n\right)\left((\Delta W_m)^2-\Delta t_m\right)\right]\\&=\mathbb{E}\left[\beta^{(n)}\beta^{(m)}\left((\Delta W_n)^2-\Delta t_n\right)\right]\underbrace{\mathbb{E}\left[\left((\Delta W_m)^2-\Delta t_m\right)\right]}_{=0}=0\end{aligned}$$

and vice versa for $n > m$. Hence, only the terms with $n = m$ have to be considered, and we obtain

$$\begin{aligned}\left\|\sum_{n=0}^{N-1}\beta^{(n)}\left((\Delta W_n)^2-\Delta t_n\right)\right\|^2_{L^2(d\mathbb{P})}&=\mathbb{E}\left[\sum_{n=0}^{N-1}\left(\beta^{(n)}\right)^2\left((\Delta W_n)^2-\Delta t_n\right)^2\right]\\&=\sum_{n=0}^{N-1}\mathbb{E}\left[\left(\beta^{(n)}\right)^2\right]\mathbb{E}\left[\left((\Delta W_n)^2-\Delta t_n\right)^2\right]\to 0\end{aligned}$$

because $\mathbb{E}\left[\left((\Delta W_n)^2-\Delta t_n\right)^2\right]=2\Delta t_n^2$ according to Exercise 4.

(vi) With essentially the same arguments, it can be shown that

$$\lim_{N\to\infty}\frac{1}{2}\sum_{n=0}^{N-1}\partial_t^2F^{(n)}\cdot(\Delta t_n)^2=0$$

$$\lim_{N\to\infty}\sum_{n=0}^{N-1}\partial_t\partial_xF^{(n)}\cdot\Delta t_n\Delta X_n=0$$

and that the remainder term from the Taylor expansion can be neglected when the limit is taken. ■

2.7 The Feynman-Kac formula

Let X_t be the solution of the SDE

$$dX_t=f(t,X_t)dt+g(t,X_t)dW_t,\qquad t\in[t_0,T],\qquad X_{t_0}=\xi$$

with suitable functions f and g. Let $u(t,x)$ be the solution of the (deterministic) partial differential equation (PDE)

$$\partial_tu(t,x)+f(t,x)\partial_xu(t,x)+\frac{1}{2}g^2(t,x)\partial_x^2u(t,x)=0,\qquad t\in[t_0,T],\qquad x\in\mathbb{R}$$

with terminal condition

$$u(T,x)=\psi(x)$$

for some $\psi : \mathbb{R} \longrightarrow \mathbb{R}$. Apply the Itô formula (Theorem 2.6.2) to $u(t, X_t)$:

$$du(t, X_t) = \underbrace{\Big(\partial_t u(t, X_t) + f(t, X_t)\partial_x u(t, X_t) + \frac{1}{2} g^2(t, X_t)\partial_x^2 u(t, X_t)\Big)}_{=0}\, dt + g(t, X_t)\partial_x u(t, X_t)\, dW_t$$

Equivalent:

$$\underbrace{u(T, X_T)}_{\psi(X_T)} = u(t_0, \underbrace{X_{t_0}}_{\xi}) + \int_{t_0}^{T} g(t, X_t)\partial_x u(t, X_t)\, dW_t$$

Taking the expectation and applying Lemma 2.5.2 yields the **Feynman-Kac formula** (Richard Feynman, Mark Kac)

$$\mathbb{E}\Big(\psi(X_T)\Big) = u(t_0, \xi).$$

Remark: This derivation is informal, because we have tacitly assumed that all terms exist. See, e.g., Chapter 15 in [Ste01] for a correct proof.

2.8 Extension to higher dimensions

In order to model options on several underlying assets (e.g. basket options), we have to consider vector-valued Itô integrals and SDEs. A d-dimensional SDE takes the form

$$X_j(t) = X_j(0) + \int_0^t f_j(s, X(s))\, ds + \sum_{k=1}^{m} \int_0^t g_{jk}(s, X(s))\, dW_k(s) \qquad (j = 1, \ldots, d) \tag{2.17}$$

for $d, m \in \mathbb{N}$ and suitable functions

$$f_j : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \qquad g_{jk} : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}.$$

$W_1(s), \ldots, W_m(s)$ are one-dimensional scalar Wiener processes which are pairwise independent. (2.17) is equivalent to

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s) \tag{2.18}$$

with vectors

$$W(t) = \big(W_1(t), \ldots, W_m(t)\big)^T \in \mathbb{R}^m$$
$$f(t, x) = \big(f_1(t, x), \ldots, f_d(t, x)\big)^T \in \mathbb{R}^d$$

and a matrix

$$g(t,x) = \begin{pmatrix} g_{11}(t,x) & \cdots & g_{1m}(t,x) \\ \vdots & & \vdots \\ g_{d1}(t,x) & \cdots & g_{dm}(t,x) \end{pmatrix} \in \mathbb{R}^{d\times m}$$

Theorem 2.8.1 (Multi-dimensional Itô formula) *Let X_t be the solution of the SDE (2.18) and let $F : [0,\infty) \times \mathbb{R}^d \longrightarrow \mathbb{R}^n$ be a function with continuous partial derivatives $\partial_t F$, $\partial_{x_j} F$, and $\partial_{x_j}\partial_{x_k} F$. Then, the process $Y(t) := F(t, X_t)$ satisfies*

$$\begin{aligned} dY_\ell(t) &= \partial_t F_\ell(t, X_t)\, dt \\ &\quad + \sum_{i=1}^{d} \partial_{x_i} F_\ell(t, X_t) \cdot f_i(t, X_t)\, dt \\ &\quad + \frac{1}{2} \sum_{i=1}^{d}\sum_{j=1}^{d} \partial_{x_i}\partial_{x_j} F_\ell(t, X_t) \cdot \left(\sum_{k=1}^{m} g_{ik}(t, X_t) g_{jk}(t, X_t) \right) dt \\ &\quad + \sum_{i=1}^{d} \partial_{x_i} F_\ell(t, X_t) \cdot \sum_{k=1}^{m} g_{ik}(t, X_t)\, dW_k \end{aligned}$$

or equivalently

$$dY_\ell = \left\{ \partial_t F_\ell + f^T \nabla F_\ell + \frac{1}{2}\mathrm{tr}\Big(g^T (\nabla^2 F_\ell) g \Big) \right\} dt + (\nabla F_\ell)^T g\, dW(t)$$

where ∇F_ℓ is the gradient and $\nabla^2 F_\ell$ is the Hessian of F_ℓ, and where $\mathrm{tr}(A) = \sum_{j=1}^{m} a_{jj}$ is the trace of a matrix $A = (a_{ij})_{i,j} \in \mathbb{R}^{m\times m}$.

Proof: Similar to the case $d = m = 1$.

Final remark: Itô vs. Stratonovich. The Itô integral is not the only stochastic integral, and the Stratonovich integral is a famous alternative. The Stratonovich integral has the advantage that the ordinary chain rule remains valid, i.e. the additional term in the Itô formula does not appear when the Stratonovich integral is used. Their disadvantage is the fact that Stratonovich integrals are not martingales, whereas Itô integrals are. Stratonovich integrals can be transformed into Itô integrals and vice versa. See 3.1, 3.3 in [Øks03] and 3.5, 4.9 in [KP99].
Actually, the SDEs (2.10) or (2.17) should be called "Itô SDE" or "SDE of the Itô type". Since only Itô SDEs and no Stratonovich SDEs will appear in this lecture, however, we simply use the term "SDE" for "Itô SDE".

Chapter 3

The Black-Scholes equation

References: [BK04, GJ10, Sey09]

Goal: Find equations to determine the value of an option on a single underlying asset. Throughout this chapter, we make the assumptions (A1)-(A5) from 1.3 unless otherwise stated.

3.1 Geometric Brownian motion

First step: Model the price of the underlying by a suitable process S_t.
For the value of a bond with interest rate $r > 0$, we have $B_t = B_0 e^{rt}$. Try to "stochastify" this equation with a Wiener process in order to model the underlying.

First attempt: $S_t = S_0 e^{at} + \sigma W_t$ for some $a, \sigma \in \mathbb{R}$. Problem: S_t can have negative values. Not good.

Second attempt: For the bond we have

$$\ln B_t = \ln B_0 + rt.$$

which motivates the ansatz

$$\ln S_t = \ln S_0 + at + \sigma W_t$$

for some $a, \sigma \in \mathbb{R}$ to model the underlying. The parameter σ is called the **volatility**. Applying $\exp(\dots)$ gives

$$S_t = S_0 \exp(at + \sigma W_t)$$

and hence $S_t \geq 0$ if $S_0 \geq 0$. In fact, S_t is the geometric Brownian motion from 2.6 and solves the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with $\mu = a + \sigma^2/2$. Interpretation:

$$\begin{array}{rcccc} \dfrac{dS_t}{S_t} & = & \mu dt & + & \sigma dW_t \\ \text{relative change} & = & \text{deterministic trend} & + & \text{random fluctuations} \end{array}$$

Lemma 3.1.1 (moments of GBM) *The geometric Brownian motion*

$$S_t = S_0 \exp(at + \sigma W_t), \qquad a = \mu - \sigma^2/2$$

with $\mu \in \mathbb{R}$, $\sigma \in \mathbb{R}$ and fixed (deterministic) initial value S_0 has the following properties:

1. $\mathbb{E}(S_t) = S_0 e^{\mu t}$
2. $\mathbb{E}(S_t^2) = S_0^2 e^{(2\mu+\sigma^2)t}$
3. $\mathbb{V}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)$

Proof: Exercise.

Definition 3.1.2 (log-normal distribution) *A vector-valued random variable $X(\omega) \in \mathbb{R}^d$ is **log-normal** (=log-normally distributed) if $\ln X = (\ln X_1, \ldots, \ln X_d)^T \in \mathbb{R}^d$ is normally distributed, i.e. $\ln X \sim \mathcal{N}(\xi, \Sigma)$ for some $\xi \in \mathbb{R}^d$ and a symmetric, positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$. The expectation and the covariance matrix have the entries*

$$\begin{aligned} \mathbb{E}_i(X) &= e^{\xi_i + \Sigma_{ii}/2}, \\ \mathbb{V}_{ij}(X) &= \mathbb{E}\Big(\big(\mathbb{E}_i(X) - X_i\big)\big(\mathbb{E}_j(X) - X_j\big)\Big) = e^{\xi_i+\xi_j+\frac{1}{2}(\Sigma_{ii}+\Sigma_{jj})}\left(e^{\Sigma_{ij}} - 1\right). \end{aligned}$$

For $d = 1$ and $\Sigma = \sigma^2$ the corresponding density is

$$\phi(x) = \phi(x, \xi, \sigma^2) = \begin{cases} \dfrac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\dfrac{(\ln x - \xi)^2}{2\sigma^2}\right) & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}$$

Proof: Exercise.

Example. The (one-dimensional) geometric Brownian motion

$$S_t = S_0 \exp(at + \sigma W_t)$$

is log-normal, because

$$\ln S_t = \ln S_0 + at + \sigma W_t \sim \mathcal{N}(\ln S_0 + at, \sigma^2 t)$$

3.2 Derivation of the Black-Scholes equation

Situation: $S_t > 0$ value of an underlying, $B_t > 0$ value of a bond.

Goal: Determine the fair price V_t of a European call or put option.

Replication strategy: Consider a portfolio containing $a_t \in \mathbb{R}$ underlyings and $b_t \in \mathbb{R}$ bonds such that

$$V_t = a_t S_t + b_t B_t$$

(cf. section 1.5). Assume that the portfolio is self-financing: no cash inflow or outflow, i.e. buying an item must be financed by selling another one. This means that

$$a_{t+\delta} S_\star + b_{t+\delta} B_\star = a_t S_\star + b_t B_\star + \mathcal{O}\big(\delta^2\big) \tag{3.1}$$

for all $S_\star > 0$ and $B_\star > 0$. Consequence:

$$\begin{aligned} V_{t+\delta} - V_t &= (a_{t+\delta} S_{t+\delta} - a_t S_t) + (b_{t+\delta} B_{t+\delta} - b_t B_t) \\ &= a_t (S_{t+\delta} - S_t) + b_t (B_{t+\delta} - B_t) + \underbrace{(a_{t+\delta} - a_t) S_{t+\delta} + (b_{t+\delta} - b_t) B_{t+\delta}}_{=\mathcal{O}(\delta^2)} \end{aligned}$$

for all $t \geq 0$ and small $\delta > 0$. For $\delta \longrightarrow 0$ we obtain (in an integral sense)

$$dV_t = a_t dS_t + b_t dB_t.$$

Now suppose that

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad\qquad dB_t = r B_t dt \tag{3.2}$$

with $\mu, \sigma, r \in \mathbb{R}$. This yields

$$\begin{aligned} dV_t &= a_t \big(\mu S_t dt + \sigma S_t dW_t\big) + b_t \big(r B_t dt\big) \\ &= \big(a_t \mu S_t + b_t r B_t\big) dt + a_t \sigma S_t dW_t. \end{aligned} \tag{3.3}$$

Assume that the value of the option is a function of t and S_t, i.e. $V_t = V(t, S_t)$. Apply the Itô formula:

$$\begin{aligned} dV(t, S_t) = &\left(\partial_t V(t, S_t) + \partial_S V(t, S_t) \cdot \mu S_t + \frac{1}{2} \partial_S^2 V(t, S_t) \cdot \sigma^2 S_t^2 \right) dt \\ &+ \partial_S V(t, S_t) \cdot \sigma S_t dW_t \end{aligned} \tag{3.4}$$

Equating the dW_t-terms in (3.3) and (3.4) yields

$$a_t = \partial_S V(t, S_t),$$

while equating the dt-terms yields

$$\begin{aligned} & a_t\mu S_t + b_t r B_t = \partial_t V(t,S_t) + \partial_S V(t,S_t)\cdot \mu S_t + \frac{1}{2}\partial_S^2 V(t,S_t)\cdot \sigma^2 S_t^2 \\ \Longrightarrow \quad & b_t r B_t = \partial_t V(t,S_t) + \frac{1}{2}\partial_S^2 V(t,S_t)\cdot \sigma^2 S_t^2 \\ \Longrightarrow \quad & b_t = \frac{1}{B_t r}\left(\partial_t V(t,S_t) + \frac{1}{2}\partial_S^2 V(t,S_t)\cdot \sigma^2 S_t^2\right) \end{aligned}$$

for $B_t r \neq 0$. The formulas for a_t and b_t yield

$$V(t,S_t) = a_t S_t + b_t B_t = \partial_S V(t,S_t)\cdot S_t + \frac{1}{B_t r}\left(\partial_t V(t,S_t) + \frac{1}{2}\partial_S^2 V(t,S_t)\cdot \sigma^2 S_t^2\right) B_t.$$

Since this is true for **every** value of S_t, we can consider $S = S_t$ as a parameter. Multiplying with r yields the **Black-Scholes equation**

$$\boxed{\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0.}$$

Fischer Black and Myron Scholes 1973, Robert Merton 1973
Nobel Prize in Economics 1997

The Black-Scholes equation is a partial differential equation (PDE): It involves partial derivatives with respect to t and S. This PDE must be solved **backwards** in time: instead of an initial condition, we have the **terminal condition**

$$V(T,S) = \psi(S)$$

where T is the expiration time and $\psi(S)$ is the payoff function, i.e. $\psi(S) = (S-K)^+$ for a call and $\psi(S) = (K-S)^+$ for a put.
The Black-Scholes equation must be solved for $S \in \mathbb{R}_+ := [0,\infty)$ because only non-negative prices make sense. At the boundary $S = 0$, no boundary condition is required, because

$$\lim_{S\to 0} \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) = 0, \qquad \lim_{S\to 0} rS\partial_S V(t,S) = 0$$

if V is sufficiently smooth. For $S = 0$, we obtain

$$0 = \partial_t V(t,0) - rV(t,0) \qquad \Longrightarrow \qquad V(t,0) = e^{-r(T-t)} V(T,0). \tag{3.5}$$

This yields $V(t,0) = 0$ for calls and $V(t,0) = e^{-r(T-t)}K$ for puts.

Remark. Surprisingly, the parameter μ from (3.2) does **not** appear in the Black-Scholes equation. A similar observation has been made for the simple discrete model from 1.5.

3.3 Black-Scholes formulas

First goal: Solve the Black-Scholes equation for an European call, i.e.

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$
$$V(T,S) = (S-K)^+$$

with parameters $r, \sigma, K, T > 0$.

Step 1: Transformation to the heat equation

Define new variables:

$$\begin{aligned} x(S) &= \ln(S/K) && x : (0,\infty) \longrightarrow (-\infty,\infty) \\ \tau(t) &= \frac{\sigma^2}{2}(T-t) && \tau : [0,T] \longrightarrow [0,\sigma^2 T/2] \\ w(\tau,x) &= \frac{V(t,S)}{K} && w : [0,\sigma^2 T/2] \times (-\infty,\infty) \longrightarrow \mathbb{R} \end{aligned}$$

Derivatives in new variables:

$$\partial_t V(t,S) = K\partial_t w(\tau,x) = K\partial_\tau w(\tau,x)\frac{d\tau}{dt} = -K\frac{\sigma^2}{2}\partial_\tau w(\tau,x)$$
$$\partial_S V(t,S) = K\partial_x w(\tau,x)\frac{dx}{dS} = \frac{K}{S}\partial_x w(\tau,x) \qquad \left(\text{because } \frac{dx}{dS} = \frac{1}{S/K}\cdot\frac{1}{K} = \frac{1}{S}\right)$$
$$\partial_S^2 V(t,S) = \quad \ldots \quad = \frac{K}{S^2}\left(\partial_x^2 w(\tau,x) - \partial_x w(\tau,x)\right)$$

Insert into the Black-Scholes equation:

$$\begin{aligned} 0 &= \partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) \\ &= -K\frac{\sigma^2}{2}\partial_\tau w(\tau,x) + \frac{\sigma^2}{2} S^2 \frac{K}{S^2}\left(\partial_x^2 w(\tau,x) - \partial_x w(\tau,x)\right) + rS\frac{K}{S}\partial_x w(\tau,x) - rKw(\tau,x) \end{aligned}$$

Divide by $K\frac{\sigma^2}{2}$ and let $c := \frac{2r}{\sigma^2}$:

$$\partial_\tau w(\tau,x) = \partial_x^2 w(\tau,x) - \partial_x w(\tau,x) + c\partial_x w(\tau,x) - cw(\tau,x)$$

Next, we eliminate the last three terms. Ansatz:

$$u(\tau,x) = e^{-\alpha x - \beta\tau} w(\tau,x), \qquad \alpha,\beta \in \mathbb{R}$$

Compute $\partial_\tau u(\tau,x)$:

$$\begin{aligned} \partial_\tau u(\tau,x) &= -\beta u(\tau,x) + e^{-\alpha x-\beta\tau}\partial_\tau w(\tau,x) \\ &= -\beta u(\tau,x) + e^{-\alpha x-\beta\tau}\left(\partial_x^2 w(\tau,x) + (c-1)\partial_x w(\tau,x) - cw(\tau,x)\right) \end{aligned}$$

Since

$$\partial_x w(\tau,x) = \partial_x\left(e^{\alpha x+\beta\tau}u(\tau,x)\right) = \alpha e^{\alpha x+\beta\tau}u(\tau,x) + e^{\alpha x+\beta\tau}\partial_x u(\tau,x)$$
$$\partial_x^2 w(\tau,x) = \alpha^2 e^{\alpha x+\beta\tau}u(\tau,x) + 2\alpha e^{\alpha x+\beta\tau}\partial_x u(\tau,x) + e^{\alpha x+\beta\tau}\partial_x^2 u(\tau,x)$$

it follows that

$$\begin{aligned}\partial_\tau u(\tau,x) =& -\beta u(\tau,x) + \Big(\alpha^2 u(\tau,x) + 2\alpha\partial_x u(\tau,x) + \partial_x^2 u(\tau,x)\Big)\\ &+ (c-1)\Big(\alpha u(\tau,x) + \partial_x u(\tau,x)\Big) - cu(\tau,x)\\ =&\partial_x^2 u(\tau,x) + \Big(2\alpha + (c-1)\Big)\partial_x u(\tau,x) + \Big(-\beta + \alpha^2 + (c-1)\alpha - c\Big)u(\tau,x)\end{aligned}$$

Hence, the terms including $u(\tau,x)$ and $\partial_x u(\tau,x)$ vanish if

$$-\beta + \alpha^2 + (c-1)\alpha - c = 0 \qquad \text{and} \qquad 2\alpha + (c-1) = 0.$$

The solution is

$$\alpha = -\frac{1}{2}(c-1), \qquad\qquad \beta = -\frac{1}{4}(c+1)^2 = -(1-\alpha)^2.$$

With these parameters, $u(\tau,x)$ solves the **heat equation**

$$\partial_\tau u(\tau,x) = \partial_x^2 u(\tau,x), \qquad x\in\mathbb{R}, \tau\in[0,\sigma^2T/2]$$

with initial condition

$$u(0,x) = e^{-\alpha x}w(0,x) = e^{-\alpha x}\frac{V(T,S)}{K} = e^{-\alpha x}\frac{(S-K)^+}{K} = e^{-\alpha x}(e^x-1)^+$$

since $S = Ke^x$.

Step 2: Solving the heat equation

Lemma 3.3.1 (solution of the heat equation) *Let $u_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function which satisifes the growth condition*

$$|u_0(x)| \le Me^{\gamma x^2}. \tag{3.6}$$

with constants $M > 0$ and $\gamma \ge 0$. If $\tau_{\max} \le 1/4\gamma$, then the function

$$u(\tau,x) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi$$

is the unique solution $u \in C([0,\tau_{\max})\times\mathbb{R}) \cap C^2((0,\tau_{\max})\times\mathbb{R})$ of the heat equation

$$\partial_\tau u(\tau,x) = \partial_x^2 u(\tau,x), \qquad x\in\mathbb{R}, \tau\in(0,\tau_{\max}] \tag{3.7}$$

and for all $x\in\mathbb{R}$ we have

$$\lim_{\tau\to 0} u(\tau,x) = u_0(x).$$

Proof. Straightforward computations yield

$$\begin{aligned}
\partial_\tau u(\tau,x) &= -\frac{1}{4\sqrt{\pi\tau^3}}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi \\
&\quad+\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\frac{(x-\xi)^2}{4\tau^2}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\frac{(x-\xi)^2}{4\tau^2}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi-\frac{1}{2\tau}u(\tau,x),
\end{aligned}$$

$$\partial_x u(\tau,x) = -\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\frac{x-\xi}{2\tau}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi,$$

$$\begin{aligned}
\partial_x^2 u(\tau,x) &= -\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\left(\frac{1}{2\tau}-\frac{(x-\xi)^2}{4\tau^2}\right)\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,d\xi \\
&= \frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\frac{(x-\xi)^2}{4\tau^2}\exp\left(-\frac{(x-\xi)^2}{4\tau}\right)u_0(\xi)\,ds-\frac{1}{2\tau}u(\tau,x) \\
&= \partial_\tau u(\tau,x).
\end{aligned}$$

Next, we show that $\lim_{\tau\to 0} u(\tau,x) = u_0(x)$. With the substitution

$$\eta(\xi) := \frac{\xi-x}{2\sqrt{\tau}}, \qquad \frac{d\eta}{d\xi} = \frac{1}{2\sqrt{\tau}}$$

we obtain

$$\begin{aligned}
\lim_{\tau\to 0} u(\tau,x) &= \lim_{\tau\to 0}\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}\exp(-\eta^2)u_0(x+2\eta\sqrt{\tau})2\sqrt{\tau}\,d\eta \\
&= \lim_{\tau\to 0}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\eta^2)u_0(x+2\eta\sqrt{\tau})\,d\eta \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\eta^2)u_0(x)\,d\eta \\
&= \frac{1}{\sqrt{\pi}}u_0(x)\int_{-\infty}^{\infty}\exp(-\eta^2)\,d\eta \\
&= u_0(x).
\end{aligned}$$

Uniqueness follows from the maximum principle. Here, the growth condition (3.6) is required. ∎

By a tedious[1] calculation, it can be shown that

$$u(\tau,x) = \exp\left((1-\alpha)x + (1-\alpha)^2\tau\right)\Phi(d_1) - \exp\left(-\alpha x + \alpha^2\tau\right)\Phi(d_2)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-s^2/2}\,ds \tag{3.8}$$

$$d_{1/2} = \frac{\ln\frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.9}$$

Remark: $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

Step 3: Inverse transform

Since $\beta = -(1-\alpha)^2$ and $\beta + \alpha^2 = 2\alpha - 1 = -c$ it follows that

$$\begin{aligned} V(t,S) &= Kw(\tau,x) = K\exp(\alpha x + \beta\tau)u(\tau,x) \\ &= K\exp(\alpha x + \beta\tau)\exp\left((1-\alpha)x + (1-\alpha)^2\tau\right)\Phi(d_1) \\ &\quad - K\exp(\alpha x + \beta\tau)\exp\left(-\alpha x + \alpha^2\tau\right)\Phi(d_2) \\ &= \underbrace{K\exp(x)}_{=S}\Phi(d_1) - K\exp\Big(\underbrace{(\beta+\alpha^2)}_{-c}\tau\Big)\Phi(d_2) \\ &= S\Phi(d_1) - K\exp(-r(T-t))\Phi(d_2) \end{aligned}$$

Check behaviour on the boundary: Since $\lim_{S\to 0}\Phi\big(d_{1/2}(S)\big) = 0$ we obtain

$$\lim_{S\searrow 0} V(t,S) = \lim_{S\searrow 0}\Big[S\Phi\big(d_1(S)\big) - Ke^{-r(T-t)}\Phi\big(d_2(S)\big)\Big] = 0 \iff (3.5)\ \checkmark$$

Check terminal condition:

$$V(T,S) = S\Phi(d_1) - K\Phi(d_2)$$

By definition of $d_{1/2} = d_{1/2}(t)$

$$\lim_{t\to T} d_{1/2}(t) = \lim_{t\to T}\frac{\ln\frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = \lim_{t\to T}\frac{\ln\frac{S}{K}}{\sigma\sqrt{T-t}} = \begin{cases}\infty & \text{if } S > K\\ 0 & \text{if } S = K\\ -\infty & \text{if } S < K\end{cases}$$

and hence

$$\lim_{t\to T}\Phi(d_{1/2}(t)) = \begin{cases}1 & \text{if } S > K\\ 1/2 & \text{if } S = K\\ 0 & \text{if } S < K\end{cases} \quad\Longrightarrow\quad \lim_{t\to T} V(t,S) = \begin{cases}S-K & \text{if } S > K\\ 0 & \text{if } S = K\\ 0 & \text{if } S < K\end{cases} \quad \checkmark$$

[1] ... so tedious that we do not even dare to ask the reader to prove this as an exercise.

