REGULAR LOCAL RINGS AND ALGEBRAIC GEOMETRY

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1. Introduction

Commutative algebra and algebraic geometry are two very fascinating subjects. But the reader who treats them separately will no doubt be at a disadvantage. One should realize that they constitute ideas that are ultimately connected. That one uses the preciseness of language from the first to prove theorems in the second and that one has to keep the geometry from the second in mind to motivate the proofs of the first.

The purpose of this paper is to show how regular local rings relate to local (and global) properties of varieties. The material presented below can be found in Serre's elegant book on local algebra [**Serre**](as well as the standard [**A&M**]) and in various books on algebraic geometry (we have chosen [**Bump**] as a simpler version of [**Mum**] as the main reference). The author has tried to include as many proofs as possible, but in numerous instances only references to the proofs have been given. After we present the necessary material from commutative algebra we move onto seeing what it tells us about varieties. Finally we have included some examples to try and give the reader a more visual picture as to what is going on.

2. General Notions from Commutative Algebra

Let A be a noetherian commutative ring with unity.

An ideal $\mathfrak{p}$ of A is *prime* if and only if $A/\mathfrak{p}$ is a domain. We denote the set of proper prime ideals of A by $\mathtt{Spec}(A)$.

An ideal $\mathfrak{m}$ of A is *maximal* if and only if $A/\mathfrak{m}$ is a field. We denote the set of maximal ideals of A by $\mathtt{Max}(A)$

A ring A is *local* if and only if it contains a unique maximal ideal $\mathfrak{m}$.

Let $\mathtt{Jac}(A)$ be the intersection of all maximal ideals in A.

$$\mathtt{Jac}(A) = \bigcap_{\mathfrak{m}\in\mathtt{Max}(A)} \mathfrak{m}.$$

One can show that $x \in \mathtt{Jac}(A)$ if and only if $1 - xy \in A^*$ for all $y \in A$.

A fundamental result that we will use later is:

Theorem A1.1: (**Nakayama's Lemma 1**) Let M be a finitely generated A-module and $\mathfrak{q}$ be an ideal of A contained in $\mathtt{Jac}(A)$. If $\mathfrak{q}M = M$ then $M = 0$.

Theorem A1.2: (**Nakayama's Lemma 2**) If N is a submodule of an A-module M such that

$$M = N + \mathfrak{q}M$$

with $\mathfrak{q}$ an ideal of A contained in $\mathtt{Jac}(A)$ then $N = M$.

One way of getting local rings out of general rings is the process of localization which we now describe.

Let S be a subset of A that contains 1 and is closed under multiplication. Let M be any A-module.

Define

$$S^{-1}M = \left\{ \frac{m}{s} : m \in M, s \in S \right\},$$

where

$$\frac{m}{s} = \frac{m'}{s'} \iff \text{there exists } s'' \in S \text{ such that } s''(s'm - sm') = 0.$$

As it stands this is just a set of equivalence classes , but one can make it into an additive group by defining

$$\frac{m}{s} + \frac{n}{t} = \frac{tm + sn}{st}.$$

It is not difficult to check that this operation is well-defined. We have natural maps

$$A \to S^{-1}A \qquad a \mapsto \frac{a}{1},$$

$$M \to S^{-1}M \qquad m \mapsto \frac{m}{1}.$$

Now $S^{-1}A$ is a ring if we define

$$\frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

Also $S^{-1}M$ is a $S^{-1}A$ module (and an A-module). In fact we have canonically

$$S^{-1}A \otimes_A M = S^{-1}M.$$

One can also show that the functor $M \mapsto S^{-1}M$ is exact (See [**A&M**]).

Focusing on the ring $S^{-1}A$ one sees that the prime ideals of this ring are of the form $S^{-1}\mathfrak{p}$ for $\mathfrak{p} \subset A$ such that $\mathfrak{p} \cap S = \emptyset$. Actually we have a bijection between prime ideals of $S^{-1}A$ and prime ideals of A that do not meet S. If we let $\psi : A \to S^{-1}A$ be the natural map above we get

$$\psi^{-1}(S^{-1}\mathfrak{p}) = \mathfrak{p}.$$

We will be interested in a special case of this construction. Namely, for any $\mathfrak{p}$ prime ideal of A let $S = A - \mathfrak{p}$. By the definition of a prime ideal this is a multiplicative set. Then we write $A_{\mathfrak{p}} = S^{-1}A$ and $M_{\mathfrak{p}} = S^{-1}M$.

Note that by the correspondence between prime ideals of $A_{\mathfrak{p}}$ and A we immediately see that $A_{\mathfrak{p}}$ is a local ring. We will denote its maximal ideal by $\mathfrak{m}_{\mathfrak{p}}$, or just $\mathfrak{m}$ if there is no confusion. We will usually denote the residue field for $k_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{m}_{\mathfrak{p}}$, or just k.

A *filtered ring* is a ring A together with a family $(A_n)_{n\in\mathbb{Z}}$ of ideals such that

$$A_0 = A, \qquad A_{n+1} \subset A_n, \qquad A_pA_q \subset A_{p+q}.$$

A *filtered module* over a filtered ring A is an A-module M and a family $(M_n)_{n\in\mathbb{Z}}$ of submodules satisfying

$$M_0 = M, \qquad M_{n+1} \subset M_n, \qquad A_pM_q \subset M_{p+q}.$$

We say that A is *Hausdorff* if and only if $\cap_n A_n = 0$.

