概率统计

习题课 三

一、填空题

1. 设 $P\{X \ge 0, Y \ge 0\} = \dfrac{3}{7}$, $P\{X \ge 0\} = P\{Y \ge 0\} = \dfrac{4}{7}$,

则 $P\{\max\{X,Y\} \ge 0\} = \underline{5/7}$.

解 $\max\{X,Y\} < 0 \Leftrightarrow X < 0, Y < 0.$

$$P\{\max\{X,Y\} < 0\} = P\{X < 0, Y < 0\}$$

因为 $P\{X \ge 0\} = P\{Y \ge 0\} = \dfrac{4}{7}$,

所以 $P\{X < 0\} = P\{Y < 0\} = \dfrac{3}{7}$,

又因为 $P\{(X < 0) \cup (Y < 0)\} = 1 - P\{X \ge 0, Y \ge 0\} = \dfrac{4}{7}$

故 $P\{X < 0, Y < 0\} = \dfrac{3}{7} + \dfrac{3}{7} - \dfrac{4}{7} = \dfrac{2}{7}$

2. 已知 *X*、*Y* 的分布律为

Y \ X	0	1
0	1/3	b
1	a	1/6

且 $\{X = 0\}$ 与 $\{X + Y = 1\}$ 独立，则 $a = \underline{1/3}$ ，$b = \underline{1/6}$.

解 $P\{X = 0, X + Y = 1\} = P\{X = 0, Y = 1\} = a$

$$P\{X = 0\} = P\{X = 0, Y = 0\} + P\{X = 0, Y = 1\} = a + \frac{1}{3}$$

$$P\{X + Y = 1\} = P\{X = 0, Y = 1\} + P\{X = 1, Y = 0\} = a + b$$

因为 $\{X=0\}$ 与 $\{X+Y=1\}$ 独立，所以

$$P\{X=0, X+Y=1\} = P\{X=0\} \cdot P\{X+Y=1\}$$

即

$$a = (a + \frac{1}{3})(a+b)$$

联立

$$a + b + \frac{1}{3} + \frac{1}{6} = 1$$

得到

$$a = \frac{1}{3}, b = \frac{1}{6}.$$

二、选择题

1. 已知 X_1、X_2 相互独立，且分布律为

$$\begin{array}{c|cc} X_i & 0 & 1 \\ \hline P & 1/2 & 1/2 \end{array} \quad (i=1,2)$$

那么下列结论正确的是 ___C___.

A. $X_1 = X_2$

B. $P\{X_1 = X_2\} = 1$

C. $P\{X_1 = X_2\} = 1/2$

D. 以上都不正确

解 $\{X_1 = X_2\} = \{X_1 = 0, X_2 = 0\} + \{X_1 = 1, X_2 = 1\}$

因为 X_1、X_2 相互独立，所以

$$P\{X_1 = 0, X_2 = 0\} = P\{X_1 = 0\} \cdot P\{X_2 = 0\} = 1/4$$

$$P\{X_1 = 1, X_2 = 1\} = P\{X_1 = 1\} \cdot P\{X_2 = 1\} = 1/4$$

故 $\quad P\{X_1 = X_2\} = 1/2$

概率论

2. 设离散型随机变量(X,Y) 的联合分布律为

(X,Y)	$(1,1)$	$(1,2)$	$(1,3)$	$(2,1)$	$(2,2)$	$(2,3)$
P	$1/6$	$1/9$	$1/18$	$1/3$	α	β

且 X、Y 相互独立, 则 _____ A _____ .

A． $\alpha = 2/9, \beta = 1/9$ B． $\alpha = 1/9, \beta = 2/9$

C． $\alpha = 1/6, \beta = 1/6$ D． $\alpha = 8/15, \beta = 1/18$

解 因为 X、Y 相互独立，所以

$$P\{X=1, Y=3\} = P\{X=1\} \cdot P\{Y=3\}$$

即

$$\frac{1}{18} = \left(\frac{1}{6} + \frac{1}{9} + \frac{1}{18}\right)\left(\frac{1}{18} + \beta\right)$$

解得

$$\beta = 1/9$$

又因为

$$\alpha + \beta + \frac{1}{6} + \frac{1}{9} + \frac{1}{18} + \frac{1}{3} = 1 \quad \text{或者} \quad \alpha + \beta = \frac{1}{3}$$

故

$$\alpha = \frac{2}{9}$$

3. 设 $X \sim N\left(\mu_1, \sigma_1^2\right)$, $Y \sim N\left(\mu_2, \sigma_2^2\right)$, 那么 X 和 Y 的联合分布为 ___**C**___.

$A.$ 二维正态分布, 且 $\rho = 0$

$B.$ 二维正态分布, 且 ρ 不定

$C.$ 来必是二维正态分布

$D.$ 以上都不对

当 X、Y 相互独立时, 则 X 和 Y 的联合分布为 A .

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}\right.\right.$$

$$\left.\left. -2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(x-\mu_2)^2}{\sigma_2^2}\right]\right\}$$

三、解答题

1. 把一枚均匀硬币抛掷三次,设 X 为三次抛掷中正面出现的次数,而 Y 为正面出现次数与反面出现次数之差的绝对值,求 (X,Y) 的分布律与边缘分布.

解 (X,Y) 可取值 $(0,3)$, $(1,1)$, $(2,1)$, $(3,3)$

$P\{X=0,\ Y=3\}=(1/2)^3=1/8$

$P\{X=1,\ Y=1\}=\binom{3}{1}\cdot\dfrac{1}{2}\cdot\left(\dfrac{1}{2}\right)^2=3/8$

$P\{X=2,\ Y=1\}=\binom{3}{2}\cdot\left(\dfrac{1}{2}\right)^2\cdot\dfrac{1}{2}=3/8$

$P\{X=3,\ Y=0\}=(1/2)^3=1/8.$

X \ Y	1	3
0	0	1/8
1	3/8	0
2	3/8	0
3	0	1/8

(X,Y) 关于 X 的边缘分布

$P\{X=0\}=P\{X=0, Y=1\}+P\{X=0, Y=3\}=1/8,$

$P\{X=1\}=P\{X=1, Y=1\}+P\{X=1, Y=3\}=3/8,$

$P\{X=2\}=P\{X=2, Y=1\}+P\{X=2, Y=3\}=3/8,$

$P\{X=3\}=P\{X=3, Y=1\}+P\{X=3, Y=3\}=1/8.$

(X,Y) 关于 Y 的边缘分布

$P\{Y=1\}=\sum_{k=0}^{3} P\{X=k, Y=1\}=3/8+3/8=6/8,$

$P\{Y=3\}=\sum_{k=0}^{3} P\{X=k, Y=3\}=1/8+1/8=2/8.$

2. 设二维连续型随机变量(X,Y)的联合分布函数为

$$F(x,y) = A(B + \arctan\frac{x}{2})(C + \arctan\frac{y}{3})$$

(1) 求A、B、C 的值，

(2) 求(X,Y)的联合密度，

(3) 判断 X、Y 的独立性．

解 (1) 由 $F(+\infty, -\infty) = 0$, $F(-\infty, +\infty) = 0$,

$F(+\infty, +\infty) = 0$, 得到

$$A(B + \frac{\pi}{2})(C - \frac{\pi}{2}) = 0$$

$$A(B - \frac{\pi}{2})(C + \frac{\pi}{2}) = 0$$

$$A(B + \frac{\pi}{2})(C + \frac{\pi}{2}) = 1$$

解得 $A = \frac{1}{\pi^2}$, $B = C = \frac{\pi}{2}$.

