GRAPH THEORY HW 5

10.3 Solution:

(1) If some vertex v is known to be an end-vertex of a tree T, then $T - v$ is a tree on the remaining $n - 1$ (labeled) vertices. We know that there are $(n-1)^{(n-3)}$ such trees, and for each tree our vertex v may adjoin any of the $n - 1$ vertices and be an end-vertex. Thus, there are $(n-1)^{(n-2)}$ (labeled) trees on n vertices in which v is an end vertex.

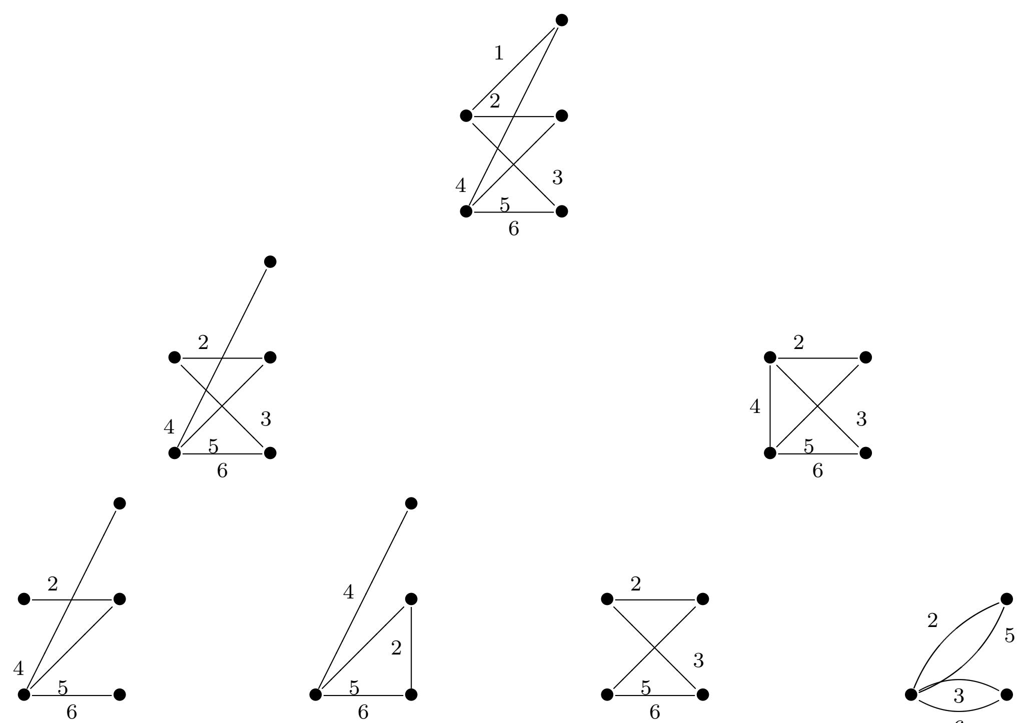
(2) We need to calculate $\lim_{n\to\infty} \frac{(n-1)^{(n-2)}}{n^{(n-2)}}$, since this is the fraction of trees with given end vertex over all trees.

$$\begin{aligned} \lim_{n\to\infty} \frac{(n-1)^{(n-2)}}{n^{(n-2)}} &= \lim_{n\to\infty} \left(1 + \frac{-1}{n}\right)^n \lim_{n\to\infty} (1 - 1/n)^{-2} \\ &= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1}. \quad \square \end{aligned}$$

10.5 Solution:

(1) Given a spanning tree T of G, there are two possibilities: either T contains e or it does not. The number of spanning trees of G not containing e is clearly $\tau(G - e)$. Furthermore, the number of spanning trees of G containing e is $\tau(G \backslash e)$. Indeed, a tree T in G containing e corresponds to a tree $T \backslash e$ for $G \backslash e$, and conversely.

