More on Grassmannians

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1 Introduction

Grassmannians are fundamental objects in algebraic geometry: they are simultaneously objects of interests in their own right and basic tools in the construction and study of other varieties. In this paper, we will fix an algebraically closed field $\mathbf{k}$ and assume all varieties and vector spaces are defined over $\mathbf{k}$ if not mentioned explicitly.

For any natural numbers k and n, we define $Gr(k,n)$ to be the set of all k-dimensional subspaces of $\mathbf{k}^n$. More generally, for any vector space V, we define $Gr(k,V)$ to be the set of all k-dimensional subspaces of V. When $k = 1$, the Grassmannian $Gr(k,n)$ or $Gr(k,V)$ are just the projective space $\mathbf{P}^n$ or $\mathbf{P}(V)$. Since a k-dimensional subspace of V is naturally identified with a $(k-1)$-plane in $\mathbf{P}(V)$, we can also think of $Gr(k,n)$ or $Gr(k,V)$ as $(k-1)$-planes in $\mathbf{P}^n$ or $\mathbf{P}(V)$; and when we think of the Grassmannian in this way, we will write it as $\mathbf{G}(k-1,n-1)$ or $\mathbf{G}(k-1,\mathbf{P}(V))$.

In this paper, we first describe the variety structure of the Grassmannians in a different way from what we did in class, i.e., by giving explicit affine open covers, which is essentially the same as what we learned from differential geometry course when we describe the Grassmannians as manifolds. This can be found in [2], [4] and [3]. Next, we talk about a natural generalization of the homogeneous coordinates on projective spaces, which we call Stiefel coordinates on Grassmannians, and discuss the relation between Stiefel coordinates and Plücker coordinates. We also go further by studying the Stiefel coordinate rings, which are analogs to homogeneous coordinate rings. This part comes from [2] mostly. Then we look at the Grassmannians in different point of views: as homogeneous spaces and as moduli spaces. This part comes from [1] and [4]. Finally, we introduce a number of other varieties related to Grassmannians. This part mostly comes from [4].

2 The Variety Structure

So far, a Grassmannian is just a set. It can be given a natural variety structure in the following way.

Let Λ be a k-dimensional subspace of $\mathbf{k}^n$, or equivalently, an element of $Gr(k,n)$, then Λ is determined by k linearly independent vectors $v_i = (a_{i,1}, \dots, a_{i,n})$, $i = 1, \dots, k$ in Λ, hence is represented by a matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots \\ a_{k,1} & a_{k,2} & \dots & a_{k,n} \end{pmatrix}$$

where the ith row is the vector v_i. The condition that the rows are linearly independent is equivalent to the matrix is of full rank, or equivalently, at least one $k \times k$ submatrix of the matrix is invertible. Two such matrices represent the same element of $Gr(k,n)$ if and only if they defer by a left multiplication of an invertible $k \times k$ matrix. Here we introduce a notation for submatrix: for any $k \times n$ matrix A, we write $A_{i_1,\dots,i_k}$ for the submatrix of A formed by columns $i_1, \dots, i_k$, and call it the $(i_1, \dots, i_k)$th submatrix:

$$A_{i_1,\dots,i_k} = \begin{pmatrix} a_{1,i_1} & \dots & a_{1,i_k} \\ \dots & \dots & \dots \\ a_{k,i_1} & \dots & a_{k,i_k} \end{pmatrix}$$

To make notations simpler, we write $A_{\underline{i}}$ for $A_{i_1,\dots,i_k}$, and call it the $\underline{i}$th submatrix of A, where $\underline{i}$ denote the multi-index $(i_1, \dots, i_k)$.

If Λ has a matrix representative whose $\underline{i}$th submatrix is invertible, then any other matrix representative of Λ has the same property, i.e., the $\underline{i}$th submatrix is invertible. Now choose a matrix representative A of Λ with $A_{\underline{i}}$ being invertible, then we can "normalize" the representative by setting

$$B = A_{\underline{i}}^{-1} A$$

whose $\underline{i}$th submatrix is I. Suppose after deleting the $\underline{i}$th submatrix from B, we get a $k \times (n-k)$ submatrix of B:

$$B_{[n]-\underline{i}} = \begin{pmatrix} b_{1,1} & \dots & b_{1,n-k} \\ \dots & \dots & \dots \\ b_{k,1} & \dots & b_{k,n-k} \end{pmatrix}$$

where $[n] = (1, 2, \dots, n)$ and $[n] - \underline{i}$ denotes the multi-index formed by deleting $\underline{i}$ from $[n]$. These $b_{i,j}$'s can be thought of as local coordinates for Λ. Thus we can cover $Gr(k, n)$ by affine open subsets and make it into an abstract variety.

To be more precise, we will proceed as follows. We need to give a Zariski topology to $Gr(k, n)$ first. We identify the set of all $k \times n$ matrices with the affine space $\mathbf{A}^{kn}$. Let $S(k, n) \subseteq \mathbf{A}^{kn}$ be the subset of all $k \times n$ matrices of full rank, then $S(k, n)$ is as an open subvariety of $\mathbf{A}^{kn}$, which we call *Steifel Variety.* Then we get a map $\pi : S(k, n) \to Gr(k, n)$ by sending any full rank matrix to the subspace of $\mathbf{k}^n$ that the matrix represents. This map π is surjective, and hence induces a quotient topology on $Gr(k, n)$, which we call the Zariski topology.

For $1 \leq i_1 < \cdots < i_k \leq n$, let $U_{\underline{i}} \subseteq S(k, n)$ be the open subset

$$U_{\underline{i}} = \{A \in S(k, n) \mid \det A_{\underline{i}} \neq 0\}$$

Then $\overline{U}_{\underline{i}} := \pi(U_{\underline{i}})$ are open subsets of $Gr(k, n)$, which form an open cover. We have bijective maps

$$\begin{aligned} \varphi_{\underline{i}} : \overline{U}_{\underline{i}} &\longrightarrow \mathbf{A}^{k(n-k)} \\ \pi(A) &\mapsto (A_{\underline{i}}^{-1} A)_{[n]-\underline{i}} \end{aligned}$$

which give $\overline{U}_{\underline{i}}$ structure of affine varieties (actually affine spaces). We need to check the compatibility of the structures, i.e., we need to check that the maps

$$\varphi_{\underline{i}} \circ \varphi_{\underline{j}}^{-1} : \varphi_{\underline{j}}(\overline{U}_{\underline{i}} \cap \overline{U}_{\underline{j}}) \to \varphi_{\underline{i}}(\overline{U}_{\underline{i}} \cap \overline{U}_{\underline{j}})$$

are morphisms of varieties. Let $B = (b_{i,j}) \in \varphi_{\underline{j}}(\overline{U}_{\underline{i}} \cap \overline{U}_{\underline{j}})$. Then $\varphi_{\underline{j}}^{-1}(B)$ is represented by a $k \times n$ matrix $\widetilde{B}$ whose $(\underline{j})$th submatrix is identity while the rest columns form the matrix B. Since $\varphi_{\underline{j}}^{-1}(B)$ is also in $\overline{U}_{\underline{i}}$, we know $\widetilde{B}_{\underline{i}}$ is also invertible, and

$$\varphi_{\underline{i}}(\varphi_{\underline{j}}^{-1}(B)) = (\widetilde{B}_{\underline{i}}^{-1} \widetilde{B})_{[n]-\underline{i}}$$

So $\varphi_{\underline{i}} \circ \varphi_{\underline{j}}^{-1}$ is a composition of the following maps: $\mathbf{A}^{k(n-k)} \to U_{\underline{j}} : B \mapsto \widetilde{B}$, $U_{\underline{i}} \to U_{\underline{i}} : A \to A_{\underline{i}}^{-1}A$ and $U_{\underline{i}} \to \mathbf{A}^{k(n-k)} : A \mapsto A_{[n]-\underline{i}}$, each of which is regular. Hence $\varphi_{\underline{i}} \circ \varphi_{\underline{j}}^{-1}$ is regular.

