623 Computational Finance, Fall 2014.
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Chapter IV. Fixed income markets.
Partial information can be found in Chapters 3.3.1, 3.3.2, 3.6.1, 3.7.1, 3.7.2 in "Monte Carlo Methods in Financial Engineering" by Glasserman 2004; and in Chapters 3.3, 5.1, 6.1–6.3 in "Interest Rate Models – Theory and Practice" by Brigo and Mercurio 2006.

1 Interest rate derivatives

Basic products in the fixed income markets.

- **Zero coupon zero default risk bond**, such as a treasury bond. Let

$$P_t(T) \;=\; \text{value at time } t < T \text{ of the bond with face value 1 which pays 1 at maturity time } T. \quad (1)$$

- **Forward Rate Agreement (FRA)** allows one to lock in the desired borrowing or lending rate K over a future time period $[T_0, T_1]$. Denote by $L_{T_0}(T_1)$ the (future) borrowing/lending rate, given by the market, assuming simple compounding with time step $T_1 - T_0$:

$$P_{T_0}(T_1)(1 + (T_1 - T_0)L_{T_0}(T_1)) = 1, \qquad L_{T_0}(T_1) = \frac{1}{(T_1 - T_0)P_{T_0}(T_1)} - \frac{1}{T_1 - T_0}$$

Then, the total payoff to the FRA buyer at time T_1 is

$$1 + (T_1 - T_0)K - 1 - (T_1 - T_0)L_{T_0}(T_1) = (T_1 - T_0)(K - L_{T_0}(T_1))$$

So, the FRA buyer loses when $L_{T_0}(T_1) > K$ and gains when $L_{T_0}(T_1) < K$. The payoff of FRA can be replicated by the following strategy:

  - at time t,
    * buy $(T_1 - T_0)K + 1$ bonds with maturity T_1,
    * sell 1 bond with maturity T_0;
  - at time T_0, borrow 1 unit of currency to repay at time T_1 (i.e. sell a bond with maturity T_1 at time T_0).

The total payoff of this strategy at time T_1 is

$$(T_1 - T_0)K - (T_1 - T_0)L_{T_0}(T_1),$$

which is the same as the payoff of FRA. Hence, the present prices of the two must coincide. In particular, the price of FRA is

$$(T_1 - T_0)KP_t(T_1) - P_t(T_0) + P_t(T_1)$$

In fact, in FRA contracts, there is no cash flow at the initial time T: the value of K is chosen so that the price of the contract is zero. Thus,

$$K = F_t(T_0, T_1) = \frac{P_t(T_0) - P_t(T_1)}{(T_1 - T_0)P_t(T_1)},$$

where we have introduced the notion of **forward rate** $F_t(T_0, T_1)$.

When $T_1 \downarrow T_0$, the above formula produces the **instantaneous forward rate**

$$f_t(T) = \partial_T \log P_t(T)$$

- More generally, we can consider **Interest-Rate Swap (IRS)**, which gives one a right to lend or borrow one unit of currency at rate K over each period $[T_{i-1}, T_i]$, with repayment at T_i, for $i = 1, \ldots, N$ and some fixed $T_0 < T_1 < \cdots < T_N$. The **swap rate** at time $t \leq T_0$ is then denoted by S_t and it is the fair value of K that makes the initial price of IRS zero. It can be shown by, essentially, the same no-arbitrage argument that

$$S_t = \frac{P_t(T_0) - P_t(T_N)}{\sum_{j=1}^{N} (T_j - T_{j-1}) P_t(T_j)} \tag{2}$$

  Observe that the forward rate $F_t(T_0, T_1)$ and the swap rate S_t are known today (time $t = 0$) from today's term structure (i.e. from today's bond prices or forward rates).

- Next, we consider interest rate derivatives, whose prices cannot be determined uniquely from today's term structure. We start with the interest rate **cap**. This agreement gives its holder the right to borrow one unit of currency at rate K, at every future timeT_{j-1}, with repayment at T_j, for $j = 1, \ldots, N$. The decision (whether to borrow or not) for the period $[T_{j-1}, T_j]$ is made at time T_{j-1}. A cap can be divided into several **caplets**, corresponding to different time periods $[T_{j-1}, T_j]$. The payoff of one caplet is the benefit that this contract holder receives by borrowing at rate K and lending at the market rate at time T_j. In other words, the caplet corresponding to $[T_{j-1}, T_j]$ pays

$$(T_j - T_{j-1}) \left(L_{T_{j-1}}(T_j) - K \right)^+ \quad \text{at time } T_j, \tag{3}$$

  where $L_t(T)$ is the market forward rate available at time t for borrowing at t with repayment at T:

$$L_t(T) = F_t(t, T) = \frac{1 - P_t(T)}{(T - t) P_t(T)}$$

  A **cap** is a portfolio of N caplets, corresponding to the periods $[T_{j-1}, T_j]$ for all $j = 1, \ldots, N$. In the cap contract, the borrowing rate is 'capped' at level K, hence the name.

