FUNCTION FIELDS AND PROJECTIVE SMOOTH CURVES

JOSÉ ALEJANDRO LARA RODRÍGUEZ

Contents

Abstract. Let K a field. We will show that K always has valuation rings. We will show that a valuation ring A induces a valuation v on K in such a way that the corresponding valuation ring of v is A. Conversely, we will show that given a valuation v on a field K, the corresponding valuation ring of v is a valuation ring. Next we only deal with discrete valuation rings because all valuations on function fields of one variable must be discrete. The discrete valuation rings will be the connection between function fields of one variable and algebraic geometry. Indeed, if C_K is the set of all discrete valuation rings of the function field K/k, we make C_K into an irreducible variety of dimension one over k.

Date: January 14, 2009.

1. Commutative Algebra

1.1. **Modules.**

Proposition 1.1.1. *[Nakayama's lemma] Let M be a finitely generated A-module and I be an ideal of A contained in the Jacobson radical $\mathfrak{R}$ of A.*

i) *Then $IM = M$ implies $M = 0$.*
ii) *If N is a submodule of M such that $M = IM+N$, then $M = N$.*

Proof. Assume $M \neq 0$ and let $m_1, \dots, m_n$ be a minimal system of generators of M. Since $m_n \in M = IM$, we can choose $a_1, \dots, a_n \in I$ such that $m_n = \sum_{i=1}^{n} a_i m_i$. This implies $(1 - a_n)m_n = \sum_{i=1}^{n-1} a_i m_i$. Since $a_n \in \mathfrak{R}$, $1 - a_n$ is a unit in A and, therefore, $m_1, \dots, m_{n-1}$ generate M, which is a contradiction to the minimality of the chosen system of generators.

Since $M = IM + N$ we have $M/N = (IM + N)/N = I(M/N)$. By first part it follows $M/N = 0$ and so $M = N$. □

Let A be a local ring, $\mathfrak{m}$ its maximal ideal, $k = A/\mathfrak{m}$ its residue field. Let M be a finitely generated A-module. $M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, hence is naturally an $A/\mathfrak{m}$-module, i.e. a k-vector space, and as such is finite-dimensional.

Proposition 1.1.2. *Let $x_i (1 \leq i \leq n)$ be elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then the x_i generete M.*

Proof. Let N be the submodule of M generated by the x_i. Then $M = \mathfrak{m}M + N$. Indeed, if $y \in M$, we can write $\bar{y} = \bar{a}_1\bar{x}_1 + \cdots + \bar{a}_n\bar{x}_n$. Then $y = (y-x)+x$, where $x = \sum a_i x_i \in N$ and $y-x \in \mathfrak{m}M$. By (1.1.1(ii)) we have $M = N$. □

1.2. **Localizations at a multiplicative set.**

Let M be an A-module and let S be a multiplicative subset of A. Define a relation $\sim$ on $M \times S$ as follows:

$$(m, s) \sim (m', s') \Leftrightarrow \exists t \in S \text{ such that } t(sm' - s'm) = 0.$$

Let m/s denote the equivalence class of the pair (m, s), let $S^{-1}M$ denote the set of such fractions, and make $S^{-1}M$ into an $S^{-1}A$-module with the obvious definitions of addition and scalar multiplication.

If $\mathfrak{p}$ is a prime ideal of A, we denote $M_\mathfrak{p}$ the localization of M at $S = A \setminus \mathfrak{p}$.

Let $f\colon M' \to M$ be an A-module homomorphism. Then $S^{-1}f\colon S^{-1}M' \to S^{-1}M$ given by $(S^{-1}f)(m'/s) = f(m')/s$ is an $S^{-1}A$-module homomorphism. If $g\colon M \to M''$ is an A-module homomorphism, then

$S^{-1}(g \circ f) = (S^{-1}g) \circ (S^{-1}f)$:

$$\begin{aligned} S^{-1}(g \circ f)(m'/s) &= g(f(m'))/s = (S^{-1}g)(f(m')/s) \\ &= (S^{-1}g)((S^{-1}f)(m'/s)) \\ &= ((S^{-1}g) \circ (S^{-1}f))(m'/s). \end{aligned}$$

Proposition 1.2.1. *The operation S^{-1} is exact, i.e., if* $M' \xrightarrow{f} M \xrightarrow{g} M''$ *is exact at M, then* $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ *is exact at $S^{-1}M$.*

Proof. Since $g \circ f = 0$, we have $(S^{-1}g) \circ (S^{-1}f) = S^{-1}(g \circ f) = 0$ and so $\operatorname{Im} S^{-1}f \subseteq \ker S^{-1}g$. Conversely, let $m/s \in \ker S^{-1}g$, then $g(m)/s = (S^{-1}g)(m/s) = 0/1$. Thus $tg(m) = 0$ for some $t \in S$. But $tg(m) = g(tm)$ since g is an A-module homomorphism. Thus $tm \in \ker g = \operatorname{Im} f$ and so $tm = f(m')$ for some $m' \in M'$. Hence in $S^{-1}M$ we have $m/s = tm/ts = f(m')/ts = (S^{-1}f)(m'/ts) \in \operatorname{Im} S^{-1}f$. Hence $\ker S^{-1}g \subset \operatorname{Im} S^{-1}f$. □

Proposition 1.2.2. *Let M be an A-module. Then the following are equivalent:*

i) $M = 0$
ii) $M_{\mathfrak{p}} = 0$ *for all prime ideals $\mathfrak{p}$ of A.*
iii) $M_{\mathfrak{m}} = 0$ *for all maximal ideals $\mathfrak{m}$ of A.*

Proof. Clearly i) $\Rightarrow$ ii) $\Rightarrow$ iii) is trivial. Assume iii) but suppose $M \neq 0$. Take $x \in M \setminus \{0\}$. The ideal $\mathfrak{a} = \mathtt{Ann}(x) = \{y \in A | yx = 0\}$ is proper because $1x = x \neq 0$. Thus $\mathfrak{a}$ is contained in some maximal ideal $\mathfrak{m}$. Since $M_{\mathfrak{m}} = 0$, we have $x/1 = 0/1$, that is, there exists $y \in A \setminus \mathfrak{m}$ such that $yx = 0$. But $yx = 0$ implies $y \in \mathfrak{a} \subseteq \mathfrak{m}$ which is a contradiction. □

Proposition 1.2.3. *Let $\phi\colon M \to N$ be an A-module homomorphism. Then the following are equivalent:*

i) ϕ *is injective.*
ii) $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *is injective for each prime ideal $\mathfrak{p}$.*
iii) $\phi_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ *is injective for each maximal ideal $\mathfrak{m}$.*

Similarly with "injective" replaced by "surjective" throughout.

