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A comparison of different parameter estimation methods for exponentially modified Gaussian distribution

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Abstract

The convolution of the independent Gaussian and exponential distribution is known as the exponentialy modified Gaussian (EMG) distribution. The main feature of this distribution is its differential behavior on the right and left tails. The distribution exhibits a normally-distributed left tail and an exponentially-distributed right tail. This distribution has found practical applications in a variety of scientific disciplines such as chromatography, cellular biology, radiotherapy, microarray preprocessing, and macroeconomics. With the precise measurement of different natural phenomena, fitting an appropriate distribution and estimation of its parameters is becoming challenging. This study discusses the complexity of parameter estimation of the EMG distribution. In particular, to estimate the parameters of the EMG, eleven different methods, namely the method of moment estimation (MME), approximated L-moment estimation (ALME), the maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), the maximum product spacing (MPS), the minimum spacing absolute distance estimation (MSADE), the minimum spacing absolute log-distance estimation (MSALDE), Cramér-Von-Mises (CVM), Anderson–Darling method (AD), and right-tail Anderson–Darling method (RAD) are considered. Besides a comprehensive simulation study, a real data on time in hours to detect cancer cells using 3um erlotinib is also a part of this study.

Keywords Anderson–Darling method · Cramér-Von-Mises estimation method · Exponentially modified Gaussian distribution · Maximum likelihood estimation · Root mean squared error · Weighted least squared estimation

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1 Introduction

In probability literature, the sum of independent normal and exponential random variables is known as the exponentially modified Gaussian (EMG) distribution or exGaussian distribution, i.e., EMG random variable Z may be expressed as $Z = X + Y$, where X and Y are independent, X is Gaussian with mean μ and variance σ^2, while Y is exponential with rate λ. The distribution exhibits a normally-distributed left tail and an exponentially-distributed right tail. This distribution has found practical applications in a variety of scientific disciplines such as chromatography [9, 18, 24], cellular biology [16, 17], radiotherapy [31], microarray preprocessing [23, 34] and macroeconomics [35].

In the literature, Grushka [18] used the EMG distribution to examine its utility in the analysis of strongly overlapped chromatographic peaks. Delley [9] suggested that the EMG distribution is more accurate model than the simple Gaussian function for analysis of chromatography. In another study, Kalambet et al. [24] motivated by the usage of the EMG distribution for peak approximation in chromatography and studied the EMG peak deconvolution routine for chromatography. In particular, they used a combination of two EMG formulas and linear optimization methods. Haney [20] derived the properties of EMG distribution by re-parameterization of the cumulative distribution function of the EMG distribution. Further, a new consistent parameter estimation method was also proposed which always returns valid parameter estimates. Howerton et al. [23] used the EMG distribution to analyze the experimental zone profiles. They devised a simulation study to generate a series of profiles assuming fixed values for retention time and different values for σ and λ. Statistical moments are used to analyze each profile while an iterative method was used to fit the EMG. It is noticed that the statistical moment method is much more susceptible to error when the degree of asymmetry is large, or the integration limit is inappropriately chosen, or the number of points is small, or when the signal-to-noise ratio is small.

Nicolaescu et al. [31] presented multi-parameter characterization results for clustering ion beams by using the EMG function. Moret-Tatay et al. [30] described that reaction times in psychology are usually modeled through the EMG distribution, as it provides a good fit to multiple empirical data. Silver et al. [34] introduced the ”normexp" method to observe intensities of exponentially distributed signals with normally distributed background values. The authors developed an algorithm for exact maximum likelihood estimation (MLE) using an optimization method that uses saddle-point estimates as the starting values. They have shown numerically that the MLE performs better in terms of estimation accuracy.

Golubev [16] suggested that the EMG is suitable for the analysis of variabilities featured by a number of biological phenomena which are often thought to be associated with the lognormal distributions. The author used EMG to estimate deterministic and probabilistic parts of the transition probability model of cell cycles. Later, Golubev [17] discussed the applicability of EMG and exponentially modified gamma-distribution (EMGD) in biomedical sciences and related disciplines such as data on cell cycle, gene expression, physiological responses to stimuli. Recently, Ara et al. [4] introduced a control chart to monitor schedule time using the EMG distribution.

The EMG distribution is also used to capture the main features of exports [35]. The authors stated the sales data fits better to EMG than either of the often assumed log-normal distribution or Pareto distribution. Li [26] used a modified hyperbolic tangent function for the error function to describe the chromatographic peaks and proposed a simplified analytical form of the EMG function.

Parameters estimation for different distributions, (for example, generalized exponential distribution [19], generalized Rayleigh distribution [25], Gompertz distribution [12], Marshall Olkin exponential distribution [5], weighted Lindley distribution [29], Rayleigh distribution [11], extended exponential geometric distribution [27], transmuted Rayleigh distribution [13], three-parameter log-normal distribution [6], and weighted exponential distribution [10] among others), assuming different estimation methods to select the most appropriate method is a very popular topic in probability theory.

With the precise measurement of different natural phenomena, fitting an appropriate distribution and estimation of its parameters is becoming challenging. Motivated by the importance of EMG distribution in different biological and chemical fields, this study discusses and compares eleven different methods of estimation, namely the method of moment estimation (MME), approximated L-moment estimation (ALME), the maximum likelihood estimation (MLE), least squares estimation (LSE), weighted least squares estimation (WLSE), the maximum product spacing (MPS), the minimum spacing absolute distance estimation (MSADE), the minimum spacing absolute log-distance estimation (MSALDE), Cramer-Von-Mises (CVM), Anderson–Darling method (AD), and right-tail Anderson–Darling method (RAD).

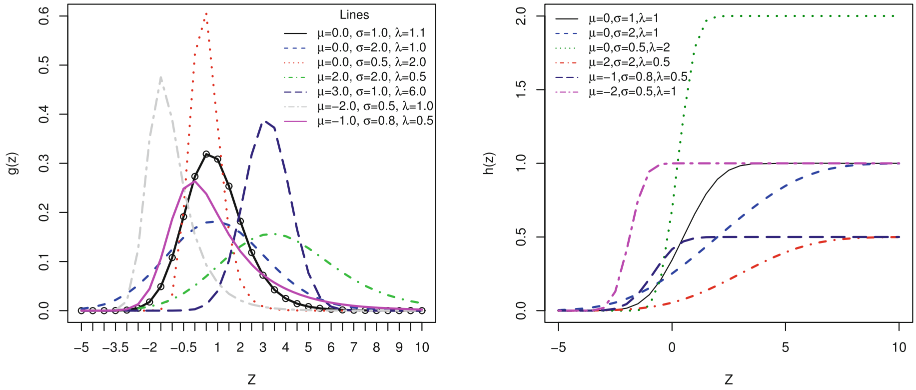
2 EMG distribution

Assuming X and Y are two independent random variables, let define a random variable Z as $Z = X + Y$, where $X \sim N(\mu, \sigma^2)$, and $Y \sim Exp(\lambda)$. To this end, the random variable Z is the sum of normal and exponential random variables and is said to follow an EMG distribution with parameters (μ, σ, λ). The cumulative distribution function (CDF) and probability density function (PDF), derived by convolution, are given in Eqs. (1) and (2), respectively.

$$G(z) = \frac{1}{2} - \frac{1}{2}\exp\left(\frac{\lambda^2\sigma^2}{2} + \lambda\mu - z\lambda\right)\operatorname{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right) + \frac{1}{2}\operatorname{erf}\left(z - \frac{\mu}{\sqrt{2}\sigma}\right) \tag{1}$$

$$g(z) = \frac{\lambda}{2}\exp\left(\frac{\lambda^2\sigma^2}{2} + \lambda\mu - z\lambda\right)\operatorname{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right) \tag{2}$$

where $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)dt$ and $\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-t^2)dt = 1 - \operatorname{erf}(x)$. Hence, the EMG distribution generalizes both the normal and exponential distributions. To justify this, let the variance of an exponential distribution with rate parameter λ is λ^{-2}. It is noticed in the literature that the variance can be made arbitrarily small by increasing the value of λ, which corresponds to a point mass distribution. Similarly, the normal distribution also becomes a point mass distribution at μ as σ decreases. In other words, as the exponential part has zero variance, i.e., $\lambda \to \infty$, the EMG distribution behavior is like a normal distribution or an exponential distribution when its normal component has zero variance, i.e., $\sigma \to 0$. Furthermore, the exponential distribution controls the mass on the right tail. The graphical depiction of the PDF of EMG is given in Fig. 1a. Figure 1b shows the hazard rate of EMG distribution which allows for increasing, decreasing and constant hazard rates depending upon the different values of μ, σ and λ.



(A) PDF **(B) Hazard function**

Fig. 1 PDF and hazard function of EMG distribution for different choices of μ, σ and λ parameters

3 Parameter estimation

This section describes eleven methods of parameter estimation for the EMG distribution. We assume that $\mathbf{Z} = (z_1, \cdots, z_n)$ is a random vector of size n from the EMG distribution and interested to estimate the μ, σ and λ using the information of the observed sample.

3.1 Method of moment estimation (MME)

To obtain the MME of the three-parameter EMG distribution, equate the first three theoretical moments with the sample moments, i.e., $\frac{1}{n}\sum_{i=1}^{n} z_i$, $\frac{1}{n}\sum_{i=1}^{n} z_i^2$ and $\frac{1}{n}\sum_{i=1}^{n} z_i^3$ respectively. In this case,

$$\frac{1}{n}\sum_{i=1}^{n} z_i = \mu + \frac{1}{\lambda} \tag{3}$$

$$\frac{1}{n}\sum_{i=1}^{n} z_i^2 = \sigma^2 + \mu^2 + \frac{2\mu}{\lambda} + \frac{2}{\lambda^2} \tag{4}$$

$$\frac{1}{n}\sum_{i=1}^{n} z_i^3 = 3\mu\sigma^2 + \frac{3\sigma^2}{\lambda} + \mu^3 + \frac{3\mu^2}{\lambda} + \frac{6\mu}{\lambda^2} + \frac{6}{\lambda^3} \tag{5}$$

Solving these equations simultaneously one can obtain the moment estimators $\hat{\mu}_{MME}$, $\hat{\sigma}_{MME}$ and $\hat{\lambda}_{MME}$ of the parameters μ, σ and λ, respectively.

3.2 Method of L-moments estimation (LME)

The L-moments estimators can be obtained as the linear combinations of order statistics. The L-moments estimators were originally proposed by Hosking [21] and they are known for their robustness compared to the usual moment estimators. The L-moments estimators are also obtained in the same way as the ordinary moment estimators, i.e., by equating the sample L-moments with the population L-moments. Another advantage of these estimators is that these exist whenever the mean of the distribution exists, even though some higher moments may not exist, hence these are relatively robust to the effects of outliers [22]. The first three sample L-moments are

$$l_1 = \frac{1}{n}\sum_{i=1}^{n} z_i \tag{6}$$

$$l_2 = \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-1)z_i - l_1 \tag{7}$$

where $z_i = z_{(i)}$ denote the order statistics, i.e., values arranged in ascending order $z_1 < z_2 < \cdots, z_n$ and

$$\begin{aligned} l_3 \;&= \frac{5}{n(n-1)(n-2)}\sum_{i=1}^{n}(i-1)(i-2)z_i - \frac{4}{n(n-1)}\sum_{i=1}^{n}(i-1)z_i + \frac{(n-2)}{(n-1)}l_1 - \frac{1}{(n-1)}l_1 \\ &+\frac{1}{n(n-1)(n-2)}\sum_{i=1}^{n}(i-2)^2 z_i - \frac{2}{n(n-1)}\sum_{i=1}^{n}(i-2)z_i + \frac{1}{n(n-1)(n-2)}\sum_{i=1}^{n}(i-2)z_i \end{aligned} \tag{8}$$

whereas the first three population L-moments are

$$\psi_1 = E(Z_{1:1}) = E(Z) = \mu + \frac{1}{\lambda} \tag{9}$$

$$\psi_2 = \frac{1}{2}[E(Z_{2:2}) - E(Z_{1:2})] \tag{10}$$

$$\psi_3 = \frac{1}{3}[E(Z_{3:3}) - 2E(Z_{2:3}) + E(Z_{1:3})] \tag{11}$$

where

$$\begin{aligned} E(Z_{2:2}) &= \frac{2!}{(2-1)!(2-2)!}\int_{-\infty}^{\infty} zG(z)^{2-1}\{1-G(z)\}^{2-2}dG(z) \\ &= 2\int_{-\infty}^{\infty} z[1-\exp\big(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2\big)] \\ &\quad \times \{(\lambda/2)\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)\} \times \operatorname{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right)dz \\ &= -2\left\{\frac{\exp(\lambda^2\sigma^2(1+2\lambda(\mu-\lambda\sigma^2))) - 4(1+\lambda\mu)}{4\lambda}\right\} \end{aligned} \tag{12}$$

and

$$\begin{aligned} E(Z_{1:2}) &= \frac{2!}{(1-1)!(2-1)!}\int_{-\infty}^{\infty} zG(z)^{1-1}\{1-G(z)\}^{2-1}dG(z) \\ &= 2\int_{-\infty}^{\infty} z[1-1+\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)] \\ &\quad \times \{\lambda/2\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)\}\operatorname{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right)dz \\ &= -2\left\{\frac{\exp\big(\lambda^2\sigma^2(-1-2\lambda\mu+2\lambda^2\sigma^2)\big)}{4\lambda}\right\} \end{aligned} \tag{13}$$

Replacing the above results in ψ_2, we get

$$\psi_2 = \frac{1}{\lambda} + \mu + \exp(\lambda^2\sigma^2)\left(\lambda\sigma^2 - \mu - \frac{1}{2\lambda}\right) \tag{14}$$

Now, for ψ_3

$$\begin{aligned}
E(Z_{3:3}) &= \frac{3!}{(3-1)!(3-3)!}\int_{-\infty}^{\infty} zG(z)^{3-1}\{1-G(z)\}^{3-3}dG(z) \\
&= 3\int_{-\infty}^{\infty} z[1-\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)]^2 \\
&\quad \times \{(\lambda/2)\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)\}\,\mathrm{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right)dz \\
&= 3\lambda\sigma^2 - 3\exp(\lambda^2\sigma^2)(\lambda\sigma^2 + \mu^2) + \exp(3\lambda^2\sigma^2)(\lambda\sigma^2 + \mu) + 3\mu
\end{aligned} \tag{15}$$

Similarly,

$$\begin{aligned}
E(Z_{2:3}) &= \frac{3!}{(2-1)!(3-2)!}\int_{-\infty}^{\infty} zG(z)^{2-1}\{1-G(z)\}^{3-2}dG(z) \\
&= 6\int_{-\infty}^{\infty} z[1-\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)][\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)] \\
&\quad \times \{(\lambda/2)\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)\}\,\mathrm{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right)dz \\
&= \frac{3\exp(\lambda^2\sigma^2)}{2\lambda} + 3\exp(\lambda^2\sigma^2)(\mu - \lambda\sigma^2) - \frac{2\exp(2\lambda^2\sigma^2)}{3\lambda} \\
&\quad - 2\exp(2\lambda^2\sigma^2)(\mu - 2\lambda\sigma^2)
\end{aligned} \tag{16}$$

$$\begin{aligned}
E(Z_{1:3}) &= \frac{3!}{(1-1)!(3-1)!}\int_{-\infty}^{\infty} zG(z)^{1-1}\{1-G(z)\}^{3-1}dG(z) \\
&= 3\int_{-\infty}^{\infty} z[1-1+\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)]^2 \\
&\quad \times \{(\lambda/2)\exp(-\lambda z + \mu\lambda + 0.5\lambda^2\sigma^2)\}\,\mathrm{erfc}\left(\frac{\lambda\sigma^2 + \mu - z}{\sqrt{2}\sigma}\right)dz \\
&= \frac{\exp(3\lambda^2\sigma^2)}{3\lambda} - (\mu + 2\lambda\sigma^2)\exp(3\lambda^2\sigma^2)
\end{aligned} \tag{17}$$

Again replacing the above results in ψ_3, we get

$$\begin{aligned}
\psi_3 &= \lambda\sigma^2 + \mu + \exp(\lambda^2\sigma^2)(\lambda\sigma^2 - 3\mu - 1/\lambda) + \exp(2\lambda^2\sigma^2)\left(\frac{4}{9\lambda} + \frac{4\mu}{3} - \frac{8\lambda\sigma^2}{3}\right) \\
&\quad + \exp(3\lambda^2\sigma^2)\left(\frac{1}{9\lambda} - \frac{\lambda\sigma^2}{3}\right)
\end{aligned} \tag{18}$$

The resultant L-moments are called as the approximated L-moments. The approximated L-moments estimators $\hat{\mu}_{ALME}$, $\hat{\sigma}_{ALME}$ and $\hat{\lambda}_{ALME}$ of the parameters μ, σ and λ can be obtained numerically by solving the following equations.

$$\psi_1 = l_1, \quad \psi_2 = l_2 \quad \text{and} \quad \psi_3 = l_3 \tag{19}$$

3.3 Method of maximum likelihood estimation (MLE)

The MLE method is the most widely used parameter estimation method and its success stems from its many desirable properties, including, consistency, asymptotic efficiency, invariance property as well as its intuitive appeal. In our case, the likelihood function of the density is given by:

$$L(\mu, \sigma, \lambda, \mathbf{z}) = \prod_{i=1}^{n} g(z_i) = \prod_{i=1}^{n} \{\lambda/2e^{-z_i\lambda+\mu\lambda+0.5\lambda^2\sigma^2}\} \operatorname{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right) \tag{20}$$

$$\log L(\mu, \sigma, \lambda, \mathbf{z}) = n\log(\lambda/2) + (\lambda/2)\left\{n(2\mu) + n(\lambda\sigma^2) - 2\sum_{i=1}^{n} z_i\right\}$$
$$+\sum_{i=1}^{n}\log\left(\operatorname{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right) \tag{21}$$

Taking partial derivatives of the logarithmic L(μ, σ, λ;$\mathbf{z}$) with respect to μ, σ and λ and equating the resulting equation to zero we obtain the ML estimators for μ, σ, and λ as follows.

$$\frac{\partial \log L(\mu, \sigma, \lambda, \mathbf{z})}{\partial \mu} = \sum_{i=1}^{n} - \frac{\sqrt{\frac{2}{\pi}}\, e^{-\frac{(-z_i+\lambda\sigma^2+\mu)^2}{2\sigma^2}}}{\sigma \operatorname{erfc}\left(\frac{-z_i+\lambda\sigma^2+\mu}{\sqrt{2}\sigma}\right)} + \lambda n = 0 \tag{22}$$

$$\frac{\partial \log L(\mu, \sigma, \lambda, \mathbf{z})}{\partial \sigma} = \sum_{i=1}^{n} \frac{\sqrt{\frac{2}{\pi}}\,(-z_i+\lambda\sigma^2+\mu)\, e^{-\frac{(-z_i+\lambda\sigma^2+\mu)^2}{2\sigma^2}}}{\sigma^2 \operatorname{erfc}\left(\frac{-z_i+\lambda\sigma^2+\mu}{\sqrt{2}\sigma}\right)} + \lambda^2 n\sigma = 0 \tag{23}$$

and

$$\frac{\partial \log L(\mu, \sigma, \lambda, \mathbf{z})}{\partial \lambda} = \frac{1}{2}\left(-2\sum_{i=1}^{n} z_i + \lambda n\sigma^2 + 2\mu n\right) + \frac{1}{2}\lambda n\sigma^2 + \frac{n}{\lambda} = 0 \tag{24}$$

The solution of the above normal equations cannot be obtained in closed form, hence the MLEs can be obtained numerically by an iterative method, e.g., the Newton Raphson.

