PHYS 324 HW #1

1. a) hydrogen atom

$$F_{centrifugal} = F_{coulombic}$$

$$-m\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_0 r^2} \Rightarrow r = \frac{e^2}{4\pi\epsilon_0 m v^2}$$

while. $L = n\hbar = mvr$.

$$\Rightarrow r = \frac{me^2r^2}{4\pi\epsilon_0 m^2v^2r^2} = \frac{me^2r^2}{4\pi\epsilon_0 (L)^2} = \frac{me^2 r^2}{4\pi\epsilon_0 (n\hbar)^2}$$

$$\Rightarrow r = \frac{4\pi\epsilon_0 n^2\hbar^2}{me^2}$$

On the other hand,

$$E = E_{kinetic} + E_{potential} = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r}$$

while. $m\frac{v^2}{r} = \frac{e^2}{4\pi\epsilon_0 r^2} \Rightarrow mv^2 = \frac{e^2}{4\pi\epsilon_0 r}$

$$\Rightarrow E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_0 r} = \frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r} - \frac{e^2}{4\pi\epsilon_0 r} = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 r}$$

$$= -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0}\frac{me^2}{4\pi\epsilon_0 n^2\hbar^2} = -\frac{me^4}{2(4\pi\epsilon_0)^2 n^2\hbar^2}$$

while the free stationary electron energy is

$$E_{free} = mc^2$$

$$\Rightarrow E/E_{free} = -\frac{e^4}{32\pi^2\epsilon_0^2\hbar^2c^2}\frac{1}{n^2}$$

b). $$\hbar\omega = E_n - E_m = -\frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2 n^2} + \frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2 m^2}$$

(here m_e is the mass of the electron. n & m are integers)

$$\Rightarrow \hbar\omega = \frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2}\left(\frac{1}{m^2} - \frac{1}{n^2}\right) = \hbar\frac{2\pi c}{\lambda}$$

$$\Rightarrow \frac{1}{\lambda} = \underbrace{\frac{m e e^4}{64\pi^3 \epsilon_0^2 \hbar^3 c}}_{R_\infty} \left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

c). $$R_\infty = \frac{m e e^4}{64\pi^3 \epsilon_0^2 \hbar^3 c} = \frac{m_e e^4}{8\epsilon_0^2 h^3 c}$$

$$\Rightarrow hcR_\infty = \frac{m e e^4}{8\epsilon_0^2 h^2} = m_e c^2 \frac{\alpha^2}{2}, \quad \text{where } \alpha = \frac{1}{4\pi\epsilon_0}\frac{e^2}{\hbar c} = \frac{e^2}{2\epsilon_0 h c}$$

is the fine structure constant

$$\alpha \approx \frac{1}{137}.$$

$$\Rightarrow hcR_\infty = \underbrace{m_e c^2}_{0.511\,MeV} \frac{\alpha^2}{2} = 0.511\times 10^6 eV \cdot \frac{1}{137^2}\cdot\frac{1}{2} = 13.6\,eV$$

Since. $hc = 1.24\times 10^{-8}\ ev\cdot m.$

$$\Rightarrow R_\infty = \frac{13.6\,eV}{1.24\times 10^{-8}\,ev\cdot m} = 1.1\times 10^7\,m^{-1}$$

d) $$\frac{1}{\lambda} = R_\infty\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}R_\infty \Rightarrow \lambda = \frac{4}{3R_\infty} = \frac{4}{3\times 1.1\times 10^7\,m^{-1}} = 1.21\times 10^{-7}\,m$$

2(a) $e^{h\nu/kT}$ Taylor expansion $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots$

$$\Rightarrow e^{h\nu/kT} = 1 + \frac{h\nu}{kT} + \frac{1}{2!}\left(\frac{h\nu}{kT}\right)^2 + \cdots \approx 1 + \frac{h\nu}{kT}$$

$$\Rightarrow u_\nu = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1} \approx \frac{8\pi h}{c^3}\frac{\nu^3}{h\nu/kT} = \frac{8\pi kT}{c^3}\nu^2 = \frac{8\pi kT}{c^3}\nu^2 \propto \nu^2$$