The compatibility of the variety structure on these open sets allows us to define regular functions on $Gr(k,n)$. For any open set $U \subseteq Gr(k,n)$, we have $U = \bigcup_{\underline{i}}(U \cap \overline{U}_{\underline{i}})$, and we can define regular functions on each $U \cap \overline{U}_{\underline{i}}$. By compatibility, these functions glue together to regular functions on U. So $Gr(k,n)$ has a structure of variety. Indeed, we can conclude

Theorem 2.1. *The Grassmannian $Gr(k,n)$ is a smooth, irreducible, projective algebraic variety of dimension $k(n-k)$ with an open cover by $\binom{n}{k}$ affine spaces.*

Since $\pi^{-1}(\pi(A))$ is exactly the orbit of A in $S(k,n)$ under the action of $\mathrm{GL}(k)$, we have $Gr(k,n) = S(k,n)/\,\mathrm{GL}(k)$. This generalizes the representation of $\mathbf{P}^n$ as a quotient $(\mathbf{k}^{n+1} \setminus 0)/\mathbf{k}^{\times}$.

3 Stiefel Coordinates

In the case $k = 1$, $Gr(1,n)$ is the projective space $\mathbf{P}^{n-1}$, and any point Λ of $Gr(1,n)$ is represented by a $1 \times n$ matrix, whose entries are just the homogeneous coordinates of the point Λ. So for general k, we can think of the entries of the $k \times n$ matrix that represents a point of $Gr(k,n)$ as certain generalized homogeneous coordinates, which we will call *Steifel coordinates.* As with the homogeneous coordinates on a projective space, the Stiefel coordinates on a Grassmannian are not unique. Indeed, $A, B \in S(k,n)$ represent the same point of $Gr(k,n)$ if and only if $A = gB$ for some $g \in \mathrm{GL}(k)$ (the analog of a constant factor for usual homogeneous coordinates). The Stiefel coordinates of a point p in $Gr(k,n)$ are essentially the affine coordinates of a point in $S(k,n)$ above p.

We now compare Stiefel coordinates and Plücker coordinates. Recall that the Pücker coordinates are defined through the Pücker embedding:

$$Gr(k,n) \hookrightarrow \mathbf{P}\Big(\bigwedge^k \mathbf{k}^n\Big) : \Lambda \mapsto [v_1 \wedge \cdots \wedge v_k]$$

where $v_1, \dots, v_k$ span Λ. Write in terms of the standard basis of $\bigwedge^k \mathbf{k}^n$:

$$v_1 \wedge \cdots \wedge v_k = \sum_{1 \le i_1 < \cdots < i_k \le n} p_{i_1,\dots,i_k} e_{i_1} \wedge \cdots \wedge e_{i_k}$$

The coefficients $p_{\underline{i}} := p_{i_1,\dots,i_k}$ are called *Plücker* coordinates of Λ. Using the standard basis of $\bigwedge^k \mathbf{k}^n$, we have an identification $\mathbf{P}\big(\bigwedge^k \mathbf{k}^n\big) \cong \mathbf{P}^N$ where $N = \binom{n}{k}$, and the Plücker embedding is

$$Gr(k,n) \hookrightarrow \mathbf{P}^N : \Lambda \mapsto [p_{[k]} : \cdots : p_{\underline{i}} : \cdots : p_{[n]-[k]}]$$

Let A be the $k \times n$ matrix whose rows are $v_1, \dots, v_n$, then $\Lambda \in Gr(k,n)$ is represented by A, and the Plücker coordinates are just all the $k \times k$ minors of A: $p_{\underline{i}} = \det A_{\underline{i}}$ for all $\underline{i}$. Moreover, the Plücker relations become identities in the Stiefel coordiantes that all the $k \times k$ minors satisfy. Consider the affine open set $\overline{U}_{1,2,\dots,k}$. Any point in it is represented by a matrix A of the form

$$A = (I, B) = \begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & \cdots & b_{2,k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & \cdots & b_{k,k} \end{pmatrix}$$

Then we have $p_{1,2,\dots,k} = 1$ and all other Plücker coordinates appear as the minors of the matrix $B = (b_{i,j})$ of all sizes. In particular, the expansion of these determinants along any row or column yields a quadratic relation among these minors. In this way, we can write down all the Plücker relations on $Gr(k,n)$ explicitly.

Conversely, if we are given Plücker coordinates, then we can reconstruct the Stiefel coordinates. We take $Gr(2,4)$ as an example to illustrate this. Suppose we are given $p_{i,j}$ for $1 \le i < j \le 4$ with $p_{1,2} = 1$. Thus we can choose $A_{1,2} = I$ and we get

$$A = \begin{pmatrix} 1 & 0 & -p_{2,3} & -p_{2,4} \\ 0 & 1 & p_{1,3} & p_{1,4} \end{pmatrix}$$

The equality $p_{3,4} = \det A_{3,4}$ is guaranteed by the Plücker relation $p_{1,2}p_{3,4} - p_{1,3}p_{2,4} + p_{1,4}p_{2,3} = 0$ on $Gr(2,4)$.

4 Stiefel Cooridnate Rings

Stiefel coordinates and Plücker coordinates can be viewed as two possible generalizations of homogeneous coordinates on projective spaces, yet in two different directions. The Stiefel coordinates are more closer to the homogeneous coordinates in spirit.