  Analogously, one defines **floorlet**, which pays

$$(T_j - T_{j-1}) \left(K - L_{T_{j-1}}(T_j) \right)^+ \quad \text{at time } T_j. \tag{4}$$

  A **floor** is defined as a portfolio of N floorlets, corresponding to the periods $[T_{j-1}, T_j]$ for all $j = 1, \ldots, N$.

- **Swaption** is a very common form of interest rate derivative since most mortgages have prepayment options, which can be viewed as swaptions.

  A payer (call) swaption gives its holder the right, but not the obligation, to enter IRS with rate K at a future time T_0, with repayment times $T_1, \ldots, T_N$. The decision of borrowing at each T_{j-1} or not (i.e. of entering the IRS or not) has to be made at time T_0. Therefore, a payer (call) swap with rate K pays

$$(S_{T_0} - K)^+ \sum_{j=1}^{N} (T_j - T_{j-1}) P_{T_0}(T_j) = \left(\sum_{j=1}^{N} P_{T_0}(T_j)(T_j - T_{j-1})(F_{T_0}(T_{j-1}, T_j) - K) \right)^+$$

at time T_0.

Analogously, we define the receiver (put) swaption, which pays

$$(K - S_{T_0})^+ \sum_{j=1}^{N} (T_j - T_{j-1}) P_{T_0}(T_j) = \left(\sum_{j=1}^{N} P_{T_0}(T_j)(T_j - T_{j-1})(K - F_{T_0}(T_{j-1}, T_j)) \right)^+$$

at time T_0, where S_{T_0} is the swap rate at time T_0.

2 Hull-White extended Vasicek Model

In the HW-Vasicek model (1990), the short rate r_t is given by the following SDE under the risk neutral measure $\mathbb{Q}$:

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t \ , \quad r_0 \tag{5}$$

There are two parameters $a, \sigma > 0$, which we assume to be constant, and a function $\theta(s)$, for $s \geq 0$. The parameters a, σ are first determined from yield curve correlation and volatility data. The function $\theta(s)$ is determined from **calibrating the model to today's term structure of bond prices (or froward rates)**.

The linear SDE (5) is exactly solvable just like in the case of constant θ.

$$r_t = e^{-at} r_0 + \int_0^t e^{-a(t-s)} \theta(s) ds + \sigma \int_0^t e^{-a(t-s)} dW_s \tag{6}$$

It follows from (6) that for $t > 0$ the random variable r_t is Gaussian with

$$\mathbb{E} r_t = e^{-at} r_0 + \int_0^t e^{-a(t-s)} \theta(s) ds, \tag{7}$$

$$\mathrm{Var}(r_t) = \sigma^2 \int_0^t e^{-2a(t-s)} ds = \frac{\sigma^2}{2a} \left[1 - e^{-2at} \right], \tag{8}$$

$$\mathrm{Var}[r_t] \sim \frac{\sigma^2}{2a} \text{ at large } t.$$

Since variance is bounded for all time the short term rate is bounded with high probability for all time, a desirable property since in practice we would not expect large interest rates (certainly beyond 10%) to play an important role in our model.

2.1 Estimating volatility and mean reversion rate

To estimate the parameters a and σ, we find a formula for the **autocorrelation function** of (r_t), denoted $\rho(t_1, T_2)$, which is the correlation of r_{t_1} and r_{t_2}, for all $0 \leq t_1 \leq t_2 < \infty$. From (6) we have that

$$\begin{aligned} \mathrm{cov}(r_{t_1}, r_{t_2}) \ &= \ \sigma^2 E \left[\left(\int_0^{t_1} e^{-a(t_1-s)} \ dB_s \right) \left(\int_0^{t_2} e^{-a(t_2-s)} \ dB_s \right) \right] \\ &= \ \sigma^2 \int_0^{t_1} e^{-a(t_1-s)-a(t_2-s)} ds \ = \ \frac{\sigma^2}{2a} \left[e^{-a(t_2-t_1)} - e^{-a(t_2+t_1)} \right] \end{aligned} \tag{9}$$

From (8), (9) we conclude that at large t_1 the autocorrelation $\rho(t_1, t_2)$ is given by

$$\rho(t_1, t_2) \sim e^{-a(t_2 - t_1)} . \tag{10}$$

We can now estimate the autocorrelation and variance from the historical data (assuming that a is the same under both pricing and physical measures) and use (8), (10) to find the reasonable values for the parameters a, σ of the HW-Vasicek model.

2.2 Pricing bonds and calibrating to the initial term structure

In order to price interest rate derivatives we need to give a formula for bond prices in terms of the current short rate r_t. The bond price is given by the expectation of its discounted payoff:

$$P_t(T) = E\left(\exp\left(-\int_t^T r_s\, ds\right)\right) . \tag{11}$$

To calibrate the short rate model to today's term structure (i.e. bond prices or forward rates), we need to choose θ, such that

$$P^{mkt}(T) = P_t(T) \quad \text{for all } T, \tag{12}$$

where the LHS of (12) is current market price of a zero coupon bond with face value 1 and maturity T, and the right hand side contains the model price of this bond. The value of r_0 is today's short rate, and a and σ are determined for historical data.

