# MATH 168 Midterm

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1)

(a)

$$B = \begin{array}{c} \\ \left(\begin{array}{ccccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right) \\ \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \end{array} \begin{array}{c} A \\ B \\ C \\ D \\ \\ \end{array}$$

(b) We know the diagnol elements of $W = B^T B$ are $W_{ii} = \sum_{k=1}^{9} B_{ki}^2 = \sum_{k=1}^{9} B_{ki}$

$\because B_{ki}$ is either $0$ or $1$.

$\therefore$ The diagnol entries of $W$ are:

$$W_{11} = \sum_{k=1}^{4} B_{k1} = 1+0+0+0 = 1$$

$$W_{22} = \sum_{k=1}^{4} B_{k2} = 1+1+1+0 = 3$$

$$W_{33} = \sum_{k=1}^{4} B_{k3} = 1+1+0+0 = 2$$

$$W_{44} = \sum_{k=1}^{4} B_{k4} = 1+0+0+0 = 1$$

$$W_{55} = \sum_{k=1}^{4} B_{k5} = 0+0+1+0 = 1$$

$$W_{66} = \sum_{k=1}^{4} B_{k6} = 0+1+0+1 = 2$$

$$W_{77} = \sum_{k=1}^{4} B_{k7} = 0+0+1+0 = 1$$

$$W_{88} = \sum_{k=1}^{4} B_{k8} = 0+0+0+1 = 1$$

$$W_{99} = \sum_{k=1}^{4} B_{k9} = 0+0+1+0 = 1$$

$\therefore$ W looks something like

$$\begin{pmatrix} 1 & & & & & & & & \\ & 3 & & & & & & & \\ & & 2 & & & & & & \\ & & & 1 & & & & & \\ & & & & 1 & & & & \\ & & & & & 2 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & \\ & & & & & & & & 1 \end{pmatrix}$$

Moreover, $W$ represents the one-mode projection of our bipartite network onto the nodes (in contrast to $BB^T$ being the one-mode projection onto the groups). In particular, the off diagnol elements represent weights in this network, which is equal to the number of common groups shared by a node pair. If we were to set the diagnol of $W$ to 0s, then it would have no self-edges and define the adjacency matrix for the one-mode projection.

(c) If we can relabel the nodes appropriately and rewrite the adjacency matrix in block diagnol form, the number of "blocks" is the number of components in our network, with each block representing its respective component. This would look like $\begin{pmatrix} \square & & 0 \\ & \square & \\ 0 & & \ddots \end{pmatrix}$

with the squares representing blocks and 0s everywhere else. An adjacency matrix rewritten this way with only one block represents a network with a single component.

A more practical method to find the number of components using a program would be to use Breadth or Depth First Search. In this method, we start with a single node and traverse all the nodes connected to it using BFS, recording each one. Then choosing a node that our algorithm hasn't yet reached, we traverse with BFS again. The number of distinct starting nodes we have are the number of components in the network.
(the process is repeated until we reach all nodes)

2)
(a) We know that the Graph Laplacian is a positive semi-definite matrix i.e. all its eigenvalues are non-negative (see eq 6.53 in Newman for details). Moreover, $\because L = D - A$, $D$ is the diagnol matrix with node degrees on the diagnol and rows of $A$ sum to the degree of the respective node they represent $\Rightarrow$ every row of $L$ sums to 0. Consequently, $\vec{1}$ is an eigenvector of $L$, with associated eigenvalue 0 $\because L\vec{1} = 0\vec{1}$.

In particular then, since $L$ has no negative eigenvalues, $0$ is its smallest eigenvalue i.e. $\lambda_1 = 0$. Additionally, we also know that $G$ is connected i.e. it only has one component. If we were to represent $L$ in block diagnol form (as discussed in 1(c)), it would only have one block i.e. the eigenvalue $0$ has multiplicity 1. Then we know that in a network with only one component there is only one eigenvector with eigenvalue $0$ i.e. all other eigenvectors have non-zero and in particular, positive eigenvalues. $\therefore$ The second smallest eigenvalue is positive i.e. $\lambda_2 > 0$.