All in all, we have shown the following

Theorem 3.3.2 (Black-Scholes formula for calls) *If $r, \sigma, K, T > 0$, then the* ***Black-Scholes formula***

$$V(t,S) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

with Φ and $d_{1/2}$ from (3.8) and (3.9), respectively, is the (unique) solution of the Black-Scholes equation for European calls, i.e.

$$\partial_t V(t,S) + \frac{\sigma^2}{2}S^2\partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$
$$V(T,S) = (S-K)^+.$$

Corollary 3.3.3 (Black-Scholes formula for puts) *The Black-Scholes equation for a European put*

$$\partial_t V(t,S) + \frac{\sigma^2}{2}S^2\partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$
$$V(T,S) = (K-S)^+$$

with $r, \sigma, K, T > 0$ has the unique solution

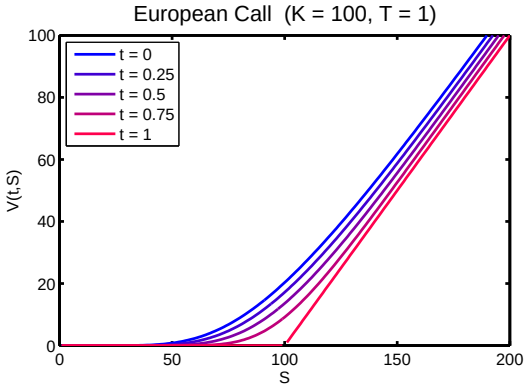
$$V(t,S) = Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1)$$

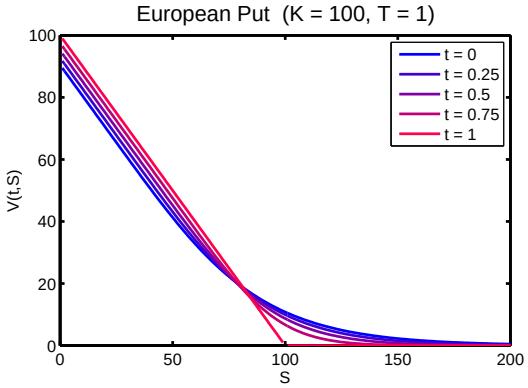
with Φ and $d_{1/2}$ from (3.8) and (3.9), respectively.

Proof. Let $V_C(t,S)$ be the value of a call with the same T and K. The put-call-parity (Lemma 1.4.1) and Theorem 3.3.2 imply

$$\begin{aligned} V(t,S) &= e^{-r(T-t)}K + V_C(t,S) - S \\ &= e^{-r(T-t)}K + S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2) - S \\ &= e^{-r(T-t)}K\big(1-\Phi(d_2)\big) + S\big(\Phi(d_1)-1\big) \\ &= e^{-r(T-t)}K\Phi(-d_2) - S\Phi(-d_1) \end{aligned}$$

because $\Phi(x) + \Phi(-x) = 1$. ∎





Definition 3.3.4 (Greeks) *For a European option with value $V(t,S)$ we define* ***"the greeks"***

delta:	Δ	$=$	$\partial_S V$	*theta:*	θ	$=$	$\partial_t V$
gamma:	Γ	$=$	$\partial_S^2 V$	*rho:*	ρ	$=$	$\partial_r V$
vega/kappa:	κ	$=$	$\partial_\sigma V$				

These partial derivatives can be considered as "condition numbers" which measure the sensitivity of $V(t,S)$ with respect to the corresponding parameters. This information is important for stock broker.

Remark: Explicit formulas for the greeks can be derived from the Black-Scholes formulas (exercise).

3.4 Risk-neutral valuation and equivalent martingale measures

In 1.5 we have seen that in the simplified two-scenario model the value of an option can be priced by replication. The same strategy was applied to the refined model in the previous section. In the simple situation considered in 1.5, the value of an option turned out to be the discounted expectation of the payoff under the risk-neutral probability. In this subsection, we will see that this is also true for the refined model from 3.2.

Theorem 3.4.1 (Option price as discounted expectation) *If $V(t,S)$ is the solution of the Black-Scholes equation*

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0 \qquad t \in [0,T], \quad S > 0$$
$$V(T,S) = \psi(S)$$

with payoff function $\psi(S)$, then

$$V(t_\star, S_\star) = e^{-r(T-t_\star)} \int_0^\infty \psi(x)\phi(x,\xi,\beta^2)\, dx \tag{3.10}$$

for all $t_\star \in [0,T]$ and $S_\star > 0$. The function ϕ is the density of the log-normal distribution (cf. Definition 3.1.2) with parameters

$$\xi = \ln S_\star + \left(r - \frac{\sigma^2}{2}\right)(T - t_\star), \qquad \beta^2 = \sigma^2(T - t_\star). \tag{3.11}$$

The assertion can be shown by showing that the above representation coincides with the Black-Scholes formulas for puts and calls. Such a proof, however, involves several changes

of variables in the integral representations and rather tedious calculations. We give a shorter and more elegant proof:

Proof. Step 1: In our derivation of the Black-Scholes model, we have assumed that

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

i.e. that the price of the underlying is a geometric Brownian motion with drift μS_t; cf. (3.2). It turned out, however, that the parameter μ does not appear in the Black-Scholes equation. Hence, we can choose $\mu = r$ and consider the SDE

$$\begin{aligned} d\widehat{S}_t &= r\widehat{S}_t dt + \sigma \widehat{S}_t dW_t, \qquad t \in [t_\star, T] \\ \widehat{S}_{t_\star} &= S_\star \end{aligned}$$

as a model for the stock price.
Step 2: The function $u(t,S) := e^{r(T-t)}V(t,S)$ solves the PDE

$$\partial_t u(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 u(t,S) + rS\partial_S u(t,S) = 0, \qquad t \in [0,T]$$

because

$$\begin{aligned} &\partial_t u(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 u(t,S) + rS\partial_S u(t,S) \\ =& -re^{r(T-t)}V(t,S) + e^{r(T-t)}\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 e^{r(T-t)}\partial_S^2 V(t,S) + rSe^{r(T-t)}\partial_S V(t,S) \\ =& e^{r(T-t)} \underbrace{\left(-rV(t,S) + \partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) \right)}_{=0 \quad \text{(Black-Scholes equation)}} = 0. \end{aligned}$$

Moreover, u satisfies the terminal condition

$$u(T,S) = V(T,S) = \psi(S).$$

Step 3: Applying the Feynman-Kac formula (cf. 2.7) with $f(t,S) = rS$ and $g(t,S) = \sigma S$ yields

$$\mathbb{E}\left(\psi(\widehat{S}_T)\right) = u(t_\star, S_\star) = e^{r(T-t_\star)}V(t_\star, S_\star)$$

and thus

$$V(t_\star, S_\star) = e^{-r(T-t_\star)}\mathbb{E}\left(\psi(\widehat{S}_T)\right).$$

We know that $\widehat{S}_T$ is log-normal, i.e.

$$\mathbb{E}\left(\psi(\widehat{S}_T)\right) = \int_0^\infty \psi(x)\phi(x,\xi,\beta^2)\, dx$$

where $\phi(x,\xi,\beta)$ is the density of the log-normal distribution with parameters (3.11); cf. the example after Definition 3.1.2. ∎

Interpretation. We know from Definition 3.1.2 that

$$\begin{aligned}\mathbb{E}\left(\widehat{S}_T\right) &= \int_0^\infty x\phi(x,\xi,\beta^2)\,dx \\ &= \exp\left(\xi + \frac{\beta^2}{2}\right) \\ &= \exp\left(\ln S_\star + \left(r - \frac{\sigma^2}{2}\right)(T-t_\star) + \frac{1}{2}\sigma^2(T-t_\star)\right) \\ &= \exp\left(\ln S_\star + r(T-t_\star)\right) \\ &= S_\star \exp\left(r(T-t_\star)\right).\end{aligned}$$

This means that for $\mu = r$ the expected value of the stock is exactly the money obtained by investing $S_\star$ into a bond at time $t_\star$ and waiting until T. Hence, the log-normal distribution with parameters (3.11) defines the **risk-neutral probability**; cf. 1.5. The integral in (3.10) is precisely the expected payoff under the risk-neutral probability, and (3.10) states that the price of the option is obtained by discounting the expected payoff.

A different perspective. Consider now the geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

with $\mu \neq r$. Since $\mathbb{E}(S_t) = S_0 e^{\mu t}$, an investor expects that $\mu > r$ as a compensation for the risk, because otherwise he might prefer to invest into the riskless bond $B_t = B_0 e^{rt}$. The term

$$\gamma = \frac{\mu - r}{\sigma}$$

is called **market price of risk**, and we have

$$dS_t = rS_t dt + \sigma S_t(\gamma\, dt + dW_t) = rS_t dt + \sigma S_t dW_t^\gamma, \qquad \text{with } W_t^\gamma = \gamma\, t + W_t.$$

Problem: W_t^γ is not a Wiener process under the probability measure $\mathbb{P}$, because $\mathbb{E}(W_t^\gamma) = \gamma\, t + \mathbb{E}(W_t) = \gamma\, t \neq 0$ for $t > 0$ and $\mu \neq r$.

Question: Is there another probability measure $\mathbb{Q}$ such that W_t^γ is a Wiener process under $\mathbb{Q}$?

Definition 3.4.2 (equivalent martingale measure) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\{\mathcal{F}_t : t \geq 0\}$. A probability measure $\mathbb{Q}$ is called an **equivalent martingale measure** or **risk-neutral probability** if there is a random variable $Y > 0$ such that*

- *$\mathbb{Q}(A) = \mathbb{E}(\mathbf{1}_A \cdot Y) = \int_A Y(\omega) d\mathbb{P}(\omega)$ for all events $A \in \mathcal{F}$, and*
- *$e^{-rt} S_t$ is a martingale under $\mathbb{Q}$ with respect to the filtration $\{\mathcal{F}_t : t \geq 0\}$.*

Remark. The first property implies that $\mathbb{P}(A) > 0 \iff \mathbb{Q}(A) > 0$ ("equivalent").

Now let

$$Y_T := \exp\left(-\gamma W_T - \frac{\gamma^2}{2}T\right) \qquad \text{and} \qquad \mathbb{Q}(A) = \mathbb{E}(\mathbf{1}_A \cdot Y_T).$$

Then, Girsanov's theorem states that $W_t^\gamma = \gamma\, t + W_t$ is a Wiener process under $\mathbb{Q}$ (see e.g. 4.4 in [Ben04], 8.6 in [Øks03]). Moreover, the Itô formula yields

$$d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt}\Big(rS_t dt + \sigma S_t dW_t^\gamma\Big) = \sigma e^{-rt}S_t dW_t^\gamma.$$

Hence, $e^{-rt}S_t$ is a martingale under $\mathbb{Q}$, and $\mathbb{Q}$ is an equivalent martingale measure. All in all, we have exchanged

$$\mu \longrightarrow r, \qquad \mathbb{P} \longrightarrow \mathbb{Q}, \qquad W_t \longrightarrow W_t^\gamma.$$

Now we are back in the situation of Theorem 3.4.1, and it follows that

$$V(t_\star, S_\star) = e^{-r(T-t_\star)}\mathbb{E}_{\mathbb{Q}}\left(\psi(S_T)\right) \tag{3.12}$$

where S_t is the solution of

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t^\gamma, \qquad t \in [t_\star, T] \\ S_{t_\star} &= S_\star. \end{aligned}$$

If $\mu = r$, then $\mathbb{P} = \mathbb{Q}$ and $W_t = W_t^\gamma$.

General pricing formula. Up to now, we have only considered European options, i.e. options with a payoff that depends only on the value of the underlying at maturity. For Asian or barrier options, the pricing formula (3.12) can be generalized to

$$V_t = e^{-r(T-t_\star)}\mathbb{E}_{\mathbb{Q}}\left(V_T|\mathcal{F}_t\right)$$

(cf. 5.2.4 in [Shr04]).

Fundamental theorems of option pricing:

- If a market model has at least one equivalent martingale measure, then there is no arbitrage possibility (cf. Theorem 5.4.7 in [Shr04]).
- Consider a market model with at least one equivalent martingale measure. Then, the equivalent martingale measure is unique if and only if the model is complete, i.e. if every derivative (options, forwards, futures, swaps, ...) can be replicated (hedged) (cf. Theorem 5.4.9 in [Shr04]).

3.5 Extensions

The "standard" Black-Scholes model can be generalized in several ways:

- Options with $d > 1$ underlyings (e.g. basket options) are modeled by the d-dimensional Black-Scholes equation

$$\partial_t V + \frac{1}{2}\sum_{i,j=1}^{d} \rho_{ij}\sigma_i\sigma_j S_i S_j \partial_{S_i}\partial_{S_j} V + r\sum_{i=1}^{d} S_i \partial_{S_i} V - rV = 0$$

  where $V = V(t, S_1, \ldots, S_d)$, $r, \sigma_i > 0$ and $\rho_{ij} \in [-1, 1]$ are the correlation coefficients.
- Non-constant interest rate and volatility: $r = r(t, S)$, $\sigma = \sigma(t, S)$
- Stochastic volatility: Either $\sigma = \sigma(\omega)$ is a random variable with known distribution or $\sigma = \sigma_t(\omega)$ is a stochastic process.
- Dividends: When a dividend $\delta \cdot S_t$ with $\delta \geq 0$ is paid at time t, the price of the underlying drops by the same amount due to the no-arbitrage assumption. Hence, a **continuous** flow of dividends can be modeled by

$$dS_t = (\mu - \delta)S_t dt + \sigma S_t dW_t,$$

  which yields the Black-Scholes equation

$$\partial_t V(t, S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t, S) + (r - \delta) S \partial_S V(t, S) - rV(t, S) = 0.$$

  Black-Scholes formulas with dividends: 4.5.1 in [GJ10].
- Nonzero transaction costs $\Longrightarrow$ nonlinear Black-Scholes equation
- Discontinuous underlyings: Jump-diffusion models, Black-Scholes PDE with additional integral term

Remark: Some of these extensions will be considered in the lecture *Numerical methods in mathematical finance II* (summer term).

Chapter 4

Binomial methods

Situation: Let $S(t)$ be the value of an underlying, and let $V(t,S)$ be the value of an option with maturity $T > 0$.

Assumptions: Assume (A1)-(A5) from Section 1.3.

Goal: Approximate $V(t,S)$ by a numerical method.

Idea: Refine the simple discrete model from 1.5 such that it approximates the continuous-time Black-Scholes model.

Remark: For European calls/puts, such an approximation is not necessary, because $V(t,S)$ can be computed via the Black-Scholes formula. Nevertheless, such options will serve as a model problem. The numerical method can be extended to other types of options.

4.1 Derivation

Discretize the time-interval $[0,T]$: Choose $N \in \mathbb{N}$, let $\tau = T/N$ and $t_n = n \cdot \tau$.
Let S_n be the price of the underlying at time t_n. Bond: $B(t) = B(0)e^{rt}$
Additional assumptions:

1. For a given price S_n, the price at $t_{n+1} = t_n + \tau$ is

$$S_{n+1} = \begin{cases} u \cdot S_n & \text{with probability } p \\ d \cdot S_n & \text{with probability } 1-p \end{cases}$$

   with (unknown) $u > 1$, $0 < d < 1$, $p \in [0,1]$.

2. The expected profit from investing into the underlying is the same as for the bond:

$$\mathbb{E}\big(S_{n+1}|S_n\big) = e^{r\tau} S_n.$$

3. $\mathbb{E}\big(S_{n+1}^2|S_n\big) = e^{(2r+\sigma^2)\tau} S_n^2$ with given volatility $\sigma \in \mathbb{R}$.

Remark: In the continuous-time model where $S(t)$ is modeled by a geometric Brownian motion, the last two conditions hold for $S_n = S(t_n)$ if $\mu = r$ (risk-neutral pricing).

Compute u, d, p:

1. $e^{r\tau} S_n = \mathbb{E}\big(S_{n+1}|S_n\big) = uS_n p + dS_n(1-p) \Longleftrightarrow p = \dfrac{e^{r\tau} - d}{u - d}$
   Since $p \in [0,1]$, it follows that $d \le e^{r\tau} \le u$.
2. $e^{(2r+\sigma^2)\tau} S_n^2 = \mathbb{E}\big(S_{n+1}^2|S_n\big) = \big(uS_n\big)^2 p + \big(dS_n\big)^2 (1-p)$

Only two conditions for three unknowns u, d, p. Choose third condition:

3. $u \cdot d \overset{!}{=} 1$

Solution of 1.-3.:

$$\begin{aligned} u &= \beta + \sqrt{\beta^2 - 1} & \beta &:= \frac{1}{2}\left(e^{-r\tau} + e^{(r+\sigma^2)\tau}\right) \\ d &= \frac{1}{u} = \beta - \sqrt{\beta^2 - 1} & p &= \frac{e^{r\tau} - d}{u - d}. \end{aligned} \tag{4.1}$$

Replication strategy: Consider a portfolio containing $a \in \mathbb{R}$ underlyings and $b \in \mathbb{R}$ bonds such that

$$aS_n + bB(t_n) \overset{!}{=} V\big(t_n, S_n\big) =: V_n$$

It follows that

$$\begin{aligned} \mathbb{E}\big(V_{n+1} \mid V_n\big) &= a\mathbb{E}(S_{n+1} \mid S_n) + be^{r\tau} B(t_n) \\ &= e^{r\tau}\big(aS_n + bB(t_n)\big) \\ &= e^{r\tau} V_n \end{aligned} \tag{4.2}$$

4.2 Algorithm

Cox, Ross & Rubinstein 1979

1. Forward phase: initialization of the tree. For all $n = 0, \dots, N$ and $j = 0, \dots, n$ let

$$S_{jn} = u^j d^{n-j} S(0) = \text{(approximate) price of the underlying at time } t_n \text{ after } j \text{ "ups" and } n-j \text{ "downs".}$$

The condition $d \cdot u = 1$ implies that

$$\begin{aligned} S(0) = S_{00} = S_{12} = S_{24} = \dots \\ S_{11} = S_{23} = S_{35} = \dots \\ S_{01} = S_{13} = S_{25} = \dots \end{aligned}$$

At t_n there are only $n+1$ possible values $S_{0n}, \dots, S_{nn}$ of the underlying.

```
S_00 = S(0)
for n = 0, 1, 2, ..., N − 1
    S_{0,n+1} = d S_{0,n}
    for j = 0, ..., n
        S_{j+1,n+1} = u S_{j,n}
    end
end
```

2. Backward phase: compute the option values. Let V_{jn} be the value of the option after j "ups" and $n-j$ "downs" of the underlying. At maturity, we have

$$V_{jN} = \psi(S_{jN}), \qquad \psi(S_{jN}) = \begin{cases} (S_{jN} - K)^+ & \text{(call)} \\ (K - S_{jN})^+ & \text{(put)} \end{cases}$$

Use (4.2):

$$\begin{aligned} & e^{r\tau} V_{jn} = \mathbb{E}\big(V(t_{n+1}) \mid V_{jn}\big) = pV_{j+1,n+1} + (1-p)V_{j,n+1} \\ \Longrightarrow \qquad & V_{jn} = e^{-r\tau}\Big(pV_{j+1,n+1} + (1-p)V_{j,n+1}\Big) \end{aligned}$$

(a) European options:

```
for j = 0, ..., N
    V_jN = ψ(S_jN)
end
for n = N − 1, N − 2, ..., 0
    for j = 0, ..., n
        V_jn = e^{−rτ}( p V_{j+1,n+1} + (1 − p) V_{j,n+1} )
    end
end
Result:  V_00
```

(b) American options: Check in each step if early exercise is advantageous. At time t_n the value of the option must not be less than $\psi(S_{jn})$ for all j.

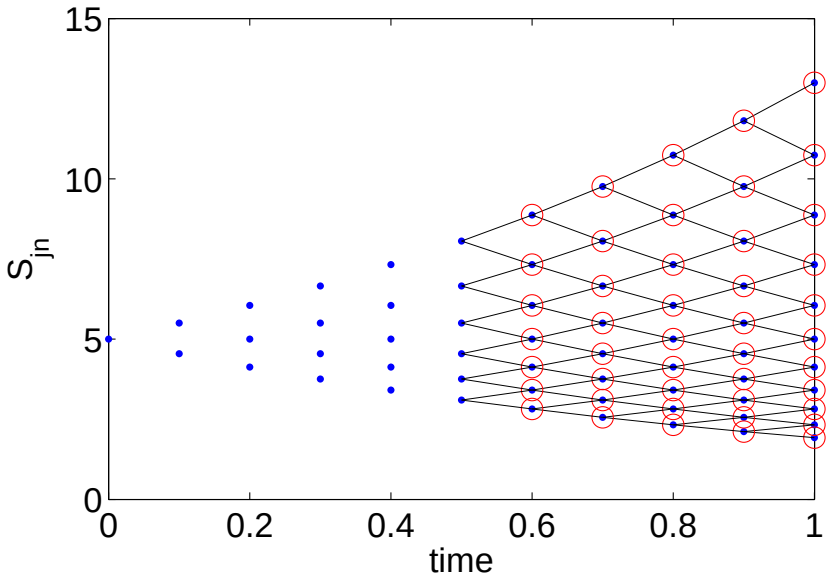


Figure 4.1: Illustration of the binomial method ($S(0) = 5$, $T = 1$, $N = 10$).

```
for j = 0, ..., N
    V_jN = ψ(S_jN)
end
for n = N − 1, N − 2, ..., 0
    for j = 0, ..., n
        Ṽ_jn = e^{−rτ} ( p V_{j+1,n+1} + (1 − p) V_{j,n+1} )
        V_jn = max{Ṽ_jn, ψ(S_jn)}
    end
end
Result:  V_00
```

Remarks.

1. The result V_{00} is only an **approximation** for the true value $V(0, S(0))$ of the option, because the price process has been approximated.
2. The result $V_{00} \approx V(0, S(0))$ depends on the initial value $S(0)$. For a different value of $S(0)$, the entire computation must be repeated.
3. An efficient implementation of the binomial method requires only $\mathcal{O}(N)$ operations; see [Hig02].

Examples: see slides

4.3 Discrete Black-Scholes formula

Lemma 4.3.1 *Let $V(t,S)$ be the value of a European option with payoff function $\psi(S)$ and maturity $T>0$. Then, the binomial method yields the approximation*

$$V_{00} = e^{-rT} \sum_{j=0}^{N} \mathcal{B}(j,N,p)\psi(S_{jN})$$

where

$$\mathcal{B}(j,N,p) = \binom{N}{j} p^j (1-p)^{N-j}$$

is the binomial distribution.

Proof: exercise.

Remarks:

1. This result explains the name "binomial method".
2. Interpretation: V_{00} is the discounted expected payoff under a suitable probability; cf. 1.5 and 3.4.

Proposition 4.3.2 (Discrete Black-Scholes formula) *For a European call with maturity T and strike K, the binomial method yields the approximation*

$$V_{00} = S(0)\Psi(m,N,q) - Ke^{-rT}\Psi(m,N,p)$$

where

$$\begin{aligned} q &= upe^{-r\tau} \\ m &= \min\{0 \le j \le N : (S_{jN} - K) \ge 0\}. \\ \Psi(m,N,p) &= \sum_{j=m}^{N} \mathcal{B}(j,N,p) \end{aligned}$$

Proof: exercise.

Question: What happens if we let $N \longrightarrow \infty$ and $\tau = T/N \longrightarrow 0$?

For simplicity, we consider a slightly different binomial method. For given $\sigma > 0$ let

$$\hat{u} = \hat{u}(\tau) = e^{\sigma\sqrt{\tau}} \qquad \text{and} \qquad \hat{d} = \hat{d}(\tau) = \frac{1}{\hat{u}} = e^{-\sigma\sqrt{\tau}} \tag{4.3}$$

and suppose that u and d from (4.1) are replaced by $\hat{u}$ and $\hat{d}$. It can be shown, however, that

$$\hat{u}(\tau) - u = \mathcal{O}\left(\tau^{3/2}\right). \tag{4.4}$$

If $\tau \le (\sigma/r)^2$, then we have $\hat{d} \le e^{r\tau} \le \hat{u}$ as before.

Proposition 4.3.3 (Convergence of the discrete Black-Scholes formula)
Consider a European call with maturity T and strike K. Let $V_{00} = V_{00}^{(N)}$ be the approximation given by the binomial method with $\tau = T/N$, and for u and d replaced by $\hat{u}$ and $\hat{d}$, respectively. Then, $V_{00}^{(N)}$ converges to the value given by the (continuous) Black-Scholes formula:

$$\lim_{N\to\infty} V_{00}^{(N)} = S(0)\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds$$

$$d_{1/2} = \frac{\ln\left(\frac{S(0)}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}$$

Proof: See 3.3 in [GJ10] (use central limit theorem).

Chapter 5

Numerical methods for stochastic differential equations

5.1 Motivation

According to 3.4 the value of a European option is the discounted expected payoff under the risk-neutral probability:

$$V(0, S_0) = e^{-rT}\mathbb{E}_{\mathbb{Q}}\Big(\psi\big(S(T)\big)\Big)$$

For the standard Black-Scholes model:

$$V(0, S_0) = e^{-rT} \int_0^\infty \psi(x)\phi(x, \xi, \beta^2)\, dx$$

with log-normal density ϕ and parameters

$$\xi = \ln S_0 + \left(r - \frac{\sigma^2}{2}\right) T, \qquad \beta = \sigma\sqrt{T}.$$

Two ways to price the option:

1. **Quadrature formula.** Let $w(x) := \psi(x)\phi(x, \xi, \beta^2)$.
   Choose $0 \leq x_{min} < x_{\max}$ such that $w(x) \approx 0$ for $x \notin [x_{\min}, x_{\max}]$.
   $x_{\min} = K$ and $x_{\max}$ sufficiently large for calls, $x_{\min} = 0$ and $x_{\max} = K$ for puts.
   Choose large $N \in \mathbb{N}$, let $h = (x_{\max} - x_{\min})/N$ and $x_k = x_{\min} + kh$. Approximate

   $$\int_0^\infty w(x)\, dx \approx \int_{x_{\min}}^{x_{\max}} w(x)\, dx = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} w(x)\, dx \approx \sum_{k=0}^{N-1} h \sum_{j=1}^{s} b_j w(x_k + c_j h)$$

   with suitable nodes $c_j \in [0, 1]$ and weights b_j.

2. **Monte Carlo method.** In the Black-Scholes model, $S(t)$ is defined by the SDE

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \qquad t \in [0,T], \qquad S_0 \text{ given}$$

(risk-neutral, $\mu = r$)
Solution: Geometric Brownian motion

$$S(t) = S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right).$$

This is the process which corresponds to $\phi(x, \xi, \beta^2)$, because $S(T)$ is log-normal with the same parameters. Estimate the expectated payoff as follows:

- Generate many realizations $S(T,\omega_1), \ldots, S(T,\omega_m)$, $m \in \mathbb{N}$ "large".
- Approximate

$$V(0,S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\psi\big(S(T,\omega_j)\big)$$

Consider now a more complicated price process:

$$dS(t) = rS(t)dt + \sigma(t)S(t)dW^1(t) \tag{5.1a}$$

$$d\sigma^2(t) = \kappa\big(\theta - \sigma^2(t)\big)dt + \nu\left(\rho dW^1(t) + \sqrt{1-\rho^2}dW^2(t)\right) \tag{5.1b}$$

Heston model with parameters $r, \kappa, \theta, \nu > 0$, initial values S_0, σ_0, independent scalar Wiener processes $W^1(t)$, $W^2(t)$, correlation $\rho \in [-1,1]$
Steven L. Heston 1993
Now the volatility is not a parameter, but a stochastic process defined by a second SDE. We do not have an explicit formula for $S(t)$ and $\sigma(t)$, but the Monte Carlo approach is still feasible:

- Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$. For each $\omega_1, \ldots, \omega_m$ compute approximations

$$X_n^{(1)}(\omega_j) \approx S(t_n,\omega_j), \qquad X_n^{(2)}(\omega_j) \approx \sigma^2(t_n,\omega_j), \qquad n = 0,\ldots,N$$

by solving the SDEs (5.1a), (5.1b) numerically.