The calibration of the HW-Vasicek model to today's term structure is fairly simple because the short rate r_t can be written as a sum

$$r_t = r_t^* + \alpha(t), \quad \text{where } dr_t^* = -ar_t^* dt + \sigma dW_t,\;\; r_0^* = 0. \tag{13}$$

The function $\alpha(t),\ t \geq 0$, is deterministic and satisfies

$$d\alpha(t) \;=\; [\theta(t) - a\alpha(t)]\, dt\,, \quad \alpha(0) \;=\; r_0, \tag{14}$$

$$\alpha(t) = e^{-at} r_0 + \int_0^t e^{-a(t-s)} \theta(s) ds$$

It is easy to see that if $r^*(\cdot)$ satisfies (13) and $\alpha(\cdot)$ satisfies (14), then (r_t) satisfies (5).

Rem 1. *We can view (13) as the decomposition into a part r^*, which is entirely due to yield curve volatility, and a deterministic part obtained from today's yield curve. In particular, if there is* **zero** *yield curve volatility, then the forward rate curve is frozen in time, the instantaneous forward rate at time t is $\alpha(t)$, and today's bond price is given by the discount formula*

$$P_0(T) \;=\; \exp\left\{-\int_0^T \alpha(s)\, ds\right\} . \tag{15}$$

From (12)–(13), we have that

$$P_0(T) = A(T) \exp\left\{-\int_0^T \alpha(s)\, ds\right\} \tag{16}$$

where the function $A(T)$ is the price of a bond with time to maturity T, in a Vasicek model with $\theta = 0$ and $r_0 = 0$:

$$A(T) \ = \ E\left[\exp\left\{-\int_0^T r_s^*\, ds\right\} \,\Big|\, r_0^* = 0\right] = \exp\left(-\frac{\sigma^2}{2a^2}(B(T)-T) - \frac{\sigma^2}{4a}(B(T))^2\right), \tag{17}$$

$$B(T) = \frac{1}{a}\left(1 - e^{-aT}\right) \tag{18}$$

Thus,

$$P_0(T) = \exp\left\{-\frac{\sigma^2}{2a^2}(B(T)-T) - \frac{\sigma^2}{4a}(B(T))^2 - r_0 B(T) - \int_0^T \int_0^s e^{-a(s-u)}\theta(u) du ds\right\}$$

More generally,

$$P_t(T) = \exp\left\{-C(t,T) - r_t B(T-t)\right\},$$

where

$$C(t,T) = \frac{\sigma^2}{2a^2}(B(T-t) - T + t) + \frac{\sigma^2}{4a}(B(T-t))^2 + \int_t^T \int_t^s e^{-a(s-u)}\theta(u) du ds \tag{19}$$

Recall the **calibration** problem:

$$P^{mkt}(T) = P_0(T), \quad \text{for all } T$$

Its solution is given by

$$\alpha(T) = \partial_T \log A(T) + f^{mkt}(T),$$

or, equivalently,

$$\theta(T) = \partial_T f^{mkt}(T) + a f^{mkt}(T) + \frac{\sigma^2}{2a}(1 - e^{-2aT}),$$

where f^{mkt} is the current **instantaneous forward rate curve** given by the market:

$$f^{mkt}(T) = -\partial_T \log P^{mkt}(T)$$

2.3 Pricing caps and floors via change of numeraire

Assume that the current time is zero. Consider future times $t < T$ and a caplet which pays

$$(T-t)(L_t(T) - K)^+ = (T-t)(F_t(t,T) - K)^+ = \left(\frac{1}{P_t(T)} - 1 - K(T-t)\right)^+$$

at time T.

Notice that

$$\frac{1}{P_t(T)} = \exp\left\{C(t,T) + r_t B(T-t)\right\},$$

where, under $\mathbb{Q}$,

$$r_t \sim N\left(e^{-at} r_0 + \int_0^t e^{-a(t-s)}\theta(s) ds,\ \frac{\sigma^2}{2a}\left[1 - e^{-2at}\right]\right)$$

The above inspires us to **price a caplet in HW-Vasicek model via the standard BS formula!** However, we obtain a path-dependent payoff if we use the standard discounting with the **money market** account:

$$V_0 = \mathbb{E}^{\mathbb{Q}} \left(\exp\left(-\int_0^T r_u du \right) \left(\exp\left\{ C(t,T) + r_t B(T-t) \right\} - \tilde{K} \right)^+ \right),$$

where $\tilde{K} = 1 + K(T-t)$.

This problem can be fixed if we apply a **change of numeraire** and use the **bond with maturity** T **as a discount factor**:

$$\begin{aligned} \frac{V_0}{P_0(T)} &= \mathbb{E}^{\mathbb{Q}^T} \left(\frac{1}{P_T(T)} \left(\exp\left\{ C(t,T) + r_t B(T-t) \right\} - \tilde{K} \right)^+ \right) \\ &= \mathbb{E}^{\mathbb{Q}^T} \left(\exp\left\{ C(t,T) + r_t B(T-t) \right\} - \tilde{K} \right)^+, \end{aligned} \tag{20}$$

where $\mathbb{Q}^T$ is a **new pricing measure, under which the prices of all traded (non-dividend-paying) assets, divided by** $P_t(T)$**, are martingales**. It is often referred to as a **forward measure**.

