The Functor of Points

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Abstract

We study the functor of points. After showing that the functor of points is a well defined notion, we establish a criteria for determining whether a given functor is the functor of points for some scheme. We conclude with applications to fiber products, tangent bundles, and moduli spaces.

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1 Introduction

Schemes have a great deal of structure that cannot be studied through the underlying point-set. As a consequence, set-theoretic constructions such as fiber products are tricky for schemes. Remarkably, we can associate a functor h_X to each scheme X restoring some set-theoretic data. This functor h_X is called the functor of points. After giving its definition, we will show that a scheme is uniquely determined by its corresponding functor of points. We will also determine a criterion that can be used to determine wether a given functor is the functor of points for some scheme. We will conclude with several applications.

2 Basic Definitions

One of the exotic features of schemes is that the points of the underlying topological spaces look nothing alike. There are closed and non-closed points, there are singular and non-singular points, each point may have a different residue field and consequently the standard interpretation of the structure sheaf as a sheaf of functions breaks down. Introducing the functor of points is an attempt to remedy this situation, at least in part.

In order to motivate the definition that follows, consider the following examples. In each example, we have a concrete category and for an object X in that category, the underlying point set is denoted by $|X|$.

Example 2.1. In the category of sets, if Z is the set consisting of a single point, then $|X| = \mathrm{Hom}(Z, X)$.

Example 2.2. In the category of abelian groups, $|X| = \mathrm{Hom}(\mathbb{Z}, X)$.

Example 2.3. In the category of commutative rings with identity and identity-preserving homomorphisms, $|X| = \mathrm{Hom}(\mathbb{Z}[x], X)$.

In general, for any fixed object Z in a category $\mathcal{C}$, we can define a functor $\varphi_Z : \mathcal{C} \to \mathfrak{Sets}$ by

$$\varphi_Z(X) = \mathrm{Hom}_{\mathcal{C}}(Z, X).$$

As we will see in the next section, in order to make sense of $\varphi_Z(X)$ as the set of points of X we need the functor φ_Z to be faithful. This may not always be the case and indeed is not the case for schemes.

In order to do something useful for schemes, we have to take this one step further. Instead of considering $\mathrm{Hom}(Z, X)$ for some fixed scheme Z, we consider all such sets simultaneously. In other words, we define a functor from the category of schemes to a category of functors.

Definition 2.4. Let $\mathfrak{Schemes}^{\circ}$ denote the opposite category of schemes and let $\mathfrak{Sets}$ denote the category of sets. Let X be a scheme. The **functor of points of X** is the functor

$$h_X : \mathfrak{Schemes}^{\circ} \to \mathfrak{Sets}$$

defined by $h_X(Y) = \mathrm{Hom}(Y, X)$. If $f : Y \to Z$ is a morphism of schemes, the induced morphism of sets $h_X(Z) \to h_X(Y)$ is given by $h_X(f)(g) = g \circ f \in h_X(Y)$.

Definition 2.5. Let $\mathfrak{Fun}(\mathfrak{Schemes}^{\circ}, \mathfrak{Sets})$ denote the category of functors from $\mathfrak{Schemes}^{\circ}$ to $\mathfrak{Sets}$. Then the **functor of points** is a functor

$$h : \mathfrak{Schemes} \to \mathfrak{Fun}(\mathfrak{Schemes}^{\circ}, \mathfrak{Sets})$$

given by $h(X) = h_X$. Given a morphism $f : X \to X'$, $h(f)$ is the natural transformation $h_X \to h_{X'}$ defined on $g \in h_X(Y)$ by $h(f)(g) = f \circ g$.

Definition 2.6. For a scheme X, the elements of $h_X(Y)$ are called the **Y-valued points of** X.

If $Y = \mathrm{Spec}(R)$ for a commutative ring R, this is usually abbreviated the R-valued points of X.

Definition 2.7. If $F : \mathfrak{Schemes}^o \to \mathfrak{Sets}$ is a functor, then F is **representable** if there is a scheme X such that $F \cong h_X$, where the isomorphism is in the category $\mathfrak{Fun}(\mathfrak{Schemes}^o, \mathfrak{Sets})$ (that is, a natural transformation from F to h_X with an inverse natural transformation h_X to F).

Problems such as moduli space problems usually involve determining whether some given functor is representable.

These definitions can be adjusted to make sense for relative schemes. Recall that if S is a scheme, X is a scheme over S if X is a scheme and there is a morphism of schemes $f : X \to S$. In this context, X is also called an S-scheme. If $S \cong \mathrm{Spec}(R)$ is an affine scheme, we say that X is a scheme over R, or an R-scheme. If X' is another S-scheme, with morphism $f' : X' \to S$, a morphism $F : X \to X'$ of S-schemes is a morphism of schemes such that $f' \circ F = f$. The functor of points can then be defined on S-schemes to be the functor $F_X(X') = \mathrm{Hom}_S(X', X)$ where X' is also an S-scheme and the morphisms are taken to be morphisms of S-schemes. The other definitions are obtained analogously. When there is ambiguity, I will specify that we are considering the functor over S-schemes by using the adjective "relative" before the appropriate terms.

For these definitions to make sense and be of any use to us, we need to prove that a scheme X is determined by its functor of points. This is the content of Yoneda's lemma, which will be stated and proved in the next section. For now, we will assume this to be the case and look at some examples.

Example 2.8. Suppose R is a finitely generated $\mathbb{Z}$-algebra. Specifically, suppose

$$R = \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m).$$

Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(\mathbb{Z})$. We want to consider $h_X(Y) = \mathrm{Hom}(Y, X)$. Specifying a morphism from Y to X is equivalent to specifying a ring homomorphism from R to $\mathbb{Z}$, which is equivalent to specifying images $x_i \mapsto a_i$ for $i = 1, \ldots, n$ subject to the condition that $f_j(a_1, \ldots, a_n) = 0$ for $j = 1, \ldots, m$. In other words, $h_X(Y)$ is identified with all integral solutions to the equations $f_j = 0$. These are the $\mathbb{Z}$-points of X.

Example 2.9. Suppose Y is an arbitrary scheme. Let A be a ring and let $X = \mathrm{Spec}(A)$. Let B be the ring of global sections of the structure sheaf on

Y. Then there is a natural bijection

$$\alpha : \mathrm{Hom}_{\mathfrak{Schemes}}(Y, X) \to \mathrm{Hom}_{\mathfrak{Rings}}(A, B).$$

Therefore we can identity $h_X(Y)$ with the set of ring homomorphisms from A to B.