- Approximate

$$V(0,S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\psi\big(\underbrace{X_N^{(1)}(\omega_j)}_{\approx S(T,\omega_j)}\big)$$

The Monte Carlo approach even works for other types of options. As an example, consider an Asian option with payoff

$$\psi\big(t \mapsto S(t)\big) = \left(S(T) - \frac{1}{T}\int_0^T S(t)\,dt\right)^+ \qquad \text{(average strike call).}$$

Now the payoff depends on the entire path $t \mapsto S(t)$. We approximate

$$S(t_n, \omega_j) \approx X_n^{(1)}(\omega_j), \qquad \frac{1}{T}\int_0^T S(t, \omega_j)\, dt \approx \frac{1}{N+1}\sum_{n=0}^{N} X_n^{(1)}(\omega_j)$$

and hence

$$V(0, S_0) \approx e^{-rT}\frac{1}{m}\sum_{j=1}^{m}\left(X_N^{(1)}(\omega_j) - \frac{1}{N+1}\sum_{n=0}^{N} X_n^{(1)}(\omega_j)\right)^+$$

Remark: In the original paper, Heston derives an explicit Black-Scholes-type formula for European options by means of characteristic functions. Hence, European options in the Heston model can also be priced by quadrature formulas, but for Asian options this is impossible.

Goal: Construct and analyze numerical methods for SDEs.

5.2 Euler-Maruyama method

5.2.1 Derivation

Consider the one-dimensional SDE

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t), \qquad t \in [0, T], \qquad X(0) = X_0$$

with suitable functions f and g and a given initial value X_0. Choose $N \in \mathbb{N}$, define step-size $\tau = T/N$ and $t_n = n\tau$.

$$\begin{aligned} X(t_{n+1}) &= X(t_n) + \int_{t_n}^{t_{n+1}} f\big(s, X(s)\big)\, ds + \int_{t_n}^{t_{n+1}} g\big(s, X(s)\big)dW(s) \\ &\approx X(t_n) + \underbrace{(t_{n+1} - t_n)}_{=\tau} f\big(t_n, X(t_n)\big) + g\big(t_n, X(t_n)\big)\underbrace{\big(W(t_{n+1}) - W(t_n)\big)}_{=:\Delta W_n} \end{aligned}$$

Replacing $X(t_n) \longrightarrow X_n$ and "$\approx$" $\longrightarrow$ "$=$" yields the

Euler-Maruyama method (Gisiro Maruyama 1955, Leonhard Euler 1768-70):
For $n = 0, \ldots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$X_{n+1} = X_n + \tau f\big(t_n, X_n\big) + g\big(t_n, X_n\big)\Delta W_n.$$

If $g(t, x) \equiv 0$, then the Euler-Maruyama method reduces to the explicit Euler method applied to the ordinary differential equation $X'(t) = f\big(t, X(t)\big)$.

Hope that $X_n \approx X(t_n)$:

	SDE		recursion
$X(t)$	exact	$\Longrightarrow$	approx.
X_n	approx.	$\overset{?}{\Longleftarrow}$	exact

The exact solution $X(t_n)$ and the numerical approximation X_n are random variables. For every path $t \mapsto W(t,\omega)$ of the Wiener process, a different result is obtained. $X(t)$ is called **strong solution** if $t \mapsto W(t,\omega)$ is given, and **weak solution** if $t \mapsto W(t,\omega)$ can be chosen. Approximations of weak solutions: For each n, generate a random number $Z_n \sim \mathcal{N}(0,1)$ and let

$$\Delta W_n = \sqrt{\tau} Z_n.$$

Question: Does X_n really approximate $X(t_n)$? In which sense? How accurately?

5.2.2 Weak and strong convergence

Definition 5.2.1 (strong and weak convergence) *Let $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$ and $t_n = n\tau$. An approximation $X_n(\omega) \approx X(t_n,\omega)$ converges*

- ***strongly** with order $\gamma > 0$, if there is a constant $C > 0$ independent of τ such that*

$$\max_{n=0,\dots,N} \mathbb{E}\Big(|X(t_n) - X_n|\Big) \leq C\tau^\gamma$$

  for all sufficiently small τ, and

- ***weakly** with order $\gamma > 0$ with respect to a function $F : \mathbb{R} \longrightarrow \mathbb{R}$, if there is a constant $C > 0$ independent of τ such that*

$$\max_{n=0,\dots,N} \Big|\mathbb{E}\big[F\big(X(t_n)\big)\big] - \mathbb{E}\big[F(X_n)\big]\Big| \leq C\tau^\gamma$$

  for all sufficiently small τ.

Remarks:

- Strong convergence $\Longrightarrow$ path-wise convergence
  Weak convergence $\Longrightarrow$ convergence of moments (if $F(x) = x^k$) or probabilities (if $F(x) = \mathbf{1}_{[a,b]}$).
- Strong convergence of order γ implies weak convergence of order $\geq \gamma$ with respect to $F(x) = x$ (exercise).

5.2.3 Strong convergence of the Euler-Maruyama method

For simplicity, we only consider the autonomous SDE

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dW(t), \qquad t \in [0,T]$$

and the Euler-Maruyama approximation

$$X_{n+1} = X_n + \tau f\big(X_n\big) + g\big(X_n\big)\Delta W_n.$$

with $X(0) = X_0$, $T > 0$, $N \in \mathbb{N}$, $\tau = T/N$, $t_n = n\tau$.
We assume that $f = f(x)$ and $g = g(x)$ satisfy the Lipschitz condition (2.12): There is a constant $L \geq 0$ such that

$$|f(x) - f(y)| \leq L|x-y|, \qquad |g(x) - g(y)| \leq L|x-y| \tag{2.12}$$

for all $x, y \in \mathbb{R}$. In the autonomous case, this implies the linear growth condition (2.13) (exercise).

Theorem 5.2.2 (strong error of the Euler-Maruyama method) *Under these conditions, there is a constant $\hat{C}$ such that*

$$\max_{n=0,\dots,N} \mathbb{E}\Big(|X(t_n) - X_n|\Big) \leq \hat{C}\tau^{1/2}$$

for all sufficiently small τ. $\hat{C}$ does not depend on τ.

For the proof we need the following

Lemma 5.2.3 (Gronwall) *Let $\alpha : [0,T] \longrightarrow \mathbb{R}_+$ be a positive integrable function. If there are constants $a > 0$ and $b > 0$ such that*

$$0 \leq \alpha(t) \leq a + b\int_0^t \alpha(s)\, ds$$

for all $t \in [0,T]$, then $\alpha(t) \leq ae^{bt}$.

Proof: exercise.

Proof of Theorem 5.2.2.
Strategy:

- Define the step function

$$Y(t) = \sum_{n=0}^{N-1} \mathbf{1}_{[t_n,t_{n+1})}(t)X_n \qquad \text{for } t \in [0,T), \qquad Y(T) := X_N.$$

  For $n = 0, \dots, N-1$ this means that

$$Y(t) = X_n \quad \Longleftrightarrow \quad t \in [t_n, t_{n+1}).$$

- Define $\alpha(s) := \sup_{r\in[0,s]} \mathbb{E}\big(|Y(r) - X(r)|^2\big)$ and prove the Gronwall inequality

$$0 \le \alpha(t) \le C\tau + b \int_0^t \alpha(s)\, ds. \tag{5.2}$$

- Apply Gronwall's lemma. This yields $\alpha(t) \le \tau \hat{C}^2$ with $\hat{C}^2 = Ce^{bT}$ for all $t \in [0,T]$
- Since[1] $\mathbb{E}(Z) \le \sqrt{\mathbb{E}(Z^2)}$ for random variables Z, it follows that

$$\begin{aligned}\max_{n=0,\dots,N} \mathbb{E}\big(|X_n - X(t_n)|\big) &\le \sup_{t\in[0,T]} \mathbb{E}\big(|Y(t) - X(t)|\big) \\ &\le \sup_{t\in[0,T]} \sqrt{\mathbb{E}\big(|Y(t) - X(t)|^2\big)} \\ &= \sqrt{\alpha(T)} \le \sqrt{\tau}\hat{C}\end{aligned}$$

Main challenge: Prove Gronwall inequality (5.2). Choose fixed $t \in [0,T]$ and let n be the index with $t \in [t_n, t_{n+1})$.

Derive integral representation of the error:

$$\begin{aligned}Y(t) = X_n &= X_0 + \sum_{k=0}^{n-1} (X_{k+1} - X_k) \\ &= X_0 + \sum_{k=0}^{n-1} \Big(\tau f(X_k) + g(X_k)\Delta W_k\Big) \\ &= X_0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f(X_k)\, ds + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} g(X_k)\, dW(s) \\ &= X_0 + \int_0^{t_n} f\left(Y(s)\right)\, ds + \int_0^{t_n} g\left(Y(s)\right)\, dW(s)\end{aligned}$$

This is *not* an SDE, because we have $\int_0^{t_n} \dots$ instead of $\int_0^t \dots$. Comparing with the exact solution

$$X(t) = X(0) + \int_0^t f\big(X(s)\big) ds + \int_0^t g\big(X(s)\big) dW(s)$$

[1] Elementary calculation: $0 \le \mathbb{V}(Z) = \mathbb{E}\left[(Z - \mathbb{E}(Z))^2\right] = \mathbb{E}\left[Z^2 - 2Z\mathbb{E}(Z) + \mathbb{E}(Z)^2\right] = \mathbb{E}\left(Z^2\right) - \big(\mathbb{E}(Z)\big)^2$ and hence $\big(\mathbb{E}(Z)\big)^2 \le \mathbb{E}\left(Z^2\right)$.

yields the error representation

$$\begin{aligned} Y(t) - X(t) &= \underbrace{\int_0^{t_n} \Big[f\left(Y(s)\right) - f\big(X(s)\big) \Big] \, ds}_{=:\mathcal{T}_1} + \underbrace{\int_0^{t_n} \Big[g\left(Y(s)\right) - g\big(X(s)\big) \Big] \, dW(s)}_{=:\mathcal{T}_2} \\ &\quad - \underbrace{\int_{t_n}^{t} f\big(X(s)\big) ds}_{=:\mathcal{T}_3} - \underbrace{\int_{t_n}^{t} g\big(X(s)\big) dW(s)}_{=:\mathcal{T}_4} \\ &= \mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_4. \end{aligned}$$

Setting $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4)^T$ and applying the Cauchy-Schwarz inequality gives

$$(\mathcal{T}_1 + \mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_4)^2 = \big((1,1,-1,-1)\mathcal{T}\big)^2 \leq 4\|\mathcal{T}\|_2^2 = 4 \cdot (\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 + \mathcal{T}_4^2)$$

and hence

$$\mathbb{E}\big(|Y(t) - X(t)|^2\big) \leq 4 \cdot \mathbb{E}\left(\mathcal{T}_1^2 + \mathcal{T}_2^2 + \mathcal{T}_3^2 + \mathcal{T}_4^2\right).$$

First term: For functions $u \in L^2([0, t_n])$ the Cauchy-Schwarz inequality yields

$$\left(\int_0^{t_n} u(s) \cdot 1 \, ds \right)^2 \leq \int_0^{t_n} |u(s)|^2 \, ds \cdot \underbrace{\int_0^{t_n} 1^2 \, ds}_{=t_n}. \tag{5.3}$$

Using the Lipschitz bound (2.12), we obtain

$$\begin{aligned} \mathbb{E}\left(\mathcal{T}_1^2\right) &= \mathbb{E}\left[\left(\int_0^{t_n} \Big[f\left(Y(s)\right) - f\big(X(s)\big) \Big] \, ds \right)^2 \right] \\ &\leq t_n \mathbb{E}\left(\int_0^{t_n} \Big| f\left(Y(s)\right) - f\big(X(s)\big) \Big|^2 \, ds \right) \\ &\leq TL^2 \int_0^{t_n} \mathbb{E}\left(\big|Y(s) - X(s)\big|^2 \right) ds \\ &\leq TL^2 \int_0^{t} \alpha(s) \, ds \qquad (t \text{ instead of } t_n) \end{aligned}$$

because $t \geq t_n$ by assumption.

Second term: It follows from the Itô isometry (Theorem 2.3.5) and the Lipschitz bound (2.12) that

$$\begin{aligned}
\mathbb{E}\left(\mathcal{T}_2^2\right) &= \mathbb{E}\left[\left(\int_0^{t_n} \left[g\left(Y(s)\right) - g\big(X(s)\big)\right] dW(s)\right)^2\right] \\
&= \mathbb{E}\left(\int_0^{t_n} \left|g\left(Y(s)\right) - g\big(X(s)\big)\right|^2 ds\right) \\
&\leq L^2 \int_0^{t_n} \mathbb{E}\left(\left|Y(s) - X(s)\right|^2\right) ds \\
&\leq L^2 \int_0^{t} \alpha(s)\, ds
\end{aligned}$$

because $t \geq t_n$ by assumption.

Third term: Equation (5.3) and the linear growth bound (2.13) yield

$$\begin{aligned}
\mathbb{E}\left(\mathcal{T}_3^2\right) &= \mathbb{E}\left[\left(\int_{t_n}^{t} f\big(X(s)\big) ds\right)^2\right] \\
&\leq (t - t_n)\mathbb{E}\left(\int_{t_n}^{t} \left|f\big(X(s)\big)\right|^2 ds\right) \\
&\leq \tau K \cdot \mathbb{E}\left(\int_{t_n}^{t} \left(1 + |X(s)|^2\right) ds\right) \leq c\tau^2
\end{aligned}$$

because Theorem 2.6.3 states that $\mathbb{E}\left(1 + |X(s)|^2\right)$ remains bounded on $[t_n, t]$.

Last term: Using the Itô isometry and the linear growth bound (2.13) it follows that

$$\begin{aligned}\mathbb{E}\left(\mathcal{T}_4^2\right) &= \mathbb{E}\left[\left(\int_{t_n}^{t} g\big(X(s)\big)dW(s)\right)^2\right] \\ &= \mathbb{E}\left(\int_{t_n}^{t} \big|g\big(X(s)\big)\big|^2 ds\right) \\ &\leq K\cdot\mathbb{E}\left(\int_{t_n}^{t}\left(1+|X(s)|^2\right)ds\right) \leq c\tau\end{aligned}$$

These bounds yield the Gronwall inequality (5.2) with $b = 4(T+1)L^2$ and with C depending on K and $\sup_{s\in[0,T]}\mathbb{E}\left(1+|X(s)|^2\right)$. ■

5.2.4 Weak convergence of the Euler-Maruyama method

Theorem 5.2.4 (weak error of the Euler-Maruyama method) *Under the conditions of 5.2.3, there is a constant $\hat{C}$ such that*

$$\max_{n=0,\dots,N}\left|\mathbb{E}\big[F\big(X(t_n)\big)\big] - \mathbb{E}\big[F(X_n)\big]\right| \leq \hat{C}\tau$$

for all sufficiently small τ and all smooth functions F. $\hat{C}$ does not depend on τ.

Proof. Define piecewise linear interpolation: In addition to the piecewise constant $Y(t)$, we define the piecewise linear interpolation

$$Z(t) = X_n + (t-t_n)f\big(X_n\big) + g\big(X_n\big)\big(W(t)-W(t_n)\big) \qquad \text{for } t\in[t_n,t_{n+1}).$$

Since $Y(t) = X_n$ for $t\in[t_n,t_{n+1})$, this is equivalent to

$$Z(t) = X(0) + \int_0^t f\big(Y(s)\big)ds + \int_0^t g\big(Y(s)\big)dW(s)$$

$$\textbf{or} \qquad dZ = f(Y)dt + g(Y)dW(t).$$

Properties:

- $Z(t_n) = X_n = Y(t_n)$ for all $n = 0,\dots,N$.
- For every $\delta\in[0,\tau]$, $Z(t_n+\delta)$ is the Euler-Maruyama approximation after one step with step-size δ and initial value $Z(t_n) = Y_n$.
- $t\mapsto Z(t,\omega)$ is continuous with probability 1.

Choose $n \in \{1, \dots, N\}$ and consider the error at time t_n.
Apply the Feynman-Kac formula: Let $u(t,x)$ be the solution of the PDE

$$\partial_t u(t,x) + f(x)\partial_x u(t,x) + \frac{1}{2}g^2(x)\partial_x^2 u(t,x) = 0, \qquad t \in [0,t_n], \qquad x \in \mathbb{R}$$

with terminal condition $u(t_n,x) = F(x)$. Apply the Itô formula to $u(t,Z(t))$:

$$\begin{aligned} du(t,Z) &= \Big(\underbrace{\partial_t u(t,Z)}_{= \ \dots \text{ (PDE)}} + f(Y)\partial_x u(t,Z) + \frac{1}{2}g^2(Y)\partial_x^2 u(t,Z) \Big)\, dt \\ &\quad + g(Y)\partial_x u(t,Z)\, dW(t) \\ &= \Big(\big[f(Y) - f(Z)\big]\partial_x u(t,Z) + \frac{1}{2}\big[g^2(Y) - g^2(Z)\big]\partial_x^2 u(t,Z) \Big)\, dt \\ &\quad + g(Y)\partial_x u(t,Z)\, dW(t) \end{aligned}$$

Equivalent:

$$\begin{aligned} u(t_n,Z(t_n)) = u(0,Z(0)) &+ \int_0^{t_n} \big[f(Y) - f(Z)\big]\partial_x u(t,Z)\, dt \\ &+ \frac{1}{2}\int_0^{t_n} \big[g^2(Y) - g^2(Z)\big]\partial_x^2 u(t,Z)\, dt \\ &+ \int_0^{t_n} g(Y)\partial_x u(t,Z)\, dW(t) \end{aligned}$$

By construction: $u(t_n,Z(t_n)) = u(t_n,X_n) = F(X_n)$
Feynman-Kac (see 2.7): $u(0,Z(0)) = u(0,X(0)) = \mathbb{E}\big[F(X(t_n))\big]$
This yields

$$\begin{aligned} \big|\mathbb{E}[F(X_n)] - \mathbb{E}\big[F(X(t_n))\big]\big| \le &\underbrace{\int_0^{t_n} \Big|\mathbb{E}\Big(\big[f(Y) - f(Z)\big]\partial_x u(t,Z)\Big)\Big|\, dt}_{=:\mathcal{T}_1} \\ &+ \underbrace{\frac{1}{2}\int_0^{t_n} \Big|\mathbb{E}\Big(\big[g^2(Y) - g^2(Z)\big]\partial_x^2 u(t,Z)\Big)\Big|\, dt}_{=:\mathcal{T}_2} \end{aligned}$$

Bounds for $\mathcal{T}_1$ and $\mathcal{T}_2$: Define

$$G(t,x) = \big[f(Y(t)) - f(x)\big]\partial_x u(t,x)$$

and apply the Itô formula to $G(t,Z)$:

$$dG(t,Z) = \left(\partial_t G(t,Z) + \partial_x G(t,Z)\cdot f(Y) + \frac{1}{2}\partial_x^2 G(t,Z)\cdot g^2(Y)\right)dt + \partial_x G(t,Z)\cdot g(Y)dW(t)$$

Equivalent:

$$G(t,Z(t)) = \underbrace{G(t_{n-1},Z(t_{n-1}))}_{=0\ \text{(Def.)}} + \int_{t_{n-1}}^{t} \partial_t G(s,Z)\,ds + \int_{t_{n-1}}^{t} \partial_x G(s,Z)\cdot f(Y)\,ds + \frac{1}{2}\int_{t_{n-1}}^{t} \partial_x^2 G(t,Z)\cdot g^2(Y)\,ds + \int_{t_{n-1}}^{t} \partial_x G(s,Z)\cdot g(Y)\,dW(s)$$

where $Z = Z(s)$ and $Y = Y(s)$. Consider $\mathbb{E}(\ldots)$:

$$\mathbb{E}\big[G(t,Z(t))\big] = \int_{t_{n-1}}^{t} \mathbb{E}\Big(\partial_t G(s,Z)\Big)\,ds + \int_{t_{n-1}}^{t} \mathbb{E}\Big(\partial_x G(s,Z)\cdot f(Y)\Big)\,ds + \frac{1}{2}\int_{t_{n-1}}^{t} \mathbb{E}\Big(\partial_x^2 G(t,Z)\cdot g^2(Y)\Big)\,ds \;+\; 0$$

It can be shown that all three integrands remain bounded. It follows that

$$\big|\mathbb{E}\big[G(t,Z(t))\big]\big| \le C\cdot(t-t_{n-1}) \le C\tau.$$

Consequence:

$$\mathcal{T}_1 = \int_0^{t_n} \Big|\mathbb{E}\Big(\big[f(Y)-f(Z)\big]\partial_x u(t,Z)\Big)\Big|\,dt = \int_0^{t_n} \big|\mathbb{E}\big[G(t,Z(t))\big]\big|\,dt \le Ct_n\tau \le CT\tau.$$

In a similar way, it can be shown that $\mathcal{T}_2 \le Ct_n\tau \le CT\tau$. This proves that

$$\Big|\mathbb{E}\big[F\big(X(t_n)\big)\big] - \mathbb{E}\big[F(X_n)\big]\Big| \le \hat{C}\tau.$$

■

5.3 Higher-order methods

Consider again the one-dimensional SDE

$$dX(t) = f\big(X(t)\big)dt + g\big(X(t)\big)dW(t), \qquad t\in[0,T], \qquad X(0) = X_0$$

with suitable functions f and g and a given initial value X_0.
Goal: Construct numerical methods with higher order.

Caution! Numerical methods for ordinary differential equations can in general not be extended to stochastic differential equations!

Example: Heun's method.

Heun's method for the ODE $\dot{y}(t) = f(y)$ with initial value $y(0) = y_0$ takes the form

$$\begin{aligned}\widetilde{y}_{n+1} &= y_n + \tau f(y_n)\\ y_{n+1} &= y_n + \frac{\tau}{2}\left(f(y_n) + f(\widetilde{y}_{n+1})\right).\end{aligned}$$

Similar to trapezoidal rule, but explicit. The natural modification of this method for SDEs is

$$\begin{aligned}\widetilde{X}_{n+1} &= X_n + \tau f(X_n) + g(X_n)\Delta W_n\\ X_{n+1} &= X_n + \frac{\tau}{2}\Big(f(X_n) + f(\widetilde{X}_{n+1})\Big) + \frac{1}{2}\Big(g(X_n) + g(\widetilde{X}_{n+1})\Big)\Delta W_n.\end{aligned}$$

Consider the special case $f(x) \equiv 0$, $g(x) = x$. For the exact solution $X(t)$, it follows that

$$\mathbb{E}(X(t)) = \mathbb{E}(X_0) + \int_0^t \mathbb{E}\big[\underbrace{f\big(X(s)\big)}_{=0}\big]\, ds + \underbrace{\mathbb{E}\left[\int_0^t g\big(X(s)\big)dW(s)\right]}_{=0},$$

i.e. that $\mathbb{E}(X(t))$ is constant. The method simplifies to

$$\begin{aligned}\widetilde{X}_{n+1} &= X_n + X_n\Delta W_n\\ X_{n+1} &= X_n + \frac{1}{2}\Big(X_n + \widetilde{X}_{n+1}\Big)\Delta W_n.\end{aligned}$$

or equivalently

$$X_{n+1} = X_n + X_n\Delta W_n + \frac{1}{2}X_n(\Delta W_n)^2.$$

This yields

$$\mathbb{E}(X_{n+1}) = \mathbb{E}(X_n) + \underbrace{\mathbb{E}(X_n\Delta W_n)}_{=0} + \frac{1}{2}\mathbb{E}\left(X_n(\Delta W_n)^2\right) = \mathbb{E}(X_n) + \frac{\tau}{2}\mathbb{E}(X_n)$$

and thus for $N \longrightarrow \infty$ and $\tau = T/N \longrightarrow 0$

$$\lim_{\tau\to 0}\mathbb{E}(X_N) = \lim_{\tau\to 0}(1+\tau/2)^N X_0 = \lim_{N\to\infty}\left(1 + \frac{T}{2N}\right)^N X_0 = e^{T/2}X_0.$$

Hence, the method is not consistent! No convergence!

Stochastic Taylor expansions

Important tool for the construction of higher-order methods.
For smooth $F : \mathbb{R} \longrightarrow \mathbb{R}$, the Itô formula yields

$$dF(X) = \underbrace{\Big(F'(X) \cdot f(X) + \frac{1}{2} F''(X) \cdot g^2(X) \Big)}_{=:\mathcal{L}_0 F(X)} dt + \underbrace{F'(X) \cdot g(X)}_{=:\mathcal{L}_1 F(X)} dW(t)$$

(no time derivative, because $F = F(x)$ does not depend on t) or equivalently

$$F\big(X(s)\big) = F\big(X(t_n)\big) + \int_{t_n}^{s} \mathcal{L}_0 F\big(X(r)\big)\, dr + \int_{t_n}^{s} \mathcal{L}_1 F\big(X(r)\big)\, dW(r) \tag{5.4}$$

Choose $F(x) = f(x)$ and $F(x) = g(x)$, respectively, and substitute into the SDE:

$$\begin{aligned} X(t) &= X(t_n) + \int_{t_n}^{t} f\big(X(s)\big) ds + \int_{t_n}^{t} g\big(X(s)\big) dW(s) \\ &= X(t_n) + \underbrace{\int_{t_n}^{t} f\big(X(t_n)\big)\, ds}_{=(t-t_n) f\left(X(t_n)\right)} + \underbrace{\int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_0 f\big(X(r)\big)\, dr\, ds}_{=\mathcal{T}_{11}} + \underbrace{\int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_1 f\big(X(r)\big)\, dW(r)\, ds}_{=\mathcal{T}_{12}} \\ &\quad + \underbrace{\int_{t_n}^{t} g\big(X(t_n)\big)\, dW(s)}_{g\left(X(t_n)\right)\left[W(t)-W(t_n)\right]} + \underbrace{\int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_0 g\big(X(r)\big)\, dr\, dW(s)}_{=\mathcal{T}_{21}} + \underbrace{\int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_1 g\big(X(r)\big)\, dW(r)\, dW(s)}_{=\mathcal{T}_{22}} \end{aligned} \tag{5.5}$$

with

$$\mathcal{L}_0 f = f' \cdot f + \frac{1}{2} f'' \cdot g^2 \qquad \mathcal{L}_1 f = f' \cdot g$$
$$\mathcal{L}_0 g = g' \cdot f + \frac{1}{2} g'' \cdot g^2 \qquad \mathcal{L}_1 g = g' \cdot g$$

If all double integrals $\mathcal{T}_{ij}$ are ignored and if $t = t_{n+1}$, then we obtain the Euler-Maruyama method.