If P is an A-submodule of a filtered module M then we make P into a filtered A-module via the grading $P_n = P \cap M_n$. In the same spirit we have a quotient filtration on $N = M/P$, namely $N_n = (M_n + P)/P$. Morphisms between filtered A-module M and N are A-linear maps $u : M \to N$ such that $u(M_n) \subset N_n$. If we have an exact sequence of filtered A-modules

$$0 \to P \to M \to N \to 0,$$

then we get for all n an induced exact sequence on each term of the filtration

$$0 \to P_n \to M_n \to N_n \to 0.$$

In particular, let A be a local ring with maximal ideal $\mathfrak{m}$. The $\mathfrak{m}$-*adic filtration* of A is the filtration for which $A_n = \mathfrak{m}^n$ for $n \geq 0$.

A *graded ring* is a ring A with a direct sum decomposition

$$A = \bigoplus_{n\in\mathbb{Z}} A_n,$$

where the A_n are additive subgroups of A such that $A_n = \{0\}$ if $n < 0$ and $A_pA_q \subset A_{p+q}$.

Let A be a filtered ring. We define the *associated graded ring to A* to be the graded ring

$$\mathtt{gr}(A) = \bigoplus_n A_n/A_{n+1}.$$

Theorem A2: If A is Hausdorff and if $\mathtt{gr}(A)$ is an integrally closed domain, then A is an integrally closed domain.

Proof: See Serre's book [**Serre**].

As usual let A be a local ring with maximal ideal $\mathfrak{m}$ then we have the associated graded ring

$$\mathtt{gr}_{\mathfrak{m}}(A) = \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1} = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots .$$

Assume that A is generated by A_0 and by a finite set of elements $x_1, \ldots, x_r$ of A_1. Define a function

$$n \mapsto \chi(A, n) = l(A_n) = l(\mathfrak{m}^n/\mathfrak{m}^{n+1}),$$

where $l(A_n)$ is the length of the A-module A_n.

Theorem A3: (Hilbert) There exists a polynomial $H_A(X)$ of degree $\leq r-1$ such that $\chi(A, n) = H_A(n)$ for all large enough n.

Proof: See [**Serre**].

It can be shown ([**Serre**]) that the Hilbert polynomial $H_A(X)$ can be expressed as a $\mathbb{Z}$-linear combination of the *binomial polynomials* $Q_0(X) = 1$, $Q_1(X) = X$, $Q_k(X) = \binom{X}{k}$. That is

$$H_A = \sum e_k Q_k, \qquad\qquad e_k \in \mathbb{Z}.$$

If one defines the *difference operator* Δ to be

$$\Delta f(n) = f(n+1) - f(n),$$

we see that $\Delta Q_k = Q_{k-1}$.

Returning to the Hilbert polynomial H_A we get that because it has degree $\leq r-1$ that

$$\Delta^{r-1} H_A = e_{r-1}.$$

We will use this in our proofs in the next section.

3. Dimension Theory of Rings

Let A be a ring. A finite increasing sequence

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$$

of prime ideal of A such that $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $0 \leq i \leq r-1$ is called a *chain of prime ideals* in A. The integer r is called the *length* of the chain, $\mathfrak{p}_0$ is called the *origin* of the chain and $\mathfrak{p}_r$ is called the *extremity* of the chain. We say that a chain like the one above *joins* $\mathfrak{p}_0$ to $\mathfrak{p}_r$.

Note that chains in A with origin $\mathfrak{p}_0$ correspond bijectively to chains in $A/\mathfrak{p}_0$ with origin 0. Also chains in A with extremity $\mathfrak{p}_r$ correspond to chains in the local ring $A_{\mathfrak{p}_r}$ with extremity the maximal ideal of the ring.

Definition: Let A be a ring. We define the *Krull dimension* of A (or just the dimension of A) to be

$$\mathtt{dim}(A) = \mathtt{sup}\{r : \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r \text{ is a chain in } A\ \}.$$

Definition: If $\mathfrak{p}$ is a prime ideal of A then the *height* of $\mathfrak{p}$ is the dimension of the local ring $A_{\mathfrak{p}}$. Symbolically

$$\mathtt{ht}(\mathfrak{p}) = \mathtt{dim}(A_{\mathfrak{p}})$$

In other words it is the supremum of lengths of chains with extremity $\mathfrak{p}$ in A. We extend this definition to any ideal $\mathfrak{a}$ of A by setting

$$\mathtt{ht}(\mathfrak{a}) = \mathtt{inf}_{\mathfrak{p}\in\mathcal{V}(\mathfrak{a})}\ \mathtt{ht}(\mathfrak{p})$$

for $\mathfrak{a} \neq A$ and $\mathtt{ht}(A) = \mathtt{dim}(A)$.

Let B be a commutative A-algebra. We say that an element $x \in B$ is *integral* over A if and only if

$$x^n + a_1x^{n-1} + \cdots + a_n = 0, \qquad a_i \in A$$

for some $n \geq 1$. This is the same as saying that $A[x] \subset B$ is finitely generated over A.

For the rest of this paper assume that $A \subset B$ and that all $x \in B$ are integral over A (then we say that B is *integral* over A).

Note that

$$\mathtt{ht}(\mathfrak{p}) + \mathtt{dim}(A/\mathfrak{p}) \leq \mathtt{dim}(A)$$

but equality may not hold even for well-behaved rings. However we have the following result (See [**A&M**]):

Theorem B1: Let A be a domain that is a finitely generated algebra over a field. Then for any prime $\mathfrak{p}$ of A we have

$$\mathtt{ht}(\mathfrak{p}) + \mathtt{dim}(A/\mathfrak{p}) = \mathtt{dim}(A).$$

We say that a domain A is *integrally closed* if and only if every element in the field of fractions of A that is integral over A is already in A.