We conclude this section by showing that in the relative context, the K points of a scheme X are the same as the K-rational points of X.

Definition 2.10. If X is a scheme over S and $S = \mathrm{Spec}(K)$ for a field K, the **K-rational points of** X are the points $p \in X$ whose residue field $\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p$ is K.

Proposition 2.11. Suppose X is a scheme over K, where K is a field. Then the relative K-points of X are precisely the closed K-rational points of X.

Proof. Let $f : X \to \mathrm{Spec}(K)$ be the given morphism and note that $\mathrm{Spec}(K)$ is over K via the identity map. A K-point of X is therefore a morphism $F : \mathrm{Spec}(K) \to X$ such that $f \circ F$ is the identity. The image of F is a single point $p \in X$, so we may reduce to an affine scheme $p \in \mathrm{Spec}(T) \subset X$. Since $\mathrm{Spec}(T)$ is a scheme over K, T is a K-algebra and F corresponds to a K-algebra homomorphism $F^* : T \to k$. Since $f \circ F$ is the identity and $\mathrm{Spec}(K) = \{(0)\}$, it follows respectively that F^* is surjective and the ideal $\mathfrak{p}$ corresponding to the point p is the kernel of F^*. Since F^* is surjective, the kernel is a maximal ideal, hence p is a closed point, and its residue field is $T/\mathfrak{p} \cong K$.

Now suppose $p \in X$ is a closed K-rational point. Again, let $p \in \mathrm{Spec}(T) \subset X$. Then p corresponds to a maximal ideal $\mathfrak{m}_p \subset T$ with $T/\mathfrak{m}_p \cong K$. Thus there is a surjective ring homomorphism $F : T \to K$ whose kernel is $\mathfrak{m}_p$, which induces a map $F^* : \mathrm{Spec}(K) \to \mathrm{Spec}(T)$ given by $F^*(0) = p$. Composing with the inclusion gives the desired K-point of X. □

Example 2.12. If we replace $\mathbb{Z}$ in the definitions of R and T in Example 2.8 above with an algebraically closed field k, we recover precisely the closed points of the scheme $\mathrm{Spec}(R)$, which is the affine variety corresponding to R. These are the k-points of X.

Example 2.13. If we replace $\mathbb{Z}$ in the definition of R by an algebraically closed field k of characteristic zero, and let T be $\mathbb{Z}$ or $\mathbb{Q}$, we get the closed points of X that correspond to the integral or rational points on the algebraic variety X.

Note that viewing X as a K-scheme is essential. For example, suppose $X = \mathrm{Spec}(\mathbb{C})$. As a $\mathbb{C}$-scheme (with the identity map) $h_X(X)$ is just the identity. However, as an arbitrary scheme we have:

$$h_X(X) = \mathrm{Hom}_{\mathfrak{Schemes}}(X, X) \cong \mathrm{Hom}_{\mathfrak{Rings}}(\mathbb{C}, \mathbb{C})$$

where the isomorphism is in the category of sets. The set of ring endomorphisms of $\mathbb{C}$ contains, in particular, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and is therefore highly nontrivial.

3 Yoneda's Lemma and Representability

3.1 Yoneda's Lemma and Consequences

In order to consider a functor $\Phi : \mathfrak{C} \to \mathfrak{Sets}$ of the form $X \mapsto \mathrm{Hom}_{\mathfrak{C}}(Z, X)$ for some fixed object Z of $\mathfrak{C}$ as something describing X as a point-set, we want Φ to be faithful. In other words, given $f \in \mathrm{Hom}_{\mathfrak{C}}(X, X')$ then f should be determined uniquely by the induced map $f^* : \Phi(X) \to \Phi(X')$ defined by $f^*(\alpha) = f \circ \alpha$. This is necessary in order to interpret the morphism f as a "function" on the point-sets.

As an example, consider the category $\mathfrak{Hot}$ whose objects are CW complexes and whose morphisms are homotopy classes of continuous maps. Let Z be the CW complex consisting of a single point. Then

$$\Phi(X) := \mathrm{Hom}_{\mathfrak{Hot}}(Z, X) = \pi_0(X)$$

where $\pi_0(X)$ is the set of connected components (which are the same as path connected components since CW complexes are locally path connected). Let $X = \mathbb{S}^1$ and let X' be a connected CW complex with a nontrivial fundamental group. Fix a basepoint $x_0 \in X$. Then

$$\pi_1(X, x_0) \subset \mathrm{Hom}_{\mathfrak{Hot}}(X, X')$$

and is therefore nontrivial. However, since X and X' are connected, $\Phi_Z(X)$ and $\Phi_Z(X')$ are both singleton sets. Hence morphisms from X to X' (of which there are more than one) correspond to the same map of sets from $\Phi_Z(X)$ to $\Phi_Z(X')$. Therefore this functor is not faithful. Moreover, this functor would not be faithful for any choice of a CW complex Z.

The same situation happens in the category of schemes. See Chapter II, section 6 of [Mum88]. This is why the definition of the functor of points for schemes is defined as landing in the category of functors. In order to show that this serves our purposes we prove the following theorem, known as Yoneda's lemma.

Theorem 3.1. Let $\mathfrak{C}$ be a category and let X and X' be objects of $\mathfrak{C}$.

1. If F is any contravariant functor from $\mathfrak{C}$ to the category of sets, the natural transformations from $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ to F are in natural correspondence with the elements $F(X)$.

2. The maps of functors from $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ to $\mathrm{Hom}_{\mathfrak{C}}(-, X')$ are the same as morphisms from X to X'. In other words, the functor

$$h : \mathfrak{C} \to \mathfrak{Fun}(\mathfrak{C}^o, \mathfrak{Sets})$$

   sending X to h_X is an equivalence of $\mathfrak{C}$ with a full subcategory of the category of functors. In particular, if the functors $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ and $\mathrm{Hom}_{\mathfrak{C}}(-, X')$ are isomorphic then $X \cong X'$.

