GRAPH THEORY HW 6

22.4 Solution:

(1) $\bullet \longrightarrow \bullet$

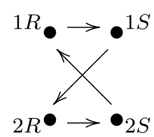
(2) If M is the adjacency matrix for D, and $\tilde{M}$ is the adjacency matrix for $\tilde{D}$, then $M^T = \tilde{M}$. $\square$

22.5 Solution:

(1) Let G be a Hamiltonian graph, and consider $C = v_0 v_1 \cdots v_0$ a Hamiltonian cycle in G. Orient the edges in C by their order in the cycle C, i.e. $\{v_0, v_1\} \to (v_0, v_1)$, and direct the remaining edges in any arbitrary way. Now G is strongly connected. Indeed, for any pair of vertices (a, b), starting at a we can get to b via the cycle C. Thus G is orientable.

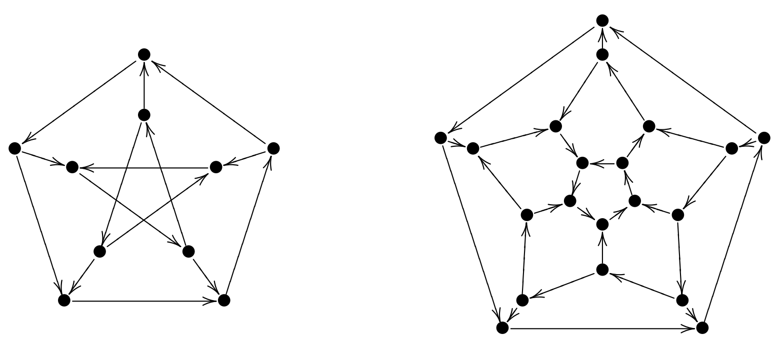
(2) K_n, $(n \geq 3)$ can be oriented by orienting the cycle $(1, 2, \ldots, n, 1)$ as above, and arbitrarily directing the other edges (for instance, directing $\{a, b\}$ to (a, b) if $a < b$).

Consider $K_{r,s}$ $(r, s \geq 2)$. Label the vertices in the first partite set R as $\{1R, \ldots, rR\}$, and the vertices in the second partite set S as $\{1S, \ldots, sS\}$. Direct the following edges: $(1R, 1S), (1S, 2R), (2R, 2S), (2S, 1R)$, as in the picture.



Now suppose v is another vertex in R; direct the edges $(vR, 1S), (2S, vR)$. Do this for all vertices in R, and similarly for S, but reversing the roles of R and S. Orient all other edges arbitrarily. Then G is strongly connected, as is easily verified. So G is orientable.

(3)



$\square$

22.7 Solution: We list the largest weight path to each vertex: $B-30$, $C-50$, $D-62$, $E-97$, $F-85$, $G-105$. Note that there are two longest paths from A to G, namely: $ACDEG$, and $ACDFG$. $\square$

23.4 Solution:

(1) Let m be the number of edges in T. Then $\sum_{v\in V(T)} out(v) = m = \sum_{v\in V(T)} in(v)$, since each edge contributes exactly one out degree and one in degree. This is true for a general digraph.

(2)

$$\begin{aligned}\sum_v out(v)^2 - \sum_v in(v)^2 &= \sum_v \left(out(v)^2 - in(v)^2\right)\\ &= \sum_v \left[(out(v)+in(v))(out(v)-in(v))\right]\\ &= (n-1)\sum_v (out(v)-in(v))\\ &= 0,\end{aligned}$$

where $out(v) + in(v) = n - 1$ since T is a tournament, and the last equation holds by (1). $\square$

23.5 Solution: Solutions are not unique. Here is an Eulerian trail, with 3 cycles starting at (12):

(1221)(2113)(1331)(3112)
(1222)(2222)(2223)(2332)(3223)(2331)(3113)(1332)(3222)(2221)(2112)
(1223)(2333)(3333)(3331)(3111)(1111)(1113)(1333)(3332)(3221)(2111)(1112)

