GRAPH THEORY HW 3

6.3 Solution:

(1) K_n is Eulerian when all degrees are even, so when n is odd.
(2) $K_{m,n}$ is Eulerian when m and n are even, since then the degrees of all vertices will be even (either m or n).
(3) All of the platonic graphs are $m-$regular, but m is even only for the octahedron. So this is the only Eulerian platonic graph.
(4) W_n is never Eulerian, since every vertex on the “rim” has degree 3, which is odd.
(5) Q_k is Eulerian when k is even. Indeed, Q_k is $k-$regular. □

6.6 Solution:

(1) If $e = v_i v_j$ is an edge in G, then $v_e \in L(G)$ has degree $\deg v_i - 1 + \deg v_j - 1 = \deg v_i + \deg v_j - 2$. Since $\deg v_i$ is always even, we see that $\deg v_e$ is even for all $v_e \in L(G)$. Thus $L(G)$ is Eulerian.
(2) No. $L(K_{1,3})$ is K_3, which is Eulerian, but $K_{1,3}$ is not. □

6.7 Solution:

(1) Starting from v, we are obligated to return to v after traversing a length 4 cycle. After three such trips, we will have completed an Eulerian trail.
(2) This same graph is not randomly traceable if we start from a different vertex, say any vertex w of degree 2. In this case, we can take a cycle of length 4 and return to w; however, it is impossible to return without reusing one of the edges we have already used.
(3) A randomly traceable graph would be useful so that exhibit-goers could see each exhibit without needing to carefully plan out their route, and without repeating exhibits. □

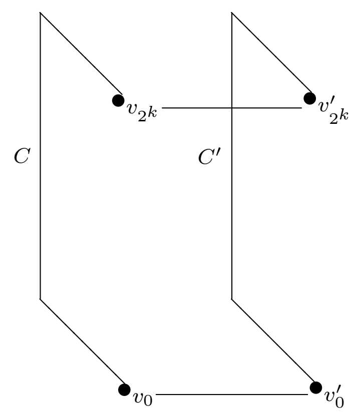
7.3 Solution:

(1) K_n has C_n as a subgraph, and so is Hamiltonian for all $n > 2$.
(2) $K_{m,n}$ is Hamiltonian when $m = n > 1$. Indeed, traversing a Hamiltonian cycle C in $K_{m,n}$ means alternating between bipartite sets. This creates a one-one correspondence between vertices in the bipartite sets, so they must have the same number. To see that there actually is a hamiltonian cycle, label the vertices $1a, 2a, \ldots$ in the first partite set, and $1b, 2b, \ldots$ in the second. Then the path $1a, 1b, 2a, 2b, \ldots, na, nb, 1a$ is a Hamiltonian cycle.
(3) Every platonic graph is Hamiltonian, as is easily verified.
(4) W_n is Hamiltonian for every $n > 2$.
(5) Q_k is Hamiltonian for every $k > 1$. This can be proved inductively. Clearly Q_2 is Hamiltonian. Inductively, assume that Q_{k-1} is hamiltonian. Q_k has vertices consisting of binary strings of length k. Partition Q_k into two sets of vertices by the last bit of the string. For notation, we will write v if the last digit is 0, and w' if the last

digit is 1, and v' if v and v' differ only in the last digit. By definition, v_i is adjacent to v_i'. Then each component consists of 2^{k-1} vertices and the induced subgraph is isomorphic to Q_{k-1}. By assumption, each component is Hamiltonian; if we find some Hamiltonian cycle $C = v_0 \ldots v_{2^{k-1}} v_0$ in the first component, then we can form a Hamiltonian cycle $C' = v_0' \ldots v_{2^{k-1}}' v_0'$ in the other component. From these, we can form a Hamiltonian cycle in Q_k via

$$v_0 \cdots v_{2^k} v_{2^k}' v_{2^k-1}' \cdots v_0' v_0.$$

Pictorially,



The result then follows by induction. ☐

7.5 Solution:

(1) This follows from the proof above.
(2) It is bipartite, but has 13 vertices.
(3) If n is odd, then there is an odd number of squares on the chessboard. We will adjoin two vertices (squares) with an edge if and only if there is a nights move from one vertex to the next. If we fix an origin, and label the squares by the coordinates, then we can partition the vertex set by having $(i, j) \in E$ if $i + j$ is even, and $(i, j) \in O$ if $i + j$ is odd. We need to check that from any point (x, y), a knight's move always changes the parity of the sum; but this is clear since we're moving 2 in one direction and 1 in another, for an odd total change. Thus the board is an odd bipartite graph, so it contains no Hamiltonian cycle. ☐

7.7 Solution:

(1) Typo: G must be a simple graph. Suppose that v, w are non-adjacent vertices. We want to prove that $\deg(v) + \deg(u) \geq n$. Consider G as a subgraph of K_n. Since G has $\frac{(n-1)(n-2)}{2} + 2$ edges, we know $G = K_n - E$, where $|E| = n - 3$. Since v, w are not adjacent, we know $vw \in E$. Let $E' = E \setminus \{vw\}$, so $|E'| = n - 2$. Suppose that at most k edges who are incident v are in E', so that at most $n - 2 - k$ edges who are incident w are in E. Then $\deg(v) + \deg(w) \geq (n - 1 - k) + (n - 1 - (n - 2 - k)) \geq n$. Theorem 7.1 then guarantees that G is Hamiltonian.
(2) If $n = 3$, then we want to find a non-Hamiltonian graph with 2 edges. $K_{1,2}$ is such a graph.

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8.2 Solution: We get the same path as outlined in figure 8.2, except in reverse direction, $LKIFHEBA$. Thus the weights add up to 17. $\square$

8.3 Solution: There is a maximum weight path of weight 44. It is given by $AECFIKHJL$, and also $ABDGECFIKHJL$. The algorithm is the similar except the permanent labels are kept only after all paths have been considered; this makes the algorithm impractical for large graphs. $\square$

8.7 Solution: The maximal cycles are $AEBCD$, and $AEBDC$, with total weight 32. One can find this by searching exhaustively. $\square$