Proof. i) $\Rightarrow$. Since $0 \to M \to N$ is exact, by (1.2.1) $0 \to M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is exact, i.e., $\phi_{\mathfrak{p}}$ is injective.

ii)$\Rightarrow$ iii). Trivial because every maximal ideal is a prime ideal.

iii) $\Rightarrow$ i). Let $M' = \ker \phi$, then the sequence $0 \to M' \to M \to N$ is exact, hence $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to N_m$ is exact by (1.2.1), and therefore

$M'_{\mathfrak{m}} \cong \ker \phi_{\mathfrak{m}} = 0$ since $\phi_{\mathfrak{m}}$ is injective. Hence $M' = 0$ by (1.2.2) and ϕ is injective. $\square$

1.3. **Integrality.** Let A be a subring of the ring B. An element x of B is integral over A if there is a monic polynomial $f(T) \in A[T]$ such that $f(x) = 0$. The integral closure of A in B is the ring $\bar{A} = \{x \in B | x \text{ is integral over } A\}$. A is said to be integrally closed in B if $A = \bar{A}$.

Let $A \subseteq B$ be rings. Let $\mathfrak{b}$ be an ideal of B and put $\mathfrak{a} = A \cap \mathfrak{b}$. Then $\iota \colon A/\mathfrak{a} \to B/\mathfrak{b}$ given by $\iota(a + \mathfrak{a}) = a + \mathfrak{b}$ is a monomorphism because $\iota(a + \mathfrak{a}) = \mathfrak{b}$ implies $a \in A \cap \mathfrak{b} = \mathfrak{a}$. So we identify $A/\mathfrak{a}$ with its image in $B/\mathfrak{b}$ and write $A/\mathfrak{a} \subseteq B/\mathfrak{b}$.

Proposition 1.3.1. *Let $A \subseteq B$ be rings, B integral over A.*

i) *If $\mathfrak{b}$ is an ideal of B and $\mathfrak{a} = A \cap \mathfrak{b}$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.*
ii) *If S is a multiplicative closed subset of A, then $S^{-1}B$ is integral over $S^{-1}A$.*

Proof. If $x \in B$, by hypothesis x satisfies a monic equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0, \quad n \geq 1,$$

with coefficients $a_i \in A$. Reducing this equation mod $\mathfrak{b}$ and using that $a_i + \mathfrak{b} = \iota(a_i + \mathfrak{a})$ the result follows.

Now, let $x/s \in S^{-1}B$. The equation above gives:

$$(x/s)^n + (a_1/s)(x/s)^{n-1} + \cdots + a_n/s^n = 0/1$$

which shows that x/s is integral over $S^{-1}A$. $\square$

Proposition 1.3.2. *Let $A \subseteq B$ be rings, C the integral clousure of A in B. Let S^{-1} be a multiplicative closed subset of A. Then $S^{-1}C$ is the integral clousure of $S^{-1}A$ in $S^{-1}B$.*

Proof. By (1.3.1), $S^{-1}C$ is integral over $S^{-1}A$. Conversely, if $b/s \in S^{-1}B$ is integral over $S^{-1}A$, then we have an equation of the form

$$(b/s)^n + (a_1/s_1)(b/s)^{n-1} + \cdots + a_n/s_n = 0,$$

where $a_i \in A, s_i \in S$. Multiplying the equation above by $(st)^n/1$, where $t = s_1 \cdots s_n$ we see that bt in integral over A and so $bt \in C$. Then $(b/s) = (bt/st) \in S^{-1}C$. $\square$

The following proposition shows that integral clousure is a local property.

Proposition 1.3.3. *Let A be an integral domain. Then the following are equivalent:*

i) *A is integrally closed.*

ii) $A_{\mathfrak{p}}$ *is integrally closed for each prime ideal* $\mathfrak{p}$.
iii) $A_{\mathfrak{m}}$ *is integrally closed for each maximal ideal* $\mathfrak{m}$.

Proof. Let K be the field of fractions of A, let C be the integral clousure of A in K, and let $f\colon A \to C$ be the inclusion map of A into C. i) $\Leftrightarrow$ ii) If $A = C$, f is surjective and so $f_{\mathfrak{p}}\colon A_{\mathfrak{p}} \to C_{\mathfrak{p}}$ is surjective for each prime ideal $\mathfrak{p}$. By (1.3.2) $C_{\mathfrak{p}}$ is the integral clousure of $A_{\mathfrak{p}}$ so $A_{\mathfrak{p}} = C_{\mathfrak{p}}$ and $A_{\mathfrak{p}}$ is integrally closed. Conversely, if $A_{\mathfrak{p}}$ is integrally closed, $f_{\mathfrak{p}}\colon A_{\mathfrak{p}} \to C_{\mathfrak{p}}$ is surjective and by (1.2.3) $f\colon A \to C$ is surjective, that is $A = C$.

ii) $\Leftrightarrow$ iii) One implication is trivial. If $A_{\mathfrak{m}}$ is integrally closed, $f_{\mathfrak{m}}\colon A_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is surjective and by (1.2.3) $f\colon A \to C$ is surjective.

□

1.4. **Valuations Rings.**

Definition 1.4.1. Let B be an integral domain, K its field of fractions. B is a *valuation ring* of K if, for each $x \neq 0$, either $x \in B$ or $x^{-1} \in B$ (or both).

Proposition 1.4.2. *If B is a valuation ring, then*
i) *B is a local ring.*
ii) *If B' is a ring such that $B \subseteq B' \subseteq K$, then B' is a valuation ring of K.*
iii) *B is integrally closed in K.*

Proof. i) Let $\mathfrak{m}$ be the set of non-units of B. If $x \in \mathfrak{m}$, $x \neq 0$ then $x \in B$ but $x^{-1} \notin B$ (If x^{-1} were in B, x could be a unit). Conversely, if $x \neq 0$ is such that $x^{-1} \notin B$, then $x \in B$ because B is a valuation ring, and x is not a unit, so $x \in \mathfrak{m}$. Thus $x \in \mathfrak{m}$ if and only if $x = 0$ or $x^{-1} \notin B$. Let's see that $\mathfrak{m}$ is an ideal. If $a \in B$ and $x \in \mathfrak{m}$ but $ax \notin \mathfrak{m}$, then $(ax)^{-1} \in B$ and $x^{-1} = a(ax)^{-1} \in B$ which contradicts $x \in \mathfrak{m}$. Thus $ax \in \mathfrak{m}$. Now, let x, y be non-zero elements of $\mathfrak{m}$. Since $x^{-1}, y^{-1} \notin B$, $x, y \in B$. Now $xy^{-1} \in B$ or $x^{-1}y \in B$. If $xy^{-1} \in B$, then $x + y = (xy^{-1} + 1)y \in B\mathfrak{m} \subseteq \mathfrak{m}$. Similarly, if $x^{-1}y \in B$. Hence $\mathfrak{m}$ is an ideal and therefore B is a local ring.

ii) Obvious.

iii) Let $x \in K$ integral over B. Then we have

$$x^n + b_1 x^{n-1} + \cdots + b_n = 0, \quad n \geq 1,$$

with $b_i \in B$. If $x \in B$ there is nothing to prove. If not, since B is a valuation ring we have $x^{-1} \in B$. Then

$$x = -(b_1 + b_2 x^{-1} + \cdots + b_n x^{1-n}) \in B.$$

□

Let K be a field and Ω a fixed algebraically closed field. Let Σ be the set of all pair (A, f) where A is a subring of K and f is a homomorphism of A into Ω. We partially order this set as follows:

$$(A, f) \leq (A', f') \Leftrightarrow A \subseteq A' \text{ and } f'|A = f.$$

Let $\{(A_i, f_i)\}_{i\in I}$ be a chain in Σ. Let $A = \cup A_i$ and define $f\colon A \to \Omega$ as $f(a) = f_i(a)$ if $a \in A_i$. If $a \in A_i$ and $a \in A_j$, then $(A_i, f_i) \leq (A_j, f_j)$ or $(A_j, f_j) \leq (A_i, f_i)$. In any case we have $f_i(a) = f_j(a)$. This shows that f is well defined. Thus (A, f) is an upper bound for the chain. By Zorn's lemma Σ has at least one maximal element.

