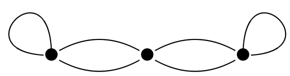
GRAPH THEORY HW 2

5.3 Solution: It is sufficient to consider the case when G is connected. Choose any vertex v_0 in G. If w is another vertex in G, then label w R (red) or B (blue) if there is a path from v_0 to w of even or odd length, respectively. It remains to check that no vertex is labeled both red and blue. If w is some vertex labeled R and B, then there is a closed walk from w to w including v_0. This walk necessarily contains a cycle; since every path from a red vertex to a blue vertex has odd length, this cycle has odd length, contrary to our hypothesis. It follows that our partition is proper. ☐

5.6 Solution:

(1) If G has minimum degree k, then find a vertex v such that $\deg v = k$. Now delete the k edges (or less if there are loops) incident v, so that v is isolated. Thus we have produced a cutset with (at most) k elements, so that $\lambda(G) \leq k$.

(2)



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5.7 Solution:

(1) $\Rightarrow$: If $|G| = 2$ then the result is trivial, so assume $|G| > 2$. We will prove the equivalent result: if P is a path from a to b which has no cut vertex, then there is a cycle from a to b. This is easy to verify if a and b are adjacent, and when $d(a, b) = 2$. Indeed, if a is adjacent to b, then let $u \neq a$ be a neighbor of b. Then since G is 2-connected, there is a path P from a to u in $G - b$. This path cannot use any edge incident b, so P, ub, ba is a cycle. If $d(a, b) = 2$, then there is a path avb; since this path has no cut vertex, $G - v$ is connected. Thus there is some path P from a to b so that $Pbva$ is a cycle. If a and b are adjacent, then choose any vertex $v \neq a$ adjacent to b.

We will proceed inductively on $d(a, b)$. Let P be a path of minimum length from a to b, say $P = av_1 \cdots v_k b$. Then there is a cycle C_1 enclosing v_1 and b and a cycle C_2 enclosing a, v_1, by the induction hypothesis. If C_1 contains a or C_2 includes b, we are done. Otherwise, if these two cycles have any vertices in common, then we can construct a closed walk which includes vertices a and b only once. Then there is a cycle enclosing a and b. Finally, if C_1 and C_2 do not intersect, we can construct a cycle using edge v_1v_2 in P and the two cycles to construct a cycle enclosing a and b. Thus, in any case, we have a cycle enclosing a and b, and the result follows by induction.

$\Leftarrow$: Consider any vertex v, and the graph $G - v$. If a, b are vertices in $G - v$, then by hypothesis there is a cycle C in G containing a and b. Hence, a and b are still connected in $G - v$, since there is a path from a to b in $C - v$.

(2) A graph G is 2-edge-connected if and only if each edge is contained in a cycle. ☐

Date: September 1, 2006.

5.8 Solution:

(1) There is a typo: G must be restricted to be a simple graph. We have that $(A^2)_{ij} = (r_i \cdot c_j)$, where r_i is the i^{th} row and c_j is the j^{th} column of A. Notice that all entries are either 0 or 1 since G is simple. Then we have that $r_{ik}c_{kj} = 1$ if and only if v_i is adjacent to v_k and v_k is adjacent to v_j, i.e. if and only if there is a walk of length two between v_i and v_j through vertex v_k. This proves the result.

(2) We already know that $2m$ is the sum of the degrees of the vertices of G. Then it suffices to show that $(A^2)_{ii} = \deg v_i$. We have already shown that $(A^2)_{ii}$ is the number of length 2 walks from v_i to itself; if we take v_i to any other adjacent vertex v_k, then since G is simple the only way to return to v_i is to go back along the same edge. Thus the number of length 2 walks from v_i to itself is just the number of edges incident v_i, i.e. the $\deg v_i$.

(3) Consider that $(A^3)_{ij} = (r_i \cdot c_j)$, where r_i is the i^{th} row of A^2, and c_j is the j^{th} column of A. If v_k is adjacent to v_j, then $r_{ik}c_{kj} = w \cdot 1$, where w is the number of length 2 walks from i to k. If v_k is not adjacent to v_j, then $r_{ik}c_{kj} = 0$. Thus $(A^3)_{ij}$ is the number of length 3 walks from i to j.

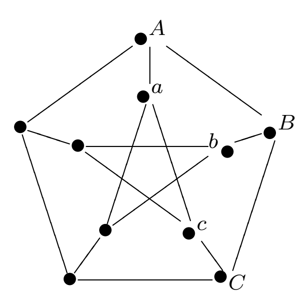
Consider a triangle T in G. If v is a vertex in the triangle, then T determines exactly two length 3 cycles (intuitively, clockwise and counter-clockwise) beginning and ending at v. Thus each triangle determines exactly six length 3 walks which begin and end at the same vertex. We have just shown that $(A^3)_{ii}$ is the number of length 3 walks which begin and end at v_i. Thus, $\text{Trace}(A^3) = 6t$. $\square$

5.9 Solution:

(1) Let P be a path from w to v of minimal length. Since $d(w,v) \geq 2$, there is a vertex z in P which is adjacent to w, and distinct from v. By the minimality of P, we have that $d(w,z) + d(z,v) = d(w,v)$. Indeed, if $d(z,v) < d(w,v) - d(w,z)$, then there would be some path Q of length less than $d(w,v) - 1$ from z to v which we could union with the length 1 path $v \to z$, contradicting the minimality of P. But clearly $d(z,v) \leq d(w,v) - d(w,z)$, since $P - wz$ is a path of the given length. The result follows.

