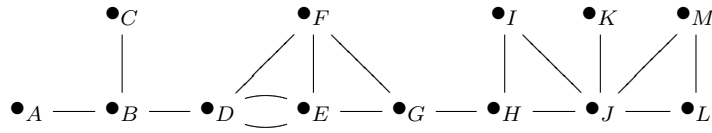


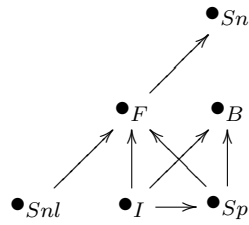
GRAPH THEORY HW 1

1.5 Solution:



□

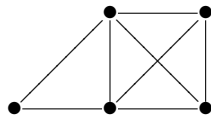
1.7 Solution: The arrows may go in either direction.



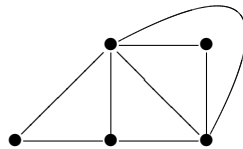
□

2.2 Solution:

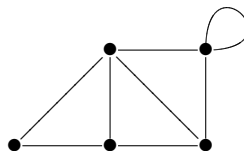
(i)



(ii)



(iii)



□

2.5 Solution:

Date: September 1, 2006.

- (1) Between any two distinct vertices v_1, v_2 in a simple graph G , there are exactly two possibilities: v_1 is adjacent with v_2 , or v_1 is not adjacent with v_2 . There are $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs of distinct vertices. The isomorphism class of a labeled graph on n vertices

is completely determined by the edge set; thus, there are $2^{\binom{n}{2}}$ distinct graphs.

It is worth emphasizing that G being labeled is critical; for unlabeled graphs, the general problem is very difficult.

- (2) There are $\binom{n}{2}$ many edges available; of these we want to choose precisely m . There are $\binom{\binom{n}{2}}{m}$ ways of doing this. \square

2.11 Solution:

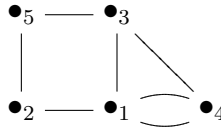
$G - e$: $G - e$ contains the same vertices as G , for a total of n . We have deleted one edge from the m edges, so $m - 1$ remain.

$G - v$: $G - v$ contains one fewer vertex than G , for a total of $n - 1$. We have deleted $\deg v = k$ edges as well, so there remain only $m - k$. (We assume that G has no loops at v .)

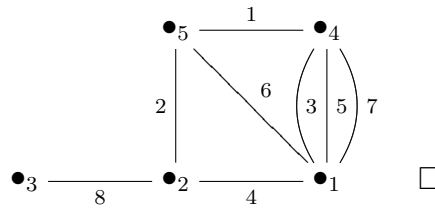
$G \setminus e$: $G \setminus e$ contains one fewer vertex than G , for a total of $n - 1$. We have deleted only one edge, so there remain $m - 1$. (We assume that e is not a loop.) \square

2.13 Solution:

(1)



(2)



\square

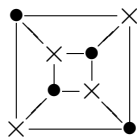
2.14 Solution:

- (1) The sum of the i^{th} row (or the i^{th} column) is $\deg v_i$, the degree of the i^{th} vertex. This is true because we have counted the number of edges incident v_i , and no edge was counted twice because there are no loops. Notice that an (undirected) graph always has symmetric adjacency matrix, regardless of loops or multiple edges.
- (2) The sum of the i^{th} row of the incidence matrix is the $\deg v_i$, since we are counting the number of edges incident v_i .
- (3) Since there are no loops, each edge is incident on exactly 2 vertices; thus the sum of the j^{th} column is 2. \square

3.5 Solution: The following contains particular examples; there may be more.

- (1) $K_{5,5}$.