Higher strong order

Since

$$\mathbb{E}\left[\big(W(t_{n+1}) - W(t_n) \big)^2 \right] = t_{n+1} - t_n,$$

we conjecture that for $t \longrightarrow t_n$ the dominant integral term is

$$\mathcal{T}_{22} = \int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_1 g\big(X(r)\big)\, dW(r) dW(s).$$

Ignoring $\mathcal{T}_{11}, \mathcal{T}_{12}, \mathcal{T}_{21}$ yields the approximation

$$X(t) \approx X(t_n) + (t - t_n) f\big(X(t_n)\big) + g\big(X(t_n)\big)\big(W(t) - W(t_n)\big) + \mathcal{T}_{22}.$$

In order to approximate $\mathcal{T}_{22}$, we apply (5.4) with $F(x) = \mathcal{L}_1 g(x)$ and ignore higher-order terms:

$$\begin{aligned}
\mathcal{T}_{22} &= \int_{t_n}^{t} \int_{t_n}^{s} \mathcal{L}_1 g\big(X(r)\big)\, dW(r) dW(s) \\
&\approx \mathcal{L}_1 g\big(X(t_n)\big) \int_{t_n}^{t} \int_{t_n}^{s} dW(r) dW(s) \\
&= g'\big(X(t_n)\big) \cdot g\big(X(t_n)\big) \int_{t_n}^{t} \int_{t_n}^{s} dW(r) dW(s) \qquad (5.6)
\end{aligned}$$

The integral can be explicitly computed:

$$\begin{aligned}
\int_{t_n}^{t} \int_{t_n}^{s} dW(r) dW(s) &= \int_{t_n}^{t} [W(s) - W(t_n)] dW(s) \\
&= \int_{t_n}^{t} W(s) dW(s) - W(t_n) \int_{t_n}^{t} dW(s) \\
&= \int_{0}^{t} W(s) dW(s) - \int_{0}^{t_n} W(s) dW(s) - W(t_n)[W(t) - W(t_n)] \\
&= \frac{1}{2}\left(W^2(t) - t\right) - \frac{1}{2}\left(W^2(t_n) - t_n\right) - W(t_n)[W(t) - W(t_n)] \\
&= \frac{1}{2} W^2(t) + \left(1 - \frac{1}{2}\right) W^2(t_n) - W(t_n) W(t) + \frac{1}{2}(t_n - t) \\
&= \frac{1}{2}\big[W(t) - W(t_n)\big]^2 - \frac{1}{2}(t - t_n)
\end{aligned}$$

For $t = t_{n+1} = t_n + \tau$, this yields the

Milstein method (Grigori N. Milstein 1974):

For $n = 0, \ldots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$X_{n+1} = X_n + \tau f\big(X_n\big) + g\big(X_n\big)\Delta W_n + g'(X_n) \cdot g(X_n)\frac{1}{2}\left[\big(\Delta W_n\big)^2 - \tau\right].$$

Strong order 1, weak order 1 (cf. 10.3 in [KP99])

Remark: Milstein = Euler-Maruyama + additional term
If $g'(x) = 0$ ("additive noise") then Milstein = Euler-Maruyama

Problem: Have to compute derivative g'. Difficult if g is complicated or not explicitly given (i.e. no formula for g).
Idea: Avoid g' by using a (sufficiently accurate) approximation. Let

$$\widetilde{X}_{n+1} = X_n + \tau f\big(X_n\big) + \sqrt{\tau} g\big(X_n\big).$$

(similar to Euler-Maruyama, but with $\sqrt{\tau}$ instead of ΔW_n). Taylor yields

$$\begin{aligned} g(\widetilde{X}_{n+1}) &= g(X_n) + g'(X_n)\left[\widetilde{X}_{n+1} - X_n\right] + \mathcal{O}\left(\left|\widetilde{X}_{n+1} - X_n\right|^2\right) \\ &= g(X_n) + g'(X_n)\left[\tau f\big(X_n\big) + g\big(X_n\big)\sqrt{\tau}\right] + \mathcal{O}(\tau) \\ &= g(X_n) + g'(X_n) g(X_n)\sqrt{\tau} + \mathcal{O}(\tau) \end{aligned}$$

and hence

$$g'(X_n)g(X_n) = \frac{g(\widetilde{X}_{n+1}) - g(X_n)}{\sqrt{\tau}} + \mathcal{O}\big(\sqrt{\tau}\big).$$

This yields the

Stochastic Milstein-Runge-Kutta method:
For $n = 0, \ldots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$\begin{aligned} \widetilde{X}_{n+1} &= X_n + \tau f\big(X_n\big) + \sqrt{\tau} g\big(X_n\big) \\ X_{n+1} &= X_n + \tau f\big(X_n\big) + g\big(X_n\big)\Delta W_n + \frac{g(\widetilde{X}_{n+1}) - g(X_n)}{\sqrt{\tau}} \cdot \frac{1}{2}\left[\big(\Delta W_n\big)^2 - \tau\right]. \end{aligned}$$

Strong order 1, weak order 1 (in spite of the additional approximation).

Higher weak order

Go back to (5.5), i.e.

$$\begin{aligned} X(t) = X(t_n) &+ (t - t_n) f\big(X(t_n)\big) + \mathcal{T}_{11} + \mathcal{T}_{12} \\ &+ g\big(X(t_n)\big)\big[W(t) - W(t_n)\big] + \mathcal{T}_{21} + \mathcal{T}_{22} \end{aligned}$$

and approximate each double integral by freezing the integrand at t_n:

$$\mathcal{T}_{11} = \int_{t_n}^{t}\int_{t_n}^{s} \mathcal{L}_0 f\big(X(r)\big)\, dr\, ds \;\approx\; \int_{t_n}^{t}\int_{t_n}^{s} \mathcal{L}_0 f\big(X(t_n)\big)\, dr\, ds = \frac{(t-t_n)^2}{2}\mathcal{L}_0 f\big(X(t_n)\big)$$

$$\mathcal{T}_{12} = \int_{t_n}^{t}\int_{t_n}^{s} \mathcal{L}_1 f\big(X(r)\big)\, dW(r)\, ds \;\approx\; \mathcal{L}_1 f\big(X(t_n)\big)\mathcal{I}_{(1,0)}(t)$$

$$\mathcal{T}_{21} = \int_{t_n}^{t}\int_{t_n}^{s} \mathcal{L}_0 g\big(X(r)\big)\, dr\, dW(s) \;\approx\; \mathcal{L}_0 g\big(X(t_n)\big)\mathcal{I}_{(0,1)}(t)$$

$$\mathcal{T}_{22} \;\approx\; \mathcal{L}_1 g\big(X(t_n)\big)\frac{1}{2}\Big(\big[W(t)-W(t_n)\big]^2 - (t-t_n)\Big) \qquad \text{(cf. (5.6))}$$

with

$$\mathcal{I}_{(1,0)}(t) = \int_{t_n}^{t}\int_{t_n}^{s} dW(r)\, ds = \int_{t_n}^{t} W(s) - W(t_n)\, ds$$

$$\mathcal{I}_{(0,1)}(t) = \int_{t_n}^{t}\int_{t_n}^{s} dr\, dW(s) = \int_{t_n}^{t} (s-t_n)\, dW(s).$$

Applying the integration by parts formula

$$\int_a^b u(s)\, dW(s) = u(b)W(b) - u(a)W(a) - \int_a^b W(s)u'(s)\, ds \tag{5.7}$$

(see exercises) with $a = t_n$, $b = t$, $u(s) = s$ gives

$$\begin{aligned}\int_{t_n}^{t} s\, dW(s) &= tW(t) - t_nW(t_n) - \int_{t_n}^{t} W(s)\, ds \\ &= tW(t) - t_nW(t_n) - \int_{t_n}^{t} \big[W(s) - W(t_n)\big]\, ds - (t-t_n)W(t_n) \\ &= t\big[W(t) - W(t_n)\big] - \mathcal{I}_{(1,0)}(t)\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{I}_{(0,1)}(t) = \int_{t_n}^{t} (s-t_n)\, dW(s) &= \int_{t_n}^{t} s\, dW(s) - t_n\big[W(t) - W(t_n)\big] \\ &= (t-t_n)\big[W(t) - W(t_n)\big] - \mathcal{I}_{(1,0)}(t).\end{aligned}$$

Hence, only $\mathcal{I}_{(1,0)}(t)$ has to be computed. For weak convergence all random variables can be replaced by other random variables with the same moments. It can be shown (exercise) that

$$\mathbb{E}\left(\mathcal{I}_{(1,0)}(t)\right) = 0, \qquad \mathbb{E}\left(\mathcal{I}^2_{(1,0)}(t)\right) = \frac{(t-t_n)^3}{3}, \qquad \mathbb{E}\Big(\mathcal{I}_{(1,0)}(t)\big[W(t)-W(t_n)\big]\Big) = \frac{(t-t_n)^2}{2}.$$

Let $Z_1 \sim \mathcal{N}(0,1)$ and $Z_2 \sim \mathcal{N}(0,1)$ be independent and $W(t) - W(t_n) = \sqrt{t-t_n}Z_1$. If we let

$$Y_n := \frac{\tau^{3/2}}{2}\left(Z_1 + \frac{1}{\sqrt{3}}Z_2\right),$$

then Y_n has the same properties as $\mathcal{I}_{(1,0)}(t_{n+1})$, i.e.

$$\mathbb{E}\left(Y_n\right) = 0, \qquad \mathbb{E}\left(Y_n^2\right) = \frac{\tau^3}{3}, \qquad \mathbb{E}\Big(Y_n \Delta W_n\Big) = \frac{\tau^2}{2}$$

(exercise). For $t = t_{n+1}$, this yields the following method:

$$\begin{aligned} X_{n+1} = X_n + \tau f\big(X_n\big) + \underbrace{\frac{\tau^2}{2}\mathcal{L}_0 f\big(X_n\big)}_{\approx \mathcal{T}_{11}} + \underbrace{\mathcal{L}_1 f\big(X_n\big)Y_n}_{\approx \mathcal{T}_{12}} \\ + g\big(X_n\big)\Delta W_n + \underbrace{\mathcal{L}_0 g(X_n)\big[\tau \Delta W_n - Y_n\big]}_{\approx \mathcal{T}_{21}} + \underbrace{\mathcal{L}_1 g\big(X_n\big)\frac{1}{2}\left(\Delta W_n^2 - \tau\right)}_{\approx \mathcal{T}_{22}} \end{aligned}$$

Weak order 2 (no proof).

Simplification: The weak order is not reduced if ΔW_n is replaced by a "cheaper" random variable with the same moments. Let $\Delta V_n \in \left\{\sqrt{3\tau}, -\sqrt{3\tau}, 0\right\}$ with

$$\mathbb{P}\left(\Delta V_n = \sqrt{3\tau}\right) = \mathbb{P}\left(\Delta V_n = -\sqrt{3\tau}\right) = \frac{1}{6}, \quad \mathbb{P}(\Delta V_n = 0) = \frac{2}{3}.$$

It can be checked (exercise) that

$$\mathbb{E}\left((\Delta V_n)^k\right) = \mathbb{E}\left((\Delta W_n)^k\right) \qquad \text{for } k = 1, \ldots, 5.$$

Moreover, we can replace Y_n by

$$\widetilde{Y}_n = \frac{\tau}{2}\Delta V_n.$$

because

$$\mathbb{E}\left(\widetilde{Y}_n\right) = 0, \qquad \mathbb{E}\left(\widetilde{Y_n}^2\right) = \frac{\tau^3}{4} = \frac{\tau^3}{3} + \mathcal{O}(\tau^3), \qquad \mathbb{E}\Big(\widetilde{Y}_n \Delta V_n\Big) = \frac{\tau^2}{2}.$$

5.4 Numerical methods for systems of SDEs

Consider now the vector-valued SDE

$$X(t) = X(0) + \int_0^t f(s, X(s))\, ds + \int_0^t g(s, X(s))\, dW(s)$$

with

$$X(t) \in \mathbb{R}^d, \quad W(t) \in \mathbb{R}^m, \quad f : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad \text{and} \quad g : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times m}.$$

Notation as in 2.8.
The **Euler-Maruyama method** can be readily extended to vector-valued SDEs: For $n = 0, \dots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

$$X_{n+1} = X_n + \tau f\big(t_n, X_n\big) + g\big(t_n, X_n\big)\Delta W_n.$$

Strong order 1/2, weak order 1, similar proof.
What about the Milstein method?

- **Case $d \geq 1$ and $m = 1$:** Straightforward extension.
  For $n = 0, \dots, N-1$ let $\Delta W_n = W(t_{n+1}) - W(t_n)$ and

  $$X_{n+1} = X_n + \tau f\big(t_n, X_n\big) + g\big(t_n, X_n\big)\Delta W_n + J_g(t_n, X_n) g(t_n, X_n) \frac{1}{2}\Big[\big(\Delta W_n\big)^2 - \tau\Big]$$

  where $J_g = \big[\partial_{x_k} g_j\big]_{j,k}$ is the Jacobian. Strong order 1, weak order 1.

- **Case $d \geq 1$ and $m > 1$:** More complicated. Adapting the derivation via stochastic Taylor expansions yields

  $$X_{n+1}^{(j)} = X_n^{(j)} + \tau f_j\big(t_n, X_n\big) + \sum_{k=1}^{m} g_{jk}\big(t_n, X_n\big)\Delta W_n^{(k)} + \sum_{i,k=1}^{m} \sum_{l=1}^{d} \partial_{x_l} g_{jk}\big(t_n, X_n\big) \cdot g_{li}\big(t_n, X_n\big) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_i(\theta) dW_k(s)$$

  where $X_n^{(j)}$ and $\Delta W_n^{(j)}$ denote the j-th entry of X_n and ΔW_n, respectively. Similar to the scalar case, the derivatives of g can be avoided by a Runge-Kutta-type approach.
  The stochastic integrals cannot be computed analytically. These integrals are solutions of small systems of SDEs, which have to be approximated numerically. Details: 5.3 in [GJ10].

5.5 Mean-square-error of the Monte Carlo simulation

Consider a European option with payoff function ψ and price process

$$dS(t) = f\big(t, S(t)\big)dt + g\big(t, S(t)\big)dW(t), \qquad t \in [0, T]. \tag{5.8}$$

Standard Monte Carlo method (cf. section 5.1):

- Choose $N \in \mathbb{N}$, let $\tau = T/N$ and $t_n = n\tau$ for $n = 0, \dots, N$.
  Generate $m \in \mathbb{N}$ paths $t \mapsto W(t, \omega_j)$ of the Wiener process ($j = 1, \dots, m$).
  For each path, compute approximations $S_n(w_j) \approx S(t_n, \omega_j)$ by solving (5.8) with a numerical method of weak order γ.
- Approximate the discounted expectation:

$$\mathbb{E}_{\mathbb{Q}}\Big[\psi\big(S(T)\big)\Big] \approx \frac{1}{m}\sum_{j=1}^{m} \psi\big(S(T, \omega_j)\big) \approx \frac{1}{m}\sum_{j=1}^{m} \psi\big(S_N(\omega_j)\big) =: \widehat{V}$$

Two sources of error:

- Estimate the expectation from finitely many samples.
- Approximate the exact $S(T, \omega_j)$ by a numerical method.

Both errors are measured by the mean-square-error.

Definition 5.5.1 (Mean-square-error) *Let $\hat{\theta}$ be an estimator for an unknown (deterministic) quantity θ. Then, the mean-square-error of $\hat{\theta}$ is*

$$MSE(\hat{\theta}) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] \overset{(\star)}{=} \mathbb{V}(\hat{\theta}) + \mathbb{E}(\hat{\theta} - \theta)^2.$$

The term $\mathbb{E}(\hat{\theta} - \theta) = \mathbb{E}(\hat{\theta}) - \theta$ is called the bias. Notation: $\mathbb{E}(X)^2 := \big(\mathbb{E}(X)\big)^2 \neq \mathbb{E}(X^2)$.
Proof of $(\star)$:

$$\begin{aligned}\mathbb{V}(\hat{\theta}) + \mathbb{E}(\hat{\theta} - \theta)^2 &= \Big(\mathbb{E}(\hat{\theta}^2) - \mathbb{E}(\hat{\theta})^2\Big) + \Big(\mathbb{E}(\hat{\theta})^2 - 2\mathbb{E}(\hat{\theta})\theta + \theta^2\Big) \\ &= \mathbb{E}\Big(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2\Big) = \mathbb{E}\left[(\hat{\theta} - \theta)^2\right].\end{aligned}$$

Applying this with $\mathbb{E} = \mathbb{E}_{\mathbb{Q}}$ and

$$\theta = \mathbb{E}\Big(\psi\big(S(T)\big)\Big), \qquad \hat{\theta} = \widehat{V} = \frac{1}{m}\sum_{j=1}^{m} \psi\big(S_N(\omega_j)\big)$$

yields

$$\begin{aligned}\mathrm{MSE}(\widehat{V}) &= \frac{1}{m^2}\mathbb{V}\left(\sum_{j=1}^{m} \psi\big(S_N(\omega_j)\big)\right) + \left(\frac{1}{m}\sum_{j=1}^{m} \mathbb{E}\Big[\psi\big(S_N(\omega_j)\big)\Big] - \mathbb{E}\Big[\psi\big(S(T)\big)\Big]\right)^2 \\ &= \frac{\mathbb{V}\big(\psi(S_N)\big)}{m} + \Big(\mathbb{E}\big[\psi(S_N)\big] - \mathbb{E}\big[\psi\big(S(T)\big)\big]\Big)^2 \\ &\leq \frac{C}{m} + C\tau^{2\gamma}.\end{aligned}$$

Consequence: $\sqrt{\mathrm{MSE}(\widehat{V})} \sim C\sqrt{m^{-1} + \tau^{2\gamma}}$.
Slow convergence with respect to m!

Example: Euler-Maruyama method. If $\varepsilon > 0$ is a given error tolerance, then

$$\mathrm{MSE}(\widehat{V}) = \varepsilon^2 \leq \frac{C}{m} + C\tau^2 \qquad \Longleftrightarrow \qquad m = \mathcal{O}\big(\varepsilon^{-2}\big) \text{ and } \tau = \mathcal{O}(\varepsilon)\,.$$

Since $\tau = T/N$, we have to compute $m = \mathcal{O}(\varepsilon^{-2})$ simulations with $N = \mathcal{O}(\varepsilon^{-1})$ time-steps. Hence, the total numerical work (= total number of time-steps) is $\mathcal{O}(\varepsilon^{-3})$.

The computational costs can be reduced by Multi-Level Monte Carlo methods, cf. part 2 of the lecture.

Chapter 6

Pseudo-random numbers and Monte Carlo integration

6.1 Pseudo-random numbers

Stochastic simulations are based on random variables. In order to approximate the weak solution of a SDE, for example, the Wiener increment $\Delta W_n = W(t_{n+1}) - W(t_n)$ is simulated by drawing a random number $Z_n \sim \mathcal{N}(0,1)$ and letting $\Delta W_n = \sqrt{\tau} Z_n$; cf. 5.2.

Question: What does it mean to "draw a random number"? How can a computer generate a random number?

Computers can only generate **pseudo-random numbers**, i.e. sequences of numbers which seem to be random, but which are actually generated by a deterministic algorithm. Hence, simulation results can be reproduced if necessary. Every such sequence is periodic, but with a very large period.

6.1.1 Uniform pseudo-random numbers

First goal: generate uniformly distributed pseudo-random numbers $X_i \in [0,1]$.
Notation: $X_i \sim \mathcal{U}(0,1)$.
MATLAB command: `rand(...)`

Method 1: Linear congruential generator

Choose $M \in \mathbb{N}$ and $a, b, X_0 \in \{0, 1, \ldots, M-1\}$ and let

$$X_i = (aX_{i-1} + b) \mod M, \qquad U_i = \frac{X_i}{M} \qquad (i = 1, 2, 3, \ldots)$$

Reminder: $x \mod y = z \iff x = ny + z$ for some $n \in \mathbb{N}$ and $z \in \{0, 1, \ldots, y-1\}$.
The entire sequence depends on the "seed" X_0.
"Bad" parameters must be avoided:

- $a \neq 0$
- If $b = 0$, then we must choose $X_0 \neq 0$.
- $a \neq 1$ (too predictable)

By definition $U_i \in \left\{0, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M-1}{M}\right\}$, i.e. M =number of possible values of U_i. Hence, M should be choosen very large.
Since $X_i \in \{0, 1, \ldots, M-1\}$, the sequence $(X_i)_i$ is periodic with period $\leq M$.

MATLAB routine `msg16807`: $a = 7^5$, $b = 0$, $M = 2^{31} - 1$, period $= 2^{31} - 2 \approx 2 \cdot 10^9$. Too small!

Method 2: Fibonacci generator

Choose $k, l, M \in \mathbb{N}$, let $m = \max\{k, l\}$, generate $X_1, \ldots, X_{m-1}$ with Method 1

$$X_i = (X_{i-k} + X_{i-l}) \mod M, \qquad U_i = \frac{X_i}{M} \qquad (i = m, m+1, m+2, \ldots)$$

MATLAB routine `mlfg6331_64`: $k = 31$, $l = 63$, $M = 2^{64}$, period $\approx 2^{124} > 2 \cdot 10^{37}$

Method 3: Combined multiple recursive generator

Choose $M_1, M_2, M \in \mathbb{N}$ very large, $a, b, c, d \in \mathbb{N}$ large

$$\begin{aligned} X_i &= (aX_{i-2} - bX_{i-3}) \mod M_1 \\ Y_i &= (cY_{i-1} - dY_{i-3}) \mod M_2 \\ Z_i &= (X_i - Y_i) \mod (M-1) \end{aligned} \qquad U = \begin{cases} Z_i/M & \text{if } Z_i \neq 0 \\ (M-1)/M & \text{if } Z_i = 0 \end{cases}$$

MATLAB routine `mrg32k3a`, parameters on p. 108 in [GJ10].

Remark: There are many more methods. The default algorithm in MATLAB is called Mersenne Twister `mt19937ar`.

6.1.2 Normal pseudo-random numbers

Idea: Transform uniform pseudo-random numbers to obtain normal pseudo-random numbers.

Method 1: Inversion

Let $U \sim \mathcal{U}(0,1)$, i.e. $\mathbb{P}(U \leq x) = x$ for all $x \in [0,1]$.
Let $F : \mathbb{R} \longrightarrow [0,1]$ be a strictly increasing probability distribution. Hence, $F^{-1} : [0,1] \longrightarrow \mathbb{R}$ exists, and if $X := F^{-1}(U)$, then

$$\mathbb{P}(X \leq x) = \mathbb{P}\big(U \leq F(x)\big) = F(x).$$

Hence, F is the distribution of X. Apply this to the normal distribution

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-s^2/2}\, ds.$$

Problem: No explicit formula for F or F^{-1}. Numerical inversion with Newton's method is ill-conditioned: If $u \approx 0$ or $u \approx 1$, then small perturbations cause large perturbations of $F^{-1}(u)$.

Method 2: Box-Muller method

Let $X \in \mathbb{R}^d$ be a random variable with density $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, and let $A := \{x \in \mathbb{R}^d : f(x) > 0\}$. Let $g : A \longrightarrow B := g(A) \subset \mathbb{R}^d$ be invertible with continuously differentiable inverse g^{-1}. If $Y = g(X)$, then

$$\mathbb{P}(Y \in C) = \mathbb{P}\big(g(X) \in C\big) = \mathbb{P}\big(X \in g^{-1}(C)\big) = \int_{g^{-1}(C)} f(x)\, dx$$

$$= \int_C f\Big(g^{-1}(y)\Big) \cdot |\det J_{g^{-1}}(y)|\ dy$$

for all Borel sets $C \subset B$. $J_{g^{-1}}$ denotes the Jacobian of g^{-1}. Hence, the function

$$y \mapsto f\Big(g^{-1}(y)\Big) \cdot |\det J_{g^{-1}}(y)|$$

is the density of $Y = g(X)$.

Use this to transform the uniform distribution to the normal distribution. Let $d = 1$, $A = [0,1]$, $f(x) = \mathbf{1}_A(x)$ and seek g such that for all $y \in B$

$$\underbrace{f\Big(g^{-1}(y)\Big)}_{=1 \text{ since } g^{-1}(y) \in A} \cdot |\det J_{g^{-1}}(y)| = \left|\frac{dg^{-1}}{dy}(y)\right| \stackrel{!}{=} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Problem: No explicit formula for g. Idea: Transform in $\mathbb{R}^2$ instead of $\mathbb{R}$.
Let $A = (0,1) \times (0,1)$, $f(x) = \mathbf{1}_A$ and

$$g(x) = \begin{pmatrix} \sqrt{-2\ln x_1}\cos(2\pi x_2) \\ \sqrt{-2\ln x_1}\sin(2\pi x_2) \end{pmatrix}, \qquad x = (x_1, x_2) \in A.$$

The inverse is (exercise)[1]

$$g^{-1}(y) = \begin{pmatrix} \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right) \\ \frac{1}{2\pi} \arctan\left(\frac{y_2}{y_1}\right) \end{pmatrix}$$

[1]The arctan cannot yield negative values because y_1 and y_2 are not arbitrary – these values are coupled according to the definition of g.

and it can be shown (exercise) that

$$|\det J_{g^{-1}}(y)| = \frac{1}{2\pi} \exp\left(-\frac{1}{2}(y_1^2 + y_2^2)\right)$$

is the density of the standard normal distribution in $\mathbb{R}^2$. Hence:

$$g_1(X), g_2(X) \sim \mathcal{N}(0,1) \quad \Longleftrightarrow \quad X_1, X_2 \sim \mathcal{U}(0,1).$$

Box-Muller algorithm: Generate uniformly distributed random numbers $U_1, U_2 \sim \mathcal{U}(0,1)$ and let

$$Z_1 = \sqrt{-2\ln U_1}\cos(2\pi U_2) \sim \mathcal{N}(0,1), \qquad Z_2 = \sqrt{-2\ln U_1}\sin(2\pi U_2) \sim \mathcal{N}(0,1)$$

G. E. P. Box and M. E. Muller 1958

Method 3: Polar method

Goal: Avoid trigonometric functions.
If $U_i \sim \mathcal{U}(0,1)$, then $V_i = 2U_i - 1 \sim \mathcal{U}(-1,1)$. Reject (V_1, V_2) if $V := V_1^2 + V_2^2 \geq 1$. The accepted samples are uniformly distributed on the unit circle (with density $f(x) = 1/\pi$), and it can be shown (exercise) that

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} V \\ \frac{1}{2\pi}\arctan\left(\frac{V_2}{V_1}\right) \end{pmatrix}$$

is uniformly distributed on $(0,1)\times(0,1)$. Hence

$$Z_1 = \sqrt{-2\ln W_1}\cos(2\pi W_2) \sim \mathcal{N}(0,1), \qquad Z_2 = \sqrt{-2\ln W_1}\sin(2\pi W_2) \sim \mathcal{N}(0,1)$$

and by definition

$$\cos(2\pi W_2) = \frac{V_1}{\sqrt{V}}, \qquad\qquad \sin(2\pi W_2) = \frac{V_2}{\sqrt{V}}.$$

Polar method: For $i \in \{1,2\}$ generate uniform random numbers $U_i \in (0,1)$ and let $V_i = 2U_i - 1$.