Proof. If α is a natural transformation:

$$\alpha : \mathrm{Hom}_{\mathfrak{C}}(-, X) \to F$$

then α induces a map of sets

$$\alpha : \mathrm{Hom}_{\mathfrak{C}}(X, X) \to F(X).$$

Since $\mathfrak{C}$ is a category, there is an identity element $1_X \in \mathrm{Hom}_{\mathfrak{C}}(X, X)$. Therefore $\alpha(1_X)$ is a point in the set $F(X)$. Accordingly, denoting the category of contravariant functors from $\mathfrak{C}$ to $\mathfrak{Sets}$ by $\mathfrak{F}$, we can define a map of sets:

$$\begin{aligned} \rho : \mathrm{Hom}_{\mathfrak{F}}(\mathrm{Hom}_{\mathfrak{C}}(-, X), F) &\to F(X) \\ \alpha &\mapsto \alpha(1_X). \end{aligned}$$

We will show that ρ is a bijection by defining an inverse. Given $p \in F(X)$, for each $Y \in \mathrm{Ob}(\mathfrak{C})$ define a map of sets

$$\begin{aligned} \alpha_{Y,p} : \mathrm{Hom}_{\mathfrak{C}}(Y, X) &\rightarrow F(Y) \\ f &\mapsto F(f)(p). \end{aligned}$$

Since F is a contravariant functor, $f : Y \rightarrow X$ in $\mathfrak{C}$ implies $F(f) : F(X) \rightarrow F(Y)$ is a map of sets, hence this definition makes sense.

First we show that for a fixed $p \in F(X)$, the collection of maps $\alpha_{Y,p}$, where Y varies over the objects of $\mathfrak{C}$, defines a natural transformation. In other words, if $g : Y \rightarrow Z$ is a morphism in $\mathfrak{C}$, we need to check that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_{\mathfrak{C}}(Z, X) \arrow[r, "\alpha_{Z,p}"] \arrow[d, "g^*"'] & F(Z) \arrow[d, "F(g)"] \\ \mathrm{Hom}_{\mathfrak{C}}(Y, X) \arrow[r, "\alpha_{Y,p}"] & F(Y) \end{tikzcd}$$

where g^* is the usual pull back on morphisms. To see this, we compute and use the fact that F is contravariant:

$$\begin{aligned} \alpha_{Y,p}(g^* f) &= \alpha_{Y,p}(f \circ g) \\ &= F(f \circ g)(p) \\ &= F(g)(F(f)(p)) \\ &= F(g)(\alpha_{Z,p}(f)). \end{aligned}$$

Hence the collection defines a natural transformation, which we will denote by α_p. The map of sets

$$\begin{aligned} F(X) &\rightarrow \mathrm{Hom}_{\mathfrak{F}}(\mathrm{Hom}_{\mathfrak{C}}(-, X), F) \\ p &\mapsto \alpha_p \end{aligned}$$

will be denoted by σ.

Finally we check that ρ and σ are inverses. First consider $\sigma \circ \rho$. Suppose α is a natural transformation from $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ and F. Then for any $f \in \mathrm{Hom}_{\mathfrak{C}}(Y, X)$ the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}}(X,X) \arrow[r, "\alpha"] \arrow[d, "f^*"'] & F(X) \arrow[d, "F(f)"] \\
\mathrm{Hom}_{\mathfrak{C}}(Y,X) \arrow[r, "\alpha"] & F(Y)
\end{tikzcd}$$

Therefore:

$$\begin{aligned}
\sigma \circ \rho(\alpha)(f) &= \sigma(\alpha(1_X))(f) \\
&= F(f)(\alpha(1_X)) \\
&= \alpha(f^*(1_X)) \\
&= \alpha(f).
\end{aligned}$$

Thus $\sigma \circ \rho(\alpha) = \alpha$. On the other hand:

$$\begin{aligned}
\rho \circ \sigma(p) &= \rho(\alpha_p) \\
&= \alpha_p(1_X) \\
&= F(1_X)(p) \\
&= p.
\end{aligned}$$

It follows that ρ and σ are inverses, hence ρ is a bijection. This proves the first statement.

For the second statement, apply the first statement to the functor $F = \mathrm{Hom}_{\mathfrak{C}}(-, X')$. Then the natural transformations from $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ to $\mathrm{Hom}_{\mathfrak{C}}(-, X')$ are in bijection with

$$F(X) = \mathrm{Hom}_{\mathfrak{C}}(X, X')$$

which is the content of the second statement. □

Corollary 3.2. The functor of points is well defined.

Proof. Let $\mathfrak{C}$ be the category of schemes. By the second statement of Theorem 3.1, for two schemes X, X' a morphism $X \to X'$ is uniquely specified by a natural transformation from h_X to $h_{X'}$, and vice versa. Hence the functor of points is faithful and therefore well defined. $\square$

We can improve Theorem 3.1. We will do this in the relative context. Let R be a commutative ring with identity and $S = \mathrm{Spec}(R)$ be the associated affine scheme. Suppose X is an S-scheme. We lose no generality since all schemes can be regarded as schemes over $\mathrm{Spec}(\mathbb{Z})$. The improvement is that h_X is determined by the restriction $\mathrm{Hom}_S(-, X)$ to affine S-schemes. This is really just a statement that schemes are build up from affine schemes. Note that a contravariant functor on the category of affine S-schemes is the same as a covariant functor on the category of R-algebras.

Corollary 3.3. If R is a commutative ring with identity, a scheme over R is determined by the restriction of its functor of points to affine schemes over R. In fact:

$$h : \mathfrak{R} - \mathfrak{Schemes} \to \mathfrak{Fun}(\mathfrak{R} - \mathfrak{algebras}, \mathfrak{Sets})$$

is an equivalence of the category of R-schemes with a full subcategory of the category of functors.

Proof. Write h'_X for the restricted functor $\mathrm{Hom}_R(-, X)$. Suppose X and X' are both R-schemes. It suffices to show that every natural transformation $\varphi : h'_X \to h'_{X'}$ is induced by a unique morphism $f : X \to X'$ in the sense of Theorem 3.1.

Let $\{U_\alpha\}$ be a cover of X by affine R-schemes. Let $\iota_\alpha \in h'_X(U_\alpha)$ be the inclusions. Then $\varphi(\iota_\alpha) \in h'_{X'}(U_\alpha)$. I claim that if $U_\alpha \cap U_\beta \neq \varnothing$, then

$$\varphi(\iota_\alpha)\big|_{U_\alpha \cap U_\beta} = \varphi(\iota_\beta)\big|_{U_\alpha \cap U_\beta}.$$

Observe that

$$\iota_\alpha\big|_{U_\alpha \cap U_\beta} = \iota_\beta\big|_{U_\alpha \cap U_\beta}.$$

Since restriction is a map of sets $h'_X(U_\alpha) \to h'_X(U_{\alpha \cap U_\beta})$ (and likewise for $h'_{X'}$) and since φ is a natural transformation:

$$\begin{aligned} \varphi(\iota_\alpha)\big|_{U_\alpha \cap U_\beta} &= \varphi\left(\iota_\alpha\big|_{U_\alpha \cap U_\beta}\right) \\ &= \varphi\left(\iota_\beta\big|_{U_\alpha \cap U_\beta}\right) \\ &= \varphi(\iota_\beta)\big|_{U_\alpha \cap U_\beta}. \end{aligned}$$

It follows that there is a unique morphism $f : X \to X'$ such that $f \circ \iota_\alpha = \varphi(\iota_\alpha)$. Since the ι_α's glue together to form the identity map, it follows that $f = \varphi(1_X)$. $\square$

3.2 Characterization of a Scheme among Functors

In many applications, a construction in the category of schemes is defined in terms of functors. It is then necessary to check that the resulting functor is the functor of points for some scheme. Here we give a criterion to check whether a given functor is induced by a scheme. Unfortunately some vocabulary is necessary before we can state and prove the theorem.