(b) Let us first note that $\because$ the inverse of a diagnol matrix is simply the inverse of each of its elements i.e. $D^{-1} = \text{diag}(k_1^{-1}, k_2^{-1}, \ldots, k_n^{-1})$ where $k_i$ is the degree of the $i^{th}$ node and $n$ is the number of nodes. $\therefore$ In $L_{rw} = D^{-1}A$, multiplying $A$ by $D^{-1}$ serves to normalize its rows such that they all sum to 1 ($\because$ the sum of the row $i$ is the degree of node $i$ i.e. $k_i$).

Consequently, $\vec{1}$ is an eigenvector of $L_{rw}$ with associated eigenvalue 1 i.e. $L_{rw}\vec{1} = 1(\vec{1})$ $\because$ every row of $L_{rw}$ sums to 1. Now let $\lambda$ be an eigenvalue of $L_{rw}$ with associated eigenvector $\vec{v}$ i.e. we have $L_{rw}\vec{v} = \lambda\vec{v}$. Let $v_k$ be the maximal element of $\vec{v}$ i.e. $|v_k| = \max\{|v_1|, \ldots, |v_n|\}$, then $|v_k| > 0$ $\because$ eigenvectors are non-zero. $\because$ $L_{rw}\vec{v} = \lambda\vec{v}$, we then have:

$$|L_{rw_{k1}} v_1 + L_{rw_{k2}} v_2 + \ldots + L_{rw_{kn}} v_n| = |\lambda||v_k|$$

where $L_{rw_{ij}}$ is the $i,j^{th}$ element of $L_{rw}$.

But then, $|\lambda|\,|v_k| \leq L_{rw_{k1}}|v_1| + L_{rw_{k2}}|v_2| + \ldots + L_{rw_{kn}}|v_n|$ (by the triangle inequality).

Furthermore, $|\lambda|\,|v_k| \leq L_{rw_{k1}}|v_k| + \ldots + L_{rw_{kn}}|v_k|$ $\because$ $|v_k|$ is $\max\{|v_1|, \ldots, |v_n|\}$

Then, $|\lambda|\,|v_k| \leq |v_k|(L_{rw_{k1}} + \ldots + L_{rw_{kn}})$

But as discussed above all rows of $L_{rw}$ sum to 1 so $|\lambda|\,|v_k| \leq |v_k|$ and since $|v_k| > 0$, we finally have $|\lambda| \leq 1$ as wanted $\square$.

(c) We first note that $\because$ $G$ is a $k$-regular graph, all rows of $A$ sum to $k$ and thus we have $A\vec{1} = k\vec{1}$ i.e $A$ has eigenvector $\vec{1}$ with eigenvalue $k$. Now let $\vec{v}$ be an arbitrary unit eigenvector of $A$ with associated eigenvalue $\lambda$ i.e. $A\vec{v} = \lambda\vec{v}$.

$$\Rightarrow |\lambda| = |v^T A v| = \Big|2 \sum_{(i,j)\in E} v_i v_j\Big|$$

$$\leq \sum_{(i,j)\in E} (v_i^2 + v_j^2)$$

$$\parallel \quad \leftarrow \text{since each node has deg. } k$$

$$k \sum_{i\in V} v_i^2 = k \;(\because \vec{v} \text{ unit eigenvector})$$

$\therefore$ $|\lambda| \leq k$ for an arbitrary $\lambda$ so using the notation in the problem, $|\nu_i| \leq k \;\; \forall i \in \{1, \ldots, n\}$.

3)

(a) A looks something like

$$\begin{pmatrix} \square & & & 0 \\ & \square & & \\ & & \ddots & \\ 0 & & & \square \end{pmatrix}$$

in block diagnol form.