Rem 2. **Risk neutral measure** *is a pricing measure with the* **money market** *as a numeraire.* **Forward measure** *is a pricing measure with the* **bond price** *as a numeraire.*

Q 1. *What is the* **distribution of** r_t **under the forward measure** $\mathbb{Q}^T$*?*

The short rate satisfies the same SDE:

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t$$

However, W, which is a BM under the risk neutral measure $\mathbb{Q}$, may not be a BM under $\mathbb{Q}^T$! In fact, we have

$$dW_t = dW_t^T - \lambda_t dt,$$

where W^T is a BM under $\mathbb{Q}^T$ and λ is a stochastic process (the drift of W under $\mathbb{Q}^T$). Thus, we have

$$dr_t = (\theta(t) - \lambda_t \sigma - ar_t)dt + \sigma dW_t^T$$

Once we know λ, if it is not too complicated (e.g. deterministic), we may be able to determine the distribution of r_t under $\mathbb{Q}^T$.

Q 2. $\lambda =?$

The money market account, discounted with $P_t(T)$, should be a mtg under $\mathbb{Q}$. Computing the differential

$$d\frac{\exp\left(\int_0^t r_u du\right)}{P_t(T)} = (\cdots \lambda_t \cdots)dt + (\cdots)dW_t^T,$$

and setting the drift to zero, we can obtain the right value of λ_t. We start by observing that

$$X_t = \frac{P_t(T)}{\exp\left(\int_0^t r_u du\right)} = \frac{\exp\left(-C(t,T) - r_t B(T-t)\right)}{\exp\left(\int_0^t r_u du\right)}$$

is a mtg under $\mathbb{Q}$. Hence, it is not hard to deduce (from the above decomposition of X_t) that

$$dX_t = -X_t B(T-t)\sigma dW_t$$

Applying Ito's lemma, we obtain:

$$d(1/X_t) = \frac{1}{X_t}B^2(T-t)\sigma^2 dt + \frac{1}{X_t}B(T-t)\sigma dW_t$$

$$= \frac{1}{X_t}(B^2(T-t)\sigma^2 - \lambda_t B(T-t)\sigma)dt + \frac{1}{X_t}B(T-t)\sigma dW_t^T$$

Setting the drift of the above to zero, we obtain

$$\lambda_t = \sigma^2 B(T-t),$$

and, hence,

$$\begin{aligned} dr_t &= (\theta(t) - \sigma^2 B(T-t) - ar_t)dt + \sigma dW_t^T \\ &= (\tilde{\theta}(t) - ar_t)dt + \sigma dW_t^T, \end{aligned}$$

where

$$\tilde{\theta}(t) = \theta(t) - \sigma^2 B(T-t) = \theta(t) - \sigma^2 \frac{1}{a}\left(1 - e^{-aT}\right)$$

In particular, under $\mathbb{Q}^T$, we have

$$r_t \sim N\left(e^{-at}r_0 + \int_0^t e^{-a(t-s)}\tilde{\theta}(s)ds,\ \frac{\sigma^2}{2a}\left[1 - e^{-2at}\right]\right),$$

and we can **price caplets in terms of zero coupon bonds, using (20) and the standard BS formula**:

$$V_0 = P_0(T)\mathbb{E}^{\mathbb{Q}^T}\left(\exp\left\{C(t,T) + r_t B(T-t)\right\} - \tilde{K}\right)^+ = P_0(T)C^{BS}(\tilde{S},\tilde{K},t,\tilde{\sigma},\tilde{r},\tilde{q})$$

Exercise 1. *Find the auxiliary parameters $\tilde{S}$, $\tilde{\sigma}$, $\tilde{r}$, $\tilde{q}$.*

A cap is a portfolio of caplets, so its price is obtained by a simple summation. Floorlets and floors can be treated similarly.

Rem 3. *In the affine models, such as HW-Vasicek (and in some other models), one can reduce the problem of pricing a* **swaption** *to that of pricing a cap or a floor. Hence, closed form expressions for swaptions are also available in HW-Vasicek. This reduction is known as the Jamshidian's trick. It is a very useful technique, which, however, goes beyond the scope of this course.*

2.4 Approximation with a lattice

Although we can price caps and floors (and even swaptions) explicitly in the HW-Vasicek model, there are many other (more exotic) products, for which we don't have explicit formulas even in this simple model. An example of such a product is a Bermudan swaption, which can be exercised prior to its maturity T_0 (T_0 is the time at which the tow parties enter into a swap (IRS) agreement). One can, in principle, price such contracts using MC, but a tree is more convenient for such a purpose (recall that HW-Vasicek is a one-dimensional model).

Rem 4. *There is another reason why the tree (or lattice) approximation may be desirable. Recall that the exact calibration of HW-Vasicek model to the initial term structure is done in a continuous time setting. However, if we use MC to simulate the paths of (r_t), we will discretize the path, and, in particular, use a discrete approximation for $\int_0^t r_u du$, for the discount factor. All this will create a discretization error, so the bond prices (or forward rates) computed in this model using MC may not exactly coincide with the initial term structure to which the model was calibrated. A tree model does not have this problem.*

Here, we approximate the process (r_t) with a trinomial tree.

