Thermal model of Marscrete extrusion

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Abstract

Short abstract

Contents

1 Objective

This document describes a (hopefully) simple lumped-mass thermal model of Marscrete extrusion. By modeling Marscrete extrusion as a system of differential equations, I hope to be able to draw conclusions about the controllability and observability of the thermal system.

In particular, there are two questions to address:

- **Controllability:** is it possible to pump heat into the system (as a function of time) in a way that drives the temperature profile (as a function of position) from an initial profile to a desired one?
- **Observability:** is it possible to determine the current temperature profile (as a function of position), given the history of heat added to the system and the history of temperature sensor measurements?

Depending on whether the thermal system is controllable and/or observable, it may be possible to further design controllers and estimators:

- It may be possible to optimize the heat input (control) signal to minimize various objective functions while satisfying certain constraints.
- Observability can be used as a criterion to determine the viability of different temperature sensor placements.
- It may be possible to estimate the temperature-vs-position profile using an observer such as a Kalman filter and a reduced sensor set.

2 Motivating example - pipe flow

3 Modeling

3.1 Control input: heat flux

The heating mechanism (i.e. the control input) for the extruder operates via Joule heating:

$$q_{\text{control}} = i^2 R, \tag{1}$$

where the heat delivered to the extruder is a consequence of Ohm's law, with current i and wire resistance R. Over a finite area A, this results in a heat flux

$$q^{''} = \frac{i^2 R}{A}. \tag{2}$$

3.2 One-dimensional heat equation - first pass

As a basic example, consider the heat equation, in one dimension (x), with an internal heat source $\dot{q}$.

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = \dot{q}, \tag{3}$$

where u is the temperature of the material and $x \in [0, L]$, where L is the length of the material.

3.2.1 Homogeneous solution

The homogeneous one-dimensional heat equation[1] is

$$\frac{\partial u}{\partial t} - \alpha^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{4}$$

with homogeneous boundary conditions

$$u(0, t) = u(L, t) = 0 \text{ and } u(x, 0) = f(x). \tag{5}$$

Start off with the separation of variables ansatz:

$$u(x, t) = X(x)T(t), \tag{6}$$

so that

$$\frac{\partial u}{\partial t} = X(x)T'(t), \tag{7a}$$

$$\frac{\partial u}{\partial x} = X'(x)T(t) \rightarrow \frac{\partial^2 u}{\partial x^2} = X''(x)T(t), \tag{7b}$$

and substitute this expression for u back into the homogeneous 1-D heat equation:

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}, \tag{8}$$

and because the left-hand side and right-hand side are independent, both sides must be equal to a constant:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\alpha^2 T(t)} = -\lambda \rightarrow \begin{cases} X''(x) + \lambda X(x) = 0, \\ T'(t) + \lambda \alpha^2 T(t) = 0. \end{cases} \tag{9}$$

We can now solve for $X(x)$ and $T(t)$ separately. First, the general solution for $X(x)$ is

$$X(x) = c_1 \sin\left(\sqrt{\lambda} x\right) + c_2 \cos\left(\sqrt{\lambda} x\right). \tag{10}$$

[1] This solution is from Bill Goodwine's *Engineering Differential Equations: Theory and Applications*.

Enforcing the left-hand boundary condition $u(0,t) = 0$ implies $c_2 = 0$. From the right-hand boundary condition,

$$u(L,t) = X(L) = c_1 \sin\left(\sqrt{\lambda} L\right) = 0 \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \; n = 1, 2, 3, ..., \tag{11}$$

and therefore

$$X_n(x) = c_n \sin\left(\frac{n\pi x}{L}\right). \tag{12}$$

Now, the solution for $T(t)$ is

$$T(t) = A\, \mathrm{e}^{-(\alpha n\pi/L)^2 t}, \tag{13}$$

where A is some constant yet to be determined. In fact, since both A and c_n are unknown at this point, we can group them together in a new series of constants c_n, resulting in the following general solution:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{e}^{-(\alpha n\pi/L)^2 t}} \tag{14}$$

The last step to finding the general solution is to determine c_n by applying the remaining boundary condition $u(x,0) = f(x)$:

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x). \tag{15}$$

To express the initial temperature profile $f(x)$ as a Fourier sine series, first multiply both sides by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from 0 to L:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) c_n \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \int_0^L \sin\left(\frac{m\pi x}{L}\right) f(x) \mathrm{d}x, \tag{16}$$

which evaluates to 0 everywhere except when $m = n$, resulting in

$$c_n \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) \mathrm{d}x, \tag{17}$$

which evaluates to

$$\boxed{c_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) \mathrm{d}x} \tag{18}$$

3.2.2 Nonhomegeneous solution