condition: $h\nu/kT \ll 1 \Rightarrow \nu \ll kT/h.$

b). $$\frac{du}{d\nu} = \frac{8\pi h}{c^3}\frac{3\nu^2(e^{h\nu/kT}-1) - \nu^3 e^{h\nu/kT}\frac{h}{kT}}{(e^{h\nu/kT}-1)^2} = 0, \quad \frac{du}{d\nu} = 0$$

$$\Rightarrow 3(e^{h\nu/kT}-1) = \nu e^{h\nu/kT}\frac{h}{kT}.$$

Let $x = \frac{h\nu}{kT}$, then $X = X_{max}$ when $V = V_{max}$

$\Rightarrow 3(e^x - 1) = xe^x$ — This can be solved by Mathematica,

$$\Rightarrow \quad x \approx 2.8$$

$$\Rightarrow \quad \nu_{max} = \frac{kT x_{max}}{h} = \frac{2.8kT}{h}$$

We can see that $\nu_{max} \propto T$, thus with a higher temperature, we tend to have an increasing ν, in other words, a bluer object.

c). $u_\nu d\nu = u_\lambda d\lambda \qquad \lambda = \frac{c}{\nu} \Rightarrow d\lambda = d\frac{c}{\nu} = -\frac{c}{\nu^2} d\nu \quad , \nu = \frac{c}{\lambda}$

while $\quad u_\nu = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT}-1}$

$$\Rightarrow \quad u_\lambda d\lambda = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1} d\nu = u_\lambda\left(-\frac{c}{\nu^2}\right) d\nu.$$

$$\Rightarrow \quad u_\lambda = \frac{8\pi h\nu^5}{c^4}\frac{1}{e^{h\nu/kT}-1} = \frac{8\pi hc}{\lambda^5}\frac{1}{e^{hc/\lambda kT}-1}$$

4. $\quad p = -i\hbar\frac{\partial}{\partial x} \quad \Rightarrow \quad \langle p\rangle = -i\hbar\int \psi^* \frac{\partial\psi}{\partial x} dx$

while $\quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi = i\hbar\frac{\partial\psi}{\partial t}$

$$\Rightarrow \quad \frac{\partial\langle p\rangle}{\partial t} = -i\hbar\int[\psi^*_t\psi_x + \psi^*\psi_{xt}]dx$$

$$= \int\left[\left(-\frac{\hbar^2}{2m}\psi^*_{xx} + V\psi^*\right)\psi_x + \psi^*\left(\frac{\hbar^2}{2m}\psi_{xxx} - V_x\psi - V\psi_x\right)\right]dx$$

$$= \underbrace{\int\left[\left(-\frac{\hbar^2}{2m}\psi^*_{xx}\right)\psi_x\right.}_{\text{integrate by part.}} \left. + \psi^*\left(\frac{\hbar^2}{2m}\psi_{xxx} - V_x\psi\right)\right]dx.$$

$$\Rightarrow \quad \int\psi^*_{xx}\psi_x dx = \int\psi_x d\psi^*_x = \overset{0}{\cancel{\psi_x\psi^*_x\Big|_{-\infty}^{\infty}}} - \int\psi^*_x\psi_{xx}dx.$$

boundary conditions of ψ. & all its derivatives. (→ 0 at infinity)

$$= -\int\psi^*_x\psi_{xx}dx = -\int\psi_{xx}d\psi^* = -\overset{0}{\cancel{\psi_{xx}\psi^*\Big|_{-\infty}^{\infty}}} + \int\psi^*\psi_{xxx}dx$$

$$\Rightarrow \quad \frac{\partial \langle p \rangle}{\partial t} = \int \left[-\frac{\hbar^2}{2m}\Psi^{*}\Psi_{xxx} + \Psi^{*}\frac{\hbar^2}{2m}\Psi_{xxx} - \Psi^{*}V_x\Psi \right] dx$$

$$= -\int \Psi^{*} V_x \Psi\, dx = \langle -\frac{\partial V}{\partial x} \rangle.$$

