Math/CS 443/543
Fall 2006

Solution to exercise 3.10(iii): If G is a simple graph with line graph $L(G)$ then there is a homomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(L(G))$ which sends a permutation of the vertices of G to the induced permutation of the edges of G (which are just the vertices of $L(G)$). In general this homomorphism is neither injective nor surjective. (It is surjective but not injective for $G = K_2$, an isomorphism fro $G = K_3$ and injective but not surjective for $G = K_4$.) We show that for $G = K_5$ this homomorphism is an isomorphism. This with earlier exercises shows that $\mathrm{Aut}(P) \cong \mathrm{Aut}(L(K_5)) \cong \mathrm{Aut}(K_5) \cong S_5$.

Label the vertices of K_5 with labels $\{1, 2, 3, 4, 5\}$. Then the edges of K_5 and the vertices of $L = L(K_5)$ are labeled by the 2-element subsets of $\{1, 2, 3, 4, 5\}$. If $\sigma \in S_5$ is a permutation of $\{1, 2, 3, 4, 5\}$ and σ fixes every 2-element set, then σ must fix each element. Indeed, if $\sigma(\{1, 2\}) = \{1, 2\}$ and $\sigma(\{3, 4\}) = \{3, 4\}$, then clearly $\sigma(5) = 5$; similarly for the other elements of $\{1, 2, 3, 4, 5\}$. This shows that $S_5 \cong \mathrm{Aut}(K_5) \to \mathrm{Aut}(L)$ is injective.

To see that it is surjective, let τ be a permutation of the 2-element subsets of $\{1, 2, 3, 4, 5\}$. To say that τ is an automorphism of L is to say that two 2-element subsets are adjacent in L if and only if their images under τ are adjacent. In particular, if H is a complete subgraph of L, then so is $\tau(H)$. We argue below that the complete subgraphs of L with 4 vertices are in 1-1 correspondence with $\{1, 2, 3, 4, 5\}$ and so an automorphism τ of L gives a permutation of $\{1, 2, 3, 4, 5\}$ whose image under $\mathrm{Aut}(K_5) \to \mathrm{Aut}(L)$ is τ.

If $i \in \{1, 2, 3, 4, 5\}$, then the four vertices of L which include i are all adjacent in L and so induce a complete subgraph. (So if $i = 1$, we're talking about $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$.) Conversely, if we have four 2-element subsets of $\{1, 2, 3, 4, 5\}$ any two of which intersect in one element, then in fact they all intersect in one element. To see this, write one of the sets as $\{i, j\}$ and another as $\{i, k\}$ ($j \neq k$). If a third were $\{j, k\}$ then there could be no fourth, so the third must be $\{i, l\}$ and similarly with the fourth. (Note that this argument does not work with 5 replaced by 4.)

This argument can be used to show that $\mathrm{Aut}(L(K_n)) \cong \mathrm{Aut}(K_n) \cong S_n$ for all $n \geq 5$.