GRAPH THEORY HW 7

24.3 Solution:

(1) It is clear that PQ has non-negative entries since P and Q both have non-negative entries. It is sufficient to prove that p_iQ is a probability vector whenever p_i is a probability vector and Q is a transition matrix. Indeed, we have that $(p_iQ)_j = \sum_{k=1}^{n} p_i Q_{jk}$. Summing over j, we have

$$\begin{aligned}\sum_{j=1}^{n}(p_iQ)_j &= \sum_{j=1}^{n}\sum_{k=1}^{n} p_i Q_{jk} \\ &= \sum_{k=1}^{n} p_i \sum_{j=1}^{n} Q_{jk} \\ &= \sum_{k=1}^{n} p_i \cdot 1 = 1.\end{aligned}$$

Hence, p_iQ is indeed a probability vector, since the sum of its (non-negative) components is 1.

(2) The digraph of PQ has an adjacency matrix that is the product of the adjacency matrices of P and Q. □

24.4 Solution:

(2) Let s_i be a state in M, an irreducible chain. From each s_j, there exists a non-zero probability p_j of executing a path from $s_j \to s_i$ since the underlying digraph is strongly connected. Let $0 < x = \mathrm{Min}\{p_j\}$. The probability of never re-visiting s_i is less than $\lim_{n\to\infty}(1-x)^n = 0$. Hence s_i is persistent.

(1) By part (2) above, if M is irreducible, we are done. Otherwise, we can partition the vertex set of the underlying digraph into non-empty sets V_1, V_2, where the digraph induced by V_1 is irreducible and each edge from goes from V_2 to V_1. By part (2), we know that there exists a persistent state in V_2.

(3) Infinite Markov chains can be defined similarly, where instead of demanding that the matrix P be finite, we allow it to be countable. We still demand that the entries be non-negative, and that the sum of the row entries be one.

The most obvious example is sometimes called a right-shift on $\mathbb{N}$. Here is a graph:

$$\bullet_0 \longrightarrow \bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \cdots$$

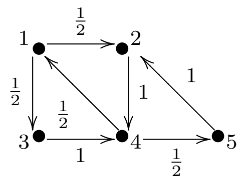
Each state n is transient, because after time $t = n+1$, it is impossible to return to state n regardless of the initial state. □

Problem. *Consider the Markov chain M with 5 vertices and the following transition probabilities:*

$$p_{12} = \frac{1}{2},\ p_{13} = \frac{1}{2},\ p_{24} = 1,\ p_{34} = 1,\ p_{41} = \frac{1}{2},\ p_{45} = \frac{1}{2},\ p_{52} = 1,$$

and all other $p_{ij} = 0$. Draw the corresponding weighted graph. Is M irreducible? What is the period of vertex v_1?

Solution:



M is irreducible since the digraph is strongly connected. We claim that the period of vertex v_1 is 3. To see this, notice that starting from v_1, we must pass through v_4 before returning to v_1. It takes 2 moves to get to v_4; from v_4 we can either return to v_1 or take a trip around the cycle $(5, 2, 4)$, which has length 3. In either case, at v_4 we have used $2 + 3c$ moves, where c is the number of trips around the cycle; so we use $3 + 3c$ moves to return to v_1. $\square$

Extra Credit

Problem. *For the Markov chain M above, find a stationary distribution. Prove that the powers of the transition matrix T do not converge to any matrix Q.*

Solution: Let T be the transition matrix for M. Then we want to find a vector x such that $xT = T$. Equivalently, we want to find $(T - I)^T x^T = 0$, or $x^T \in N\left((T - I)^T\right)$. It follows from linear algebra that $x = r \cdot (1, \frac{3}{2}, \frac{1}{2}, 2, 1)$. Since we also require that x is a probability vector (i.e. the sum of the entries is 1), we must choose $r = \frac{1}{6}$. Thus

$$x = \left(\frac{1}{6}, \frac{1}{4}, \frac{1}{12}, \frac{1}{3}, \frac{1}{6}\right)$$

is (the only) stationary distribution.

We want to show that the powers of P do not converge to any matrix Q. Notice that $T^5 = T^2$. It follows that $T^n \cdot T^3 = T^n$ for all $n \geq 2$, so that the powers T^n is one of T^2, T^3, T^4 depending on the reduction of n modulo 3 ($n \geq 2$). Since these are all distinct, we know that $\lim_{n\to\infty} T^n$ does not exist. $\square$