- (2) The cube:



- (3) K_4 .
 (4) Any cubic graph must have an even number of vertices. Indeed, consider that by the handshaking lemma the sum of all vertex degrees must be even. If $|V(G)|$ is odd, then the sum of the vertex set is $3 \cdot |V(G)|$, also odd. This is a contradiction. Hence, there is no cubic graph on 11 vertices.
 (5) Note: in the book there is a typo: it should say “(other than $K_5, K_{4,4}, Q_4$)”. The octahedron graph is a connected example. Since G need not be connected, we may take the disjoint union of K_5 and say $K_{4,4}$, or more generally the disjoint union of any number of 4-regular graphs. \square

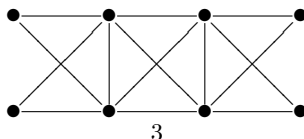
3.8 Solution:

- (1) If G is a simple graph on n vertices, then $|E(G)| + |E(\overline{G})| = \binom{n}{2}$, since this is how many edges are in the complete graph. If G is self-complementary, so that $G = \overline{G}$, then $4|E(G)| = n(n-1)$. Reading this equation mod 4, we have that $n(n-1) \equiv 0 \pmod{4}$. The only solutions are $n \equiv 0 \pmod{4}$ or $n-1 \equiv 0 \pmod{4}$, so that $n = 4k$ or $n = 4k+1$ for some $k \in \mathbb{N}$.
 (2) Notice that the complement of a disconnected graph is connected. Indeed, if G can be written as the disjoint union of two graphs G_1, G_2 , then \overline{G} contains $K_{|V(G_1)|, |V(G_2)|}$ as a subgraph with the same vertex set as \overline{G} . But $K_{|V(G_1)|, |V(G_2)|}$ is connected, so that \overline{G} is connected. Thus any self-complementary graph must be connected, since connectedness is preserved under graph-isomorphism.

If G is self-complementary on 4 vertices, then $|E(G)| = \frac{4(4-1)}{4} = 3$. If we scan (page 11) for the simple connected graphs on 4 vertices with 3 edges, which are the only candidates, we see that there are only 2. The graph #5 contains a vertex of degree 3; if it was self-complementary, then its complement would have a vertex of degree 0 - impossible. Thus it is not self-complementary. Since the complement of #6 is a simple connected graph with the same number of edges and vertices, #6 must be self-complementary, since we have shown that it cannot be the complement of #5.

If G is self-complementary on 5 vertices, then $|E(G)| = \frac{5(5-1)}{4} = 5$. The only candidates are graphs #14-#18. We can rule out #14, since it has a vertex of degree 4. In fact, #16 is complementary to #17, so that either #15 and #18 are both self-complementary or they are complementary to each other. It is perhaps easiest to see that #18 is self-complementary, since its complement is clearly regular of degree 2, and thus equal to C_5 . Thus both #15 and #18 are self-complementary.

- (3) One can check that



is self-complementary; in fact, this is a particular case of a general construction which produces self complementary graphs of order $4n$:

$$\overline{K_n} - K_n - K_n - \overline{K_n} \quad \square$$

Problem. Prove that the graphs C_n , P_n and W_n are connected.

Solution. We will present two solutions for C_n , based on the concepts of *connectedness* and *path-connectedness*, respectively; it is true that for graphs these two notions coincide, but this requires proof.

- (1) Suppose that C_n is not connected, so that C_n can be written as the disjoint union of two non-empty graphs G_1, G_2 . Consider the smallest vertex label v_i in G_2 . By definition of C_n , there is an edge $v_{i-1}v_i$; if $i > 1$, then $v_{i-1} \in G_1$, and we have a contradiction, since there are no edges between G_1 and G_2 . So $i = 1$. Interchanging G_1, G_2 in the above argument proves that $v_i \in G_1$ and $v_i \in G_2$, a final contradiction.
- (2) Consider two vertices $v_i, v_j \in V(C_n)$; we may assume that $i < j$. Then the path $v_i \rightarrow v_{i+1} \rightarrow \dots \rightarrow v_j$ is a path adjoining v_i with v_j . Thus, C_n is connected.

It follows immediately that W_n is connected since W_n is formed by adjoining a vertex v_n and all edges from v_n to C_{n-1} , which is connected.