It turns out that a domain A is integrally closed if and only if $A_{\mathfrak{p}}$ is integrally closed for all prime ideals $\mathfrak{p}$ if and only if $A_{\mathfrak{m}}$ is integrally closed for all maximal ideals $\mathfrak{m}$. For a proof the reader can consult [**A&M**].

4. Regular Local Rings

Theorem/Definition C1: Let A be a noetherian local ring of dimension r with maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. The following are equivalent:

(1) A is *regular*.

(2) $\mathfrak{m}$ can be generated by r elements.

(3) The dimension over k of $\mathfrak{m}/\mathfrak{m}^2$ is r.

(4) The graded ring $\mathtt{gr}_{\mathfrak{m}}(A)$, associated to the $\mathfrak{m}$-adic filtration of A, is isomorphic to $k[X_1, \ldots, X_r]$.

Proof: **(2)** $\iff$ **(3)**. Suppose that $x_1, \ldots, x_r$ are elements of A such that $(x_1, \ldots, x_r) = \mathfrak{m}$. Look at the classes $\overline{x_1}, \ldots, \overline{x_r} \in \mathfrak{m}/\mathfrak{m}^2$. These clearly generated $\mathfrak{m}/\mathfrak{m}^2$ and therefore

$$r \geq \mathtt{dim}_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

Conversely, let $\mathfrak{m}/\mathfrak{m}^2 = (\overline{x_1}, \ldots, \overline{x_s})$ for some $x_1, \ldots, x_s$. Then we see that

$$\mathfrak{m} = (x_1, \ldots, x_s) + \mathfrak{m} \cdot \mathfrak{m}.$$

By Nakayama (theorem **A1.2**) we see that $\mathfrak{m} = (x_1, \ldots, x_s)$. Thus if r is the number of generators required to generate $\mathfrak{m}$ we get

$$r \leq \mathtt{dim}_k \left(\mathfrak{m}/\mathfrak{m}^2\right).$$

This establishes the equivalence of the two conditions. †

(4) $\Rightarrow$ **(3)** Assume that

$$\bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1} \cong (A/\mathfrak{m})[X_1, \ldots, X_r].$$

Note that we are thinking of $(A/\mathfrak{m})[X_1, \ldots, X_r] = k[X_1, \ldots, X_r]$ as graded by degree. Under the isomorphism $\mathfrak{m}/\mathfrak{m}^2$ corresponds to polynomials of degree 1 modulo polynomials of degree 2. But this is precisely $kX_1 \oplus \cdots \oplus kX_r$. We can conclude that $\mathfrak{m}/\mathfrak{m}^2$ has dimension r over k. †

(2) $\Rightarrow$ **(4)** Assume that A is a local ring such that $\mathfrak{m} = (x_1, \ldots, x_r)$. Consider

$$\mathtt{gr}_{\mathfrak{m}}(A) = A/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots = k \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \cdots .$$

Then $\mathtt{gr}_{\mathfrak{m}}(A)$ is generated by k and $x_1, \ldots, x_r \in \mathfrak{m}$. Look at $k[X_1, \ldots, X_r]$ graded by degree. We have a natural map

$$k[X_1, \ldots, X_r] \to \mathtt{gr}_{\mathfrak{m}}(A), \qquad X_i \to x_i.$$

The map is obviously surjective and it respects both gradings. Let R be the kernel then we have the exact sequence

$$0 \to R \to k[X_1, \ldots, X_r] \to \mathtt{gr}_{\mathfrak{m}}(A) \to 0$$

which gives for any $n \geq 0$ an exact sequence

$$0 \to R_n \to k[X_1, \ldots, X_r]_n \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to 0.$$

Thus we get

$$l(A_n) + l(R_n) = \binom{n+r-1}{r-1}.$$

By comparing highest coefficients, we get

$$e_{r-1}(H_A) = \Delta^{r-1} H_A = 1 - \Delta^{r-1} H_R.$$

Suppose in order to get a contradiction that $R \neq 0$. One can show that any nonzero graded submodule of $k[X_1, \ldots, X_r]$ has $\Delta^{r-1} H_R > 0$. Indeed let $f \in R$, $f \neq 0$ and thus

$$f \cdot k[X_1, \ldots, X_r] \subset R.$$

But $lf \cdot k[X_1, \ldots, X_r]_n = \binom{n+r-1}{r-1}$ so $\Delta^{r-1} H_R \geq 1$.

We conclude that unless $R = 0$ we have $\Delta^{r-1} H_A < 1$. But if $\mathfrak{m}$ is generated by r elements then $\Delta^{r-1} H_A \geq 1$. A contradiction that shows that $R = 0$. This proves what we need. †

We have established all equivalences. □

Corollary C2: A regular local ring A is normal.

Proof: Indeed by the previous theorem $\mathtt{gr}_{\mathfrak{m}}(A)$ is normal, so theorem **A2** A is normal. □

We need a result that we will not prove.

Corollary C3 (Auslander - Buchsbaum): A regular local ring A is factorial.

Proof: See [**Serre**].

If A is a regular local ring, a *regular system of parameters of* A is any set $\{x_1, \ldots, x_r\}$ that generates $\mathfrak{m}$. We have the following theorem that characterizes regular systems of parameters

Theorem C4: If $\{x_1, \ldots, x_p\}$ are p elements of the maximal ideal $\mathfrak{m}$ of a regular local ring A, the following three properties are equivalent:

(1) $x_1, \ldots, x_p$ is a subset of a regular system of parameters of A.

(2) The images of $x_1, \ldots, x_p$ in $\mathfrak{m}/\mathfrak{m}^2$ are linearly independent over k.