Let (B, g) be a maximal element of Σ. We want to prove that B is a valuation ring of K.

Lemma 1.4.3. *B is a local ring and $\mathfrak{m} = \ker g$ is its maximal ideal.*

Proof. Since $\operatorname{Im} g \subseteq \Omega$ is a subring of a field, it is an integral domain. Thus $B/\ker g$ is an integral domain and therefore $\mathfrak{m} = \ker g$ is prime. We extend g to a homomorphism $\bar{g}\colon B_{\mathfrak{m}} \to \Omega$ by putting $\bar{g}(b/s) = g(b)/g(s)$. Now $\bar{g}$ is well defined because

$$\frac{b}{s} = \frac{b'}{s'} \Rightarrow \frac{g(b)}{g(s)} = \frac{g(b)g(sb')}{g(s)g(bs')} = \frac{g(b')}{g(s')}.$$

Since $\bar{g}(b/1) = g(b)/g(1) = g(b)/1$, we have $\bar{g}|B = g$, that is, $(B, g) \leq (B_{\mathfrak{m}}, \bar{g})$. Because of maximality of (B, g) it follows that $B = B_{\mathfrak{m}}$. Then B is a local ring and $\mathfrak{m}$ its maximal ideal. $\square$

Lemma 1.4.4. *Let $x \in K$, $x \neq 0$. Let $B[x]$ be the subring of K generated by x over B, and let $\mathfrak{m}[x]$ be the extension of $\mathfrak{m}$ in $B[x]$. Then either $\mathfrak{m}[x] \neq B[x]$ or $\mathfrak{m}[x^{-1}] \neq B[x^{-1}]$.*

Proof. Suppose that $\mathfrak{m}[x] = B[x]$ and $\mathfrak{m}[x^{-1}] = B[x^{-1}]$. Since $1 \in B[x]$ and $1 \in B[x^{-1}]$ we can write

$$1 = u_0 + u_1 x + \cdots + u_m x^m, \quad (u_i \in \mathfrak{m}) \tag{1}$$

$$1 = v_0 + v_1 x^{-1} + \cdots + v_n x^{-n}, \quad (v_j \in \mathfrak{m}) \tag{2}$$

in which we may assume that the degrees m, n are as small as possible. Suppose that $m \geq n$. Multiplying both sides of (2) by x^n we get $x^n = v_0 x^n + v_1 x^{n-1} + \cdots + v_n$. Then

$$(1 - v_0)x^n = v_1 x^{n-1} + \cdots + v_n.$$

Since $v_0 \in \mathfrak{m}$ and $\mathfrak{m}$ is maximal by (1.4.3), we have that $(1 - v_0)$ is a unit in B. The last equation above becomes

$$x^n = w_1 x^{n-1} + \cdots + w_n. \tag{3}$$

Substitution of (3) in (1) yields a contradiction of the minimality of the exponent m. $\square$

Theorem 1.4.5 (Existence of valuation rings)**.** *Let (B, g) be a maximal element of Σ. Then B is a valuation ring of the field K.*

Proof. Let $x \in K$. By (1.4.4) we may assume that $\mathfrak{m}[x]$ is not the whole ring $B' = B[x]$. Then $\mathfrak{m}[x]$ is contained in a maximal ideal $\mathfrak{m}'$ of B', and we have $\mathfrak{m}' \cap B = \mathfrak{m}$ because $\mathfrak{m}' \cap B$ is a proper ideal of B and contains $\mathfrak{m}$. Hence the embedding of B in B' induces an embedding of the field $k = B/\mathfrak{m}$ in the field $k' = B'/\mathfrak{m}'$; also $k' \cong k[\bar{x}]$ where $\bar{x}$ is the image of x in k' (the natural map $b_0 + b_1 x + \cdots + b_s x^s \to (b_0 + \mathfrak{m}) + \cdots + (b_s + \mathfrak{m})(x + \mathfrak{m})^s$ has kernel $\mathfrak{m}'$). Since $1 = \sum u_i x^i$ it follows $\bar{x}$ is algebraic over k, and therefore k' is a finite algebraic extension of k.

$$\begin{tikzcd} B \arrow[r] \arrow[d] & B[x] \arrow[r] & B[x]/\mathfrak{m}' = (B/\mathfrak{m})[\bar{x}] = k' \\ k = B/\mathfrak{m} \arrow[ur] & & \end{tikzcd}$$

Now the homomorphism g induces an embedding $\bar{g}$ of k in Ω, since by (1.4.3) $\mathfrak{m}$ is the kernel of g. Since Ω is algebraically closed, $\bar{g}$ can be extended to an embedding $\bar{g}'$ of k' into Ω in the following way. Let p be the minimum polynomial of $\bar{x}$, let p' the irreducible polynomial obtained by applying the map $\bar{g}$ to the coefficients of p and let β be a root of p' in Ω. It follows we have an isomorphism between $k[\bar{x}] = k/(p)$ and $\operatorname{Im} g/(p') \subset \Omega$. Composing $\bar{g}'$ with the natural homomorphism $B' \to k'$, we have $g' \colon B' \to \Omega$ which extends g. Since the pair (B, g) is maximal, it follows that $B' = B$ and therefore $x \in B$. $\square$

Definition 1.4.6. Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ two local rings. We say that B dominates A if $A \subseteq B$ and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$.

Notice that B dominates A if and only if $A \subseteq B$ and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$. The definition implies that $\mathfrak{m}_A \subseteq A \cap \mathfrak{m}_B$. Suppose $x \in A \cap \mathfrak{m}_B$ but $x \notin \mathfrak{m}_A$; then $\mathfrak{m}_A + (x) = A$ and so there exist $y \in A$ and $z \in \mathfrak{m}_A$ such that $z + xy = 1$ and we get the contradiction $1 - z, z \in \mathfrak{m}_B$. Thus we have the equality.

From the definition is clear that the *domination* relation is a parcial order in the set Σ of the local rings contained in a field K.