$$(3)\, F_X(x) = F(x,+\infty) = \frac{1}{\pi^2}(\frac{\pi}{2} + \arctan\frac{x}{2})(\frac{\pi}{2} + \frac{\pi}{2})$$

$$= \frac{1}{\pi}(\frac{\pi}{2} + \arctan\frac{x}{2}) \quad (-\infty < x < +\infty)$$

$$F_Y(y) = F(+\infty, y) = \frac{1}{\pi^2}(\frac{\pi}{2} + \arctan\frac{y}{3})(\frac{\pi}{2} + \frac{\pi}{2})$$

$$= \frac{1}{\pi}(\frac{\pi}{2} + \arctan\frac{y}{3}) \quad (-\infty < y < +\infty)$$

可见 $\quad F(x,y) = F_X(x)F_Y(y),\, (x,y) \in R^2.$

故 X、Y 相互独立.

(2) (X,Y) 的联合密度为

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{6}{\pi^2(4+x^2)(9+y^2)}$$

(3) $f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$

$$= \frac{6}{\pi^2(4+x^2)} \int_{-\infty}^{+\infty} \frac{1}{9+y^2}dy$$

$$= \frac{2}{\pi^2(4+x^2)}[\arctan\frac{y}{3}]_{-\infty}^{\infty} = \frac{2}{\pi(4+x^2)}$$

$$(-\infty < x < +\infty)$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y) dx$$

$$= \frac{6}{\pi^2(9+y^2)} \int_{-\infty}^{+\infty} \frac{1}{4+x^2} dx$$

$$= \frac{3}{\pi^2(9+y^2)} [\arctan\frac{x}{2}]_{-\infty}^{\infty}$$

$$= \frac{3}{\pi(9+y^2)} \quad (-\infty < y < +\infty)$$

可见 $\quad f(x,y) = f_X(x)f_Y(y) \quad (x,y) \in R^2.$

故 X、Y 相互独立.

3. 设 X、Y 相互独立且服从 $U[-b,b]$，求方程 $t^2 + tX + Y = 0$ 有实根的概率，并求当 $b \to \infty$ 时这概率的极限.

解 X、Y 相互独立且服从 $U[-b,b]$，所以 X、Y 的联合密度为

$$f(x,y) = \begin{cases} \dfrac{1}{4b^2}, & |x| \le b, |y| \le b \\ \\ 0, & \text{其它} \end{cases}$$

方程 $t^2 + tX + Y = 0$ 有实根的概率为

$$P\{X^2 - 4Y \ge 0\}$$

$$= P\left\{Y \le \frac{X^2}{4}\right\} = \iint_D f(x,y)dxdy, \quad \text{其中} \quad D : y \le \frac{x^2}{4}.$$

$$\begin{cases} y = x^2 / 4 \\ x = b \end{cases} \Leftrightarrow \begin{cases} x = b \\ y = b^2 / 4 \le b \,(> b) \end{cases}$$
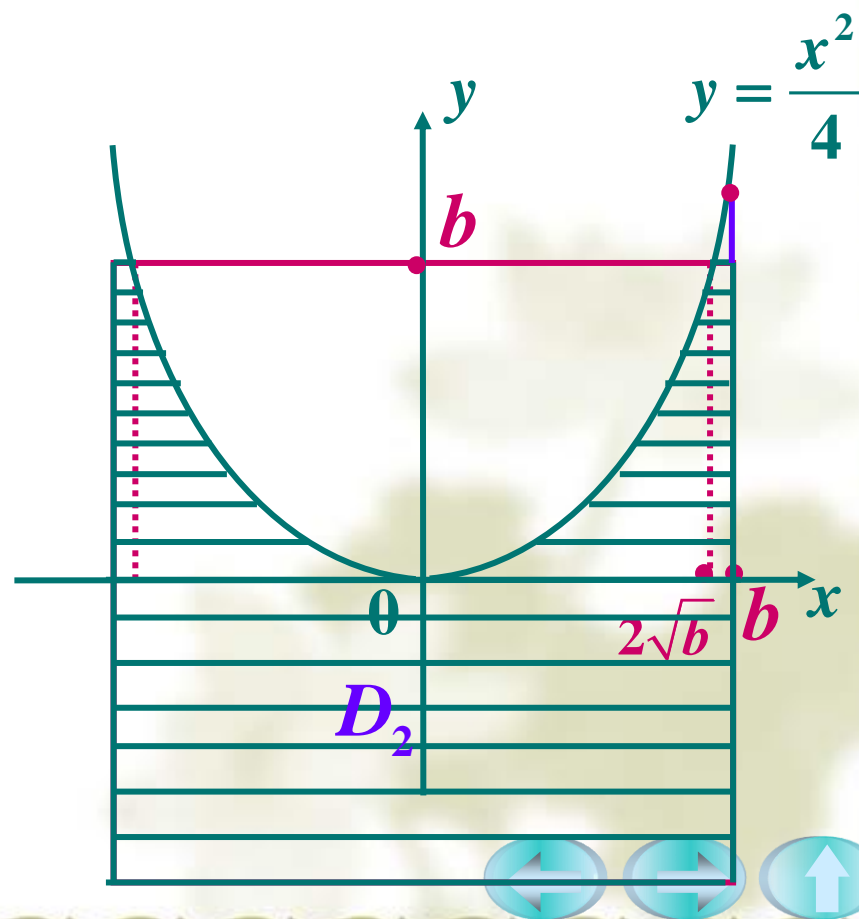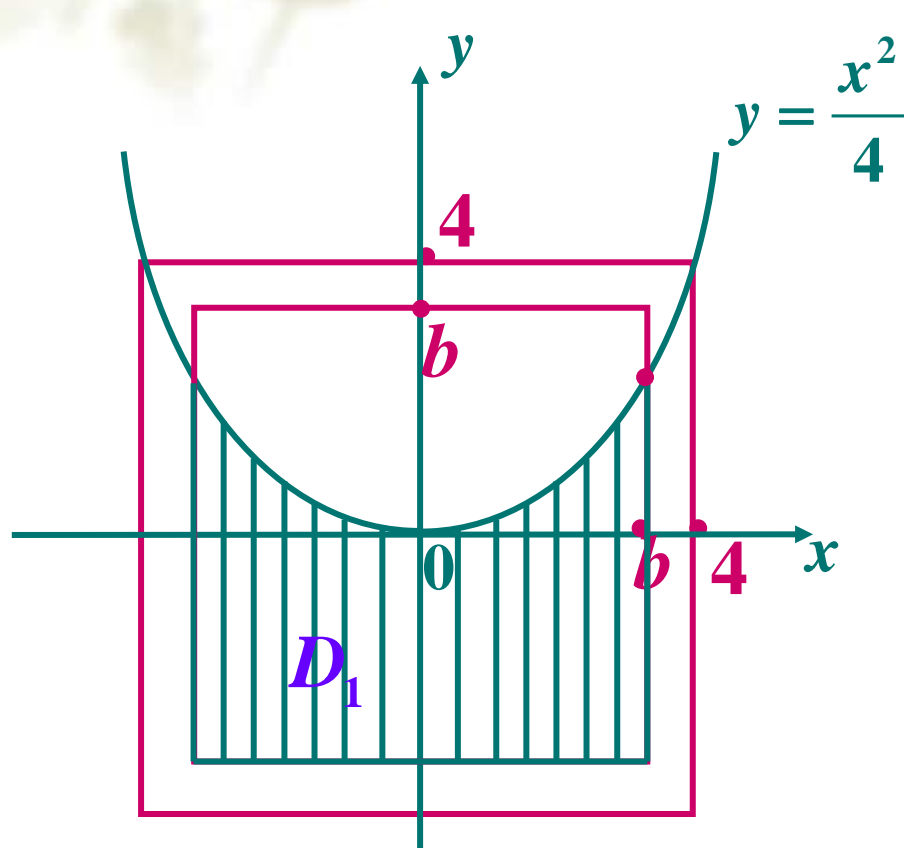
当 $b \le 4$ 时,