(3) The local ring $A/(x_1, \ldots, x_r)$ is regular, and has dimension $\mathtt{dim}(A) - p$. (In particular, $(x_1, \ldots, x_p)$ is a prime ideal.)

Proof: **(1)** $\iff$ **(2)** Indeed we have shown that regular systems of parameters of A correspond to k-bases of $\mathfrak{m}/\mathfrak{m}^2$. †

(2) $\implies$ **(3)** Let $\mathfrak{p} = (x_1, \ldots, x_p)$ and $\mathfrak{n} = \mathfrak{m}/\mathfrak{p}$. Notice that $\mathfrak{n}$ is the maximal ideal of $A/\mathfrak{p}$. We have the exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Therefore

$$\mathbf{(2)} \iff [\mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 : k] = p \iff [\mathfrak{n}/\mathfrak{n}^2, k] = \mathtt{dim}(A) - p.$$

But $x_1, \ldots, x_p$ is a subset of a system of generators of $\mathfrak{m}$ of A, so $A/(x_1, \ldots, x_p)$ has dimension $\mathtt{dim}(A) - p$. Now the result follows from our characterization of regular local rings. †

(3) $\implies$ **(2)**. Indeed **(3)** gives

$$[\mathfrak{n}/\mathfrak{n}^2 : k] = \mathtt{dim} A/\mathfrak{p} \qquad \text{and} \qquad \mathtt{dim} A/\mathfrak{p} = \mathtt{dim}(A) - p.$$

And the result follows from the exact sequence. †

This finishes the proof. □

Corollary C5: If $\mathfrak{p}$ is an ideal of a regular local ring A, the following are equivalent:

(1) $A/\mathfrak{p}$ is a regular local ring.

(2) $\mathfrak{p}$ is generated by a subset of a regular system of parameters of A.

Proof: In light of the previous theorem we only need to prove **(1)** $\implies$ **(2)**. As before set $\mathfrak{n} = \mathfrak{m}/\mathfrak{p}$ and write down the exact sequence:

$$0 \to \mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0,$$

and since $[\mathfrak{n}/\mathfrak{n}^2 : k] = \mathtt{dim} A/\mathfrak{p}$, we have $[\mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 : k] = \mathtt{ht}_A \mathfrak{p}$.

Thus if $x_1, \ldots, x_p$ are elements of $\mathfrak{p}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a k-basis of $\mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2$, then the ideal $(x_1, \ldots, x_p) \subset \mathfrak{p}$ is prime and of height $p = \mathtt{ht}_A \mathfrak{p}$; hence $\mathfrak{p} = (x_1, \ldots, x_p)$. $\square$

Theorem C6: If $\mathfrak{p}$ is a prime ideal of a regular local ring A, then the local ring $A_{\mathfrak{p}}$ is regular.

Proof: See Serre's book [**Serre**].

Theorem C7: If A is a complete local ring, and if A and $k = A/\mathfrak{m}$ have the same characteristic then the following are equivalent:

(1) A is regular of dimension r.

(2) A is isomorphic to the formal power series ring $k[[X_1, \ldots, X_r]]$.

Proof: **(2)** $\Longrightarrow$ **(1)** follows from the defining theorem for regular local rings. Indeed, assume $A \cong k[[X_1, \ldots, X_r]]$. Compute $\mathtt{gr}_{\mathfrak{m}}(A)$, where $\mathfrak{m} = (X_1, \ldots, X_r)$. Since any polynomial in $f \in k[X_1, \ldots, X_r]$ can be written as

$$f = f_0 + f_1 + \cdots + f_d, \qquad \text{where } f_d \in \mathfrak{m}^d/\mathfrak{m}^{d+1}$$

we see that $\mathtt{gr}_{\mathfrak{m}}(A) \cong k[X_1, \ldots, X_r]$.

To finish the proof of the implication we must show that $k[[X_1, \ldots, X_r]]$ has Krull dimension r. To see this notice that $k[[X_1, \ldots, X_r]]$ is the completion of the polynomial ring $k[X_1, \ldots, X_r]$ under the degree grading. Dimension does not change with completion, and $\mathtt{dim}(k[X_1, \ldots, X_r]) = r$, so we see that $\mathtt{dim}(A) = r$.

Now our defining theorem of regular local rings applies to A and we have proved the implication. †

(1) $\Longrightarrow$ **(2)** For this we refer the reader to [**Serre**]. $\square$

Regular local rings of dimension one (also known as *discrete valuation rings*) have the nice property that there is an element π such that for any $f \in A$ we have $f = u\pi^\nu$ with u unit, $\nu \in \mathbb{Z}$. We have the following results for discrete valuation rings that we will need in the next section:

Theorem C8: Let A be a ring. The following are equivalent:

(1) A is a discrete valuation ring, i.e. a regular local ring of dimension one.

(2) A is a local ring of dimension one that is integrally closes.

Proof: See [**A&M**].

Theorem C9: Let A be a discrete valuation ring, F its field of fractions. Let B be any subring of F such that $A \subset B$. Then either $B = F$ or $B = A$.

Proof: Suppose that $B \neq A$, say $a \in B - A$. Let v be the discrete valuation on A then $v(a) < 0$. Now let $x \in F$ be arbitrary. For sufficiently large n we get $v(xa^{-n}) = v(x) - nv(a) > 0$, so $xa^{-n} \in A \subset B$. Therefore $x = xa^{-n}a^n \in B$ and the proof is complete. $\square$

Theorem C10: Let A be a domain, F its field of fractions, B a subring of F such that $A \subset B$, which is finitely generated as an A-algebra. Let $\mathfrak{P}$ be a prime of B lying over $\mathfrak{p}$ i.e. $\mathfrak{p} = A \cap \mathfrak{P}$. Suppose that $A_{\mathfrak{p}}$ is a discrete valuation ring. Then there exists $g \in A - \mathfrak{p}$ such that $A_g = B_g$.
Proof: The proof is not difficult, but it involves notions that have not been introduced. See [**Bump**].