Theorem 1.4.7. *Let K be a field. Then*

i) *A local ring R contained in K is a valuation ring of K if and only if it is a maximal element of the set of local rings contained in K, with respect to the relation of domination.*

ii) *Every local ring contained in K is dominated by some valuation ring of K.*

Proof. i) Let Σ as above. Σ is nonempty because $K \in \Sigma$. Let $\{(A_i, \mathfrak{m}_i)\}$ be a chain of elements of Σ. Let $A = \cup A_i$ and $\mathfrak{m}' = \cup \mathfrak{m}_{A_i}$. Then A is subring of A and $\mathfrak{m}'$ is an ideal of A. Let $\mathfrak{m}_A$ be the set of non-units of A. If $x \in \mathfrak{m}_A$, then $x \in A_i$ for some i. Now, x is a non-unit of A_i (If were a unit of A_i it would be a unit of A), that is $x \in \mathfrak{m}_{A_i}$ and hence $x \in \mathfrak{m}'$. Conversely, if $x \in \mathfrak{m}_{A_i}$ for some i, then x is a non-unit of A. Indeed, if $xy = 1$ for some y in some A_j, then $x, y \in A_k$ where $k = \max\{i, j\}$ which says that x is a unit in A_k. This contradicts $x \in \mathfrak{m}_{A_k}$. Then $\mathfrak{m}_A = \mathfrak{m}'$ is an ideal and thus A is a local ring with maximal ideal $\mathfrak{m}_A$, i.e., $(A, \mathfrak{m}_A) \in \Sigma$. By Zorn's lemma Σ has maximal elements.

Let $(B, \mathfrak{m}_B)$ be a maximal element. We want to prove that B is a valuation ring. Let Ω be an algebraic clousure of $k = B/\mathfrak{m}_B$. Let $g\colon B \to \Omega$ the compositum of $B \to k$ and $k \to \Omega$. Thus (B, g) belongs the set of S of the pairs (A, f) where A is a subring of K and f is a homomorphism of A into Ω. Let us see that (B, g) is maximal element of S. Let $(B', g') \in S$ such that $B \subseteq B'$ and $g'|B = g$. Let $\mathfrak{m} = \ker g'$. Since $B'/\ker g' \cong \operatorname{Im} g' \subseteq \Omega$, $\mathfrak{m}$ is prime. We extend g' to $\bar{g}'\colon B'_{\mathfrak{m}} \to \Omega$ as in (1.4.3). Then $(B'_{\mathfrak{m}}, \bar{g}') \in \Sigma$. Since we have $B \subseteq B' \subseteq B'_{\mathfrak{m}}$ and $\mathfrak{m}_B \subset \mathfrak{m}' \subseteq \mathfrak{m}B'_{\mathfrak{m}}$ and B is a maximal element of Σ it follows that $B = B' = B'_{\mathfrak{m}}$. Thus (B, g) is maximal element of S. By (1.4.5) B is a valuation ring.

Conversely, if B is a valuation ring, by (1.4.2) B is a local ring. Let us see that $(B, \mathfrak{m}_B)$ is maximal element of Σ. Suppose that $B \subseteq B'$ and $\mathfrak{m}_B \subseteq \mathfrak{m}_{B'}$, where B' is a local ring. By (1.4.2) is also a valuation ring. Then $\mathfrak{m}_B = \mathfrak{m}_{B'}$. Indeed, if $x \in \mathfrak{m}_{B'}$ but $x \notin B$, then $x^{-1} \in B \subseteq B'$ and x would be a unit in B'. Thus $\mathfrak{m}_{B'} \subseteq B$. Therefore $\mathfrak{m}_B = B \cap \mathfrak{m}_{B'} = \mathfrak{m}_{B'}$. It follows that $B = B'$, and hence B is a maximal element.

ii) If R is a local ring contained in K, let Σ be the set of all local rings contained in K which contain R. Proceed as in i). □

1.5. **Valuations and Valuations Rings.** A *totally ordered group* G is an abelian group $(G, +)$ with a relation $\alpha < \beta$ which is trichotomic, transitive and such that preserves the group operation, i.e. if $\alpha < \beta$ then $\alpha + \gamma < \beta + \gamma$.

Definition 1.5.1. Let K be a field and let G be a totally ordered group. A valuation v of K with values in G is a surjective function $v\colon K^\times \to G$ (where $K^\times = K - \{0\}$ is the multiplicative group of K) satisfying:

(1) For $x, y \in K^\times$, $v(xy) = v(x) + v(y)$.
(2) For $x, y \in K^\times$ such that $x + y \neq 0$, $v(x+y) \geq \min\{v(x), v(y)\}$.

Note that the first condition says that v is a group homomorphism.

We extend the valuation v to the whole of K by defining $v(0) = \infty$, where ∞ is a symbol not in G, such that $\alpha < \infty$ and $\infty+\infty = \alpha+\infty = \infty + \alpha = \infty$ for all $\alpha \in G$. The new v also satisfies (1) and (2).

The following statements are simple consequences of the definition.

(1) $v(1) = 0$.
(2) $v(x^{-1}) = -v(x)$ for all $x \neq 0$.
(3) $v(x) = v(-x)$
(4) If $v(x) \neq v(y)$, then $v(x+y) = \min\{v(x), v(y)\}$.

Proposition 1.5.2. *Let v be a valuation on a field K and define*

$$\mathcal{O}_v = \{x \in K | v(x) \geq 0\}$$
$$\mathcal{P}_v = \{x \in K | v(x) > 0\}.$$

Then $\mathcal{O}_v$ is a valuation ring of K, called the valuation ring of v. *$\mathcal{P}_v$ is the maximal ideal of $\mathcal{O}_v$.*

Proof. Since $v(xy) = v(x) + v(y)$ and $v(x-y) \geq \min\{v(x), v(-y)\} = \min\{v(x), v(y)\}$ it follows that $\mathcal{O}_v$ is a subring and $\mathcal{P}_v$ is an ideal. Now, if $x \in K$, but $x \notin \mathcal{O}_v$, then $v(x) < 0$. Thus $v(x^{-1}) = -v(x) > 0$ and so $x^{-1} \in \mathcal{O}_v$. Therefore, given $x \in K$, we have $x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$. It follows that K is the field of fractions of $\mathcal{O}_v$ because $x = x/1 = 1/x^{-1}$. Hence $\mathcal{O}_v$ is a valuation ring. Now, $x \in \mathcal{O}_v$ is a unit if and only if $x^{-1} \in \mathcal{O}_v$, i.e., if and only if $v(x) \geq 0$ and $-v(x) = v(x^{-1}) \geq 0$. Therefore $\mathcal{O}_v^\times = \ker v = \{x \in K | v(x) = 0\}$. It follows that $\mathcal{P}_v$ is the set of nonunits and hence $\mathcal{P}_v$ is the maximal ideal of $\mathcal{O}_v$. □

Since $v\colon (K^\times, \cdot) \to (G, +)$ is a group epimorphism with $\ker v = \mathcal{O}_v^\times$, we have $(K^\times/\mathcal{O}_v^\times, \cdot) \cong (G, +)$. We have the following proposition.