These concepts are useful in their own right, since they extend topological concepts to our category of functors. The first term we need to define is an open subfunctor. This definition requires some preliminary preparation.

Definition 3.4. A natural transformation $\alpha : G \to F$ of functors from a category $\mathfrak{C}$ to the category $\mathfrak{Sets}$ is **injective** if for every object X the induced map of sets $G(X) \to F(X)$ is injective. In this case we will say G is a **subfunctor** of F.

In particular, if U is an open subscheme of X then the natural transformation $\iota : h_U \to h_X$ induced by the inclusion by Theorem 3.1 is injective since the Hom functor is left exact. This example is our model for open subfunctors of functors from $\mathfrak{Rings}$ to $\mathfrak{Sets}$.

Since not every functor is associated to a scheme as a functor of points, we need to give a more general definition. This requires the definition of a fiber product of functors. This is piggy-backed on the fiber product in the category of sets.

Definition 3.5. If F, G, H are functors from a category $\mathfrak{C}$ to the category $\mathfrak{Sets}$ and if $g : G \to F$ and $h : H \to F$ are natural transformations, the **fiber product of** G **and** H **over** F, denoted $G \times_F H$, is the functor from $\mathfrak{C}$ to $\mathfrak{Sets}$ defined by setting, for any object Z of $\mathfrak{C}$:

$$(G \times_F H)(Z) = \{(x, y) \in G(Z) \times H(Z) \mid g(x) = h(y) \text{ in } F(Z)\}$$

and defined on morphisms in the obvious way.

Remark 3.6. It is clear that there is a natural isomorphism $G \times_F H \cong H \times_F G$.

We can now define an open subfunctor.

Definition 3.7. A subfunctor $\alpha : G \to F$ in $\mathfrak{Fun}(\mathfrak{Rings}, \mathfrak{Sets})$ is an **open subfunctor** if for each natural transformation $\psi : h_{\mathrm{Spec}(R)} \to F$ (i.e., by Theorem 3.1, each element $\psi \in F(R)$) the fiber product

$$\begin{tikzcd} G_\psi \arrow[r] \arrow[d] & h_{\mathrm{Spec}(R)} \arrow[d, "\psi"] \\ G \arrow[r, "\alpha"] & F \end{tikzcd}$$

(where G_ψ stands for $G \times_F h_{\mathrm{Spec}(R)}$) yields a natural transformation $G_\psi \to h_{\mathrm{Spec}(R)}$ that is isomorphic to the injection from the functor represented by some open subscheme of $\mathrm{Spec}(R)$.

Example 3.8. For a scheme X, the open subschemes of h_X are precisely the natural transformations $h_U \hookrightarrow h_X$ induced by the inclusion $\iota : U \hookrightarrow X$ for open subschemes $U \subset X$.

We also need to define an open cover of a functor.

Definition 3.9. Suppose F is a functor from $\mathfrak{Rings}$ to $\mathfrak{Sets}$ and that $\{\alpha_i : F_i \hookrightarrow F\}$ is a collection of open subfunctors. Then for every ring R and natural transformation $\eta : h_{\mathrm{Spec}(R)} \to F$ there is a collection of open subschemes $U_i \subset \mathrm{Spec}(R)$ such that $F_i \times_F h_{\mathrm{Spec}(R)} \cong h_{U_i}$. We say that the F_i's form an **open cover of** F if $\mathrm{Spec}(R)$ is covered by the open subschemes U_i for every ring R.

The following lemma will be useful.

Lemma 3.10. A collection of open subfunctors $\{\alpha_i : F_i \hookrightarrow F\}$ is an open cover if and only if for every field K we have

$$F(K) = \bigcup \alpha_i(F_i(K)).$$

Proof. We sketch the proof.

Suppose that our collection is an open cover. Let K be an arbitrary field and let $X = \mathrm{Spec}(K)$. It is clear that

$$\bigcup \alpha_i(F_i(K)) \subset F(K).$$

Therefore we only need to show the other inclusion. If $F(K) = \varnothing$ we are done. Otherwise suppose $\eta \in F(K)$. By Theorem 3.1, η corresponds to a natural transformation $h_X \to F$, which we will also call η. For each i we have $F_i \times_F h_X \cong h_{U_i}$ for some open subscheme $U_i \subset X$. Since X is the prime spectrum of a field, either $U_i = \varnothing$ or $U_i = X$ for each i. Since we have an open cover, there is some i such that $U_i = X$. This implies that there is a natural transformation $\eta' : h_X \to F_i$ such that the following diagram commutes:

$$\begin{tikzcd} & h_X \arrow[dl, "\eta'"'] \arrow[dr, "\eta"] & \\ F_i \arrow[rr, "\alpha_i"'] & & F \end{tikzcd}.$$

Again, by Theorem 3.1, η' corresponds to an element of $F_i(K)$, which we will also call η', such that $\alpha_i(\eta') = \eta$. This shows the other inclusion and completes the forward direction.