3.4 The method of least-squares (LS)

A standard approach in parameter estimation is the least squares estimators. For estimation of parameters μ, σ and λ of EMG distribution, the least squares estimator can be obtained by minimizing.

$$\sum_{i=1}^{n}\left(G(z_{(i)}) - \frac{i}{n+1}\right)^2 \tag{25}$$

with respect to the unknown parameters μ, σ and λ. For the simplicity of notation, let $z_i = z_{(i)}$ denote the ith order statistics. Then, $\hat{\mu}_{LSE}$, $\hat{\sigma}_{LSE}$ and $\hat{\lambda}_{LSE}$ are obtained by minimizing

$$\sum_{i=1}^{n}\left[\frac{1}{2} - \frac{1}{2}\exp(\lambda^2\sigma^2/2 + \lambda\mu - z_i\lambda)\operatorname{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{z_i-\mu}{\sqrt{2}\sigma}\right) - \frac{i}{n+1}\right]^2 \tag{26}$$

with respect to μ, σ and λ. Taking partial derivatives with respect to μ, σ and λ, we obtained the following equations.

$$\sum_{i=1}^{n} 2\left[-\frac{e^{\frac{-(z_i-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} + \frac{e^{\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i-\frac{(\lambda\sigma^2+\mu-z_i)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma}\right]\left[1/2 - \frac{i}{(n+1)}\right.$$
$$\left.+ 1/2\,\mathrm{erf}\left(\frac{z_i-\mu}{\sqrt{2}\sigma}\right) - 1/2e^{\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i}\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right] = 0 \tag{27}$$

$$\sum_{i=1}^{n} 2\left[1/2 - \frac{i}{(n+1)} + 1/2\,\mathrm{erf}\left(\frac{z_i-\mu}{\sqrt{2}\sigma}\right) - 1/2\exp(\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i)\right.$$
$$\left.\times\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right]\left[-1/2\exp(\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i)(\lambda^2\sigma)\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right.$$
$$\left.-\frac{\exp\left(\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i-\frac{(\lambda\sigma^2+\mu-z_i)^2}{2\sigma^2}\right)(\lambda\sigma^2+\mu-z_i)}{\sqrt{2\pi}\sigma^2} - \frac{\exp\left(\frac{-(z_i-\mu)^2}{2\sigma^2}(z_i-\mu)\right)}{\sqrt{2\pi}\sigma^2}\right] = 0 \tag{28}$$

$$\sum_{i=1}^{n} \exp(\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i)(\lambda^2\sigma)\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\left[1/2 - \frac{i}{(n+1)}\right.$$
$$\left.+ 0.5\,\mathrm{erf}\left(\frac{z_i-\mu}{\sqrt{2}\sigma}\right) - 0.5\exp(\lambda\mu+\lambda^2\sigma^2/2-\lambda z_i)\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right] = 0 \tag{29}$$

respectively. The weighted least squares estimators of the parameters of EMG distribution are obtained by minimizing

$$\sum_{i=1}^{n} w_i\left[G(z_{(i)}) - \frac{i}{n+1}\right]^2 \tag{30}$$

with respect to parameters, where $w_i = 1/V(Z_i) = \frac{(n+1)^2(n+2)}{i(n-i+1)}$ [12, 29]. Thus, $\hat{\mu}_{WLSE}$, $\hat{\sigma}_{WLSE}$ and $\hat{\lambda}_{WLSE}$ can be obtained by minimizing

$$\sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{1}{2} - \frac{1}{2}\exp(\lambda^2\sigma^2/2+\lambda\mu-z_i\lambda)\,\mathrm{erfc}\left(\frac{\lambda\sigma^2+\mu-z_i}{\sqrt{2}\sigma}\right)\right.$$
$$\left.+\frac{1}{2}\,\mathrm{erf}\left(\frac{z_i-\mu}{\sqrt{2}\sigma}\right) - \frac{i}{n+1}\right]^2 \tag{31}$$

with respect to μ, σ and λ, respectively.

3.5 Method of the maximum product of spacings (MPS)

An alternative to the MLE method is the maximum product of spacing (MPS) for parameter estimation suggested by Cheng and Amin [8]. The uniform spacing from the EMG distribution is given as

$$D_i = G(z_i|\mu,\sigma,\lambda) - G(z_{i-1}|\mu,\sigma,\lambda) \tag{32}$$

where z_i denote the ordered values for $i = 1, 2, \cdots, n$, with the upper and lower limits of D_i are 0 and 1, respectively. The estimators under the MPS, i.e., $\hat{\mu}_{MPS}$, $\hat{\sigma}_{MPS}$ and $\hat{\lambda}_{MPS}$, are obtained by maximizing Eq. (33) with respect to the parameters, i.e., the geometric mean of the spacings

$$G(\mu, \sigma, \lambda) = \left[\prod_{i=1}^{n+1} D_i(\mu, \sigma, \lambda)\right]^{\frac{1}{n+1}} \tag{33}$$

or equivalently by maximizing the function

$$H(\mu, \sigma, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\mu, \sigma, \lambda) \tag{34}$$

The $\hat{\mu}_{MPS}$, $\hat{\sigma}_{MPS}$ and $\hat{\lambda}_{MPS}$ can be obtained by solving the following non-linear equations

$$\frac{\partial}{\partial\mu} H(\mu, \sigma, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{\delta_1(z_i|\mu, \sigma, \lambda)}{D_i(\mu, \sigma, \lambda)} = 0 \tag{35}$$

$$\frac{\partial}{\partial\sigma} H(\mu, \sigma, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{\delta_2(z_i|\mu, \sigma, \lambda)}{D_i(\mu, \sigma, \lambda)} = 0 \tag{36}$$

$$\frac{\partial}{\partial\lambda} H(\mu, \sigma, \lambda) = \frac{1}{n+1}\sum_{i=1}^{n+1} \frac{\delta_3(z_i|\mu, \sigma, \lambda)}{D_i(\mu, \sigma, \lambda)} = 0 \tag{37}$$

where

$$\begin{aligned}
\delta_1(z_i|\mu, \sigma, \lambda) = \sum_{i=1}^{n} \Bigg\{ & -\frac{1}{2}\lambda \exp\left(-\lambda z_i + \frac{\lambda^2\sigma^2}{2} + \lambda\mu\right)\text{erfc}\left(\frac{-z_i + \lambda\sigma^2 + \mu}{\sqrt{2}\sigma}\right) \\
& + \frac{\exp(-\frac{(-z_i + \lambda\sigma^2 + \mu)^2}{2\sigma^2} - \lambda z_i + \frac{\lambda^2\sigma^2}{2} + \lambda\mu)}{\sqrt{2\pi}\sigma} - \frac{\exp(-\frac{(z_i-\mu)^2}{2\sigma^2})}{\sqrt{2\pi}\sigma} \Bigg\} \\
& - \sum_{i=i-1}^{n} \Bigg\{ -\frac{1}{2}\lambda \exp\left(-\lambda z_{i-1} + \frac{\lambda^2\sigma^2}{2} + \lambda\mu\right)\text{erfc}\left(\frac{-z_{i-1} + \lambda\sigma^2 + \mu}{\sqrt{2}\sigma}\right) \\
& + \frac{\exp(-\frac{(-z_{i-1} + \lambda\sigma^2 + \mu)^2}{2\sigma^2} - \lambda z_{i-1} + \frac{\lambda^2\sigma^2}{2} + \lambda\mu)}{\sqrt{2\pi}\sigma} \\
& - \frac{\exp\left(-\frac{(z_{i-1} - \mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma} \Bigg\}
\end{aligned} \tag{38}$$

$$\delta_2(z_i|\mu, \sigma, \lambda) = \sum_{i=1}^{n} \Bigg\{ -\frac{1}{2}\lambda^2\sigma \exp\left(-\lambda z_i + \frac{\lambda^2\sigma^2}{2} + \lambda\mu\right)\text{erfc}\left(\frac{-z_i + \lambda\sigma^2 + \mu}{\sqrt{2}\sigma}\right)$$

$$+\frac{(\sqrt{2}\lambda-\frac{-z_i+\lambda\sigma^2+\mu}{\sqrt{2}\sigma^2})\exp(-\frac{(-z_i+\lambda\sigma^2+\mu)^2}{2\sigma^2}-\lambda z_i+\frac{\lambda^2\sigma^2}{2}+\lambda\mu)}{\sqrt{\pi}}$$

$$\left.-\frac{(z_i-\mu)\exp\left(-\frac{(z_i-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma^2}\right\}$$

$$-\sum_{i=i-1}^{n}\left\{-\frac{1}{2}\lambda^2\sigma\exp\left(-\lambda z_{i-1}+\frac{\lambda^2\sigma^2}{2}+\lambda\mu\right)\operatorname{erfc}\left(\frac{-z_{i-1}+\lambda\sigma^2+\mu}{\sqrt{2}\sigma}\right)\right.$$

$$+\frac{\left(\sqrt{2}\lambda-\frac{-z_{i-1}+\lambda\sigma^2+\mu}{\sqrt{2}\sigma^2}\right)}{\sqrt{\pi}}$$

$$\times\exp\left(-\frac{(-z_{i-1}+\lambda\sigma^2+\mu)^2}{2\sigma^2}-\lambda z_{i-1}+\frac{\lambda^2\sigma^2}{2}+\lambda\mu\right)$$

$$\left.-\frac{(z_{i-1}-\mu)\exp\left(-\frac{(z_{i-1}-\mu)^2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma^2}\right\} \tag{39}$$

$$\delta_3(z_i|\mu,\sigma,\lambda)=\sum_{i=1}^{n}\left\{\frac{\sigma\exp\left(-\frac{(-z_i+\lambda\sigma^2+\mu)^2}{2\sigma^2}-\lambda z_i+\frac{\lambda^2\sigma^2}{2}+\lambda\mu\right)}{\sqrt{2\pi}}\right.$$

$$\left.-\frac{1}{2}\exp\left(-\lambda z_i+\frac{\lambda^2\sigma^2}{2}+\lambda\mu\right)(-z_i+\lambda\sigma^2+\mu)\operatorname{erfc}\left(\frac{-z_i+\lambda\sigma^2+\mu}{\sqrt{2}\sigma}\right)\right\}$$

$$-\sum_{i=i-1}^{n}\left\{\frac{\sigma\exp\left(-\frac{(-z_{i-1}+\lambda\sigma^2+\mu)^2}{2\sigma^2}-\lambda z_{i-1}+\frac{\lambda^2\sigma^2}{2}+\lambda\mu)\right.}{\sqrt{2\pi}}\right.$$

$$-\frac{1}{2}\exp\left(-\lambda z_{i-1}+\frac{\lambda^2\sigma^2}{2}+\lambda\mu\right)(-z_{i-1}+\lambda\sigma^2+\mu)$$

$$\left.\operatorname{erfc}\left(\frac{-z_{i-1}+\lambda\sigma^2+\mu}{\sqrt{2}\sigma}\right)\right\} \tag{40}$$

Under different conditions, Cheng and Amin [8] showed that maximizing MPS estimators is as efficient and consistent as the MLE estimators.

3.6 Method of minimum spacing absolute distance estimator (MSADE)

The minimum spacing distance estimators [36] $\hat{\mu}_{MSADE}$, $\hat{\sigma}_{MSADE}$ and $\hat{\lambda}_{MSADE}$ of μ, σ and λ are obtained by minimizing

$$T(\mu, \sigma, \lambda) = \sum_{i=1}^{n+1} h\left(D_i(\mu, \sigma, \lambda), \frac{1}{n+1}\right) \tag{41}$$

where $h(a, b)$ is an appropriate distance. A choice of $h(a, b)$ is the absolute distance, i.e., $\mid a - b \mid$. These estimators are called the minimum spacing absolute distance estimators (MSADE). The MSADE for parameters μ, σ and λ can be obtained by minimizing

$$T(\mu, \sigma, \lambda) = \sum_{i=1}^{n+1} \left| D_i(\mu, \sigma, \lambda) - \frac{1}{n+1} \right| \tag{42}$$

with respect to μ, σ and λ, respectively. The estimators $\hat{\mu}_{MSADE}$, $\hat{\sigma}_{MSADE}$ and $\hat{\lambda}_{MSADE}$ of μ, σ and λ can be obtained by solving the following nonlinear equations

$$\frac{\partial T(\mu, \sigma, \lambda)}{\partial \mu} = \sum_{i=1}^{n+1} \frac{D_i(\mu, \sigma, \lambda) - \frac{1}{n+1}}{\mid D_i(\mu, \sigma, \lambda) - \frac{1}{n+1} \mid} \delta_1(z_i | \mu, \sigma, \lambda) = 0 \tag{43}$$

$$\frac{\partial T(\mu, \sigma, \lambda)}{\partial \sigma} = \sum_{i=1}^{n+1} \frac{D_i(\mu, \sigma, \lambda) - \frac{1}{n+1}}{\mid D_i(\mu, \sigma, \lambda) - \frac{1}{n+1} \mid} \delta_2(z_i | \mu, \sigma, \lambda) = 0 \tag{44}$$

$$\frac{\partial T(\mu, \sigma, \lambda)}{\partial \lambda} = \sum_{i=1}^{n+1} \frac{D_i(\mu, \sigma, \lambda) - \frac{1}{n+1}}{\mid D_i(\mu, \sigma, \lambda) - \frac{1}{n+1} \mid} \delta_3(z_i | \mu, \sigma, \lambda) = 0 \tag{45}$$

where $\delta_1(\cdot | \mu, \sigma, \lambda)$, $\delta_2(\cdot | \mu, \sigma, \lambda)$ and $\delta_3(\cdot | \mu, \sigma, \lambda)$ are given in Eqs. (38)–(40), respectively.

3.7 Method of minimum spacing absolute log-distance estimator (MSALDE)

The minimum spacing distance estimators of $\hat{\mu}_{MSALDE}$, $\hat{\sigma}_{MSALDE}$ and $\hat{\lambda}_{MSALDE}$ of μ, σ and λ are obtained by minimizing

$$T(\mu, \sigma, \lambda) = \sum_{i=1}^{n+1} h(D_i(\mu, \sigma, \lambda), \frac{1}{n+1}) \tag{46}$$

where $h(a, b)$ is an appropriate distance and $\mid \log a - \log b \mid$ is called the absolute-log distance. These estimators are called the minimum spacing absolute-log distance estimators (MSALDE). The MSALDE of parameters μ, σ and λ, are obtained by minimizing

$$T(\mu, \sigma, \lambda) = \sum_{i=1}^{n+1} \left| \log D_i(\mu, \sigma, \lambda) - \log \frac{1}{n+1} \right| \tag{47}$$

with respect to μ, σ and λ respectively. The estimators $\hat{\mu}_{MSALDE}$, $\hat{\sigma}_{MSALDE}$ and $\hat{\lambda}_{MSALDE}$ of μ, σ and λ can be obtained by solving the below nonlinear equations

$$\frac{\partial T(\mu, \sigma, \lambda)}{\partial \mu} = \sum_{i=1}^{n+1} \frac{\log D_i(\mu, \sigma, \lambda) - \log \frac{1}{n+1}}{\mid \log D_i(\mu, \sigma, \lambda) - \log \frac{1}{n+1} \mid} . \frac{\delta_1(z_i | \mu, \sigma, \lambda)}{D_i(\mu, \sigma, \lambda)} = 0 \tag{48}$$

$$\frac{\partial T(\mu,\sigma,\lambda)}{\partial \sigma}=\sum_{i=1}^{n+1}\frac{\log D_i(\mu,\sigma,\lambda)-\log\frac{1}{n+1}}{|\log D_i(\mu,\sigma,\lambda)-\log\frac{1}{n+1}|}.\frac{\delta_2(z_i|\mu,\sigma,\lambda)}{D_i(\mu,\sigma,\lambda)}=0 \tag{49}$$

$$\frac{\partial T(\mu,\sigma,\lambda)}{\partial \lambda}=\sum_{i=1}^{n+1}\frac{\log D_i(\mu,\sigma,\lambda)-\log\frac{1}{n+1}}{|\log D_i(\mu,\sigma,\lambda)-\log\frac{1}{n+1}|}.\frac{\delta_3(z_i|\mu,\sigma,\lambda)}{D_i(\mu,\sigma,\lambda)}=0 \tag{50}$$

where the sum is taken over the indices for which $\log D_i(\mu,\sigma,\lambda)\neq\log\frac{1}{n+1}$ and $\delta_1(\cdot|\mu,\sigma,\lambda)$, $\delta_2(\cdot|\mu,\sigma,\lambda)$ and $\delta_3(\cdot|\mu,\sigma,\lambda)$ are given in the previous subsection.