$$P\left\{X^2 - 4Y \ge 0\right\}$$

$$= \iint\limits_{D_1} f(x,y)dxdy$$

$$= \frac{1}{4b^2}\iint\limits_{D_1} dxdy$$

$$= \frac{2}{4b^2}\left(b^2 + \int_0^b dx \int_0^{x^2/4} dy\right)$$

$$= \frac{1}{2} + \frac{b}{24}$$

$y = \dfrac{x^2}{4}$

b

D_1

b 4
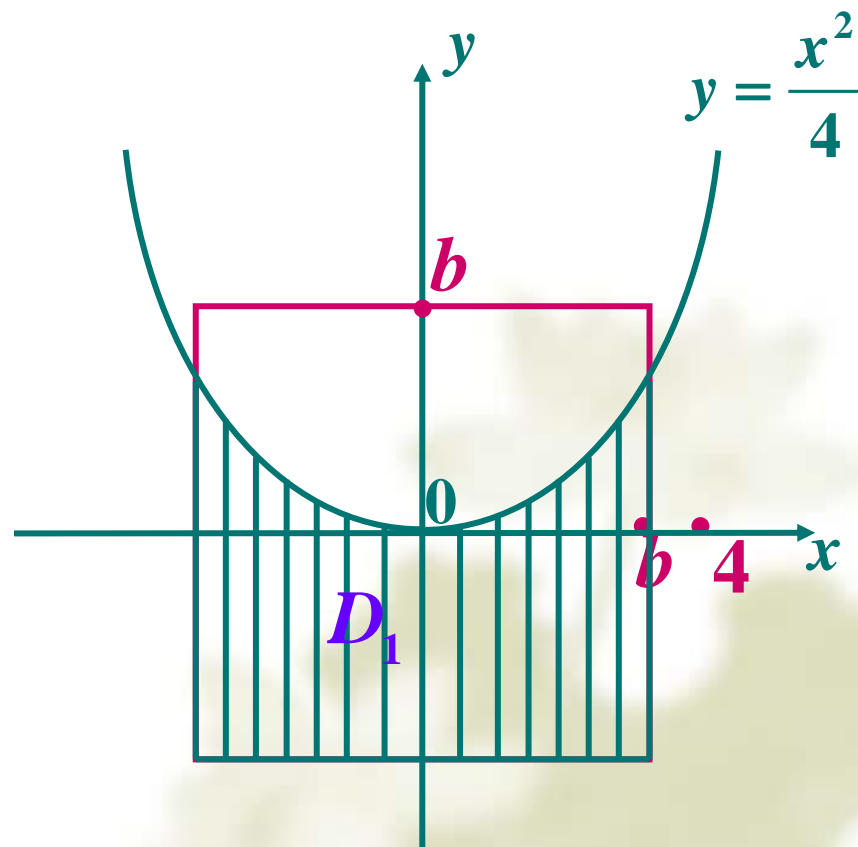
当 $b > 4$ 时，

$$P\left\{X^2 - 4Y \geq 0\right\} = \iint\limits_{D_2} f(x,y)dxdy = \frac{1}{4b^2}\iint\limits_{D_2} dxdy$$

$$= \frac{2}{4b^2}\left\{b^2 + [b(b - 2\sqrt{b}) + \int_0^{2\sqrt{b}} \frac{x^2}{4}dx\right\}$$

