RATIONAL CURVES ON ELLIPTIC SURFACES

DOUGLAS ULMER

Abstract

We prove that a very general elliptic surface $\mathcal{E} \to \mathbb{P}^1$ over the complex numbers with a section and with geometric genus $p_g \geq 2$ contains no rational curves other than the section and components of singular fibers. Equivalently, if $E/\mathbb{C}(t)$ is a very general elliptic curve of height $d \geq 3$ and if L is a finite extension of $\mathbb{C}(t)$ with $L \cong \mathbb{C}(u)$, then the Mordell-Weil group $E(L) = 0$.

1. Introduction

Fix a field k and consider elliptic curves E over $K = k(t)$. When k is a finite field, we showed in [Ulm07] that there are often finite extensions L of K which are themselves rational function fields (i.e., $L \cong k(u)$) such that the rank of $E(L)$ is as large as desired. Indeed, under a mild parity hypothesis on the conductor of E (which should hold roughly speaking in one half of all cases), we may take extensions of the form $L = k(t^{1/d})$ with d varying through integers prime to p. More generally, for any elliptic curve E over K with $j(E) \notin k$ there is a finite extension of K of the form $k'(u)$ with k' a finite extension of k such that E obtains unbounded rank in the layers of the tower $k'(u^{1/d})$. Our aim in this note is to show that the situation is completely different when k is the field of complex numbers.

From now on we take $k = \mathbb{C}$. If E is an elliptic curve over $\mathbb{C}(t)$, the *height* of E is the smallest non-negative integer d such that E has a Weierstrass equation

$$y^2 = x^3 + a(t)x + b(t)$$

where $a(t)$ and $b(t)$ are polynomials of degree $\leq 4d$ and $\leq 6d$ respectively. Our results concern elliptic curves of height $d \geq 3$.

1.1. Theorem. *A very general elliptic curve E over $\mathbb{C}(t)$ of height $d \geq 3$ has the following property: For every finite rational extension $L \cong \mathbb{C}(u)$ of $\mathbb{C}(t)$, the Mordell-Weil group $E(L) = 0$.*

Here "very general" means that in the relevant moduli space, the statement holds on the complement of a countable union of proper closed subsets. See Subsection 3.4 below for more details.

The theorem shows in particular that there is no hope of producing elliptic curves of large rank over $\mathbb{C}(u)$ by starting with a general curve over $\mathbb{C}(t)$ and iteratively making rational field extensions. What can be done with special elliptic curves remains a very interesting open question about which we speculate in the last section.

Now we connect the theorem with the title of the paper. Attached to $E/\mathbb{C}(t)$ is a unique elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ with the properties that $\mathcal{E}$ is smooth over $\mathbb{C}$, and π is proper and relatively minimal with generic fiber isomorphic to $E/\mathbb{C}(t)$. Conversely, given a relatively minimal elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$, its generic fiber is an elliptic curve over $\mathbb{C}(t)$. The height of E is then equal to the degree of the invertible sheaf $\omega = (R^1\pi_*\mathcal{O}_\mathcal{E})^{-1}$ on $\mathbb{P}^1$. It is well known that the height d of E is 0 if and only if $\mathcal{E}$ is a product $E_0 \times \mathbb{P}^1$; $d = 1$ if and only if $\mathcal{E}$ is a rational surface; $d = 2$ if and only if $\mathcal{E}$ is a K3 surface; and $d \geq 3$ if and only if the Kodaira dimension of $\mathcal{E}$ is 1. See [Ulm11, Lecture 3] for details on the assertions in this paragraph.

The Mordell-Weil group $E(\mathbb{C}(t))$ is canonically isomorphic to the group of sections of $\pi : \mathcal{E} \to \mathbb{P}^1$ via a map which sends a section to its image in the generic fiber. Suppose $L = \mathbb{C}(u)$ is a finite extension of $\mathbb{C}(t)$ over which E has a rational point. Then we have a diagram

$$
\begin{array}{ccc}
\mathcal{E}' & \xrightarrow{\tilde{f}} & \mathcal{E} \\
{\scriptstyle s}\Big\uparrow\Big\downarrow{\scriptstyle \pi'} & & \Big\downarrow{\scriptstyle \pi} \\
\mathbb{P}^1_u & \xrightarrow{f} & \mathbb{P}^1_t.
\end{array}
$$

Here $\pi' : \mathcal{E}' \to \mathbb{P}^1_u$ is the elliptic surface attached to E/L; it is birational to (but not in general isomorphic to) the fiber product $\mathcal{E} \times_{\mathbb{P}^1_t} \mathbb{P}^1_u$. The section s corresponds to the hypothesized rational point in $E(L)$. It is clear that $\tilde{f}(s(\mathbb{P}^1_u))$ is a rational curve on $\mathcal{E}$ (i.e., a reduced and irreducible subscheme of dimension 1 whose normalization is $\mathbb{P}^1$) which is not contained in any fiber of π. Conversely, if $C \subset \mathcal{E}$ is a rational curve not contained in any fiber of π with normalization $g : \mathbb{P}^1 \to C$, and if $L = \mathbb{C}(u)$ is the extension of $\mathbb{C}(t)$

corresponding to $f = \pi \circ g : \mathbb{P}^1 \to \mathbb{P}^1$, then we have the following diagram:

$$
\begin{array}{ccccc}
\mathcal{E}' & \longrightarrow & \mathbb{P}^1_u \times \mathcal{E} & \longrightarrow & \mathcal{E} \\
\Big\uparrow\!{\scriptstyle s}\;\Big\downarrow{\scriptstyle \pi'} & & \Big\uparrow\;\Big\downarrow\;{\scriptstyle g}\nearrow & & \Big\downarrow{\scriptstyle \pi} \\
\mathbb{P}^1_u & =\!=\!= & \mathbb{P}^1_u & \xrightarrow{\;\;f\;\;} & \mathbb{P}^1_t.
\end{array}
$$

Here the section in the middle comes from the universal property of the fiber product, $\mathcal{E}'$ is a relatively minimal desingularization of $\mathbb{P}^1_u \times \mathcal{E}$, and the section on the left is the unique lift of the section in the middle. Since a rational curve in $\mathcal{E}$ not contained in a fiber of π maps finite-to-one to the base $\mathbb{P}^1_t$, we call it a "rational multisection". This discussion shows that the theorem above is equivalent to the following.

1.2. Theorem. *The only rational curves on a very general elliptic surface $\mathcal{E} \to \mathbb{P}^1$ of height $d \geq 3$ (i.e., geometric genus $p_g = d - 1 \geq 2$) are the zero section and components of singular fibers. Equivalently, $\mathcal{E}$ has no rational multisections other than the zero section.*

The theorem is obviously false for $d \leq 1$ since every rational or ruled surface contains infinitely many rational curves. The same is known to be true in the case $d = 2$, the case of elliptic K3 surfaces; see [BHT11, Rmk. 6]. (Thanks to Remke Kloosterman and Brendan Hassett for clarifying this point.) The case $d = 1$ has been used to construct elliptic curves of the highest known rank over $\mathbb{C}(t)$. See [Shi86] and [Sti87].

Note that the theorem is strictly stronger than the theorem of [Cox90] saying that a very general $\mathcal{E}$ as above has no sections other than the zero section. There are a number of results in the literature along the lines of our result, saying for example that a general hypersurface in $\mathbb{P}^n$ of sufficiently large degree contains no rational curves. Our result is related, but necessarily trickier because the surfaces in question certainly do contain rational curves, namely the zero section and the components of bad fibers. Nevertheless, a variant of an idea of Voisin [Voi96] will allow us to show that generically there are no others.