Let $x_{i,j}(1 \leq i \leq k,\ 1 \leq j \leq n)$ be $k \cdot n$ indeterminants, viewed as entries of an indeterminant matrix $\mathfrak{X} = (x_{i,j})$. For convenience, we write $F(\mathfrak{X})$ for any polynomial F in $x_{i,j}$'s and $k[\mathfrak{X}]$ for the polynomial ring $k[\, x_{i,j} \,|\, 1 \leq i \leq k,\ 1 \leq j \leq n]$. This ring will be the analog of the homogeneous polynomial ring for a projective space. Since Stiefel coordinates, viewed as entries of a $k \times n$ matrix, are determined up to a left multiplication by an invertible $k \times k$ matrix, any equation $F(\mathfrak{X}) = 0$ with $F \in k[\mathfrak{X}]$ defines a closed subset of $Gr(k, n)$ if F satisfies

$$F(g\mathfrak{X}) = C(g)F(\mathfrak{X}),\ \forall\ g \in \mathrm{GL}(k)$$

where C is a non-vanishing function on $\mathrm{GL}(k)$. Let $\mathcal{A}$ denote the k-algebra generated by all the polynomials satisfying the above property. Then we have the following

Theorem 4.1. *Any polynomial F satisfying the above property can be written as a homogeneous polynomial f in all the $k \times k$ minors of $\mathfrak{X}$, and $C(g) = (\det g)^d$ where $d = \deg f$. In other words, $\mathcal{A}$ is a k-algebra generated by all the $k \times k$ minors of $\mathfrak{X}$.*

This is essentially the *First Fundamental Theorem of Invariant Theory*. We see that any homogeneous polynomial of $\mathcal{A}$ must be of degree kd for some d. So $\mathcal{A}$ is naturally a graded k-algebra:

$$\mathcal{A} = \bigoplus_{d=0}^{\infty} \mathcal{A}_d, \quad \mathcal{A}_d = \mathcal{A} \cap k[\mathfrak{X}]_{kd}.$$

We will call $\mathcal{A}$ the *Stiefel coordinate ring* on $Gr(k, n)$.

Let $\mathcal{B}$ denote the homogeneous coordinate ring of $Gr(k, n)$ as a subvariety of $\mathbf{P}^N$ under the Plücker embedding, i.e, $\mathcal{B}$ is the quotient of the coordinate ring of $\mathbf{P}^N$ by the ideal generated by all Plücker relations. Then we have a natural isomorphism:

$$\mathcal{B} \xrightarrow{\sim} \mathcal{A} : p_{\underline{i}} \mapsto \det(\mathfrak{X}_{\underline{i}})$$

which is also an isomorphism of k-algebras: $\mathcal{B}_d \cong \mathcal{A}_d$. This isomorphism is actually the Plücker embedding on the coordinate rings.

Let $\{F_\alpha\}$ be a set of homogeneous polynomials in $\mathcal{A}$. This set defines a closed subset of $Gr(k,n)$:

$$Z(\{F_\alpha\}) := \{\pi(A) \in Gr(k,n) \,|\, F_\alpha(A) = 0, \ \forall\ \alpha\}$$

We write $D(F) := Gr(k,n) \backslash Z(F)$, and call it the *distinguished open set* of $Gr(k,n)$. It is easy to see that $D(\det \mathfrak{X}_{\underline{i}}) = \overline{U}_{\underline{i}}$.

If $Z \subseteq Gr(k,n)$ is a closed subset of $Gr(k,n)$, define

$$I(Z) := \{F \in \mathcal{A} \,|\, F(A) = 0, \ \forall\ A \in \pi^{-1}(Z)\}$$

Then $I(Z)$ is a homogeneous radical ideal of $\mathcal{A}$:

$$I(Z) = \bigoplus_{d \geq 0} \big(I(Z) \cap \mathcal{A}_d\big)$$

and it is generated by homogeneous elements of $\mathcal{A}$.

Define the *irrelevant ideal* as

$$\mathcal{A}_+ := \bigoplus_{d > 0} \mathcal{A}_d$$

We have the "Grassmannian" version of *Nullstellensatz* as follows:

Proposition 4.1. (i) *If $Z = Z(\{F_\alpha\})$ and $F \in I(Z)$, then for some $m > 0$, $F^m \in \sum_\alpha \mathcal{A} \cdot F_\alpha$, i.e.,*

$$I(V(\{F_\alpha\})) = \sqrt{\sum \mathcal{A} \cdot F_\alpha}$$

(ii) *The operations I and Z give a bijection*

$$\{\text{closed subsets of } Gr(k,n)\} \longleftrightarrow \{\text{homogeneous radical ideals of } \mathcal{A} \text{ except } \mathcal{A}_+\}$$

Some thoughts. Although the homogeneous polynomials in $\mathbf{k}[\mathfrak{X}]$ which define a subvariety of a Grassmannian can chosen from $\mathcal{A}$, it is not necessary to do so. For example, on $Gr(2,4)$ the equations $x_{1,1} = x_{1,2} = x_{2,1} = x_{2,2} = 0$ defines a subvariety of $Gr(2,4)$ in a well manner: the homogeneous ideal $(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$ of $\mathbf{k}[\mathfrak{X}]$

is actually invariant under the action of GL(2), although none of the $x_{i,j}$'s are in $\mathcal{A}$. So, in general, instead of looking at those defining equations each of which is invariant under the action, we should look at the homogeneous ideals of $\mathbf{k}[\mathfrak{X}]$ which are invariant under the action of GL(k), and I suspect we can get back to the ideal of $\mathcal{A}$ by intersecting this GL(k)-invariant ideal of $\mathbf{k}[\mathfrak{X}]$ with $\mathcal{A}$. This will allow us to check smoothness of a subvariety easily by looking at the Jacobian matrix. I think we have the following: if the GL(k)-invariant ideal of a subvariety $X \subseteq Gr(k,n)$ is generated by $F_1, \ldots, F_t$, then the ideal generated by all the partial derivatives $\partial F_\alpha/\partial x_{i,j}$ is again GL(k)-invariant, and a point $p \in X$ is smooth if and only if $\operatorname{rank}(\partial F_\alpha/\partial x_{i,j}(p)) = \dim Gr(k,n) - \dim_p X$. This should follow easily from the picture of the map $\pi : S(k,n) \to Gr(k,n)$. First, π is a smooth map implies for any subvariety $X \subseteq Gr(k,n)$, X is singular iff $\pi^{-1}(X)$ is singular, and the singularties of $\pi^{-1}(X)$ are just the preimages of the singularties of X. So to find the singularities of X is equivalent to find the singularties of $\pi^{-1}(X)$. Second, $\pi^{-1}(X)$ is a subvariety of an affine space $\mathbf{A}^{kn}$, so it is easy to check smoothness by looking at the Jacobian matrix, which is the same Jacobian matrix in terms of Stiefel coordinates. The problem is that I don't know how to describe the GL(k)-invariant homogeneous ideals associated to a subvariety in general, and don't know what properties they have. For example, are they finitely generated? how do they behave after taking radicals? I wonder if there are any references talking about this.