$$= 1 - \frac{2}{3\sqrt{b}}$$
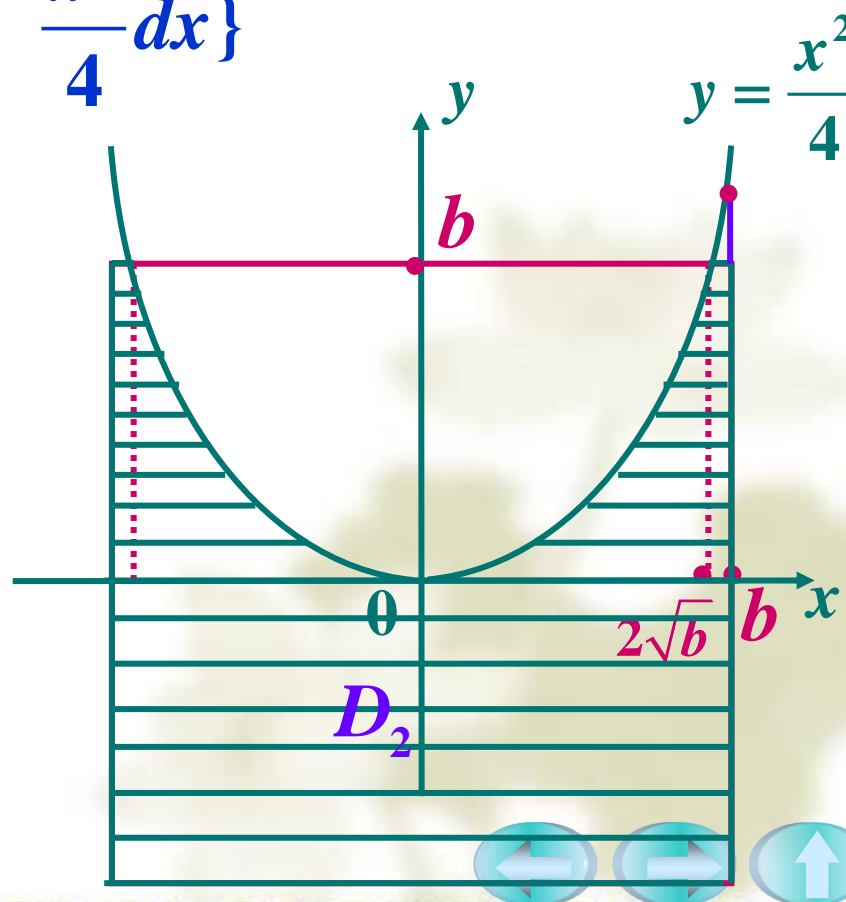
因而

$$P\left\{X^2 - 4Y \geq 0\right\} = \begin{cases} \dfrac{1}{2} + \dfrac{b}{24}, & 0 \leq b \leq 4 \\[4mm] 1 - \dfrac{2}{3\sqrt{b}}, & b > 4 \end{cases}$$

可见 $\displaystyle\lim_{b \to \infty} P\left\{X^2 - 4Y \geq 0\right\}$

$$= \lim_{b \to \infty}\left(1 - \frac{2}{3\sqrt{b}}\right) = 1.$$

4. 设**(X, Y)**的概率密度是

$$f(x,y) = \begin{cases} Ay(1-x), & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{其它} \end{cases}$$

求 **(1)** A 的值 **(2)** **(X,Y)**的分布函数 **(3)** 两个边缘密度.

解 **(1)** $1 = \iint\limits_{R^2} f(x,y)dxdy$

$$= \int_0^1 dx \int_0^x Ay(1-x)dy$$

$$= \frac{A}{2} \int_0^1 (x^2 - x^3)dx$$

$$= A/24$$
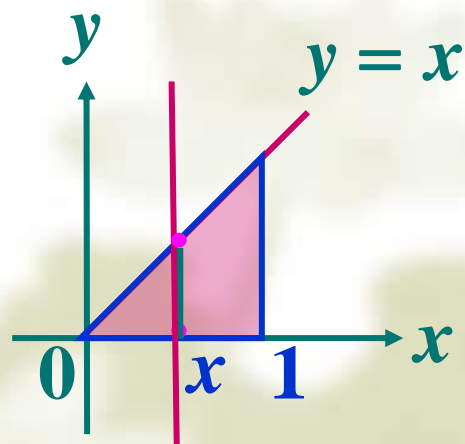
故 $A = 24.$

解 (2) $F(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x,y)dxdy$

积分区域 $D = (-\infty, x] \times (-\infty, y]$

$f(x,y) \neq 0$ 区域 $\{(x,y) | 0 \leq x \leq 1, 0 \leq y \leq x\}$

当 $x < 0$ 时, 不论 $y < 0$ 还是 $y \geq 0$, 都有 $F(x,y) = 0.$

暂时固定
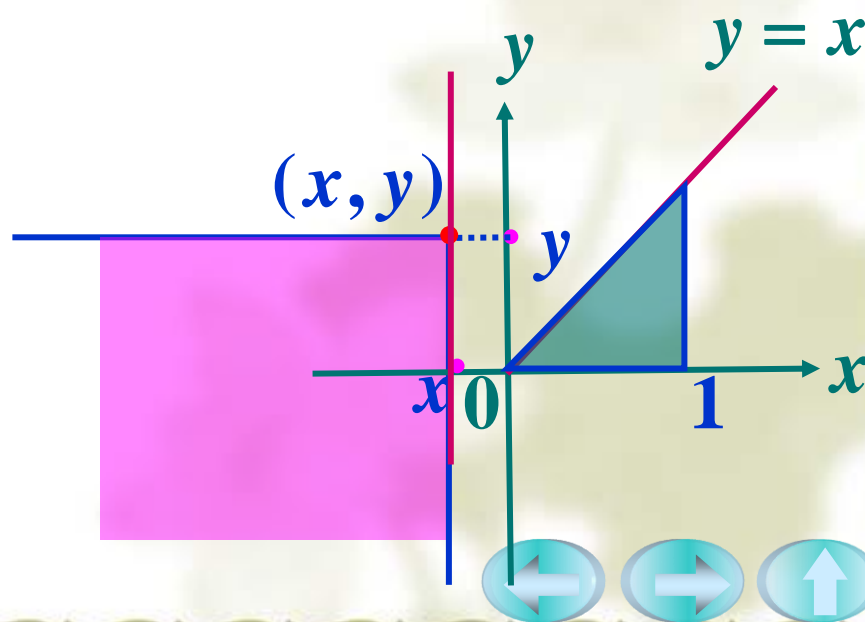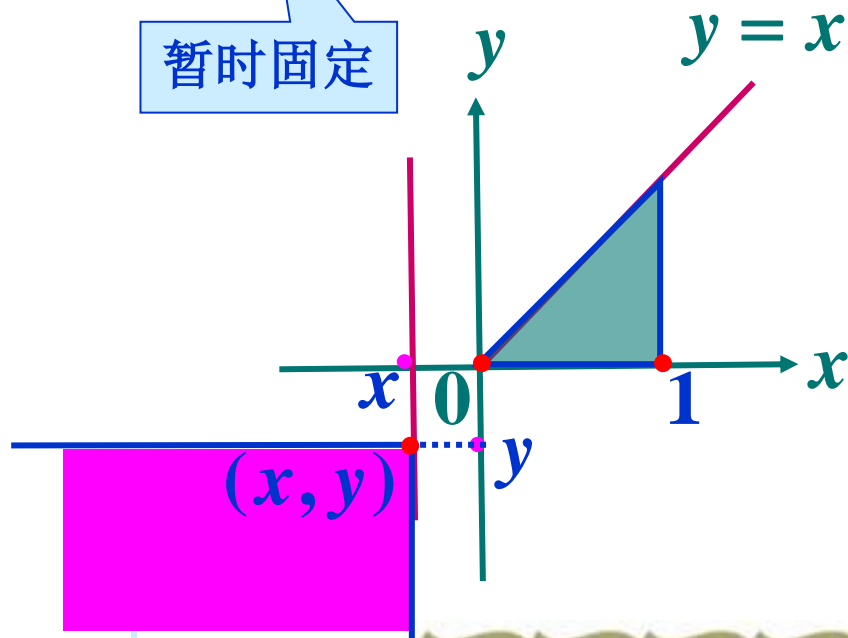
当 $0 \leq x < 1, y < 0$ 时，$F(x, y) = 0.$