3.8 Cramer-Von-Mises method of estimation

Macdonald [28] showed that the Cramer-von-Mises has a smaller bias as compared to other minimum distance type estimators. Assuming $\hat{\mu}_{CME}$, $\hat{\sigma}_{CME}$ and $\hat{\lambda}_{CME}$, the Cramer-von-Mises estimators are obtained by minimizing

$$C(\mu,\sigma,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left[G(z_i)-\frac{2_i-1}{2n}\right]^2 \tag{51}$$

where $z_i=z_{(i)}$ denote the ith order statistics. To obtain the estimators of the EMG distribution, the following non-linear equations need to be solved numerically.

$$\sum_{i=1}^{n}\left[G(z_i)-\frac{2_i-1}{2n}\right]\delta_1(z_i|\mu,,\sigma\lambda)=0 \tag{52}$$

$$\sum_{i=1}^{n}\left[G(z_i)-\frac{2_i-1}{2n}\right]\delta_2(z_i|\mu,\sigma,\lambda)=0 \tag{53}$$

$$\sum_{i=1}^{n}\left[G(z_i)-\frac{2_i-1}{2n}\right]\delta_3(z_i|\mu,\sigma,\lambda)=0 \tag{54}$$

where $\delta_1(\cdot|\mu,\sigma,\lambda)$, $\delta_2(\cdot|\mu,\sigma,\lambda)$ and $\delta_3(\cdot|\mu,\sigma,\lambda)$ are given in Eqs. (38)–(40), respectively.

3.9 Method of Anderson–Darling and right-tail Anderson–Darling

Anderson and Darling [2] introduced the Anderson–Darling (AD) test to detect sample distribution departure from the normality and a nice feature of this test is that it converges towards the asymptote very quickly [3, 32]. Denoting by $\hat{\mu}_{ADE}$, $\hat{\sigma}_{ADE}$ and $\hat{\lambda}_{ADE}$, the AD estimators are obtained by minimizing the following equation with respect to μ, σ and λ.

$$A(\mu,\sigma,\lambda)=-n-\frac{1}{n}\sum_{i=1}^{n}(2_i-1)[\log G(z_{i:n}|\mu,\sigma,\lambda)+\log\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)] \tag{55}$$

These estimators can also be obtained by solving the following system of non-linear equations.

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\delta_1(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}-\frac{\delta_1(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}\right]=0 \tag{56}$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\delta_2(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}-\frac{\delta_2(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}\right]=0 \tag{57}$$

$$\sum_{i=1}^{n}(2i-1)\left[\frac{\delta_3(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}-\frac{\delta_3(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}\right]=0 \tag{58}$$

where $\bar{G}(\cdot) = 1 - G(\cdot)$ denotes the survival function, $z_{i:n}$ denote the order statistics, and $\delta_1(\cdot|\mu,\sigma,\lambda)$, $\delta_2(\cdot|\mu,\sigma,\lambda)$ and $\delta_3(\cdot|\mu,\sigma,\lambda)$ are given in Eqs. (38)–(40).

The estimates of the Right-Tail Anderson–Darling, $\hat{\mu}_{RTADE}$, $\hat{\sigma}_{RTADE}$ and $\hat{\lambda}_{RTADE}$ of the parameters μ, σ and λ are obtained by minimizing

$$R(\mu,\sigma,\lambda)=\frac{n}{2}-2\sum_{i=1}^{n}G(z_{i:n}|\mu,\sigma,\lambda)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda) \tag{59}$$

or by solving the following system of non-linear equations.

$$-2\sum_{i=1}^{n}\frac{\delta_1(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_1(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}=0 \tag{60}$$

$$-2\sum_{i=1}^{n}\frac{\delta_2(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_2(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}=0 \tag{61}$$

$$-2\sum_{i=1}^{n}\frac{\delta_3(z_{i:n}|\mu,\sigma,\lambda)}{G(z_{i:n}|\mu,\sigma,\lambda)}+\frac{1}{n}\sum_{i=1}^{n}(2i-1)\frac{\delta_3(z_{n+1-i:n}|\mu,\sigma,\lambda)}{\bar{G}(z_{n+1-i:n}|\mu,\sigma,\lambda)}=0 \tag{62}$$

where $\delta_1(\cdot|\mu,\sigma,\lambda)$, $\delta_2(\cdot|\mu,\sigma,\lambda)$ and $\delta_3(\cdot|\mu,\sigma,\lambda)$ are given in Eqs. (38)–(40).

4 Monte Carlo simulation study

This section presents a Monte Carlo simulation study to compare the performance of the different methods of estimation. To assess the performance of different estimators, we used bias, root mean squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. In particular, different sample sizes, like $n = 20, 50, 100, 200$ and 250 are used to compare the performance. Further, we considered $\mu = 0, -1, -2, 2, 3$, $\sigma = 0.5, 0.8, 1, 2$ and $\lambda = 0.5, 1, 2, 6$. The simulation study was repeated 5000 times to get stable results. To begin, we estimated the parameters using the method of maximum likelihood which are further used as initial values for all other methods. To generate random numbers from EMG for simulation, one can use the fact that $Z = X + Y$, i.e., sum of the random numbers generated from exponential and normal distributions follows the EMG. Similarly, R language [33] function *uniroot()* can also be used, which in fact find the root of the equation $F(z) - u = 0$, where $F(z)$ is the CDF of the EMG distribution and $u \sim Uniform(0, 1)$. However, in this study, we used the R package 'emg' [15] to generate the random numbers from the EMG distribution.

For each estimator under each method of estimation, the bias, root mean-squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions were calculated. The formulae to calculate these statistics are given as follows:

$$Bias(\hat{\mu})=\frac{1}{R}\sum_{i=1}^{R}(\hat{\mu}_i-\mu), \qquad Bias(\hat{\sigma})=\frac{1}{R}\sum_{i=1}^{R}(\hat{\sigma}_i-\sigma)$$

$$Bias(\hat{\lambda}) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\lambda}_i - \lambda), \qquad RMSE(\hat{\mu}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\mu}_i - \mu)^2}$$

$$RMSE(\hat{\sigma}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\sigma}_i - \sigma)^2}, \quad RMSE(\hat{\lambda}) = \sqrt{\frac{1}{R}\sum_{i=1}^{R}(\hat{\lambda}_i - \lambda)^2}$$

$$D_{abs}(\hat{\mu}) = \frac{1}{nR}\sum_{i=1}^{R}\sum_{j=1}^{n}|G(z_{ij}|\mu, \sigma, \lambda) - G(z_{ij}|\hat{\mu}, \hat{\sigma}, \hat{\lambda})|$$

$$D_{max}(\hat{\mu}) = \frac{1}{nR}\sum_{i=1}^{R}\max_{j}|G(z_{ij}|\mu, \sigma, \lambda) - G(z_{ij}|\hat{\mu}, \hat{\sigma}, \hat{\lambda})|$$

Using the above accuracy measures, the results of a simulation study are presented in Tables 1, 5, 6, 7, 8, 9 and 10. Partial sum of the ranks are also shown in row with label $\sum Ranks$. A superscript shows the rank of each estimate among all the considered eleven estimation methods. For instance, Table 1 presents the MME, $Bias(\hat{\mu})$ as 0.280^3 for sample size $n = 20$, which tells that $Bias(\hat{\mu})$ is calculated using the MME method of estimation is 0.280 and it ranked $3rd$ among all consider estimators used in simulation study. The following conclusion are drawn from the Tables 1, 5, 6, 7, 8, 9 and 10. The key findings are the following.

1. It is observed that all the eleven estimators have consistency property, i.e., the bias and RMSE decrease as the sample size n increased.
2. Comparing the D_{abs} and D_{max} for all eleven considered estimation methods, it is noticed that these both get smaller as sample size n becomes larger.
3. As far as the performance of different estimation methods is concerned, it is observed that the MPS method performs better as compared to other considered estimation methods on the basis of least biases and RMSEs. The MLE method is the next best method of estimation, followed by the MME. The AMLE ranked 4th while LS method ranked 5th, Anderson–Darling ranked 11 and the MSADE ranked 10 among the eleven estimation methods.
4. For fixed λ, the bias of μ increases with its nominal value. For example, assuming $n = 20$ in Table 1 the bias of MLE is 0.216 for $\mu = 0, \sigma = \lambda = 1$, whereas the bias is 0.114 for $\mu = -2, \sigma = 0.5, \lambda = 1$ (Table 5). However, the bias, as reported in Table 8, is 0.432 for $\mu = \sigma = 2, \lambda = 0.5$. The same trend is noticed for the RMSE.
5. For fixed λ, the bias and RMSE decrease when μ and λ decreased.
6. It is also observed that the bias of λ is larger than μ and σ. We attribute this to the exponential distribution behavior, which is a skewed distribution, as the normal distribution is a symmetrical distribution. Since the EMG is the sum of the exponential and normal random variables, the effect of estimation is more serious on the rate parameter of the EMG.
7. Comparing Tables 1 and 6 it is noticed that the biases and RMSEs become large as σ gets large. Furthermore, the RMSE of λ decreases by increasing σ for a small sample size. As the sample size increases, the RMSE also increases.
8. For fixed λ, when σ and μ decrease, the RMSE and bias decrease for the MLE, MME, AMLE, whereas for the other methods the values of these measure increase.
9. The MPS and MLE perform more consistently than the other methods.
10. In general, the MSADE, MSALDE, and AD produce the largest biases and RMSEs. The same trend can be noticed in D_{abs} and D_{max}.

Table 1 Simulation results for $\mu = 0, \sigma = 1$ and $\lambda = 1$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.280^{3}	-0.291^{4}	0.216^{2}	$-1.140\text{e}02^{6}$	$-5.278\text{e}02^{7}$	0.032^{1}	$-2.353\text{e}03^{9}$	8321.443^{10}	$-3.637\text{e}01^{5}$	16406.884^{11}	$-7.010\text{e}02^{8}$
	RMSE($\hat{\mu}$)	0.577^{2}	0.396^{1}	0.651^{4}	$2.471\text{e}03^{6}$	$8.850\text{e}03^{8}$	0.619^{3}	$2.135\text{e}04^{9}$	74169.220^{10}	$6.847\text{e}02^{5}$	118718.843^{11}	$7.977\text{e}03^{7}$
	Bias($\hat{\sigma}$)	0.035^{2}	-0.998^{4}	-0.081^{3}	$2.718\text{e}01^{6}$	$1.748\text{e}02^{8}$	0.018^{1}	$2.186\text{e}03^{10}$	2395.117^{11}	$1.026\text{e}01^{5}$	639.204^{9}	$1.004\text{e}02^{7}$
	RMSE($\hat{\sigma}$)	0.297^{1}	0.999^{4}	$0.398^{2.5}$	$6.744\text{e}02^{6}$	$3.265\text{e}03^{8}$	$0.398^{2.5}$	$2.025\text{e}04^{11}$	8835.365^{10}	$3.000\text{e}02^{5}$	3558.826^{9}	$1.265\text{e}03^{7}$
	Bias($\hat{\lambda}$)	527060.569^{11}	-0.030^{1}	7.067^{2}	$2.799\text{e}02^{5}$	$6.273\text{e}02^{7}$	7.239^{3}	$9.438\text{e}02^{8}$	9215.620^{10}	$8.345\text{e}01^{4}$	8851.163^{9}	$3.243\text{e}02^{6}$
	RMSE($\hat{\lambda}$)	7639526.297^{11}	0.081^{1}	13.845^{2}	$8.311\text{e}03^{7}$	$8.958\text{e}03^{8}$	18.120^{3}	$5.976\text{e}03^{6}$	42625.254^{10}	$2.915\text{e}03^{4}$	41091.820^{9}	$4.582\text{e}03^{5}$
	D_{abs}	0.987^{10}	0.999^{11}	$0.061^{1.5}$	0.897^{5}	0.908^{6}	$0.061^{1.5}$	0.954^{8}	0.464^{3}	0.587^{4}	0.976^{9}	0.945^{7}
	D_{max}	0.982^{10}	0.989^{11}	0.107^{2}	0.926^{5}	0.934^{6}	0.102^{1}	0.965^{8}	0.897^{3}	0.903^{4}	0.976^{9}	0.945^{7}
	$\sum$Ranks	50^{6}	33^{3}	20^{2}	46^{5}	58^{8}	16^{1}	59^{9}	67^{10}	36^{4}	73^{11}	54^{7}
50	Bias($\hat{\mu}$)	0.169^{3}	-0.290^{4}	0.128^{2}	$-2.641\text{e}01^{5}$	$-8.173\text{e}02^{7}$	0.012^{1}	$-3.252\text{e}03^{10}$	-1362.890^{9}	$-9.659\text{e}01^{6}$	18056.062^{11}	$-9.915\text{e}02^{8}$
	RMSE($\hat{\mu}$)	0.420^{3}	0.338^{1}	0.453^{4}	$2.677\text{e}03^{6}$	$9.178\text{e}03^{7}$	0.408^{2}	$3.694\text{e}04^{10}$	23644.620^{9}	$1.003\text{e}03^{5}$	139381.766^{11}	$1.228\text{e}04^{8}$
	Bias($\hat{\sigma}$)	$0.032^{2.5}$	-1.000^{4}	-0.009^{1}	$1.939\text{e}01^{5}$	$3.600\text{e}02^{8}$	$0.032^{2.5}$	$2.999\text{e}03^{11}$	1851.410^{10}	$2.349\text{e}01^{6}$	676.504^{9}	$1.519\text{e}02^{7}$
	RMSE($\hat{\sigma}$)	0.237^{2}	1.000^{4}	0.239^{3}	$5.087\text{e}02^{6}$	$3.454\text{e}03^{8}$	0.233^{1}	$4.868\text{e}04^{11}$	6158.604^{10}	$5.066\text{e}02^{5}$	5594.818^{9}	$2.086\text{e}03^{7}$
	Bias($\hat{\lambda}$)	28226.792^{11}	-0.035^{1}	2.764^{3}	$2.366\text{e}02^{4}$	$9.384\text{e}02^{7}$	2.302^{2}	$1.631\text{e}03^{8}$	2895.041^{9}	$3.147\text{e}02^{5}$	7193.613^{10}	$4.147\text{e}02^{6}$
	RMSE($\hat{\lambda}$)	613339.112^{11}	0.058^{1}	7.999^{2}	$6.992\text{e}03^{5}$	$9.591\text{e}03^{7}$	8.852^{3}	$9.629\text{e}03^{8}$	19243.880^{9}	$8.337\text{e}03^{6}$	35817.556^{10}	$5.034\text{e}03^{4}$
	D_{abs}	0.976^{10}	0.998^{11}	0.038^{1}	0.917^{4}	0.943^{6}	0.039^{2}	0.956^{8}	0.475^{3}	0.932^{5}	0.967^{9}	0.951^{7}
	D_{max}	0.987^{10}	0.998^{11}	0.067^{2}	0.912^{4}	0.953^{6}	0.065^{1}	0.965^{8}	0.906^{3}	0.923^{5}	0.976^{9}	0.963^{7}
	$\sum$Ranks	52.5^{6}	37^{3}	18^{2}	39^{4}	56^{8}	15.5^{1}	74^{10}	64^{9}	43^{5}	78^{11}	54^{7}
100	Bias($\hat{\mu}$)	0.088^{3}	-0.293^{4}	0.063^{2}	$-9.089\text{e}01^{5}$	$-9.809\text{e}02^{7}$	-0.008^{1}	$-1.240\text{e}04^{10}$	-3231.035^{9}	$-9.707\text{e}01^{6}$	$1.536\text{e}04^{11}$	$-1.166\text{e}03^{8}$
	RMSE($\hat{\mu}$)	0.303^{2}	0.317^{4}	0.306^{3}	$8.695\text{e}02^{5}$	$8.733\text{e}03^{7}$	0.275^{1}	$3.795\text{e}04^{10}$	12845.262^{8}	$9.012\text{e}02^{6}$	$1.487\text{e}05^{11}$	$1.393\text{e}04^{9}$
	Bias($\hat{\sigma}$)	0.016^{2}	-1.000^{4}	0.0005^{1}	$1.841\text{e}01^{5}$	$3.700\text{e}02^{8}$	0.021^{3}	$7.164\text{e}03^{11}$	1831.493^{10}	$2.189\text{e}01^{6}$	$5.909\text{e}02^{9}$	$1.640\text{e}02^{7}$
	RMSE($\hat{\sigma}$)	0.197^{3}	1.000^{4}	0.168^{2}	$5.026\text{e}02^{5}$	$3.751\text{e}03^{8}$	0.162^{1}	$2.066\text{e}04^{11}$	5523.077^{10}	$5.582\text{e}02^{6}$	$4.626\text{e}03^{9}$	$2.118\text{e}03^{7}$