Fedor Bogomolov points out that a very simple construction contained in a paper with Yuri Tschinkel [BT07, Rmk. 6.11] yields elliptic surfaces of every height $d > 2$ with no rational multisections. More generally, for every $g \geq 0$ and all sufficiently large d, they produce an elliptic fibration of degree d without multiple fibers and with no multisection of geometric genus $\leq g$. Their construction is easily modified to produce elliptic fibrations with section but without other multisections of low genus. Our method is completely different, and we hope it and the conjecture it suggests cast enough light to be of independent interest.

In the following section, we give a heuristic sketch of why the theorem should be true which is very suggestive. The following three sections contain a proof of the theorem which is logically independent of the heuristic sketch. The final section of the paper presents further speculations based on the heuristics in Section 2.

2. Heuristics

It is known ([Cox90]) that a very general elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ of height $d \geq 3$ has no sections other than the zero section, and its Néron-Severi group has rank 2, generated by the zero section and the class of a fiber of π: $NS(\mathcal{E}) = \langle O, F \rangle$. (This is also implied by Theorem 1.2. Our proof, although related, is independent of that of [Cox90].) Under the intersection pairing on $NS(\mathcal{E})$, we have $O^2 = -d$, $O.F = 1$, $F^2 = 0$. The canonical class $K = K_{\mathcal{E}}$ is $(d-2)F$.

Suppose $C \subseteq \mathcal{E}$ is a rational curve which is not the zero section and which is not supported in a fiber of π. In $NS(\mathcal{E})$, we have $C \sim eO + fF$ for certain integers e and f, namely $e = C.F$ and $f = C.O + de$. Since C is not the zero section, $C.O \geq 0$ and so $f \geq de$. Also, since C is not the zero section, it is not a section and so $e > 1$.

Now let L be the invertible sheaf $\mathcal{O}_{\mathcal{E}}(eO + fF)$ with integers $e > 1$ and $f \geq de$. It is easy to see that

$$(2.1) \qquad \pi_* L \cong \mathcal{O}_{\mathbb{P}^1}(f) \oplus \mathcal{O}_{\mathbb{P}^1}(f - 2d) \oplus \mathcal{O}_{\mathbb{P}^1}(f - 3d) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(f - de)$$

and $R^1\pi_* L = 0$. Since we assumed that $f \geq de$, we find that $H^1(\mathcal{E}, L) = H^2(\mathcal{E}, L) = 0$, and so, by Riemann-Roch,

$$h^0(\mathcal{E}, L) = \chi(\mathcal{E}, L) = \frac{1}{2}L.(L - K) + \chi(\mathcal{E}, \mathcal{O}).$$

(Alternatively, it is easy to compute $h^0(\mathcal{E}, L)$ directly from equation (2.1).)

This shows that the dimension of the linear series $|L| = \mathbb{P}H^0(\mathcal{E}, L)$ is $(1/2)L.(L-K)+\chi(\mathcal{E},\mathcal{O})-1$. It is also clear that this linear series is base-point free.

The characteristic zero Bertini theorem [Har77, Cor. 10.9] implies that the general member of the linear series $|L|$ is a smooth curve of genus $p_a = (1/2)L.(L + K) + 1$.

Generically, one expects that the singular members of the linear series $|L|$ are curves with at worst nodes as singularities and that the locus of curves with δ nodes has codimension δ. Since

$$\dim |L| - p_a = \chi(\mathcal{E}, \mathcal{O}) - L.K - 2 = -(d-2)(e-1) < 0,$$

generically we do not expect to find any rational curves in $|L|$, and therefore generically $\mathcal{E}$ should have no rational multisections other than the zero section.

More speculatively, one might expect that the codimension in moduli of the locus of elliptic surfaces $\mathcal{E}$ of height $d \geq 3$ which do admit a rational multisection C with $C.F = e > 1$ might be $(d-2)(e-1)$. But one reason for caution here is that the locus of $\mathcal{E}$ with $NS(\mathcal{E}) = \langle O, F \rangle$ is not Zariski open. In fact it is the complement of a divisor and a countable union of closed subvarieties of codimension $d - 1$ which is dense in the classical topology. Also, when the Néron-Severi group is more complicated, rational curves may appear in classes other than $eO + fF$ and $\mathcal{E}$ may even have smooth rational multisections.

3. Set up and strategy

In this section, we establish notation and explain the strategy to prove Theorem 1.2. We consider the moduli space of elliptic surfaces of degree d and the Hilbert scheme of rational curves on these surfaces. Arguing by contradiction, we suppose that a general such surface carries a rational curve and deduce geometric consequences of this. In the following two sections, we construct a distribution tangent to the rational curves which, roughly speaking, implies that they all come from one universal rational curve. We then deduce a contradiction from this exotic situation.

3.1. We review the construction of the moduli space of elliptic fibrations over $\mathbb{P}^1$, as in [Mir81].

Consider the affine space $\mathbb{A}^{10d+2}$ of pairs of homogeneous polynomials (A, B) in $\mathbb{C}[t_0, t_1]$ of degrees $4d$ and $6d$ respectively. Let T_d be the open subset consisting of pairs satisfying the conditions: (i) $4A^3 + 27B^2 \neq 0$; and (ii) for every place v of $P^1_\mathbb{C}$, $\mathrm{ord}_v(A) < 4$ or $\mathrm{ord}_v(B) < 6$. Associated to the pair (A, B) we have $a = A(t, 1)$ and $b = B(t, 1)$ and the elliptic curve over $\mathbb{C}(t)$ with affine Weierstrass equation

$$(3.1.1) \qquad\qquad y^2 = x^3 + ax + b.$$

The condition (i) guarantees that this is a smooth curve, and condition (ii) guarantees that it has height d in the sense of Section 1.

Let $\mathcal{E}$ be the minimal elliptic surface associated to E. More explicitly, let U be the closed subset of $\mathbb{A}^1 \times \mathbb{P}^2$ defined by the vanishing of $x^3 + axz^2 + bz^3 - y^2 z$, let $a' = A(1, t')$, $b' = B(1, t')$, and let U' be the closed subset of $\mathbb{A}^1 \times \mathbb{P}^2$ defined by the vanishing of $x'^3 + a'x'z'^2 + b'z'^3 - y'^2 z'$. Let $\mathcal{W}$ be the result of gluing

$U \setminus \{t = 0\}$ to $U' \setminus \{t' = 0\}$ via the map

$$(t', [x', y', z']) = (t^{-1}, [t^{-2d}x, t^{-3d}y, z]).$$

The surface $\mathcal{W}$ has an obvious projection to $\mathbb{P}^1$ which is proper and relatively minimal with generic fiber E, but $\mathcal{W}$ may have singularities. Indeed, it has a rational double point in each fiber where $\Delta = -16(4A^3 + 27B^2)$ has a zero of order > 1. Resolving these double points yields a smooth, relatively minimal elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$.