当 $0 \leq x < 1, 0 \leq y < x$ 时，
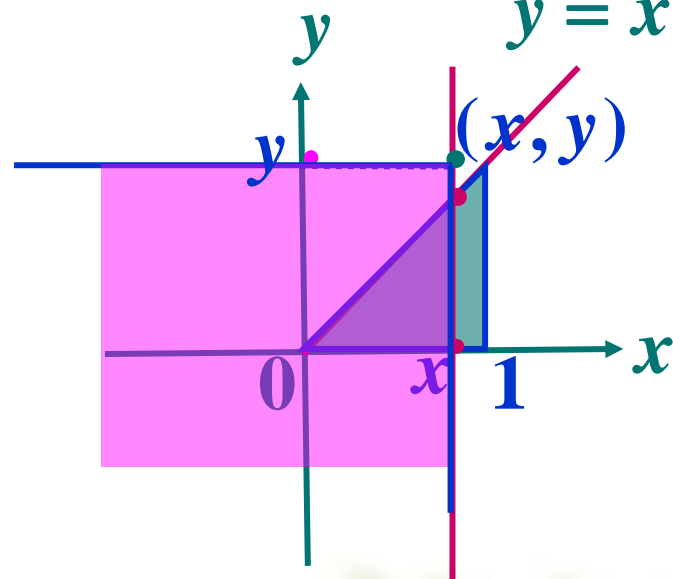
$$F(x, y) = 24 \int_0^y y dy \int_y^x (1-x) dx$$

$$= 24 \int_0^y [(x - \frac{x^2}{2})y - y^2 + \frac{y^3}{2}] dy$$

$$= 3y^4 - 8y^3 + 12(x - x^2/2)y^2.$$

$y = x$

y

0 x 1 x

y (x, y)

$y = x$

y

y (x, y)
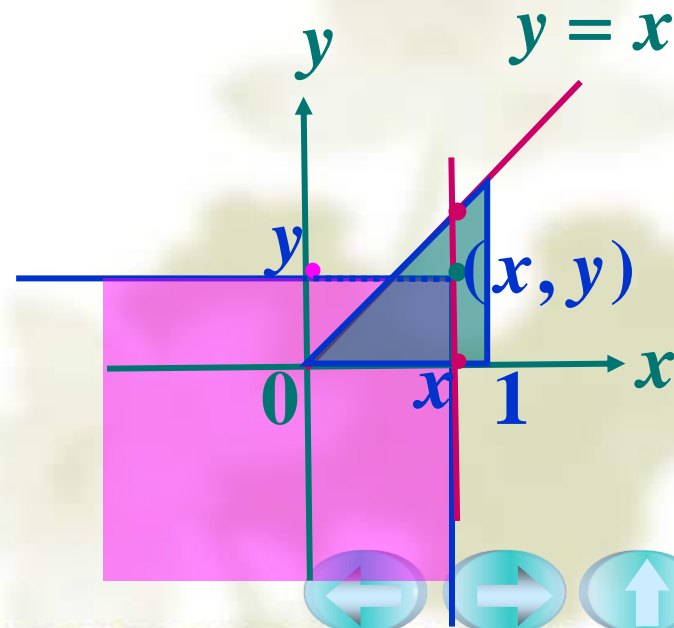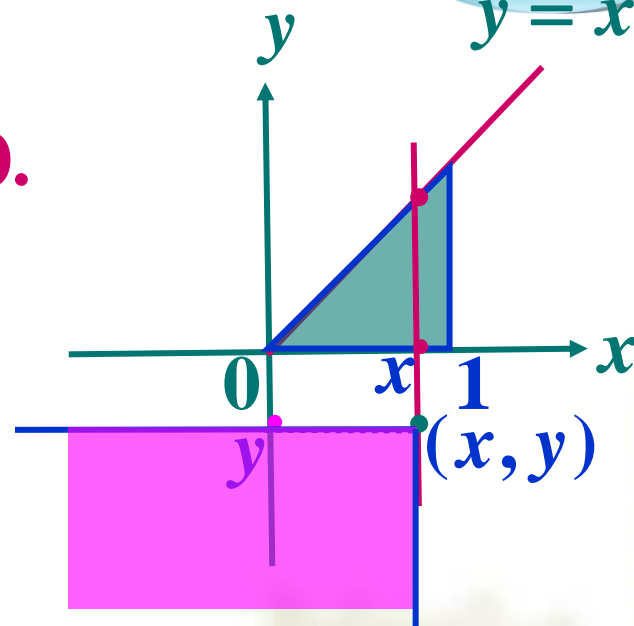
0 x 1 x