In particular, A contains $m$ blocks, with $m$ being the number of communities and is 0 everywhere else. Also, the blocks themselves are sparse dependendent on their size since the probability with which two nodes in the same community are connected is $p_m = R(n_m - 1)^{-\beta}$ where whether sparsity of a block is directly or inversly proportional to the block's size depends on the constant $\beta$.

(b) We know the expected degree of a node in community $m$ is $\langle k \rangle_m = \dfrac{2|E_m|}{n_m}$ where $|E_m|$ is the expected number of edges in community $m$. $\because$ Each node is adjacent to every other node in community $m$ with independent probability $p_m = R(n_m - 1)^{-\beta}$, each edge in $m$ exists with probability $p_m$.

$\therefore$ The expected number of edges = total possible edges $\times$ $p_m$

$$= \binom{n_m}{2} p_m$$

$$|E_m| = \frac{n_m!}{2!(n_m-2)!} R(n_m-1)^{-\beta}$$

$$\therefore \langle k \rangle_m = \frac{2\binom{n_m}{2} p_m}{n_m} = \frac{\cancel{2}}{\cancel{n_m}}\left(\frac{\overset{(n_m-1)!}{\cancel{n_m!}}}{\cancel{2}!(n_m-2)!}\right) R(n_m-1)^{-\beta}$$

$$= (n_m-1) \cdot R(n_m-1)^{-\beta}$$

$$\therefore \langle k \rangle_m = R(n_m-1)^{-\beta+1}$$

(c) We calculate the expected value of $\bar{C}_m$ for an arbitrary node in community $m$. From above, $\langle k \rangle_m = R(n_m-1)^{-\beta+1}$ so the possible number of pairs of neighbors for an arbitrary node are $\binom{\langle k \rangle_m}{2}$ and the number of such pairs of neighbors we expect to be adjacent are $\binom{\langle k \rangle_m}{2} p_m$ since each such edge exist with independent probability $p_m$.

$$\therefore \bar{C}_m = \frac{(\text{num pair of neighbors connected})}{(\text{num pairs of neighbors})}$$

$$= p_m \cancel{\binom{\langle k \rangle_m}{2}} \Big/ \cancel{\binom{\langle k \rangle_m}{2}}$$

$$\therefore \bar{C}_m = p_m$$

$$\bar{C}_m = R(n_m-1)^{-\beta}$$

But $\langle k \rangle_m = R(n_m-1)^{1-\beta} \Rightarrow \langle k \rangle_m^{-\beta/1-\beta} = \underbrace{R^{-\beta/1-\beta}(n_m-1)^{-\beta}}_{\propto p_m}$

$$\therefore \bar{C}_m \propto \langle k \rangle^{-\beta/1-\beta}$$

(d) For the expected value of the local clustering coefficient to fall off with increasing degree $\langle k \rangle^{-3/4}$, $\beta$ must equal $-3$ $\because$ $\bar{C}_m \propto \langle k \rangle^{-(-3)/1-(-3)} = \langle k \rangle^{3/4}$. So when $\beta = -3$, $\bar{C}_m$ falls falls off with increasing degree $\langle k \rangle^{-3/4}$.

(e) While not the worst choice, I wouldn't say this is a good model for social networks. First of all, within social networks we typically expect there to be some connections between communities or some sort of giant component structure to display the small world effect which doesn't necessarily happen here. For the right choice of $\beta$ the model does encode our expectation of nodes (people) in larger communities having more connections. However, instead of all edges being determined completely at random, for a better model of a social network showing preferential attachment, we would want $p_m$ to also depend on the degrees of the pair of nodes being considered such that nodes with higher degrees are more likely to connect with other nodes of higher degree. $\therefore$ We could make this a better model for social networks by making $p_m$ directly proportional to the degrees of the pair of nodes under consideration.