(2) Notice that the Peterson graph contains no triangles, and is 3-regular. So beginning with any v, we see that 3 vertices are distance one away, and 9 vertices are distance 2 away (distinct since there are no triangles). Since there are only 10 vertices, we have finished the proof.

Alternatively:



By the reflectional and rotational symmetries, we only need to consider the values of $d(A,*)$ and $d(a,\times)$ where $* \in \{a,b,c,B,C\}$, $\times \in \{b,c\}$. Since there are only 7 cases, we may check

$$\begin{array}{llll} d(A,a)=1 & d(A,b)=2 & d(A,c)=2 & d(A,B)=1 \\ d(A,C)=2 & d(a,b)=2 & d(a,c)=1 & \end{array} \qquad \square$$

5.11 Solution:

(1) Suppose C_1 and C_2 both contain edge $e = v_x v_y$. Suppose $C_x = v_x \to v_{v1} \to \cdots \to v_{vk} \to v_y \to v_x$, and $C_y = v_x \to v_{w1} \to \cdots \to v_{wj} \to v_y \to v_x$. Then we can construct the walk $v_x \to v_{v1} \to \cdots \to v_{vk} \to v_y \to v_{wj} \to \cdots \to v_{w1} \to v_x$, which clearly omits edge e. Since every closed walk contains a cycle, we are done. [This is a consequence of the fact that every walk between two vertices $v \neq w$ contains a path from v to w]

(2) If S and T are distinct cutsets of G, and $e \in S \cap T$, then there is a cutset R which does not contain e. Indeed, it is sufficient to produce a disconnecting set which does not contain e. Consider that $G - S$ is disconnected into non-empty graphs G_1 and G_2; since cutsets are minimal, $e = v_1 v_2$ is a bridge in $G - (S \setminus \{e\})$, with $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. Since S and T are distinct cutsets, G has more than 2 vertices. Without loss of generality we may suppose G_1 contains more than one vertex. Then if $I = \{\text{edges } e \in G - S : e \text{ is incident } v_1\}$, we have that $(S \cup I) \setminus \{e\}$ is a disconnecting set not containing e. Hence it contains a cutset not containing $\{e\}$. $\square$

5.12 Solution:

(1) G has a cutset C^*, then $G - C^*$ consists of two distinct components G_1, G_2. If C is a cycle in G, we may suppose that C contains a vertex v in G_1. Notice that as we traverse the cycle C, we may alternate between components G_1 and G_2. Each edge in C which changes components is an edge in the cutset C^*; since we must return to G_1 after traversing the cycle, there must be an even number of such edges of C in common with C^*.

(2) Consider the set S, and the subgraph H of G induced by S. It suffices to prove the result for a connected component of H, so suppose that H is connected. If $v \in H$, then let $E_v = \{e \in S : e \text{ is incident to } v\}$. Evidently, E_v is a disconnecting set of H; furthermore, E_v is the disjoint union of cutsets. Indeed, $H - E_v$ has some number of components, $\{H_i\}$; fixing any H_i, we see that $E(H_i) \cap E_v$ are precisely the edges needed to disconnect H_i. Since $E_v \subset S$, we see that $|E_v|$ is even by the hypothesis. Since v is arbitrary, it follows that H is Eulerian (each vertex has even degree). Notice that every Eulerian graph has an edge set which can be written as the edge-disjoint union of cycles. Since H is Eulerian, we are done. $\square$

5.13 Solution:

(1) Suppose I is independent. If $S \subset I$ is not independent, then I contains a cycle; contradiction.

(2) We may suppose that the edges of G consist of only $I \cup J$. Assume that for each edge $e \in J \setminus I$, $I \cup \{e\}$ is dependent, i.e. contains a cycle. Then I is a spanning tree for G; hence I has $n-1$ edges. Thus J contains at least n edges, and so is not a tree, i.e. J contains a cycle. This contradiction establishes the result.

(3) Same as above, cycle replaced by cutset.

(4) Consider that $G' = G - I \cap J$ is connected. It is sufficient to consider $I' = I - I \cap J$, and $J' = J - I \cap J$. Assume that each edge $e \in J'$ is a bridge in $G' - I'$,

adjoining components G_1, G_2. Since $G' - J'$ is connected, there must be (at least) a corresponding edge $f \in I'$ adjoining components G_1, G_2. If e_1 and e_2 may correspond to the same edge $f \in I'$, then e_1 is not a bridge in $G' - I'$ since e_2 connects the same components. But then $|J| \leq |I|$, a contradiction. $\square$

Problem. *Show that for all odd $n \geq 1$, there is a simple graph with $\frac{n^2-1}{4}$ edges without triangles and for all even $n \geq 2$ there is a simple graph with n vertices and $\frac{n^2}{4}$ edges without triangles.*

Solution. Consider that every bipartite graph contains only even cycles. Then if n is odd, we have that $G = K_{\frac{n-1}{2},\frac{n+1}{2}}$ has $\left(2 \cdot \frac{n-1}{2} \cdot \frac{n+1}{2}\right)/2 = \frac{n^2-1}{4}$ edges. Since a triangle is an odd length cycle, G contains no triangles. Similarly for n even, we know that $K_{\frac{n}{2},\frac{n}{2}}$ has $\frac{n^2}{4}$ edges, and contains no triangles. $\square$