当 $0 \le x < 1, y \ge x$ 时,

$$F(x,y) = 24\int_0^x (1-x)dx \int_0^x y\,dy$$

$$= 12\int_0^x (x^2 - x^3)dx$$

$$= 4x^3 - 3x^4.$$

当 $x \ge 1, y \ge 1$ 时,

$$F(x,y) = 24\int_0^1 (1-x)dx \int_0^x y\,dy$$
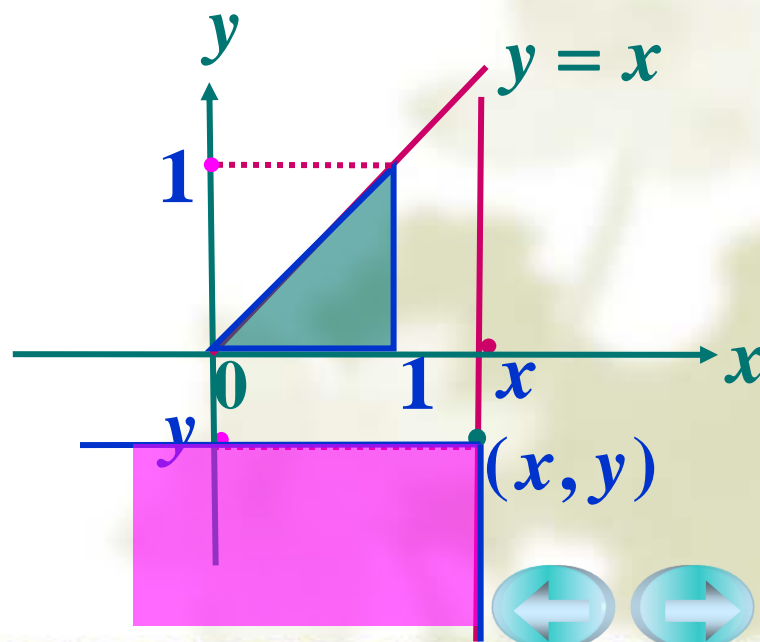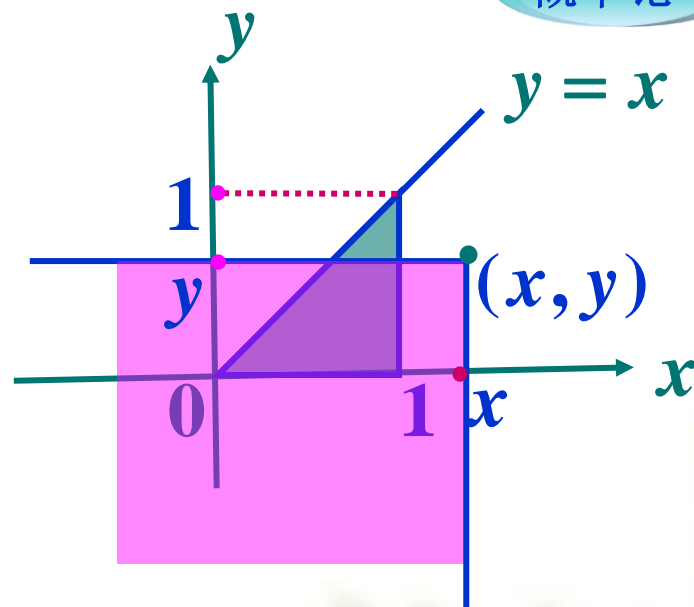
$$= 1.$$
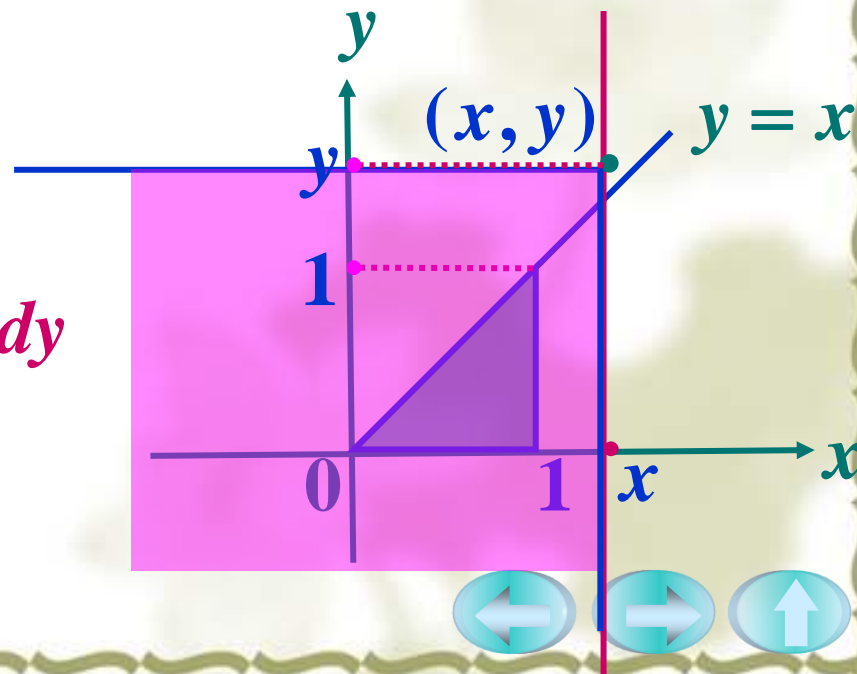
当 $x \geq 1, 0 \leq y < 1$ 时,

$$F(x, y) = 24 \int_0^y y \, dy \int_y^1 (1-x) \, dx$$

$$= 24 \int_0^y \left(\frac{y}{2} - y^2 + \frac{y^3}{2} \right) dy$$

$$= 3y^4 - 8y^3 + 6y^2.$$

当 $x \geq 1, y < 0$ 时,

$$F(x, y) = 0.$$

综上

$$F(x,y) = \begin{cases} 0, & x > 0 \text{ 或 } y < 0 \\ 3y^4 - 8y^3 + 12(x - x^2/2)y^2, & 0 \le x < 1, 0 \le y < x \\ 4x^3 - 3x^4, & 0 \le x < 1, y \ge x \\ 3y^4 - 8y^3 + 6y^2, & x \ge 1, 0 \le y < 1 \\ 1, & x \ge 1, y \ge 1 \end{cases}$$
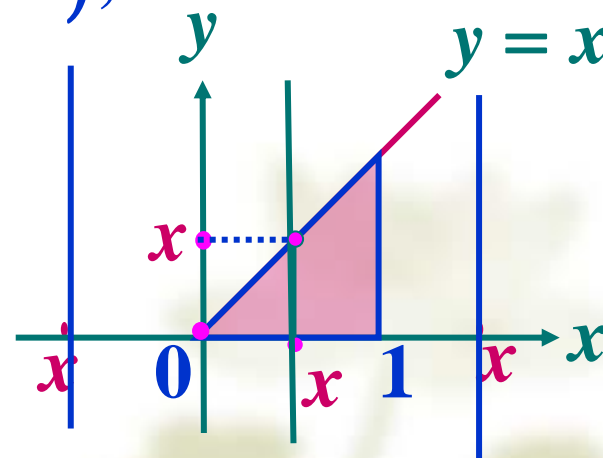
暂时固定

解 **(3)** $\quad f_X(x) = \int_{-\infty}^{+\infty} f(x,y)dy$

当 $x > 1$ 或 $x < 0$ 时，$\quad \forall y \in (-\infty, +\infty)$,
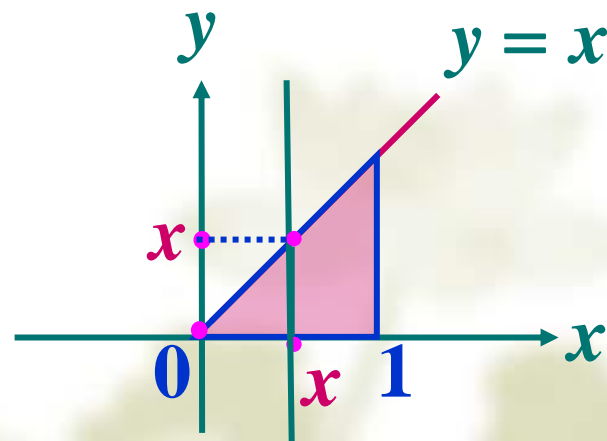
都有 $f(x,y) = 0$, 故 $f_X(x) = 0$.