Now suppose our collection of open subfunctors satisfies

$$\bigcup \alpha_i(F_i(K)) = F(K) \tag{3.1}$$

for every field K. Let $X = \mathrm{Spec}(R)$ be an affine scheme. If $F(R)$ is empty, the definition of open cover is vacuous. So assume $F(R) \neq \varnothing$. Let $\eta \in F(R)$ (so that we have a natural transformation $\eta : h_X \to F$), and let $U_i \subset X$ be open subschemes such that $F_i \times_F h_X \cong h_{U_i}$ for each i. We must show that

$$X = \bigcup U_i.$$

One inclusion is obvious. For the other inclusion, let $p \in X$ be a point. Let $\kappa(p)$ be the residue field of p. Then for each i we have the following commutative diagram of sets:

$$\begin{tikzcd}
h_{U_i}(\mathrm{Spec}(\kappa(p))) \arrow[r] \arrow[d] & h_X(\mathrm{Spec}(\kappa(p))) \arrow[d, "\eta"] \\
F_i(\kappa(p)) \arrow[r, "\alpha_i"'] & F(\kappa(p)).
\end{tikzcd}$$

By definition of the fiber product of functors:

$$h_{U_i}(\kappa(p)) = \{(f, g) \in F_i(\kappa(p)) \times h_X(\mathrm{Spec}(\kappa(p))) \bigm| \alpha_i(f) = \eta(g).\}$$

Note that there is a sequence of ring homomorphisms

$$R \to \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p = \kappa(p).$$

This gives rise to a morphism of schemes $g : \mathrm{Spec}(\kappa(p)) \to X$. Hence $g \in h_X(\mathrm{Spec}(\kappa(p)))$. By equation (3.1) there exists some i and some $f \in F_i(\kappa(p))$ such that $\alpha_i(f) = \eta(g)$. Thus $(f, g) \in h_{U_i}(\mathrm{Spec}(\kappa(p)))$. This implies that if ι is the incluion $U_i \hookrightarrow X$, there is a morphism $g_i : \mathrm{Spec}(\kappa(p)) \to U_i$ such that $\iota g_i = g$. This implies that $p \in U_i$, completing the proof. $\square$

Finally, the other definition we need is the functorial notion of a sheaf.

Definition 3.11. A functor $F : \mathfrak{Rings} \to \mathfrak{Sets}$ is a **sheaf in the Zariski topology** if for each ring R and each open covering of $X = \mathrm{Spec}(R)$ by principal open affines $U_i = \mathrm{Spec}(R_{f_i})$, the functor F satisfies the sheaf axiom for the open covering: for each collection of elements $\alpha_i \in F(R_{f_i})$ such that α_i and α_j map to the same element in $F(R_{f_i f_j})$ (under the image of the natural maps $R_{f_i}, R_{f_j} \to R_{f_i f_j}$) there is a unique element $\alpha \in F(R)$ mapping to each of the α_i's (under the image of the natural localization map $R \to R_{f_i}$).

These definitions are the beginning of a larger story. For example, closed subfunctors can be defined in a manner similar to the definition of open subfunctors. Care must be taken with these definitions, since some very basic facts in an actual topology are no longer true in this context. See Section VI.1.1 of [EH00] for the caveats. A more general construction mimicking topology and defined on a category is called the Grothendieck Topology.

We can now state and prove the desired theorem. In what follows, if R is a ring, h_R is the functor of points associated with $\mathrm{Spec}(R)$.

Theorem 3.12. A functor $F : \mathfrak{Rings} \to \mathfrak{Sets}$ is of the form h_Y for some scheme Y if and only if

1. F is a sheaf in the Zariski topology; and
2. there exists rings R_i and open subfunctors
$$\alpha_i : h_{R_i} \to F$$
such that for every field K, $F(K)$ is the union of the images of $h_{R_i}(K)$ under the maps α_i.

Proof. We sketch the proof. The forward direction is obvious. The functor of points h_Y is a sheaf in the Zariski topology, and if $Y_i = \operatorname{Spec}(R_i)$ yields a cover of Y by open affines, we have an open cover of open subfunctors $\alpha_i : h_{Y_i} \hookrightarrow h_Y \cong F$. To show the converse, we must (i) construct the scheme Y, (ii) construct a natural transformation $\alpha : h_Y \to F$ and (iii) show α is an isomorphism.

First we use the open subfunctors to construct a scheme. Let $Y_i = \operatorname{Spec}(R_i)$. Since we have open subfunctors, in the following diagram:

$$\begin{tikzcd} h_{Y_i} \times_F h_{Y_j} \arrow[r] \arrow[d] & h_{Y_i} \arrow[d, "\alpha_i"] \\ h_{Y_j} \arrow[r, "\alpha_j"] & F \end{tikzcd}$$

there is an open subscheme $U_j \subset Y_j$ such that $h_{Y_i} \times_F h_{Y_j} \cong h_{U_j}$ and symmetrically there is an open subscheme $U_i \subset Y_i$ such that $h_{Y_j} \times_F h_{Y_i} \cong h_{U_i}$. Hence by Remark 3.6 $h_{U_i} \cong h_{U_j}$ so by Theorem 3.1 we have a unique isomorphism $\varphi_{ij} : U_i \cong U_j$ that carries the natural transformation $h_{U_i} \hookrightarrow h_{Y_i} \to F$ to the natural transformation $h_{U_j} \hookrightarrow h_{Y_j} \to F$. One then checks that these isomorphism are compatible in the sense that

1. for $i \neq j$, $\varphi_{ij} = \varphi_{ji}^{-1}$, and
2. for i, j, k pairwise distinct, on the appropriate intersections $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$.

It follows that we can glue the Y_i's together to form a scheme Y. See Exercise 2.12 in Chapter II of [Har77] or Section 1.2.4 of [EH00].

Next we use the first condition to glue the natural transformations α_i together. First, by the construction in Corollary 3.3 the functor F induces uniquely a contravariant functor from the category of schemes to the category of sets. We will call this functor F. Definition 3.11 extends naturally by the gluing procedure. Since principal opens form a basis for the topology of affine schemes, we can cover each Y_i by principal opens, and we can assume each U_{ij} is principal. By Theorem 3.1, each α_i corresponds to an element (which will carry the same symbol) $\alpha_i \in F(Y_i)$. By construction, α_i and α_j map to the same element $\alpha_{ij} \in F(U_{ij})$. Hence since F is a sheaf in the Zariski topology we can glue the natural transformations together to obtain a natural transformation $\alpha : h_Y \to F$.