3. $\Psi = A e^{-x^2/a^2 + i\omega t}$

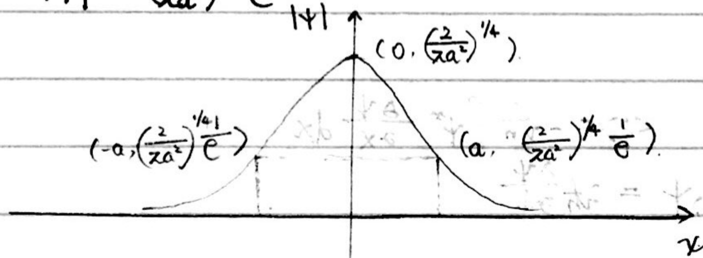
a) $\rho = |\Psi|^2 = A^2 e^{-2x^2/a^2}$

$\int_{-\infty}^{\infty} \rho\, dx = 1 \Rightarrow A^2 \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx = 1.$ while $\int_{-\infty}^{\infty} e^{-2x^2/a^2} dx = a\sqrt{\frac{\pi}{2}}.$

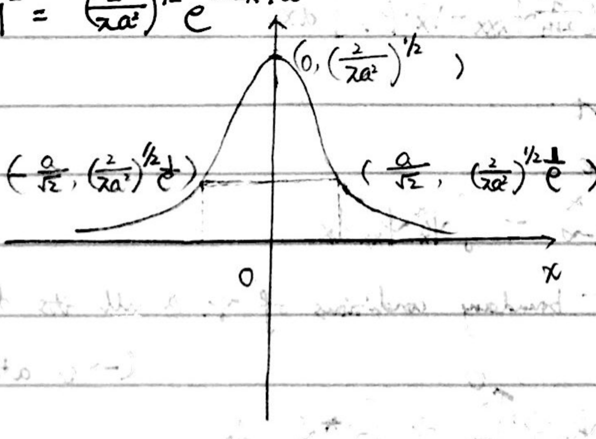
$\Rightarrow A^2 = 1 / \int_{-\infty}^{\infty} e^{-2x^2/a^2} dx = \sqrt{\frac{2}{\pi a^2}}$

$\Rightarrow A = \left(\frac{2}{\pi a^2}\right)^{1/4} \qquad \Rightarrow \rho = |\Psi(x)|^2 = \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2x^2/a^2}$

b) $|\Psi| = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2}$



$\rho = |\Psi|^2 = \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2x^2/a^2}$



c) odd parity: the wave function is an odd function.

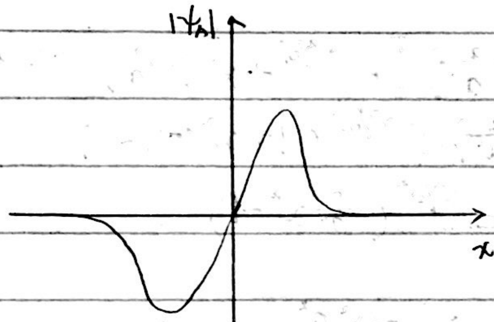
$$\rho_A = |\psi_A|^2 = B^2 x^2 e^{-2x^2/a^2}$$

$$\int_{-\infty}^{+\infty} \rho_A \, dx = \int_{-\infty}^{\infty} B^2 x^2 e^{-2x^2/a^2} dx = B^2 \int_{-\infty}^{\infty} x^2 e^{-2x^2/a^2} dx = B^2 \frac{27}{4}\sqrt{\frac{\pi}{2}} = 1$$

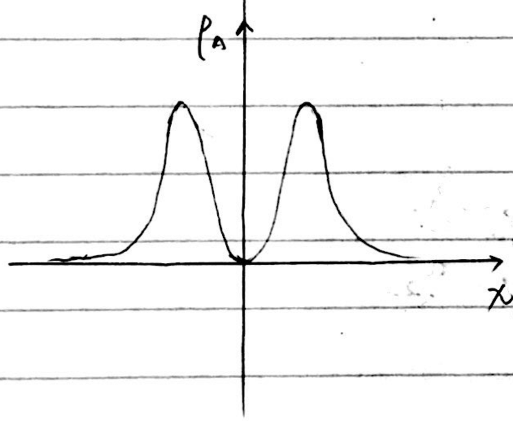
$$\Rightarrow \quad B^2 = \sqrt{\frac{32}{729\pi}} \quad \Rightarrow \quad B = \left(\frac{32}{729\pi}\right)^{1/4}$$

$$\Rightarrow \quad \psi_A = \left(\frac{32}{729\pi}\right)^{1/4} x e^{-x^2/a^2} \quad , \quad \rho_A = \left(\frac{32}{729\pi}\right)^{1/2} x^2 e^{-2x^2/a^2}$$