Proposition 1.5.3. *If A is a valuation ring and K is the fraction field of A, then $K^\times/A^\times$ is an ordered group and the natural projection is a valuation with valuation ring A and value group $K^\times/A^\times$.*

Proof. We know that $K^\times/A^\times$ is an abelian group. If $x, y \in K^\times$, define $x \bmod A^\times \leq y \bmod A^\times$ if $yx^{-1} \in A$ and $x \bmod A^\times < y \bmod A^\times$ if $yx^{-1} \in A - A^\times$. One easily check the definition does not depend on the representatives and that $K^\times/A^\times$ is an ordered gruop.

In order to prove that $v\colon K^\times \to K^\times/A^\times$ is valuation, we note that

$$v(xy) = xy \bmod A^\times = (x \bmod A^\times)(y \bmod A^\times)$$

for any $x, y \in K^\times$. On the other hand, if $x + y \neq 0$, let us assume that $x \bmod A^\times \leq y \bmod A^\times$, so that $yx^{-1} \in A$. Then $(x+y)x^{-1} = 1 + yx^{-1} \in A$, that is

$$v(x+y) = (x+y) \bmod A^\times \geq x \bmod A^\times = \min\{x \bmod A^\times, y \bmod A^\times\}$$

Therefore v is a valuation. The valuation ring of v is given by

$$\mathcal{O}_v = \{x \in K^\times | v(x) \geq \bar{1}\} \cup \{0\} = A.$$

□

Remark 1.5.4. Propositions (1.5.2) and (1.5.3) shows that the concepts of valuation rings and valuation are essentially the same.

1.6. **Discrete Valuations Rings.**

Definition 1.6.1.

i) A valuation v is discrete if its value group G is the integers. The corresponding valuation ring is called a *discrete valuation ring of v*.
ii) An integral domain A is a *discrete valuation ring* if there is a discrete valuation v of its field of fractions K such that $A = \mathcal{O}_v$, that is, A is the valuation ring of v.

Recall that the dimension (Krull dimension) of a ring A is the maximum possible length of a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of distinct prime ideals in A.

Proposition 1.6.2. *If A is a discrete valuation ring, then A is a Noetherian local domain of dimension one.*

Proof. Let v a valuation on the field of fractions K such that $A = \mathcal{O}_v$. By (1.5.2) and (1.4.2) is a local ring. If I is an ideal in A, there is a least positive integer k such that $v(x) = k$ for some $x \in I$. Then I contains every $y \in A$ with $v(y) \geq k$, and therefore the only nonzero ideals in A are the ideals $\mathcal{P}_k = \{y \in A | v(y) \geq k\}$, $(k \geq 1)$. It follows that A is Noetherian. Since v is surjective, there exists $x \in \mathcal{P}$ such that $v(x) = 1$, and then $\mathcal{P} = (x)$, and thus $\mathcal{P}_k = (x^k)$. Hence $\mathcal{P}$ is the unique nonzero prime ideal of A. Therefore $\dim A = 1$. □

1.7. **Regular local rings.**

Definition 1.7.1. Let A be a Noetherian local ring of dimension d, $\mathfrak{m}$ its maximal ideal and $k = A/\mathfrak{m}$ its residue field. A is a *regular local ring* if $\dim A = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$.

Theorem 1.7.2 (6.2A). *Let A be a Noetherian local domain of dimension one, with maximal ideal $\mathfrak{m}$ and field of quotients K. Then the following conditions are equivalent.*

(i) *A is a discrete valuation ring;*
(ii) *A is integrally closed;*
(iii) *$\mathfrak{m}$ is a principal ideal;*
(iv) *A is a regular local ring.*

Proof. i) $\Rightarrow$ ii). Let v be a valuation of K sucht that $A = \mathcal{O}_v$. By (1.5.2) A is valuation ring, and by (1.4.2) A is integrally closed.

ii) $\Rightarrow$ iii). Suppose A is integrally closed. Let $a \in \mathfrak{m}$, $a \neq 0$. Since A is a local domain of dimension one, $\mathfrak{m}$ is the only prime ideal of A, so the radical ideal of (a) is $\mathfrak{m}$. Since A is Noetherian, $\mathfrak{m}$ is finitely generated, say $\mathfrak{m} = (x_1, \ldots, x_d)$. For each $i\,(1 \leq i \leq d)$, there exist $n_i > 0$ such that $x_i^{n_i} \in (a)$. Let $n = 1 + \sum_{i=1}^{d}(n_i - 1)$. The ideal $\mathfrak{m}^n$ is genereated by the products $x_1^{r_1} \cdots x_d^{r_d}$ where $\sum_{i=1}^{d} r_i = n$. Observe that $r_i \geq n_i$ for at least one index i (otherwise, $r_i < n_i$ for all i implies $r_i \leq n_i - 1$ and hence $n \leq n - 1$). So each monomial $x_1^{r_1} \cdots x_d^{r_d} \in (a)$ and therefore $\mathfrak{m}^n \subseteq (a)$.

If $\mathfrak{m} \subset (a)$, then $\mathfrak{m} = (a)$ and $\mathfrak{m}$ is principal. Otherwise, choose n such that $\mathfrak{m}^n \subseteq (a)$ but $\mathfrak{m}^{n-1} \nsubseteq (a)$. Choose $b \in \mathfrak{m}^{n-1}$ and $b \notin (a)$ and let $x = a/b \in K$. If x^{-1} were in A, then $b = ax^{-1}$ would be in (a), so $x^{-1} \notin A$. Therefore x^{-1} is not integral over A. We claim that $x^{-1}\mathfrak{m} \nsubseteq \mathfrak{m}$. Otherwise, $\mathfrak{m}$ would be an $A[x^{-1}]$-module with $\mathtt{Ann}(\mathfrak{m}) = 0$, that is, it would be a faithful $A[x^{-1}]$-module, finitely generated as an A-module. But it is a fact that an element $x \in B$ is integral over A if and only if $A[x]$ is a finitely generated A-module. We have $x^{-1}\mathfrak{m} = (b/a)\mathfrak{m} \subseteq A$ because $bt \in \mathfrak{m}^n \subseteq (a)$ for any $t \in \mathfrak{m}$. Then $\mathfrak{m} \subseteq Ax$ and hence $\mathfrak{m} = Ax = (x)$.

iii) $\Rightarrow$ iv) By (1.1.2) we have $\dim_k \mathfrak{m}/\mathfrak{m}^2 \leq 1$, where $k = A/\mathfrak{m}$. But $\mathfrak{m} \neq \mathfrak{m}^2$ (By (1.1.1) $\mathfrak{m} = \mathfrak{m}^2$ would imply $\mathfrak{m} = 0$). Thus $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1 = \dim A$.

iv) $\Rightarrow$ i) Let $x \in \mathfrak{m} - \mathfrak{m}^2$. It follows that $\mathfrak{m} = (x)$ and $(x^k) \neq (x^{k+1})$. So, given any nonzero element $a \in A$, we have $(a) = (x^k)$ for some exactly one value of k. Define $v(a) = k$. We extend v to $K^\times$ by defining $v(ab^{-1}) = v(a) - v(b)$. It is easy to check that v is well defined and that it is a discrete valuation. Therefore A is the valuation ring of v. $\square$

1.8. **Dedekind domains.**

Proposition 1.8.1. *Let A be a Noetherian domain of dimension one. Then the following are equivalent:*

i) *A is integrally closed.*
ii) *For each nonzero prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ is a discrete valuation ring.*

Proof. If A is integrally closed, by (1.3.3) $A_\mathfrak{p}$ is integrally closed for each nonzero prime ideal $\mathfrak{p}$ and by (1.7.2) $A_\mathfrak{p}$ is a discrete valuation ring. Conversely, if ii) holds, by (1.7.2) $A_\mathfrak{p}$ is integrally closed and by (1.3.3) A is integrally closed. □

Definition 1.8.2. A *Dedekind domain* is a Noetherian, integrally closed, integral domain of dimension one.