Finally, we use the last part of the second condition to show that α is an isomorphism. If $\alpha : h_Y(S) \to F(S)$ is a bijection for each ring S, one can use abstract nonsense to show that the collection of inverses $\alpha^{-1} : F(S) \to h_Y(S)$ is a natural transformation that is an inverse for α, and thus α is an isomorphism. It is clear from the construction that α is injective. It therefore suffices to show that for each S, the induced map of sets is surjective. Let S be a ring and let $X = \mathrm{Spec}(S)$. If $F(S) = \varnothing$ there is nothing to show. Accordingly, suppose $\eta \in F(S)$. By Lemma 3.10, the last part of the second condition implies that the functors h_{Y_i} form an open cover of F. Hence using η as a natural transformation $h_X \to S$ we have a cover of open subschemes $\iota_i : V_i \hookrightarrow X$ such that $h_{Y_i} \times_F h_X \cong h_{V_i}$. The fiber product induces a natural transformation $\eta_i : h_{V_i} \to h_{Y_i}$ such that $\alpha_i \eta_i = \eta \iota_i$. By Theorem 3.1 we view $\eta_i \in h_{Y_i}(V_i)$, i.e. as a morphism $V_i \to Y_i$. Next one checks compatibility, and concludes that we can glue the η_i's to obtain a morphism $X \to Y$, i.e. an element of $h_Y(X)$. The image of this morphism under α is, by construction, precisely the original element $\eta \in F(S)$. Hence the maps of sets induced by α are surjective, therefore bijective, and hence α is an isomorphism. Thus F is represented by the scheme Y. $\square$

As an application, one can use Theorem 3.12 to show that the Grassmanian functor

$$g(k, n) : \mathfrak{Rings} \to \mathfrak{Sets}$$

defined by

$$g(k,n)(T) = \{\text{rank } \ k \ \text{ direct summands of } \ T^n\}$$

is representable. As one would expect, it is the functor of points of the Grassmanian scheme $G_{\mathbb{Z}}(k,n)$. See Exercise VI-18 of [EH00] for a sketch of the argument.

4 Applications

4.1 Fiber Products and Group Schemes

Our first application of the functor of points is to describe the fiber product of schemes. First we will review the basic construction. In the process, we will see that there isn't a satisfying way of relating the fiber product of two schemes to the fiber product of the underlying point sets. The functor of points will remedy this problem to some extent. The functor of points also provides a more elegant way to define the fiber product of two schemes. Finally, we will give a sensible definition of a group scheme.

Begin with the affine case. Suppose $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$. Suppose further that there are morphisms of affine schemes $f : X \to S$ and $g : Y \to S$. These morphisms induce ring homomorphisms $f^* : R \to A$ and $g^* : R \to B$ making A and B intro R-algebras. This allows us to construct the tensor product $A \otimes_R B$. Define

$$X \times_S Y := \operatorname{Spec}(A \otimes_R B).$$

It is easy to see using the correspondence between affine scheme morphisms and ring homomorphisms along with the universal property of tensor products that this yields a fiber product.

Now suppose X, Y and S are schemes with morphisms $f : X \to S$ and $g : Y \to S$. Cover S with open affines $\operatorname{Spec}(R_\rho)$. Then $f^{-1}(\operatorname{Spec}(R_\rho))$ and $g^{-1}(\operatorname{Spec}(R_\rho))$ are open subsets of X, and therefore are open subschemes. See Exercise 2.2 in Chapter II of [Har77]. Accordingly, cover $f^{-1}(\operatorname{Spec}(R_\rho))$ with open affines $\operatorname{Spec}(A_{\rho\alpha})$ and cover $g^{-1}(\operatorname{Spec}(R_\rho))$ with open affines $\operatorname{Spec}(B_{\rho\beta})$.

Then for each ρ, α and β we have by construction the restricted morphisms

$$\begin{aligned} f_{\rho\alpha} : \mathrm{Spec}(A_{\rho\alpha}) &\rightarrow \mathrm{Spec}(R_\rho) \\ g_{\rho\beta} : \mathrm{Spec}(B_{\rho\beta}) &\rightarrow \mathrm{Spec}(R_\rho). \end{aligned}$$

This yields a fiber product

$$Z_{\rho\alpha\beta} := \mathrm{Spec}(A_{\rho\alpha} \otimes_{R_\rho} B_{\rho\beta}).$$

With some tedium, one can check that these schemes agree on overlaps and can therefore be glued together to from a scheme Z, which satisfies the universal property of the fiber product in the category of schemes.

The functor of points provides a more elegant, although perhaps less constructive, proof that the fiber product exists in the category of schemes. First one shows that if F, G and H are functors from the category of rings to the category of sets and all are sheaves in the Zariski topology, and if we have natural transformations $G \to F$ and $H \to F$, then the fiber product $G \times_F H$ is a sheaf in the Zariski topology. With this lemma and Theorem 3.12, one can show that if X, Y and S are schemes and there are morphisms $X \to S$, $Y \to S$, then the fiber product $h_X \times_{h_S} h_Y$ is representable. Finally, using Theorem 3.1 this implies that the scheme representing this functor satisfies the universal property of the fiber product in the category of schemes. By uniqueness of universal objects, this yields a scheme isomorphic to the scheme constructed above.

Note that a fiber product over a terminal object is a product. In the category of schemes, $\mathrm{Spec}(\mathbb{Z})$ is a terminal object. Hence given any two schemes X and Y, we can form a product by taking their fiber product over $\mathbb{Z}$. If S is a scheme, in the category of S-schemes any fiber product over S is a product.

Let us consider two examples to see what can go wrong with trying to associate the fiber products of schemes with the fiber products of the underlying point sets.

Example 4.1. Let $m, n \in \mathbb{Z}$ be relatively prime and let $A = \mathbb{Z}/m\mathbb{Z}$, $B = \mathbb{Z}/n\mathbb{Z}$ and $R = \mathbb{Z}$. Let $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Then

$$X \times_S Y = \mathrm{Spec}(A \otimes_R B) = \mathrm{Spec}(0) = \varnothing.$$

This certainly appears to be a strange result. Since the fiber product over $\mathrm{Spec}(\mathbb{Z})$ is the product in the category of schemes, the product is empty. The product of the underlying point sets is bijective with the set of ordered pairs of primes, the first dividing m and the second dividing n. On the other hand, the fiber product of the underlying sets is empty since each such prime p maps to itself in $\mathbb{Z}$ and the primes of m and n are distinct. This apparent pathology merely manifests the fact that the point set underlying the terminal object in the category of schemes is not the same as the terminal object in the category of sets.

Example 4.2. Let $A = \mathbb{C}$, $B = \mathbb{C}$ and $R = \mathbb{R}$, with $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$. Then:

$$X \times_S Y = \mathrm{Spec}(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathrm{Spec}(\mathbb{C}) \coprod \mathrm{Spec}(\mathbb{C})$$

where the coproduct symbol stands for disjoint union. Therefore the underlying point set of this fiber product is a pair of disjoint points. On the other hand, the fiber product of the underlying sets is a single ordered pair (p_1, p_2) where p_i is the singleton set underlying the i^{th} copy of $\mathrm{Spec}(\mathbb{C})$. There is no meaningful way to relate these two fiber products.

The functor of points remedies these problems to some extent. This is the content of the next proposition.