5 Grassmannians as Homogeneous Spaces

It is easy to see that the action of GL(n) on $\mathbf{k}^n$ induces an action on the Grassmannian $Gr(k,n)$:

$$\mathrm{GL}(n) \times Gr(k,n) \to Gr(k,n) : (g, \Lambda) \mapsto g(\Lambda)$$

The map can also be described in terms of Stiefel coordinates: if Λ is represented by $A \in S(k,n)$, then $g(\Lambda)$ is represented by Ag^T. This action is transitive. Indeed, let Λ_1 be spanned by $e_1, \ldots, e_k$, and for any other Λ, choose a basis $v_1, \ldots, v_k$, and extend to a basis of $\mathbf{k}^n$: $\{v_1, \ldots, v_k, \ldots, v_n\}$. Let g be the $n \times n$ matrix whose ith row is v_i for $i = 1, \ldots, n$. Then $v_i = e_i g$ for $i = 1, \ldots, k$ and hence $g^T(\Lambda) = \Lambda'$. So $Gr(k,n) \cong \mathrm{GL}(n)/H_\Lambda$ where H_Λ is the stablizer of any point Λ. If we choose Λ to be the subspace spanned by $e_1, \ldots, e_k$, then H_Λ consists of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is an invertible $k \times k$ matrix and C is an invertible $(n-k) \times (n-k)$ matrix. So $Gr(k,n)$ is a homogeneous space, which also shows the smoothness. Moreover, this provides us way to describe the tangent space of $Gr(k,n)$ in an intrinsic way. Let $\Lambda \in Gr(k,n)$. Then we can identify the tangent space $T_\Lambda Gr(k,n)$ with the quotient of the Lie algebra of $\mathrm{GL}(V)$ by the Lie algebra of H_Λ. The Lie algebra of $\mathrm{GL}(V)$ is just $\mathrm{End}(V)$ and the Lie algebra of H_Λ is $\{\varphi \in \mathrm{End}(V) \mid \varphi(\Lambda) \subseteq \Lambda\}$. The quotient is naturally identified with $\mathrm{Hom}(\Lambda, \mathbf{k}^n/\Lambda)$. Hence we get an intrinsic description of the tangent space:

$$T_\Lambda Gr(k,n) = \mathrm{Hom}(\Lambda, \mathbf{k}^n/\Lambda)$$

If the Grassmannian is given as $Gr(k,V)$, then

$$T_\Lambda Gr(k,V) = \mathrm{Hom}(\Lambda, V/\Lambda).$$

This importance of this description is more evident in the description of the tangent bundle of a Grassmannian. Indeed, we have

$$TGr(k,V) = \mathrm{Hom}(E,Q)$$

where E is the tautological vector bundle on $Gr(k,V)$:

$$E = \{(v,\Lambda) \in V \times Gr(k,V) \mid v \in \Lambda\}$$

and Q is the quotient bundle of the trivial bundle $V \times Gr(k,V)$ by E.

6 Grassmannians as Moduli Spaces

Let's review the definition of Grassmannians. We started with defining $Gr(k,V)$ as a set of k-linear subspaces of V, and then we define topology and affine open covers to make it into a variety. Here we take a different point of view: $Gr(k,V)$ is the variety that parameterizes the set of all k-linear subspaces of V. In this point of view, the Grassmannian $Gr(k, V)$ is called a *moduli space*. We now give the precise definitions.

We define a *family of k-dimensional subspaces of V parameterized* by a variety S is a rank k vector sub-bundle of the trivial bundle $V_S := V \times S$. This allows us to define a contravariant functor $\mathfrak{Gr}_{k,V}$ from the category of varieties to the category of sets by

$$\mathfrak{Gr}_{k,V}(S) = \{\text{all families of } k\text{-subspaces of } V \text{ parameterized by } S\}$$

If $f : T \to S$ is a morphism of two varieties, and $E \to S$ is a rank k vector sub-bundle of V_S, then the pull-back f^*E is a rank k vector sub-bundle of V_T. Thus f induces $\mathfrak{Gr}_{k,V}(f) : \mathfrak{Gr}_{k,V}(S) \to \mathfrak{Gr}_{k,V}(T)$ by $E \mapsto f^*E$.

Theorem 6.1. *The functor $\mathfrak{Gr}_{k,V}$ is* represented *by the variety $Gr(k, V)$, i.e., there is a natural isomorphism of functors $\mathfrak{Gr}_{k,V} \cong \mathrm{Mor}(-, Gr(k, V))$.*

We should emphasize that the Grassmannian $Gr(k, V)$ here is not taken as the set of all k-dimensional linear subspaces of V but a variety glued by $\binom{n}{k}$ affine spaces $\mathbf{A}^{k(n-k)}$ in the way we described above.

Let $\{\mathrm{pt}\}$ denote the variety consisting of a single point. Then we get a bijection between $\mathfrak{Gr}_{k,V}(\{\mathrm{pt}\})$ and $\mathrm{Mor}(\{\mathrm{pt}\}, Gr(k, V))$. Since a family of k-dimensional subspaces of V parameterized by $\{\mathrm{pt}\}$ is just a k-dimensional linear subspace of V, $\mathfrak{Gr}_{k,V}(\{\mathrm{pt}\})$ is just the set of all k-dimensional linear subspaces of V. The set $\mathrm{Mor}(\{\mathrm{pt}\}, Gr(k, V))$ can be identified with $Gr(k, V)$ as a set. Therefore we recover the bijection between the set of all points of $Gr(k, V)$ and the set of all k-dimensional linear subspaces of V.

For any variety S and any morphism $f : S \to Gr(k, V)$, there is a family of linear subspaces of V parameterized by S corresponding to f through the bijection $\mathfrak{Gr}_{k,V}(S)$ and $\mathrm{Mor}(S, Gr(k, V))$. In particular, the identity map $\mathrm{id} : Gr(k, V) \to Gr(k, V)$ corresponds to a family of k-dimensional subspaces of V parameterized by $Gr(k, V)$, called the *universal family*. It is nothing but the tautological vector bundle over $Gr(k, V)$. Moreover, for any variety S and any morphism $f : S \to Gr(k, V)$, the corresponding family parameterized by S is just the pull-back of the tautological bundle by f.

In terms of functors, the definition of Grassmannians can be generalized to the category of schemes. Let S be a fixed scheme, E a vector bundle on X, and k a positive integer less than $\operatorname{rank} E$. Define a contravariant functor

$$\mathfrak{Gr}_{k,E/S} : \{\text{Schemes}/S\} \to \{\text{Sets}\}$$

by associating to a S-scheme X the set of all vector sub-bundles of rank k of the vector bundle $E \times_S X$ over X. It can be shown that the functor $\mathfrak{Gr}_{k,E/S}$ is represented by an S-scheme, which is denoted by $Gr(k, E/S)$ and called a *Grassmannian.* It is unique up to unique isomorphism.

7 Varieties Related to Grassmannians

7.1 Subvarieties of Grassmannians

To begin with, an inclusion of vector spaces $W \hookrightarrow V$ induces an inclusion of Grassmannians $Gr(k, W) \hookrightarrow Gr(k, V)$; likewise, a quotient map $V \to V/W$ to the quotient of V by an l-dimensional subspace W of V induces an inclusion $Gr(k-1, V/W) \to Gr(k, V)$. The image of such maps are called *sub-Grassmannians*, and are subvarieties of $Gr(k, V)$.

