Problems

1. (80 pt. in total)

Assume that we have n data points, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let the degree of polynomial be d. Then, we want to find $w_0, w_1, w_2, \cdots, w_d$ of the polynomial such that

$$\begin{aligned}
\hat{f}(x_1) &= w_0 + w_1 x_1 + w_2 x_1^2 + \cdots + w_d x_1^d = y_1, \\
\hat{f}(x_2) &= w_0 + w_1 x_2 + w_2 x_2^2 + \cdots + w_d x_2^d = y_2, \\
\hat{f}(x_3) &= w_0 + w_1 x_3 + w_2 x_3^2 + \cdots + w_d x_3^d = y_3, \\
\hat{f}(x_4) &= w_0 + w_1 x_4 + w_2 x_4^2 + \cdots + w_d x_4^d = y_4, \\
\hat{f}(x_5) &= w_0 + w_1 x_5 + w_2 x_5^2 + \cdots + w_d x_5^d = y_5, \\
&\vdots \\
\hat{f}(x_n) &= w_0 + w_1 x_n + w_2 x_n^2 + \cdots + w_d x_n^d = y_n.
\end{aligned}$$

Now, we reformulate the equations into the vector and matrix form. First, let $\mathbf{w} = [w_0, w_1, \cdots, w_d]^T$ and $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T$. Then, the above equations can be rewritten as

$$\hat{f}(x_1) = [1, x_1, x_1^2, x_1^3, \cdots, x_1^d] \cdot \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix} = [1, x_1, x_1^2, x_1^3, \cdots, x_1^d]\mathbf{w} = y_1$$

Similarly, we have,

$$\begin{aligned}
[1, x_2, x_2^2, x_2^3, \cdots, x_2^d]\mathbf{w} &= y_2, \\
[1, x_3, x_3^2, x_3^3, \cdots, x_3^d]\mathbf{w} &= y_3, \\
[1, x_4, x_4^2, x_4^3, \cdots, x_4^d]\mathbf{w} &= y_4, \\
[1, x_5, x_5^2, x_5^3, \cdots, x_5^d]\mathbf{w} &= y_5, \\
&\vdots \\
[1, x_n, x_n^2, x_n^3, \cdots, x_n^d]\mathbf{w} &= y_n.
\end{aligned}$$

Then, all equations can be written as the form of linear equation,

$$A\mathbf{w} = \mathbf{y},$$

where A is the stack of $[1, x_i, x_i^2, x_i^3, \cdots, x_i^d]$ for $i = 1, \cdots, n$. Under this setting, answer the following questions.

1-(a) What is the size of vector w and y? (10pt)

$$W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_d \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

w is a $(d+1)\times1$ matrix, a vector size is $d+1$

Y is a $n\times1$ matrix, a vector size is n

1-(b) What is the size of matrix A? Write A. (10pt)

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^d \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^d \end{bmatrix}$$

A is $n \times (d+1)$ matrix.

1-(c) Let $d = n$, then, A becomes a square matrix. Compute the determinant of A. (40pt in total, Derivation: 30pt, Answer: 10pt, Hint: Vandermonde Matrix.)

Given $d+1 = n$, Hence we have

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \cdots & x_3^n \\ 1 & x_4 & x_4^2 & x_4^3 & \cdots & x_4^n \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{bmatrix}$$

we claim $\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$

we prove the result by induction on n

$n=2$ / In this case $A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$ Thus $\det(A) = (x_2 - x_1)$

Let the result hold for $n=k$. Then we prove the result for $n=k+1$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ 1 & x_3 & x_3^2 & \cdots & x_3^n \\ 1 & x_4 & x_4^2 & \cdots & x_4^n \\ 1 & x_5 & x_5^2 & \cdots & x_5^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^{k+1} \end{bmatrix}$$

As the determinant of a matrix is not affected by elementary column operations, we have denoting the i-th column by C_i

$C_{k+1} - x_1 C_k$ implies

1-(d) What is the condition that makes the determinant of A non-zero? (10pt)

As $\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$ we have $\det A \ne 0 \Longleftrightarrow x_j \ne x_i$ for all $1 \le i < j \le n$

1-(e) Assume that the determinant of A is non-zero, then, what is the solution of linear equation, $A\mathbf{w} = \mathbf{y}$, with respect to $\mathbf{w}$? (10pt)

If $\det A \ne 0$ then we have A^{-1} exists and hence $Aw = y \Rightarrow w = A^{-1} y$

$\therefore$ the solution is given by $w = A^{-1} y$

1-c.