Lemma C11: (Nagata) Let A be a domain, and let $x \in A$ be an element such that (x) is prime. If $A[1/x]$ is factorial, then A is factorial.
Proof: See [**Mum**].

5. Algebraic Geometry

Let $(X, \mathcal{O}_X)$ be an affine or projective variety. Let $Y \subset X$ be an irreducible closed subvariety of X. Define

$$\mathcal{O}_{X,Y} = \varinjlim \mathcal{O}_X(U),$$

where the direct limit is formed using open subsets U, such that $U \cap Y \neq \emptyset$.

To get an algebraic description let U be an irreducible affine open set such that $Y \cap U \neq \emptyset$ with coordinate ring $A = \mathcal{O}_X(U)$ (in the case when X is affine just take U to be an irreducible component of X). Note that A is a domain. Set $\mathfrak{p} = \mathcal{I}(Y \cap U)$. Since each open set contains a principal open we have

$$\mathcal{O}_{X,Y} = \varinjlim \mathcal{O}_X(U_f) = \cup A_f = A_{\mathfrak{p}},$$

where U_f's are open such that $f \in A$, $f \notin \mathfrak{p}$.

In particular, we have $\mathcal{O}_{X,x} = A_{\mathfrak{m}}$ where $\mathfrak{m}$ corresponds to the point $x \in X$.

Lemma D1:

$$\mathcal{O}_{X,Y} = (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}.$$

Proof: Remember that $A_{\mathfrak{p}} = \{(a/s) : a \in A, s \in A - \mathfrak{p}\}$, $A_{\mathfrak{m}} = \{(b/t) : b \in A, t \in A - \mathfrak{m}\}$, and $(A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}} = \{(\alpha/\tau) : \alpha = b/t, b \in A, t \in$

$A - \mathfrak{m}, \tau = c/z, c \in A - \mathfrak{p}, z \in A - \mathfrak{m}\}$. Define a homomorphism

$$\psi : A_{\mathfrak{p}} \to (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}, \qquad \frac{a}{s} \mapsto \frac{(a/1)}{(s/1)}.$$

ψ is easily seen to be onto. In fact, let $(b/t)/(c/z) \in (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}$ then too at image of $(bz)/(ct)$ under ψ we have

$$\frac{(bz)/1}{ct/1} = \frac{bz/1}{ct/1} \cdot \frac{1/z}{1/z} \cdot \frac{1/t}{1/t} = \frac{(b/t)}{(c/z)}.$$

Also one can see that the map is one-to-one. Indeed, suppose $\phi(a/s) = 0$ i.e. $(a/1)/(s/1) = 0$ in $(A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}$. This happens if and only if there exist $c \in A - \mathfrak{p}$ and $z \in A - \mathfrak{m}$ so that $(c/z) \cdot (a/1) = 0$ in $A_{\mathfrak{m}}$. Now this happens if and only if there exists $t \in A - \mathfrak{m}$ so that $tca = 0$ in A. But since $\mathfrak{p} \subset \mathfrak{m} \iff A - \mathfrak{m} \subset A - \mathfrak{p}$ we have $tc \in A - \mathfrak{p}$ and by definition of $A_{\mathfrak{p}}$ we have $a/s = 0$ there.

This shows that $A_{\mathfrak{p}} \cong (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}$ and finishes the proof. □

Lemma D2:

$$\mathtt{dim}(\mathcal{O}_{X,Y}) = \mathtt{dim}(X) - \mathtt{dim}(Y).$$

Proof: This is equivalent to

$$\mathtt{dim}(A_{\mathfrak{p}}) + \mathtt{dim}(A/\mathfrak{p}) = \mathtt{dim}(A),$$

which is true, since A is a domain. □

Theorem D3: Let X be as above then $X_{\mathtt{Sing}}$ (the subset of X that consists of singular points) is a proper closed set of X.

Proof: A slightly stronger result was shown in class.

We are now in a position to prove some properties of $\mathcal{O}_{X,Y}$.

Theorem D4: Let X and Y be as above with X irreducible. Then $\mathcal{O}_{X,Y}$ is a regular local ring if and only if $X_{\mathtt{Sing}} \not\subset Y$.

Proof: ($\Leftarrow$) Let $x \in Y - X_{\mathtt{Sing}}$ then $\mathcal{O}_{X,x} = A_{\mathfrak{m}}$ is a regular local ring. And by the first lemma $\mathcal{O}_{X,Y} = (A_{\mathfrak{m}})_{\mathfrak{p}A_{\mathfrak{m}}}$. Now by theorem **C6** we see that $\mathcal{O}_{X,Y}$ is a regular local ring. †

($\Rightarrow$) Let $d = \mathtt{dim}(X)$, $r = \mathtt{dim}(Y)$ and $s = d - r = \mathtt{dim}(\mathcal{O}_{X,Y})$. Let $\mathfrak{p}$ be the prime ideal of A that corresponds to Y.

We know that Y contains a regular point x. Let $\mathfrak{m}$ be the maximal ideal of A that corresponds to $x \in Y \subset X$. We will show that $A_{\mathfrak{m}}$ is a regular local ring of dimension d.