There is an action of $\mathbb{C}^* \times \mathrm{PGL}_2(\mathbb{C})$ on T_d: $\lambda \in \mathbb{C}^*$ acts by $\lambda(A, B) = (\lambda^4 A, \lambda^6 B)$, and $\mathrm{PGL}_2(\mathbb{C})$ acts through its standard action on $\mathbb{C}[t_0, t_1]$. Two pairs (A, B) give isomorphic elliptic surfaces $\mathcal{E}$ if and only if they lie in the same orbit of $\mathbb{C}^* \times \mathrm{PGL}_2(\mathbb{C})$ acting on T_d. For $d \geq 2$, the orbit space $T_d/(\mathbb{C}^* \times \mathrm{PGL}_2(\mathbb{C}))$ has the structure of a quasi-projective variety and is the coarse moduli space of elliptic surfaces over $\mathbb{P}^1$ with a section [Mir81, Cor. 5.5]. We denote it by $\mathcal{M}_d$; it is irreducible of dimension $10d - 2$.

It will be convenient to introduce the Zariski open subset $T_d^o \subseteq T_d$ consisting of pairs (A, B) where

$$\Delta = -16(4A^3 + 27B^2)$$

and

$$\Gamma = (\partial A/\partial t_0)(\partial B/\partial t_1) - (\partial A/\partial t_1)(\partial B/\partial t_0)$$

have distinct, simple zeroes and are not divisible by t_1. (That this is indeed a non-empty Zariski open set follows from [CD86, Lemma 3.1].) The image of T_d^o in $\mathcal{M}_d$ contains a Zariski open which will be denoted $\mathcal{M}_d^o$.

3.2. The construction above of $\mathcal{E} \to \mathbb{P}^1$ globalizes easily: Over T_d we have a natural family of elliptic surfaces

where the fiber of Φ over (A, B) is the elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ associated to the elliptic curve (3.1.1). Let $\mathcal{S}^o = \Phi^{-1}(T_d^o)$ and note that on T_d^o the discriminant Δ has simple zeroes, so the fibers of Φ are just the surfaces $\mathcal{W}$ written down above (i.e., these are already smooth surfaces and no further blowing up is needed to arrive at a smooth model).

3.3. Now consider the Hilbert scheme $\mathrm{Hom}_{T_d}(\mathbb{P}^1_{T_d}, \mathcal{S})$ of T_d-morphisms from the projective line over T_d to $\mathcal{S}$. We refer to [Kol96] for the construction and general background. Let $\mathrm{Hom}^{bir}_{T_d}(\mathbb{P}^1_{T_d}, \mathcal{S})$ be the open subscheme representing morphisms which are birational onto their image (i.e,, generically of degree 1), and let MS_d be the open subscheme of $\mathrm{Hom}^{bir}_{T_d}(\mathbb{P}^1_{T_d}, \mathcal{S})$ representing morphisms whose image does not lie in a fiber of Π and is not the zero section. (MS stands for multisection, although "parameterized multisection not lying in the zero section" would be more accurate.)

We have a diagram

$$
\begin{array}{ccc}
MS \times \mathbb{P}^1_{T_d} & \longrightarrow & \mathcal{S} \\
\downarrow & & \downarrow{\scriptstyle \Phi} \\
MS & \longrightarrow & T_d \\
& & \downarrow \\
& & \mathcal{M}_d
\end{array}
$$

where the left and bottom horizontal arrows are the obvious projections, the top horizontal arrow is the universal parameterized multisection, and the other arrows were constructed above.

3.4. Theorem 1.2 is equivalent to the assertion that no component of MS dominates $\mathcal{M}_d$. Indeed, MS is a countable union of quasi-projective varieties and so has countably many components. The image of each component in $\mathcal{M}_d$ is a constructible set and if no component dominates, then there is a countable union of closed subvarieties of $\mathcal{M}_d$ which contains the image of $MS \to \mathcal{M}_d$. The surfaces $\mathcal{E}$ whose moduli point lies outside this union are then the "very general" surfaces of the theorem.

We will eventually prove Theorem 1.2 by contradiction. Thus, assume that some component of MS dominates $\mathcal{M}_d$. Taking a slice of MS and passing to Zariski opens in MS and $\mathcal{M}_d$, we arrive at a diagram

$$
\begin{array}{ccc}
\mathcal{B} \times \mathbb{P}^1 & \longrightarrow & \mathcal{S}^o \\
\downarrow & & \downarrow{\scriptstyle \Phi} \\
\mathcal{B} & \longrightarrow & T^o_d \\
& & \downarrow \\
& & \mathcal{M}^o_d
\end{array}
$$

where $\mathcal{B}$ is irreducible and $\mathcal{B} \to \mathcal{M}^o_d$ is étale, dominant, and lies over the smooth locus of $\mathcal{M}^o_d$.

To simplify notation for the rest of the proof, we write $\mathcal{X}$ for the fiber product $\mathcal{B} \times_{T_d^o} \mathcal{S}^o$ so that we have a diagram

(3.4.1)

$$
\begin{array}{ccc}
\mathcal{B} \times \mathbb{P}^1 = \mathcal{C} & \xrightarrow{\;\;i\;\;} & \mathcal{X} \\
& \searrow^{pr_\mathcal{B}} \quad \swarrow^{\Phi} & \\
& \mathcal{B}. &
\end{array}
$$

The following proposition sums up the discussion in this section.

3.5. Proposition. *If Theorem 1.2 is false, then there exists a smooth, irreducible, quasi-projective variety $\mathcal{B}$ of dimension $10d-2$ which is étale over a closed subset of T_d^o and a diagram* (3.4.1) *such that the fiber of Φ over each $b \in \mathcal{B}$ is a Weierstrass fibration $\pi_b : \mathcal{E}_b \to \mathbb{P}^1$ of height d, the induced morphism $\mathcal{B} \to \mathcal{M}_d$ is étale and dominant, and the map $\{b\} \times \mathbb{P}^1 \to \mathcal{E}_b$ is a multisection of π_b.*

In the next section we will collect some differential information which severely restricts the image of the upper horizontal map i. In the following section, we will see that the restrictions are so severe that we arrive at a contradiction.

4. Constructing a distribution

Throughout this section, we assume that Theorem 1.2 is false and we let $\mathcal{X} \to B$ be the family of elliptic surfaces constructed in Proposition 3.5.

Let $N = \dim \mathcal{B} = 10d - 2$. In this section we will first show that $\Phi_*(\Omega_\mathcal{X}^{N+1})$ is locally free on $\mathcal{B}$ of rank $(d-2)N$. Then we will show that for all x in a certain open subset of $\mathcal{X}$, the sections of $\Phi_*(\Omega_\mathcal{X}^{N+1})$ generate a codimension 2 subspace of the fiber of $\Omega_\mathcal{X}^{N+1}$ at x. This subspace determines a codimension 2 subspace of the tangent space $T_{\mathcal{X},x}$; i.e., we have a distribution of rank N over an open subset of $\mathcal{X}$.

If ω is a section of $\Phi_*(\Omega_\mathcal{X}^{N+1})$ over some open in B, then $i^*\omega$ is a section of

$$
pr_{\mathcal{B}*}(\Omega_\mathcal{C}^{N+1}) \cong K_\mathcal{B} \otimes pr_{\mathcal{B}*}(\Omega_{\mathbb{P}^1}^1) = 0.
$$

The equality $i^*\omega = 0$ amounts to saying that the image of i is tangent to the distribution above (i.e., $di(T_{\mathcal{C},c})$ contains the subspace). This imposes strong differential conditions on the image of $i : \mathcal{C} \to \mathcal{X}$ and will eventually lead to a contradiction.