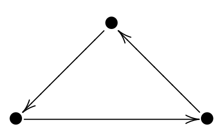
The circular arrangement asked for can be found by taking the last numbers from each edge. In this case, we get:

$$131222323132212333111332112.$$

It is clear that every possible triple occurs once since we have used every edge from our graph, and each edge $(abbc)$ is a triple of numbers. $\square$

23.6 Solution:

(1)

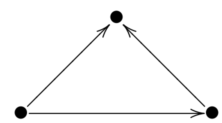


(2) We will prove the equivalent statement: T is reducible if and only if T is not strongly connected. $\Rightarrow$: If V_1, V_2 are two components with arrows only directed from V_1 to V_2, then consider a vertices $v \in V_2$, and $w \in V_1$. Then evidently there is no path from v to w. Thus T is not strongly conected.

$\Leftarrow$: Since T is not strongly connected, consider a vertex $v \in T$ which is not path connected to every vertex. Let V be the set of vertices to which v is path-connected. Then $W = T \setminus V$, V are two non-empty disjoint sets of vertices such that each edge from V to W is directed from W to V. Indeed, if $v'w$ were an edge for some $v' \in V$, $w \in W$, then w would belong to V, contrary to our assumption. Thus T is reducible. $\square$

23.7 Solution:

(1)



(2) The graph defines a partial order $\leq$ on the set of vertices: $v \leq w$ means $vw \in E(T)$. This is an order relation precisely because it is transitive, and it is a total order because T is a tournament, so that all vertices are comparable. The teams can thus be ranked according to their order. In particular, there is a maximum element, i.e. a team which beats all other teams.

(3) Let w be the greatest team. Then $V = \{w\}$ and $T \setminus W$ partition the vertex set so that each edge is from V to $T \setminus W$. Hence, T is reducible and not strongly connected. $\square$

Extra Credit

23.8 Solution:

(1) There are $\frac{n(n-1)}{2}$ edges, since the underlying graph is complete. Each edge contributes 1 to the sum $s_1 + \cdots + s_n$, by adding one outdegree to some vertex.

(2) For $k < n$, let V be the vertices corresponding to $s_1, \ldots, s_k$. Then V induces a subgraph of T, which is also a tournament. Hence $s_1 + \cdots + s_k \geq \frac{k(k-1)}{2}$, by part (1). We note that the strict inequality happens precisely when there exists an edge $vw \in E(T)$ for $v \in V$, and $w \in T \setminus V$. If T is strongly connected, strict inequality is clearly necessary for each k, for otherwise V and $T \setminus V$ would prove that T is reducible.

On the otherhand, suppose that T is reducible into components $V_1 = T \setminus V_2, V_2$, where each edge goes from V_1 to V_2. If $|V_2| = k$, then $s_1 + \cdots + s_k = \frac{k(k-1)}{2}$ provided that these scores are taken from vertices in V_2. Indeed, consider any $v_2 \in V_2$, and $v_1 \in V_1$. Then $out(v_2) \leq k - 1$, while $out(v_1) \geq k$. This proves that if T is reducible, there exists a k such that $s_1 + \cdots + s_k = \frac{k(k-1)}{2}$; if we have strict inequality for all k, then T is irreducible (strongly connected).

(3) If T is transitive, then we can order the teams from worst to best such that each team beats all worse teams. This means that $s_k = k - 1$ for all k.

Conversely, suppose that $s_k = k - 1$ for each k. We will prove that T is transitive by induction on $|T| = n$. Clearly the result is true for small n. Suppose $|T| = n + 1$. By hypothesis, there exists a vertex v with $out(v) = n$. The subgraph $T \setminus \{v\}$ is a tournament that satisfies the hypothesis, since we did not change the outdegree of any remaining vertex. By the induction hypothesis, $T \setminus \{v\}$ is transitive. Hence T is transitive, since v dominates every other vertex in T. The result follows by induction. $\square$