4) Newman Exercise 13.8

(a) The probability this new node makes a citation to some previously added node $i$ is $q_i + a_i$. This citation must be made to some node so we normalize the probability s.t. its sum over all $i$ is 1. $\therefore$ The normalized probability that the citation is made to node $i$ is

$$\frac{q_i + a_i}{\sum_i (q_i + a_i)} = \frac{q_i + a_i}{\sum_i q_i + \sum_i a_i} = \frac{q_i + a_i}{n\langle q\rangle + n\int a p(a)\, da}$$

(in the limit of large $n$)

$\therefore$ probability citation is made to particular node with in degree $q_i$ $= \dfrac{q_i + a_i}{nc + n\bar{a}}$ $\because \langle q\rangle = c$

But a newly added paper cites $c$ previously existing papers on average, thus the probability that the $(n+1)^{th}$ node attaches to some node with in-degree $q_i$ is $\dfrac{c(q_i + a_i)}{n(c+\bar{a})}$.

(b) Noting that there are $n p_q(a)$ nodes with in degree $q$ and parameter $a$, and using the probability above, we know that for large $n$, the expected number of new citations the paper makes to all nodes with in-degree $q$ is $\frac{c(q+a)\cdot n\, p_q(a)}{n(c+\bar{a})}$ $= \frac{c(q+a)}{c+\bar{a}} p_q(a)$

(for expressions on right and directly below $p_q(a)$ shorthand $\int p_q(a)\,da$ i.e. simply $p_q$)

Then the expected number of nodes with in-degree $q$ after the addition of the new node is

$$\underbrace{n p_q(a)}_{\text{num nodes with in degree } q \text{ initially}} + \underbrace{\frac{c(q-1+a)}{c+\bar{a}} p_{q-1}(a)}_{\text{nodes gained}} - \underbrace{\frac{c(q+a)}{c+\bar{a}} p_q(a)}_{\text{nodes lost}}$$

So we have $(n+1)\, p_q(a) = n p_q(a) + \frac{c(q-1+a)}{c+\bar{a}} p_{q-1}(a) - \frac{c(q+a)}{c+\bar{a}} p_q(a)$

And for $q=0$: $(n+1)\, p_0(a) = n p_0(a) + 1 - \frac{ca}{c+\bar{a}} p_0(a)$

Taking the limit $n \longrightarrow \infty$, we get:

$$p_q(a) = \frac{c}{c-\bar{a}}\left[(q-1+a)\, p_{q-1}(a) - (q-a)\, p_q(a)\right] \quad \text{for } q \geq 1$$

and $p_0(a) = 1 - \frac{ca}{c+\bar{a}} p_0(a)$ for $q = 0$.

$$\Downarrow$$

$$p_0(a)\left[1 + \frac{ca}{c+\bar{a}}\right] = 1 \Rightarrow p_0(a) = \frac{c+\bar{a}}{c+\bar{a}+ca}$$

and $p_q(a)\left[1 + \frac{c(q+a)}{c+\bar{a}}\right] = \frac{c(q-1+a)}{c+\bar{a}} p_{q-1}(a)$

$$\Rightarrow p_q(a) = \frac{c(q-1+a)}{c+\bar{a}+c(q+a)} p_{q-1}(a)$$

(c) We observe this leads to the pattern described by

$$p_q = \frac{(q-1+a)(q-2+a)\ldots a}{(q+a+1+\bar{a}/c)\ldots(a+2+\bar{a}/c)} \cdot \frac{(1+\bar{a}/c)}{(a+1+\bar{a}/c)}$$

Then using the gamma function,

$$p_q = (1+\bar{a}/c)\, \frac{\Gamma(q+a)\,\Gamma(a+1+\bar{a}/c)}{\Gamma(a)\,\Gamma(q+a+2+\bar{a}/c)}$$

and in particular $p_q(a) = p_q\, p(a)$

so using the Euler Beta fcn i.e. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$,

we get $p_q(a) = \frac{B(q+a, 2+\bar{a}/c)}{B(a, 1+\bar{a}/c)} p(a)$ as required.

We can see this distribution has a power-law tail by using Stirling's approximation $\Gamma(x) \simeq \sqrt{2\pi}\, e^{-x} e^{-x-1/2}$ which upon application to the formula above gives an exponent of $2 + \frac{\bar{a}}{c}$.