Table 1 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	Bias($\hat{\lambda}$)	303.298^{6}	-0.036^{1}	0.808^{3}	$7.975e01^{4}$	$6.313e02^{8}$	0.534^{2}	$4.731e03^{10}$	1796.839^{9}	$1.005e02^{5}$	$5.968e03^{11}$	$4.478e02^{7}$
	RMSE($\hat{\lambda}$)	11426.028^{9}	0.049^{1}	3.949^{3}	$7.225e02^{4}$	$5.925e03^{7}$	3.754^{2}	$1.473e04^{10}$	8388.760^{8}	$1.453e03^{5}$	$3.121e04^{11}$	$4.276e03^{6}$
	D_{abs}	0.985^{10}	0.997^{11}	$0.027^{1.5}$	0.934^{5}	0.956^{7}	$0.027^{1.5}$	$3.473e220^{9}$	0.475^{3}	0.952^{6}	0.907^{4}	0.967^{8}
	D_{max}	0.987^{10}	0.998^{11}	$0.046^{1.5}$	$0.897^{4.5}$	0.954^{7}	$0.046^{1.5}$	0.970^{9}	0.869^{3}	0.943^{6}	$0.897^{4.5}$	0.963^{8}
	$\sum$Ranks	45^{5}	40^{4}	17^{2}	37.5^{3}	59^{7}	13^{1}	80^{11}	$60^{8.5}$	46^{6}	70.5^{10}	$60^{8.5}$
200	Bias($\hat{\mu}$)	0.048^{3}	-0.292^{4}	0.029^{2}	-138.239^{5}	$-1.067e03^{8}$	-0.010^{1}	$-1.288e04^{11}$	-3598.212^{9}	-139.517^{6}	$5.639e03^{10}$	$-9.993e02^{7}$
	RMSE($\hat{\mu}$)	0.207^{3}	0.304^{4}	0.196^{2}	406.801^{6}	$8.779e03^{7}$	0.180^{1}	$3.842e04^{10}$	10059.239^{8}	378.691^{5}	$5.859e04^{11}$	$1.296e04^{9}$
	Bias($\hat{\sigma}$)	$0.012^{2.5}$	-1.000^{4}	0.001^{1}	16.092^{6}	$2.689e02^{8}$	$0.012^{2.5}$	$7.321e03^{11}$	1775.331^{10}	15.856^{5}	$3.185e02^{9}$	$1.509e02^{7}$
	RMSE($\hat{\sigma}$)	0.137^{3}	1.000^{4}	0.114^{2}	55.649^{6}	$2.485e03^{9}$	0.110^{1}	$2.025e04^{11}$	5018.426^{10}	50.735^{5}	$1.824e03^{7}$	$2.154e03^{8}$
	Bias($\hat{\lambda}$)	118.747^{6}	-0.036^{1}	0.136^{3}	91.335^{4}	$6.902e02^{8}$	0.040^{2}	$5.188e03^{11}$	1592.432^{9}	93.026^{5}	$4.402e03^{10}$	$4.151e02^{7}$
	RMSE($\hat{\lambda}$)	8383.076^{9}	0.043^{1}	1.162^{3}	270.822^{5}	$6.217e03^{8}$	0.642^{2}	$1.550e04^{10}$	5620.190^{7}	260.535^{4}	$2.477e04^{11}$	$3.715e03^{6}$
	D_{abs}	0.976^{10}	0.986^{11}	$0.019^{1.5}$	0.485^{5}	0.956^{7}	$0.019^{1.5}$	0.965^{9}	0.465^{3}	0.485^{4}	0.943^{6}	0.964^{8}
	D_{max}	0.978^{10}	0.997^{11}	$0.033^{1.5}$	0.915^{5}	0.943^{7}	$0.033^{1.5}$	0.965^{9}	0.900^{3}	0.904^{4}	0.921^{6}	0.954^{8}
	$\sum$Ranks	46.5^{6}	41^{4}	16^{2}	42^{5}	62^{9}	12.5^{1}	82^{11}	59^{7}	38^{3}	70^{10}	60^{8}
250	Bias($\hat{\mu}$)	0.034^{3}	-0.293^{4}	0.019^{2}	-170.245^{5}	$-1.505e03^{8}$	-0.011^{1}	$-1.307e04^{11}$	-3289.436^{9}	$-1.742e02^{6}$	$6.942e03^{10}$	$-1.017e03^{7}$
	RMSE($\hat{\mu}$)	0.178^{3}	0.303^{4}	0.161^{2}	501.423^{5}	$1.098e04^{8}$	0.151^{1}	$3.933e04^{10}$	9289.105^{7}	$5.239e02^{6}$	$1.051e05^{11}$	$1.306e04^{9}$
	Bias($\hat{\sigma}$)	0.008^{2}	-1.000^{4}	0.002^{1}	20.257^{5}	$3.806e02^{6}$	0.011^{3}	$7.710e03^{9}$	1581.650^{8}	$2.381e01^{7}$	$3.987e02^{11}$	$1.559e02^{10}$
	RMSE($\hat{\sigma}$)	0.131^{3}	1.000^{4}	0.099^{2}	69.844^{5}	$3.006e03^{8}$	0.097^{1}	$2.344e04^{11}$	4564.647^{10}	$2.734e02^{6}$	$3.529e03^{9}$	$2.197e03^{7}$
	Bias($\hat{\lambda}$)	0.092^{4}	-0.037^{1}	0.062^{3}	111.490^{5}	$8.964e02^{8}$	0.013^{2}	$5.790e03^{11}$	1488.802^{9}	$1.305e02^{6}$	$4.481e03^{10}$	$4.226e02^{7}$
	RMSE($\hat{\lambda}$)	0.943^{4}	0.042^{1}	0.493^{3}	322.505^{5}	$6.621e03^{9}$	0.335^{2}	$1.767e04^{10}$	4736.482^{8}	$1.263e03^{6}$	$2.514e04^{11}$	$3.640e03^{7}$
	D_{abs}	0.987^{10}	0.998^{11}	$0.017^{1.5}$	0.487^{4}	0.956^{7}	$0.017^{1.5}$	0.975^{9}	0.459^{3}	0.943^{6}	0.934^{5}	0.967^{8}
	D_{max}	0.970^{10}	0.989^{11}	$0.029^{1.5}$	0.796^{4}	0.921^{7}	$0.029^{1.5}$	0.942^{9}	0.738^{3}	0.901^{6}	0.897^{5}	0.932^{8}
	$\sum$Ranks	39^{4}	40^{5}	16^{2}	38^{3}	61^{8}	13^{1}	80^{11}	57^{7}	49^{6}	72^{10}	63^{9}

5 Real data analysis

This section presents an application for illustration purposes of the EMG distribution to a real data set is presented. This real life application will highlight the EMG distribution flexibility in modelling practical data. For comparison purposes, we also fitted normal (N), exponential (E), lognormal, and Weibull distributions to the same data. The PDFs of these distributions are given as:

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right); -\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0 \tag{63}$$

$$f(x; \lambda) = \lambda \exp(-\lambda x); x, \lambda > 0 \tag{64}$$

$$f(x, \mu, \sigma) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right); x > 0, -\infty < \mu < \infty, \sigma > 0 \tag{65}$$

$$f(x, \beta, \lambda) = \beta\lambda^{\beta} x^{\beta-1}\left(\exp(-\lambda x)^{\beta}\right); x, \lambda, \beta > 0 \tag{66}$$

To fit the distributions, R package *fitdistrplus* is used. In term of performance, we considered Akaike information criterion (AIC) [1] and Bayesian information criterion (BIC) [7] for comparing the distributions. Note that $AIC = 2m - 2LL$ whereas $BIC = m\log(n) - 2LL$, where m is the number of parameters in the statistical model, n is the sample size, and LL is the maximized value of the log-likelihood function for the estimated model.

The data set given in Table 2 is taken from Gabriel et al. [14]. The data set is about the PC9 cancer cell observations with 3um erlotinib applied at time zero. The experiment was performed on 2011/9/9 in the Vito Quaranta laboratory at Vanderbilt University Cancer Biology Center by Darren Tyson. Cells were tracked by nuclear labeling with histone H2B and imaged on a BD Pathway 855 for several days. There are 341 observations and all the numerical values are in hours. The parameter estimates along with the model selection measures are given in Table 3. On the basis of AIC, BIC and log-likelihood, it is observed that the EMG is the best compared to normal, exponential, lognormal and Weibull as the EMG model has the lowest AIC and BIC compared to other models. The graphical illustration of the results is given in Fig. 2. Furthermore, parameters of the EMG distribution are also estimated using different estimation methods and results are listed in Table 4. It is clear from the table that the MLE, RAD, AD and MPS methods outperform the other methods, as these methods have the lowest sum of ranks. Hence, the conclusions of simulation study are supplemented by the real data estimates.

Table 2 Time in hours to detect cancer cells with 3um erlotinib

16.80	18.50	19.90	21.20	19.10	20.40	16.30	16.70	25.10	23.30	20.70	18.70
17.90	19.80	19.80	19.50	17.60	18.40	25.50	193.80	30.10	72.40	21.90	22.60
23.00	22.80	191.60	191.60	159.70	146.40	26.80	30.10	59.30	126.60	83.60	189.20
26.40	113.60	19.80	20.50	188.60	126.00	188.10	25.50	81.70	30.70	67.30	186.80
182.40	24.10	26.90	78.70	114.80	27.00	28.40	96.10	133.60	175.90	183.60	21.90
24.80	182.40	91.20	60.20	80.10	61.90	88.60	25.50	94.10	160.70	98.70	25.20
65.50	18.30	20.80	120.80	76.20	23.00	60.20	112.60	56.80	75.10	88.20	175.80
34.20	28.80	36.00	70.70	135.20	22.00	24.50	71.40	121.90	54.10	103.40	173.10
113.30	160.90	23.70	45.00	38.70	25.10	96.50	105.40	85.40	147.90	56.80	91.80
154.20	167.80	90.40	108.10	168.40	168.40	141.70	167.20	31.10	35.20	162.80	162.80
49.30	162.60	64.20	68.60	71.70	131.10	40.20	63.30	36.60	32.10	159.30	62.60
27.10	31.00	134.20	157.10	127.90	40.40	156.40	156.40	97.10	52.00	24.60	26.90
64.50	148.60	50.50	153.30	26.10	34.10	127.10	129.90	89.30	149.20	39.40	36.00
28.10	32.90	31.70	139.60	137.90	137.90	49.80	51.40	130.20	130.20	2.00	130.20
127.90	127.90	67.60	127.30	67.40	42.40	126.30	126.30	106.50	125.60	51.80	52.00
75.90	92.00	50.40	57.10	2.70	74.00	53.20	65.10	117.50	117.50	117.00	76.70
117.40	117.40	25.40	72.90	12.60	55.10	113.30	113.30	112.10	52.30	26.70	72.40
1.70	22.30	29.40	31.10	16.90	107.90	108.70	108.70	95.60	105.50	106.20	106.20
103.10	36.50	56.20	93.30	102.80	62.60	100.50	95.80	98.40	98.40	98.20	98.20
96.70	16.70	0.40	0.50	91.70	91.70	63.80	80.50	90.20	90.20	87.60	76.40
85.70	85.70	84.60	67.50	84.60	84.60	57.40	59.00	80.10	80.10	81.00	81.00
79.70	79.70	77.40	77.40	75.40	75.40	63.80	63.80	41.90	54.90	66.60	66.60
64.90	64.90	65.30	58.30	62.40	46.30	37.30	7.20	53.00	53.00	51.70	51.70
50.20	50.20	45.90	45.90	43.90	43.90	11.90	23.60	40.00	40.00	32.00	4.90
31.00	31.00	29.90	4.30	27.80	27.80	27.40	27.40	25.50	25.50	14.70	24.40
25.40	25.40	8.70	22.90	23.70	23.70	21.60	21.60	22.10	22.10	19.10	19.10
12.90	17.80	16.10	16.10	17.10	17.10	13.70	13.70	11.30	5.10	1.50	10.00
10.00	10.90	10.90	10.70	10.70	10.40	10.40	10.40	10.40	8.90	8.90	7.70
7.70	4.80	4.80	3.60	3.60							

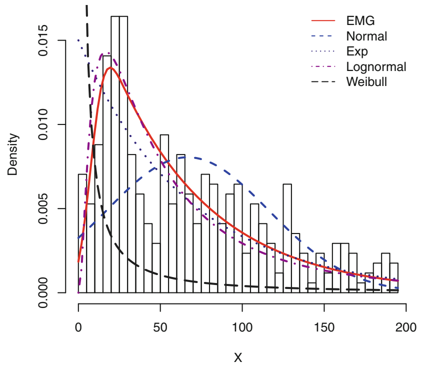
Table 3 Parameter estimation of EMG and comparison with some existing distributions for the real data

Model	Parameters estimates	Loglikelihood	AIC	BIC
Normal	$\hat{\mu} = 66.65543$, $\hat{\sigma} = 49.55686$ (2.683655, 1.897630)	– 1814.822	3633.644	3641.308
Exponential	$\hat{\lambda} = 0.01500253$, (0.0008088)	– 1773.042	3548.084	3551.916
lognormal	$\hat{\mu} = 3.826037$, $\hat{\sigma} = 1.007248$ (0.05454551, 0.03856933)	– 1790.999	3585.998	3593.662
Weibull	$\hat{\beta} = 1.308466$, $\hat{\lambda} = 0.01385474$ (0.05681653, 0.318151)	– 24883.625	49767.254	49767.26166
EMG	$\hat{\mu} = 7.92076067$, $\hat{\sigma} = 6.57293954$, $\hat{\lambda} = 0.01704493$ (1.51175331, 1.264117615, 0.001012831)	– 1763.412	**3532.824**	**3534.422**

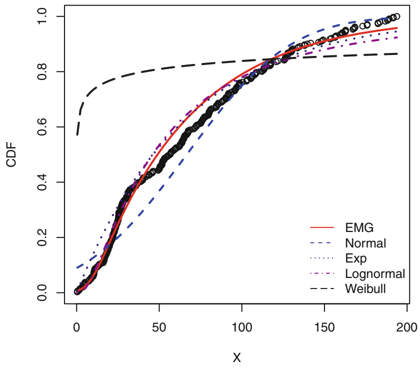
Bold values denote the minimum values of AIC and BIC

Table 4 Comparison of estimation methods for real data set

Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
$\hat{\mu}$	30.9681	0.4892	7.9112	− 3.4051	− 6.6000	0.6451	0.1001	0.0065	− 6.6000	10.0743	8.4206
SD($\hat{\mu}$)	23.0481^{11}	7.4308^{5}	1.5118^{2}	9.9725^{8}	$14.5200^{9.5}$	7.2749^{4}	7.8199^{6}	7.9135^{7}	$14.5200^{9.5}$	2.1543^{3}	0.5006^{1}
$\hat{\sigma}$	34.3846	0.0035	6.5652	0.2356	0.3000	1.2429	1.0499	0.9949	0.3000	4.8322	0.00003
SD($\hat{\sigma}$)	27.8146^{11}	6.5665^{9}	1.2641^{1}	11.3251^{10}	$6.2700^{7.5}$	5.3271^{4}	5.5201^{5}	5.5751^{6}	$6.2700^{7.5}$	1.7378^{2}	0.00001^{1}
$\hat{\lambda}$	0.0280	0.0341	0.0170	9.9895	5.9000	0.0126	0.9000	1.1021	5.9000	0.2064	0.2091
SD($\hat{\lambda}$)	0.0110^{4}	0.01771^{5}	0.0010^{2}	0.00002^{1}	$5.8830^{10.5}$	0.0044^{3}	0.8830^{8}	1.0857^{9}	$5.8830^{10.5}$	0.1894^{7}	0.1849^{6}
$\sum$Ranks	26^{9}	19^{6}	5^{1}	19^{6}	$27.5^{10.5}$	$12^{3.5}$	19^{6}	22^{8}	$27.5^{10.5}$	$12^{3.5}$	8^{2}



(a) Density Plot



(b) Empirical CDF

Fig. 2 Visual check for the EMG fitted model to observed data

Conclusion

In this study, the EMG distribution is considered due to its practical application in biology and chemistry and parameters of the distribution are estimated by eleven different methods of estimation, namely, the method of moment estimation (MME), approximated L-moment estimation (ALME), the maximum likelihood estimation (MLE), the least squares estimation (LSE), the weighted least squares estimation (WLSE), the maximum product spacing (MPS), the minimum spacing absolute distance estimation (MSADE), the minimum spacing absolute log-distance estimation (MSALDE), Cramer-Von-Mises (CVM), Anderson–Darling (AD) method, and the right-tail Anderson–Darling (RAD) method. The performance of different estimators are assessed using a comprehensive Monte Carlo simulation study. The estimators are compared on the basis of bias, root mean-squared error, the average absolute difference between the theoretical and empirical estimate of the distribution functions, and the maximum absolute difference between the theoretical and empirical distribution functions. Results showed that the performance of the maximum product spacing estimation method is better than the rest methods. To show an application of the EMG distribution, a real data set has also been analyzed.

Appendix

This section presents some additional tables of biases, root mean squared errors, and other measures under different estimation methods (Tables 5, 6, 7, 8, 9, 10).

Table 5 Simulation results for $\mu = -2, \sigma = 0.5$ and $\lambda = 1$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.232^{3}	0.548^{4}	0.114^{2}	-4.032^{5}	$2.411e01^{8}$	0.017^{1}	$-2.193e03^{9}$	50932.510^{11}	-5.621^{6}	19878.857^{10}	-7.320^{7}
	RMSE($\hat{\mu}$)	0.419^{3}	0.558^{4}	0.404^{2}	13.227^{5}	$1.618e03^{7}$	0.362^{1}	$4.826e04^{9}$	2313433.025^{11}	16.783^{6}	227216.550^{10}	$2.651e03^{8}$
	Bias($\hat{\sigma}$)	0.119^{4}	-0.095^{3}	-0.056^{2}	0.182^{5}	6.076^{7}	0.023^{1}	$2.299e03^{10}$	2763.911^{11}	0.372^{6}	574.507^{9}	$1.537e01^{8}$
	RMSE($\hat{\sigma}$)	0.276^{3}	0.101^{1}	0.282^{4}	1.324^{5}	$2.582e02^{7}$	0.270^{2}	$6.014e04^{10}$	62713.850^{11}	1.765^{6}	6465.474^{9}	$4.442e02^{8}$
	Bias($\hat{\lambda}$)	5.164^{6}	1.303^{1}	3.435^{3}	4.077^{4}	5.640^{7}	2.978^{2}	$7.054e02^{9}$	8239.500^{11}	4.541^{5}	7463.072^{10}	$5.694e01^{8}$
	RMSE($\hat{\lambda}$)	24.135^{6}	1.425^{1}	11.628^{2}	17.884^{4}	$2.553e03^{8}$	13.295^{3}	$5.940e03^{9}$	44685.927^{11}	18.647^{5}	43967.572^{10}	$1.972e03^{7}$
	D_{abs}	0.978^{10}	0.989^{11}	0.062^{2}	$0.449^{3.5}$	0.921^{6}	0.061^{1}	0.954^{8}	0.463^{5}	$0.449^{3.5}$	0.968^{9}	0.942^{7}
	D_{max}	0.967^{10}	0.973^{11}	0.112^{2}	$0.939^{4.5}$	0.938^{6}	0.103^{1}	0.953^{8}	0.928^{3}	$0.939^{4.5}$	0.965^{9}	0.942^{7}
	$\sum$Ranks	45^{6}	$36^{3.5}$	19^{2}	$36^{3.5}$	56^{7}	12^{1}	72^{9}	74^{10}	42^{5}	76^{11}	60^{8}
50	Bias($\hat{\mu}$)	0.119^{3}	0.556^{4}	0.032^{2}	$3.275e01^{7}$	$9.410e-01^{5}$	-0.010^{1}	$-3.401e03^{9}$	4108.892^{10}	$1.412e01^{6}$	$1.073e04^{11}$	-89.195^{8}
	RMSE($\hat{\mu}$)	0.261^{3}	0.560^{4}	0.225^{2}	$2.001e03^{8}$	$1.179e03^{5}$	0.204^{1}	$2.063e04^{9}$	94314.401^{10}	$1.572e03^{7}$	$9.506e04^{11}$	1412.414^{6}
	Bias($\hat{\sigma}$)	0.066^{3}	-0.092^{4}	-0.020^{1}	$7.061e00^{7}$	$4.340e00^{5}$	0.022^{2}	$2.583e03^{11}$	1233.613^{10}	$4.458e00^{6}$	$3.316e02^{9}$	10.151^{8}
	RMSE($\hat{\sigma}$)	0.232^{4}	0.095^{1}	0.164^{3}	$3.322e02^{8}$	$1.885e02^{6}$	0.154^{2}	$2.082e04^{11}$	5080.594^{10}	$2.669e02^{7}$	$2.892e03^{9}$	173.083^{5}
	Bias($\hat{\lambda}$)	0.503^{3}	1.294^{4}	0.343^{2}	$5.009e01^{8}$	$2.592e01^{6}$	0.178^{1}	$2.309e03^{9}$	3042.137^{10}	$2.289e01^{5}$	$8.491e03^{11}$	34.215^{7}
	RMSE($\hat{\lambda}$)	4.736^{4}	1.365^{1}	2.793^{3}	$2.294e03^{8}$	$1.862e03^{7}$	2.398^{2}	$1.649e04^{9}$	22137.822^{10}	$1.202e03^{6}$	$4.576e04^{11}$	425.571^{5}
	D_{abs}	0.998^{11}	0.100^{3}	$0.039^{1.5}$	0.923^{7}	0.942^{8}	$0.039^{1.5}$	0.967^{10}	0.473^{5}	0.920^{6}	0.953^{9}	0.471^{4}
	D_{max}	0.997^{11}	0.186^{3}	0.069^{2}	0.979^{8}	0.978^{6}	0.067^{1}	0.989^{10}	0.965^{4}	0.978^{6}	0.980^{9}	0.977^{5}
	$\sum$Ranks	42^{4}	24^{3}	16.5^{2}	61^{8}	53^{6}	11.5^{1}	78^{10}	69^{9}	49^{6}	80^{11}	48^{5}
100	Bias($\hat{\mu}$)	0.063^{3}	0.554^{4}	0.014^{2}	-10.791^{5}	-37.902^{7}	-0.008^{1}	$-9.556e03^{11}$	174.075^{9}	-12.248^{6}	9314.169^{10}	-97.644^{8}
	RMSE($\hat{\mu}$)	0.191^{3}	0.556^{4}	0.141^{2}	24.675^{5}	111.346^{7}	0.134^{1}	$3.083e04^{10}$	25565.034^{9}	27.027^{6}	124531.117^{11}	1014.504^{8}
	Bias($\hat{\sigma}$)	0.027^{3}	-0.093^{4}	-0.007^{1}	0.940^{5}	3.462^{7}	0.019^{2}	$5.955e03^{11}$	1087.139^{10}	1.136^{6}	280.189^{9}	10.984^{8}
	RMSE($\hat{\sigma}$)	0.210^{4}	0.095^{1}	0.105^{3}	2.943^{5}	10.059^{7}	0.103^{2}	$1.856e+04^{11}$	4177.207^{10}	3.253^{6}	3525.889^{9}	126.270^{8}