- If $V := V_1^2 + V_2^2 \geq 1$: reject and start again.
- Else: Let

$$Z_1 = \frac{V_1}{\sqrt{V}}\sqrt{-2\ln V} \sim \mathcal{N}(0,1), \qquad Z_2 = \frac{V_2}{\sqrt{V}}\sqrt{-2\ln V} \sim \mathcal{N}(0,1).$$

G. Marsaglia

The probability that $V < 1$ is $\pi/4$. Hence, about 21.46% of the random tuples (V_1, V_2) are rejected. Nevertheless, the polar method is usually more efficient than the standard Box-Muller method.

6.1.3 Correlated normal random vectors

Let $X(\omega) \in \mathbb{R}^d$, $\mu \in \mathbb{R}^d$ and let $\Sigma \in \mathbb{R}^{d\times d}$ be symmetric and positive definite.

Goal: Generate random vectors $X \sim \mathcal{N}(\mu, \Sigma)$, i.e.

$$\mathbb{P}(X \in B) = \int_B \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\Big(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\Big)\, dx$$

for all Borel sets $B \subset \mathbb{R}^d$; cf. Definition 2.2.1. The matrix

$$\rho \in \mathbb{R}^{d\times d} \quad \text{with entries} \quad \rho_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$$

is called the correlation matrix.

Reminder: Every symmetric, positive definite matrix $\Sigma \in \mathbb{R}^{d\times d}$ has a **Cholesky decomposition**

$$\Sigma = LL^T, \qquad L = \begin{pmatrix} \star & 0 \cdots & \cdots & \cdots & 0 \\ \star & \star & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ \star & \cdots & \cdots & \cdots & \star \end{pmatrix}, \qquad L_{ij} = 0 \text{ if } i < j.$$

Proof by induction (exercise).

If $z \in \mathbb{R}^d$ and $x = Lz$, then

$$z^T z = (L^{-1}x)^T(L^{-1}x) = x^T(LL^T)^{-1}x = x^T\Sigma^{-1}x.$$

For $A \subset \mathbb{R}^d$ and $B := \{z = L^{-1}x, x \in A\}$ we have

$$\begin{aligned}\int_B \frac{1}{\sqrt{(2\pi)^d}} \exp\Big(-\frac{1}{2}z^T z\Big)\, dz &= \int_A \frac{1}{\sqrt{(2\pi)^d}} \cdot \frac{1}{|\det(L)|} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x\Big)\, dx \\ &= \int_A \frac{1}{\sqrt{(2\pi)^d \det(\Sigma)}} \exp\Big(-\frac{1}{2}x^T\Sigma^{-1}x\Big)\, dx\end{aligned}$$

because $(\det(L))^2 = \det(L)\cdot\det(L^T) = \det(\Sigma)$. Consequence:

$$Z \sim \mathcal{N}(0, I) \quad\Longrightarrow\quad X = LZ \sim \mathcal{N}(0, \Sigma) \quad\Longrightarrow\quad X + \mu \sim \mathcal{N}(\mu, \Sigma).$$

6.2 Monte Carlo integration and variance reduction

Example: European basket call with $d \in \mathbb{N}$ underlyings modelled by geometric Brownian motion

$$dS(t) = rSdt + \operatorname{diag}(\sigma)\operatorname{diag}(S)L\,dW(t), \qquad S(t_\star) = S_\star$$

with $\sigma = (\sigma_1, \dots, \sigma_d)$, $r > 0$, $S = (S_1, \dots, S_d)$, $dW = (dW_1, \dots, dW_d)$ and a lower triangular matrix $L \in \mathbb{R}^{d\times d}$.
Payoff function (cf. 1.2):

$$\psi(x) = \left(\sum_{i=1}^{d} c_i x_i - K\right)^+, \qquad c_i > 0$$

As in the scalar case (cf. 3.4), it can be shown that the value of the option is the discounted expected payoff

$$V(t_\star, S_\star) = e^{-r(T-t_\star)}\mathbb{E}_{\mathbb{Q}}\left(\psi(S_T)\right) = e^{-r(T-t_\star)} \int_0^\infty \cdots \int_0^\infty \underbrace{\psi(x)\phi(x,\xi,\beta)}_{=:g(x)}\, dx_1 \cdots dx_d$$

where ϕ is the multivariate log-normal distribution. The parameters ξ and β depend on $S_\star$, $T - t_\star$, and on the covariance matrix $\Sigma = LL^T$. In order to price the option, we thus have to approximate the d-dimensional integral

$$\int_0^\infty \cdots \int_0^\infty g(x)\, dx_1 \cdots dx_d$$

Approximation by quadrature as in 5.1:

- Truncation: $0 \le x_{\min}^{(i)} < x_{\max}^{(i)}$ such that

$$\int_{x_{\min}^{(d)}}^{x_{\max}^{(d)}} \cdots \int_{x_{\min}^{(1)}}^{x_{\max}^{(1)}} g(x)\, dx_1 \cdots dx_d \approx \int_0^\infty \cdots \int_0^\infty g(x)\, dx_1 \cdots dx_d$$

- Discretization: For every $i = 1, \dots, d$ choose large $N^{(i)} \in \mathbb{N}$, let $h^{(i)} = (x_{\max}^{(i)} - x_{\min}^{(i)})/N^{(i)}$ and $x_k^{(i)} = x_{\min}^{(i)} + kh^{(i)}$
- Approximate by quadrature (here: midpoint rule):

$$\int_{x_{\min}^{(d)}}^{x_{\max}^{(d)}} \cdots \int_{x_{\min}^{(1)}}^{x_{\max}^{(1)}} g(x)\, dx_1 \cdots dx_d$$

$$\approx h^{(1)} \cdot \ldots \cdot h^{(d)} \sum_{k_1=1}^{N^{(1)}} \cdots \sum_{k_d=1}^{N^{(d)}} g\left(x_{k_1}^{(1)} + \tfrac{1}{2}h^{(1)}, \ \dots\ , x_{k_d}^{(d)} + \tfrac{1}{2}h^{(d)}\right)$$

Problem: Need function evaluations at $N^{(1)} \cdot \ldots \cdot N^{(d)}$ points, e.g. N^d evaluations if $N^{(i)} = N$ for all i. Exponential growth for $d \longrightarrow \infty$, "curse of dimension". Very expensive or impossible for $d \gg 1$.

Solutions? Sparse grids ($\longrightarrow$ summer term) or (Quasi-)Monte Carlo integration.

Monte Carlo integration

Consider bounded domain $\mathcal{D} \subset \mathbb{R}^d$, function $f : \mathcal{D} \longrightarrow \mathbb{R}$, density $\phi : \mathcal{D} \longrightarrow \mathbb{R}$. As in 5.1 approximate

$$\mathbb{E}_\phi(f) := \int_{\mathcal{D}} f(x)\phi(x)\, dx \approx \frac{1}{m} \sum_{j=1}^{m} f(X_j) \tag{6.1}$$

where $X_j \in \mathcal{D}$ are random vectors with

$$\mathbb{P}(X_j \in A) = \int_A \phi(x)\, dx \qquad \text{for all measurable } A \subset \mathcal{D}.$$

Question: How accurate is this approximation?

For simplicity consider only the case $d = 1$. If $d > 1$, then the following result can be applied to each entry of the random vector.

Lemma 6.2.1 (Chebyshev's inequality) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $\delta > 0$. If $Z : \Omega \longrightarrow \mathbb{R}$ is square integrable, i.e. if $\int_\Omega |Z|^2 d\mathbb{P}$ exists, then*

$$\mathbb{P}\big(|Z - \mathbb{E}(Z)| \geq \delta\big) \leq \frac{\mathbb{V}(Z)}{\delta^2}$$

where $\mathbb{V}(Z) = \mathbb{E}(|Z - \mathbb{E}(Z)|^2)$ is the variance of Z.

Proof. Define

$$\chi_\delta(\omega) = \begin{cases} 1 & \text{if } |Z(\omega) - \mathbb{E}(Z)| \geq \delta \\ 0 & \text{else.} \end{cases}$$

Then $\chi_\delta(\omega) \leq \frac{|Z(\omega) - \mathbb{E}(Z)|}{\delta}$ for all $\omega \in \Omega$ by construction, and hence

$$\begin{aligned} \mathbb{P}\big(|Z - \mathbb{E}(Z)| \geq \delta\big) &= \int_\Omega \chi_\delta(\omega)\, d\mathbb{P}(\omega) = \int_\Omega \chi_\delta^2(\omega)\, d\mathbb{P}(\omega) \\ &\leq \frac{1}{\delta^2} \int_\Omega |Z(\omega) - \mathbb{E}(Z)|^2\, d\mathbb{P}(\omega) = \frac{\mathbb{V}(Z)}{\delta^2}. \end{aligned}$$

■

Now let $Y_m := \frac{1}{m}\sum_{j=1}^m f(X_j) \approx \mathbb{E}_\phi(f)$ and

$$\mathbb{E}(f(X_j)) = \mathbb{E}_\phi(f), \qquad \mathbb{V}(f(X_j)) = \sigma^2$$

for all $j = 1, \dots, m$ and some $\sigma > 0$. Consequence:

$$\begin{aligned}\mathbb{E}(Y_m) &= \mathbb{E}_\phi(f)\\ \mathbb{V}(Y_m) &= \mathbb{V}\left(\frac{1}{m}\sum_{j=1}^m f(X_j)\right) = \frac{1}{m^2}\sum_{j=1}^m \underbrace{\mathbb{V}\big(f(X_j)\big)}_{=\sigma^2} = \frac{\sigma^2}{m}.\end{aligned}$$

Applying Lemma 6.2.1 to Y_m yields for all $\delta > 0$

$$\mathbb{P}\big(|Y_m - \mathbb{E}_\phi(f)| \geq \delta\big) \leq \frac{\mathbb{V}(Y_m)}{\delta^2} = \frac{\sigma^2}{\delta^2 m}.$$

Now choose $\varepsilon > 0$ and let $\delta := \frac{\sigma}{\sqrt{\varepsilon m}}$:

$$\mathbb{P}\left(|Y_m - \mathbb{E}_\phi(f)| \geq \frac{\sigma}{\sqrt{\varepsilon m}}\right) \leq \varepsilon$$

or equivalently

$$\mathbb{P}\left(|Y_m - \mathbb{E}_\phi(f)| < \frac{\sigma}{\sqrt{\varepsilon m}}\right) > 1 - \varepsilon.$$

Interpretation:

$$\text{Good approximation with high probability} \quad \Longleftrightarrow \quad \varepsilon \text{ small, } \frac{\sigma}{\sqrt{\varepsilon m}} \text{ small.}$$

Slow convergence: For fixed ε and σ, reducing the error by a factor of 10 comes at the cost of increasing the number of samples by a factor of 100.

Variance reduction

Idea: Try to decrease σ to improve the accuracy.

Method 1: Decomposition

Let $g : \Omega \longrightarrow \mathbb{R}$ be a function such that

$$\mathbb{E}_\phi(g) = \int_\Omega g(x)\phi(x)\,dx \approx \int_\Omega f(x)\phi(x)\,dx = \mathbb{E}_\phi(f)$$

and such that $\mathbb{E}_\phi(g)$ can be computed analytically. Let

$$Y_m = \frac{1}{m}\sum_{j=1}^{m} f(X_j) \approx \mathbb{E}_\phi(f) \qquad \text{(as before)}$$
$$Y_m^\star = \frac{1}{m}\sum_{j=1}^{m} g(X_j) \approx \mathbb{E}_\phi(g)$$

and approximate

$$\mathbb{E}_\phi(f) \approx \widehat{Y}_m := Y_m - Y_m^\star + \mathbb{E}_\phi(g).$$

Let $\mathrm{Cov}(Z_1, Z_2)$ be the covariance of two random variables Z_1 and Z_2, i.e.

$$\mathrm{Cov}(Z_1, Z_2) = \mathbb{E}\Big(\big(Z_1 - \mathbb{E}(Z_1)\big)\big(Z_2 - \mathbb{E}(Z_2)\big)\Big).$$

Since

$$0 \le \mathbb{V}(Z_1 - Z_2) = \mathbb{V}(Z_1) + \mathbb{V}(Z_2) - 2\mathrm{Cov}(Z_1, Z_2) \tag{6.2}$$

it follows that

$$\mathrm{Cov}(Z_1, Z_2) \le \frac{1}{2}\big(\mathbb{V}(Z_1) + \mathbb{V}(Z_2)\big). \tag{6.3}$$

Idea: If $g \approx f$, then we expect that $\mathrm{Cov}(Y_m, Y_m^\star)$ is nearly maximal, i.e.

$$\mathrm{Cov}(Y_m, Y_m^\star) \approx \frac{1}{2}\big(\mathbb{V}(Y_m) + \mathbb{V}(Y_m^\star)\big).$$

Hence, the new estimator $\widehat{Y}_m$ has a smaller variance:

$$\mathbb{V}(\widehat{Y}_m) \;=\; \mathbb{V}\left(Y_m - Y_m^\star\right) + \underbrace{\mathbb{V}\left(\mathbb{E}_\phi(g)\right)}_{=0} \overset{(6.2)}{=} \mathbb{V}\left(Y_m\right) + \mathbb{V}\left(Y_m^\star\right) - 2\mathrm{Cov}(Y_m, Y_m^\star) \;\approx\; 0.$$

Method 2: Antithetic variates

Assumption: Y_m is generated with normal random variables $X_j \sim \mathcal{N}(0,1)$.
Since $-X_j \sim \mathcal{N}(0,1)$, too, we define $Y_m^- := \frac{1}{m}\sum_{j=1}^m f(-X_j) \approx \mathbb{E}_\phi(f)$ and put $\widehat{Y}_m = \frac{1}{2}(Y_m + Y_m^-)$. Applying

$$\mathbb{V}(Z_1 + Z_2) = \mathbb{V}(Z_1) + \mathbb{V}(Z_2) + 2\mathrm{Cov}(Z_1, Z_2)$$

we obtain

$$\begin{aligned}\mathbb{V}(\widehat{Y}_m) = \frac{1}{4}\mathbb{V}(Y_m + Y_m^-) &= \frac{1}{4}\Big(\mathbb{V}(Y_m) + \underbrace{\mathbb{V}(Y_m^-)}_{=\mathbb{V}(Y_m)} + 2\mathrm{Cov}(Y_m, Y_m^-)\Big) \\ &= \frac{1}{2}\Big(\mathbb{V}(Y_m) + \mathrm{Cov}(Y_m, Y_m^-)\Big).\end{aligned} \tag{6.4}$$

If $\mathrm{Cov}(Y_m, Y_m^-) > 0$, then (6.3) yields

$$\mathrm{Cov}(Y_m, Y_m^-) \le \frac{1}{2}\big(\mathbb{V}(Y_m) + \mathbb{V}(Y_m^-)\big) = \mathbb{V}(Y_m)$$

and hence it follows from (6.4) that

$$\mathbb{V}(\widehat{Y}_m) \le \mathbb{V}(Y_m) \qquad (\Longrightarrow \text{at least not worse}).$$

If $\mathrm{Cov}(Y_m, Y_m^-) \le 0$, then (6.4) yields

$$\mathbb{V}(\widehat{Y}_m) \le \frac{1}{2}\mathbb{V}(Y_m) \qquad (\Longrightarrow \text{smaller variance}).$$

Variance reduction by antithetic variates for SDEs

For SDE-based Monte Carlo methods (cf. section 5.5), the mean-square-error depends on $\mathbb{V}\big(\psi(S_N)\big)$. This can be reduced by approximating the SDE

$$dS(t) = f\big(t, S(t)\big)dt + g\big(t, S(t)\big)dW(t), \qquad t \in [0, T].$$

with Euler-Maruyama and antithetic variates:

$$\begin{aligned} S_{n+1}^+ &= S_n^+ + \tau f\big(t_n, S_n^+\big) + g\big(t_n, S_n^+\big)\Delta W_n \\ S_{n+1}^- &= S_n^- + \tau f\big(t_n, S_n^-\big) - g\big(t_n, S_n^-\big)\Delta W_n, \qquad n = 0, \ldots, N-1, \end{aligned}$$

and then use the values $S_n = \frac{1}{2}\big(S_n^+ + S_n^-\big)$.

6.3 Quasi-Monte Carlo methods

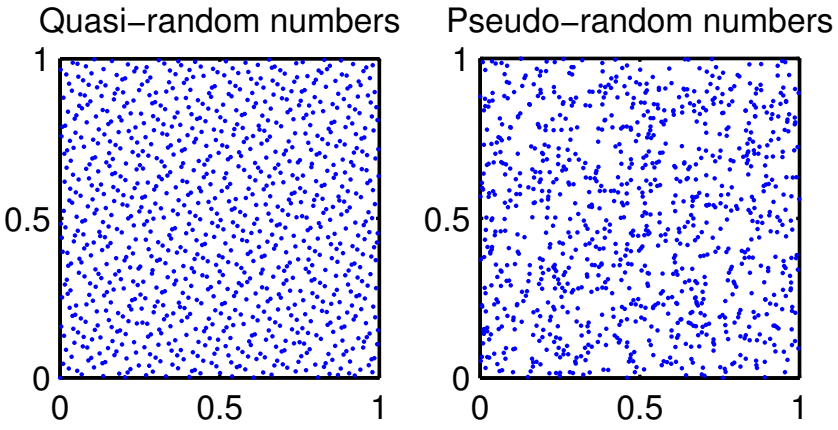
Let $\mathcal{D} = [0,1]^d$ be the d-dimensional unit cube. Monte Carlo integration:

$$\mathbb{E}(f) := \int_{\mathcal{D}} f(x)\, dx \approx \frac{1}{m}\sum_{j=1}^{m} f(X_j)$$

with uniformly distributed random vectors $X_j \in \mathcal{D}$. This is the special case $\phi(x) \equiv 1$ in (6.1).

Problem: Tuples of uniform random numbers are usually not homogeneously distributed in space.
Quasi-Monte Carlo methods use the same formula, but replace the random vectors X_j by deterministic low-discrepancy point sequences.

Definition 6.3.1 (Discrepancy)



1. *Let $\mathcal{R}$ be the set of all axially parallel d-dimensional rectangles $R \subset \mathcal{D}$. The* ***discrepancy*** *of the points $x_1, \dots, x_m \in \mathcal{D} \subset \mathbb{R}^d$ is*

$$D_m := \sup_{R \in \mathcal{R}} \left| \frac{\#of\ x_i\ in\ R}{m} - vol(R) \right|$$

where $vol(R) = \int_R 1\, dx$ denotes the volume of R.

2. *The* ***star discrepancy*** *$D_m^\star$ is defined as D_m, but the supremum is only taken over those R for which $(0, \dots, 0)$ is one of the corners.*

3. *A sequence $(x_k)_{k \in \mathbb{N}}$ of points $x_k \in \mathcal{D}$ is called* ***low-discrepancy sequence*** *if*

$$D_m = \mathcal{O}\left(\frac{(\log m)^d}{m} \right).$$

In this case, the x_k are called ***quasi-random vectors****.*

Properties (without proofs):

- $D_m^\star \leq D_m \leq 2^d D_m^\star$
- The Koksma-Hlawka-Theorem provides the deterministic error bound

$$\left| \mathbb{E}(f) - \frac{1}{m} \sum_{j=1}^{m} f(x_j) \right| \leq TV(f) \cdot D_m^\star$$

  where $TV(f)$ is the total variation of f; cf. (2.2). Numerical tests show that this bound is often too pessimistic.

- It can be shown that

$$\mathbb{E}(D_m) = \mathcal{O}\left(\sqrt{\frac{\log \log m}{m}} \right)$$

  for *randomly* chosen sequences.

Example: For $d = 1$ the sequence with

$$x_j = \frac{2j-1}{2m}, \qquad j = 1, \ldots, m$$

has $D_m^\star = 1/(2m)$. This value is optimal. But: This sequence can only be used if m is known *a priory*, and if m is changed, then all values change.

Van der Corput sequence:

$$\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$$

Algorithm: Represent the index $j \in \mathbb{N}$ as a binary number

$$j = \sum_{k=0}^{L} d_k 2^k = (d_L d_{L-1} \ldots d_1 d_0)_2, \qquad d_k \in \{0, 1\}$$

and define

$$\eta_2(j) = \sum_{k=0}^{L} d_k 2^{-k-1} = (.d_0 d_1 \ldots d_L)_2, \qquad d_k \in \{0, 1\}.$$

Interpretation: Reverse binary digits and put the radix point in front of the sequence. Example: $j = 6$ yields $d_2 = d_1 = 1$, $d_0 = 0$ and hence $\eta_2(6) = \frac{0}{2} + \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$.

Generalization: For an arbitrary base $b \in \mathbb{N}$ define the **radical-inverse function**

$$\eta_b(j) = \sum_k d_k b^{-k-1}$$

where $d_k \in \{0, 1, \ldots, b-1\}$ are the coefficients from the representation $j = \sum_k d_k b^k$.

The **Halton sequence** generates quasi-random vectors in the hypercube $\mathcal{D} = [0, 1]^d$ by letting

$$x_j = \big(\eta_{p_1}(j), \ldots, \eta_{p_d}(j)\big)$$

where $p_1, \ldots, p_d$ are prime numbers with $p_i \neq p_j$ for $i \neq j$.

Other possibility: Sobol sequence

Chapter 7

Finite-difference methods for parabolic differential equations

7.1 Motivation and model problem

Reminder (cf. 3.2): The value of an European option is the solution of the Black-Scholes equation

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0$$

with terminal condition

$$V(T,S) = \psi(S) \qquad \text{(payoff function)}.$$

Notation: $T > 0$ maturity, $r > 0$ interest rate, $\sigma \in \mathbb{R}$ volatility, S price of the underlying.

More complicated market models (e.g. with volatility $\sigma = \sigma(t,S)$) lead to similar PDEs for which no solution formulas are available; cf. 3.5.

Question: Numerical methods?

Basic types of PDEs

- **Elliptic PDEs:** Poisson equation

$$-\Delta u(x) = f(x), \qquad f(x) \text{ given}, \quad \Delta u = \sum_{k=1}^{d} \partial_{x_k}^2 u(x) \text{ Laplace operator}$$

- **Parabolic PDEs:** Heat equation

$$\partial_t u(t,x) = \Delta u(t,x)$$

- **Hyperbolic PDEs:** Wave equation

$$\partial_t^2 u(t,x) = \Delta u(t,x)$$

The Black-Scholes equation is a parabolic PDE and can be transformed to the heat equation; cf. 3.3. Therefore, we will consider the following

Model problem: Heat equation on an interval with Dirichlet boundary conditions.

$$\begin{aligned}
\partial_t u(t,x) &= \partial_x^2 u(t,x) && t \in (0, t_{\text{end}}], x \in (a,b) && \text{PDE} && (7.1a)\\
u(t,a) = u_a(t), \quad u(t,b) &= u_b(t) && t \in [0, t_{\text{end}}] && \text{boundary conditions} && (7.1b)\\
u(0,x) &= u_0(x) && x \in [a,b] && \text{initial condition} && (7.1c)
\end{aligned}$$

The parameters a, b, t_{end}, the boundary values $u_a(t)$ and $u_b(t)$ and the initial data $u_0(x)$ are given.

Notation: We say that $g \in C^j([a,b])$ if and only if $x \mapsto g(x)$ is j times continuously differentiable on (a,b), and all derivatives can be extended to $[a,b]$. Moreover, let

$$\|g\|_\infty = \max_{x \in [a,b]} |g(x)|$$

denote the maximum norm on $[a,b]$.

7.2 Space discretization with finite differences

Choose $1 < m \in \mathbb{N}$, let $h = (b-a)/m$ and $x_k = a + k \cdot h$ for $k = 0, \ldots, m$.
Goal: Find $w : [0, t_{\text{end}}] \longrightarrow \mathbb{R}^{m-1}$ such that the entries $w_k(t)$ of the vector $w(t)$ approximate the values of the exact solution at the inner grid points, i.e.

$$w_k(t) \approx u(t, x_k) \qquad (k = 1, \ldots, m-1)$$

for all $t \in [0, t_{\text{end}}]$. Approximate spatial derivatives by difference quotients.

Lemma 7.2.1 (difference quotients) *For arbitrary $k \in \{1, \ldots, m-1\}$, the (first or second) derivative of a function $y : [a,b] \longrightarrow \mathbb{R}$ in x_k can be approximated as follows:*

- *If $y \in C^2([a,b])$, then*

$$\max_{k=1,\ldots,m-1} \left| y'(x_k) - \frac{y(x_{k+1}) - y(x_k)}{h} \right| \leq Ch \|y''\|_\infty \tag{7.2}$$

$$\max_{k=1,\ldots,m-1} \left| y'(x_k) - \frac{y(x_k) - y(x_{k-1})}{h} \right| \leq Ch \|y''\|_\infty \tag{7.3}$$

- *If* $y \in C^3([a,b])$, *then*

$$\max_{k=1,\dots,m-1} \left| y'(x_k) - \frac{y(x_{k+1}) - y(x_{k-1})}{2h} \right| \leq Ch^2 \left\| \frac{d^3 y}{dx^3} \right\|_\infty \tag{7.4}$$

- *If* $y \in C^4([a,b])$, *then*

$$\max_{k=1,\dots,m-1} \left| y''(x_k) - \frac{y(x_{k+1}) - 2y(x_k) + y(x_{k-1})}{h^2} \right| \leq Ch^2 \left\| \frac{d^4 y}{dx^4} \right\|_\infty \tag{7.5}$$

The constant C *is independent of* h *and can have different values in each case.*

Proof: Use Taylor's theorem (exercise).

Applying (7.5) to the heat equation yields

$$\partial_t u(t,x_k) = \partial_x^2 u(t,x_k) \approx \frac{u(t,x_{k+1}) - 2u(t,x_k) + u(t,x_{k-1})}{h^2} \tag{7.6}$$

for all $k = 1,\dots,m-1$ and $t \in (0, t_{\text{end}}]$. The boundary values $u(t,x_0) = u_a(t)$ and $u(t,x_m) = u_b(t)$ are known from (7.1b).

Reformulation in matrix-vector notation: Define

$$\bar{u}(t) = \Big(u(t,x_1),\dots,u(t,x_{m-1})\Big)^T \in \mathbb{R}^{m-1} \tag{7.7}$$

$$g(t) = \Big(u_a(t),0,0,\dots,0,0,u_b(t)\Big)^T \in \mathbb{R}^{m-1} \tag{7.8}$$

and the matrix

$$A = \begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 1 & -2 & 1 & 0 & & & \vdots \\ 0 & 1 & -2 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & 0 & 1 & -2 & 1 & 0 \\ \vdots & & & 0 & 1 & -2 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix} \in \mathbb{R}^{(m-1)\times(m-1)}. \tag{7.9}$$

Then (7.6) is equivalent (check!) to

$$\bar{u}'(t) \approx \frac{1}{h^2} A\bar{u}(t) + \frac{1}{h^2} g(t).$$

The approximation $w(t) \approx \bar{u}(t)$ is now defined as the solution of the initial value problem

$$w'(t) = \frac{1}{h^2} A w(t) + \frac{1}{h^2} g(t) \tag{7.10a}$$

$$w(0) = \bar{u}(0). \tag{7.10b}$$

Hence, the space discretization turns the PDE into an ordinary differential equation (ODE).