4.1. Start with the exact sequence

$$
0 \to T_{\mathcal{X}/\mathcal{B}} \to T_\mathcal{X} \to \Phi^*(T_\mathcal{B}) \to 0
$$

of locally free sheaves on $\mathcal{X}$, tensor with $K_{\mathcal{X}} \cong \Phi^*(K_{\mathcal{B}}) \otimes \Omega^2_{\mathcal{X}/\mathcal{B}}$, and take the direct image under Φ. Noting that $\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}}) = 0$, we obtain an exact sequence

$$0 \to \Phi_*(\Omega^{N+1}_{\mathcal{X}}) \to K_{\mathcal{B}} \otimes \Phi_*(\Omega^2_{\mathcal{X}/\mathcal{B}}) \otimes T_{\mathcal{B}} \to K_{\mathcal{B}} \otimes R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}})$$

of sheaves on $\mathcal{B}$. Since for all $b \in \mathcal{B}$, $h^0(\mathcal{E}_b, \Omega^2_{\mathcal{E}_b/\mathbb{C}}) = d-1$ and $h^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_b/\mathbb{C}}) = 10d$, the middle and right sheaves are locally free of ranks $(d-1)N$ and $10d$ respectively. The map between them is the identity on $K_{\mathcal{B}}$ tensored with the Kodaira-Spencer map $\Phi_*(\Omega^2_{\mathcal{X}/\mathcal{B}}) \otimes T_{\mathcal{B}} \to R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}})$.

For each $b \in \mathcal{B}$, the classes of the zero section and a fiber of π_b are linearly independent in $H^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_b/\mathbb{C}})$ and we get two everywhere independent sections of $R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}})$ over $\mathcal{B}$. We write $R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}})_0$ for their orthogonal complement. This is a locally free subsheaf of $R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}})$ of rank $N = 10d-2$, and it contains the image of the Kodaira-Spencer map.[1] Thus we have an exact sequence

$$0 \to \Phi_*(\Omega^{N+1}_{\mathcal{X}}) \to K_{\mathcal{B}} \otimes \Phi_*(\Omega^2_{\mathcal{X}/\mathcal{B}}) \otimes T_{\mathcal{B}} \to K_{\mathcal{B}} \otimes (R^1\Phi_*(\Omega^1_{\mathcal{X}/\mathcal{B}}))_0.$$

Next, we will show that the right hand map is surjective and so the kernel is locally free.

4.2. Proposition. *The sheaf $\Phi_*(\Omega^{N+1}_{\mathcal{X}})$ is locally free of rank $(d-2)N$.*

Proof. It is known [CD86] that elliptic surfaces are birational to hypersurfaces in a weighted projective space. We will use this fact and the well-understood deformation theory of such hypersurfaces to compute the Kodaira-Spencer map above.

Fix a point $b \in \mathcal{B}$ mapping to a pair $(A, B) \in T_d^o$ and let $\mathcal{E} = \Phi^{-1}(b)$, the elliptic surface associated to (A, B).

Consider the graded rings

$$S_0 = \mathbb{C}[t_0, t_1] \subseteq S = \mathbb{C}[t_0, t_1, x, y]$$

where the degrees of t_0 and t_1 are 1, the degree of x is $2d$, and the degree of y is $3d$. Let $\mathbb{P} = \mathbb{P}(1, 1, 2d, 3d)$ be the weighted projective space $\operatorname{Proj}(S)$. (See [Dol82] for general background on weighted projective varieties.)

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[1]To see this, we use standard facts from IVHS; see for example [CGGH83, §1]. Focus attention at a point $b \in \mathcal{B}$, let $v \in T_{\mathcal{B},b}$, and let $\partial_{v,0} : H^0(\mathcal{E}_b, \Omega^2_{\mathcal{E}_c/\mathbb{C}}) \to H^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_c/\mathbb{C}})$ and $\partial_{v,1} : H^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_c/\mathbb{C}}) \to H^2(\mathcal{E}_b, \mathcal{O}_{\mathcal{E}_c/\mathbb{C}})$ be the linear maps associated to v by the Gauss-Manin connection, so that $v \mapsto \partial_{v,0}$ is essentially the fiber of the Kodaira-Spencer map at b. If $Q(\cdot, \cdot)$ is the polarization on $H^2(\mathcal{E}_b, \mathbb{C})$, we have $Q(\partial_{v,0}(\alpha), \beta) + Q(\alpha, \partial_{v,1}(\beta)) = 0$ for $\alpha \in H^0(\mathcal{E}_b, \Omega^2_{\mathcal{E}_c/\mathbb{C}})$ and $\beta \in H^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_c/\mathbb{C}})$. If β is equal to the class of the fiber or the zero section, $\partial_{v,1}(\beta) = 0$ since these classes are algebraic, so lie in the lattice $H^2(\mathcal{E}_b, \mathbb{Z})$, and extend to a neighborhood of b. Thus for all $\alpha \in H^0(\mathcal{E}_b, \Omega^2_{\mathcal{E}_c/\mathbb{C}})$, we have $\partial_{v,0}(\alpha) \in H^1(\mathcal{E}_b, \Omega^1_{\mathcal{E}_c/\mathbb{C}})_0$, which is our claim.

Let $F = x^3 + Ax + B - y^2$, a homogeneous element of S of degree $6d$. It is easy to see that the weighted hypersurface $\mathrm{Proj}(S/(F))$ is isomorphic to $\mathcal{E}$ with its zero section collapsed to a point. (Here we use that (A, B) is in T_d^o so that Δ has distinct roots and $\mathcal{E} = \mathcal{W}$ in the notation of Subsection 3.1.) This construction globalizes (i.e., we have a family over T_d^o), so the deformations of $\mathcal{E}$ preserving the zero section are the same as those of the hypersurface.

To make this more precise, we use subscripts to denote partial derivatives, so, e.g., $F_{t_0} := \partial F / \partial t_0$. Let R be the Jacobian ring $S/(F_{t_0}, F_{t_1}, F_x, F_y)$ and use superscripts to denote the graded pieces. Then works of Carlson-Griffiths and Saito explained in [CD86] show that $H^1(\mathcal{E}, T_{\mathcal{E}})_0$, the subspace of $H^1(\mathcal{E}, T_{\mathcal{E}})$ corresponding to first order deformations which preserve the classes of the zero section and the fiber, is isomorphic to R^{6d}. Explicitly, an element of R^{6d} can be written in the form $\tilde{A}x + \tilde{B}$ where $\tilde{A} \in S_0^{4d}$ and $\tilde{B} \in S_0^{6d}$. The corresponding first order deformation of $\mathrm{Proj}(S/(F))$ is the subscheme of $\mathbb{P} \times \mathrm{Spec}\,\mathbb{C}[s]/(s^2)$ defined by the vanishing of

$$x^3 + (A + s\tilde{A})x + (B + s\tilde{B}) - y^2.$$

It is also explained in [CD86] that $H^0(\mathcal{E}, \Omega_{\mathcal{E}}^2) \cong R^{d-2}$. Explicitly, $p(t_0, t_1) \in S_0^{d-2} = R^{d-2}$ corresponds to the 2-form on $\mathcal{E}$ which on the affine piece $\mathrm{Spec}\,\mathbb{C}[t, x, y]/(x^3 + ax + b - y^2)$ is equal to $p(t, 1)dt\,dx/2y$.

One also knows by [CD86] that $H^1(\mathcal{E}, \Omega_{\mathcal{E}}^1)_0 \cong R^{7d-2}$ and that the Kodaira-Spencer map

$$H^0(\mathcal{E}, \Omega_{\mathcal{E}}^2) \otimes H^1(\mathcal{E}, T_{\mathcal{E}})_0 \to H^1(\mathcal{E}, \Omega_{\mathcal{E}}^1)_0$$

can be identified with the map

$$R^{d-2} \otimes R^{6d} \to R^{7d-2}$$

induced by multiplication in S. Thus to compute the fiber at b of the sheaf $\Phi_*(\Omega_{\mathcal{X}}^{N+1})$, we should compute the kernel of the multiplication map $R^{d-2} \otimes R^{6d} \to R^{7d-2}$.