Table 5 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	Bias($\hat{\lambda}$)	0.142^{3}	1.285^{4}	0.056^{2}	7.552^{6}	-2.115^{5}	0.010^{1}	$4.544e03^{10}$	1926.094^{9}	8.031^{7}	5041.132^{11}	54.581^{8}
	RMSE($\hat{\lambda}$)	1.898^{4}	1.325^{3}	0.539^{2}	15.831^{5}	168.973^{7}	0.509^{1}	$1.829e04^{10}$	17577.318^{9}	19.050^{6}	34998.159^{11}	805.147^{8}
	D_{abs}	0.987^{11}	0.098^{3}	$0.027^{1.5}$	0.478^{5}	0.498^{9}	$0.027^{1.5}$	0.965^{10}	0.470^{4}	0.478^{7}	0.478^{6}	0.482^{8}
	D_{max}	0.999^{11}	0.180^{3}	$0.047^{1.5}$	0.989^{6}	0.989^{9}	$0.047^{1.5}$	0.997^{10}	0.960^{4}	0.989^{7}	0.987^{5}	0.989^{8}
	$\sum$Ranks	$42^{4.5}$	26^{3}	15^{2}	$42^{4.5}$	58^{7}	11^{1}	83^{11}	$64^{8.5}$	51^{6}	72^{10}	$64^{10.5}$
200	Bias($\hat{\mu}$)	0.038^{3}	0.555^{4}	0.007^{2}	-16.716^{6}	-69.871^{7}	-0.003^{1}	$-1.421e04^{11}$	-1217.978^{9}	-18.041^{5}	$1.133e04^{10}$	-857.242^{8}
	RMSE($\hat{\mu}$)	0.142^{4}	0.556^{3}	0.096^{2}	34.093^{5}	167.904^{7}	0.093^{1}	$3.664e04^{9}$	10253.707^{8}	35.751^{6}	$1.749e05^{11}$	52347.810^{10}
	Bias($\hat{\sigma}$)	0.011^{2}	-0.094^{4}	-0.003^{1}	1.713^{5}	5.981^{7}	0.013^{3}	$8.886e03^{11}$	1022.749^{10}	1.888^{6}	$3.726e02^{9}$	120.525^{8}
	RMSE($\hat{\sigma}$)	0.180^{4}	0.094^{3}	$0.069^{1.5}$	4.197^{5}	14.471^{7}	$0.069^{1.5}$	$2.254e04^{11}$	3788.368^{8}	4.398^{6}	$5.221e03^{9}$	7596.314^{10}
	Bias($\hat{\lambda}$)	0.063^{3}	1.269^{4}	0.021^{2}	10.649^{5}	13.132^{7}	-0.001^{1}	$6.215e03^{11}$	824.872^{9}	11.346^{6}	$1.735e03^{10}$	60.982^{8}
	RMSE($\hat{\lambda}$)	0.184^{3}	1.289^{4}	0.124^{2}	19.922^{5}	245.303^{7}	0.116^{1}	$1.924e04^{11}$	7005.587^{9}	20.836^{6}	$1.722e04^{10}$	753.007^{8}
	D_{abs}	0.993^{11}	0.097^{3}	$0.019^{1.5}$	$0.485^{5.5}$	0.499^{8}	$0.019^{1.5}$	0.987^{10}	0.457^{4}	$0.485^{5.5}$	0.967^{9}	0.489^{7}
	D_{max}	0.986^{11}	0.178^{3}	$0.033^{1.5}$	0.955^{7}	0.955^{7}	$0.033^{1.5}$	0.976^{10}	0.935^{4}	0.955^{7}	0.967^{9}	0.944^{5}
	$\sum$Ranks	41^{4}	28^{3}	13.5^{2}	43.5^{5}	49^{7}	11.5^{1}	76^{10}	61^{8}	47.5^{6}	77^{11}	64^{9}
250	Bias($\hat{\mu}$)	0.028^{3}	0.554^{4}	0.004^{2}	-19.818^{5}	-81.684^{7}	-0.004^{1}	$-1.491e04^{10}$	-919.662^{9}	-20.989^{6}	16343.007^{11}	-136.256^{8}
	RMSE($\hat{\mu}$)	0.127^{3}	0.556^{4}	0.082^{2}	38.285^{5}	184.564^{7}	0.080^{1}	$3.676e04^{10}$	11221.147^{9}	39.731^{6}	219661.636^{11}	1677.062^{8}
	Bias($\hat{\sigma}$)	0.002^{1}	-0.094^{4}	-0.003^{2}	2.120^{5}	6.957^{7}	0.011^{3}	$9.618e03^{11}$	881.656^{10}	2.288^{6}	499.308^{9}	15.357^{8}
	RMSE($\hat{\sigma}$)	0.176^{4}	0.094^{3}	$0.061^{1.5}$	4.738^{5}	15.834^{7}	$0.061^{1.5}$	$2.347e04^{11}$	3345.513^{9}	4.923^{6}	6489.642^{10}	204.669^{8}
	Bias($\hat{\lambda}$)	0.047^{3}	1.266^{4}	0.014^{2}	12.412^{5}	17.961^{7}	-0.003^{1}	$6.240e03^{11}$	856.658^{9}	12.778^{6}	1242.352^{10}	71.609^{8}
	RMSE($\hat{\lambda}$)	0.158^{3}	1.282^{4}	0.103^{2}	22.278^{5}	274.050^{7}	0.098^{1}	$1.572e04^{11}$	13318.168^{9}	28.012^{6}	14801.396^{10}	930.392^{8}
	D_{abs}	0.997^{11}	0.097^{3}	$0.017^{1.5}$	0.487^{6}	0.50^{9}	$0.017^{1.5}$	0.987^{10}	0.453^{4}	0.487^{6}	0.487^{6}	0.491^{8}
	D_{max}	0.999^{11}	0.178^{3}	0.029^{1}	0.996^{6}	0.996^{9}	0.030^{2}	0.998^{10}	0.927^{4}	0.996^{7}	0.996^{8}	0.995^{5}
	$\sum$Ranks	39^{5}	29^{3}	14^{2}	37^{4}	60^{7}	12^{1}	84^{11}	63^{9}	49^{6}	75^{10}	61^{8}

Table 6 Simulation results for $\mu = 0, \sigma = 2$ and $\lambda = 1$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.169^{2}	-0.698^{4}	0.212^{3}	$-6.343\text{e}01^{5}$	$-3.288\text{e}02^{8}$	-0.077^{1}	$-2.272\text{e}03^{7}$	3251.097^{9}	$-2.134\text{e}02^{6}$	6978.946^{11}	$-5.218\text{e}03^{10}$
	RMSE($\hat{\mu}$)	0.763^{1}	0.855^{2}	0.995^{3}	$8.274\text{e}02^{5}$	$6.555\text{e}03^{7}$	1.061^{4}	$5.111\text{e}04^{8}$	101956.964^{9}	$2.847\text{e}03^{6}$	127585.485^{10}	$1.806\text{e}05^{11}$
	Bias($\hat{\sigma}$)	-0.074^{2}	-2.001^{4}	-0.210^{3}	$1.449\text{e}01^{5}$	$1.702\text{e}02^{7}$	-0.029^{1}	$2.322\text{e}03^{8}$	2236.760^{11}	$5.029\text{e}01^{6}$	657.427^{9}	$7.914\text{e}02^{10}$
	RMSE($\hat{\sigma}$)	0.447^{1}	2.001^{4}	0.619^{2}	$2.259\text{e}02^{5}$	$2.225\text{e}03^{7}$	0.632^{3}	$6.432\text{e}04^{11}$	9377.016^{8}	$9.581\text{e}02^{6}$	16427.930^{9}	$2.678\text{e}04^{10}$
	Bias($\hat{\lambda}$)	898142.188^{11}	-0.265^{1}	6.803^{2}	$1.506\text{e}02^{4}$	$6.817\text{e}02^{8}$	8.214^{3}	$6.068\text{e}02^{7}$	10368.602^{10}	$1.564\text{e}02^{5}$	8414.913^{9}	$3.836\text{e}02^{6}$
	RMSE($\hat{\lambda}$)	13008254.404^{11}	0.330^{1}	10.441^{2}	$4.079\text{e}03^{5}$	$8.575\text{e}03^{8}$	15.151^{3}	$4.992\text{e}03^{7}$	44599.747^{10}	$2.309\text{e}03^{4}$	39675.187^{9}	$4.836\text{e}03^{6}$
	D_{abs}	0.989^{10}	0.999^{11}	$0.0.061^{1.5}$	0.845^{4}	0.943^{6}	$0.061^{1.5}$	0.967^{8}	0.466^{3}	0.876^{5}	0.976^{9}	0.954^{7}
	$D_{\max}$	0.965^{10}	0.976^{11}	0.103^{2}	0.854^{4}	0.897^{6}	0.099^{1}	0.934^{8}	0.831^{3}	0.886^{5}	0.954^{9}	0.923^{7}
	$\sum$Ranks	48^{6}	38^{4}	18.5^{2}	37^{3}	57^{7}	17.5^{1}	64^{9}	63^{8}	43^{5}	75^{11}	67^{10}
50	Bias($\hat{\mu}$)	0.184^{2}	-0.704^{4}	0.233^{3}	$-1.750\text{e}02^{5}$	$-8.173\text{e}02^{8}$	0.015^{1}	$-6.154\text{e}03^{11}$	-615.156^{7}	$-2.091\text{e}02^{6}$	$4.760\text{e}03^{10}$	$-2.835\text{e}03^{9}$
	RMSE($\hat{\mu}$)	0.633^{1}	0.769^{3}	0.763^{2}	$1.642\text{e}03^{5}$	$7.350\text{e}03^{7}$	0.770^{4}	$2.709\text{e}04^{9}$	102376.706^{11}	$2.784\text{e}03^{6}$	$3.991\text{e}04^{10}$	$2.669\text{e}04^{8}$
	Bias($\hat{\sigma}$)	-0.014^{2}	-2.000^{4}	-0.059^{3}	$3.859\text{e}01^{5}$	$4.591\text{e}02^{8}$	0.013^{1}	$3.773\text{e}03^{11}$	2400.437^{10}	$6.212\text{e}01^{6}$	$3.496\text{e}02^{7}$	$4.891\text{e}02^{9}$
	RMSE($\hat{\sigma}$)	0.314^{1}	2.000^{4}	0.356^{2}	$5.909\text{e}02^{5}$	$4.217\text{e}03^{8}$	0.366^{3}	$2.066\text{e}04^{11}$	7896.269^{10}	$9.187\text{e}02^{6}$	$2.576\text{e}03^{7}$	$5.019\text{e}03^{9}$
	Bias($\hat{\lambda}$)	158076.460^{11}	-0.263^{1}	5.223^{2}	$4.083\text{e}02^{5}$	$1.041\text{e}03^{7}$	5.526^{3}	$2.069\text{e}03^{8}$	3902.231^{9}	$4.064\text{e}02^{4}$	$6.589\text{e}03^{10}$	$6.222\text{e}02^{6}$
	RMSE($\hat{\lambda}$)	3414577.059^{11}	0.271^{1}	8.701^{2}	$8.478\text{e}03^{6}$	$9.814\text{e}03^{8}$	11.292^{3}	$9.480\text{e}03^{7}$	23367.743^{9}	$6.681\text{e}03^{5}$	$3.426\text{e}04^{10}$	$4.559\text{e}03^{4}$
	D_{abs}	0.965^{10}	0.976^{11}	$0.038^{1.5}$	0.856^{4}	0.921^{6}	$0.038^{1.5}$	0.961^{9}	0.474^{3}	0.897^{5}	0.943^{7}	0.954^{8}
	$D_{\max}$	0.965^{10}	0.976^{11}	$0.063^{1.5}$	0.765^{4}	0.921^{6}	$0.063^{1.5}$	0.954^{9}	0.765^{3}	0.897^{5}	0.932^{7}	0.943^{8}
	$\sum$Ranks	48^{6}	$39^{3.5}$	17^{1}	$39^{3.5}$	58^{7}	18^{2}	75^{11}	62^{9}	43^{5}	68^{11}	61^{8}
100	Bias($\hat{\mu}$)	0.170^{2}	-0.709^{4}	0.211^{3}	$-1.979\text{e}02^{5}$	$-9.869\text{e}02^{7}$	0.040^{1}	$-1.442\text{e}04^{10}$	-3941.980^{8}	$-2.135\text{e}02^{6}$	5326.053^{9}	$-5.721\text{e}04^{11}$
	RMSE($\hat{\mu}$)	0.561^{1}	0.741^{4}	0.638^{3}	$1.238\text{e}03^{5}$	$9.706\text{e}03^{7}$	0.621^{2}	$4.039\text{e}04^{9}$	13284.646^{8}	$1.628\text{e}03^{6}$	78744.975^{10}	$2.732\text{e}06^{11}$
	Bias($\hat{\sigma}$)	0.005^{1}	-2.000^{4}	-0.011^{2}	$3.656\text{e}01^{5}$	$6.531\text{e}02^{8}$	0.016^{3}	$8.406\text{e}03^{10}$	2387.263^{9}	$3.885\text{e}01^{6}$	313.041^{7}	$9.978\text{e}03^{11}$
	RMSE($\hat{\sigma}$)	0.248^{1}	2.000^{4}	0.255^{2}	$3.842\text{e}02^{5}$	$5.474\text{e}03^{8}$	0.265^{3}	$2.589\text{e}04^{10}$	7182.353^{9}	$4.607\text{e}02^{6}$	2936.491^{7}	$4.716\text{e}05^{11}$
	Bias($\hat{\lambda}$)	5827.851^{11}	-0.266^{1}	3.878^{3}	$3.088\text{e}02^{4}$	$1.523\text{e}03^{7}$	3.668^{2}	$4.432\text{e}03^{9}$	1726.367^{8}	$5.920\text{e}02^{5}$	5666.633^{10}	$7.448\text{e}02^{6}$
	RMSE($\hat{\lambda}$)	356065.399^{11}	0.275^{1}	7.244^{2}	$5.507\text{e}03^{5}$	$1.309\text{e}04^{8}$	8.462^{3}	$1.230\text{e}04^{7}$	8024.442^{6}	$1.428\text{e}04^{9}$	30489.634^{10}	$5.326\text{e}03^{4}$