Properties of the discretization matrix

Lemma 7.2.2 *For $k = 1, \ldots, m-1$, the vectors*

$$\nu_k := \left(\sin\left(\frac{k\pi}{m}\right), \sin\left(\frac{2k\pi}{m}\right), \ldots, \sin\left(\frac{(m-1)k\pi}{m}\right) \right)^T \in \mathbb{R}^{m-1}$$

are eigenvectors of the matrix A, and the corresponding eigenvalues are

$$\lambda_k := 2\left(\cos\left(\frac{k\pi}{m}\right) - 1\right) \in (-4, 0).$$

Proof: Exercise.

Notation. Henceforth, $|v| := \sqrt{v^T v}$ denotes the Euclidean norm of a vector v or the induced matrix norm, respectively. (If v is a scalar, then the norm coincides with the absolute value.) Let I be the identity matrix. The exponential of a matrix M is defined by

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!}$$

with convergence in the matrix norm.

Lemma 7.2.3 *For any $s \geq 0$, $1 < m \in \mathbb{N}$ and $h = (b-a)/m$ we have*

$$\left|e^{sA}\right| \leq 1, \qquad \left|(I - sA)^{-1}\right| \leq 1.$$

Proof. Since A is symmetric, there is an orthogonal matrix Q such that $A = Q\Lambda Q^T$ with

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{m-1}), \qquad \lambda_k = 2\left(\cos\left(\frac{k\pi}{m}\right) - 1\right) < 0.$$

Hence,

$$A^2 = Q\Lambda Q^T Q\Lambda Q^T = Q\Lambda^2 Q^T, \qquad A^k = Q\Lambda^k Q^T$$

and as a consequence

$$e^{sA} = \sum_{k=0}^{\infty} Q \frac{(s\Lambda)^k}{k!} Q^T = Q e^{s\Lambda} Q^T.$$

Since $e^{s\Lambda}$ is a diagonal matrix and $s\lambda_k \leq 0$, we have

$$\left|e^{s\Lambda}\right| = \sup_{k=1,\ldots,m-1} |e^{s\lambda_k}| \leq 1.$$

This yields the bound

$$\left|e^{sA}\right| \leq |Q| \cdot \left|e^{s\Lambda}\right| \cdot |Q^T| \leq 1$$

because $|Q| = 1$. In a similar way, it can be checked that

$$(I - sA)^{-1} = Q\,(I - s\Lambda)^{-1}\,Q^T$$

which yields the bound

$$\left|(I - sA)^{-1}\right| \leq |Q| \cdot \left|(I - s\Lambda)^{-1}\right| \cdot |Q^T| = \max_{k=1,\ldots,m-1} \frac{1}{1 - s\lambda_k} \leq 1$$

since $s\lambda_k \leq 0$. ∎

Next goal: Prove an error bound for the approximation $w(t) \approx \bar{u}(t)$. In which norm? In order to study convergence, we consider the limit $m \to \infty$, but since $w(t), \bar{u}(t) \in \mathbb{R}^{m-1}$, the length of the error vector $w(t) - \bar{u}(t)$ increases. The disadvantage of the Euclidean norm is the fact that the norm of a constant vector depends on its length: If $c > 0$ and $(c, \ldots, c)^T \in \mathbb{R}^{m-1}$, then

$$|(c, \ldots, c)^T| = \left(\sum_{i=1}^{m-1} c^2\right)^{1/2} = c\sqrt{m-1}. \tag{7.11}$$

Therefore, it is advantageous to consider a scaled norm:

Definition 7.2.4 (scaled norm) *Let $1 < m \in \mathbb{N}$ and $h = (b-a)/m$. The scaled norm $|\cdot|_h$ on $\mathbb{R}^{m-1}$ is defined by*

$$|v|_h = \sqrt{h}|v| \qquad\qquad v \in \mathbb{R}^{m-1}.$$

The induced matrix norm of a matrix $M \in \mathbb{R}^{(m-1)\times(m-1)}$ is again denoted by $|M|_h$. Note that

$$|M|_h = \sup_{v \neq 0} \frac{|Mv|_h}{|v|_h} = \sup_{v \neq 0} \frac{\sqrt{h}|Mv|}{\sqrt{h}|v|} = |M|. \tag{7.12}$$

Motivation: If $f \in C([a,b])$ is continuous with $f(a) = f(b) = 0$ and $\hat{f}_k = f(x_k)$ for $k = 1, \dots, m-1$, then

$$\|f\|_{L^2([a,b])} = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} \approx \left(h \sum_{k=1}^{m-1} |f(x_k)|^2 \right)^{1/2} = |\hat{f}|_h$$

for sufficiently large m.

Theorem 7.2.5 (error of the space discretization) *Let $u = u(t,x)$ be the solution of the model problem (7.1a), (7.1b), (7.1c), and let $w(t)$ be the solution of (7.10). Assume that $x \mapsto u(t,x) \in C^4([a,b])$ for all $t \in [0, t_{end}]$ with*

$$\sup_{s \in [0,t_{end}]} \left\| \partial_x^4 u(s,\cdot) \right\|_\infty < \infty.$$

Then, there is a constant C such that

$$|\bar{u}(t) - w(t)|_h \le Cth^2 \sup_{s \in [0,t]} \left\| \partial_x^4 u(s,\cdot) \right\|_\infty$$

for all $t \in [0, t_{end}]$.

Proof. By definition of $\bar{u}$ we have for $k = 1, \dots, m-1$

$$\bar{u}_k'(t) = \partial_t u(t,x_k) = \partial_x^2 u(t,x_k) = \frac{u(t,x_{k+1}) - 2u(t,x_k) + u(t,x_{k-1})}{h^2} + r_k(t)$$

with remainder term $r_k(t)$ bounded by

$$|r_k(t)| \le Ch^2 \left\| \partial_x^4 u(t,\cdot) \right\|_\infty$$

according to Lemma 7.2.1. Setting $r := (r_1, \dots, r_{m-1})^T$, this is equivalent to

$$\bar{u}'(t) = \frac{1}{h^2} A\bar{u}(t) + \frac{1}{h^2} g(t) + r(t).$$

Comparing with (7.10) shows that the error solves the ODE

$$\frac{d}{dt}\big(\bar{u}(t) - w(t)\big) = \frac{1}{h^2} A\big(\bar{u}(t) - w(t)\big) + r(t).$$

with initial value $\bar{u}(0) - w(0) = 0$. The solution is given by the variation-of-constants formula

$$\big(\bar{u}(t) - w(t)\big) = e^{tA/h^2} \underbrace{\big(\bar{u}(0) - w(0)\big)}_{=0} + \int_0^t e^{(t-s)A/h^2} r(s) \, ds.$$

Hence, Lemma 7.2.2 and (7.12) yield

$$\left|\bar{u}(t) - w(t)\right|_h \leq \int_0^t \underbrace{\left|e^{(t-s)A/h^2}\right|_h}_{\leq 1} |r(s)|_h \, ds,$$

and with

$$|r(s)|_h = \sqrt{h}|r(s)| \leq \sqrt{h}\sqrt{m-1} \max_{i=1,\dots,m-1} |r_i(s)| \leq \sqrt{b-a} \cdot Ch^2 \left\|\partial_x^4 u(s,\cdot)\right\|_\infty$$

it follows that

$$\begin{aligned}\left|\bar{u}(t) - w(t)\right|_h &\leq Ch^2\sqrt{b-a} \int_0^t \left\|\partial_x^4 u(s,\cdot)\right\|_\infty \, ds \\ &\leq Ch^2\sqrt{b-a}\, t \sup_{s\in[0,t]} \left\|\partial_x^4 u(s,\cdot)\right\|_\infty .\end{aligned}$$

■

Remark. The regularity assumptions are very strong and exclude payoff functions as initial data.

7.3 Time discretization

Space discretization turns the heat equation (cf. (7.10)) into the ODE

$$w'(t) = \frac{1}{h^2} A w(t) + \frac{1}{h^2} g(t)$$

with initial value $w(0) = \bar{u}(0)$.

Time-discretization: Choose a step-size $\tau > 0$, let $t_n = n\tau$ and compute approximations $w^n \approx w(t_n)$ with a Runge-Kutta method (see Appendix 7.14). Examples:

- Explicit Euler method (order 1)

$$w^{n+1} = w^n + \frac{\tau}{h^2} A w^n + \frac{\tau}{h^2} g(t_n)$$

- Implicit Euler method (order 1)

$$\begin{aligned} w^{n+1} &= w^n + \frac{\tau}{h^2} A w^{n+1} + \frac{\tau}{h^2} g(t_{n+1}) \\ \Leftrightarrow \quad \left(I - \frac{\tau}{h^2} A\right) w^{n+1} &= w^n + \frac{\tau}{h^2} g(t_{n+1}) \end{aligned}$$

- Trapezoidal rule (order 2)

$$\begin{aligned} w^{n+1} &= w^n + \frac{\tau}{2h^2} A(w^{n+1} + w^n) + \frac{\tau}{2h^2}(g(t_{n+1}) + g(t_n)) \\ \Leftrightarrow \left(I - \frac{\tau}{2h^2} A\right) w^{n+1} &= \left(I + \frac{\tau}{2h^2} A\right) w^n + \frac{\tau}{2h^2}(g(t_{n+1}) + g(t_n)) \end{aligned}$$

A-stability

For simplicity, assume homogeneous boundary conditions in (7.1b), i.e. $u(t,a) = u(t,b) = 0$ and hence $g(t) = 0$, such that

$$w'(t) = \frac{1}{h^2} A w(t), \qquad w(0) = \bar{u}(0). \tag{7.13}$$

As in the proof of Lemma 7.2.3 we consider the eigendecomposition

$$A = Q \Lambda Q^T, \qquad \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_{m-1}), \qquad \lambda_k < 0 \tag{7.14}$$

with $|Q|_h = |Q| = 1$ and $\lambda_k \in (-4, 0)$. Then, the exact solution of (7.13) is

$$w(t) = \exp\left(tA/h^2\right) w(0) = Q \exp\left(t\Lambda/h^2\right) Q^T w(0).$$

The solution remains bounded for all $t \geq 0$ because

$$|w(t)|_h \leq \underbrace{\left|\exp\left(t\Lambda/h^2\right)\right|_h}_{\leq 1} |w(0)|_h.$$

Does the numerical approximation have the same property?

Explicit Euler method:

$$w^{n+1} = w^n + \frac{\tau}{h^2} A w^n = \left(I + \frac{\tau}{h^2} A\right) w^n.$$

The approximations are bounded by

$$|w^n|_h \leq \left|I + \frac{\tau}{h^2} A\right|_h^n |w^0|_h = \left|I + \frac{\tau}{h^2} \Lambda\right|_h^n |w^0|_h = \max_{k=1,\dots,m-1} \left|1 + \frac{\tau \lambda_k}{h^2}\right|^n |w^0|_h.$$

Hence, the numerical solution remains bounded for all $n \in \mathbb{N}$ if

$$\left|1 + \frac{\tau \lambda_k}{h^2}\right| \leq 1 \quad \Longleftrightarrow \quad \tau \leq \frac{2h^2}{|\lambda_k|}$$

for all $k = 1, \dots, m-1$. Since $\max_{k=1,\dots,m-1} |\lambda_k| \approx 4$, we obtain the **stability condition**

$$\tau \leq \frac{h^2}{2}.$$

This is a severe restriction, because $h \ll 1$ must be small to ensure an acceptable accuracy of the spatial approximation. For larger step-sizes, the norm of the numerical solution may tend to ∞ whereas the exact solution remains bounded. Reducing the step size τ, however, increases the number of steps and hence the numerical costs. Inefficient!

Definition 7.3.1 (A-stability) *The initial value problem*

$$y' = \lambda y, \quad t \geq 0, \qquad\qquad y(0) = y_0, \tag{7.15}$$

is called ***Dahlquist's test equation****. A one-step method (e.g. Runge-Kutta) is called* ***A-stable*** *if the numerical solution* $(y^n)_{n\in\mathbb{N}}$ *of* (7.15) *with arbitrary* $\lambda \in \mathbb{C}$, $Re(\lambda) \leq 0$ *and arbitrary step-size* $\tau > 0$ *remains bounded for all* $n \in \mathbb{N}$.

Remark. If ν_k is the k-th eigenvector of A, then

$$w'(t) = \frac{1}{h^2} A w(t), \qquad\qquad w(0) = \nu_k$$

is equivalent to (7.15) with $\lambda := \lambda_k / h^2 < 0$.
The explicit Euler method (and every other explicit Runge-Kutta method) is not A-stable.

Implicit Euler method: The implicit Euler

$$w^{n+1} = w^n + \frac{\tau}{h^2} A w^{n+1} \qquad \text{or equivalently} \qquad \left(I - \frac{\tau}{h^2} A\right) w^{n+1} = w^n$$

is the simplest A-stable method. Hence, we expect that the corresponding numerical approximation of

$$w'(t) = \frac{1}{h^2} A w(t)$$

remains bounded without any step-size restrictions. This is indeed the case: Lemma 7.2.3 implies that

$$|w^n|_h \leq \left| \left(I - \frac{\tau}{h^2} A\right)^{-1} \right|_h |w^{n-1}|_h \leq |w^{n-1}|_h$$

and hence $|w^n|_h \leq |w^0|_h$ for all $n \in \mathbb{N}$.

Trapezoidal rule: The trapezoidal rule is A-stable (exercise).

Error bound for the implicit Euler method

Instead of (7.10) we consider a general (and possibly nonlinear) initial value problem

$$y' = f(t, y), \quad t \in [0, t_{\text{end}}], \qquad y(0) = y_0. \tag{7.16}$$

Let $\langle z, \tilde{z} \rangle = z^T \tilde{z}$ denote the Euclidean scalar product on $\mathbb{R}^d$ with norm $|z| = \sqrt{\langle z, z \rangle}$.

Theorem 7.3.2 (error bound for the implicit Euler method) *Let $f : [0, t_{end}] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be C^1 and assume that f satisfies the one-sided Lipschitz condition*

$$\Big\langle f(t,z) - f(t,\tilde{z}), z - \tilde{z} \Big\rangle \leq \ell |z - \tilde{z}|^2 \qquad \text{for all } t \in [0, t_{end}] \text{ and } z, \tilde{z} \in \mathbb{R}^d \tag{7.17}$$

with a constant $\ell \in \mathbb{R}$. Let $y(t)$ be the exact solution of the initial value problem (7.16). Let $y^n \approx y(t_n)$ be the approximations computed with the implicit Euler method

$$y^{n+1} = y^n + \tau f(t_{n+1}, y^{n+1}), \qquad y^0 = y_0$$

with step-size $\tau = t_{end}/N$. If $\tau\ell < 1$, then the global error is bounded by

$$\max_{n=0,\dots,N} |y^n - y(t_n)| \leq \frac{C}{2} \max_{t \in [0, t_{end}]} |y''(t)| \cdot \tau \tag{7.18}$$

with

$$C := \begin{cases} 1/|\ell| & \text{if } \ell < 0 \\ t_{end} & \text{if } \ell = 0 \\ \dfrac{e^{\ell t_{end}/(1-\tau\ell)} - 1}{\ell} & \text{if } \ell > 0. \end{cases}$$

Remarks:

- The assumption $\tau\ell < 1$ guarantees that the nonlinear problem which has to be solved in each time-step has indeed a solution; cf. Satz 75.1, p. 561, in [HB09].
- For large positive $\ell \gg 1$, the implicit Euler method is usually not suited, because then the condition $\tau\ell < 1$ imposes very small time-steps. The error bound also shows that this condition is not sufficient for an accurate approximation: If $\tau\ell < 1$ but $\tau\ell \approx 1$, then $e^{\ell t_{\text{end}}/(1-\tau\ell)} \gg 1$ and hence $C \gg 1$.

  In many applications, however, we have $\ell \leq 0$. In this case, the assumption $\tau\ell < 1$ is not a restriction.
- For Dahlquist's test equation (7.15) with real $\lambda < 0$, we can choose $\ell = \lambda$.
- If $\ell < 0$ and $\ell \approx 0$, then the constant $C = 1/|\ell|$ in the error bound is very large. In this case, however, we can simply choose the slightly larger $\ell = 0$ in the one-sided Lipschitz condition (7.17). This yields an error bound with $C = t_{\text{end}}$.

Proof. For simplicity consider only autonomous ODEs, i.e. $y' = f(y)$. Let Φ_τ be the numerical flow of the implicit Euler method, i.e.

$$y^{n+1} = \Phi_\tau(y^n) = y^n + \tau f\Big(\Phi_\tau(y^n)\Big). \tag{7.19}$$

Step 1: Local error. Prove a bound for $\big|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\big|$, i.e. for the error after one step starting with exact data. Taylor expansion of the exact solution:

$$y(t_n) = y(t_{n+1}) - \tau \underbrace{y'(t_{n+1})}_{=f(y(t_{n+1}))} + d_{n+1} \tag{7.20}$$

$$\text{with} \quad |d_{n+1}| \le \underbrace{\frac{1}{2} \max_{t\in[0,t_{\text{end}}]} |y''(t)|}_{=:\hat{C}} \tau^2.$$

(7.19), (7.20) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
&|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)|^2 \\
&= \Big\langle \underbrace{y(t_{n+1})}_{=\dots\ (7.20)} - \underbrace{\Phi_\tau\big(y(t_n)\big)}_{=\dots\ (7.19)}, y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big) \Big\rangle \\
&= \Big\langle y(t_n) + \tau f\big(y(t_{n+1})\big) - y(t_n) - \tau f\Big(\Phi_\tau\big(y(t_n)\big)\Big), y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big) \Big\rangle \\
&\quad - \big\langle d_{n+1}, y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big) \big\rangle \\
&\overset{(7.17)}{\le} \tau\ell \left|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\right|^2 + |d_{n+1}| \cdot \left|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\right|
\end{aligned}$$

and hence

$$(1 - \tau\ell) \left|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\right|^2 \le |d_{n+1}| \cdot \left|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\right|.$$

Since $1 - \tau\ell > 0$ by assumption, it follows that

$$\left|y(t_{n+1}) - \Phi_\tau\big(y(t_n)\big)\right| \le \frac{1}{(1-\tau\ell)} |d_{n+1}| \le \frac{\hat{C}}{(1-\tau\ell)} \tau^2.$$

Step 2: Stability. For any $z, \hat{z} \in \mathbb{R}^d$ we have

$$\begin{aligned}
|\Phi_\tau(z) - \Phi_\tau(\hat{z})|^2 &= \big\langle z + \tau f\big(\Phi_\tau(z)\big) - \hat{z} - \tau f\big(\Phi_\tau(\hat{z})\big), \Phi_\tau(z) - \Phi_\tau(\hat{z}) \big\rangle \\
&\overset{(7.17)}{\le} |z - \hat{z}| \cdot |\Phi_\tau(z) - \Phi_\tau(\hat{z})| + \tau\ell |\Phi_\tau(z) - \Phi_\tau(\hat{z})|^2.
\end{aligned}$$

Using again that $1 - \tau\ell > 0$ by assumption, this yields

$$|\Phi_\tau(z) - \Phi_\tau(\hat{z})| \le \frac{1}{1-\tau\ell} |z - \hat{z}|.$$

For the error after k steps, this implies

$$|\Phi_\tau^k(z) - \Phi_\tau^k(\hat{z})| \le \frac{1}{(1-\tau\ell)^k} |z - \hat{z}|.$$

Step 3: Error accumulation and global error. We represent the global error by the telescoping sum ("Lady Windermere's fan")

$$\begin{aligned} y(t_n) - y^n &= \Phi_\tau^0\big(y(t_n)\big) - \Phi_\tau^n(y^0) \\ &= \sum_{k=0}^{n-1} \Big(\Phi_\tau^k\big(y(t_{n-k})\big) - \Phi_\tau^{k+1}\big(y(t_{n-k-1})\big)\Big)\,. \end{aligned} \tag{7.21}$$

Picture: Lady Windermere's fan

From step 1 and 2, we know that

$$\begin{aligned} \big|\Phi_\tau^k\big(y(t_{n-k})\big) - \Phi_\tau^{k+1}\big(y(t_{n-k-1})\big)\big| &\le \frac{1}{(1-\tau\ell)^k}\,\big|y(t_{n-k}) - \Phi_\tau\big(y(t_{n-k-1})\big)\big| \\ &\le \frac{\hat{C}}{(1-\tau\ell)^{k+1}}\tau^2. \end{aligned}$$

Taking norms in (7.21) and applying the triangle inequality thus gives for $\ell \neq 0$

$$\begin{aligned} |y(t_n) - y^n| &\le \frac{\hat{C}}{1-\tau\ell}\tau^2 \sum_{k=0}^{n-1} \frac{1}{(1-\tau\ell)^k} = \frac{\hat{C}}{1-\tau\ell}\tau^2 \cdot \frac{(1-\tau\ell)^{-n}-1}{(1-\tau\ell)^{-1}-1} \\ &= \hat{C}\tau \cdot \frac{(1-\tau\ell)^{-n}-1}{\ell} \end{aligned}$$

- If $\ell < 0$, then

  $$\frac{(1-\tau\ell)^{-n}-1}{\ell} = \frac{1-(1+\tau|\ell|)^{-n}}{|\ell|} \le \frac{1}{|\ell|}$$

  because $(1+\tau|\ell|)^{-n} > 0$.

- If $\ell > 0$, then $1-\tau\ell < 1$. By assuption, we also know that $1-\tau\ell > 0$, i.e. $1-\tau\ell \in (0,1)$ and hence $(1-\tau\ell)^{-n} > 1$. This yields

  $$(1-\tau\ell)^{-n} = \left(1 + \frac{\tau\ell}{1-\tau\ell}\right)^n \le \left(e^{\tau\ell/(1-\tau\ell)}\right)^n = e^{n\tau\ell/(1-\tau\ell)} \le e^{\ell t_{\text{end}}/(1-\tau\ell)}$$

  because $1+\xi \le e^\xi$ for all $\xi \in \mathbb{R}$ and $n\tau \le t_{\text{end}}$ for all $n = 0, \dots, N$.

- The case $\ell = 0$ is left as an exercise. ∎

General principle: Consistency + stability $\Longrightarrow$ convergence

7.4 Approximation of the heat equation in time and space

Back to the heat equation:

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x) && t \in (0, t_{\text{end}}],\ x \in (a,b) \\ u(t,a) &= u_a(t), \quad u(t,b) = u_b(t) && t \in [0, t_{\text{end}}] \\ u(0,x) &= u_0(x) && x \in [a,b], \end{aligned}$$

cf. (7.1a)-(7.1c). Semi-discretization in space leads to the initial value problem

$$\begin{aligned} w'(t) &= \frac{1}{h^2} A w(t) + \frac{1}{h^2} g(t) \\ w(0) &= \bar{u}(0) \end{aligned}$$

with a vector $w(t) = \big(w_1(t), \dots, w_{m-1}(t)\big) \in \mathbb{R}^{m-1}$.
Time-discretization: Apply the implicit Euler method

$$w^{n+1} = w^n + \frac{\tau}{h^2} A w^{n+1} + \frac{\tau}{h^2} g(t_{n+1}).$$

Remark: To avoid confusion, we point out that $w^n \in \mathbb{R}^{m-1}$ is a **vector** for each n, whereas $w_k(t) \in \mathbb{R}$ is a **scalar** for each t, namely the k-th entry of the vector $w(t)$. The k-th entry of w^n is denoted by w_k^n. By definition, we expect that

$$w^n \approx w(t_n), \qquad w_k^n \approx w_k(t_n) \approx u(t_n, x_k).$$

In order to apply Theorem 7.3.2, we need the following

Lemma 7.4.1 *For every $1 < m \in \mathbb{N}$ and $h = (b-a)/m$ there is a constant $\ell_m < 0$ such that the function*

$$F(t,z) = \frac{1}{h^2} A z + \frac{1}{h^2} g(t)$$

satisfies the one-sided Lipschitz condition

$$\Big\langle F(t,z) - F(t,\tilde{z}), z - \tilde{z} \Big\rangle \leq \ell_m |z - \tilde{z}|^2$$

for all $t \in [0, t_{end}]$ and $z, \tilde{z} \in \mathbb{R}^{m-1}$.

Proof: Exercise.

Remark. The constant ℓ_m depends on m, but we can simply choose $\ell = 0$ in order to obtain a one-sided Lipschitz bound with a constant independent of m.

Corollary 7.4.2 (total error in time and space) *Let*

$$\bar{u}(t) = \Big(u(t,x_1),\dots,u(t,x_{m-1})\Big)^T \in \mathbb{R}^{m-1} \tag{7.22}$$

be the exact solution of (7.1a)-(7.1c) at the grid points, and let $w^n \in \mathbb{R}^{m-1}$ be the approximation obtained with the implicit Euler method in time and finite differences in space ($n = 0,\dots,N$). In addition to the assumptions of Theorem 7.2.5 it is assumed that

$$\max_{t\in[0,t_{end}]} |w''(t)|_h \le C \tag{7.23}$$

with a constant C independent of τ and h. Then, the error of the approximation is bounded by

$$\max_{n=1,\dots,N} |\bar{u}(t_n) - w^n|_h \le c t_{end}(h^2+\tau)$$

The constant c does not depend on h or τ, but on the regularity of the solution.

Proof. For all $n \in \mathbb{N}$ and $1 < m \in \mathbb{N}$, Theorem 7.2.5 and 7.3.2 imply

$$\begin{aligned}
|\bar{u}(t_n) - w^n|_h &\le |\bar{u}(t_n) - w(t_n)|_h + \underbrace{|w(t_n) - w^n|_h}_{\le\sqrt{h}|w(t_n)-w^n|} \\
&\le C t_{\text{end}} h^2 \sup_{t\in[0,t_{\text{end}}]} \left\|\partial_x^4 u(t,\cdot)\right\|_\infty + \frac{t_{\text{end}}}{2} \max_{t\in[0,t_{\text{end}}]} \underbrace{\sqrt{h}|w''(t)|}_{|w''(t)|_h} \cdot \tau
\end{aligned}$$

■

If the initial value problem for $w(t)$ is solved with the trapezoidal rule instead of the implicit Euler and slightly stronger regularity assumptions are made, then we obtain the error bound

$$\max_{n=1,\dots,N} |\bar{u}(t_n) - w^n|_h \le c t_{\text{end}}(h^2+\tau^2)$$

because the order of the trapezoidal rule is 2. The combination of the trapezoidal rule in time and finite differences in space is called *Crank-Nicolson method* in the literature. The results obtained with the implicit Euler method, however, are sometimes better because this method is L-stable such that errors committed in previous steps are damped; cf. IV.3 in [HW10]

If a Runge-Kutta method of order p is applied, then one would expect a total error of $ct_{\text{end}}(h^2+\tau^p)$. Unfortunately, this is in general **not** the case! The reason is that the error analysis for ODEs is based on regularity assumptions of the right-hand side which are not given in case of the heat equation. Roughly speaking, this is due to the fact that for $m \longrightarrow \infty$ we have $|A|/h^2 \longrightarrow \infty$, and the problem becomes "infinitely stiff". This leads to **order reduction**; cf. IV.15 in [HW10].