First, we construct a trinomial tree for r^*:

$$dr_t^* = -ar_t^* dt + \sigma dW_t$$

- Our lattice will consist of integer points (n, m) with $n = 0, 1, 2, ..., N$ and $m = -N, \ldots, N$, with some positive integers N. We associate with each (n, m) a time $t = n\Delta t$ and a value of $r^* = m\Delta r$, with some Δt and Δr.
- In one time step Δt a walk on the lattice can go from (n, m) to one of three points $(n + 1, m),\ (n + 1, m + 1),\ (n + 1, m - 1)$. The transition probabilities are defined by

$$\begin{cases} (n,m) \to (n+1, m+1) & \text{with probability } \ p_u(m)\ , \\ (n,m) \to (n+1, m) & \text{with probability } \ p_s(m)\ , \\ (n,m) \to (n+1, m-1) & \text{with probability } \ p_d(m) \end{cases} \tag{21}$$

- We find values for the probabilities p_u, p_s, p_d by equating the first and second moments of the increment $r^*(t + \Delta t) - r^*(t)$ in the discrete and continuous models. For the discrete model, starting at $r_t^* = m\Delta r$, the two moments are given by

$$\begin{cases} \text{first moment :} & [p_u(m) - p_d(m)]\Delta r\ , \\ \text{second moment :} & [p_u(m) + p_d(m)](\Delta r)^2\ . \end{cases} \tag{22}$$

To get the first and second moments in the continuous case we observe from (13) that

$$r^*_{t+\Delta t} - r^*_t \ \simeq\ -ar_t^*\Delta t + \sigma\sqrt{\Delta t}\,\xi \quad \text{where } \xi \text{ is standard normal.} \tag{23}$$

Based on this approximation, we have

$$\begin{cases} \mathbb{E}\left(r^*_{t+\Delta t} - r^*_t \,|\, r^*_t = m\Delta r\right) & \approx\ -am\Delta r\Delta t\ , \\ \mathbb{E}\left((r^*_{t+\Delta t} - r^*_t)^2 \,|\, r^*_t = m\Delta r\right) & \approx\ \sigma^2\Delta t + [am\Delta r\Delta t]^2\ , \end{cases} \tag{24}$$

Equating the first and second moments in the discrete case (22) and in the continuous case (24) we have that

$$\begin{cases} [p_u(m) - p_d(m)]\Delta r & =\ -am\Delta r\Delta t\ , \\ [p_u(m) + p_d(m)](\Delta r)^2 & =\ \sigma^2\Delta t + [am\Delta r\Delta t]^2\ . \end{cases} \tag{25}$$

Evidently (25) has generally a unique solution for $p_u(j), p_d(j)$, and then we can find $p_s(j)$ from

$$p_u(m) + p_s(m) + p_d(m) = 1$$

- We still need to decide on suitable values for $\Delta r, \Delta t$. Note that, since $p_u(m) + p_d(m) \leq 1$, the second equation of (25) implies that

$$\sigma^2 \Delta t / (\Delta r)^2 \leq 1,$$

so, as in the explicit Euler scheme for PDEs, we must take time discretization to be the same order as the square of the space discretization. Hull and White choose a value consistent with this, namely

$$\text{HW choice}: \quad \sigma^2 \Delta t / (\Delta r)^2 \; = \; 1/3 \, . \tag{26}$$

Rem 5. *This is actually a minor thing but the reason the value (26) is taken is so that third order discrete and continuous moments agree if (26) holds.*

- Using (26), we solve (25) to obtain

$$\begin{cases} p_u(m) & = \; 1/6 + [(am\Delta t)^2 - am\Delta t]/2 \, , \\ p_d(m) & = \; 1/6 + [(am\Delta t)^2 + am\Delta t]/2 \, , \\ p_s(m) & = \; 2/3 - (am\Delta t)^2 \, . \end{cases} \tag{27}$$

- Notice that, when m becomes large, $p_u(m)$ may still become negative. Similar problem occurs for $p_d(m)$ when m is negative. To avoid negative probabilities, we have to 'trim' the tree by limiting the nodes to (n, m) that satisfy $|m| \leq M < N$. This corresponds to a process that has boundaries at $M\Delta r$ and $-M\Delta r$. Then, we need to provide boundary conditions at those points. As M will be chosen very large, so that there is a very small probability of reaching it, there is almost no difference in the choice of boundary conditions. Here, we choose an absorbing boundary condition:

$$p_u(M) = p_u(-M) = 0, \qquad p_d(M) = p_d(-M) = 0, \qquad p_s(M) = p_s(-M) = 1$$

- Finally, we need to choose the appropriate value of M. The value of J has to be large enough so that there is a very small probability of r^* reaching it. We use the "3 standard deviations rule" that we used in Chapter II to reduce the infinite interval $-\infty < x < \infty$ for the parabolic PDE which we needed to solve for pricing stock options to a finite interval $a < x < b$. From (8) we see that the standard deviation of $r^*(t)$ is less than $\sigma/\sqrt{2a}$ no matter how large t is, and the mean of $r^*(t)$ is zero. Hence $r^*(t)$ takes values larger than $3\sigma/\sqrt{2a}$ in absolute value with very small probability-less than $.1\%$. Thus a good interval for the range of r^* is $-3\sigma/\sqrt{2a} < r < 3\sigma/\sqrt{2a}$, whence we should set

$$M\Delta r = 3\sigma/\sqrt{2a},$$

$$M = \left\{ \frac{3}{2a\Delta t} \right\}^{1/2} \tag{28}$$

The next step is to find α, such that the tree model for $r_{n\Delta t} = r^*_{n\Delta t} + \alpha(n\Delta t)$ reproduces the initial term structure.