We will make this more explicit following Cox and Donagi. Recall that since $(A, B) \in T_d^o$, we have that t_1 does not divide Γ. Lemma 2.4 of [CD86] says that for $\lambda \in \mathbb{C}$ and $6d - 2 \leq j \leq 8d - 1$, the kernel of the multiplication map $(t_0 - \lambda t_1) : R^j \to R^{j+1}$ is non-zero if and only if $t_0 - \lambda t_1$ divides $\Gamma = A_{t_0}B_{t_1} - A_{t_1}B_{t_0}$. Let λ_i, $i = 1, \ldots, 10d - 2$, be the roots of $\Gamma(t, 1)$ and choose a non-zero vector $v_i \in R^{6d}$ in the kernel of $t_0 - \lambda_i t_1$. Since t_1 does not divide Γ, multiplication by t_1 is an invertible map $R^{6d} \to R^{6d+1}$, and we may also describe v_i as an eigenvector of $t_1^{-1}t_0$ with eigenvalue λ_i. As the λ_i are distinct, the v_i are linearly independent, and since the dimension of R^{6d} is $10d - 2$, $\{v_i\}$ is a basis of R^{6d}. Let $w_i = t_1^{d-2}v_i \in R^{7d-2}$. Since t_1 does not divide Γ, it is clear that $\{w_i\}$ is a basis of R^{7d-2}. In terms of these bases, it is evident

that the multiplication map

$$R^{d-2} \otimes R^{6d} \to R^{7d-2}$$

sends $p(t_0, t_1) \otimes v_i$ to $p(\lambda_i, 1) w_i$. Therefore,

$$\ker\left(R^{d-2} \otimes R^{6d} \to R^{7d-2}\right) \cong \bigoplus_{i=1}^{N} W_i \otimes \mathbb{C} v_i$$

where $W_i = \{p \in R^{d-2} | p(\lambda_i, 1) = 0\}$, a codimension 1 subspace of R^{d-2}. (It is an interesting question whether the directions v_i in moduli have any geometric significance.)

This calculation shows that for every $b \in \mathcal{B}$, the map $R^{d-2} \otimes R^{6d} \to R^{7d-2}$ is surjective and that the dimension of the fiber of $\Phi_*(\Omega_{\mathcal{X}}^{N+1})$ at b is $(d-2)N$. Thus the fibers of the coherent sheaf $\Phi_*(\Omega_{\mathcal{X}/\mathcal{B}}^N)$ have constant dimension, and it is therefore locally free of rank $(d-2)N$. $\qquad\square$

4.3. Our next task is to write down explicitly sections of $\Phi_*(\Omega_{\mathcal{X}}^{N+1})$ in a neighborhood of any $b \in B$.

Fix a point $b \in \mathcal{B}$ mapping to $(A, B) \in T_d^o$. We constructed above a basis $v_1, \ldots, v_N$ of the tangent space $T_{B,b}$ indexed by the roots $\lambda_1, \ldots, \lambda_N$ of Γ. Let $s_1, \ldots, s_N$ be a system of parameters at b such that $\partial/\partial s_i|_b = v_i$.

For notational simplicity, we choose a root λ_i and consider the 1-parameter deformation of $\mathcal{E}_b$ in the corresponding direction v_i. Thus, let $\lambda = \lambda_i$ and $s = s_i$. Since λ is a root of Γ, there is a non-trivial solution (α, β) of

$$\begin{pmatrix} A_{t_0}(\lambda, 1) & A_{t_1}(\lambda, 1) \\ B_{t_0}(\lambda, 1) & B_{t_1}(\lambda, 1) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0.$$

Fix a solution and define $\tilde{A}x + \tilde{B} \in R^{6d}$ by

$$\tilde{A}x + \tilde{B} = t_1^2 \frac{\alpha F_{t_0}(\lambda, 1) + \beta F_{t_1}(\lambda, 1)}{t_0 - \lambda t_1}$$

where, as usual, $F = x^3 + Ax + B - y^2$.

The 1-parameter deformation of $\mathcal{E}_b$ in the direction $v = v_i$ corresponds to the family of hypersurfaces $\mathcal{Y}$ defined by

$$y^2 = x^3 + (A + s\tilde{A})x + (B + s\tilde{B})$$

in $\mathbb{A}_s^1 \times \mathbb{P}$. Let $\phi : \mathcal{Z} \to \mathbb{A}_s^1$ be the corresponding family of elliptic surfaces with section, so that $\phi^{-1}(0) = \mathcal{E}_b$.

It will also be convenient to consider the affine open subset $\mathcal{Z}^o \subset \mathcal{Z}$ defined by

$$y^2 = x^3 + (a + s\tilde{a})x + (b + s\tilde{b})$$

in the $\mathbb{A}^4$ with coordinates x, y, s, t where $t = t_0/t_1$, $a(t) = A(t, 1)$, etc.

4.4. Proposition. *Let $p(t) \in \mathbb{C}[t]$ be a polynomial of degree $\leq d - 3$ and let*

$$\omega = p(t) \left(\frac{(t - \lambda)dtdx}{2y} + \frac{(\alpha - \beta t)dsdx}{2y} + \frac{(2d\,\beta x)dsdt}{2y} \right).$$

Then ω is a regular 2-form on a neighborhood of $s = 0$ in $\mathcal{Z}^o$, and it extends to a section of $\phi_*(\Omega^2_{\mathcal{Z}})$ over a neighborhood of $0 \in \mathbb{A}^1_s$. Similarly,

$$ds_1 \cdots \widehat{ds_i} \cdots ds_N \wedge \omega$$

extends to a section of $\Phi_*(\Omega^{N+1}_{\mathcal{X}})$ in a neighborhood of $b \in B$.

Proof. Let $f = x^3 + (a + s\tilde{a})x + (b + s\tilde{b}) - y^2$. It is obvious that $f_y\omega = -2y\omega$ is everywhere regular on $\mathcal{Z}^o$. Straightforward calculations (sketched below) show that $f_x\omega$, $f_s\omega$, and $f_t\omega$ are also everywhere regular on $\mathcal{Z}^o$. Since $\mathcal{Z}^o$ is smooth in a neighborhood of $s = 0$, this shows that ω is a regular 2-form on that neighborhood. That it extends to a neighborhood of $s = 0$ in $\mathcal{Z}$ follows from similar calculations. The last sentence of the proposition follows immediately from what precedes it.