If we view the Grassmannians as the set of linear subspaces of the projective space $\mathbf{P}(V)$, the sub-Grassmannains are just the subsets of planes contained in a fixed subspace and/or containing a fixed subspace. In general, we can consider the subset $\Sigma(W) \subseteq \mathbf{G}(k, \mathbf{P}(V))$ of k-planes that meet a given m-dimensional linear subspace $W \subseteq \mathbf{P}(V)$, or more generally, the subset $\Sigma_l(W)$ of k-planes that meet a given W in a subspace of dimension at least l. These are again subvarieties of the Grassmannian; $\Sigma_l(W)$ may be described as the locus

$$\Sigma_l(W) = \{[\omega] \in \mathbf{G}(k, \mathbf{P}(V)) : \omega \wedge v_1 \wedge \cdots \wedge v_{m-l+1} = 0,\ \forall\ v_1, \ldots, v_{m-l+1} \in W\}$$

from which we see that it is the intersection of the Grassmannain with a linear subspace of $\mathbf{P}(\bigwedge^k V)$. These are special cases of a subvariety of $\mathbf{G}(k, \mathbf{P}(V))$ called *Schubert varieties.*

There are also analogs for Grassmannians of projections maps on projective spaces. Specifically, suppose $W \subseteq V$ is a subspace of codimension l in the n-dimensional vector space V. For $k \leq l$, we have a map $\pi : U \to Gr(k, V/W)$ defined on the open set $U \subseteq Gr(k, V)$ of k-planes meeting W only in 0 simply by taking the image; for $k \geq l$, we have a map $\eta : U' \to Gr(k-l, W)$ defined on the open subset $U' \subseteq Gr(k, V)$ of planes transverse to W by taking the intersection. Note that both these maps may be realized, via the Plücker embeddings of both target and domain, by a linear projection on the ambient projective space $\mathbf{P}(\bigwedge^k V)$. For example, the map π is the restriction to $Gr(k, V)$ of the linear map $\mathbf{P}(\bigwedge^k V) \to \mathbf{P}(\bigwedge^k (V/W))$ induced by the projection $V \to V/W$.

7.2 The Varieties of Incidence Planes

Let $\mathbf{G} = \mathbf{G}(k, n)$ be the Grassmannian of k-planes in $\mathbf{P}^n$. We define a subset $\Sigma \subset \mathbf{G} \times \mathbf{P}^n$ by settting

$$\Sigma = \{(\Lambda, p) \,|\, p \in \Lambda\}$$

and call it the *incidence correspondence*. Σ is simply the subvariety of $\mathbf{G} \times \mathbf{P}^n$ whose fiber over a given point $\Lambda \in \mathbf{G}$ is the k-plane $\Lambda \subseteq \mathbf{P}^n$ itself; in the language of moduli spaces, it is the "universal family" of k-planes. It is not hard to see that Σ is a closed subvariety of the projective variety $\mathbf{G} \times \mathbf{P}^n$, hence also a projective variety. It comes with two projection maps

$$\pi_1 : \Sigma \to \mathbf{G}(k, n), \quad \pi_2 : \Sigma \to \mathbf{P}^n$$

The fibers of π_1 are all isomorphic to $\mathbf{P}^k$, hence Σ is irreducible, and

$$\dim \Sigma = \dim \mathbf{G} + \dim \mathbf{P}^k = (k+1)(n-k) + k = nk + n - k^2$$

The construction of Σ is just the paradigm for a general construction that will arise over and over in elementary algebraic geometry. One example of its usefulness is the following proposition

Proposition 7.1. (i) *If $Z \subseteq \mathbf{G}(k, n)$ is a (closed) subvariety, then the union*

$$X = \bigcup_{\Lambda \in Z} \Lambda \subseteq \mathbf{P}^n$$

is also a (closed) subvariety.

(ii) *If $X \subseteq \mathbf{P}^n$ is a (closed) subvariety, then the locus*

$$\mathscr{I}_k(X) := \{\Lambda \in \mathbf{G}(k,n) \mid \Lambda \cap X \neq \varnothing\}$$

is a (closed) subvariety of $\mathbf{G}(k,n)$.

Proof. Indeed, $X = \pi_2(\pi_1^{-1}(Z))$ and $\mathscr{I}_k(X) = \pi_1(\pi_2^{-1}(X))$. □

$\mathscr{I}_k(X)$ is called the *variety of incidence planes*. The fiber of π_2 over a point $p \in \mathbf{P}^n$ is the Grassmannian $\mathbf{G}(k-1, \mathbf{P}(V/p)) \cong \mathbf{G}(k-1, n-1)$. It follows that $\pi_2^{-1}(X)$ is of dimension $\dim X + \dim \mathbf{G}(k-1, n-1)$, and irreducible if so is X. Since the map $\pi_2^{-1}(X) \to \mathscr{I}_k(X)$ is generically finite (in fact generically one-to-one), we conclude that $\mathscr{I}_k(X)$ is of dimension $k(n-k) + \dim X$, and irreducible if so is X.

7.3 The Join of Two Varieties

Let $X, Y \subseteq \mathbf{P}^n$ be any two disjoint projective varieties. We define the *join* of X and Y as

$$J(X,Y) = \bigcup_{p \in X, q \in Y} \overline{p,q}$$

where $\overline{p,q}$ denotes the unique line passing through p and q in $\mathbf{P}^n$.

Let $\Sigma \subseteq \mathbf{G}(1,n) \times \mathbf{P}^n$ be the incidence correspondence, and $\pi_1 : \Sigma \to \mathbf{G}(1,n), \pi_2 : \Sigma \to \mathbf{P}^n$ be the projections. Then it is easy to see that

$$J(X,Y) = \pi_2(\pi_1^{-1}(Z)) \quad \text{where } Z = \mathscr{I}_1(X) \cap \mathscr{I}_1(Y).$$

from which it follows that $J(X,Y)$ is a closed subvariety of $\mathbf{P}^n$.

How do we define $J(X,Y)$ when X and Y meet? First, we have a morphism

$$j : (X \times Y) \setminus \Delta \to \mathbf{G}(1,n)$$

by sending the pair (p,q) to the line $\overline{p,q}$, where $\Delta = \{(p,p) \mid p \in \mathbf{P}^n\}$ is the diagonal of $\mathbf{P}^n \times \mathbf{P}^n$. This morphism defines a rational map

$$j : X \times Y \dashrightarrow \mathbf{G}(1,n)$$

since $(X \times Y) \setminus \Delta$ is a dense open subset of $X \times Y$. Let $\mathscr{J}(X,Y) \subseteq \mathbf{G}(1,n)$ be the closure of the image of j. We define the join of X and Y as

$$J(X,Y) = \pi_2(\pi_1^{-1}(\mathscr{J}(X,Y))$$

It is just the closure of the union of all lines $\overline{p,q}$ with $p \in X$, $q \in Y$ and $p \neq q$. When X and Y are disjoint, this coincides with our earlier definition the of the join.

It is a natural question to ask: if X and Y meet at a point p, what lines through p lie in $\mathscr{J}(X,Y)$? We do not have the language, let alone the techniques, to answer this question here, but we do have a partial answer: if X and Y are smooth and meet transversely at p, lines through p lying in the span of the projective tangent planes to X and Y at p will be in $\mathscr{J}(X,Y)$.

We now compute the dimension of $J(X,Y)$. The map j is generically finite. Thus $\dim \mathscr{J}(X,Y) = \dim(X \times Y) = \dim X + \dim Y$. The fibers of the map π_1 are all isomorphic to $\mathbf{P}^1$, hence $\dim \pi^{-1}(\mathscr{J}(X,Y)) = \dim X + \dim Y + 1$. We expect that the map $\pi_2 : \pi^{-1}(\mathscr{J}(X,Y)) \to J(X,Y)$ is generically finite, hence the expected dimension of $J(X,Y)$ is $\dim X + \dim Y + 1$. Indeed, it is much harder for the join of two disjoint varieties to fail to have this dimension.