About Dedekind domains, we have the following fact.

Theorem 1.8.3. *The integral clousure of a Dedekind domain in a finite extension field of its quotient field is again a Dedekind domain.*

Proof. See Propositions 8.1 and 12.8 of [4]. □

2. The projective curve of a function field

2.1. **Some results of Algebraic Geometry.** The following results were given in class.

Proposition 2.1.1. *If X is a variety, then $k[X] \cong \mathcal{O}_X(X)$ as k-algebras.*

Proposition 2.1.2. *Let X be a closed subset of $\mathbb{A}^n$. Then $p \mapsto \mathfrak{m}_p$ gives a 1-1 correspondence between the points of X and the maximal ideals of $k[X]$, where $\mathfrak{m}_p$ is the maximal corresponding to the point $p \in X$.*

Proposition 2.1.3. *Let X be a closed subset of $\mathbb{A}^n$. For each $p \in X$, $\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p}$ as k-algebras.*

Proposition 2.1.4. *Let X be an irreducible variety. Then $k(X) \cong Frac(R)$, where R is the coordinate ring of any nonempty affine open set $U \subseteq X$, and hence $k(X)$ is a finite generated extension field of k, of trascendence degree $= \dim X$.*

Theorem 2.1.5 (class)**.** *The functor which sends the affine variety $(X, \mathcal{O}_X)$ to the finitely generated reduced k-algebra $\mathcal{O}_X(X) = k[X]$ is an equivalence of categories of affine varieties over k and finitely generated reduced k-algebras.*

Proposition 2.1.6 (Class)**.** *Any finite generated field extension of k is $k(X)$ for some X. In fact we can take X to be a hypersurface in $\mathbb{A}^n$.*

Proposition 2.1.7 (class)**.** *Suppose that $X \subseteq \mathbb{A}^n$ is an affine variety. Then X is regular at p if and only if $\mathcal{O}_{X,p}$ is a regular local ring.*

Proposition 2.1.8 (class)**.** *If X and Y are varieties, then $X \times Y$ has a canonical structure of variety. If X and Y are projective (resp. quasi-projective) so is $X \times Y$.*

2.2. **The curve of a function field.** Let X be a smooth irreducible curve over k (a variety of dimension one) where k is a fixed algebraically closed base field, and let $K = k(X)$ the function field of X. Then K/k is a function field of one variable. For each $p \in X$, by (2.1.7) the local ring $\mathcal{O}_{X,p}$ is a regular local ring of dimension one, and so by (1.7.2) it is a discrete valuation ring. Thus local rings of X define a subset of the set of all discrete valuation rings of K. To distinct points correspond distinct local rings, as shows the next lemma.

Lemma 2.2.1. *Let X be an irreducible quasi-projective variety, let $p, q \in X$, and suppose that $\mathcal{O}_{X,q} \subseteq \mathcal{O}_{X,p}$ as subrings of $k(X)$. Then $p = q$.*

Proof. Realize X as a subset of $\mathbb{P}^n$ for some n. Replacing X by its clousure, we may assume that X is projective. After a suitable linear change of coordinates in $\mathbb{P}^n$, we may assume that neither p or q is in the hyperplane $V(x_0)$. Thus $p, q \in X \cap D(x_0)$ which is affine, so we may assume that X is an affine variety. Then there are maximal ideals $\mathfrak{m}_p, \mathfrak{m}_q \subseteq k[X]$ such that $\mathcal{O}_{X,p} = k[X]_{\mathfrak{m}_p}$ and $\mathcal{O}_{X,q} = k[X]_{\mathfrak{m}_q}$. If $\mathcal{O}_{X,q} \subseteq \mathcal{O}_{X,p}$, then $\mathfrak{m}_p \subseteq \mathfrak{m}_q$. Since $\mathfrak{m}_p$ is a maximal ideal, we have $\mathfrak{m}_p = \mathfrak{m}_q$, hence $p = q$ by (2.1.2). □

Let K/k be a function field of one variable, that is, K is a finitely generated field extension of k with transcendence degree one. Let C_K be the set of all discrete valuation rings of K/k. C_K is always nonempty by (1.4.5). We want to show that C_K has a structure of variety. We will sometimes call the elements of C_K points, and write $\mathcal{P} \in C_K$, where $\mathcal{P}$ is the maximal ideal of the local ring $\mathcal{O}_{\mathcal{P}}$.

Remark 2.2.2. C_K is infinite, because it contains all the local rings of any smooth curve with function field K; by (2.2.1) those local rings are all distinct and there are infinitely many of them.

We give C_K the finite complement topology: U is open in C_K if $C_K - U$ either is finite or is all of C_K. Equivalently, the closed sets are the finite subsets and the whole space. Note that by the remark above, the open sets of C_K other than the empty set are infinite.

The following lemma shows that any element of K is contained in infinitely many discrete valuation rings and that for every $x \in K$, $U_x = \{R \in C_K | x \in R\}$ is an open set.

Lemma 2.2.3. *U_x is an open set of C_K. Equivalently $C_K - U_x = \{R \in C_K | x \notin R\}$ is a finite set.*

Proof. Notice that if R is a valuation ring, then $x \notin R$ if and only if $y = 1/x \in \mathfrak{m}_R$. Thus $C_K - U_x = \{R \in C_K | x \notin R\} = \{R \in C_K | x^{-1} \in \mathfrak{m}_R\}$. If $y \in k$, there are no such R. Assume $y \notin k$. Let us consider $k[y] \subseteq K$, the subring of K generated by k and $y = 1/x$. Since k is algebraically closed, y is transcendental over k, hence $k[y]$ is a polynomial ring. Since K/k is a function field of one variable, if $z \in K$ is transcendental over k, then $K/k(z)$ is a finite extension because it is algebraic and finitely generated. Since K/k has trascendence degree one, $\{y, z\}$ can not be algebraically independent. Then there exists a nonzero polynomial $p(T_1, T_2) \in k[T_1, T_2]$ such that $p(y, z) = 0$. Since both y and z are transcendental over k, y and z must appear in the expression for $p(y, z)$. Therefore, y is algebraic over $k(z)$. Thus

$$[K \colon k(y)] = [K \colon k(y, z)][k(y, z) \colon k(y)] \leq [K \colon k(z)][k(y, z) \colon k(y)] < \infty$$

Thus $K/k(y)$ is a finite extension of $k(y)$. Now let B the integral clousure of $k[y]$ in K. Then by (1.8.3), B is a Dedekind domain, and it is also a finitely generated k-algebra.