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & 0 \\ 1 & x_2 & x_2^2 & \cdots & x_2^{k+1} - x_1 x_2^k \\ 1 & x_3 & x_3^2 & \cdots & x_3^{k+1} - x_1 x_3^k \\ 1 & x_4 & x_4^2 & \cdots & x_4^{k+1} - x_1 x_4^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^{k+1} - x_1 x_{k+1}^k \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & 0 \\ 1 & x_2 & x_2^2 & \cdots & (x_2 - x_1) x_2^k \\ 1 & x_3 & x_3^2 & \cdots & (x_3 - x_1) x_3^k \\ 1 & x_4 & x_4^2 & \cdots & (x_4 - x_1) x_4^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & (x_{k+1} - x_1) x_{k+1}^k \end{bmatrix}$$

Similarly

$C_k - x_1 C_{k-1}$

$C_{k-1} - x_1 C_{k-2}$

$\vdots$

$C_4 - x_1 C_3$

$C_3 - x_1 C_2$

$C_2 - x_1 C_1$

implies

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1) x_2 & \cdots & (x_2 - x_1) x_2^k \\ 1 & x_3 - x_1 & (x_3 - x_1) x_3 & \cdots & (x_3 - x_1) x_3^k \\ 1 & x_4 - x_1 & (x_4 - x_1) x_4 & \cdots & (x_4 - x_1) x_4^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} - x_1 & (x_{k+1} - x_1) x_{k+1} & \cdots & (x_{k+1} - x_1) x_{k+1}^k \end{bmatrix}$$

Now opening the determinant along the first row we have

$$\det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & \cdots & (x_2 - x_1) x_2^k \\ 1 & x_3 - x_1 & \cdots & (x_3 - x_1) x_3^k \\ 1 & x_4 - x_1 & \cdots & (x_4 - x_1) x_4^k \\ \vdots & \vdots & & \vdots \\ 1 & x_{k+1} - x_1 & \cdots & (x_{k+1} - x_1) x_{k+1}^k \end{bmatrix} = \det \begin{bmatrix} x_2 - x_1 & (x_2 - x_1) x_2 & \cdots & (x_2 - x_1) x_2^k \\ x_3 - x_1 & (x_3 - x_1) x_3 & \cdots & (x_3 - x_1) x_3^k \\ \vdots & \vdots & & \vdots \\ x_{k+1} - x_1 & (x_{k+1} - x_1) x_{k+1} & \cdots & (x_{k+1} - x_1) x_{k+1}^k \end{bmatrix}$$

$$= \prod_{2 \le j \le (k+1)} (x_j - x_1) \det \begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^k \\ 1 & x_3 & x_3^2 & \cdots & x_3^k \\ 1 & x_4 & x_4^2 & \cdots & x_4^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & x_{k+1}^2 & \cdots & x_{k+1}^k \end{bmatrix}$$

But we have $\det \begin{bmatrix} 1 & x_2 & \cdots & x_2^k \\ 1 & x_3 & \cdots & x_3^k \\ \vdots & \vdots & & \vdots \\ 1 & x_{k+1} & \cdots & x_{k+1}^k \end{bmatrix} = \prod_{2 \le i < j \le (k+1)} (x_j - x_i)$

$\Downarrow$

Hence $\left(\prod_{2 \le j \le (k+1)} (x_j - x_1) \right) \left(\prod_{2 \le i < j \le (k+1)} (x_j - x_i) \right) = \prod_{1 \le i < j \le (k+1)} (x_j - x_i)$

i.e. we have proved by induction that

$\det(A) = \prod_{1 \le i < j \le n} (x_j - x_i)$

2. (20pt)

Suppose that $n > d$. Then, we cannot compute the inverse of A since A is not a square matrix. In this case, how can we solve the linear equation $Aw = y$? (Hint: Pseudo Inverse)

$Aw = Y$

where $A \Rightarrow$ A matrix of order $n \times (d+1)$

$w \Rightarrow$ Matrix of order $(d+1) \times 1$ or A vector of size $(d+1)$

$Y \Rightarrow$ vector of size n

First we have to write the matrix A included with Y as

$[A|Y]$

then write the matrix A into row echelon form and the same operations are applied on Y which are applied on A

Let reduced form of A is B and reduced form of Y is Y

Then put $Bw = Y$

By equating both sides we find the elements of w.