Proposition 7.2. *Let X and Y be two disjoint subvarieties of $\mathbf{P}^n$. The join $J(X,Y)$ will have dimension exactly $d = \dim X + \dim Y + 1$ when ever $d \leq n$.*

Proof. This is immediate if X and Y lie in disjoint linear subspaces of $\mathbf{P}^n$; in this case, no two lines joining points of X and Y meet, so every point of the join lies on a unique line joining X and Y, which means the map $\pi_2 : \pi^{-1}(\mathscr{J}(X,Y)) \to J(X,Y)$ is one-to-one.

To prove the proposition in general, we reembed X and Y as subvarieties $\tilde{X}$, $\tilde{Y}$ of disjoint linear subspaces $\Lambda_0, \Lambda_1 \cong \mathbf{P}^n \subseteq \mathbf{P}^{2n+1}$, where Λ_0 is the plane defined by $Z_0 = \cdots = Z_n = 0$ and Λ_1 is defined by $Z_{n+1} = \cdots = Z_{2n+1} = 0$. Let $\tilde{J} = J(\tilde{X}, \tilde{Y})$ be the joint of these two, so that we have $\dim \tilde{J} = \dim X + \dim Y + 1$. Note that projection $\pi_L : \mathbf{P}^{2n+1} \to \mathbf{P}^n$ from the plane $L \cong \mathbf{P}^n \subseteq \mathbf{P}^{2n+1}$ defined by $Z_0 - Z_{n+1} = \cdots = Z_n - Z_{2n+1} = 0$ carries $\tilde{X}$ and $\tilde{Y}$ to X and Y, respectively; moreover, the hypothesis that $X \cap Y = \varnothing$ implies that the join $\tilde{J}$ is disjoint from L, so the projection π_L is a regular map from $\tilde{J}$ to $J(X,Y)$. But it is a general fact that the projection π_L of a projective variety from a linear space L disjoint from it is finite; if the fiber $\pi_L^{-1}(p)$ is contained in a curve C, C would be contained in the plane $\overline{p, L}$ spanned by p and L and so would necessarily meet the hyperplane $L \subseteq \overline{p, L}$. Thus $\pi_L : \tilde{J} \to J(X, Y)$ is finite and $\dim(J(X, Y)) = \dim X + \dim Y + 1$. $\square$

7.4 The Secant Varieties

In the special case $X = Y$ of the preceding construction, we get a rational map

$$s : X \times X \dashrightarrow \mathbf{G}(1, n)$$

defined on $(X \times X) \setminus \Delta$ by sending a pair of two distinct points (p, q) to the line $\overline{p, q}$. Since such a line is a secant line of X, this map is called the *secant line map*; the image of this map, denoted $\mathscr{S}(X)$, is, naturally called the *variety of secant lines*. Here, we call a line $L \in \mathscr{S}(X) \subseteq \mathbf{G}(1, n)$ a *secant line* to X. So a secant line is not necessarily spanned by its intersection with X. Actually, the tangent lines, viewed as degeneracies of secant lines, are also called secant lines here.

This can be generalized for any number of $k+1$ points; for $X \subseteq \mathbf{P}^n$ a subvariety and not contained in any $k-1$-plane, we can define a rational map

$$s_k : X^{k+1} \dashrightarrow \mathbf{G}(k, n)$$

by sending a general $(k+1)$-tuple of points of X to the plane they span. The notation is generalized in the obvious way: s_k is called the *secant plane map* and its image $\mathscr{S}_k(X)$ the *variety of secant k-planes* to X.

By Proposition 7.1, we see that the union of the secant lines to a variety X is again a variety, called the *secant variety* of X, and denoted by $S(X)$. Similarly, the union of the secant k-planes to X is again a variety, denoted $S_k(X)$.

We now compute the dimension of secant varieties. Unless X is a linear subspace of $\mathbf{P}^n$, the map $s : (X \times X) \setminus \Delta \to \mathbf{G}(1, n)$ is generically finite; the fiber over a general line $l = \overline{p, q}$ in the image will be of positive dimension if and only if $l \subseteq X$. The only variety that contains the line joining any two of its points is a linear subspace of $\mathbf{P}^n$. Thus $\mathscr{S}(X) = \dim(X \times X) = 2 \dim X$ if X is not a linear subspace.

Now consider the incidence correspondence

$$\Sigma = \{(l, p) \mid p \in l\} \subseteq \mathscr{I}(X) \times \mathbf{P}^n \subseteq \mathbf{G}(1, n) \times \mathbf{P}^n$$

whose image $\pi_2(\Sigma) \subseteq \mathbf{P}^n$ is the secant variety $S(X)$. The projection map π_1 on the first factor is surjective, with all fibers irreducible of dimension 1; thus Σ is irreducible if so is X, and of dimension $2 \dim X + 1$. The following proposition then follows.

Proposition 7.3. *The variety $\mathscr{S}(X) \subseteq \mathcal{G}(1, n)$ of secant lines to an irreducible variety $X \subseteq \mathbf{P}^n$ is irreducible of dimension $2 \dim X$. The secant variety $S(X) \subseteq \mathbf{P}^n$ is irreducible of dimension at most $2 \dim X + 1$, with equality holding if and only if the general point $p \in \overline{q, r}$ lying on a secant line to X lies on only a finite number of secant lines to X.*

For general secant varieties $S_k(X)$, note that for any $k \neq n - \dim X$, the general fiber of the secant plane map s_k is finite. From this we see that the variety of secant k-planes $\mathscr{S}_k(X)$ will always have dimension $(k+1) \dim X$. By the incidence correspondence argument then, we may also deduce that the secant variety $S_k(X)$ will have dimension at most $(k + 1) \dim X + k$, with equality holding if and only if the general point $p \in S_k(X)$ lies on only finitely many secant k-planes.

7.5 Joins of Corresponding Points

Once more, let X and Y be subvarieties of $\mathbf{P}^n$, and suppose we are given a regular map $\varphi : X \to Y$ such that $\varphi(x) \neq x$ for all $x \in X$. We may then define a map

$$k_\varphi : X \to \mathbf{G}(1, n)$$

by sending a point $x \in X$ to the line $\overline{x, \varphi(x)}$ joining x to its image under φ. This is a regular map, so its image is a variety; it follows that the union

$$K(\varphi) = \bigcup_{x \in X} \overline{x, \varphi(x)}$$

is again a variety. As before, if the condition $\varphi(x) \neq x$ is violated for some $x \in X$ but still valid on a dense open subset of X, we can still define a rational map

$$k_\varphi : X \dashrightarrow \mathbf{G}(1, n)$$

and define $K(\varphi)$ to be the union of lines corresponding to the image of k_φ; of course, we will no longer be able to describe $K(\varphi)$ naively as the union of the lines $\overline{x, \varphi(x)}$.