Table 6 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	D_{abs}	0.953^{10}	0.967^{11}	0.026^{1}	0.801^{5}	0.856^{7}	0.027^{2}	0.897^{9}	0.471^{3}	0.832^{6}	0.476^{4}	0.876^{8}
	D_{max}	0.976^{10}	0.987^{11}	$0.044^{1.5}$	0.854^{5}	0.908^{7}	$0.044^{1.5}$	0.943^{9}	0.765^{3}	0.876^{6}	0.786^{4}	0.923^{8}
	$\sum$Ranks	47^{5}	40^{4}	$17.5^{1.5}$	39^{3}	59^{8}	$17.5^{1.5}$	73^{11}	54^{7}	50^{6}	61^{9}	70^{10}
200	Bias($\hat{\mu}$)	0.151^{2}	-0.709^{4}	0.168^{3}	$-3.385e02^{6}$	$-2.336e03^{7}$	0.033^{1}	$-1.645e04^{10}$	-4177.841^{8}	$-3.842e02^{6}$	$2.336e03^{9}$	$-4.581e04^{11}$
	RMSE($\hat{\mu}$)	0.481^{1}	0.726^{4}	0.522^{3}	$1.948e03^{5}$	$1.297e04^{8}$	0.486^{2}	$4.198e04^{10}$	11522.587^{7}	$2.013e03^{6}$	$2.550e04^{9}$	$2.986e06^{11}$
	Bias($\hat{\sigma}$)	0.014^{3}	-2.000^{4}	0.006^{1}	$6.156e01^{5}$	$1.087e03^{8}$	0.009^{2}	$8.948e03^{11}$	2309.349^{9}	$6.415e01^{6}$	$2.439e02^{7}$	$8.352e03^{10}$
	RMSE($\hat{\sigma}$)	0.188^{1}	2.000^{4}	0.191^{2}	$5.013e02^{5}$	$6.544e03^{9}$	0.193^{3}	$2.308e04^{10}$	6417.002^{8}	$5.621e02^{6}$	$9.917e02^{7}$	$5.459e05^{11}$
	Bias($\hat{\lambda}$)	603.979^{6}	-0.267^{1}	2.526^{3}	$5.361e02^{5}$	$1.969e03^{9}$	1.982^{2}	$5.597e03^{11}$	1604.817^{8}	$5.093e02^{4}$	$3.815e03^{10}$	$6.842e02^{7}$
	RMSE($\hat{\lambda}$)	42300.744^{11}	0.275^{1}	5.638^{3}	$7.611e03^{7}$	$1.216e04^{8}$	5.589^{2}	$1.462e04^{9}$	5308.513^{5}	$7.184e03^{6}$	$2.359e04^{10}$	$3.664e03^{4}$
	D_{abs}	0.019^{2}	0.980^{11}	0.019^{2}	0.876^{6}	0.908^{7}	0.019^{2}	0.976^{10}	0.461^{4}	0.854^{5}	0.932^{8}	0.956^{9}
	D_{max}	0.031^{2}	0.976^{11}	0.031^{2}	$0.856^{5.5}$	0.897^{7}	0.031^{2}	0.965^{10}	0.947^{4}	$0.856^{5.5}$	0.932^{8}	0.945^{9}
	$\sum$Ranks	28^{3}	40^{4}	19^{2}	45.5^{6}	63^{8}	16^{1}	81^{11}	53^{7}	44.5^{5}	68^{9}	72^{10}
250	Bias($\hat{\mu}$)	0.138^{2}	-0.712^{4}	0.155^{3}	$-3.627e02^{5}$	$-2.894e03^{8}$	0.031^{1}	$-1.665e04^{11}$	-4138.569^{9}	$-3.662e02^{6}$	2773.486^{7}	$-4.343e03^{10}$
	RMSE($\hat{\mu}$)	0.450^{2}	0.724^{4}	0.479^{3}	$2.341e03^{5}$	$1.438e04^{8}$	0.439^{1}	$4.172e04^{9}$	12058.169^{7}	$2.637e03^{6}$	48663.190^{10}	$5.060e04^{11}$
	Bias($\hat{\sigma}$)	0.015^{3}	-2.000^{4}	$0.012^{1.5}$	$7.441e01^{6}$	$1.313e03^{9}$	$0.012^{1.5}$	$9.407e03^{11}$	2199.338^{10}	$6.418e01^{5}$	266.028^{7}	$8.495e02^{8}$
	RMSE($\hat{\sigma}$)	$0.174^{1.5}$	2.000^{4}	$0.174^{1.5}$	$6.518e02^{6}$	$6.999e03^{9}$	0.175^{3}	$2.423e04^{11}$	5867.343^{8}	$5.464e02^{5}$	1647.453^{7}	$1.028e04^{10}$
	Bias($\hat{\lambda}$)	3.300^{4}	-0.267^{1}	2.072^{3}	$4.529e02^{6}$	$2.462e03^{9}$	1.497^{2}	$5.667e03^{11}$	1704.704^{8}	$3.605e02^{5}$	3180.221^{10}	$1.078e03^{7}$
	RMSE($\hat{\lambda}$)	8.821^{4}	0.268^{1}	4.982^{3}	$5.815e03^{7}$	$1.423e04^{9}$	4.627^{2}	$1.477e04^{10}$	5002.107^{6}	$4.035e03^{5}$	19749.921^{11}	$5.943e03^{8}$
	D_{abs}	0.017^{2}	0.956^{11}	0.017^{2}	0.897^{7}	0.912^{8}	0.017^{2}	0.932^{10}	0.458^{4}	0.865^{6}	0.485^{5}	0.923^{9}
	D_{max}	$0.028^{2.5}$	0.967^{11}	0.027^{1}	0.913^{7}	0.923^{8}	$0.028^{2.5}$	0.954^{10}	0.842^{4}	0.897^{6}	0.893^{5}	0.934^{9}
	$\sum$Ranks	21^{3}	40^{4}	18^{2}	49^{6}	70^{9}	15^{1}	83^{11}	56^{7}	44^{5}	62^{8}	72^{10}

Table 7 Simulation results for $\mu = 0, \sigma = 0.5$ and $\lambda = 2$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.147^{3}	-0.406^{4}	0.108^{2}	-1.165^{6}	-3.656^{7}	0.016^{1}	$-2.102e03^{9}$	9729.993^{10}	-1.659^{5}	$1.869e04^{11}$	-21.307^{8}
	RMSE($\hat{\mu}$)	0.300^{1}	0.421^{4}	0.326^{3}	15.030^{5}	24.578^{7}	0.310^{2}	$2.587e04^{9}$	128842.126^{10}	15.757^{6}	$2.334e05^{11}$	419.863^{8}
	Bias($\hat{\sigma}$)	0.012^{3}	-0.320^{7}	-0.041^{5}	-0.031^{4}	0.275^{6}	$0.009^{1.5}$	$2.378e03^{10}$	2607.557^{11}	$0.009^{1.5}$	$5.684e02^{9}$	2.420^{8}
	RMSE($\hat{\sigma}$)	0.144^{1}	0.350^{4}	$0.199^{2.5}$	1.581^{5}	2.467^{7}	$0.199^{2.5}$	$3.279e04^{11}$	10449.553^{10}	1.661^{6}	$7.142e03^{9}$	53.060^{8}
	Bias($\hat{\lambda}$)	5999.909^{11}	-0.841^{1}	14.275^{6}	2.557^{3}	3.069^{4}	14.533^{7}	$7.153e02^{8}$	4979.779^{9}	2.278^{2}	$5.568e03^{10}$	10.155^{5}
	RMSE($\hat{\lambda}$)	737154.000^{11}	0.845^{1}	27.9276^{4}	23.666^{3}	46.663^{6}	36.246^{5}	$7.294e03^{8}$	38769.497^{9}	21.984^{2}	$4.776e04^{10}$	180.582^{7}
	D_{abs}	0.974^{10}	0.987^{11}	0.062^{2}	0.456^{4}	0.457^{6}	0.061^{1}	0.952^{9}	0.466^{7}	0.456^{4}	0.932^{8}	0.456^{4}
	$D_{\max}$	0.994^{10}	0.997^{11}	0.107^{2}	$0.941^{5.5}$	0.939^{4}	0.102^{1}	0.987^{9}	0.934^{3}	$0.941^{5.5}$	0.982^{8}	0.942^{7}
	$\sum$Ranks	50^{7}	43^{5}	29^{2}	$35.5^{3.5}$	47^{6}	21^{1}	74^{10}	69^{9}	$35^{3.5}$	76^{11}	55^{8}
50	Bias($\hat{\mu}$)	0.086^{3}	-0.414^{4}	0.064^{2}	-1.567^{5}	-7.057^{7}	0.006^{1}	$-4.184e03^{10}$	576.411^{9}	-2.462^{6}	$3.006e04^{11}$	-14.269^{8}
	RMSE($\hat{\mu}$)	0.216^{2}	0.420^{4}	0.227^{3}	20.394^{5}	39.454^{7}	0.204^{1}	$2.323e04^{9}$	60306.571^{10}	25.350^{6}	$2.762e05^{11}$	102.002^{8}
	Bias($\hat{\sigma}$)	0.014^{2}	-0.333^{6}	-0.005^{1}	0.067^{4}	0.536^{7}	0.016^{3}	$3.379e03^{11}$	1918.819^{10}	0.161^{5}	$8.916e02^{9}$	1.622^{8}
	RMSE($\hat{\sigma}$)	$0.117^{1.5}$	0.350^{4}	0.119^{3}	2.249^{5}	3.505^{7}	$0.117^{1.5}$	$2.299e04^{11}$	7725.475^{9}	2.782^{6}	$8.234e03^{10}$	12.327^{8}
	Bias($\hat{\lambda}$)	786.705^{8}	-0.864^{1}	5.697^{6}	2.033^{2}	4.137^{4}	4.503^{5}	$1.661e03^{10}$	1587.879^{9}	2.411^{3}	$7.642e03^{11}$	8.792^{7}
	RMSE($\hat{\lambda}$)	20750.270^{10}	0.866^{1}	16.467^{2}	22.000^{5}	107.662^{7}	17.255^{3}	$1.368e04^{9}$	9859.165^{8}	21.175^{4}	$4.597e04^{11}$	69.244^{6}
	D_{abs}	0.967^{10}	0.987^{11}	$0.039^{1.5}$	$0.478^{5.5}$	0.480^{7}	$0.039^{1.5}$	0.945^{9}	$0.476^{3.5}$	$0.478^{5.5}$	0.932^{8}	$0.476^{3.5}$
	$D_{\max}$	$0.986.^{10}$	0.997^{11}	0.066^{2}	$0.978^{5.5}$	$0.978^{5.5}$	0.065^{1}	0.985^{9}	0.966^{3}	$0.978^{5.5}$	0.979^{8}	$0.978^{5.5}$
	$\sum$Ranks	46.5^{6}	42^{5}	20.5^{2}	37^{3}	51.5^{7}	17^{1}	78^{10}	61.5^{9}	41^{4}	79^{11}	54^{8}
100	Bias($\hat{\mu}$)	0.044^{3}	-0.421^{5}	0.032^{2}	0.459^{6}	-10.667^{7}	-0.004^{1}	$-1.045e04^{10}$	4550.278^{9}	-0.071^{4}	$2.821e04^{11}$	-22.734^{8}
	RMSE($\hat{\mu}$)	0.154^{3}	0.424^{4}	0.153^{2}	16.704^{5}	62.440^{7}	0.137^{1}	$3.482e04^{9}$	76738.442^{10}	20.299^{6}	$2.425e05^{11}$	947.948^{8}
	Bias($\hat{\sigma}$)	0.008^{3}	-0.342^{6}	$2.39e-4^{1}$	-0.047^{5}	0.749^{7}	0.011^{4}	$6.269e03^{11}$	1373.924^{10}	-0.003^{2}	$8.172e02^{9}$	3.531^{8}
	RMSE($\hat{\sigma}$)	0.098^{3}	0.351^{4}	0.084^{2}	1.897^{5}	4.612^{7}	0.081^{1}	$2.412e04^{11}$	5282.065^{9}	2.274^{6}	$6.989e03^{10}$	168.351^{8}
	Bias($\hat{\lambda}$)	15.164^{8}	-0.876^{1}	1.641^{5}	0.398^{2}	5.833^{6}	1.053^{4}	$3.359e03^{10}$	1158.718^{9}	0.627^{3}	$5.961e03^{11}$	11.500^{7}

Table 7 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	RMSE($\hat{\lambda}$)	216.435^{7}	0.876^{1}	8.022^{3}	12.839^{4}	75.613^{6}	7.359^{2}	$1.137e04^{10}$	10283.750^{9}	14.243^{5}	$3.707e04^{11}$	420.724^{8}
	D_{abs}	0.980^{10}	0.989^{11}	$0.027^{1.5}$	0.490^{6}	0.490^{6}	$0.027^{1.5}$	0.978^{9}	0.479^{3}	0.490^{6}	0.967^{8}	0.488^{4}
	D_{max}	0.978^{10}	0.989^{11}	$0.046^{1.5}$	0.939^{5}	0.939^{5}	$0.046^{1.5}$	0.967^{9}	0.929^{3}	0.940^{7}	0.956^{8}	0.939^{5}
	$\sum$Ranks	47^{6}	43^{5}	18^{2}	38^{3}	51^{7}	16^{1}	$79^{10.5}$	62^{9}	39^{4}	$79^{10.5}$	56^{8}
200	Bias($\hat{\mu}$)	0.024^{3}	-0.424^{4}	0.014^{2}	1.873^{6}	-8.423^{7}	-0.005^{1}	$-7.488e03^{9}$	7856.221^{10}	1.872^{5}	26113.403^{11}	-17.820^{8}
	RMSE($\hat{\mu}$)	0.104^{3}	0.426^{4}	0.098^{2}	13.580^{6}	59.774^{7}	0.090^{1}	$2.795e04^{9}$	82967.770^{10}	13.331^{5}	240866.827^{11}	809.437^{8}
	Bias($\hat{\sigma}$)	$0.006^{2.5}$	-0.350^{6}	$4.28e{-}4^{1}$	-0.142^{4}	0.606^{7}	$0.006^{2.5}$	$4.755e03^{11}$	1032.266^{10}	-0.153^{5}	764.237^{9}	2.659^{8}
	RMSE($\hat{\sigma}$)	0.069^{3}	0.354^{4}	0.057^{2}	1.594^{6}	4.421^{7}	0.055^{1}	$1.936e04^{11}$	4467.330^{9}	1.540^{5}	7279.107^{10}	127.048^{8}
	Bias($\hat{\lambda}$)	0.922^{6}	-0.882^{5}	0.278^{2}	-0.752^{3}	6.452^{7}	0.087^{1}	$2.507e03^{10}$	1067.262^{9}	-0.770^{4}	5194.618^{11}	7.263^{8}
	RMSE($\hat{\lambda}$)	18.683^{6}	0.882^{1}	2.435^{3}	8.271^{5}	57.808^{7}	1.482^{2}	$9.268e03^{10}$	7549.542^{9}	8.163^{4}	32631.450^{11}	232.817^{8}
	D_{abs}	0.978^{10}	0.989^{11}	$0.019^{1.5}$	0.496^{7}	$0.495^{4.5}$	$0.019^{1.5}$	0.967^{9}	0.483^{3}	0.496^{7}	0.496^{7}	$0.495^{4.5}$
	D_{max}	0.998^{10}	0.999^{11}	$0.033^{1.5}$	$0.995^{5.5}$	$0.995^{5.5}$	$0.033^{1.5}$	0.987^{9}	0.974^{3}	$0.995^{5.5}$	0.996^{8}	$0.995^{5.5}$
	$\sum$Ranks	43.5^{5}	46^{6}	15^{2}	42.5^{4}	52^{7}	11.5^{1}	$78^{10.5}$	63^{9}	40.5^{3}	78^{11}	58^{8}
250	Bias($\hat{\mu}$)	0.017^{3}	-0.425^{4}	0.009^{2}	2.778^{6}	-5.606^{7}	-0.006^{1}	$-5.597e03^{9}$	7927.342^{10}	2.672^{5}	30534.316^{11}	-15.579^{8}
	RMSE($\hat{\mu}$)	0.089^{3}	0.426^{4}	0.081^{2}	9.500^{5}	54.552^{7}	0.076^{1}	$2.371e04^{9}$	75550.739^{10}	10.425^{6}	297402.908^{11}	1061.951^{8}
	Bias($\hat{\sigma}$)	0.004^{2}	-0.352^{7}	$7.96e{-}4^{1}$	-0.227^{5}	0.341^{6}	0.006^{2}	$3.787e03^{11}$	721.006^{9}	-0.217^{4}	857.695^{10}	2.003^{8}
	RMSE($\hat{\sigma}$)	0.065^{3}	0.355^{4}	0.049^{2}	1.115^{5}	3.914^{7}	0.048^{1}	$1.655e04^{11}$	3393.580^{9}	1.213^{6}	8271.222^{10}	128.758^{8}
	Bias($\hat{\lambda}$)	0.245^{3}	-0.883^{4}	0.123^{2}	-1.310^{6}	5.303^{8}	0.031^{1}	$1.949e03^{10}$	979.917^{9}	-1.256^{5}	5538.853^{11}	5.152^{7}
	RMSE($\hat{\lambda}$)	4.191^{4}	0.883^{2}	0.941^{3}	5.554^{5}	45.753^{7}	0.875^{1}	$8.180e03^{10}$	7213.153^{9}	6.25^{6}	35517.715^{11}	335.325^{8}
	D_{abs}	0.916^{10}	0.976^{11}	$0.017^{1.5}$	0.498^{7}	$0.497^{4.5}$	$0.017^{1.5}$	0.897^{9}	0.487^{3}	0.498^{7}	0.498^{7}	$0.497^{4.5}$
	D_{max}	0.999^{10}	1.000^{11}	$0.029^{1.5}$	$0.996^{5.5}$	$0.996^{5.5}$	$0.029^{1.5}$	0.998^{9}	0.979^{3}	$0.996^{5.5}$	0.997^{8}	$0.996^{5.5}$
	$\sum$Ranks	38^{3}	47^{6}	15^{2}	$44.5^{4.5}$	52^{7}	10^{1}	78^{10}	60^{9}	$44.5^{4.5}$	79^{11}	57^{8}