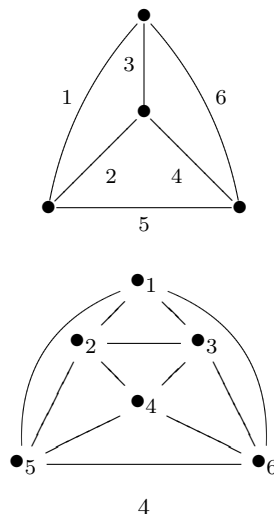
Finally, P_n is defined to be $C_n - e$ for some edge e . By relabeling the vertices, we can assume that $e = v_n v_1$. Then any two vertices are path-connected by the path defined above. \square

EXTRA CREDIT

2.9 Solution: Suppose G is a simple graph on n vertices in which the degree of each vertex is distinct. Then the range of vertices is $\{0, \dots, n-1\}$. However, if there is a vertex of degree $n-1$, then it is adjacent to every other vertex; thus there cannot be a vertex of degree 0. This contradiction proves that no such graph can exist. \square

3.9 Solution:

- (1) Clearly $L(K_3) = K_3$. Further, $K_{1,3}$ has 3 edges, which are all incident a common vertex, and therefore all adjacent; hence $L(K_{1,3}) = K_3$ as well.
- (2)



- (3) If $e = v_i v_j$ is an edge in G , then it is adjacent $k - 1$ distinct edges at v_i , as well as $k - 1$ distinct edges at v_j ; no two of these can be the same, because G is simple. Thus in $L(G)$, e (a vertex) is adjacent to $2k - 2$ other vertices. This proves that $L(G)$ is $2k - 2$ regular.
- (4) Let \mathbf{v}_e be a vertex in $L(G)$ corresponding to edge e in G . If $e = v_i v_j$, then

$$\deg \mathbf{v}_e = \deg v_i + \deg v_j - 2.$$

Since G is simple, there are no loops in $L(G)$; thus $2|E(L(G))| = \sum_{e \in E(G)} \deg \mathbf{v}_e$ by the handshaking lemma. So

$$2|E(L(G))| = \sum_{e=v_i v_j \in E(G)} ((\deg v_i - 1) + (\deg v_j - 1)) = \sum_{v \in V(G)} (\deg v (\deg v - 1)).$$

Thus our final expression is $|E(L(G))| = \frac{1}{2} \sum_{v \in V(G)} (\deg v (\deg v - 1))$.

- (5) Tedious verification...

□

3.10 Solution:

- (1) Any automorphism α of G is a permutation of the vertex set $V(G)$; each such α is also then a permutation on $V(\overline{G})$, since $V(G) = V(\overline{G})$. If a permutation π of $V(G)$ is an automorphism of G , then π is also an automorphism of $V(\overline{G})$ because automorphisms preserve non-adjacencies. Thus we may define $\phi : \mathbf{Aut}(G) \rightarrow \mathbf{Aut}(\overline{G})$ as the identity map. Clearly, ϕ is an isomorphism, thus $\Gamma(G) \cong \Gamma(\overline{G})$.
- (2) $\Gamma(K_n) = S_n$, since any permutation of vertices is an automorphism.

$\Gamma(K_{r,s}) = S_r \times S_s$ when $r \neq s$, and is $S_r \wr S_2$ when $r = s$. The latter follows since there is an extra automorphism which switches the two partite sets, but does not commute with the automorphisms in each component (in fact it switches the components): if α_i, β_i are the automorphisms of the two partite sets, and if π_i are in S_2 , then we have the multiplication

$$(\alpha_1, \beta_1, \pi_1) \cdot (\alpha_2, \beta_2, \pi_2) = \begin{cases} (\alpha_1 \alpha_2, \beta_1 \beta_2, \pi_1 \pi_2) & \text{if } \pi_1 = 1 \\ (\alpha_1 \beta_2, \beta_1 \alpha_2, \pi_1 \pi_2) & \text{if } \pi_1 = (1\ 2). \end{cases}$$

This is the group operation of the wreath product $S_r \wr S_2$.

$\Gamma(C_n) = D_n$, the dihedral group of order $2n$. This is most readily seen from the geometric interpretation of D_n , as being the group of symmetries of the regular n -gon. □