Since the map k_φ is generically one-to-one, the image $k_\varphi(X) \subseteq \mathbf{G}(1,n)$ has the same dimension as X. It follows that $\dim K(\varphi) = \dim X + 1$.

7.6 Rational Normal Scrolls

Let $k \leq l$ be two positive integers and $n = k+l+1$. Let Λ and Λ' be complementary linear subspaces of dimensions k and l in $\mathbf{P}^n$ (i.e., Λ and Λ' are disjoint and span $\mathbf{P}^n$. Choose rational normal curves $C \subseteq \Lambda$ and $C' \subseteq \Lambda'$, and an isomorphism $\varphi : C \to C'$; and let $S_{k,l}$ be the union of the lines $\overline{p, \varphi(p)}$ joining points of C to their corresponding images in C'. $S_{k,l}$ is called a *rational normal scroll.* The lines $\overline{p, \varphi(p)}$ are called the *lines of the ruling* of $S_{k,l}$; they are the only lines lying on $S_{k,l}$ unless $k = 1$. Note that $S_{k,l}$ is determined by k, l up to projective automorphisms: we can move any pair of complementary planes Λ, Λ' into any other, any rational normal curves C, C' into any other, and finally adjust φ by composing with an automorphism of Λ or Λ' inducing an automorphism of C or C'.

Proposition 7.4. (i) *The scrolls $S_{k,l}$ and $S_{k',l'} \subseteq \mathbf{P}^n$ are projectively equivalent if and only if $k = k'$.*

(ii) *In case $k < l$, the rational normal curve $C \subseteq S = S_{k,l}$ of degree k appearing in the construction of the rational normal scroll is the unique rational normal curve of degree $< l$ on S (other than the lines of the ruling of S); in particular, it is uniquely determined by S (it is called the* directrix *of S). This is not the case for the rational normal curve C' of larger degree l or for C itself in case $k = l$.*

(iii) *The image of the scroll $S = S_{k,l}$ under the projection from a point $p \in S$ is projectively equivalent to $S_{k-1,l}$ if p lies on the directrix of S; it is projectively equivalent to $S_{k,l-1}$ otherwise. In particular, all scrolls $S_{k,l}$ are rational.*

A further generalization of this notion is to allow several rational normal curves; that is, for any collection $a_1, \ldots, a_k$ natural numbers with $a_1 \leq \cdots \leq a_k$ and $\sum_i a_i = n - k + 1$, we can find complementary linear subspaces Λ_i of dimension a_i in $\mathbf{P}^n$ and rational normal curves $C_i \subseteq \Lambda_i$ in each. Choose isomorphisms $\varphi_i : C_1 \to C_i$ and let

$$S = \bigcup_{p \in C_1} \overline{p, \varphi_2(p), \ldots, \varphi_k(p)}.$$

S is called a *rational normal k-fold scroll* (or *rational normal scroll of dimension* k), and sometimes denoted $S_{a_1,\ldots,a_k}$. As before, S is determined up to projective equivalence by integers $a_1, \ldots, a_k$.

It is easy to see that the $\dim S_{k,l} = \dim C + 1$ and in general $\dim S_{a_1,\ldots,a_k} = \dim C + k - 1$.

7.7 More Incidence Correspondences

There are a number of generalizations of the incidence correspondence. For instance, we can look in the product of two Grassmannians of planes in the same space $\mathbf{P}^n$ at the locus of pairs of planes that meet: for any k and l we set

$$\Omega = \{(\Lambda, \Lambda') \mid \Lambda \cap \Lambda' \neq \varnothing\} \subseteq \mathbf{G}(k,n) \times \mathbf{G}(l,n).$$

That this is a variety is clear: we can write

$$\Omega = \{([v_0 \wedge \cdots \wedge v_k], [w_0 \wedge \cdots \wedge w_l]) \mid v_0 \wedge \cdots \wedge v_k \wedge w_0 \wedge \cdots \wedge w_l = 0\}$$

Similarly, we can for any $k < l$ consider the locus of nested pairs:

$$\mathbf{F}(k,l;n) = \{(\Lambda, \Lambda') \mid \Lambda \subseteq \Lambda'\} \subseteq \mathbf{G}(k,n) \times \mathbf{G}(l,n).$$

Note that both these constructions specializes to the construction of the incidence correspondences in case $k = 0$. A common generalization of them both in turn is the variety

$$\Psi_m = \{(\Lambda, \Lambda') \mid \dim(\Lambda \cap \Lambda') \geq m\} \subseteq \mathbf{G}(k,n) \times \mathbf{G}(l,n)$$

which gives Ω when $m = 0$ and $\mathbf{F}(k,l;n)$ when $m = k < l$.

Of course, over Ω, $\mathbf{F}(k,l;n)$ and Ψ_m there are "universal families" that play with respect to these varieties the same role as the original incidence correspondence Σ played with respect to $\mathbf{G}(k,n)$, i.e., we may set

$$\Xi_m = \{(\Lambda, \Lambda', p) \mid p \in \Lambda \cap \Lambda'\} \subseteq \Psi_m \times \mathbf{P}^n$$

7.8 Flag Varieties

The incidence correspondence $\mathbf{F}(k,l;n)$ introduced above is a special case of what is called a *flag variety*. Briefly, for any increasing sequence of integers $a_1 < a_2 < \cdots < a_k < n$ we can form the variety of *flags*

$$\begin{aligned}\mathbf{F}(a_1,\ldots,a_k;n) &= \{(\Lambda_1,\ldots,\Lambda_k) \mid \Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_k\} \\ &\subseteq \mathbf{G}(a_1,n) \times \cdots \times \mathbf{G}(a_k,n)\end{aligned}$$

That it is a subvariety of the product of Grassmannians is easy to see (in particular, it is the intersection of the inverse images of the flag varieties $\mathbf{F}(a_i,a_j;n) \subseteq \mathbf{G}(a_i,n) \times \mathbf{G}(a_j,n)$ under the corresponding map).

As with Grassmannians, we the have a second notation to write for flag varieties: for a sequence of integers $a_1 < a_2 < \cdots < a_k$ we define

$$\begin{aligned}Fl(a_1,\ldots,a_k;n) &= \{(\Lambda_1,\ldots,\Lambda_k) \mid \Lambda_1 \subseteq \cdots \subseteq \Lambda_k\} \\ &\subseteq Gr(a_1,n) \times \cdots \times Gr(a_k,n)\end{aligned}$$

It is easy to see that $Fl(a_1,\ldots,a_k;n) = \mathbf{F}(a_1-1,\ldots,a_k-1;n-1)$. The more invariant notations are $Fl(a_1,\ldots,a_k;V)$ and $\mathbf{F}(a_1-1,\ldots,a_k-1,\mathbf{P}(V))$.