Proposition 4.3. Let X, Y and K be schemes over a scheme S. Then the K-valued points of $X \times_S Y$ (over S) is canonically in bijection with the product of the K-valued points of X/S with Y/S.

Proof. We have to show that for every pair $(\varphi, \psi) \in h_X(K) \times h_Y(K)$, there is canonically a unique $\gamma \in h_{X \times_S Y}(K)$ (using the relative functors of points), and vice versa. Since S is the terminal object in the category of S-schemes, there are unique morphisms

$$\begin{aligned} \sigma : K &\rightarrow S \\ \mu : X &\rightarrow S \\ \nu : Y &\rightarrow S. \end{aligned}$$

Therefore, for any pair of morphisms $\varphi : K \rightarrow X$ and $\psi : K \rightarrow Y$ we have

$$\mu\varphi = \sigma = \nu\psi.$$

Accordingly, the universal property of fiber products gives a canonical morphism $\gamma : K \to X \times_S Y$. The converse is obvious. $\square$

This proposition shows that taking the functor of points and evaluating at some scheme K, we can consider the product of the functors to be the functor of the product. This allows us to use the product construction in the category of sets in place of the construction used in the category of schemes.

In addition, the proposition makes it easier to define group schemes. Ideally, one would like to define a group scheme to be a scheme G with a group law that is compatible with its scheme structure. The problem is that the definition of a binary operation requires using the product, hence in the category of schemes we are not able to use the product of the underlying point-sets. The construction can be done, but the definition is not pleasant. It can be found, for example, in the first section of [MFK94].

On the other hand, using the functor of points, it suffices to find a group law on the set $\mathrm{Hom}(X, G)$ for each scheme X. It is easy to see that this is equivalent to the following definition found in section VI.1.4 of [EH00].

Definition 4.4. A **group scheme** is a scheme G together with a factorization of the functor $h_G : \mathfrak{Rings} \to \mathfrak{Sets}$ through the forgetful functor $\mathfrak{Groups} \to \mathfrak{Sets}$.

Example 4.5. One can define GL_n as the affine scheme of invertible $n \times n$ matrices

$$\mathrm{Spec}(\mathbb{Z}[x_{ij}][\det(x_{ij})^{-1}]).$$

However, this is typically considered as the functor that associates to each ring T the group $\mathrm{GL}_n(T)$. This functor is representable and therefore gives a unique scheme, and since the T-valued points of this functor form a group for each T, we have a group scheme.

4.2 The Tangent Bundle to a Scheme

Fix a field K. Let $R = K[\epsilon]/(\epsilon^2)$ and let $Y = \mathrm{Spec}(R)$. The ring R is a local ring with maximal ideal (ϵ). Since ϵ is irreducible, the underlying point-set of Y consists solely of the point represented by the ideal (ϵ). The scheme Y is sometimes considered the scheme of tangent directions. In this subsection we will use the functor of points to make this precise, and we will also be able to associate Y to tangent bundles of schemes.

Let X be a scheme over K. Recall that for any K-rational point $p \in X$, the Zariski tangent space is

$$(\mathfrak{m}_p/\mathfrak{m}_p^2)^* := \mathrm{Hom}_K(\mathfrak{m}_p/\mathfrak{m}_p^2, K)$$

where $\mathfrak{m}_p$ is the maximal ideal in the local ring $\mathcal{O}_{X,p}$. We now have the following proposition.

Proposition 4.6. A Y-valued point of X is the same as a closed K-rational point of X together with an element of the Zariski tangent space to X at p.

Proof. First suppose $f \in h_X(Y)$. Evaluation at zero gives a ring homomorphism $R \to K$ which induces the inclusion

$$\begin{aligned} \iota : \mathrm{Spec}(K) &\rightarrow Y \\ (0) &\mapsto (\epsilon). \end{aligned}$$

It follows that $f \in h_X(Y)$ gives us $f' \in h_X(K)$ defined by $f' = f \circ \iota$. Then by Proposition 2.11 f' gives us a closed Y-rational point $p \in X$. It remains to construct an element of the tangent space. The morphism f' induces a local homomorphism $(f')^\sharp$ on local rings

$$(f')^\sharp : \mathcal{O}_{X,p} \to \mathcal{O}_{\mathrm{Spec}(K),(0)} = K.$$

This factors as $(f')^\sharp = \iota^\sharp \circ f^\sharp$. Therefore we have

$$f^\sharp : \mathcal{O}_{X,p} \to \mathcal{O}_{Y,(\epsilon)}.$$

In particular, $f^\sharp(\mathfrak{m}_p) = (\epsilon)$. Forget the ring structure and consider maps of K-vector spaces. Since $(\epsilon^2) = 0$ in R, $f^\sharp$ factors through a map

$$t : \mathfrak{m}_p/\mathfrak{m}_p^2 \to (\epsilon) \cong K.$$

Then t is an element of the Zariski tangent space to X at p. Therefore we associate the Y-valued point f of X to the pair (p, t) constructed above.

Conversely suppose we have a closed K-rational point $p \in X$ and an element of the Zariski tangent space $t \in (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ identified as a map

$$t : \mathfrak{m}_p/\mathfrak{m}_p^2 \to (\epsilon) \cong K.$$

On the level of the local ring, since p is a closed K-rational point we can consider the elements of $\mathcal{O}_{X,p}$ as germs of algebraic functions at p to K, with $\mathfrak{m}_p$ the ideal of those functions vanishing at p. We then have a K-algebra map:

$$\begin{aligned} \iota : K &\hookrightarrow \mathcal{O}_{X,p} \\ a &\mapsto (q \mapsto a). \end{aligned}$$

In addition, since p is a K-rational point, by Proposition 2.11 there is a morphism $\mathrm{Spec}(K) \to X$ inducing a local K-algebra map $\pi : \mathcal{O}_{X,p} \to K$ such that $\pi\iota = \mathrm{Id}_K$. We can extend ι by composing with the quotient map $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/\mathfrak{m}_p^2$. Since π is a local homomorphism, $\mathfrak{m}_p = \pi^{-1}(0)$ and hence $\pi(\mathfrak{m}_p^2) = 0$, thus π descends to a map $\mathcal{O}_{X,p}/\mathfrak{m}_p^2 \to K$. The relation $\pi\iota = \mathrm{Id}_K$ still holds when we replace $\mathcal{O}_{X,p}$ with $\mathcal{O}_{X,p}/\mathfrak{m}_p^2$. Finally we define the following K-vector space homomorphism:

$$\begin{aligned} \mathcal{O}/\mathfrak{m}_p^2 &\to \mathfrak{m}_p/\mathfrak{m}_p^2 \\ f &\mapsto f - f(p) \end{aligned}$$