It remains to sketch the calculation showing that $f_x\omega$, $f_s\omega$, and $f_t\omega$ are regular. Noting that

$$0 = df = f_x dx + f_y dy + f_s ds + f_t dt$$

on $\mathcal{Z}^o$, it follows that the 1-form

$$dy = \frac{f_x dx + f_s ds + f_t dt}{2y}$$

is everywhere regular on $\mathcal{Z}^o$. From this we deduce that the following 2-forms are regular on $\mathcal{Z}^o$:

$$(4.4.1) \quad dydx = \frac{f_s dsdx + f_t dtdx}{2y}, \quad dyds = \frac{f_x dxds + f_t dtds}{2y}, \quad \text{and}$$

$$dydt = \frac{f_x dxdt + f_s dsdt}{2y}.$$

The last ingredient is the equality defining $\tilde{a}$ and $\tilde{b}$:

$$(4.4.2) \qquad (t - \lambda)f_s = (\alpha - \beta t)f_t + (2d)\beta(3y^2 - xf_x).$$

Using (4.4.2), we may rewrite $f_x\omega$, etc., as combinations of the everywhere regular 2-forms in (4.4.1). Indeed,

$$f_x\omega = p(t) \left(\frac{(t - \lambda)f_x dtdx}{2y} + \frac{(\alpha - \beta t)f_x dsdx}{2y} + \frac{(2d\,\beta x)f_x dsdt}{2y} \right).$$

Using (4.4.2) to eliminate the third term, we have

$$f_x\omega = p(t)\left(\frac{(t-\lambda)(f_x dtdx - f_s dsdt)}{2y}\right.$$

$$\left. +\frac{(\alpha-\beta t)(f_x dsdx + f_t dsdt)}{2y} + (3d\,\beta y)dsdt\right),$$

and comparing with (4.4.1) shows that

$$f_x\omega = p(t)\left((t-\lambda)dtdy + (\alpha-\beta t)dsdy + (3d\,\beta y)dsdt\right)$$

which is visibly everywhere regular.

Similarly,

$$f_s\omega = p(t)\left(\frac{(t-\lambda)f_s dtdx}{2y} + \frac{(\alpha-\beta t)f_s dsdx}{2y} + \frac{(2d\,\beta x)f_s dsdt}{2y}\right)$$

$$= p(t)\left(\frac{(\alpha-\beta t)(f_s dsdx + f_t dtdx)}{2y}\right.$$

$$\left. +\frac{(2d\,\beta x)(f_s dsdt - f_x dtdx)}{2y} + (3d\,\beta y)dtdx\right)$$

$$= p(t)\left((\alpha-\beta t)dydx + (2d\,\beta x)dydt + (3d\,\beta y)dtdx\right)$$

and

$$f_t\omega = p(t)\left(\frac{(t-\lambda)f_t dtdx}{2y} + \frac{(\alpha-\beta t)f_t dsdx}{2y} + \frac{(2d\,\beta x)f_t dsdt}{2y}\right)$$

$$= p(t)\left(\frac{(t-\lambda)(f_t dtdx + f_s dsdx)}{2y}\right.$$

$$\left. +\frac{(2d\,\beta x)(f_t dsdt + f_x dsdx)}{2y} - (3d\,\beta y)dsdx\right)$$

$$= p(t)\left((t-\lambda)dydx + (2d\,\beta x)dsdy - (3d\,\beta y)dsdx\right),$$

making visible that $f_s\omega$ and $f_t\omega$ are everywhere regular. This completes the sketch of the calculation and the proof of the proposition. $\qquad\square$

4.5. Note that we have constructed $(d-2)N$ linearly independent sections of $\Phi_*(\Omega_{\mathcal{X}}^{N+1})$ in a neighborhood of b. The dimension count in Proposition 4.2 shows that they are a basis. It is evident that at a general point $x \in \mathcal{X}$, these sections span a subspace of the fiber of $\Omega_{\mathcal{X}}^{N+1}$ of dimension N.

For reasons which will be clear just below, we introduce $U \subset \mathcal{X}$, defined to be the open subset of $\mathcal{X}$ where Γ does not vanish. In slightly different notation, this is the open set where the coefficient $(t-\lambda)$ in ω does not vanish.

Suppose $x \in U$. Choosing an identification $\bigwedge^{N+2} T_{\mathcal{X},x}^* \cong \mathbb{C}$ yields an identification of the $\bigwedge^{N+1} T_{\mathcal{X},x}^*$, the fiber of $\Omega_{\mathcal{X}}^{N+1}$ at x, with $T_{\mathcal{X},x}$. The

subspace constructed above corresponds to a subspace of $T_{\mathcal{X},x}$ of dimension N which is independent of the choice. We write $\mathcal{D}$ for the resulting distribution of rank N on $U \subset \mathcal{X}$.

The existence of this distribution $\mathcal{D}$ of codimension 2 is the analog of the global generation result of [Voi96], taking into account the two classes of rational curves lying on every $\mathcal{E}_b$ in our family.

Note that since $t - \lambda \neq 0$ on U, we have that $\mathcal{D}_x \subset T_{\mathcal{X},x}$ projects isomorphically onto the tangent space of B. Cox and Donagi showed [CD86, Prop. 3.3] that the zeroes $t - \lambda$ of Γ are exactly the branch points away from $j = 0$ and $j = 1728$ of the j-invariant mapping $j : \mathbb{P}^1_t \to \mathbb{P}^1_j$ induced by the fibration $\mathcal{E}_b \to \mathbb{P}^1_t$. Since the image of i is a multisection, it meets U.

4.6. To finish this section, we simply note that the condition $i^*(\omega) = 0$ observed at the beginning of the section is equivalent to the assertion that $i(\mathcal{C})$ is tangent to $\mathcal{D}$. More precisely, for all c in some non-empty open subset of $\mathcal{C}$, we have $i(c) \in U$ and

$$di(T_{\mathcal{C},c}) \supset \mathcal{D}_{i(c)}.$$

The following summarizes the discussion of this section.

4.7. Proposition. *Let $U \subset \mathcal{X}$ be the open subset where Γ does not vanish. The construction above yields a distribution $\mathcal{D}$ on U of dimension N such that for all $x \in U$, $\mathcal{D}_x \subset T_{\mathcal{X},x}$ projects isomorphically onto $T_{\mathcal{B},\Phi(x)}$. The image of $i : \mathcal{C} = \mathcal{B} \times \mathbb{P}^1 \to \mathcal{X}$ meets U, and for all $c \in \mathcal{C}$ such that $i(c) \in U$, we have $di(T_{\mathcal{C},c}) \supset \mathcal{D}_{i(c)}$.*

The corollary will turn out to be such a strong restriction on i that it cannot exist.

5. Deducing a contradiction

We continue to assume that Theorem 1.2 is false, and we consider the diagram

$$
\begin{array}{ccc}
\mathcal{B} \times \mathbb{P}^1 = \mathcal{C} & \xrightarrow{\;\;i\;\;} & \mathcal{X} \\
 & {\scriptstyle pr_{\mathcal{B}}} \searrow \quad \swarrow {\scriptstyle \Phi} & \\
 & \mathcal{B} &
\end{array}
$$

and the distribution $\mathcal{D}$ on $U \subset \mathcal{X}$ constructed above.

The miraculous fact is that the distribution $\mathcal{D}$ has two first integrals. Indeed, consider the rational functions on $\mathcal{X}$ given by

$$j = \frac{-2^{12}3^3 A^3}{\Delta} = \frac{6912 A^3}{4A^3 + 27B^2} \quad \text{and} \quad k = \frac{xA}{B}.$$

5.1. Proposition. *Generically, $\mathcal{D}$ is tangent to the fibers of j and k. More precisely, for all $x \in U$, we have*

$$\mathcal{D}_x \subset T_{j^{-1}(j(x))} \quad and \quad \mathcal{D}_x \subset T_{k^{-1}(k(x))}.$$

Proof. The assertion can be checked by a straightforward calculation. Again for notational convenience we work with 1-parameter deformations in the directions corresponding to roots of Γ. Choosing such a direction and using the notation introduced just before Proposition 4.4, it suffices to show that $\omega \wedge dj = \omega \wedge dk = 0$ on $\mathcal{Z}^o$. One computes that

$$dj = \frac{2^{16}3^6 a^2 b}{\Delta^2} \left((3a'b - 2ab')dt + (3\tilde{a}b - 2a\tilde{b})ds \right)$$

and

$$dk = \frac{adx}{b} + \frac{x}{b^2} \left((a'b - ab')dt + (\tilde{a}b - a\tilde{b})ds \right)$$

where a' and b' are the derivatives of a and b with respect to t evaluated at $s = 0$, and $\tilde{a}$ and $\tilde{b}$ are as in Subsection 4.3. The desired vanishing then falls out from a simple calculation using the equation (4.4.2). $\square$

Note that generically the fibers of j and k are transverse, so the proposition gives an alternative characterization of $\mathcal{D}$.

