MATH 168: HW 4

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2) Newman 10.1

(a) $\because$ a clique is fully connected, the longest shortest path i.e. the diameter is 1.

(b) diameter of square lattice w L edges along each side $= 2L$
diameter of lattice w L edges in d dimensions $= dL$
Noting that we have $L = \sqrt[d]{n} - 1$,
diameter of hypercubic lattice w n nodes $= d(n^{1/d} - 1)$

(c) We induct on d to show that $k(k-1)^{d-1}$ nodes are reachable from the central node in a Cayley tree with parameter k after d steps.

For $d=1$: By the properties of the Cayley tree we expect num nodes reachable from central node $= k = k(k)^0$
$= k(k-1)^{d-1}$
as wanted.

Now sp our hypothesis is true after d steps st we can reach $k(k-1)^{d-1}$ nodes from the central node. Then on the d^{th} step, each of the external nodes has $(k-1)$ connections ($\because$ each node is connected to k other nodes and this node is already connected with one node after step d). $\therefore$ The num of reachable nodes increases by a factor of $k-1$ i.e. it is $k(k-1)^d = k(k-1)^{d+1-1}$ on the $(d+1)$st step which is what we want, closing the induction.

Let D be the depth of our Cayley tree i.e. at $(D+1)^{th}$ step no more nodes are reachable from the central node.
Then note that the diameter of the Cayley tree is $2D$ since to reach leaves along different branches of the central node, all paths must go through the central node (and be of depth D on both sides).

But we also have number of nodes

$$n = 1 + \sum_{d=1}^{D} k(k-1)^{d-1} = 1 + \frac{k(1-(k-1)^D)}{1-k+1} \quad \text{for } k>2$$

So $\frac{(n-1)(2-k)}{k} = 1-(k-1)^D$

$$\Rightarrow \quad (k-1)^D = 1 - \frac{(n-1)(2-k)}{k}$$

So $D = \log_{k-1}\left(1 - \frac{(n-1)(2-k)}{k}\right)$

$\therefore$ diameter $= 2\log_{k-1}\left(1 - \frac{(n-1)(2-k)}{k}\right)$ for $k \geq 2$

(d) Cliques from (a) clearly display the small world effect since their diameter is constant.

Lattices from (b) do not display small world effects since $\log(n)$ grows asymptomatically slower than n^a for any a.

Cayley trees also display the small world effect since based on the expression found in (c) their diameter also asymptomatically grows as $\log(n)$.