Forgetting the ring structure, we have a short exact sequence of K-vector spaces:

$$0 \to K \to \mathcal{O}_{X,p}/\mathfrak{m}_p^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2 \to 0.$$

Short exact sequences of vector spaces split, and the map π gives a choice of isomorphism:

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 \cong K \oplus \mathfrak{m}_p/\mathfrak{m}_p^2.$$

Pairing the inclusion $K \hookrightarrow K[\epsilon]/(\epsilon^2)$ on the first factor with t on the second factor, we have a K-algebra homomorphism

$$\mathcal{O}_{X,p}/\mathfrak{m}_p^2 \to K[\epsilon]/(\epsilon^2).$$

Precomposing with the projection gives us a K-algebra homomorphism

$$\mathcal{O}_{X,p} \to K[\epsilon]/(\epsilon^2).$$

This is sufficient to specify a morphism of K-schemes $Y \to X$ and hence a Y-valued point. $\square$

We have established a bijective correspondence between Y-valued points with pairs of closed K-rational points of X and Zariski tangent vectors to X at p. In this manner we can identify the Y-valued points with the tangent bundle over the K-rational points. If every point is K-rational, we have identified Y with the tangent bundle of X. More generally, the covariant functor $\mathrm{Hom}_K(Y, -)$ restricted to K-schemes all of whose points are K-rational is identified with the tangent bundle functor.

4.3 Parameter and Moduli Spaces

The functor of points has arguably had the greatest impact in the study of parameter spaces and moduli spaces. When one has a family of geometric objects one would like to identify and study the parameter or moduli space of such a family. One way to do so is to apply the functor of points. Then one has a family of functors, and one wants to identify this family itself as a functor. Then one asks whether the new functor is representable. In what follows, in the interests of economy, I will think of parameter spaces as moduli spaces with a trivial equivalence and hence only refer to moduli spaces.

Using the language of the functor of points we can define what it means to have a moduli space. Typically a class of objects is defined over an arbitrary scheme B. This class of objects could be along the lines of projective schemes or smooth curves, for example. Then one defines what it means to have a family $S(B)$ of such objects. For example, $S(B)$ may be all projective schemes with a fixed Hilbert polynomial, or it may be all curves over B of fixed genus g. Finally, we define an equivalence relation on the families $S(B)$ (which may be trivial if we want to consider parameter spaces). This then allows us to define the moduli functor $F : \mathfrak{Schemes}^\circ \to \mathfrak{Sets}$ by

$$F(B) = S(B)/\sim .$$

Usually one thinks of the objects in the desired moduli space to be elements of $S(\mathrm{Spec}(\mathbb{C}))$ and the elements $S(B)$ to be families of objects parameterized by the complex points of B.

Ideally we would like to identify the moduli space as a scheme in its own right. That is, we would like to find a fine moduli space, defined as follows.

Definition 4.7. Suppose a moduli functor F is representable, and the scheme M represents F. Then we say that M is a **fine moduli space** for the moduli problem defined by F.

Example 4.8. Fix a field K. We want to consider the Hilbert scheme $\mathscr{H}_P$, the moduli space of subschemes of projective n-space with a fixed Hilbert polynomial P (with a trivial equivalence). We define the Hilbert functor h_P (called the "functor of flat families of schemes in $\mathbb{P}^n_K$ with Hilbert polynomial P) to be the functor

$$h_P : \mathfrak{K} - \mathfrak{Schemes}^{\mathrm{o}} \to \mathfrak{sets}$$

given by $h_P(B)$ is the set of subschemes $\mathscr{X} \subset \mathbb{P}^n_K \times B$ flat over B whose fibers over points of B have Hilbert polynomial P. Then h_P is representable and is represented by a scheme $\mathscr{H}_P$, which is called the Hilbert scheme. This is a fine moduli space. For a proof, see Theorem 1.9 in [HM98].

Remark 4.9. In part, Murphy's Law in Algebraic Geometry states that every type of singularity, no matter how bad, occurs in some Hilbert scheme. See [Vak06].

Example 4.10. Let F be the functor where $F(B)$ is the set of smooth flat morphisms $C \to B$ whose fibers are smooth curves of genus g, with the obvious equivalence relation. Then F is not representable. In other words, the moduli space of genus g curves does not exist as a fine moduli space. See Chapter 2, section A of [HM98].

When a fine moduli space fails to exist, there are several directions in which to proceed. One direction involves enlarging the category from schemes to something along the lines of algebraic stacks. I will not consider this option here. Instead, we relax the requirements that we are looking for. Instead of demanding a scheme $\mathcal{M}$ and an isomorphism $F \cong h_{\mathcal{M}}$, we simply demand a natural transformation. This gives us what is called a coarse moduli space.

Definition 4.11. A scheme $\mathcal{M}$ and a natural transformation $\Psi_{\mathcal{M}}$ from the functor F to the functor of points $h_{\mathcal{M}}$ are a **coarse** moduli space for the functor F if

1. The map $\Psi_{\mathrm{Spec}(\mathbb{C})} : F(\mathrm{Spec}(\mathbb{C})) \to h_{\mathcal{M}}(\mathrm{Spec}(\mathbb{C}))$ is a set bijection, and
2. Given another scheme $\mathcal{M}'$ and a natural transformation $\Psi_{\mathcal{M}'}$ from F to $h_{\mathcal{M}'}$, there is a unique morphism $\pi : \mathcal{M} \to \mathcal{M}'$ such that the associated natural transformation $\Pi : h_{\mathcal{M}} \to h_{\mathcal{M}'}$ satisfies $\psi_{\mathcal{M}'} = \Pi \circ \Psi_{\mathcal{M}}$.

The first condition in the definition rules out certain trivial solutions which would be of little use, and the second condition guarantees that a coarse moduli space is unique up to a canonical isomorphism.

Example 4.12. There is a scheme $\mathcal{M}_g$ such that $(\mathcal{M}_g, F)$ is a coarse moduli space for the functor F in example 4.10. See Chapter 2 of [HM98].

5 Conclusion

Schemes present a fairly mysterious world with several pathologies. Some of these pathologies are removed by applying the functor of points. This allows certain constructions, such as fiber products, to be done in the category of sets consistent with the structures in the category of schemes. As such, the functors of points helps to place schemes back on the ground, allowing one to work with point-sets. The concept has several applications. In particular, the study of moduli spaces is an area of active research which is couched in the vocabulary of the functor of points.

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