Table 8 Simulation results for $\mu = 2, \sigma = 2$ and $\lambda = 0.5$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.564^{4}	-0.189^{2}	0.432^{3}	$-5.169e01^{5}$	-79.413^{6}	0.064^{1}	$-1.067e03^{8}$	7205.153^{11}	-81.208^{7}	3448.634^{10}	$-1.431e03^{9}$
	RMSE($\hat{\mu}$)	1.142^{2}	0.617^{1}	1.300^{4}	$3.179e03^{7}$	300.338^{5}	1.238^{3}	$1.088e04^{8}$	59785.690^{10}	386.112^{6}	111111.546^{11}	$2.766e04^{9}$
	Bias($\hat{\sigma}$)	0.048^{2}	-2.000^{4}	-0.162^{3}	$2.489e01^{7}$	6.787^{6}	0.036^{1}	$7.662e02^{11}$	568.319^{10}	6.278^{5}	384.673^{9}	$3.588e02^{8}$
	RMSE($\hat{\sigma}$)	0.573^{1}	2.000^{4}	0.796^{3}	$9.722e02^{7}$	30.473^{5}	0.795^{2}	$6.828e03^{9}$	2872.115^{8}	41.447^{6}	8221.573^{10}	$9.236e03^{11}$
	Bias($\hat{\lambda}$)	53797.411^{11}	0.543^{1}	3.448^{2}	$8.061e01^{6}$	69.607^{4}	3.643^{3}	$9.318e02^{8}$	6265.634^{10}	71.830^{5}	1647.619^{9}	$4.599e02^{7}$
	RMSE($\hat{\lambda}$)	923022.260^{11}	0.587^{1}	6.794^{2}	$1.467e03^{6}$	327.557^{4}	9.054^{3}	$7.482e03^{8}$	31507.431^{10}	388.250^{5}	14622.154^{9}	$4.101e03^{7}$
	D_{abs}	0.954^{10}	0.987^{11}	$0.061^{1.5}$	0.842^{7}	0.438^{4}	$0.061^{1.5}$	0.934^{9}	0.457^{6}	0.443^{5}	0.436^{3}	0.867^{8}
	D_{max}	0.986^{10}	0.997^{11}	0.107^{2}	0.954^{7}	0.920^{5}	0.102^{1}	0.975^{9}	0.913^{4}	0.931^{6}	0.906^{3}	0.964^{8}
	$\sum$Ranks	51^{6}	35^{3}	20.5^{2}	52^{7}	39^{4}	15.5^{1}	70^{11}	69^{10}	45^{5}	64^{8}	67^{9}
50	Bias($\hat{\mu}$)	0.333^{4}	-0.186^{2}	0.256^{3}	$-6.443e01^{5}$	$-1.184e02^{7}$	0.024^{1}	$-1.397e03^{8}$	2457.340^{9}	$-9.180e01^{6}$	4140.081^{10}	$-1.236e05^{11}$
	RMSE($\hat{\mu}$)	0.824^{3}	0.421^{1}	0.906^{4}	$1.581e03^{6}$	$2.259e03^{7}$	0.81^{2}	$1.155e04^{8}$	30804.133^{9}	$3.729e02^{5}$	72773.955^{10}	$4.438e06^{11}$
	Bias($\hat{\sigma}$)	0.057^{2}	-2.000^{4}	-0.019^{1}	$1.145e01^{5}$	$2.887e01^{6}$	0.063^{3}	$6.657e02^{9}$	283.441^{7}	$7.633e00^{4}$	382.434^{8}	$2.895e04^{11}$
	RMSE($\hat{\sigma}$)	$0.466^{1.5}$	2.000^{4}	0.477^{3}	$3.158e02^{6}$	$9.304e02^{7}$	$0.466^{1.5}$	$5.449e03^{9}$	2831.951^{8}	$5.720e01^{5}$	6487.252^{10}	$1.131e06^{11}$
	Bias($\hat{\lambda}$)	4962.528^{11}	0.520^{1}	1.356^{3}	$1.689e02^{5}$	$2.015e02^{6}$	1.151^{2}	$1.031e03^{8}$	2448.498^{10}	$1.554e02^{4}$	1741.679^{9}	$6.485e02^{7}$
	RMSE($\hat{\lambda}$)	143780.884^{11}	0.538^{1}	3.949^{2}	$5.629e03^{7}$	$3.916e03^{4}$	4.424^{3}	$7.597e03^{8}$	19246.754^{10}	$5.394e03^{6}$	13108.221^{9}	$5.309e03^{5}$
	D_{abs}	0.986^{10}	0.997^{11}	$0.039^{1.5}$	0.932^{6}	0.959^{7}	$0.039^{1.5}$	0.974^{9}	0.471^{4}	0.897^{5}	0.453^{3}	0.964^{8}
	D_{max}	0.974^{10}	0.986^{11}	0.066^{2}	$0.932^{6.5}$	$0.932^{6.5}$	0.065^{1}	0.964^{9}	0.846^{4}	0.895^{5}	0.843^{3}	0.954^{8}
	$\sum$Ranks	52.5^{7}	35^{3}	19.5^{2}	46.5^{5}	50.5^{6}	15.0^{1}	68.0^{10}	61^{8}	40.0^{4}	62.0^{9}	72.0^{11}
100	Bias($\hat{\mu}$)	0.174^{3}	-0.192^{4}	0.126^{2}	-82.726^{5}	$-1.709e02^{7}$	-0.016^{1}	$-2.299e03^{9}$	857.264^{8}	-103.851^{6}	-76938.355^{10}	$-7.836e04^{11}$
	RMSE($\hat{\mu}$)	0.598^{3}	0.326^{1}	0.612^{4}	284.301^{5}	$7.990e02^{7}$	0.549^{2}	$1.706e04^{9}$	16815.949^{8}	333.373^{6}	5555307.800^{11}	$5.045e06^{10}$
	Bias($\hat{\sigma}$)	0.030^{2}	-2.000^{4}	$9.38e-4^{1}$	5.925^{5}	$1.931e01^{7}$	0.042^{3}	$9.821e02^{9}$	137.956^{8}	7.861^{6}	19426.883^{11}	$1.881e04^{10}$
	RMSE($\hat{\sigma}$)	0.392^{3}	2.000^{4}	0.336^{2}	23.605^{5}	$2.763e02^{7}$	0.324^{1}	$6.846e03^{9}$	1262.486^{8}	28.463^{6}	1347211.203^{11}	$1.223e06^{10}$
	Bias($\hat{\lambda}$)	1.170^{4}	0.511^{3}	0.396^{2}	71.687^{5}	$4.508e02^{7}$	0.275^{1}	$1.240e03^{11}$	877.894^{9}	88.267^{6}	1198.269^{10}	$7.748e02^{8}$

Table 8 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	RMSE($\hat{\lambda}$)	58.492^{4}	0.520^{1}	1.944^{3}	265.914^{5}	$9.138e03^{10}$	1.928^{2}	$7.764e03^{8}$	12379.045^{11}	299.021^{6}	7858.218^{9}	$5.048e03^{7}$
	D_{abs}	0.954^{10}	0.976^{11}	$0.027^{1.5}$	$0.478^{5.5}$	0.854^{7}	$0.027^{1.5}$	0.908^{9}	0.471^{4}	$0.478^{5.5}$	0.460^{3}	0.876^{8}
	D_{max}	0.994^{10}	0.998^{11}	$0.046^{1.5}$	$0.980^{5.5}$	0.982^{7}	$0.046^{1.5}$	0.991^{9}	0.948^{3}	$0.980^{5.5}$	0.949^{4}	0.987^{8}
	$\sum$Ranks	$59^{3.5}$	$39^{3.5}$	17^{2}	46^{6}	58^{7}	13^{1}	73^{11}	59^{8}	47^{5}	69^{9}	72^{10}
200	Bias($\hat{\mu}$)	0.096^{3}	-0.189^{4}	0.058^{2}	-93.575^{5}	$-4.538e02^{8}$	-0.019^{1}	$-2.252e03^{9}$	102.057^{6}	-103.813^{7}	-261301.802^{11}	$-6.536e04^{10}$
	RMSE($\hat{\mu}$)	0.410^{4}	0.267^{1}	0.392^{3}	305.248^{5}	$4.482e03^{7}$	0.360^{2}	$1.690e04^{9}$	6448.070^{8}	326.819^{6}	13229032.789^{11}	$4.060e06^{10}$
	Bias($\hat{\sigma}$)	0.024^{2}	-2.000^{4}	0.002^{1}	6.741^{5}	$1.049e02^{8}$	0.025^{3}	$7.431e02^{9}$	57.745^{7}	7.784^{6}	56096.353^{11}	$1.399e04^{10}$
	RMSE($\hat{\sigma}$)	0.275^{3}	2.000^{4}	0.228^{2}	24.978^{5}	$1.518e03^{8}$	0.220^{1}	$5.586e03^{9}$	735.702^{7}	27.201^{6}	2785159.391^{11}	$8.468e05^{10}$
	Bias($\hat{\lambda}$)	0.065^{2}	0.509^{4}	0.067^{3}	79.110^{5}	$6.256e02^{8}$	0.023^{1}	$1.476e03^{11}$	322.927^{7}	88.463^{6}	1299.839^{10}	$7.409e02^{9}$
	RMSE($\hat{\lambda}$)	0.430^{2}	0.513^{3}	0.577^{4}	261.016^{5}	$1.073e04^{11}$	0.398^{1}	$9.622e03^{10}$	5824.409^{8}	281.471^{6}	7347.811^{9}	$3.592e03^{7}$
	D_{abs}	0.965^{10}	0.976^{11}	$0.019^{1.5}$	$0.485^{5.5}$	0.921^{7}	$0.019^{1.5}$	0.943^{9}	0.444^{3}	$0.485^{5.5}$	0.463^{4}	0.932^{8}
	D_{max}	0.987^{10}	0.998^{11}	$0.033^{1.5}$	$0.955^{5.5}$	0.965^{7}	$0.033^{1.5}$	0.976^{9}	0.903^{3}	$0.955^{5.5}$	0.945^{4}	0.966^{8}
	$\sum$Ranks	36^{3}	42^{5}	18^{2}	41^{4}	64^{8}	12^{1}	75^{11}	49^{7}	48^{6}	$72^{9.5}$	$72^{9.5}$
250	Bias($\hat{\mu}$)	0.068^{3}	-0.192^{4}	0.039^{2}	-86.904^{6}	$-4.788e02^{8}$	-0.023^{1}	$-2.431e03^{10}$	-7.091^{5}	-94.786^{7}	-768.354^{9}	$-6.339e05^{11}$
	RMSE($\hat{\mu}$)	0.356^{4}	0.254^{1}	0.322^{3}	295.000^{5}	$4.624e03^{8}$	0.303^{2}	$1.630e04^{9}$	4209.149^{7}	309.240^{6}	71244.395^{10}	$2.823e07^{11}$
	Bias($\hat{\sigma}$)	0.015^{2}	-2.000^{4}	0.003^{1}	6.563^{5}	$1.071e02^{8}$	0.023^{3}	$7.383e02^{9}$	23.918^{7}	7.227^{6}	873.518^{10}	$1.484e05^{11}$
	RMSE($\hat{\sigma}$)	0.261^{3}	2.000^{4}	0.199^{2}	25.408^{5}	$1.563e03^{8}$	0.193^{1}	$5.002e03^{9}$	393.006^{7}	26.341^{6}	32243.159^{10}	$6.713e06^{11}$
	Bias($\hat{\lambda}$)	0.039^{3}	0.507^{4}	0.031^{2}	74.354^{5}	$4.805e02^{8}$	0.007^{1}	$1.744e03^{11}$	167.214^{7}	81.207^{6}	1185.708^{10}	$8.972e02^{9}$
	RMSE($\hat{\lambda}$)	0.171^{2}	0.510^{4}	0.247^{3}	254.489^{5}	$5.243e03^{8}$	0.159^{1}	$1.065e04^{11}$	2905.803^{7}	266.856^{6}	6607.028^{10}	$5.296e03^{9}$
	D_{abs}	0.976^{10}	0.987^{11}	$0.017^{1.5}$	$0.487^{5.5}$	0.943^{7}	$0.017^{1.5}$	0.967^{9}	0.433^{3}	$0.487^{5.5}$	0.464^{4}	0.956^{8}
	D_{max}	0.987^{10}	0.999^{11}	$0.029^{1.5}$	$0.966^{5.5}$	0.973^{7}	$0.029^{1.5}$	0.986^{9}	0.880^{3}	$0.966^{5.5}$	0.944^{4}	0.976^{8}
	$\sum$Ranks	37^{3}	43^{5}	16^{2}	42^{4}	62^{8}	12^{1}	77^{10}	46^{6}	48^{7}	67^{9}	78^{11}

Table 9 Simulation results for $\mu = 3$, $\sigma = 1$ and $\lambda = 6$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	-0.071^{1}	0.100^{2}	$1.108e41^{11}$	2.978^{5}	3.510^{6}	-0.201^{3}	$-2.154e03^{8}$	8343.066^{10}	2.789^{4}	7607.114^{9}	-49.390^{7}
	RMSE($\hat{\mu}$)	0.353^{2}	0.100^{1}	$3.819e42^{11}$	5.814^{4}	27.822^{6}	0.496^{3}	$2.895e04^{8}$	138691.671^{10}	8.307^{5}	38416.360^{9}	1899.817^{7}
	Bias($\hat{\sigma}$)	-0.127^{3}	0.010^{1}	$-2.846e41^{11}$	-0.879^{6}	-0.525^{4}	-0.036^{2}	$5.034e03^{9}$	5075.519^{10}	-0.835^{5}	212.743^{8}	5.751^{7}
	RMSE($\hat{\sigma}$)	0.245^{2}	0.010^{1}	$9.782e42^{11}$	0.968^{4}	3.032^{6}	0.292^{3}	$5.043e04^{10}$	25690.181^{9}	1.061^{5}	1067.054^{8}	231.098^{7}
	Bias($\hat{\lambda}$)	76122.025^{10}	0.010^{1}	$1.223e41^{11}$	-0.749^{4}	-0.760^{3}	16.982^{5}	$1.124e02^{7}$	6025.720^{8}	-0.691^{2}	12422.994^{9}	23.181^{6}
	RMSE($\hat{\lambda}$)	691153.591^{10}	0.010^{1}	$4.472e42^{11}$	5.411^{2}	21.057^{4}	33.932^{5}	$1.101e03^{7}$	44813.580^{8}	6.309^{3}	63476.175^{9}	902.371^{6}
	D_{abs}	0.998^{11}	0.028^{1}	0.997^{10}	0.484^{6}	0.484^{6}	0.060^{2}	0.965^{9}	0.477^{3}	0.484^{6}	0.487^{8}	0.480^{4}
	$D_{\max}$	0.995^{11}	0.039^{1}	0.963^{10}	0.940^{6}	0.940^{6}	0.098^{2}	0.943^{9}	0.937^{3}	0.940^{6}	0.942^{8}	0.939^{4}
	$\sum$Ranks	50^{7}	9^{1}	86^{11}	$37^{3.5}$	41^{5}	25^{2}	67^{9}	61^{8}	$37^{3.5}$	68^{10}	48^{6}
50	Bias($\hat{\mu}$)	-0.069^{2}	0.100^{3}	-0.065^{1}	4.069^{5}	8.422^{7}	-0.138^{4}	$-1.347e03^{9}$	20428.569^{11}	4.126^{6}	1918.945^{10}	$-6.407e01^{8}$
	RMSE($\hat{\mu}$)	0.286^{2}	0.100^{1}	0.306^{3}	5.381^{5}	25.894^{7}	0.358^{4}	$1.511e04^{9}$	223330.010^{11}	6.018^{6}	21736.264^{10}	$4.819e03^{8}$
	Bias($\hat{\sigma}$)	-0.077^{4}	0.010^{1}	-0.068^{3}	-0.923^{7}	-0.337^{5}	-0.023^{2}	$1.601e03^{11}$	1508.209^{10}	-0.912^{6}	54.087^{9}	$1.142e01^{8}$
	RMSE($\hat{\sigma}$)	0.146^{2}	0.010^{1}	0.161^{3}	0.965^{5}	2.695^{7}	0.165^{4}	$2.094e04^{11}$	11752.962^{10}	0.987^{6}	622.411^{8}	$8.435e02^{9}$
	Bias($\hat{\lambda}$)	46997.091^{11}	0.010^{1}	12.538^{6}	$-1.602^{2.5}$	-3.795^{4}	13.934^{7}	$2.231e02^{8}$	836.829^{9}	$-1.602^{2.5}$	2512.894^{10}	6.527^{5}
	RMSE($\hat{\lambda}$)	656566.162^{11}	0.010^{1}	20.182^{5}	2.826^{2}	19.748^{4}	28.449^{6}	$2.618e03^{8}$	9142.758^{9}	3.079^{3}	35958.381^{10}	$5.631e02^{7}$
	D_{abs}	0.998^{11}	0.028^{1}	0.037^{2}	$0.499^{6.5}$	$0.499^{6.5}$	0.038^{3}	0.976^{10}	0.496^{4}	$0.499^{6.5}$	$0.499^{6.5}$	0.967^{9}
	$D_{\max}$	0.997^{11}	0.039^{1}	$0.061^{2.5}$	$0.980^{6.5}$	$0.980^{6.5}$	$0.061^{2.5}$	0.995^{10}	0.977^{4}	$0.980^{6.5}$	$0.980^{6.5}$	0.993^{9}
	$\sum$Ranks	54^{7}	10^{1}	25.5^{2}	39.5^{4}	47^{6}	32.5^{3}	76^{11}	68^{9}	42.5^{5}	70^{10}	63^{8}
100	Bias($\hat{\mu}$)	-0.059^{2}	0.100^{3}	-0.045^{1}	4.140^{5}	9.494^{8}	-0.105^{4}	$-1.654e02^{9}$	53402.787^{11}	4.177^{6}	$1.855e03^{10}$	6.303^{7}
	RMSE($\hat{\mu}$)	0.241^{2}	0.100^{1}	0.245^{3}	4.774^{5}	24.885^{8}	0.285^{4}	$5.348e03^{9}$	268320.951^{11}	4.834^{6}	$5.271e04^{10}$	11.472^{7}
	Bias($\hat{\sigma}$)	-0.054^{4}	0.010^{1}	-0.043^{3}	-0.943^{8}	-0.251^{5}	-0.020^{2}	$1.508e02^{10}$	1482.063^{11}	-0.942^{7}	$5.106e01^{9}$	-0.793^{6}
	RMSE($\hat{\sigma}$)	0.104^{2}	0.010^{1}	0.109^{3}	0.962^{6}	2.579^{8}	0.115^{4}	$6.654e03^{10}$	7402.689^{11}	0.961^{5}	$1.472e03^{9}$	1.371^{7}