Now if y is contained in a discrete valuation ring R of K/k, then $k[y] \subseteq R$, and since R is integrally closed in K, we have $B \subseteq R$. Let $\mathfrak{n} = \mathfrak{m}_R \cap B$. Then $\mathfrak{n}$ is a maximal ideal of B, and B is dominated by R. But $B_{\mathfrak{n}}$ is also a discrete valuation ring of K/k, hence $B_{\mathfrak{n}} = R$ by the maximality of valuation rings (1.4.7).

If furthermore $y \in \mathfrak{m}_R$, then $y \in \mathfrak{n}$. By (2.1.5) B is the affine coordinate ring of some affine variety X. Since B is a Dedekind domain, X has dimension one and is smooth. To say that $y \in \mathfrak{n}$ says that y, as a regular function on X, vanishes at the point of X corresponding to $\mathfrak{n}$. But $y \neq 0$, so it vanishes only at a finite set of points; these are in 1-1 correspondence with the maximal ideal of B by (2.1.2), and $R = B_{\mathfrak{n}}$ is determined by the maximal ideal $\mathfrak{n}$. Hence we conclude that $y \in \mathfrak{m}_R$ for only finitely many $R \in C_K$ as required. □

Corollary 2.2.4. *If $R \in C_K$, then there is a smooth affine curve X such that $R \cong \mathcal{O}_{X,p}$ for some $p \in X$.*

Proof. Given R, let $y \in R - k$. Then the construction used in the proof of (2.2.3) gives such a curve. □

Remark 2.2.5. For any $R \in C_K$ the residue field $R/\mathfrak{m}_R$ is (isomorphic to) k, because $\mathcal{O}_{X,p}/\mathfrak{m}_p \cong k$.

Now, we need to define a sheaf of k-algebras for C_K.

If $U \subseteq C$ is an open subset of C, we define $\mathcal{O}_C(U) = \cap_{\mathcal{P}\in U}\mathcal{O}_{\mathcal{P}}$. Given $f \in \mathcal{O}_C(U)$, let $\hat{f} \colon U \to k$ given by $\hat{f}(\mathcal{P}) = f \bmod \mathcal{P} \in \mathcal{O}_{\mathcal{P}}/\mathcal{P} = k$. Distinct points of $\mathcal{O}_C(U)$ gives distinct functions. If $f, g \in \mathcal{O}_C(U)$ are such that $\hat{f} = \hat{g}$, but $f \neq g$, then $f \bmod \mathcal{P} = g \bmod \mathcal{P}$ for all $\mathcal{P} \in U$, and hence $f - g \in \mathcal{P}$ for infinitely many $\mathcal{P} \in C$ because U is infinite. But this implies that $C - U_{1/(f-g)}$ is infinite, and by (2.2.3) this set must be finite. Thus $f = g$. We will write f instead of $\hat{f}$. Therefore we can identify the elements of $\mathcal{O}_C(U)$ with functions from U to k.

It is easy to see that $\mathcal{O}_C$ is a sheaf of k-algebras wich is a subsheaf of the sheaf of all k-valued functions.

To see that $(C, \mathcal{O}_C)$ is an abstract variety we need the following propositions.

Proposition 2.2.6. *Let X be a smooth quasi-projective irreducible curve and let $K = k(X)$. Then there is an open subset U of C_K and a homeomorphism $\varphi \colon X \to U$ such that the induced sheaf by φ and the sheaf $\mathcal{O}_C|U$ are isomorphic.*

Proof. Let K be the function field of X. By (2.1.7) and (1.7.2), it follows that for each $p \in X$, the local ring $\mathcal{O}_{X,p}$ is a discrete valuation ring. Furthermore, by (2.2.1), distinct points give rise to distinct subrings of K. Let U be the set of local rings of X and let $\varphi \colon X \to U$ be the bijective map defined by $\varphi(p) = \mathcal{O}_{X,p}$.

Let us see that U is an open set of C. If we show that that $C - U$ is contained in some finite set we are done. So we will show that U contains a nonempty open set V. Since X has a base consisting of open affine subsets, we may assume X is affine. $k[X]$ is a finitely generated k-algebra and K is the quotient field of $k[X]$. Since $k[X] = \mathcal{O}_X(X)$, U is the set of localizations of $k[X]$ at its maximal ideals. Since these local rings are all discrete valuation rins, U consists in fact of all discrete valuation rings of K/k containing $k[X]$. Let $x_1, \ldots, x_n$ be a set of generators of $k[X]$ over k. Obviously we have $k[X] \subseteq \mathcal{O}_{\mathcal{P}}$ if and only if $x_1, \ldots, x_n \in \mathcal{O}_{\mathcal{P}}$. Let U_i be the open set $\{\mathcal{O}_{\mathcal{P}} | x_i \in \mathcal{O}_{\mathcal{P}}\}$. Then $U = \cap_{i=1}^{n} U_i$ and therefore U is an open subset of C.

φ is a homeomorphism between X and U because is a closed continuous map.

□

Proposition 2.2.7. *If K/k is a function field of one variable and C_K is the set of all discrete valuation rings of K/k, then C_K is an abstract variety.*

Proof. It only remain to prove that every point $\mathcal{P} \in C$ has an open neighborhood which is isomorphic to an affine variety. Let $\mathcal{P} \in C = C_K$ be any point. Then by (2.2.4) there is an affine curve X and a point $p \in X$ with $\mathcal{O}_{\mathcal{P}} \cong \mathcal{O}_{X,p}$. It follows that the function field of X is K, and then by (2.2.6), X is isomorphic to an open subset of C. □

Proposition 2.2.8. *Let X be an smooth irreducible curve, let $p \in X$, let Y be a projective variety, and let $\varphi\colon X - \{p\} \to Y$ be a morphism. Then there exists a unique morphism $\bar{\varphi}\colon X \to Y$ extending φ.*

Proof. Embed Y as a closed subset $\mathbb{P}^n$ for some n. Then it will be sufficient to show that φ extends to a morphism of X into $\mathbb{P}^n$, because if it does, the image is necessarily contained in Y. Thus we reduce to the case $Y = \mathbb{P}^n$. Let U be an open set contained in the open set $\cap_{i=0}^n D(x_i)$, so every set of homogeneous coordinates $x_0, \ldots, x_n$ are all nonzero. We can also choose U in such a way that $\varphi(X - \{p\}) \cap U \neq \emptyset$. (We can do this by induction on n. If $\varphi(X - \{p\}) \cap U = \emptyset$, then $\varphi(X - \{p\}) \subseteq \mathbb{P}^n - U$. But $\mathbb{P}^n - U$ is the union of the hyperplanes $H_i = V(x_i)$. Since $\varphi(X - \{p\})$ is irreducible, it must be contained in H_i for some i. But $H_i \cong \mathbb{P}^{n-1}$). For each i, j, $\alpha_{i,j}(x_0, \ldots, x_n) = x_i/x_j$ is a regular function on U. We can write $\alpha_{ij} = \alpha_{i0}/\alpha_{j0}$. Pulling it back by φ, we obtain a regular function $f_{ij} = (\alpha_{i0}\varphi)/(\alpha_{j0}\varphi)$ on an open subset of X, and we can thought of it as an element of K, where $K = k(X)$.

