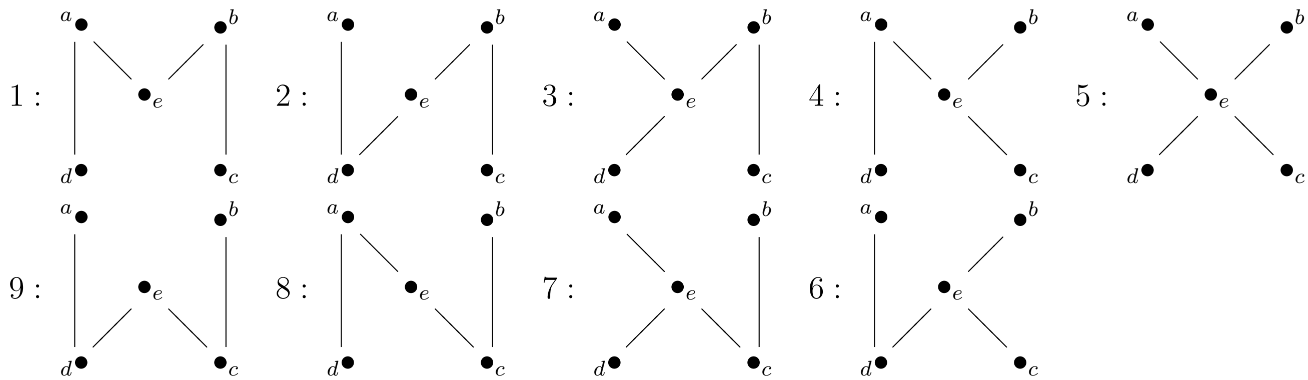
GRAPH THEORY HW 4

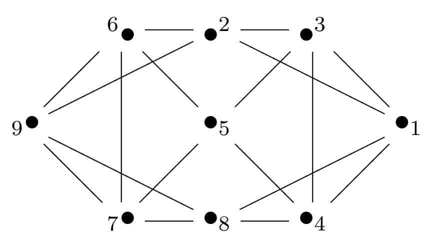
9.3 Solution:

(1) A graph G is bipartite if and only if every cycle has even length. If T is a tree, then it has no cycles; thus every tree is bipartite.
(2) The only tree that is a complete bipartite graph is $K_{1,n}$. We have shown that $K_{n,n}$ for $n > 1$ is Hamiltonian, and thus contains a cycle. Then notice that $K_{n,m}$ contains $K_{n,n}$ as a subgraph when $n \leq m$. $\square$

9.5 Solution: Each tree can be identified with the deletion of a pair of edges of the original graph, one edge from the "left" cycle and one from the "right" cycle. The resulting 9 trees are



As a side note, notice that we can form a graph by the set of all spanning trees of G, whereby two spanning trees are connected if and only if they differ by a change of one edge. Call this graph $T(G)$. We have that $T(G)$ is always connected by problem 9.11. In this case, $T(G)$ is the following 4-regular graph, which is presented so that we can easily compare automorphism groups with the original graph.

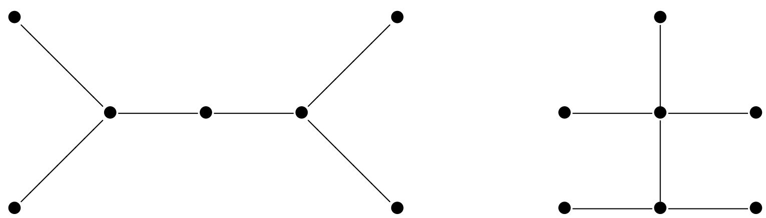


$\square$

9.7 Solution:

(1) $\gamma(K_5) = 6$, $\xi(K_5) = 4$.
(2) $\gamma(K_{3,3}) = 4$, $\xi(K_{3,3}) = 5$.
(3) $\gamma(W_5) = 4$, $\xi(W_5) = 4$.
(4) $\gamma(N_5) = 0$, $\xi(N_5) = 0$.
(5) $\gamma(P) = 6$, $\xi(P) = 9$. $\square$

9.9 Solution: Suppose that the tree T has $n \geq 3$ vertices. First we label each vertex with its maximum distance between it and any other vertex in T. Consider that no end vertex v can be in the center, because the vertex w it is adjacent to has maximum distance strictly less than v. By deleting all of the end vertices we necessarily reduce by one the label of each remaining vertex, since each maximal path necessarily goes from some vertex to an end vertex. Thus, by deleting all end vertices in stages (or pruning), we preserve the center. After repeatedly pruning, we will eventually have that $n < 3$. If $n = 1$, then this is the unique center. If $n = 2$, then these are two adjacent vertices in the center. [Notice that if $n \geq 3$, then at least one vertex must have degree greater than 1, so pruning does not create the null graph.]