当 $0 \le x \le 1$ 时,

$$f_X(x) = \int_{-\infty}^{0} f(x,y)dy$$

$$+ \int_{0}^{x} f(x,y)dy + \int_{x}^{+\infty} f(x,y)dy.$$

当 $0 \leq x \leq 1$ 时，

$$f_X(x) = \int_{-\infty}^{0} f(x,y)dy$$

$$+ \int_{0}^{x} f(x,y)dy + \int_{x}^{+\infty} f(x,y)dy .$$

$$= \int_{0}^{x} 24y(1-x)dy$$

$$= 12x^2(1-x),$$

综上，

$$f_X(x) = \begin{cases} 12x^2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{其它} \end{cases}$$

注意取值范围

解 **(2)** $\quad f_Y(y) = \displaystyle\int_{-\infty}^{+\infty} f(x,y)\,dx$
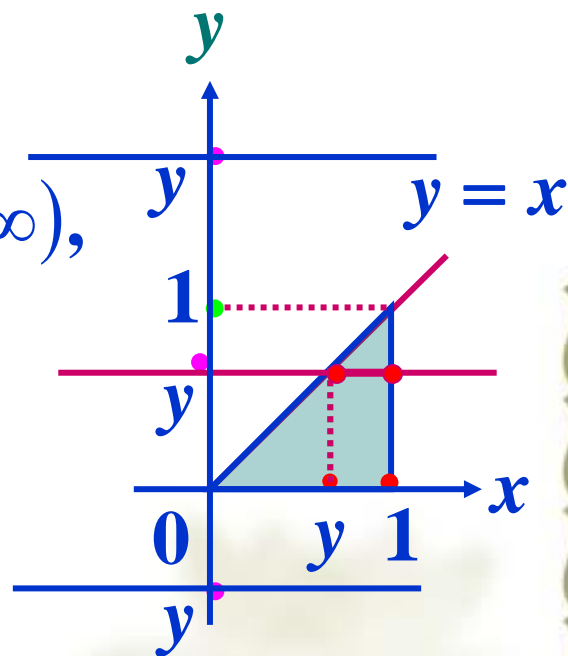
当 $y > 1$ 或 $y < 0$ 时, 对 $\forall x \in (-\infty,+\infty)$,

都有 $f(x,y) = 0$, 故 $f_Y(y) = 0$.

当 $0 \le y \le 1$ 时,

$f_Y(y) = \displaystyle\int_{-\infty}^{y} f(x,y)\,dx$

$\qquad + \displaystyle\int_{y}^{1} f(x,y)\,dx + \int_{1}^{+\infty} f(x,y)\,dx$.

$\qquad = \displaystyle\int_{y}^{1} 24y(1-x)\,dx$

$\qquad = 12y(1-y)^2,$

综上，

$$f_Y(y) = \begin{cases} 24y(1-y)^2, & 0 \le y \le 1 \\ 0, & \text{其它} \end{cases}$$

注意取值范围

5. 设 (X,Y) 的概率密度是

$$f(x,y) = \begin{cases} Ay(1-x), & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{其它} \end{cases}$$

(1) X 与 Y 是否相互独立**?**

(2) 求 $f\left(y|x\right)$ 和 $f\left(x|y\right)$；

(3) 求 $Z = X + Y$ 概率密度**.**
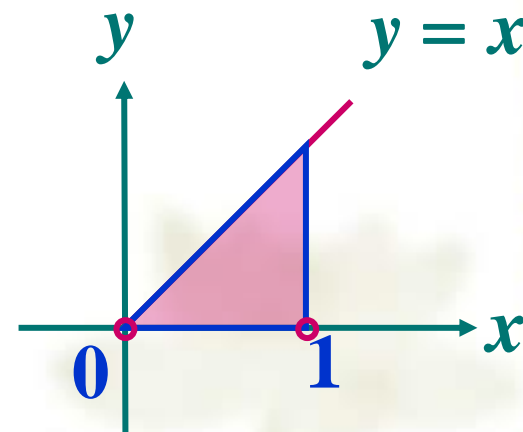
解 (1) 因为 $f(x,y) \neq f_X(x) \cdot f_Y(y)$

所以 X 与 Y 不独立 **.**

(2)

$$f(x,y) = \begin{cases} 24y(1-x), & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{其它} \end{cases}$$

$$f_X(x) = \begin{cases} 12x^2(1-x), & 0 \le x \le 1 \\ 0, & \text{其它} \end{cases}$$

当 $0 < x < 1$ 时, $f_X(x) \ne 0$.

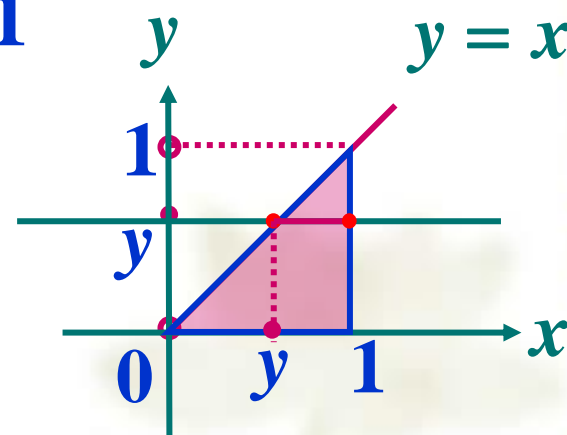故 $f(y|x) = \dfrac{f(x,y)}{f_X(x)} = \begin{cases} 2y/x^2, & 0 < x < 1, 0 < y \le x \\ 0, & \text{其它} \end{cases}$

暂时固定

$$f(x,y) = \begin{cases} 24y(1-x), & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{其它} \end{cases}$$

$$f_Y(y) = \begin{cases} 12y(1-y)^2, & 0 \le y \le 1 \\ 0, & \text{其它} \end{cases}$$

当 $0 < y < 1$ 时, $f_Y(y) \ne 0.$

故

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} 2(1-x)\big/(1-y)^2, & y \le x < 1, 0 < y < 1 \\ 0, & \text{其它} \end{cases}$$
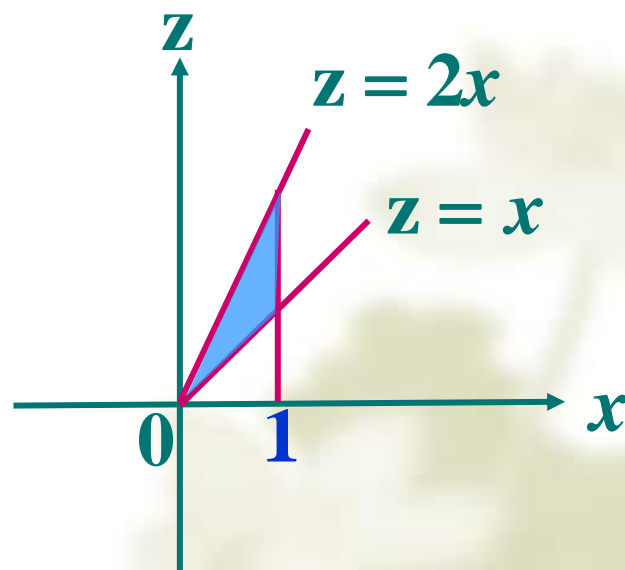
暂时固定

(3)