Now let v the valuation of K associated with the valuation ring $\mathcal{O}_{X,p}$. Let $r_i = v(f_{i0}),\ i = 0, 1 \ldots, n,\ r_i \in \mathbb{Z}$. Then

$$v(f_{ij}) = v(\alpha_{i0}\varphi) - v(\alpha_{j0}\varphi) = r_i - r_j, \quad i, j = 0, 1 \ldots, n$$

Choose k such that r_k is minimal among $r_0, \ldots, r_n$. Then $v(f_{ik}) \geq 0$ for all i, that is, $f_{0k}, \ldots, f_{nk} \in \mathcal{O}_{X,p}$. Next, define $\bar{\varphi}(p) = (f_{0k}(p), \ldots, f_{nk}(p))$, and $\bar{\varphi}(q) = \varphi(q)$ for $q \neq p$. To show that $\bar{\varphi}$ is morphism, it will be enough to show that regular functions in a neighborhood of $\bar{\varphi}(p)$ pull back to regular functions on X. Consider the open set $D(x_k)$. Then $\bar{\varphi}(p) \in D(x_k)$, since $f_{kk}(p) = 1$. Since the coordinate ring of $D(x_k)$ is $k[x_0/x_k, \ldots, x_n/x_k]$, these functions pull back to $f_{0k}, \ldots, f_{nk}$ which are regular at p by construction. It follows that for any smaller neighborhood $\bar{\varphi}(p) \in V \subset D(x_k)$, regular functions on V pull back to regular functions on X. Hence $\bar{\varphi}$ is a morphism. The uniqueness follows by the following fact: if X is irreducible, and two morphisms agree in a nonempty open subset of X, then they are equal. This completes the proof. □

Now we prove the main result of this work.

Theorem 2.2.9. *Let K be a function field of one variable over k. Then the abstract smooth curve C_K defined above is isomorphic to a smooth projective curve.*

Proof. Since C_K is a variety, then every point $\mathcal{P} \in C$ has an open neighborhood which is isomorphic to an affine variety. Since C has the finite complement topology, C is compact, so we can cover it with a finite number of open subsets U_i, each of which is isomorphic to an affine variety V_i. Embed $V_i \subseteq \mathbb{A}^{n_i}$ and let Y_i be the closure of V_i in $\mathbb{P}^{n_i}$. Then Y_i is a projective variety, and we have a morphism $\varphi_i\colon U_i \to Y_i$ which is an isomorphism of U_i onto its image. By (2.2.8) applied to the finite set of points $C - U_i$, we can find a morphism $\bar{\varphi}_i\colon C \to Y_i$ extending φ_i. By (2.1.8) $\prod Y_i$ is a projective variety. Let $\varphi\colon C \to \prod Y_i$ be the diagonal map $\varphi(\mathcal{P}) = \prod \bar{\varphi}_i(\mathcal{P})$, and let Y be the clousure of the image of φ. Then Y is a projective variety, and $\varphi\colon C \to Y$ is a morphism whose image is dense in Y.

Now let us prove that φ is an isomorphism. For any point $\mathcal{P} \in C$, we have $\mathcal{P} \in U_i$ for some i. There is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\varphi"] & Y \arrow[d, "\pi"] \\ U_i \arrow[u] \arrow[r, "\varphi_i"] & Y_i \end{tikzcd}$$

of dominant morphisms, where π is the projection map onto the ith factor. Thus we have inclusions of local rings $\mathcal{O}_{\varphi(\mathcal{P}),Y_i} \to \mathcal{O}_{\varphi(\mathcal{P}),Y} \to \mathcal{O}_{C,\mathcal{P}}$. Since the first and the two outside ones are isomorphic, so the middle one is also. Thus we see that for any $P \in C$, the map $\varphi^*_{\mathcal{P}}\colon \mathcal{O}_{\varphi(\mathcal{P}),Y} \to \mathcal{O}_{C,\mathcal{P}}$ is an isomorphism.

Let us see that φ is a bijection. Let $q \in Y$. Taking a localization of the integral clousure of $\mathcal{O}_{Y,q}$ at a maximal ideal, we see that $\mathcal{O}_{Y,q}$ is dominated by some discrete valuation ring $\mathcal{O}$ of K/k. But $\mathcal{O} = \mathcal{O}_{\mathcal{P}}$ for some $\mathcal{P} \in C$ and $\mathcal{O}_{\varphi(\mathcal{P})} \cong \mathcal{O}$, so by (2.2.1) it follows that $q = \varphi(\mathcal{P})$. On the other hand, is clear that φ is injective because distinct points of C correspond to distinct subrings of K. It follows that φ is an isomorphism. □

The following corollaries are immediate consequences of the theorem.

Corollary 2.2.10. *Every abstract smooth curve is isomorphic to a quasi-projective curve. Every smooth quasi-projective curve is isomorphic to an open subset of a smooth projective irreducible curve.*

Corollary 2.2.11. *Every irreducible curve is birationally equivalent to a smooth projective irreducible curve.*

Corollary 2.2.12. *the following three categories are equivalent:*

i) *smooth projective irreducible curves, and dominant morphisms;*
ii) *quasi-projective irreducible curves, and dominant rational maps;*
iii) *functions fields of one variable over k, and k-homomorphisms.*

Proof. There is an obvious functor from i) to ii). We know that X and Y are birationally isomorphic if and only if $k(X) \cong k(Y)$ as k-algebras, so the assignation $X \to k(X)$ is a functor from ii) to iii) which induces an equivalence of categories.

Now, let C_K the associated curve to the function field K/k which by theorem (2.2.9) is a smooth projective curve. Let $K_2 \to K_1$ be a k-homomorphism. Let X and Y such that $K_2 = k(Y)$ and $K_1 = k(X)$. Let C_{K_2}, C_{K_1} the corresponding curves. By (2.2.6) we have $X \leftrightarrow U$ and $Y \leftrightarrow U'$, where $U \subseteq C_{K_1}$ and $U' \subseteq C_{K_2}$ are open subsets; since $X \to Y$, then we have a morphism $\varphi \colon U \to C_{K_2}$. By (2.2.8) φ extends to a morphism $\bar{\varphi} \colon C_{K_1} \to C_{K_2}$. On the other hand, if $K_3 \to K_2 \to K_1$ are two morphisms, from the uniqueness part of (2.2.8) follows that the corresponding morphisms $C_1 \to C_2 \to C_3$ and $C_1 \to C_3$ are compatible. Hence the functor $K \to C_K$ is a funtor from iii) to i). Furthermore, it is the inverse of the given functor i)$\to$ii)$\to$ iii). Therefore we have an equivalence of categories. □

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