(2)



We have first taken edge 1, and found $G-1, G\backslash 1$. Then on the left side we did the same for edge 3, and on the right edge 4. The idea is to select edges so that what remains is a tree, or a tree with perhaps a disjoint union of cycles. Indeed, it is easy to calculate $\tau(G)$ for any graph in the last row. For this sequence of graphs we have $\tau(G_1, G_2, G_3, G_4) = (1, 3, 4, 4)$. Thus, $\tau(K_{2,3}) = 12$. □

10.7 Solution:

(1) There are $(n-1)T(n)$ pairs (e, T), where $e \in T$ and T is a labeled tree on n vertices. Each edge e is a bridge, and determines a partition of the n vertices. Given that one (of the two) partite set has k vertices, there are $\frac{1}{2}\binom{n}{k}$ such partitions. Since the edge e bridges these two sets, it can be one of $k(n-k)$ such edges, since we pick a vertex from the first k edges and another from the $n-k$ remaining edges. Finally T determines one of $T(k)$ subtrees on the first partite set, and one of $T(n-k)$ subtrees on the second partite set. We have shown that

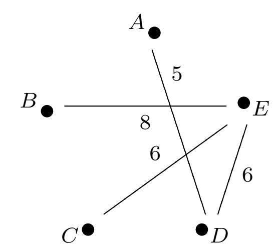
$$(n-1)T(n) = \sum_{k=1}^{n-1} \frac{1}{2}\binom{n}{k} k(n-k)T(k)T(n-k).$$

The result follows immediately.

(2) This follows immediately from Cayley's theorem, since $T(m) = m^{(m-2)}$.

□

11.2 Solution:

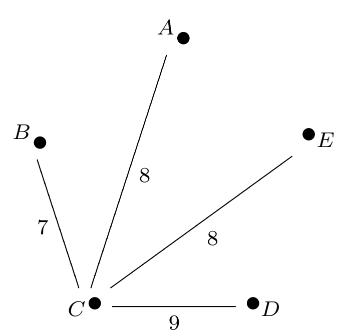


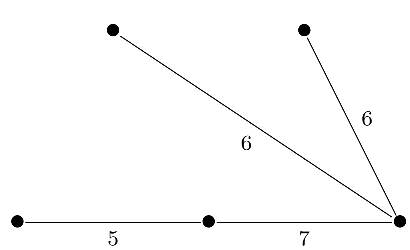
So the minimum weight spanning tree has weight 25. □

11.4 Solution:

(1) First, choose a maximum weight edge, and add the largest weight edges possible while not creating a cycle.

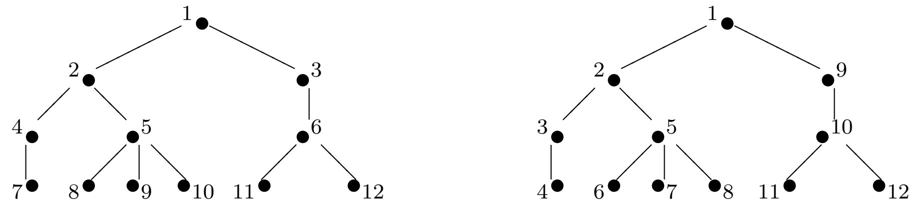
(2)





□

11.9 Solution:



These graphs represent breadth first, and then depth first searches through G. □

11.11 Solution: We can borrow much of the discussion from page 55. The only changes are the resistances, and the location of E. Indeed, we have

$$\begin{aligned} 4i_1 + i_2 + 5i_0 &= 0, \\ i_2 + 2i_5 &= -E, \\ 3i_3 + 2i_5 + 6i_7 &= 0, \\ i_2 - 6i_4 + 2i_5 + 6i_7 &= 0. \end{aligned}$$

The remaining system of equations remains unchanged, i.e.

$$\begin{aligned} i_0 - i_1 &= 0, \\ i_1 - i_2 - i_3 + i_5 &= 0, \\ i_3 - i_4 - i_7 &= 0, \\ i_5 - i_6 - i_7 &= 0. \end{aligned}$$

Now, we solve these 8 linear equations using standard techniques from linear algebra, i.e.

$$\begin{bmatrix} 5 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & -E \\ 0 & 0 & 0 & 3 & 0 & 2 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & -6 & 2 & 0 & 6 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} \overset{rref}{\longrightarrow} \left[\begin{array}{cc} I_8 & \begin{array}{c} 7E/192 \\ 7E/192 \\ -21E/64 \\ 11E/384 \\ -53E/768 \\ -43E/128 \\ -111E/256 \\ 25E/256 \end{array} \end{array} \right]. \quad \square$$