□

9.10 Solution:

(1) It is sufficient to consider G connected. So suppose that C^* has an edge in common with every spanning tree of G. If $G - C^*$ were connected, then there would exist a spanning tree T of $G - C^*$; but then T is a spanning tree of G disjoint from C^*. Since no such tree exists, $G - C^*$ is disconnected. Thus C^* contains a cutset.

(2) Suppose that C^* has an edge in the complement of every spanning tree of G. Then C^* contains a cycle. Indeed, if C^* did not contain a cycle, then it could be extended to a spanning tree T of G; but then C^* would not have an edge in the complement of T, contrary to our assumption. □

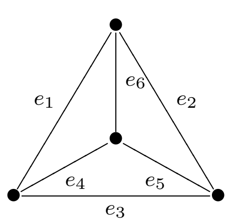
9.11 Solution:

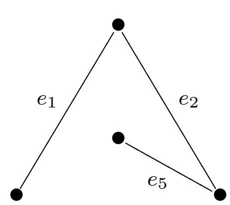
(1) Since T_1 is a tree, we know that $T_1 - e$ is disconnected, and consists of two components. Since T_2 is connected, and each edge is a bridge, there is exactly one edge f joining these two components. It follows that $(T_1 - e) \cup \{f\}$ is connected, and contains $n - 1$ edges, so is a spanning tree.

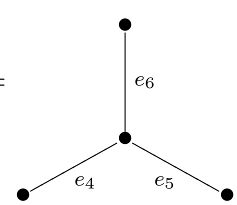
(2) If $e \in T_1 \setminus T_2$, then there is some $f \in T_2$ such that $(T_1 - e) \cup \{f\}$ is a spanning tree. We can repeat this process until every edge $e \in T_1$ is in T_2. Since all trees have the same number of edges, we then have that $T_1 = T_2$. □

Problem. *Find two distinct spanning trees in K_4 and for each of them find the corresponding fundamental sets of cycles and cutsets.*

Solution:

$K_4 =$ 

$T_1 =$ 

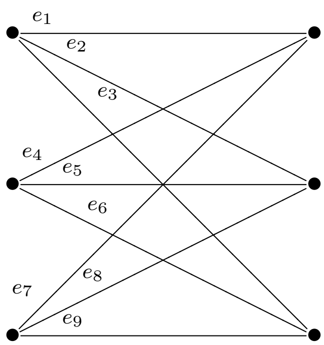
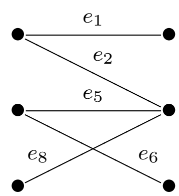
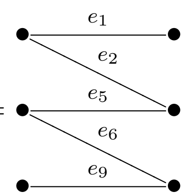
$T_2 =$ 

Denote the set fundamental sets of cycles and cutsets by $\mathcal{C}_i$, $\mathcal{K}_i$, respectively. Then

$$\begin{aligned}
\mathcal{C}_1 &= \{\{e_2, e_5, e_6\}, \{e_1, e_2, e_3\}, \{e_1, e_2, e_5, e_4\}\} \\
\mathcal{K}_1 &= \{\{e_1, e_4, e_3\}, \{e_2, e_3, e_4, e_6\}, \{e_4, e_5, e_6\}\} \\
\mathcal{C}_2 &= \{\{e_1, e_4, e_6\}, \{e_2, e_5, e_6\}, \{e_3, e_4, e_5\}\} \\
\mathcal{K}_2 &= \{\{e_1, e_4, e_3\}, \{e_1, e_6, e_2\}, \{e_2, e_5, e_3\}\}. \quad \Box
\end{aligned}$$

Problem. *Find two distinct spanning trees in $K_{3,3}$ and for each of them find the corresponding fundamental sets of cycles and cutsets.*

Solution:

$K_{3,3} =$  $T_1 =$  $T_2 =$ 

Then we have

$$\begin{aligned}
\mathcal{C}_1 &= \{\{e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_4, e_5\}, \{e_1, e_2, e_7, e_8\}, \{e_5, e_6, e_8, e_9\}\} \\
\mathcal{K}_1 &= \{\{e_1, e_4, e_7\}, \{e_2, e_3, e_4, e_7\}, \{e_3, e_4, e_5, e_7, e_8\}, \{e_3, e_6, e_7, e_8\}, \{e_7, e_8, e_9\}\} \\
\mathcal{C}_2 &= \{\{e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_4, e_5\}, \{e_1, e_2, e_5, e_6, e_7, e_9\}, \{e_5, e_6, e_8, e_9\}\} \\
\mathcal{K}_2 &= \{\{e_1, e_4, e_7\}, \{e_2, e_3, e_4, e_7\}, \{e_3, e_4, e_5, e_7, e_8\}, \{e_3, e_6, e_7, e_8\}, \{e_7, e_8, e_9\}\}.
\end{aligned}$$

Notice that $\mathcal{K}_1 = \mathcal{K}_2$ even though $T_1 \neq T_2$. $\Box$