Table 9 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	Bias($\hat{\lambda}$)	21762.592^{11}	0.010^{1}	11.618^{7}	-1.654^{2}	-4.443^{5}	10.975^{6}	$3.244e01^{8}$	1180.981^{10}	-1.663^{3}	$3.433e02^{9}$	-2.738^{4}
	RMSE($\hat{\lambda}$)	501041.659^{11}	0.010^{1}	19.043^{5}	2.260^{2}	19.498^{6}	23.275^{7}	$1.116e03^{8}$	7058.805^{9}	2.261^{3}	$7.479e03^{10}$	7.441^{4}
	D_{abs}	0.026^{2}	0.028^{4}	0.026^{2}	$0.500^{7.5}$	$0.500^{7.5}$	0.026^{2}	0.965^{11}	0.499^{5}	$0.500^{7.5}$	0.943^{10}	$0.500^{7.5}$
	D_{max}	$0.043^{3.5}$	0.039^{1}	0.042^{2}	$0.990^{7.5}$	$0.990^{7.5}$	$0.043^{3.5}$	0.999^{11}	0.989^{5}	$0.990^{7.5}$	0.996^{10}	$0.990^{7.5}$
	$\sum$Ranks	37.5^{4}	13^{1}	26^{2}	43^{5}	55^{8}	32.5^{3}	76^{10}	73^{9}	45^{6}	77^{11}	50^{7}
200	Bias($\hat{\mu}$)	-0.044^{2}	0.100^{4}	-0.030^{1}	4.109^{5}	9.939^{9}	-0.079^{3}	9.781^{8}	62453.947^{11}	4.121^{6}	$3.246e03^{10}$	6.015^{7}
	RMSE($\hat{\mu}$)	0.206^{3}	0.100^{1}	0.205^{2}	4.149^{5}	9.971^{8}	0.231^{4}	10.288^{9}	231817.468^{11}	4.170^{6}	$1.563e05^{10}$	6.568^{7}
	Bias($\hat{\sigma}$)	-0.039^{4}	0.010^{1}	-0.029^{3}	$-0.952^{8.5}$	-0.300^{5}	-0.019^{2}	0.799^{6}	1674.879^{11}	$-0.952^{8.5}$	$8.706e01^{10}$	-0.863^{7}
	RMSE($\hat{\sigma}$)	0.078^{2}	0.010^{1}	0.079^{3}	0.953^{8}	0.332^{5}	0.084^{4}	4.156^{9}	6261.079^{11}	0.952^{7}	$4.242e03^{10}$	0.903^{6}
	Bias($\hat{\lambda}$)	9187.616^{11}	0.010^{1}	10.785^{8}	-1.716^{2}	-4.912^{6}	8.572^{7}	-3.796^{5}	2478.323^{10}	-1.722^{3}	$1.854e01^{9}$	-2.874^{4}
	RMSE($\hat{\lambda}$)	243981.596^{11}	0.010^{1}	18.061^{7}	1.788^{2}	4.918^{6}	19.197^{8}	4.149^{5}	23400.272^{10}	1.795^{3}	$1.166e03^{9}$	3.062^{4}
	D_{abs}	$0.018^{1.5}$	0.028^{4}	$0.018^{1.5}$	$0.500^{7.5}$	$0.500^{7.5}$	0.019^{3}	$0.500^{7.5}$	$0.500^{7.5}$	$0.500^{7.5}$	0.923^{11}	$0.500^{7.5}$
	D_{max}	0.030^{2}	0.039^{4}	0.030^{2}	0.995^{8}	0.995^{8}	0.030^{2}	0.995^{8}	0.994^{5}	0.995^{8}	0.997^{11}	0.995^{8}
	$\sum$Ranks	36.5^{4}	17^{1}	27.5^{2}	46^{5}	54.5^{8}	33^{3}	57.5^{9}	71.5^{10}	49^{6}	80^{11}	50.5^{7}
250	Bias($\hat{\mu}$)	-0.038^{2}	0.100^{4}	-0.026^{1}	4.109^{5}	9.866^{8}	-0.073^{3}	10.249^{9}	66118.591^{11}	4.111^{6}	104.006^{10}	5.799^{7}
	RMSE($\hat{\mu}$)	0.194^{3}	0.100^{1}	0.191^{2}	4.141^{5}	9.891^{8}	0.215^{4}	10.654^{9}	223194.160^{11}	4.144^{6}	6583.304^{10}	6.285^{7}
	Bias($\hat{\sigma}$)	-0.034^{4}	0.010^{1}	-0.025^{3}	$-0.952^{8.5}$	-0.288^{5}	-0.017^{2}	0.592^{6}	1775.849^{11}	$-0.952^{8.5}$	1.951^{10}	-0.886^{7}
	RMSE($\hat{\sigma}$)	0.070^{2}	0.010^{1}	0.071^{3}	0.952^{8}	0.315^{5}	0.076^{4}	3.815^{9}	6053.939^{11}	0.952^{7}	183.626^{10}	0.920^{6}
	Bias($\hat{\lambda}$)	14177.272^{11}	0.010^{1}	10.509^{8}	-1.759^{2}	-4.930^{6}	7.480^{7}	-3.748^{5}	3090.765^{10}	-1.763^{3}	14.222^{9}	-2.830^{4}
	RMSE($\hat{\lambda}$)	544060.291^{11}	0.010^{1}	17.721^{8}	1.819^{2}	4.935^{6}	17.300^{7}	4.268^{5}	21167.612^{10}	1.822^{3}	1426.254^{9}	2.951^{4}
	D_{abs}	0.016^{2}	0.028^{4}	0.016^{2}	0.500^{8}	0.500^{8}	0.016^{2}	0.500^{8}	0.500^{8}	0.500^{8}	0.500^{8}	0.500^{8}
	D_{max}	$0.027^{2.5}$	0.039^{4}	0.026^{1}	0.996^{8}	0.996^{8}	$0.027^{2.5}$	0.996^{8}	0.996^{8}	0.996^{8}	0.996^{8}	0.996^{8}
	$\sum$Ranks	37.5^{4}	17^{1}	28^{2}	46.5^{5}	54^{8}	31.5^{3}	59^{9}	80^{11}	49.5^{6}	74^{10}	51^{7}

Table 10 Simulation results for $\mu = -1, \sigma = 0.8$ and $\lambda = 0.5$

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
20	Bias($\hat{\mu}$)	0.445^{4}	0.377^{3}	0.181^{2}	$-1.199e04^{11}$	$-5.223e01^{6}$	0.027^{1}	$-1.863e03^{8}$	14981.464^{10}	$-1.839e01^{5}$	663.441^{9}	$-2.378e02^{7}$
	RMSE($\hat{\mu}$)	0.755^{4}	0.489^{1}	0.679^{3}	$4.074e05^{10}$	$1.234e03^{6}$	0.607^{2}	$1.508e04^{8}$	86363.354^{9}	$2.960e02^{5}$	764313.533^{11}	$2.906e03^{7}$
	Bias($\hat{\sigma}$)	0.261^{3}	-0.796^{4}	-0.113^{2}	$1.985e04^{11}$	$5.851e01^{7}$	0.041^{1}	$1.646e03^{9}$	960.758^{8}	$1.635e01^{5}$	2088.085^{10}	$3.289e01^{6}$
	RMSE($\hat{\sigma}$)	0.610^{3}	0.798^{4}	0.505^{2}	$6.711e05^{11}$	$1.049e03^{7}$	0.484^{1}	$1.403e04^{9}$	4974.115^{8}	$5.628e02^{5}$	123501.662^{10}	$5.655e02^{6}$
	Bias($\hat{\lambda}$)	7112.171^{10}	0.331^{1}	1.112^{3}	$8.291e01^{5}$	$2.847e02^{7}$	0.920^{2}	$1.329e03^{8}$	14738.215^{11}	$6.037e01^{4}$	2240.609^{9}	$9.473e01^{6}$
	RMSE($\hat{\lambda}$)	381854.327^{11}	0.340^{1}	4.703^{2}	$2.269e03^{6}$	$4.897e03^{7}$	5.387^{3}	$8.566e03^{8}$	58581.314^{10}	$1.789e03^{5}$	19935.298^{9}	$1.120e03^{4}$
	D_{abs}	0.995^{10}	0.997^{11}	0.061^{2}	0.897^{6}	0.967^{7}	0.060^{1}	0.990^{9}	0.461^{4}	0.853^{5}	0.449^{3}	0.987^{8}
	D_{max}	0.970^{10}	0.976^{11}	0.111^{2}	0.955^{6}	0.957^{7}	0.103^{1}	0.968^{9}	0.922^{3}	0.942^{5}	0.934^{4}	0.961^{8}
	$\sum$Ranks	55^{7}	36^{3}	18^{2}	66^{10}	54^{6}	12^{1}	68^{11}	63^{8}	39^{4}	65^{9}	52^{5}
50	Bias($\hat{\mu}$)	0.237^{3}	0.381^{4}	0.049^{2}	$-1.344e01^{6}$	$-1.140e02^{7}$	-0.014^{1}	$-1.575e03^{9}$	7827.213^{11}	$-1.334e01^{5}$	$3.933e03^{10}$	$-1.202e03^{8}$
	RMSE($\hat{\mu}$)	0.476^{4}	0.430^{3}	0.372^{2}	$2.179e02^{6}$	$2.221e03^{7}$	0.338^{1}	$1.150e04^{8}$	59588.963^{10}	$1.662e02^{5}$	$1.086e05^{11}$	$3.028e04^{9}$
	Bias($\hat{\sigma}$)	0.169^{3}	-0.800^{4}	-0.041^{1}	8.464^{6}	$5.381e01^{7}$	0.043^{2}	$8.648e02^{11}$	534.713^{10}	4.788^{5}	$3.974e02^{9}$	$1.841e02^{8}$
	RMSE($\hat{\sigma}$)	0.477^{3}	0.800^{4}	0.287^{2}	$4.188e02^{6}$	$9.431e02^{7}$	0.274^{1}	$6.337e03^{10}$	2562.745^{8}	$3.049e02^{5}$	$1.141e04^{11}$	$5.116e03^{9}$
	Bias($\hat{\lambda}$)	0.145^{3}	0.325^{4}	0.087^{2}	$3.683e01^{6}$	$2.970e02^{8}$	0.028^{1}	$1.349e03^{10}$	7366.662^{11}	$2.885e01^{5}$	$1.142e03^{9}$	1.686e027
	RMSE($\hat{\lambda}$)	1.234^{4}	0.328^{1}	0.837^{3}	$1.338e03^{6}$	$5.281e03^{8}$	0.646^{2}	$9.770e03^{9}$	40784.365^{11}	$1.231e03^{5}$	$1.075e04^{10}$	$2.332e03^{7}$
	D_{abs}	0.923^{10}	0.990^{11}	$0.039^{1.5}$	0.821^{6}	0.843^{7}	$0.039^{1.5}$	0.907^{9}	0.474^{3}	0.796^{5}	0.743^{4}	0.876^{8}
	D_{max}	0.876^{10}	0.998^{11}	0.069^{2}	0.784^{5}	0.796^{7}	0.067^{1}	0.856^{9}	0.965^{3}	0.784^{5}	0.784^{5}	0.832^{8}
	$\sum$Ranks	$40^{3.5}$	42^{5}	16.5^{2}	47^{6}	58^{7}	10.5^{1}	75^{11}	67^{9}	$40^{3.5}$	69^{10}	64^{8}
100	Bias($\hat{\mu}$)	0.133^{3}	0.376^{4}	0.019^{2}	$-1.266e01^{5}$	$-5.558e01^{7}$	-0.012^{1}	$-2.524e03^{10}$	2209.643^{9}	$-1.392e01^{6}$	-158029.082^{11}	$-6.123e02^{8}$
	RMSE($\hat{\mu}$)	0.351^{3}	0.402^{4}	0.235^{2}	$1.386e02^{5}$	$3.368e02^{7}$	0.225^{1}	$1.533e04^{9}$	27145.010^{10}	$1.412e02^{6}$	11226157.745^{11}	$1.142e04^{8}$
	Bias($\hat{\sigma}$)	0.081^{4}	-0.800^{3}	-0.017^{1}	4.401^{5}	$1.979e01^{7}$	0.035^{2}	$1.238e03^{10}$	271.770^{9}	4.615^{6}	32328.931^{11}	$1.119e02^{8}$
	RMSE($\hat{\sigma}$)	0.427^{3}	0.800^{4}	0.185^{2}	$2.928e02^{5.5}$	$5.706e02^{7}$	0.183^{1}	$7.492e03^{10}$	1631.365^{8}	$2.928e02^{5.5}$	2215197.167^{11}	$3.782e03^{9}$
	Bias($\hat{\lambda}$)	0.058^{3}	0.323^{4}	0.019^{2}	$2.585e01^{5}$	$1.014e02^{7}$	0.000^{1}	$1.915e03^{10}$	2812.946^{11}	$2.779e01^{6}$	920.885^{9}	$1.942e02^{8}$

Table 10 continued

n	Est.	MME	ALME	MLE	LSE	WLS	MPS	MSADE	MSALDE	CVM	AD	RAD
	RMSE($\hat{\lambda}$)	0.205^{3}	0.325^{4}	0.122^{2}	$1.095e03^{5}$	$2.373e03^{7}$	0.088^{1}	$1.192e04^{10}$	21766.985^{11}	$1.096e03^{6}$	8239.861^{9}	$2.645e03^{8}$
	D_{abs}	0.843^{10}	0.897^{11}	$0.027^{1.5}$	$0.663^{5.5}$	0.743^{7}	$0.027^{1.5}$	0.796^{9}	0.475^{3}	$0.663^{5.5}$	0.477^{4}	0.765^{8}
	D_{max}	0.997^{10}	0.998^{11}	0.048^{2}	$0.985^{5.5}$	0.987^{7}	0.047^{1}	0.995^{9}	0.967^{3}	$0.985^{5.5}$	0.976^{4}	0.993^{8}
	$\sum$Ranks	39^{3}	45^{5}	14.5^{2}	41.5^{4}	56^{7}	9.5^{1}	77^{11}	64^{9}	46.5^{6}	70^{10}	57^{8}
200	Bias($\hat{\mu}$)	0.082^{3}	0.380^{4}	0.011^{2}	-10.980^{5}	$-6.073e04^{10}$	-0.003^{1}	$-2.115e03^{9}$	221.200^{7}	-12.064^{6}	-345059.428^{11}	$-8.926e02^{8}$
	RMSE($\hat{\mu}$)	0.264^{3}	0.393^{4}	0.162^{2}	29.889^{5}	$3.065e06^{10}$	0.158^{1}	$1.426e04^{9}$	12214.004^{8}	31.005^{6}	9124679.817^{11}	$9.574e03^{7}$
	Bias($\hat{\sigma}$)	0.037^{3}	-0.800^{6}	-0.007^{1}	0.190^{4}	$2.503e04^{10}$	0.025^{2}	$9.682e02^{9}$	172.402^{8}	0.308^{5}	66351.116^{11}	$1.118e02^{7}$
	RMSE($\hat{\sigma}$)	0.374^{3}	0.800^{4}	0.122^{1}	3.350^{5}	$1.287e06^{10}$	0.123^{2}	$6.647e03^{9}$	1143.282^{7}	3.452^{6}	1950260.450^{11}	$1.264e03^{8}$
	Bias($\hat{\lambda}$)	0.031^{3}	0.324^{4}	0.008^{2}	10.799^{5}	$6.042e01^{7}$	-0.001^{1}	$1.824e03^{11}$	1106.675^{10}	11.809^{6}	770.527^{9}	$3.079e02^{8}$
	RMSE($\hat{\lambda}$)	0.088^{3}	0.324^{4}	0.055^{2}	33.864^{5}	$5.895e02^{7}$	0.052^{1}	$1.246e04^{11}$	12130.283^{10}	35.048^{6}	5308.834^{9}	$4.024e03^{8}$
	D_{abs}	0.932^{10}	0.965^{11}	$0.019^{1.5}$	$0.489^{5.5}$	0.812^{7}	$0.019^{1.5}$	0.901^{9}	0.466^{3}	$0.489^{5.5}$	0.485^{4}	0.897^{8}
	D_{max}	0.993^{10}	0.996^{11}	$0.033^{1.5}$	$0.975^{5.5}$	0.983^{7}	$0.033^{1.5}$	0.987^{9}	0.949^{3}	$0.975^{5.5}$	0.973^{4}	0.986^{8}
	$\sum$Ranks	38^{3}	48^{6}	13^{2}	40^{4}	68^{9}	11^{1}	76^{11}	56^{7}	46^{5}	70^{10}	62^{8}
250	Bias($\hat{\mu}$)	0.060^{3}	0.378^{4}	0.005^{1}	-11.302^{5}	-5553.515^{9}	-0.006^{2}	$-2.309e03^{8}$	-86.771^{7}	-12.164^{6}	$-3.179e05^{11}$	$-6.741e04^{10}$
	RMSE($\hat{\mu}$)	0.236^{3}	0.388^{4}	0.138^{2}	30.659^{5}	359994.502^{9}	0.136^{1}	$1.480e04^{8}$	7350.686^{7}	34.151^{6}	$1.169e07^{11}$	$3.249e06^{10}$
	Bias($\hat{\sigma}$)	0.014^{2}	-0.800^{6}	-0.006^{1}	0.205^{4}	6259.007^{9}	0.021^{3}	$1.084e03^{8}$	130.342^{7}	0.315^{5}	$5.249e04^{11}$	$1.428e04^{10}$
	RMSE($\hat{\sigma}$)	0.365^{3}	0.800^{4}	0.107^{1}	3.429^{5}	436828.585^{9}	0.108^{2}	$6.790e03^{8}$	978.690^{7}	3.849^{6}	$1.938e06^{11}$	$7.398e05^{10}$
	Bias($\hat{\lambda}$)	0.023^{3}	0.323^{4}	0.006^{2}	11.128^{5}	55.676^{7}	-0.002^{1}	$2.108e03^{11}$	903.777^{10}	11.927^{6}	$8.920e02^{9}$	$2.857e02^{8}$
	RMSE($\hat{\lambda}$)	0.075^{3}	0.324^{4}	0.046^{2}	34.0155	148.538^{7}	0.045^{1}	$1.297e04^{11}$	9089.949^{10}	37.739^{6}	$7.126e03^{9}$	$3.513e03^{8}$
	D_{abs}	0.854^{10}	0.897^{11}	$0.017^{1.5}$	$0.490^{4.5}$	0.491^{6}	$0.017^{1.5}$	0.832^{9}	0.456^{3}	$0.490^{4.5}$	0.796^{7}	0.803^{8}
	D_{max}	0.976^{10}	0.998^{11}	$0.029^{1.5}$	0.956^{5}	0.956^{5}	$0.029^{1.5}$	0.972^{9}	0.928^{3}	0.956^{5}	0.965^{7}	0.967^{8}
	$\sum$Ranks	37^{3}	48^{6}	12^{2}	38.5^{4}	61^{8}	12^{1}	$72^{9.5}$	54^{7}	44.5^{5}	76^{11}	$72^{9.5}$

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