- First, we define the bond prices in discrete model:

$$P^n \; = \; P(0, n\Delta t) = \mathbb{E} \exp(-\sum_{i=0}^{n-1} \Delta t r_{i\Delta t}) = \mathbb{E} \exp(-\sum_{i=0}^{n-1} \Delta t (r^*_{i\Delta t} + \alpha(i\Delta t))), \;\; n = 1, 2, \ldots, N \tag{29}$$

- Next, we introduce the Arrow-Debreu (AD) securities. For each node (n, m), we define the price of AD security $\lambda(n, m)$ by

$$\lambda(n, m) = \mathbb{E}\left(\exp(-\sum_{i=0}^{n-1} \Delta t(r^*_{i\Delta t} + \alpha(i\Delta t))) \mid r^*_{n\Delta t} = m\Delta r\right) \tag{30}$$

  In the above expectation, we are restricting ourselves to paths $(n, r^*_{n\Delta t})$ which satisfy $r^*_{n\Delta t} = m\Delta r$. Then $\lambda(n, m)$ is a sum over all such paths of the discount factor times the probability of each path. In financial terms we can think of $\lambda(n, m)$ as the value of a **virtual security** which pays 1 if the short term rate at time $n\Delta t$ is $\alpha(n\Delta t) + m\Delta r$, and otherwise 0. This is very similar to the AD securities discussed in Chapter I, except that the discount factor here is random and, hence, requires a more careful treatment.

- Conditioning on the previous time step in (30), we follow the arguments presented in Chapter I to obtain the following recurrence relation, for $n = 1, \dots, N-1$ and $m = -M, \dots, M$:

$$\begin{aligned}\lambda(n+1, m) = \lambda(n, m) \exp[-(\alpha(n\Delta t) + m\Delta r)\Delta t] p_s(m) \\ + \lambda(n, m+1) \exp[-(\alpha(n\Delta t) + (m+1)\Delta r)\Delta t] p_d(m+1) \\ + \lambda(n, m-1) \exp[-(\alpha(n\Delta t) + (m-1)\Delta r)\Delta t] p_u(m-1),\end{aligned} \tag{31}$$

  with $\lambda(n, m) = 0$ for (n, m) outside of the lattice (recall that the tree has been 'trimmed'). Note also that we set $\lambda(0, 0) = 1$.

- Conditioning in (29), we can write the bond price at time $(n+1)\Delta t$ in terms of the $\lambda(n, m)$ as

$$P^{n+1} = \sum_{m=-M}^{M} \lambda(n, m) \exp[-(\alpha(n\Delta t) + m\Delta r)\Delta t], \tag{32}$$

  for $n = 1, \dots, N-1$.

- Assuming that the bond prices $\{P^n\}_{n=1}^N$ are given, we can solve the system of equations (31)–(32) to obtain $\{\lambda(n, m), \alpha(n\Delta t)\}$. In fact solving this system is much easier than in the case of calibrating a tree to call prices: notice that we can factor $\exp[-\alpha(n\Delta t)\Delta t]$ out of the right hand sides of (31) and (32). Once we find $\{\alpha(n\Delta t)\}$, we have calibrated the HW-Vasicek model to today's term structure.

3 Term structure models (aka market models, market-based models)

Even though short rate models are convenient for computations (and this is why, historically, they are the first models used for fixed income markets), they possess certain drawbacks.

Recall that HW-Vasicek model can be calibrated to fit the initial term structure. However, once calibrated, **it cannot produce a very reach family of future term structures** (given e.g. by the instantaneous forward rate curve $\{f(T)\}_{T\geq 0}$). An example of a time series of $\{f_t(T)\}_{T\geq 0}$, for various times t, is given in Figure 1.

