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Algebraic Geometry
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Here are some exercises. There are also many interesting problems in Shafarevitch.

1. Identify the set of $n \times n$ matrices over k with $\mathbf{A}^{n^2}$ in the obvious way. Are the following subsets open, closed, locally closed, or none of these (in the Zariski topology of course): diagonal matrices; diagonalizable matrices; symmetric matrices; invertible matrices fixing a given line in k^n; matrices with 6 as an eigenvalue; matrices of rank r; nilpotent matrices?

2. Identify the set of monic polynomials in one variable with coefficients in k and of degree n with $\mathbf{A}^n$ in the obvious way. Are the following subsets open, closed, locally closed or none of the above: irreducible polynomials; polys divisible by a given poly; polys relatively prime to a given poly; polys with distinct roots; odd polys?

3. Show that if k is not algebraically closed, every closed subset of $\mathbf{A}^n$ is a "hypersurface," i.e., is the zero set of just one equation.

4. (Finite sets are complete intersections.) Suppose $X \subset \mathbf{A}^n$ is a finite set. Show that there are n polynomials $f_1, \ldots, f_n$ so that $X = V(f_1, \ldots, f_n)$.

5. Let $k \subseteq k'$ be two fields. Show that the Zariski topology on $\mathbf{A}^n_k$ is the topology induced by the Zariski topology on $\mathbf{A}^n_{k'}$.

6. Identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the obvious way. Prove that the Zariski topology on $\mathbf{A}^2$ is not the product of the Zariski topologies on $\mathbf{A}^1$.

7. Show that for any subsets $X \subseteq \mathbf{A}^n$ and $J \subseteq k[x_1, \ldots, x_n]$, $V(I(V(J))) = V(J)$ and $I(V(I(X))) = I(X)$.

8. Show that a topological space is irreducible if and only if every pair of non-empty open subsets has a non-empty intersection if and only if every non-empty open subset is dense.

9. Prove that the image of an irreducible topological space under a continuous map is irreducible.

10. Prove that $X = \mathbf{A}^2 - (0,0)$ is not an affine variety. (Hint: Compute the ring of regular functions of X.)

11. (Closedness is local) One of the following two exercises is correct and the other is not. Figure out which is which and then do the correct one. (i) Prove that if $X \subseteq Y$ are topological spaces then X is closed in Y if and only if for each $y \in Y$ there exists an open subset U of Y containing y such that $X \cap U$ is closed in U. (ii) Prove that if $X \subseteq Y$ are topological spaces then X is closed in Y if and only if for each $x \in X$ there exists an open subset U of Y containing x such that $X \cap U$ is closed in U.

12. For any affine variety Z, we write $k[Z]$ for its affine coordinate ring. Prove that $k[X \times Y] \cong k[X] \otimes_k k[Y]$.

13. Let $f : X \to Y$ be a regular map of affine varieties. We have a ring homomorphism $f^* : k[Y] \to k[X]$ induced by pull back of functions. Prove that f^* is injective if and only if the image of f is dense in Y. (In particular, f need not be onto.)

14. With the notations of the previous exercise, prove that f^* is surjective if and only if f induces an isomorphism between X and a closed subvariety of Y. (In particular, it is not enough that f be injective.)

15. If $f \in k[x_1, \ldots, x_n]$ then $D(f) = \mathbf{A}^n - V(f)$ is an open subset called a principle affine open of $\mathbf{A}^n$. We proved in class (or will prove soon) that this is indeed affine, i.e., that it is isomorphic to a closed subset of some affine space. Show that every affine open subset of $\mathbf{A}^n$ is a $D(f)$. For extra credit, you can say whether the same is true with $\mathbf{A}^n$ replaced by any affine variety.