Solving the linear systems

A disadvantage of implicit Runge-Kutta methods is the fact that in each step a system of (typically nonlinear) equations has to be solved. In case of the heat equation, this system is linear. If the implicit Euler method is applied to (7.10), then we have to solve

$$Mw^{n+1} = w^n + \frac{\tau}{h^2} g(t_{n+1}) =: z \qquad \text{with} \quad M := I - \frac{\tau}{h^2} A.$$

For arbitrary τ and h the matrix M is symmetric and positive definite. Hence, the linear system can be solved via the Cholesky decomposition $M = LL^T$. It can be shown that the Cholesky decomposition does not produce any fill-in, i.e. $L_{jk} = 0$ for $j \notin \{k, k+1\}$. As a consequence, the numerical work for solving the linear system is only $\mathcal{O}(m)$ instead of $\mathcal{O}(m^3)$.

7.5 Application to the Black-Scholes equation

Consider a European capped symmetric power call, i.e. a European call option with payoff

$$V(T, S) = \min\Big(L, ((S-K)^+)^p\Big) \tag{7.24}$$

and maturity $T > 0$, strike $K > 0$ and parameters $p, L > 0$.

- "capped": $V(T, S) \leq L$ bounded
- "power": $((S-K)^+)^p$ instead of $(S-K)^+$
- "symmetric": $((S-K)^+)^p$ instead of $(S^p - K)^+$

For $p = 1$ and $L = \infty$: standard European call
The value $V(t, S)$ of the option evolves according to the Black-Scholes equation

$$\partial_t V(t,S) + \frac{\sigma^2}{2} S^2 \partial_S^2 V(t,S) + rS\partial_S V(t,S) - rV(t,S) = 0, \qquad S \in (0,\infty),\ t \in [0,T]$$

with volatility $\sigma > 0$, interest rate $r > 0$ and terminal condition (7.24). At the left boundary $S = 0$, no boundary condition is required, because the two terms $S^2\partial_S^2 V(t,S)$ and $rS\partial_S V(t,S)$ vanish for $S \to 0$ if the solution is sufficiently smooth; cf. 3.2. For $S = 0$ the PDE reduces to the ODE

$$\partial_t V(t,0) - rV(t,0) = 0,$$

and since the terminal condition (7.24) implies $V(T, 0) = 0$, it follows that

$$V(t,0) = 0 \qquad \text{for all } t \in [0,T].$$

(a) Truncation of the domain

Replace $(0,\infty)$ by $(0,S_{\max})$ with a sufficiently large $S_{\max} > K + L^{1/p}$ and boundary condition

$$V(t,S_{\max}) = L \qquad \text{for all } t \in [0,T]$$

(b) Time inversion

Let $u(t,S) := V(T-t,S)$. Then, the problem reads

$$\begin{aligned} \partial_t u &= \frac{\sigma^2}{2} S^2 \partial_S^2 u + rS\partial_S u - ru, && S \in (0,S_{\max}),\ t \in [0,T] \\ u(0,S) &= \min\Big(L, ((S-K)^+)^p\Big), && S \in [0,S_{\max}] \\ u(t,S_{\max}) &= L, && t \in [0,T] \end{aligned}$$

On the left boundary, the solution $u(t,0) = 0$ is known.

(c) Space discretization

Approximation of derivatives by finite differences yields an ordinary differential equation

$$w'(t) = Mw(t) + g(t)$$

with initial data $w(0) = \big(u(0,S_1),\dots,u(0,S_{m-1})\big)^T$; see exercise class for details. The solution yields an approximation

$$w(t) \approx \bar{u}(t) := \Big(u(t,S_1),\dots,u(t,S_{m-1})\Big)^T \in \mathbb{R}^{m-1}.$$

We only have to approximate the solution at the inner points $S_1,\dots,S_{m-1}$, because we know that $u(t,0) = 0$ and $u(t,S_{\max}) = L$.

(d) Time discretization

Solve the ODE with the implicit Euler method

$$(I - \tau M)w^{n+1} = w^n + \tau g(t_{n+1})$$

or the trapezoidal rule

$$(I - \tau M/2)w^{n+1} = (I + \tau M/2)w^n + \frac{\tau}{2}\Big(g(t_n) + g(t_{n+1})\Big).$$

Final result: $w_k^n \approx u(t_n,S_k)$ for all $n = 0,\dots,N$ and $k = 1,\dots,m-1$.

(e) Numerical experiments

See slides. The numerical examples show that the expected order of convergence is not achieved in practice because the payoff function (i.e. the initial data) does not have the required C^4 regularity. Nevertheless, we still observe convergence of the methods at some lower order. This calls for an explanation.

7.6 Non-smooth initial data

(a) Parabolic smoothing

For the our analysis, we consider again the model problem (7.1), i.e. the heat equation on an interval with Dirichlet boundary condition. This time, we choose $(a, b) = (0, \pi)$ and $u_a(t) = u_b(t) \equiv 0$ for simplicity:

$$\partial_t u(t,x) = \partial_x^2 u(t,x) \qquad t \in (0, t_{\text{end}}],\ x \in (0,\pi) \tag{7.25a}$$

$$u(t,0) = u(t,\pi) = 0 \qquad t \in [0, t_{\text{end}}] \tag{7.25b}$$

$$u(0,x) = u_0(x) \qquad x \in [0,\pi] \tag{7.25c}$$

Theorem 7.6.1 (Solution of the heat equation) *If u_0 is continuous and piecewise continuously differentiable with finitely many "kinks", then the unique solution of (7.25a)-(7.25c) is given by the Fourier series*

$$u(t,x) = \sum_{k=1}^{\infty} c_k \sin(kx) e^{-k^2 t} \quad \textit{with} \quad c_k = \frac{2}{\pi} \int_0^\pi u_0(x) \sin(kx)\, dx. \tag{7.26}$$

Sketch of the proof.

• Informal derivation of the solution formula: Separation of variables plus superposition. It can easily be checked that each term of the Fourier series (7.26) solves (7.25a) and (7.25b) (exercise).

• Initial condition: (7.25c) follows from the orthogonality relation (exercise)

$$\int_0^\pi \sin(jx)\sin(kx)\, dx = \begin{cases} 0 & \text{if } j \neq k \\ \frac{\pi}{2} & \text{if } j = k. \end{cases}$$

• Uniqueness follows from the maximum principle (see exercise class for details).

• It remains to show that derivatives of the series (7.26) are obtained by deriving term by term. Under the regularity assumptions on u_0 it can be shown that the Fourier series of the initial data

$$u_0(x) = \sum_{k=1}^{\infty} c_k \sin(kx) \quad \text{with} \quad c_k = \frac{2}{\pi} \int_0^\pi u_0(x) \sin(kx)\, dx$$

converges uniformly, and that

$$\sum_{k=1}^{\infty} |c_k| < \infty; \tag{7.27}$$

cf. §6, 2.8 in volume 2 of [FK08]. Hence, $\lim_{k\to\infty} c_k = 0$, and $\hat{c} := \sup_{k\in\mathbb{N}} |c_k| < \infty$ exists. Choose a fixed $t > 0$. Since the Fourier series

$$u(t,x) = \sum_{k=1}^{\infty} c_k \sin(kx) e^{-k^2 t}$$

is dominated by (7.27), it converges uniformly, and $u(t,\cdot)$ is thus continuous. It can be shown (see exercises) that

$$\sum_{k=1}^{\infty} k^p e^{-k^2 t} \tag{7.28}$$

converges for every fixed $p \in \mathbb{N}$. Hence, the series

$$\sum_{k=1}^{\infty} k c_k e^{-k^2 t} \cos(kx) \tag{7.29}$$

converges uniformly, because

$$|k c_k e^{-k^2 t} \cos(kx)| \le \hat{c} |k e^{-k^2 t}|. \tag{7.30}$$

This means that (7.29) coincides with $\partial_x u(t,x)$. In a similar way, we obtain from (7.28) with $p = 2$ that

$$\partial_t u(t,x) = -\sum_{k=1}^{\infty} k^2 c_k e^{-k^2 t} \sin(kx) = \partial_x^2 u(t,x).$$

■

Remark:

1. The function
$$\tilde{u}(t,x) = u_0 + \frac{u_\pi - u_0}{\pi} x + u(t,x)$$
solves (7.25a) in the special case $a = 0$, $b = \pi$ with arbitrary but constant u_0 and u_π. Solutions on arbitrary intervals can be constructed by rescaling.

2. Since (7.1a) involves $\partial_x^2 u(t,x)$, one may expect that all solutions of the PDE are twice continuously differentiable with respect to x. Theorem 7.6.1 shows, however, that solutions with lower regularity of the initial data exist.

3. All terms in the series representation (7.26) oscillate in space due to the term $\sin(kx)$, and the higher k, the faster the oscillations. The k-th term, however, is multiplied with $c_k e^{-k^2 t}$, i.e. it decays exponentially as time evolves, and the larger k, the faster the decay. This has a surprising consequence:

Theorem 7.6.2 (Parabolic smoothing) *If u_0 is continuous and piecewise continuously differentiable with finitely many "kinks", and if $u(t,x)$ is the solution of (7.25a)-(7.25c), then*

$$x \mapsto u(t,x) \in C^\infty(0,\pi) \tag{7.31}$$

for every $t > 0$.

This means that after an arbitrarily short time the solution is infinitely smooth although the initial data may have a much lower regularity. Hence, only $t \approx 0$ is critical for the approximation with finite differences.

Proof. For any $q \in \mathbb{N}$, the argument in the proof of Theorem 7.6.1 can be applied to

$$\pm \sum_{k=1}^{\infty} k^{2q+1} c_k e^{-k^2 t} \cos(kx) \qquad \text{and} \qquad \pm \sum_{k=1}^{\infty} k^{2q} c_k e^{-k^2 t} \sin(kx).$$

This yields the assertion. ∎

(b) Alternative error bound for the implicit Euler method

After discretizing (7.25) in space with finite differences, we obtain the ODE

$$w'(t) = \frac{1}{h^2} A w(t), \qquad\qquad w(0) = \bar{u}(0).$$

with $A \in \mathbb{R}^{(m-1)\times(m-1)}$ defined in (7.9). Let $w^n \approx w(t_n)$ be the approximation given by the implicit Euler method:

$$w^{n+1} = w^n + \frac{\tau}{h^2} A w^{n+1}, \qquad w^0 = w(0)$$

with $N \in \mathbb{N}$, $\tau = t_{\text{end}}/N$, $n = 0, \dots N-1$.

Theorem 7.6.3 *For all $n = 1, \dots N$, the error of the implicit Euler method is bounded by*

$$|w(t_n) - w^n|_h \le C \frac{\tau}{t_n} |w(0)|_h$$

with a constant $C \ge 0$ which does not depend on m, n or τ.

Interpretation. This result differs from the corresponding error bounds in sections 7.3 and 7.4 in several ways. First, the term $1/t_n$ is new. This term means that for small n (i.e. in the first steps), the error can be very large, but it also means that for large t_n the error will *vanish*. This is exactly what we have observed in the numerical simulations of the power option in section 7.5 (e). Moreover, the error does *not* depend on the term

$$\max_{t\in[0,t_{\text{end}}]} |w''(t)| = \max_{t\in[0,t_{\text{end}}]} \frac{|A^2 w(t)|}{h^4} \geq \frac{|A^2 w(0)|}{h^4} \tag{7.32}$$

Note that if $u_0 \notin C^4(0,\pi)$, then

$$\frac{|A^2 w(0)|}{h^4} \longrightarrow \infty \qquad \text{for} \qquad m \longrightarrow \infty, \quad h = \frac{\pi}{m} \longrightarrow 0. \tag{7.33}$$

A more general result is shown in [Tho06], Theorem 7.2, p. 117.

Numerical illustration: See slides.

Proof. As in the proof of Lemma 7.2.3 we consider the diagonalization of A, i.e. $A = Q\Lambda Q^T$ with

$$Q \text{ orthogonal}, \qquad \Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_{m-1}), \qquad \lambda_k \in (-4, 0) \text{ eigenvalues}.$$

Since $Q^T Q = I$ the function $y(t) := Q^T w(t)$ solves the ODE

$$y'(t) = Q^T w'(t) = \frac{1}{h^2} Q^T Q \Lambda Q^T w(t) = \frac{1}{h^2} \Lambda y(t),$$

and hence $y(t) = e^{t\Lambda/h^2} y(0)$. In a similar way, we transform the approximations w^n of the implicit Euler method: If we let $y^n = Q^T w^n$, then

$$y^n = y^{n-1} + \frac{\tau}{h^2}\Lambda y^n, \qquad y^n = (I - \frac{\tau}{h^2}\Lambda)^{-1} y^{n-1} = (I - \frac{\tau}{h^2}\Lambda)^{-n} y^0.$$

Hence, we only have to show that

$$\left| e^{t_n \lambda/h^2} - \left(1 - \tfrac{\tau\lambda}{h^2}\right)^{-n} \right| \leq C \frac{\tau}{t_n} \qquad \text{for all } \lambda \in (-4, 0),\ h > 0,\ n = 1, \ldots, N.$$

or, with $\mu = \tau\lambda/h^2$, $\tau/t_n = 1/n$ and $R(\mu) = \frac{1}{1-\mu}$, that

$$|e^{n\mu} - R^n(\mu)| \leq \frac{C}{n} \qquad \text{for all } \mu \in (-\infty, 0) \text{ and } n = 1, \ldots, N. \tag{7.34}$$

Case 1: $\mu \in [-1, 0)$. Comparing the Taylor expansion

$$e^{\mu} = 1 + \mu + \int_0^{|\mu|} \int_0^s e^{-r}\, dr\, ds$$

with

$$R(\mu) = 1 + \mu + \frac{\mu^2}{1-\mu}$$

yields for all $\mu \in [-1, 0)$ the "local" error bound

$$\begin{aligned} |e^\mu - R(\mu)| &\leq \int_0^{|\mu|} \int_0^s \underbrace{|e^{-r}|}_{\leq 1} \, dr \, ds + \frac{\mu^2}{1-\mu} \\ &\leq \underbrace{\int_0^{|\mu|} \int_0^s 1 \, dr \, ds}_{=\frac{\mu^2}{2}} + \mu^2 = \frac{3}{2}\mu^2. \end{aligned} \tag{7.35}$$

Moreover, it can be shown[1] that

$$R(\mu) \leq e^{\gamma\mu}, \qquad \gamma = \log(2) \approx 0.693 < 1 \qquad \text{for all } \mu \in [-1, 0). \tag{7.36}$$

Substituting (7.35) and (7.36) into the telescoping sum

$$e^{n\mu} - R^n(\mu) = \sum_{k=0}^{n-1} R(\mu)^{n-1-k} \big(e^\mu - R(\mu)\big) e^{k\mu}$$

yields

$$\begin{aligned} |e^{n\mu} - R^n(\mu)| &\leq \sum_{k=0}^{n-1} R^{n-1-k}(\mu) \, |e^\mu - R(\mu)| \, e^{k\mu} \\ &\leq \frac{3}{2}\mu^2 \sum_{k=0}^{n-1} e^{\gamma\mu(n-1-k)} e^{k\mu} \\ &\leq \frac{3}{2}\mu^2 e^{\gamma\mu(n-1)} \sum_{k=0}^{n-1} e^{k\mu(1-\gamma)}. \end{aligned}$$

Since $1 - \gamma = 1 - \log(2) > 0$ and $\mu < 0$, it follows that

$$\sum_{k=0}^{n-1} e^{k\mu(1-\gamma)} \leq \sum_{k=0}^{n-1} 1 = n$$

and hence for $\mu \in [-1, 0)$

$$|e^{n\mu} - R^n(\mu)| \leq \frac{3}{2} n\mu^2 e^{\gamma\mu n} \underbrace{e^{-\gamma\mu}}_{\leq e^\gamma} = \underbrace{\frac{3}{2}(\mu n)^2 e^{\gamma\mu n} e^\gamma}_{\leq C} \frac{1}{n}.$$

[1] To prove this, we let $f(\mu) = e^{\gamma\mu}(1-\mu) - 1$ and show that $f(\mu) \geq 0$ for all $\mu \in [-1, 0]$. Since $f(-1) = f(0) = 0$, it is enough to show that f has a maximum but no minimum in $(-1, 0)$.

Case 2: $\mu < -1$. In this case, we have

$$e^{n\mu} \leq e^{-n} \leq \frac{1}{n}$$
$$R(\mu) \leq \frac{1}{2} = e^{-\gamma} \qquad \Longrightarrow \qquad R^n(\mu) \leq e^{-n\gamma} \leq \frac{C}{n}$$

and hence

$$|e^{n\mu} - R^n(\mu)| \leq |e^{n\mu}| + |R^n(\mu)| \leq \frac{C}{n}.$$

Both cases together prove (7.34) and hence (7.32). ∎

Chapter 8

Finite-difference methods for American options

8.1 Modelling American options

American options can be exercised **before** maturity. Mathematical model?

Properties of American options: If $V_C^{Am}(t,S)$ is the value of an American call, $V_P^{Eu}(t,S)$ is the value of an European put, etc., then

$$\begin{aligned} &(K-S)^+ \le V_P^{Am}(t,S) \le K && (8.1)\\ &V_P^{Eu}(t,S) \le V_P^{Am}(t,S) \\ &V_C^{Eu}(t,S) = V_C^{Am}(t,S) \quad \text{if no dividends are paid} \\ &V_C^{Eu}(t,S) \le V_C^{Am}(t,S) \quad \text{if dividends are paid} \end{aligned}$$

Proof: Proposition 2.7 and remark 2.9 in [GJ10].

Picture

In the entire chapter, we consider American puts with no dividends and drop the indices, i.e. $V(t,S) = V_P^{Am}(t,S)$. American calls (with dividends) can be treated in a similar way.

For every $t \in [0,T)$, there is a unique $0 \le S_\star(t) < K$ such that

$$V(t,S) > (K-S)^+ \quad \text{for } S > S_\star(t) \quad \Longrightarrow \text{ no early exercise} \tag{8.2a}$$
$$V(t,S) = (K-S)^+ \quad \text{for } S \le S_\star(t) \quad \Longrightarrow \text{ early exercise} \tag{8.2b}$$

Sketch of the proof. For $S = 0$ the inequalities (8.1) imply $K \le V(t,0) \le K$ and hence

$$V(t,0) = K = (K-0)^+.$$

Since $V(t,S) > 0$ for $t < T$, it follows that

$$V(t,S) > 0 = (K-S)^+ \qquad \text{for } S \geq K.$$

A monotonicity argument yields the existence and uniqueness of $0 \leq S_\star(t) < K$. ∎

For fixed t, $S_\star(t)$ is called the **contact point**, and the function $t \mapsto S_\star(t)$ is called the **early-exercise curve**, because the option should not be exercised at time $t < T$ if $S > S_\star(t)$.

For $S \leq S_\star(t)$, the value of the option is known. For $S > S_\star(t)$ the option is not exercised and can thus be modeled by the Black-Scholes equation

$$\partial_t V + \mathcal{A}V = 0, \qquad \mathcal{A}V = \frac{\sigma^2}{2} S^2 \partial_S^2 V + rS\partial_S V - rV.$$

If $S \mapsto V(t,S)$ is C^1 for all $t \in [0,T)$, then it follows that

$$\partial_S V\big(t, S_\star(t)\big) = -1.$$

Hence, we have to solve the **free boundary value problem**

$$\begin{aligned}
\partial_t V(t,S) + \mathcal{A}V(t,S) &= 0 && \text{for } S > S_\star(t),\ t \in [0,T) && \text{(PDE)} && \text{(8.3a)}\\
V(T,S) &= (K-S)^+ && \text{for } S \geq 0 && \text{(terminal cond.)} && \text{(8.3b)}\\
V\big(t, S_\star(t)\big) &= \big(K - S_\star(t)\big)^+ && \text{for } t \in [0,T) && \text{(Dirichlet b.c.)} && \text{(8.3c)}\\
\partial_S V\big(t, S_\star(t)\big) &= -1 && \text{for } t \in [0,T) && \text{(Neumann b.c.)} && \text{(8.3d)}
\end{aligned}$$

Problem: The boundary $S_\star(t)$ changes in time and depends on the solution. Goal: Reformulate the problem without $S_\star(t)$.

We know that for $S \leq S_\star(t) < K$, the value of the option is $V(t,S) = (K-S)^+ = K - S$. This function, however, does **not** solve the Black-Scholes equation (8.3a), because then

$$\begin{aligned}
\partial_t V(t,S) + \mathcal{A}V(t,S) &= \underbrace{\partial_t(K-S)}_{=0} + \mathcal{A}(K-S)\\
&= \frac{\sigma^2}{2} S^2 \underbrace{\partial_S^2(K-S)}_{=0} + rS \underbrace{\partial_S(K-S)}_{=-1} - r(K-S) = -rK < 0
\end{aligned}$$

for all $S \leq S_\star(t)$. However, the function $V(t,S) = K - S$ satisfies the **Black-Scholes inequality**

$$\partial_t V(t,S) + \mathcal{A}V(t,S) \leq 0 \qquad \text{for } S \geq 0,\ t \in [0,T],$$

and we know:

$$\begin{aligned} S > S_\star(t) \quad &\Leftrightarrow \quad V(t,S) > (K-S)^+ \quad \Leftrightarrow \quad \partial_t V(t,S) + \mathcal{A}V(t,S) = 0 \\ S \leq S_\star(t) \quad &\Leftrightarrow \quad V(t,S) = (K-S)^+ \quad \Leftrightarrow \quad \partial_t V(t,S) + \mathcal{A}V(t,S) < 0 \end{aligned}$$

Hence, the free boundary value problem (8.3) is equivalent to the **linear complementary problem**

$$\begin{aligned} \big(V(t,S) - (K-S)^+\big)\big(\partial_t V(t,S) + \mathcal{A}V(t,S)\big) &= 0 \qquad S \geq 0, t \in [0,T] \\ -\big(\partial_t V(t,S) + \mathcal{A}V(t,S)\big) &\geq 0 \\ V(t,S) - (K-S)^+ &\geq 0 \end{aligned}$$

with terminal condition

$$V(T,S) = (K-S)^+$$

and boundary condition

$$V(t,0) = K.$$

8.2 Discretization

(a) Transformation to the heat equation.

As in 3.3 we use the following transformation:

$$\begin{aligned} x(S) &= \ln(S/K), \qquad \theta(t) = \frac{\sigma^2}{2}(T-t), \qquad c = \frac{2r}{\sigma^2} \\ E(\theta,x) &= \exp\left(\frac{1}{2}(c-1)x + \frac{1}{4}(c+1)^2\theta\right) \\ u(\theta,x) &= \frac{1}{K}E(\theta,x)V(t,S) \\ \psi(\theta,x) &= E(\theta,x)(1-e^x)^+ \end{aligned}$$

The new function $u(\theta,x)$ solves the transformed complementary problem

$$\big(u(\theta,x) - \psi(\theta,x)\big)\big(\partial_\theta u(\theta,x) - \partial_x^2 u(\theta,x)\big) = 0 \qquad x \in \mathbb{R}, \theta \in [0, \sigma^2 T/2] \tag{8.4a}$$

$$\partial_\theta u(\theta,x) - \partial_x^2 u(\theta,x) \geq 0 \tag{8.4b}$$

$$u(\theta,x) - \psi(\theta,x) \geq 0 \tag{8.4c}$$

with initial condition

$$u(0,x) = \psi(0,x)$$

and "boundary conditions"

$$\begin{aligned} \lim_{x \longrightarrow -\infty} \big(u(\theta,x) - \psi(\theta,x)\big) &= 0 \\ \lim_{x \longrightarrow \infty} u(\theta,x) &= 0 \end{aligned}$$

Proof: exercise

(b) Truncation and discretization in time and space.

Truncate the computational domain: Consider $x \in [x_{\min}, x_{\max}]$ instead of $x \in \mathbb{R}$.
Choose $1 < m \in \mathbb{N}$, let $h = (x_{\max} - x_{\min})/m$ and $x_k = x_{\min} + kh$.
Choose $N \in \mathbb{N}$, let $\tau = \frac{\sigma^2 T}{2N}$ and $\theta_n = n\tau$.

Goal: Compute approximation $w_k^n \approx u(\theta_n, x_k)$.

Discretizing the heat equation

$$\begin{aligned} \partial_\theta u(\theta,x) &= \partial_x^2 u(\theta,x), && \theta \in [0,\sigma^2 T/2],\ x \in (x_{\min}, x_{\max}) \\ u(\theta, x_{\min}) &= \psi(\theta, x_{\min}), && \theta \in [0,\sigma^2 T/2] \\ u(\theta, x_{\max}) &= 0, && \theta \in [0,\sigma^2 T/2] \\ u(0,x) &= \psi(0,x), && x \in [x_{\min}, x_{\max}] \end{aligned}$$

with the Crank-Nicolson scheme (finite differences in space, trapezoidal rule in time) yields

$$w^{n+1} = w^n + \frac{\tau}{2h^2} A(w^{n+1} + w^n) + \frac{\tau}{2h^2}\Big(g(\theta_{n+1}) + g(\theta_n)\Big)$$

with A defined in (7.9) and

$$\begin{aligned} g(\theta) &= \Big(\psi(\theta, x_{\min}), 0, \dots, 0\Big)^T \\ w^n &= (w_1^n\ ,\ \dots\ ,\ w_{m-1}^n)^T, \qquad w_k^n \approx u(\theta_n, x_k) \end{aligned}$$

Equivalent:

$$0 = \Big(I - \frac{\tau}{2h^2} A\Big) w^{n+1} - \Big(I + \frac{\tau}{2h^2} A\Big) w^n - \frac{\tau}{2h^2}\Big(g(\theta_{n+1}) + g(\theta_n)\Big)$$

For vectors $(y_1, \dots, y_d)^T$ and $(z_1, \dots, z_d)^T$ with nonnegative entries, we have

$$y_k \cdot z_k = 0 \text{ for all } k = 1, \dots, d \quad \Longleftrightarrow \quad y^T z = 0.$$

This motivates the discretization

$$\Big(w^{n+1} - \bar\psi(\theta_{n+1})\Big)^T \left(\Big(I - \frac{\tau}{2h^2} A\Big) w^{n+1} - \Big(I + \frac{\tau}{2h^2} A\Big) w^n - \frac{\tau}{2h^2}\Big(g(\theta_{n+1}) + g(\theta_n)\Big)\right) = 0 \tag{8.5a}$$

$$w^{n+1} - \bar\psi(\theta_{n+1}) \geq 0 \tag{8.5b}$$

$$\Big(I - \frac{\tau}{2h^2} A\Big) w^{n+1} - \Big(I + \frac{\tau}{2h^2} A\Big) w^n - \frac{\tau}{2h^2}\Big(g(\theta_{n+1}) + g(\theta_n)\Big) \geq 0 \tag{8.5c}$$

for the transformed linear complementary problem (8.4), with

$$\bar\psi(\theta) = \big(\psi(\theta, x_1), \dots, \psi(\theta, x_{m-1})\big)^T.$$

This has to be solved for $n = 0, \dots, N-1$.