It is easy to see that the group $\mathrm{GL}(n)$ acts naturally on $Fl(a_1,\ldots,a_k;n)$:

$$\begin{aligned}\mathrm{GL}(n) \times Fl(a_1,\ldots,a_k;n) &\to Fl(a_1,\ldots,a_k;n) \\ (g,(\Lambda_1,\ldots,\Lambda_k)) \quad &\mapsto (g(\Lambda_1),\ldots,g(\Lambda_k))\end{aligned}$$

The action is transitive. Let Λ_i^0 be spanned by $e_1,\ldots,e_{a_i}$ for $i = 1,\ldots,k$. Then $(\Lambda_1^0,\ldots,\Lambda_k^0) \in Fl(a_1,\ldots,a_k;n)$. For any other point $(\Lambda_1,\ldots,\Lambda_k)$, choose a basis $(v_1,\ldots,v_{a_1})$ for Λ_1, extend it to a basis $(v_1,\ldots,v_{a_1},\ldots,v_{a_2})$ for Λ_2, $\ldots$, extend to a basis $(v_1,\ldots,v_{a_k})$ for Λ_k, and finally extend to a basis $(v_1,\ldots,v_n)$ for $\mathbf{k}^n$. Let g be the $n \times n$ matrix whose ith row is v_i as a row vector in $\mathbf{k}^n$. Then we have $v_i = ge_i$ for $i = 1,\ldots,a_k$, or $g(\Lambda_i^0) = \Lambda_i$. This shows the action by $\mathrm{GL}(n)$ is transitive. It follows that flag varieties are smooth.

Note that we have a natural morphism

$$Fl(a_1,\ldots,a_k;n) \to Fl(a_2,\ldots,a_k;n):\ (\Lambda_1,\Lambda_2\ldots,\Lambda_k) \mapsto (\Lambda_2,\ldots,\Lambda_k)$$

The fiber over each point $(\Lambda_2, \ldots, \Lambda_k)$ is just $Gr(a_1, \Lambda_2) \cong Gr(a_1, a_2)$. Since the fibers are all smooth and irreducible of the same dimension, we can show by induction on k that all flag varieties are irreducible. This also provides us a way to compute the dimension of flag varieties:

$$\begin{aligned}
&\dim Fl(a_1, \ldots, a_k; n) = \dim Gr(a_1, a_2) + \dim Fl(a_2, \ldots, a_k; n) \\
&= \cdots = \dim Gr(a_1, a_2) + \cdots + \dim Gr(a_{k-1}, a_k) + \dim Gr(a_k, n) \\
&= a_1(a_2 - a_1) + \cdots + a_{k-1}(a_k - a_{k-1}) + a_k(n - a_k)
\end{aligned}$$

7.9 Tangential Varieties

The projective tangent space $\mathbf{T}_p(X)$ of a variety $X \subseteq \mathbf{P}^n$ at a point p is a linear subspace of $\mathbf{P}^n$. This allows us to define the *Gauss map* associated to smooth variety. If X is smooth of pure dimension k, this map is just:

$$g = g_X : X \to \mathbf{G}(k, n)$$

sending a point $p \in X$ to its tangent plane $\mathbf{T}_p(X) \subseteq \mathbf{P}^n$. That this is a regular map is clear from the description of the projective tangent space as the kernel of the linear map given by the matrix of partial derivatives $(\partial F_\alpha / \partial Z_i)$, where $\{F_\alpha\}$ is a collection of generators of the ideal of X. Note that if X is singular then g is still defined and regular on the open set of smooth points of X, which is dense, hence gives a rational map

$$g : X \dashrightarrow \mathbf{G}(k, n)$$

We will call the image $g(X) \subseteq \mathbf{G}(k, n)$ the *variety of tangent planes* to X, or *Gauss image* of X, and denote it $\mathscr{T}(X)$. Note that there is some potential confusion here if X is singular, since the projective tangent plane to X at a singular point, being a subspace of dimension strictly higher than k, will not appear in $\mathscr{T}(X)$, and conversely $\mathscr{T}(X)$ will contain k-planes that are limits of tangent planes to X but not tangent planes themselves.

The simplest example of a Gauss map is the one associates to a hyperplane $X \subseteq \mathbf{P}^n$; if X is given by the homogeneous polynomial $F(Z_0, \ldots, Z_n)$ then the Gauss map $g : X \to \mathbf{G}(n-1, n) = \mathbf{P}^{n*}$ is given by

$$g(p) = \left[\frac{\partial F}{\partial Z_0}(p), \ldots, \frac{\partial F}{\partial Z_n}(p)\right]$$

Note that in case the degree of F is 2, this is a linear map. Indeed, as we have seen a smooth quadric hypersurface $X \subseteq \mathbf{P}(V) = \mathbf{P}^n$ may be given by a symmetric bilinear form

$$Q : V \times V \to \mathbf{k}$$

and the Gauss map is then just the restriction to X of the linear map $\mathbf{P}^n \to \mathbf{P}^{n*}$ associated to the induced isomorphism

$$\tilde{Q} : V \to V^*.$$

It is elementary to see that if $X \subseteq \mathbf{P}^n$ is a smooth hypersurface of degree $d \geq 2$, then the fibers of the Gauss map $g : X \to \mathbf{P}^{n*}$ are finite: since the partial derivatives of F do not vanish simultaneously at any point of X, the map g_X cannot be a constant along a curve. It follows that the image $\mathscr{T}(X)$ of the Gauss map is again a hypersurface. It is a theorem of F. L. Zak, which we will not prove here, that for any smooth irreducible k-dimensional variety $X \subseteq \mathbf{P}^n$ other than a linear space, the Gauss map is finite, and in particular the dimension of the variety $\mathscr{T}(X) \subseteq \mathbf{G}(k, n)$ of tangent planes to X is again k.

Let $X \subseteq \mathbf{P}^n$ be an irreducible variety of dimension k. The union

$$TX = \bigcup_{\Lambda \in \mathscr{T}(X)} \Lambda$$

of the k-planes corresponding to points of the image $\mathscr{T}(X)$ of the Gauss map is then a subvariety of $\mathbf{P}^n$, called the *tangential variety* of X. Of course, if X is smooth, this is just the union of all its tangent planes; if X is singular, the description is not clear (though we can describe it as the closure of the union of the tangent planes to X at smooth points).

What is the dimension of TX? We look at the incidence correspondence

$$\Sigma = \{(\Lambda, p) \,|\, p \in \Lambda\} \subseteq \mathscr{T}(X) \times \mathbf{P}^n \subseteq \mathbf{G}(k, n) \times \mathbf{P}^n$$

The image $\pi_2(\Sigma) \subseteq \mathbf{P}^n$ is the tangential variety TX. The projection map π_1 on the first factor is surjective, with all fibers irreducible of dimension k; thus Σ is irreducible of dimension at most $2k$. It follows that

$$\dim TX \leq 2k$$

with equality holding if and only if a general point p on a general tangent plane $\mathbf{T}_p(X)$ lies on $\mathbf{T}_q(X)$ for only finitely many points $q \in X$.

It may well happen that the tangential variety of a k-dimensional variety $X \subseteq \mathbf{P}^n$ has dimension strictly less than $\min(2k, n)$. If we define the *deficiency* of a variety $X \subseteq \mathbf{P}^n$ such that $TX \neq \mathbf{P}^n$ to be the difference $2\dim X - \dim TX$, there are examples of varieties with arbitrarily large deficiency, though it is not known whether the deficiency of smooth varieties may be bounded above in general.

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