第二次作业

$f(1) = -1 < 0$，$f(2) = 14 > 0$ ∴ 根在 $(1,2)$ 之间

1. ① $e_i = \frac{1}{2^{i+1}}(2-1) = \frac{1}{2^{i+1}}$ $\quad e_{i+1} = \frac{1}{2^{i+2}}$ 二分法

∴ $\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = \frac{1}{2}$

∴ 此时线性收敛，收敛阶为1

② 割线法：$e_{i+1} = \left|\frac{f''(x)}{2f'(x)}\right| e_i e_{i-1}$

设 $\frac{e_{i+1}}{e_i^{\alpha}} = C$ ∴ $e_{i+1} = C\cdot e_i^{\alpha} = \left|\frac{f''(x)}{2f'(x)}\right| e_i e_{i-1}$

∴ $e_{i-1} = e_i^{\alpha-1}\cdot C\cdot\left|\frac{2f'(x)}{f''(x)}\right|$

∴ $e_{i+1} = \left|\frac{f''(x)}{2f'(x)}\right|^{\alpha-1} e_i^{\alpha}$ $\quad \alpha = 1.618$，收敛阶为1.618

$\cancel{f'(x^*)}$

③ $g'(x) = \frac{1}{2} - \frac{3}{x^4}$ 不动点：x^*1.189

∴ $g'(x^*) = \frac{1}{2} - \frac{3}{x^{*4}} = -1.001$

∴ $\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = g'(x^*) = -1.001$，不收敛

④ $g'(x) = \frac{5}{6} - \frac{1}{x^4}$ $\quad$ ∴ $g'(x^*) = \frac{5}{6} - \frac{1}{x^{*4}} = 0.333$

∴ $\lim_{i\to\infty} \frac{e_{i+1}}{e_i} = |g'(x^*)| = 0.333$

⑤ 牛顿法：$f'(x) = 4x^3$ $\quad f''(x) = 12x^2$

$f'(x^*) = 6.724$ $\quad f''(x^*) = 16.965$

∴ $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = \left|\frac{f''(x^*)}{2f'(x^*)}\right| = \frac{16.965}{6.724} = 2.523$，为二次收敛

∴ $5 > 2 > 1 > 4 > 3$，二阶收敛大于一阶收敛，同阶比较比值，小的更快

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2. 解: 牛顿迭代法不动点形式: $x=\varphi(x)=x-\frac{f(x)}{f'(x)}$

$\because f(r)=0,\ f'(r)\neq 0,\ f''(r)=\cdots=f^{(m-1)}(r)=0,\ f^{(m)}(r)\neq 0$

在0到x_k泰勒展开:

$\therefore f(x)=f(x_k)+f'(x_k)(x-x_k)+\frac{f''(x_k)}{2}(x-x_k)^2+\cdots+\frac{f^{(m)}(\xi)}{m!}(x-x_k)^m$

将 $x=r$ 代入 ξ在x_k和r之间

$\therefore 0=f(r)=f(x_k)+f'(x_k)(r-x_k)+\frac{f^{(m)}(\xi)}{m!}(r-x_k)^m$

$\therefore r=x_k-\frac{f(x_k)}{f'(x_k)}-\frac{f^{(m)}(\xi)}{m!\,f'(x_k)}(r-x_k)^m$

$\therefore r=x_{k+1}-\frac{f^{(m)}(\xi)}{m!\,f'(x_k)}(r-x_k)^m$

$\therefore \frac{x_{k+1}-r}{(x_k-r)^m}=\frac{f^{(m)}(\xi)}{m!\,f'(x_k)}$

$\therefore \lim\limits_{k\to\infty}\frac{x_{k+1}-r}{(x_k-r)^m}=\frac{f^{(m)}(\xi)}{m!\,f'(x_k)}$ $\therefore$ 至少有m阶收敛速度

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3. ~~$g(x)-x=0 \Rightarrow x^2-\frac{5}{2}x+\frac{3}{2}=0 \Rightarrow x=1$或$\frac{3}{2}$~~

~~$g(x)=x^2-\frac{3}{2}x+\frac{3}{2} \Rightarrow x=\frac{2}{3}x^2+1 \Rightarrow x_{k+1}=\frac{2}{3}x_k^2+1$~~

令 $g(r)=r \Rightarrow r^2-\frac{3}{2}r+\frac{3}{2}=r \Rightarrow r=1$或$\frac{3}{2}$ $\therefore$ 不动点为1和$\frac{3}{2}$