暂时固定

Z=X+Y 的密度函数为

$$f_Z(z) = \int_{-\infty}^{+\infty} f(x, z-x)dx$$

$$\begin{cases} 0 \le x \le 1 \\ 0 \le z\text{-}x \le x \end{cases}$$

$$\begin{cases} 0 \le x \le 1 \\ x \le z \le 2x \end{cases}$$
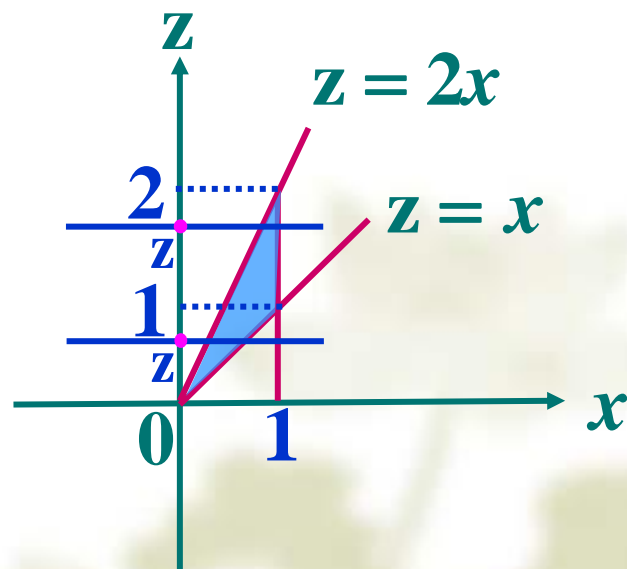
z

z = 2x

z = x

0   1   x

当 $z \leq 0$ 或 $z > 2$ 时， $f_Z(z) = 0.$

当 $0 < z \leq 1$ 时，

$$f_Z(z) = \int_{z/2}^{1} 24(z - x)(1 - x)dx$$

当 $1 < z \leq 2$ 时，

$$f_Z(z) = \int_{z/2}^{z} 24(z - x)(1 - x)dx$$

四、证明题

在区间 $[0,1]$ 上随机地投掷两点, 试证这两点间的距离的密度函数为

$$f(z) = \begin{cases} 2(1-z), & 0 \leq z \leq 1 \\ 0, & \text{其它} \end{cases}$$

证明　设这两个随机点分别为 X, Y, 则有

$X \sim U[0,1]$, $Y \sim U[0,1]$. 于是 X, Y 的概率密度分别为

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{其它} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1 \\ 0, & \text{其它} \end{cases}$$

因为 X, Y 相互独立，所以 X, Y 的联合密度为

$$f(x,y) = \begin{cases} 1, 0 \le x \le 1, 0 \le y \le 1 \\ 0, \text{其它} \end{cases}$$

这两个随机点 X, Y 的距离为 $Z = |X - Y|$.

Z 的分布函数为

$$F_Z(z) = P\{Z \le z\} = P\{|X - Y| \le z\}$$
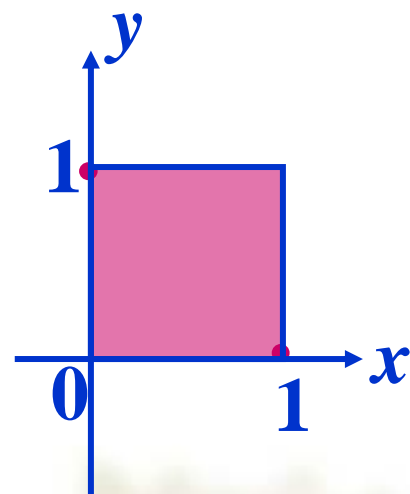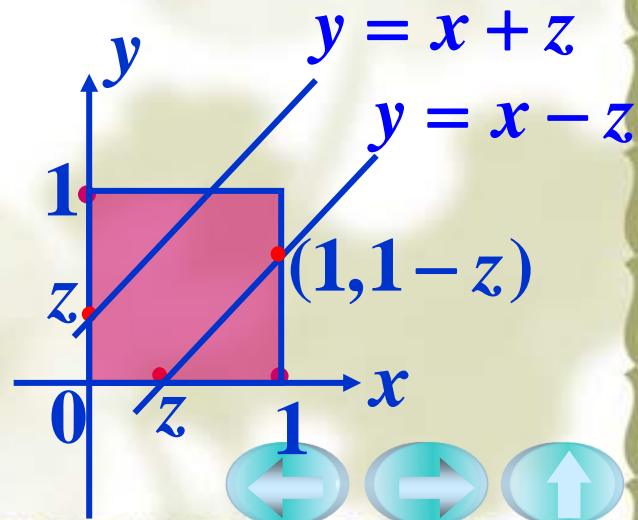
暂时固定

当 $z < 0$ 时，$F_Z(z) = 0$，$f_Z(z) = 0$.

当 $z = 0$ 时，$F_Z(z) = 0$，$f_Z(z) = 0$.
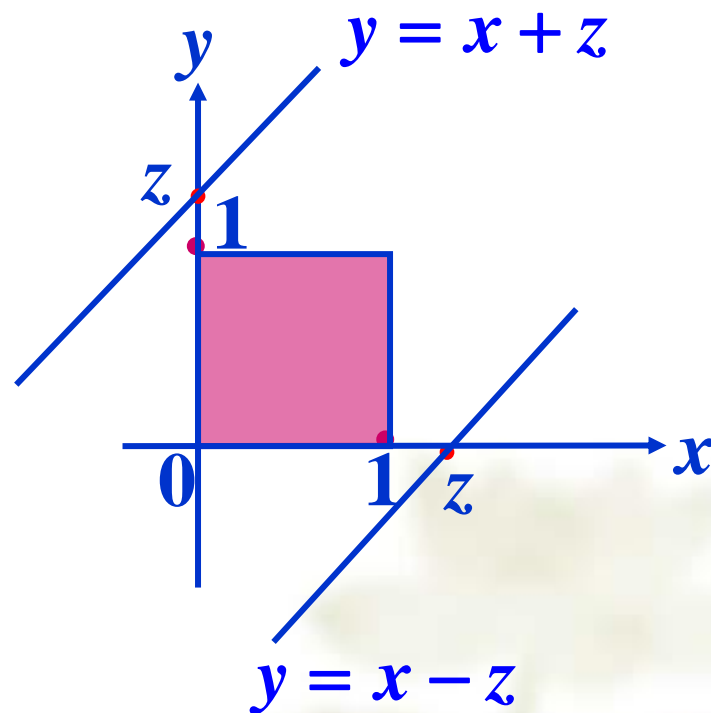
当 $z > 0$ 时，

$$F_Z(z) = \iint\limits_{|x-y| \le z} f(x,y)\,dxdy$$

当 $0 < z < 1$ 时，

$$F_Z(z) = 1 - (1-z)^2 = 2z - z^2$$

$$f_Z(z) = 2(1-z).$$

$y = x + z$

$y = x - z$

$(1, 1-z)$

当 $z \geq 1$ 时,

$$F_Z(z) = 1, \quad f_Z(z) = 0.$$

综上

$$f(z) = \begin{cases} 2\pi(1-z), & 0 \leq z \leq 1 \\ 0, & \text{其它} \end{cases}$$

$y = x + z$

$y = x - z$