Since $x \in Y$ is regular we know that $(A/\mathfrak{p})_{(\mathfrak{m}/\mathfrak{p})}$ is a regular local ring of dimension r. But localization and quotients commute so

$$(A/\mathfrak{p})_{(\mathfrak{m}/\mathfrak{p})} = A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}.$$

By corollary **C5** we get that $\mathfrak{p}A_{\mathfrak{m}} = (x_1, \ldots, x_s)$, where x_i are part of a set of generators of $\mathfrak{m}A_{\mathfrak{m}}$ and there are s of them for dimensional reason. Abusing notation let $\mathfrak{p}A_{\mathfrak{m}} = \mathfrak{p}$, $\mathfrak{m}A_{\mathfrak{m}} = \mathfrak{m}$ (this is the maximal ideal in $A_{\mathfrak{m}}$), $\mathfrak{n} = \mathfrak{m}A_{\mathfrak{m}}/\mathfrak{p}A_{\mathfrak{m}}$ (this is the maximal ideal of $(A/\mathfrak{p})_{(\mathfrak{m}/\mathfrak{p})}$) and $k = A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}}$. Recall the exact sequence that we used in the characterization of regular sequences (theorem **C4**)

$$0 \to \mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{n}/\mathfrak{n}^2 \to 0.$$

Because x is regular in Y we have that $[\mathfrak{n}/\mathfrak{n}^2 : k] = r$, because $\mathfrak{p} = (x_1, \ldots, x_s)$, where $\mathfrak{m} = (x_1, \ldots, x_s, \ldots, x_l)$ we see that $[\mathfrak{p}/\mathfrak{p} \cap \mathfrak{m}^2 : k] = s$. But then from the exact sequence $[\mathfrak{m}/\mathfrak{m}^2 : k] = s + r = d$. We can conclude that $A_{\mathfrak{m}}$ is regular and therefore x is a regular point of X. $\square$

A consequence of the theorem above is the following result that we have proved in class.

Theorem D5: Let X be a normal affine irreducible variety of dimension d. Then every irreducible component of $X_{\mathtt{Sing}}$ has dimension $\leq d - 2$.

Proof: Let Y be an irreducible component of $X_{\mathtt{Sing}}$. Then Y is closed and proper so $\mathtt{dim}(Y) \leq d - 1$. Suppose that $\mathtt{dim}(Y) = d - 1$. Then $\mathcal{O}_{X,Y}$ is a regular local ring of dimension 1. Since X is normal it is also integrally closed. This implies that $\mathcal{O}_{X,Y}$ is also integrally closed. But a local ring of dimension 1 that is integrally closed is regular by theorem **C8**, contradicting the previous theorem. $\square$

An important application of the two theorems above is that on a normal irreducible variety X one can use the usual definition of Weil divisors. Namely, for any $Y \subset X$ closed irreducible of codimension one, we see that we have a well-defined valuation function on $\mathcal{O}_{X,Y}$. Therefore it makes sense to ask about the order of $f \in A$ along Y. One than goes on to define the free group on the Y's. An element of it is called a Weil divisor

$$D = \sum \mu_Y Y, \qquad \mu_Y \in \mathbb{Z}.$$

For any $f \in A$ we get a Weil divisor (note the it is not obvious that the following definition works eventhough we have $\mathcal{O}_{X,Y}$ a discrete valuation ring because a priori the sum below is infinite)

$$\mathtt{div}(f) = \sum \mathtt{ord}_Y(f)Y.$$

The quotient group ($\mathtt{Weil\ divisor}$)/($\mathtt{Div\ of\ functions}$) is called the Picard group and is an important invariant of X.

In dimension one we have the following nice result about nonsingularity.

Theorem D6: Let C be an irreducible affine curve, and let A be its coordinate ring. A necessary and sufficient condition for C to be non-singular is that A be integrally closed.

Proof: Let A be the coordinate ring of C. A is integrally closed if and only if $A_{\mathfrak{m}}$ is integrally closed for each maximal ideal $\mathfrak{m}$. Now each $A_{\mathfrak{m}}$ is integrally closed if and only if it is regular because we are in dimension one and this happens if and only if the corresponding points in C are nonsingular. $\square$

We conclude our discussion with the following result that looks like an analog of the Implicit Function Theorem. We call a dominant morphism between varieties that have the same function field *birational*. One of the homework problems from class shows that if we have a dominant morphism then the induced homomorphism on the coordinate rings in an injection. We will use this below.

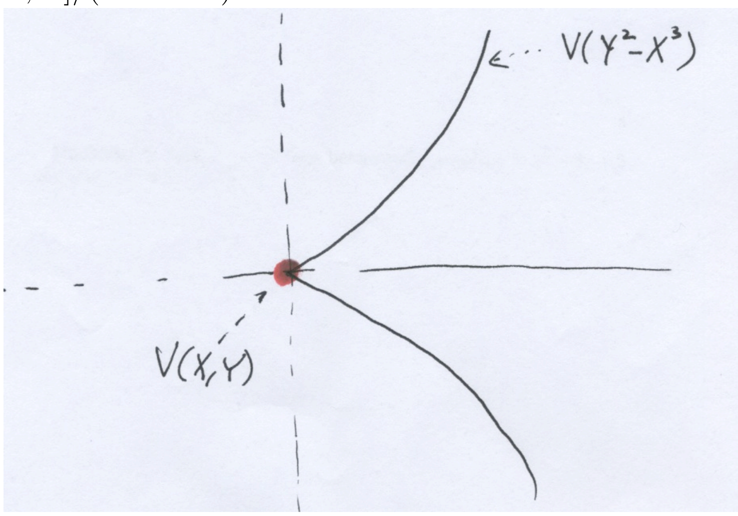
Theorem D7: Let C_1 and C_2 be irreducible affine curves, and let $f : C_2 \to C_1$ be a birational morphism. Let $y \in C_2$, and let $x = f(y) \in C_1$. Assume that x is a non-singular point of C_1. Then there exists an open set U containing x such that f induces an isomorphism $f^{-1}U \to U$.