A particular feature of the real-world dynamics of the forward rate curve that is never consistent with the short rate models, is the magnitude of changes in the long end of $f_t(T)$ (i.e. for large T). Notice that, in HW-Vasicek model, we have

$$f^{HWV}_t(T) - f^{HWV}_0(T) = e^{-aT}\left(\frac{\sigma^2}{2a^2}(e^{at} - 1)\left(2 - e^{-aT} - e^{-a(T-t)}\right) - \int_0^t e^{au}\theta(u)du + r_t e^{at} - r_0\right),$$

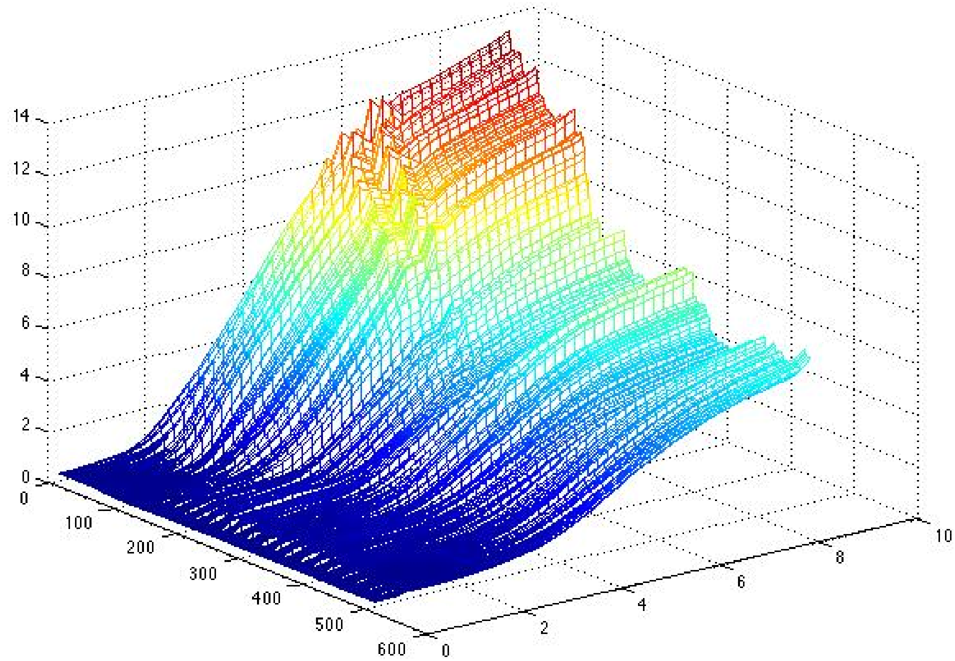

Figure 1: Instantaneous forward rate curves $f_t^{mkt}(T)$ computed from US Treasury bills and notes in 2010-2012. Time t is measured in days (from 0 to 500), and the maturity T is measured in years (from 0 to 10).

which vanishes very fast for large T. However, the real data suggests that changes in forward rates are often more significant for large maturities.

The family of possible increments of the forward rate curve, produced by the HW-Vasicek model, is simply too small to be compatible with the observed dynamics.

This is a problem for several reasons:

- Our model does not produce realistic future rates (e.g. cannot be used for risk management).
- We cannot fit the model to the historical time series of forward curves.

3.1 Heath-Jarrow-Morton (HJM) framework

Main idea: to obtain realistic dynamics of the forward rate curve, why don't we model the entire curve?

Consider the **instantaneous forward rates** at time t:

$$f_t(T) = -\partial_T \log P_t(T)$$

Recall that the short interest rate r_t is the 'left end' of the forward rate curve:

$$r_t = f_t(t)$$

Now, instead of modeling only the left end of the curve, we can model the risk neutral evolution of the entire forward rate curve. Namely, under the risk neutral measure (which corresponds to the money market asa discount factor), we assume that

$$df_t(T) = \mu_t(T)dt + (\sigma_t(T))^T dB_t, \quad \text{for all } T > 0, \tag{33}$$

where $\sigma_t(T)$ takes values in $\mathbb{R}^d$ and B is a standard d-dimensional BM.

Notice that we **cannot choose μ and σ arbitrarily**!

The reason is that, at each time t, the forward curve $\{f_t(T)\}_T$ determines the bond prices:

$$P_t(T) = \exp(-\int_t^T f_t(u)du),$$

Then, the dynamics of $\{f_t(T)\}_T$ determine uniquely the dynamics of bond prices, which, in turn, **have to be mtg's under the risk neutral measure** $\mathbb{Q}$. Therefore, we have to make sure that the dynamics of $\{f_t(T)\}_T$ are such that the resulting bond prices are martingales. This results in a **restriction on the drift of** f, which is due to **Heath-Jarrow-Morton (1987)**:

Thm 1. *Under certain regularity assumptions on σ, the model given by (33) is arbitrage-free if and only if*

$$\mu_t(T) = (\sigma_t(T))^T \int_t^T \sigma_t(u)du = \sum_{j=1}^d \sigma_t^j(T) \int_t^T \sigma_t^j(u)du$$

Thus, we can choose σ arbitrarily (provided the stochastic differential is well defined), and the forward rates can be modeled according to

$$df_t(T) = (\sigma_t(T))^T \int_t^T \sigma_t(u)dudt + (\sigma_t(T))^T dB_t,$$

e.g. via MC methods.

The **benefits** are:

- We always fit the initial term structure.
- And the model is capable of reproducing an arbitrary term structure observed in the past. Hence, it can be fitted to the historical time series of term structure (e.g. via the Principal Component Analysis).

3.2 Forward rate models (aka LIBOR market models)

Here, we simply describe the analogue of HJM framework, constructed for the more realistic market, in which only a finite number of maturities are available.