ψ_A:



ρ_A :



d) $$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \int_{-\infty}^{\infty} \underbrace{x \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2x^2/a^2}}_{\text{odd function}} dx = 0$$

$$\langle p \rangle = \int_{-\infty}^{\infty} dx \, \psi^* p \psi = -i\hbar \int_{-\infty}^{\infty} dx \, \psi^* \frac{\partial \psi}{\partial x} = -i\hbar \int_{-\infty}^{\infty} dx \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2 - i\omega t} \left(\frac{2}{\pi a^2}\right)^{1/4} e^{i\omega t} \cdot e^{-x^2/a^2} (-2x/a^2).$$

$$\propto \int_{-\infty}^{\infty} \underbrace{dx \, x e^{-2x^2/a^2}}_{\text{odd function}} = 0$$

e) $\langle x^2\rangle = \int_{-\infty}^{\infty} dx\, x^2|\psi|^2 = \int_{-\infty}^{\infty} dx\, x^2 \left(\frac{2}{\pi a^2}\right)^{1/2} e^{-2x^2/a^2} = \left(\frac{2}{\pi a^2}\right)^{1/2}\int_{-\infty}^{\infty} dx\, x^2 e^{-2x^2/a^2}$

$= a^2/4$

$\langle p^2\rangle = \int_{-\infty}^{\infty} dx\, \psi^* \left(-i\hbar\frac{\partial}{\partial x}\right)^2 \psi = -\hbar^2\int_{-\infty}^{\infty} dx\, \psi^* \frac{\partial^2\psi}{\partial x^2}$

while $\psi = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2+i\omega t}$, $\psi^* = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{-x^2/a^2-i\omega t}$

Since $\frac{\partial}{\partial x} e^{-x^2/a^2} = e^{-x^2/a^2}(-2x)\frac{1}{a^2} = -\frac{2}{a^2}xe^{-x^2/a^2}$

$\frac{\partial}{\partial x}(xe^{-x^2/a^2}) = e^{-x^2/a^2} + x^2\left(-\frac{2}{a^2}\right)e^{-x^2/a^2}$

$\Rightarrow \frac{\partial^2\psi}{\partial x^2} = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{i\omega t}\left(-\frac{2}{a^2}\right)\left[e^{-x^2/a^2} + x^2\left(-\frac{2}{a^2}\right)e^{-x^2/a^2}\right]$

$= \left(\frac{2}{\pi a^2}\right)^{1/4} e^{i\omega t}\left(-\frac{2}{a^2}\right)e^{-x^2/a^2}\left(1+x^2\left(-\frac{2}{a^2}\right)\right)$

$= \left(\frac{2}{\pi a^2}\right)^{1/4} e^{i\omega t}\left(\frac{2}{a^2}\right)e^{-x^2/a^2}\left(\frac{2}{a^2}x^2-1\right)$

$\Rightarrow \langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty} dx\left(\frac{2}{\pi a^2}\right)^{1/2}\left(\frac{2}{a^2}\right)e^{-2x^2/a^2}\left(\frac{2}{a^2}x^2-1\right)$

$= \frac{\hbar^2}{a^2}$

$$\Rightarrow \begin{cases} \Delta p^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{\hbar^2}{a^2}, & \Delta p = \frac{\hbar}{a} \\ \Delta x^2 = \langle x^2\rangle - \langle x\rangle^2 = a^2/4, & \Delta x = a/2 \end{cases}$$

$\Rightarrow \Delta p\cdot\Delta x = \frac{\hbar}{a}\cdot\frac{a}{2} = \frac{\hbar}{2}$.

The uncertainty is the minimum possible. Here a is cancelled out, thus $\Delta p\cdot\Delta x$ is independent of a. a can be anything.