$g'(x)=2x-\frac{3}{2}$ 当$r=1$时, $|g'(r)|=\frac{1}{2}<1$, 此时收敛

当$r=\frac{3}{2}$时, $|g'(r)|=\frac{3}{2}>1$, 此时不收敛

4. 解: $f'(x)=4x^3-21x^2+36x-20$ $f'''(x)=24x-42$

$f''(x)=12x^2-42x+36$

牛顿迭代: $x_{k+1}=x_k-\frac{x_k^4-7x_k^3+18x_k^2-20x_k+8}{4x_k^3-21x_k^2+36x_k-20}$

$f(r)=0,\ f'(r)=0,\ f''(r)=0,\ f'''(r)=6\neq 0$ $\therefore r=2$为三重根

将$f(x)$在x_k处展开: $f(x)=f(x_k)+f'(x_k)(x-x_k)+\frac{f''(x_k)}{2}(x-x_k)^2+\frac{f'''(\xi)}{3!}(x-x_k)^3$

ξ在x_k和r之间 代入$x=r$

$0=f(r)=f(x_k)+f'(x_k)(r-x_k)+\frac{f''(x_k)}{2}(x-x_k)^2+\frac{f'''(x_k)}{3!}(x-x_k)^3+\frac{f''''(\xi)}{4!}(x-x_k)^4$

当$k\to\infty$, x_k和ξ都$\to r$

$\therefore$ ~~$0=f(x_k)$~~ $0=\frac{f'''(x_k)}{3!}+\frac{f''''(\xi)}{4!}$

$\therefore$ 不是二次收敛 $\lim\limits_{i\to\infty}\frac{e_{i+1}}{e_i}=\frac{2}{3}$

5. (a) $f'(x)=3x^2-4$ $f''(x)=6x$. $f'(2)=8$, $f''(2)=12$

$\therefore \frac{e_{i+1}}{e_i^2}=\frac{12}{2\times 8}=\frac{3}{4}$

$\therefore e_5=\frac{3}{4}e_4^2=0.75\times 10^{-12}$

(b) $f'(0)=-4$, $f''(0)=0$ $f'''(x)=6$

$\therefore \frac{e_{i+1}}{e_i^2}=0$

$\therefore e_5=0$

6. 解: 将 ~~(x_{k+1}, y_{k+1})~~ $(u_{k+1}, v_{k+1})^T$ 进行泰勒展开 $\cancel{f_{1u}'=2\ldots}$

$0=f_1(u,v)\approx f_1(u_k,v_k)+\frac{\partial f_1}{\partial u}(u_k,v_k)(u-u_k)+\frac{\partial f_1}{\partial v}(u_k,v_k)(v-v_k)$

$0=f_2(u,v)\approx f_2(u_k,v_k)+\frac{\partial f_1}{\partial u}(u_k,v_k)(u-u_k)+\frac{\partial f_2}{\partial v}(u_k,v_k)(v-v_k)$

$\therefore \cancel{\begin{pmatrix}u_{k+1}\\v_{k+1}\end{pmatrix}=\begin{pmatrix}u_k\\v_k\end{pmatrix}}$ $f_{1u}'=2u$, $f_{1v}'=2v$

$f_{2u}'=2u-2$, $f_{2v}'=2v$

$\therefore \begin{pmatrix}u_{k+1}\\v_{k+1}\end{pmatrix}=\begin{pmatrix}u_k\\v_k\end{pmatrix}-\begin{pmatrix}2u_k & 2v_k\\2u_k-2 & 2v_k\end{pmatrix}^{-1}\begin{pmatrix}u_k^2+v_k^2-1\\(u_k-1)^2+v_k^2-1\end{pmatrix}$

$\because (u_0,v_0)=(1,1)$

$\therefore \begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}-\begin{pmatrix}0.5 & -0.5\\0 & 0.5\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0.5\\1\end{pmatrix}$

$\therefore \begin{pmatrix}u_2\\v_2\end{pmatrix}=\begin{pmatrix}0.5\\1\end{pmatrix}-\begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}^{-1}\begin{pmatrix}0.25\\0.25\end{pmatrix}=\begin{pmatrix}0.5\\1\end{pmatrix}$

$\begin{matrix}0.5 & -0.5\\0.25 & 0.25\end{matrix}$

$\therefore (u_2,v_2)=\begin{pmatrix}0.5\\1\end{pmatrix}$