LIBOR = London Inter-Bank Offered Rate,

$$L_t(T) = \frac{1 - P_t(T)}{(T-t)P_t(T)},$$

is a (simply compounded) rate to lend or borrow at time t to repay at T.

Forward LIBOR rate,

$$F_t(T_0, T_1) = \frac{P_t(T_0) - P_t(T_1)}{(T_1 - T_0)P_t(T_1)},$$

is a (simply compounded) rate to lend or borrow at future time T_0 to repay at T_1, agreed upon at time t (i.e. the FRA rate).

Notice the connection with instantaneous forward rates:

$$F_t(T_0, T_1) = \frac{1}{T_1 - T_0}\left(\exp(\int_{T_0}^{T_1} f_t(u)du) - 1\right),$$

$$f_t(T) = F_t(T, T)$$

Consider $0 < T_1 < \cdots < T_{N+1}$, and denote

$$F_t^n = F_t(T_n, T_{n+1})$$

Then, modeling the future time evolution of the term structure, with maturities $T \in \{T_1, \ldots, T_{N+1}\}$, is the same as modeling the evolution of $\{F_t^n\}_{n=\eta(t)}^N$, where

$$\eta(t) = \min\{T_i \,:\, i = 1, \ldots, N+1,\; T_i \geq t\}$$

To model the evolution of $\{F_t^n\}_{n=\eta(t)}^N$, we assume that under the risk neutral measure,

$$dF_t^n = \mu_t^n F_t^n dt + F_t^n (\sigma_t^n)^T dB_t, \quad n = 1, \ldots, N,$$

where each σ_t^n takes values in $\mathbb{R}^d$, and B is a d-dimensional standard BM.

Thm 2. *It turns out that, under mild regularity assumptions, the above model is arbitrage-free if and only if*

$$\mu_t^n = \sum_{j=\eta(t)}^{n} \frac{(T_{j+1} - T_j)F_t^j (\sigma_t^n)^T \sigma_t^j}{1 + (T_{j+1} - T_j)F_t^j}$$

In other words, the risk neutral dynamics of forward rates are given by the following system:

$$\frac{dF_t^n}{F_t^n} = \sum_{j=\eta(t)}^{n} \frac{(T_{j+1} - T_j)F_t^j (\sigma_t^n)^T \sigma_t^j}{1 + (T_{j+1} - T_j)F_t^j} dt + (\sigma_t^n)^T dB_t, \quad n = 1, \ldots, N,$$

where we are free to choose the volatility processes $\{\sigma_t^n\}_{n=1}^N$ arbitrarily (provided the above expression is well defined).

Lognormal LIBOR market model (aka lognormal forward-LIBOR model (LFM), Brace-Gatarek-Musiela (BGM) model)

Assume that $\sigma_t^n \equiv \text{const} \in \mathbb{R}^d$ for every $n = 1, \ldots, N$. This is the lognormal LIBOR market model. The motivation for such a name comes form its connection to the Black's model.

Black's model

Black's model arises from the lognormal LIBOR market model once we restrict our analysis to a single forward rate F^n and consider its dynamics under the associated forward measure $\mathbb{Q}^{T_{n+1}}$.

Notice that F_t^n is a mtg under $\mathbb{Q}^{T_{n+1}}$, as

$$F_t^n = \frac{1}{P_t(T_{n+1})} \frac{P_t(T_n) - P_t(T_{n+1})}{(T_{n+1} - T_n)}$$

is the price of tradable asset (i.e. a portfolio of bonds) discounted by the price of a bond with maturity T_{n+1}. Hence by the definition of $\mathbb{Q}^{T_{n+1}}$, it is a $\mathbb{Q}^{T_{n+1}}$-mtg.

As the volatility of F^n is constant, it cannot change under the new measure. Along with the mtg property, this implies that, under $\mathbb{Q}^{T_{n+1}}$, we have:

$$dF_t^n = F_t^n (\sigma^n)^T dB_t^{T_{n+1}} = F_t^n \tilde{\sigma}^n dW_t^{T_{n+1}}, \quad t \in [0, T_n],$$

where $\tilde{\sigma}^n \in \mathbb{R}$ and $W^{T_{n+1}}$ is a one-dimensional $\mathbb{Q}^{T_{n+1}}$-BM.

Thus, the payoff of a caplet in this model is given by

$$V_{T_{n+1}} = (T_{n+1} - T_n)(L_{T_n}(T_{n+1}) - K)^+ = (T_{n+1} - T_n)(F_{T_n}^n - K)^+$$

at time T_{n+1}. Since $F_{T_n}^n$ has a lognormal distribution under $\mathbb{Q}^{T_{n+1}}$, we obtain the caplet price as follows

$$V_0 = P_0(T_{n+1})(T_{n+1} - T_n)\mathbb{E}^{\mathbb{Q}^{T_{n+1}}}(F_{T_n}^n - K)^{=} P_0(T_{n+1})(T_{n+1} - T_n)C^{BS}(F_0^n, K, T_n, \tilde{\sigma}^n, 0, 0)$$

This is known as the Black's formula.

Similarly, we can price floorlets and floors.

For more complicated products, we use Monte Carlo.