5.1.1. Remark. We proved the proposition by calculating ω (and thus $\mathcal{D}$) explicitly in Proposition 4.4, then computing $\omega \wedge dj$ and $\omega \wedge dk$ explicitly. It would be more satisfying to have a conceptual proof.

5.2. Corollary. *The image of the composed rational map*

$$\mathcal{C} \xrightarrow{\ i\ } \mathcal{X} \xrightarrow{\ j \times k\ } \mathbb{P}^1_j \times \mathbb{P}^1_k$$

is a rational curve which meets $\mathbb{A}^1_j \times \mathbb{A}^1_k$ and maps with positive degree to $\mathbb{P}^1_j$.

Proof. Indeed, Proposition 5.1 shows that the fibers of $(j \times k) \circ i$ have dimension at least N, so its image is a curve or a point. It cannot be a point because for any $b \in B$, the image of $\{b\} \times \mathbb{P}^1$ in $\mathcal{E}_b$ is a rational multisection which is not the zero section, and the image of this curve in $\mathbb{P}^1_j \times \mathbb{P}^1_k$ maps surjectively to the factor $\mathbb{P}^1_j$. It is thus a curve which maps with positive degree to $\mathbb{P}^1_j$, and it is clearly rational. Moreover, the subset of $\mathcal{E}_b$ where $k = \infty$ is the zero section, so the image of $(j \times k) \circ i$ meets the finite part of $\mathbb{P}^1_j \times \mathbb{P}^1_k$. $\square$

Proof of Theorem 1.2. We are now ready to deduce a contradiction. Let F be the $\mathbb{P}^1$ bundle over $\mathbb{P}^1$ given by

$$F = \mathbb{P} \left(\mathcal{O}_{\mathbb{P}^1}(-2d) \oplus \mathcal{O}_{\mathbb{P}^1} \right) \to \mathbb{P}^1_t.$$

We write x for the "vertical" coordinate on F over $\mathbb{A}^1_t$, so that F is a compactification of $\mathbb{A}^2_{t,x}$. For each $b \in \mathcal{B}$ corresponding to (A, B), the elliptic surface $\mathcal{E}_b$ is the double cover of F branched along $x = \infty$ and $x^3 + Ax + B = 0$.

All this fits together into the diagram

$$
\begin{array}{ccccc}
\mathcal{X} & \xrightarrow{\;\Upsilon\;} & \mathcal{B} \times F & \dashrightarrow^{\;\Psi\;} & \mathbb{P}^1_j \times \mathbb{P}^1_k \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{B} \times \mathbb{P}^1_t & = & \mathcal{B} \times \mathbb{P}^1_t & \longrightarrow & \mathbb{P}^1_j \\
\downarrow & & \downarrow & & \\
\mathcal{B} & = & \mathcal{B} &
\end{array}
$$

whose fiber over $b \in \mathcal{B}$ is

$$
\begin{array}{ccccc}
\mathcal{E}_b & \xrightarrow{\;\Upsilon_b\;} & F & \dashrightarrow^{\;\Psi_b\;} & \mathbb{P}^1_j \times \mathbb{P}^1_k \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}^1_t & = & \mathbb{P}^1_t & \longrightarrow & \mathbb{P}^1_j .
\end{array}
$$

Let $Z \subset \mathbb{P}^1_j \times \mathbb{P}^1_k$ be the rational curve of Corollary 5.2. If Theorem 1.2 is false, then for every $b \in \mathcal{B}$, the inverse image $\Upsilon_b^{-1}\left(\Psi_b^{-1}(Z)\right)$ contains a component which is a rational curve (namely, the image of $\{b\} \times \mathbb{P}^1$ under i). Given that we have so much flexibility in choosing the branching data in the diagram above, this is too much to hope for and will lead to a contradiction. In fact, we will show that for generic b, every component of the curve $\Upsilon_b^{-1}\left(\Psi_b^{-1}(Z)\right)$ has positive geometric genus.

To see this, let us fix a $b \in \mathcal{B}$ corresponding to (A, B) and consider the diagram

$$
\begin{array}{ccc}
F & \dashrightarrow^{\;\Psi_b\;} & \mathbb{P}^1_j \times \mathbb{P}^1_k \\
\downarrow & & \downarrow \\
\mathbb{P}^1_t & \longrightarrow & \mathbb{P}^1_j .
\end{array}
$$

The map $\mathbb{P}^1_t \to \mathbb{P}^1_j$ sends t to $j = 6912 A(t)^3 / (4A(t)^3 + 27B(t)^2)$. For a generic b this has degree $12d$ and is branched in triples over $j = 0$, in pairs over $j = 1728$, and has a single ramification point with index $e = 2$ over $10d - 2$ other values of j. As b varies, these $10d - 2$ points fill out an open set in the $(10d - 2)^{th}$ symmetric power of $\mathbb{P}^1$. (See [CD86, 3.1 and 3.3].) The diagram is commutative, and we have $\Psi_b^*(k) = xA(t)/B(t)$. One sees easily that Ψ_b is finite away from the zeroes of A and B, it is undefined at the points $\{x = 0, B = 0\}$ and $\{x = \infty, A = 0\}$, it collapses the other points on $A = 0$ to $(j, k) = (0, 0)$, and it collapses the other points on $B = 0$ to $(j, k) = (1728, \infty)$.

For all b, the branch curve

$$\tilde{Y} := \{x = \infty\} \cup \{x^3 + Ax + B = 0\}$$

in F maps to the curve

$$Y := \{k = \infty\} \cup \{(6912 - 4j)k^3 + 27j(k+1) = 0\} \subset \mathbb{P}^1_j \times \mathbb{P}^1_k.$$

We now consider two cases depending on whether the degree of $Z \to \mathbb{P}^1_j$ is 1 or > 1. If the degree is 1, then Z is the graph of a rational function $k = f(j)$. Let $\tilde{Z} \subset F$ be $i(\Upsilon_b(b \times \mathbb{P}^1))$, so that $\tilde{Z}$ is the unique component of $\Psi_b^{-1}(Z)$ with positive degree over $\mathbb{P}^1_t$. Then $\tilde{Z}$ is the graph of the rational function

$$x = \frac{B(t)}{A(t)} f(j(t)).$$

We claim that for general b, $\tilde{Z}$ meets the curve $\tilde{Y}$ with odd multiplicity in > 4 points. This implies that the inverse image of $\tilde{Z}$ in $\mathcal{E}_b$ is irreducible and has positive geometric genus, a contradiction. To check the claim, assume that b is such that B has distinct zeroes and let β be the order of vanishing of f at $j = 1728$. Then $\tilde{Z}$ is the graph of

$$x = \frac{B(t)^{2\beta+1}}{A(t)} g(t)$$

where $g(t)$ is regular and non-zero at the zeroes of B. If $\beta < 0$, then $\tilde{Z}$ meets $x = \infty$ with odd multiplicity at the $6d$ zeroes of B, establishing our claim. So we may assume $\beta \geq 0$. The restriction of $x^3 + Ax + B$ to $\tilde{Z}$ is

$$\frac{B(t)^{6\beta+3}}{A(t)^3} g(t)^3 + B(t)^{2\beta+1} g(t) + B(t).$$

If $\beta > 0$, this vanishes simply at the $6d$ zeroes of B. If $\beta = 0$, it also vanishes simply at zeroes of B where $g(t) \neq -1$. Finally, if $\beta = 0$ and t is a zero of B where $g(t) = -1$, then a short calculation shows that for most A, the function under consideration vanishes to order 3. Thus, for generic b, $\tilde{Z}$ meets the branch locus $\tilde{Y}$ to odd order in $6d$ points, yielding the desired contradiction.