8.3 An iterative method for linear complementary problems

Consider the linear complementary problem

$$\begin{aligned}(w-v)^T(Mw-b) &= 0\\ w-v &\geq 0\\ Mw-b &\geq 0\end{aligned}$$

with given $M \in \mathbb{R}^{d\times d}$, given $v, b \in \mathbb{R}^d$ and unknown $w \in \mathbb{R}^d$.
The problem (8.5) is obtained for $w = w^{n+1}$, $M := I - \frac{\tau}{2h^2}A$, $v = \bar{\psi}(\theta_{n+1})$ and

$$b = \Big(I + \frac{\tau}{2h^2}A\Big)\, w^n + \frac{\tau}{2h^2}\Big(g(\theta_{n+1}) + g(\theta_n)\Big).$$

Numerical method?

(a) Iterative methods for linear systems

First, consider only the linear system $Mw = b$ with $M = (M_{ij})_{ij} \in \mathbb{R}^{d\times d}$. Instead of direct methods (e.g. Gauss elimination) we consider iterative methods based on the decomposition

$$M = D - L - U$$

with

$$D = \begin{pmatrix} M_{11} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & M_{dd}\end{pmatrix}, \qquad L = -\begin{pmatrix} 0 & \cdots & \cdots & 0\\ M_{21} & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ M_{d1} & \cdots & M_{d,d-1} & 0\end{pmatrix},$$

$$U = -\begin{pmatrix} 0 & M_{12} & \cdots & M_{1d}\\ 0 & \ddots & \ddots & \vdots\\ \vdots & & \ddots & M_{d-1,d}\\ 0 & \cdots & 0 & 0\end{pmatrix}$$

Assume that $D_{kk} = M_{kk} > 0$. This is the case, e.g., if M is symmetric and positive definite. By definition:

$$Mw = b \quad \Longleftrightarrow \quad Dw = (L+U)w + b$$

Idea: Turn this into a fixed-point iteration. This yields the **Jacobi iteration**

$$Dw^{(j+1)} = (L+U)w^{(j)} + b, \qquad j = 0, 1, 2, \ldots$$

Hope that the sequence $(w^{(j)})_{j\in\mathbb{N}_0}$ converges to a fixed-point.

Often a better convergence rate is observed with the **Gauss-Seidel iteration:**

$$(D-L)w^{(j+1)} = Uw^{(j)} + b, \qquad j = 0, 1, 2, \ldots$$

In each step, a linear system with a triangular matrix has to be solved:

$$\begin{aligned}
w_1^{(j+1)} &= \frac{1}{D_{11}}\Big(0 + \sum_{k=2}^{d} U_{1k} w_k^{(j)} + b_1\Big) \\
w_2^{(j+1)} &= \frac{1}{D_{22}}\Big(L_{21} w_1^{(j+1)} + \sum_{k=3}^{d} U_{2k} w_k^{(j)} + b_2\Big) \\
w_i^{(j+1)} &= \frac{1}{D_{ii}}\Big(\sum_{k=1}^{i-1} L_{ik} w_k^{(j+1)} + \sum_{k=i+1}^{d} U_{ik} w_k^{(j)} + b_i\Big) \qquad i = 1, \ldots, d.
\end{aligned}$$

Generalization: **Successive overrelaxation method (SOR method)**:

```
For j = 0, 1, 2, ...
    w^(j+1) = w^(j)                                   (will be overwritten)
    For k = 1, ..., d
        w̃_k^(j) = 1/D_kk [L w^(j+1) + U w^(j) + b]_k          (k-th entry)
        w_k^(j+1) = w_k^(j) + r(w̃_k^(j) − w_k^(j))
    end
end
```

with relaxation parameter $r \in \mathbb{R}$. If $r = 1$, then $w^{(j+1)} = \tilde{w}^{(j)}$, and we obtain the Gauss-Seidel method. For $r \in (0, 1)$, $w^{(j+1)}$ is an interpolation between $w^{(j)}$ and $\tilde{w}^{(j)}$.
In practice, however, typically $r > 1$ is chosen.

(b) The projected SOR method for linear complementary problems

Back to the linear complementary problem:

$$(w - v)^T (Mw - b) = 0 \tag{8.6a}$$
$$w - v \geq 0 \tag{8.6b}$$
$$Mw - b \geq 0 \tag{8.6c}$$

In general, we cannot expect that $Mw = b$. There are some entries k where $[Mw-b]_k > 0$. The complementary problem is equivalent to

$$\begin{aligned} & \min\Big\{[Mw-b]_k, [w-v]_k\Big\} = 0 \qquad \text{for all } k = 1,\dots,d \\ \iff \quad & \min\Big\{\Big[\underbrace{D^{-1}\big((D-L-U)w-b\big)}_{w-D^{-1}\left((L+U)w+b\right)}\Big]_k, [w-v]_k\Big\} = 0 \qquad \text{for all } k = 1,\dots,d \\ \iff \quad & \max\Big\{\Big[D^{-1}\big((L+U)w+b\big)\Big]_k, v_k\Big\} = w_k \qquad \text{for all } k = 1,\dots,d \end{aligned}$$

This motivates the **projected SOR method**:

```
For j = 0, 1, 2, ...
    w^(j+1) = w^(j)                                   (will be overwritten)
    For k = 1, ..., d
        w̃_k^(j) = 1/D_kk [L w^(j+1) + U w^(j) + b]_k
        w_k^(j+1) = max{ w_k^(j) + r(w̃_k^(j) - w_k^(j)) , v_k }
    end
end
```

Theorem 8.3.1 (Cryer) *Let $v, b \in \mathbb{R}^d$, $r \in (1,2)$ and assume that $M \in \mathbb{R}^{d\times d}$ is symmetric and positive definite. Then, the linear complementary problem (8.6) has a unique solution, and the iterates $w^{(j)}$ of the projected SOR method converge to the solution.*

The proof of uniqueness of the solution will be based on the following

Lemma 8.3.2 *If $M \in \mathbb{R}^{d\times d}$ is symmetric and positive definite, then the linear complementary problem (8.6) is equivalent to the problem to find a $w \in \mathbb{R}^d$ such that*

$$w \geq v \qquad \text{and} \qquad J(w) \leq J(y) \qquad \text{for all} \qquad y \geq v \tag{8.7}$$

where $J(y) = \frac{1}{2}y^T My - b^T y$.

Proof. Let w be a solution of (8.6) and let $y \geq v$. Then, we have

$$J(y) - J(w) = \frac{1}{2}\underbrace{(y-w)^T M(y-w)}_{\geq 0} + (y-w)^T(Mw-b) \geq (y-w)^T(Mw-b)$$

because M is positive definite. Since $y \geq v$ by assumption and w is the solution of (8.6), we have

$$(y-w)^T(Mw-b) = \underbrace{(y-v)^T}_{\geq 0}\underbrace{(Mw-b)}_{\geq 0} - \underbrace{(w-v)^T(Mw-b)}_{=0} \geq 0$$

which proves that $J(y) \geq J(w)$. Conversely, assume that w is the solution of (8.7). This means, in particular, that $w \geq v$. We show that $Mw-b \geq 0$ holds, too. Define $y = w+\varepsilon e_k$ for $\varepsilon > 0$, where e_k is the k-th unit vector. Then clearly $y \geq w \geq v$ and

$$0 \leq J(y) - J(w) = \frac{\varepsilon^2}{2} e_k^T M e_k + \varepsilon e_k^T (Mw - b) = \frac{\varepsilon^2}{2} M_{kk} + \varepsilon [Mw - b]_k.$$

Dividing by ε and letting $\varepsilon \to 0$ yields $[Mw-b]_k \geq 0$ for arbitrary k and hence $Mw-b \geq 0$. It remains to show that $(Mw - b)^T(w - v) = 0$. Assume that there is a k such that both

$$[Mw - b]_k > 0 \qquad \text{and} \qquad [w - v]_k > 0. \tag{8.8}$$

If we choose $\varepsilon > 0$ so small that $y := w - \varepsilon e_k \geq v$, it follows that

$$0 \leq J(y) - J(w) = \frac{\varepsilon^2}{2} M_{kk} - \varepsilon \underbrace{[Mw - b]_k}_{>0} < 0$$

for sufficiently small ε, which yields a contradiction. For every k we thus have either $[Mw - b]_k = 0$ or $[w - v]_k = 0$ and hence $(Mw - b)^T(w - v) = 0$. ∎

Proof of Theorem 8.3.1. First we show uniqueness of the solution of the linear complementary problem. Assume that w and $\hat{w}$ are both solutions of (8.6). Then

$$\begin{aligned}
0 &= J(\hat{w}) - J(w) \\
&= \frac{1}{2}(\hat{w} - w)^T M(\hat{w} - w) + (\hat{w} - w)^T (Mw - b) \\
&= \frac{1}{2}(\hat{w} - w)^T M(\hat{w} - w) + \underbrace{(\hat{w} - v)^T}_{\geq 0} \underbrace{(Mw - b)}_{\geq 0} - \underbrace{(w - v)^T (Mw - b)}_{=0} \\
&\geq \frac{1}{2}(\hat{w} - w)^T M(\hat{w} - w) \geq 0
\end{aligned}$$

and hence $\hat{w} = w$ because M is positive definite.
Next, we prove that the iterates $w^{(j)}$ of the SOR method converge to a solution of the minimization problem (8.7). This implies existence of a solution of the linear complementary problem (8.6).

Step 1: For all $j \in \mathbb{N}_0$ and $k = 1, \ldots, d$ it can be shown that there is a $r_{jk} \in [0, r]$ such that

$$w_k^{(j+1)} = w_k^{(j)} + r_{jk} \left(\tilde{w}_k^{(j)} - w_k^{(j)} \right). \tag{8.9}$$

Details: See p. 212 in [GJ10].

Step 2: Let $w^{(j,k)}$ be the vector obtained with the projected SOR method for given numbers j and k. If $k < d$, then only the entries with indices $1, \ldots, k$ have been updated in the inner loop:

$$
\begin{aligned}
w^{(j,k)} &= \left(w_1^{(j+1)}, \ldots, w_k^{(j+1)}, w_{k+1}^{(j)}, \ldots, w_d^{(j)}\right)^T && \text{for } k \in \{1, \ldots, d-1\} \\
w^{(j,0)} &:= \left(w_1^{(j)}, \ldots, w_d^{(j)}\right)^T = w^{(j)} && \text{(no update yet in step } j\text{)} \\
w^{(j,d)} &:= \left(w_1^{(j+1)}, \ldots, w_d^{(j+1)}\right)^T = w^{(j+1)} && \text{(all updates in step } j \text{ completed)}
\end{aligned}
$$

Show that the sequence $J(w^{(j,k)})$ converges when the indices are changed in the following order

$$
(j,k) = (0,1), (0,2), \ldots (0,d), \quad (1,1), (1,2), \ldots, (1,d), \quad (2,1), (2,2), \ldots . \tag{8.10}
$$

By definition, we have that

$$
w^{(j,k)} - w^{(j,k-1)} = \left(w_k^{(j+1)} - w_k^{(j)}\right) e_k \qquad (k = 1, \ldots, d). \tag{8.11}
$$

Since only the first $k-1$ entries of $w^{(j,k-1)}$ have been updated, it follows from the definition of $\tilde{w}_k^{(j)}$ in the projected SOR method that

$$
\begin{aligned}
\left[Mw^{(j,k-1)} - b\right]_k &= \left[Dw^{(j,k-1)} - Lw^{(j,k-1)} - Uw^{(j,k-1)} - b\right]_k \\
&= \left[Dw^{(j)} - Lw^{(j+1)} - Uw^{(j)} - b\right]_k \\
&= D_{kk}\left(w_k^{(j)} - \tilde{w}_k^{(j)}\right) \\
&= M_{kk}\left(w_k^{(j)} - \tilde{w}_k^{(j)}\right)
\end{aligned} \tag{8.12}
$$

For $r_{jk} > 0$ we thus obtain from (8.11) and (8.12) that

$$
\begin{aligned}
&J(w^{(j,k)}) - J(w^{(j,k-1)}) \\
&= \frac{1}{2}\big(\underbrace{w^{(j,k)} - w^{(j,k-1)}}_{(8.11)}\big)^T M\big(\underbrace{w^{(j,k)} - w^{(j,k-1)}}_{(8.11)}\big) + \big(\underbrace{w^{(j,k)} - w^{(j,k-1)}}_{(8.11)}\big)^T \left(Mw^{(j,k-1)} - b\right) \\
&= \frac{1}{2}\left(w_k^{(j+1)} - w_k^{(j)}\right)^2 \underbrace{e_k^T M e_k}_{=M_{kk}} + \left(w_k^{(j+1)} - w_k^{(j)}\right) \underbrace{e_k^T\left(Mw^{(j,k-1)} - b\right)}_{(8.12)} \\
&= \frac{M_{kk}}{2}\left(w_k^{(j+1)} - w_k^{(j)}\right)^2 + \left(w_k^{(j+1)} - w_k^{(j)}\right) M_{kk}\left(w_k^{(j)} - \tilde{w}_k^{(j)}\right).
\end{aligned}
$$

Equation (8.9) implies

$$
w_k^{(j)} - \tilde{w}_k^{(j)} = \left(w_k^{(j)} - w_k^{(j+1)}\right) / r_{jk}.
$$

Since $r_{jk} \leq r < 2$ by assumption, this yields

$$J(w^{(j,k)}) - J(w^{(j,k-1)}) = -\frac{M_{kk}}{2} \underbrace{\left(\frac{2}{r_{jk}} - 1\right)}_{\geq 0} \left(w_k^{(j+1)} - w_k^{(j)}\right)^2 \tag{8.13}$$
$$\leq -\frac{M_{kk}}{2} \left(\frac{2}{r} - 1\right) \left(w_k^{(j+1)} - w_k^{(j)}\right)^2 \leq 0.$$

If $r_{jk} = 0$, then $w_k^{(j+1)} = w_k^{(j)}$ and hence $w^{(j,k)} = w^{(j,k-1)}$. Hence, the sequence $(J(w^{(j,k)}))_{j,k}$ is monotonically decreasing. Next, we show that $J(y)$ is bounded from below. Let $\lambda_{\min}$ be the smallest eigenvalue of M. Since M is symmetric and positive definite, it follows that $\lambda_{\min} > 0$ and $y^T M y \geq \lambda_{\min} \|y\|^2$ for all y. Together with the Cauchy-Schwarz inequality, this yields that

$$J(y) = \frac{1}{2} y^T M y - b^T y \geq \frac{\lambda_{\min}}{2} \|y\|^2 - \|b\| \cdot \|y\| = \frac{1}{2\lambda_{\min}} (\lambda_{\min} \|y\| - \|b\|)^2 - \frac{\|b\|^2}{2\lambda_{\min}}$$
$$\geq -\frac{\|b\|^2}{2\lambda_{\min}}.$$

As a consequence, the sequence $(J(w^{(j,k)}))_{j,k}$ converges.

Step 3: It can be shown that for every fixed k the limit

$$w_k := \lim_{j \to \infty} w_k^{(j)}$$

exists.

Remark: In the lecture one of the students noticed that the argument in this step was not really correct. The problem is that

$$\left| w_k^{(j+1)} - w_k^{(j)} \right|^2 \to 0$$

does NOT imply that $(w_k^{(j)})_j$ is a Cauchy sequence for every k. This wrong argument, however, already appears in the reference from which I took the proof, namely [GJ10], page 213, Schritt 3. Therefore, it will take me some time to correct this. For the time being, I will not ask questions about the proof of step 3 in the exams.

Step 4: We show that $w = (w_1, \ldots, w_d)^T$ solves the linear complementary problem (8.6). Since $L + U = D - M$ we have

$$\tilde{w}_k := \lim_{j \to \infty} \tilde{w}_k^{(j)} = M_{kk}^{-1} [Lw + Uw + b]_k = w_k - M_{kk}^{-1} [Mw - b]_k$$
$$w_k = \max\{w_k + r(\tilde{w}_k - w_k), v_k\} = \max\left\{w_k - r M_{kk}^{-1} [Mw - b]_k, v_k\right\}.$$

This yields

$$\min\left\{rM_{kk}^{-1}[Mw-b]_k, w_k - v_k\right\} = 0$$

which is equivalent to (8.6). ■

8.4 Summary: Pricing American options with the projected SOR method

- Start: Free boundary problem (8.3) with solution $V(t,S)$.
- Reformulation as a linear complementary problem.
- Transformation:

  $V(t,S) \longrightarrow u(\theta,x)$, Black-Scholes inequality $\longrightarrow$ heat inequality

  $\Longrightarrow$ transformed linear complementary problem (8.4).
- Truncation: Restrict $x \in \mathbb{R}$ to $x \in [x_{\min}, x_{\max}]$, choose boundary conditions.
- Discretize time and space: $w_k^n \approx u(\theta_n, x_k)$.
- Algorithm:

```
For n = 0, 1, ..., N − 1                                  (time points)
    Solve linear complementary problem (8.5) with the
    projected SOR method
    For j = 0, 1, 2, ...                                  (iteration number)
        For k = 1, ..., m − 1                               (entry number)
            ...
        end
    end
end
```

- Transform back.

— FIN —

Appendix A

Some definitions from probability theory

Definition A.0.1 (Probability space) *The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **probability space**, if the following holds:*

1. *$\Omega \neq \emptyset$ is a set, and $\mathcal{F}$ is a **σ-algebra** (or σ-field) on Ω, i.e. a family of subsets of Ω with the following properties:*
   - *$\emptyset \in \mathcal{F}$*
   - *If $F \in \mathcal{F}$, then $\Omega \setminus F \in \mathcal{F}$*
   - *If $F_i \in \mathcal{F}$ for all $i \in \mathbb{N}$, then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{F}$*

   *The pair $(\Omega, \mathcal{F})$ is called a **measurable space**.*
2. *$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$ is a **probability measure**, i.e.*
   - *$\mathbb{P}(\emptyset) = 0$ and $\mathbb{P}(\Omega) = 1$*
   - *If $F_i \in \mathcal{F}$ for all $i \in \mathbb{N}$ are pairwise disjoint (i.e. $F_i \cap F_j = \emptyset$ for $i \neq j$), then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(F_i).$$

Definition A.0.2 (Borel σ-algebra) *If $\mathcal{U}$ is a family of subsets of Ω, then **the σ-algebra generated by $\mathcal{U}$** is*

$$\mathcal{F}_{\mathcal{U}} = \bigcap \{\mathcal{F} : \mathcal{F} \text{ is a } \sigma\text{-algebra of } \Omega \text{ and } \mathcal{U} \subset \mathcal{F}\}.$$

*If $\mathcal{U}$ is the collection of all open subsets of a topological space Ω (e.g. $\Omega = \mathbb{R}^d$), then $\mathcal{B} = \mathcal{F}_{\mathcal{U}}$ is called the **Borel σ-algebra** on Ω. The elements $B \in \mathcal{B}$ are called Borel sets.*

For the rest of this section $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Definition A.0.3 (Measurable functions, random variables)

- *A function $X : \Omega \longrightarrow \mathbb{R}^d$ is called* ***$\mathcal{F}$-measurable*** *if*

$$X^{-1}(B) := \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}$$

  for all Borel sets $B \in \mathcal{B}$. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, then every $\mathcal{F}$-measurable functions is called a ***random variable****.*

- *Random variables $X_1, \ldots, X_n$ are called* ***independent*** *if*

$$\mathbb{P}\left(\bigcap_{i=1}^{n} X_i^{-1}(A_i)\right) = \prod_{i=1}^{n} \mathbb{P}\big(X_i^{-1}(A_i)\big)$$

  for all $A_1, \ldots, A_n \in \mathcal{B}$.

- *If $X : \Omega \longrightarrow \mathbb{R}^d$ is any function, then the* ***σ-algebra generated by X*** *is the smallest σ-algebra on Ω containing all the sets*

$$X^{-1}(B) \text{ for all } B \in \mathcal{B}.$$

  Notation: $\mathcal{F}^X = \sigma\{X\}$
  $\mathcal{F}^X$ is the smallest σ-algebra where X is measurable.

Appendix B

The Riemann-Stieltjes integral

Let $f : [a, b] \to \mathbb{R}$ be bounded and let $w : [a, b] \to \mathbb{R}$ be increasing, i.e. $w(t) \geq w(s)$ for all $t \geq s$. For a partition $a = t_0 < t_1 < \ldots < t_N = b$ we define the lower and upper sums

$$\overline{S_N} := \sum_{n=0}^{N-1} \sup\{f(t) : t \in [t_n, t_{n+1}]\}\big(w(t_{n+1}) - w(t_n)\big)$$
$$\underline{S_N} := \sum_{n=0}^{N-1} \inf\{f(t) : t \in [t_n, t_{n+1}]\}\big(w(t_{n+1}) - w(t_n)\big).$$

If $\overline{S_N}$ and $\underline{S_N}$ converge to the same value as the partition is refined, then the Riemann-Stieltjes integral is defined by

$$\int_a^b f(t)dw(t) := \lim_{N\to\infty} \overline{S_N} = \lim_{N\to\infty} \underline{S_N}.$$

For $w(t) = t$, this is the standard Riemann integral.

If $w : [a, b] \to \mathbb{R}$ is not increasing but has bounded variation, then there are increasing functions $w_1 : [a, b] \to \mathbb{R}$ and $w_2 : [a, b] \to \mathbb{R}$ such that $w(t) = w - 1(t) - w_2(t)$, and the Riemann-Stieltjes integralcan be defined by

$$\int_a^b f(t)dw(t) := \int_a^b f(t)dw_1(t) - \int_a^b f(t)dw_2(t).$$

Appendix C

Runge-Kutta methods – a reminder

Consider the initial value problem

$$y'(t) = f(t, y), \qquad t \in [t_0, t_{\text{end}}], \qquad y(0) = y_0 \tag{C.1}$$

with appropriate function $f : [t_0, t_{\text{end}}] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ (e.g. $y \mapsto f(t, y)$ Lipschitz continuous). Choose $N \in \mathbb{N}$, let $\tau = (t_{\text{end}} - t_0)/N$ and $t_n = t_0 + n\tau$.
Goal: Find approximations $y_n \approx y(t_n)$, $n = 0, 1, \ldots, N$. Ansatz:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} y'(s)\, ds = y(t_n) + \int_{t_n}^{t_{n+1}} f\big(s, y(s)\big)\, ds$$

Approximate the integral by the quadrature formula

$$y(t_n + \tau) \approx y(t_n) + \tau \sum_{i=1}^{s} b_i f\big(t_n + c_i\tau, y(t_n + c_i\tau)\big)$$

with $s \in \mathbb{N}$ and coefficients b_i and c_i. Apply the same procedure to $y(t_n + c_i\tau)$: Approximate

$$\begin{aligned} y(t_n + c_i\tau) &= y(t_n) + \int_{t_n}^{t_n + c_i\tau} f\big(s, y(s)\big)\, ds \\ &\approx y(t_n) + \tau \sum_{j=1}^{s} a_{ij} f\big(t_n + c_j\tau, y(t_n + c_j\tau)\big), \qquad i = 1, \ldots, s \end{aligned}$$

with coefficients a_{ij}. This yields the **Runge-Kutta method**: For each $n = 0, \ldots, N-1$ solve the nonlinear system

$$Y_i = y_n + \tau \sum_{j=1}^{s} a_{ij} f\big(t_n + c_j\tau, Y_j\big), \qquad i = 1, \ldots, s \tag{C.2a}$$

(e.g. by a version of Newton's method) and let

$$y_{n+1} = y_n + \tau \sum_{i=1}^{s} b_i f\big(t_n + c_i\tau, Y_i\big). \tag{C.2b}$$

Each Runge-Kutta method is characterized by its coefficients a_{ij}, b_i, c_j. These are represented in the Butcher tableau:

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

Examples:

- Explicit Euler method: $y_{n+1} = y_n + \tau f(t_n, y_n)$
- Implicit Euler method: $y_{n+1} = y_n + \tau f(t_{n+1}, y_{n+1})$
- Trapezoidal rule: $y_{n+1} = y_n + \frac{\tau}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big)$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array} \qquad \begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

explicit Euler implicit Euler trapezoidal rule

The Runge-Kutta method (C.2a), (C.2b) is explicit if $a_{ij} = 0$ for all $j \geq i$.

Notation: Let $\Psi_t(t_\star, y_\star)$ be the flow of the ODE $y'(t) = f(t, y)$, i.e. $y(t) = \Psi_{t-t_\star}(t_\star, y_\star)$ is the exact solution of the initial value problem

$$y'(t) = f(t, y), \qquad t > t_\star, \qquad y(t_\star) = y_\star.$$

The approximation of a Runge-Kutta method after n steps with step-size τ and initial value $y_\star$ at time $t_\star$ is denoted by $\Phi_\tau^n(t_\star, y_\star)$; Φ_τ is called the numerical flow.

Definition C.0.4 *A Runge-Kutta method applied to the ODE $y'(t) = f(t, y)$ with a sufficiently smooth function f has order p if*

$$\|\Psi_\tau(t_\star, y_\star) - \Phi_\tau(t_\star, y_\star)\| \leq C\tau^{p+1} \tag{C.3}$$

for all $y_\star \in \mathbb{R}^d$, $t_\star \in \mathbb{R}$ and sufficiently small step-sizes $\tau > 0$. The constant may depend on f, $y_\star$, and $t_\star$ but must be independent of τ.

Remark: The term $\|\Psi_\tau(t_\star, y_\star) - \Phi_\tau(t_\star, y_\star)\|$ is the *local error*, i.e. the approximation error after only *one* step of the method. If the method is stable, then (C.3) implies the bound

$$\|y(t_n) - y_n\| \leq C\tau^p, \qquad y_n = \Phi_\tau^n(t_0, y_0)$$

for the *global error* after n steps; cf. chapter II.3 in [HNW10].

Theorem C.0.5 *A Runge-Kutta method with the property*

$$\sum_{j=1}^{s} a_{ij} = c_i \quad \textit{for all } i = 1, \ldots, s$$

has order

- $p = 1$, *if*

$$\sum_{i=1}^{s} b_i = 1,$$

- $p = 2$, *if in addition*

$$\sum_{i=1}^{s} b_i c_i = \frac{1}{2},$$

- $p = 3$, *if in addition*

$$\sum_{i=1}^{s} b_i c_i^2 = \frac{1}{3}, \qquad \sum_{i=1}^{s} \sum_{i=1}^{s} b_i a_{ij} c_j = \frac{1}{6}.$$

Proof. Taylor expansion of $y(t_0 + h)$ and y_1 about $h = 0$ + long computations. See chapter XIV, Satz 76.6, in [HB09], or chapter II.2 in [HNW10]. ■

Both Euler methods have order 1, the trapezoidal rule has order 2.

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