Proof: Let A and B be the coordinate rings of C_1 and C_2 respectively. Identify A as a subring of B by the injection $f^* : A \to B$. Let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals of y and x in B and A respectively, so that $\mathfrak{n} = A \cap \mathfrak{m}$. Since $A_{\mathfrak{n}}$ is integrally closed it is a discrete valuation ring. By theorem **C10** we see that there exists $g \in A - \mathfrak{n}$ such that $A_g = B_g$. This means that the principal open sets $U = X_g$ and $Y_g = f^{-1}U$ are isomorphic. $\square$

6. Some Examples

Example E1: A Curve in $\mathbb{A}^2$ That is Not Normal

We take the usual example $C \subset \mathbb{A}^2$, given by $C = \mathcal{V}(Y^2 = X^3)$ then by the Jacobian criterion C is singular at $(0,0)$ and so $A = k[X,Y]/(Y^2 = X^3)$ cannot be normal.



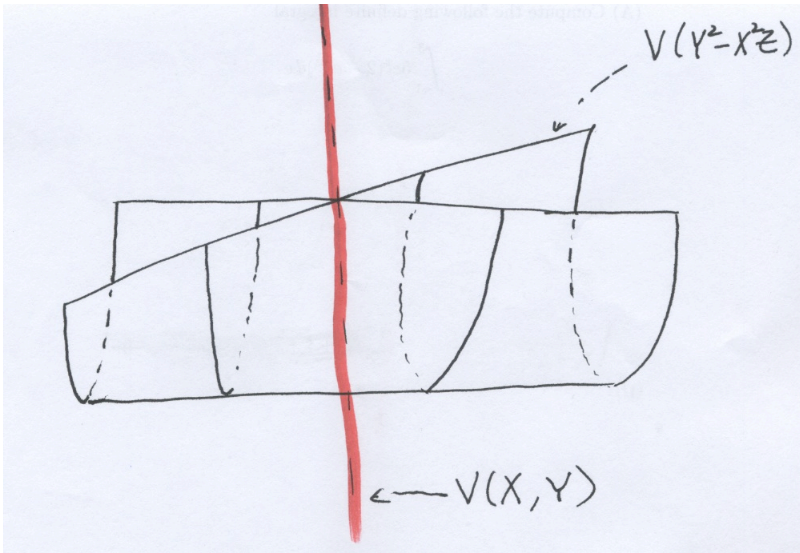
We can see that A is not normal directly by observing that

$$\overline{Y}^2 = \overline{X}^3 \text{ in } A \iff \left(\frac{\overline{Y}}{\overline{X}}\right)^2 = \overline{X} \text{ in } \mathtt{Frac}(A),$$

and so we have found an element in $\mathtt{Frac}(A)$ that is algebraic over A, but not in A. §

Example E2: A Surface in $\mathbb{A}^3$ That is Not Normal

Consider $X = \mathcal{V}(Y^2 - X^2 Z)$. Let $A = k[X,Y,Z]/(Y^2 - X^2 Z)$ be the coordinate ring of X. Notice that by the Jacobian Criterion $X_{\mathtt{Sing}} = \mathcal{V}(X,Y)$, so $\mathtt{dim} X_{\mathtt{Sing}} = 1$ and $\mathtt{dim} X = 2$. And from the results in the previous section this implies that A cannot be normal.



In fact, it is not difficult to see this from the definition of A itself. Indeed, we can exhibit an explicit element of $\mathtt{Frac}(A)$ that is algebraic, but that is not in A:

$$\overline{Y}^2 - \overline{X}^2 \cdot \overline{Z} = 0 \text{ in } A \iff \left(\frac{\overline{Y}}{\overline{X}}\right)^2 = \overline{Z} \text{ in } \mathtt{Frac}(A).$$

And we see that $(\overline{Y}/\overline{X}) \in \mathtt{Frac}(A)$ is algebraic over A, but not in A. §

Example E3: A Surface in $\mathbb{A}^3$ That is Normal and Singular Consider $X = \mathcal{V}(X^2 - YZ)$. Let $A = k[X,Y,Z]/(X^2 - YZ)$ be the coordinate ring of X. It can be shown that A is normal, however X is singular.

Indeed by the Jacobian Criterion $X_{\mathtt{Sing}} = \{(0,0,0)\}$.