Now consider the case where the degree of $Z \to \mathbb{P}^1_j$ is > 1. We claim that for generic b, $\tilde{Z}$, the component of $\Psi_b^{-1}(Z)$ with positive degree over $\mathbb{P}^1_t$, is irreducible of positive geometric genus, and therefore none of the components of $\Upsilon_b^{-1}(\Psi_b^{-1}(Z))$ with positive degree over $\mathbb{P}^1_t$ are rational, again a contradiction. To see this, we note that $\tilde{Z}$ is birational to the fiber product of $Z \to \mathbb{P}^1_j$ and $\mathbb{P}^1_t \to \mathbb{P}^1_j$. First, we show that this fiber product is irreducible.

As b varies, most of the branch locus of $F \dashrightarrow \mathbb{P}^1_j \times \mathbb{P}^1_k$ varies. More precisely, $j = 0$ and $j = 1728$ are always in the branch locus of $\mathbb{P}^1_t \to \mathbb{P}^1_j$, but the other $10d - 2$ branch points vary. It follows that for most b, the branch locus of $Z \to \mathbb{P}^1_j$ and that of $\mathbb{P}^1_t \to \mathbb{P}^1_j$ have at most two points in common. If they have none in common, it is immediate that the fiber product is irreducible. If they have one in common, then [Pak11, Prop. 3.1] shows the fiber product is irreducible. If they have two points in common, [Pak11, Prop. 3.3] shows that the fiber product is irreducible unless $f : Z \to \mathbb{P}^1_j$ and $g_b : \mathbb{P}^1_t \to \mathbb{P}^1_j$ are related as follows: There are rational functions f_1 and $g_{1,b}$ and a $d > 1$ such that $f = \mu \circ z^d \circ f_1$ and $g_b = \mu \circ z^d \circ g_{1,b}$ where μ is a Mobius transformation. It is clear that for generic b, g_b does not have this form, so the fiber product is irreducible. Finally, we let W be the normalization of $\tilde{Z}$ and use the Riemann-Hurwitz formula for $W \to Z$ to show that the geometric genus of W is positive. Since the degree is $12d$, we must show that the ramification term has degree $> 24d - 2$. We may choose b so that $10d - 2$ of the branch points of $\mathbb{P}^1_t \to \mathbb{P}^1_j$ are distinct from any critical values of $Z \to \mathbb{P}^1_j$. If δ is the degree of $Z \to \mathbb{P}^1_j$, we get a contribution of $\delta(10d - 2)$ to the ramification term. If $\delta > 2$ we are done. If $\delta = 2$, then points of W over the triple ramification points of $\mathbb{P}^1_t \to \mathbb{P}^1_j$ contribute at least $2 \cdot 4d$ more, so again we have positive geometric genus. Thus in all cases, W is an irreducible curve of positive geometric genus, and we again reach a contradiction.

This completes the proof of the theorem. □

5.3. Remark. The results of [Pak11] used above also allow one to show that for generic b and b', with corresponding maps $\mathbb{P}^1_t \to \mathbb{P}^1_j$ and $\mathbb{P}^1_{t'} \to \mathbb{P}^1_j$, the fiber product $\mathbb{P}^1_t \times_{\mathbb{P}^1_j} \mathbb{P}^1_{t'}$ is irreducible. On the other hand, given some other proof of this fact, we could deduce the irreducibility of W above by a "linear disjointness" argument. In other words, if we know that for general b and b', $\mathbb{C}(t)$ and $\mathbb{C}(t')$ are linearly disjoint extensions of $\mathbb{C}(j)$ (by which we mean that $\mathbb{C}(t) \otimes_{\mathbb{C}(j)} \mathbb{C}(t')$ is a field), then we may deduce that for general b, $\mathbb{C}(t)$ and $\mathbb{C}(Z)$ are linearly disjoint, and this is equivalent to W being irreducible.

5.4. Remark. As written, the proof of Theorem 1.2 relies on transcendental arguments via the results of [Pak11]. Nevertheless, I believe that it also holds in characteristic $p > 0$. This would not contradict the results in characteristic p mentioned in the introduction. Indeed, all the constructions of infinitely many rational curves I know of depend crucially on working over a *finite* field; i.e., they do not generalize to positive dimensional families. Thus they may all be eliminated by removing countably many proper closed subvarieties in moduli.

6. Further speculation on rational curves

A famous conjecture of Lang in [Lan86] asserts that there should be only finitely many rational curves on a surface of general type. Since these curves do not move, it is equivalent to conjecture that their degrees are bounded, where we measure degree with the canonical bundle (i.e., $\deg(C) = C.K$).

It is certainly false that there are in general only finitely many rational curves on an elliptic surface. Indeed, as soon as we have one non-torsion rational multisection, we can make infinitely many more by using the group structure. However, the heuristics in Section 2 suggest that the degree over the base of a rational curve on an elliptic surface $\mathcal{E}$ of Kodaira dimension 1 should be bounded. This is equivalent to asserting that the integer $C.K$ should be bounded as C runs through rational curves on $\mathcal{E}$.

If $\mathcal{S}$ is a surface of Kodaira dimension ≤ 0, then it is immediate that $C.K \leq 0$ for all rational curves on $\mathcal{S}$.

Combining these observations, we make the following speculation.

6.1. Conjecture. *Let $\mathcal{S}$ be a smooth projective surface over the complex numbers with canonical bundle K. Then the set of integers*

$$\{K.C \mid C \subset \mathcal{S} \ a \ rational \ curve\}$$

is bounded above.

This conjecture seems to the author to be of topological nature. Indeed, if we restrict to smooth rational curves, then it is a consequence of the Bogomolov-Miyaoka-Yau inequality. On the other hand, the results of [Ulm14] show that it is false in characteristic p for any prime $p > 0$.

Applied to the case of an elliptic surface with section and the associated elliptic curve, the conjecture would imply the following statement: Suppose that E is an elliptic curve over $K_0 = \mathbb{C}(t)$ with $j(E) \notin \mathbb{C}$ and suppose that $K_0 \subset K_1 \subset K_2 \subset \cdots$ is a sequence of finite extensions of K_0 such that each K_i is a rational function field. Then the rank of $E(K_i)$ is bounded independently of i. The special case where $K_i = \mathbb{C}(t^{1/i})$ is labelled a "conjecture(?)" in [Sil00]. The heuristics above suggest that it may hold in much greater generality.

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SCHOOL OF MATHEMATICS, GEORGIA INSTITUTE OF TECHNOLOGY, ATLANTA, GEORGIA 30332

E-mail address: ulmer@math.gatech.edu