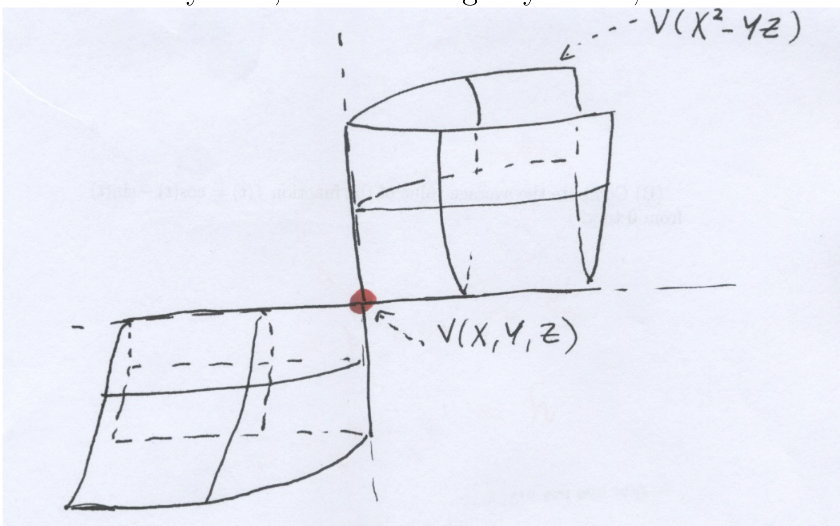
To show that A is normal observe that $X^2 - YZ = 0 \iff X \cdot \frac{X}{YZ} = 1$. That is if one has $\frac{1}{Y}$ and $\frac{1}{Z}$ already in a field containing A then $\frac{1}{X}$ is there as well. Let $K = k(Y,Z)$ and look at $L = \mathtt{Frac}(A) = k(Y,Z)[X]/(X^2 - YZ)$. Notice that $[L : K] = 2$, so any $\alpha \in L$ can be expressed in the form $\alpha = f + gX$ with $f, g \in k(Y,Z)$. The minimal polynomial of α over K looks like

$$T^2 - fgT - (f^2 + g^2X^2) = T^2 - fgT - (f^2 + g^2ZY) = 0$$

and $\alpha \in A$ when $fg, f^2 + g^2ZY \in k[Y,Z]$. Since $k[Y,Z]$ is factorial we can put

$$f = \frac{p_1}{q_2}, \quad g = \frac{p_2}{q_2}, \qquad (p_1, q_1) = 1, \quad (p_2, q_2) = 1.$$

From $fg \in k[Y, Z]$ one sees that $q_2 \mid p_1$ and $q_1 \mid p_2$. From $f^2 + g^2 ZY \in k[Y, Z]$ one sees that $q_1^2 q_2^2 \mid p_1^2 q_2^2 + p_2^2 q_1^2 ZY$. But then $q_1 \mid p_1$ and $q_2 \mid p_2$, so $q_1 = 1$ and $q_2 = 1$. This shows that any element of L that is integral over A is already in A, i.e. A is integrally closed, hence normal.



As predicted by our theorem in this case $X_{\mathtt{Sing}}$ has dimension 0. Note that, unlike the situation for curves, in higher dimensions normal and non-singular are not equivalent. According to what we have shown non-singular is weaker than normal. §

Example E4: A Variety in $\mathbb{A}^5$ That is Factorial and Singular (This is taken almost verbatim from Chapter 3 of [**Mum**].) Consider $X = \mathcal{V}(\sum_{i=1}^5 X_i^2)$. By the Jacobian Criterion $X_{\mathtt{Sing}} = \{0, 0, 0, 0, 0\}$, so X is singular.

Look at the coordinate ring $A = k[X_1, \ldots, X_5]/\sum_{i=1}^5 X_i^2$ and let $x = X_1 + iX_2$, $x' = iX_2 - X_1$. Then $\sum_{i=1}^5 X_i^2 = X_3^2 + X_4^2 + X_5^2 - x \cdot x'$. Thus

$$A/(x) \cong k[x', X_3, X_4, X_5]/(X_3^2 + X_4^2 + X_5^2)$$

which is an domain i.e. (x) is prime. But

$$A\left[\frac{1}{x}\right] \cong \frac{k[x, \frac{1}{x}, x', X_3, X_4, X_5]}{(X_3^2 + X_4^2 + X_5^2 - x \cdots x')} \cong k\left[x, \frac{1}{x}, X_3, X_4, X_5\right],$$

since the equation asserts that $x' = \frac{1}{x}(X_3^2 + X_4^2 + X_5^2)$. Now by Nagata's Lemma from the section on regular local rings we see that A is factorial and thus so is $A_{\mathfrak{m}}$ for any $\mathfrak{m}$.

Thus being factorial (see definition in the next section) is weaker than being non-singular. §

7. A Word on Good and Bad Varieties

At the end we want to say a few words about certain type of classification of varieties.

Definition: Let $(X, \mathcal{O}_X)$ be a variety.

(1) X is called *normal* if and only if for all $x \in X$ we have that $\mathcal{O}_{X,x}$ is a normal ring.

(2) X is called *factorial* if and only if for all $x \in X$ we have that $\mathcal{O}_{X,x}$ is a factorial ring.

(3) X is called *non-singular in codimension* 1 if and only if it is non-singular at all points x such that the closure of x is codimension 1.

By definition of regular local ring and from theorem **C3** and corollary **C2** we see that non-singular $\implies$ normal $\implies$ factorial. By theorem **D5** we see that normal $\implies$ non-singular in codimension 1. According to example **E3** normal is strictly weaker than non-singular and according to example **E4** factorial is weaker than non-singular. To summarize:

$$\text{non-singular} \subsetneq \text{normal} \subset \text{factorial} \subset \text{non-singular in codim 1.}$$

One can give examples to show that all of these inclusions are strict (See [**Mum**]) and so the actual picture is

$$\text{non-singular} \subsetneq \text{normal} \subsetneq \text{factorial} \subsetneq \text{non-singular in codim 1.}$$

8. References

Almost all of the algebraic results in this paper can be found in:

[**Serre**] J.-P. Serre, *Local Algebra*, Springer-Verlag (2000).

On the other hand the results on the geometric side have been taken from:

[**Bump**] D. Bump, *Algebraic Geometry*, World Scientific (1998).

Additional results mentioned in the paper can be found in the following two:

[**A&M**] M. Atiyah and I. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969).

[**Mum**] D. Mumford, *The Red Book of Varieties and Schemes* Lecture Notes in Mathematics **1358**, Springer-Verlag (1988).