PHYS. 324

HW #2

1. If the potential $V(x)$ is independent of time, the schrödinger equation can be separated into two ordinary differential equations as.

$$\begin{cases} -\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2} + V(x)\varphi(x) = E\varphi(x) & (1) \\ i\hbar\frac{d\Xi}{dt} = E\Xi(t) & (2) \end{cases}$$

where E is the separation constant

(2): $$i\hbar\frac{d\Xi}{dt} = E\Xi(t)$$
$$\Rightarrow \quad \Xi(t) = Ce^{-iEt/\hbar}$$

If we have $E = E_0 + i\Gamma$, E_0 and Γ are the real and imaginary parts, then

$$\Psi = \varphi(x)e^{-iEt/\hbar} = \varphi(x)e^{-iE_0t/\hbar}e^{\Gamma t/\hbar}$$

To normalize, we must have $\int|\Psi|^2dx = 1$

$\Rightarrow \quad e^{2\Gamma t/\hbar}\int|\varphi|^2dx = 1$ satisfied by all times.

$\Rightarrow \quad \Gamma = 0 \quad \Rightarrow \quad E = E_0$ is real

Thus $$\Psi = \varphi(x)e^{-iEt/\hbar} = \varphi(x)e^{-iE_0t/\hbar}$$

cannot be real.

$\Rightarrow$ Ψ cannot be real

On the other hand, since the Schrödinger equation is linear, so if the wave function is complex, we can separate the real φ_r and imaginary part φ_i of φ and they satisfy the equation separately.

$\Rightarrow \quad \varphi = \varphi_r + i\varphi_i$, where φ_r & φ_i are real

2 For φ. we have

$$\frac{d^2\varphi}{dx^2} = \frac{2m}{\hbar^2}(V-E)\varphi.$$

If $E < V_{min}$, then $V-E>0$ for all x

$\Rightarrow$ $\frac{d^2\varphi}{dx^2}$ & φ have the same sign everywhere.

Mathematically, if φ has its maximum. we must have $\frac{d^2\varphi}{dx^2}<0$ at the point of the maximum $\Rightarrow$ φ itself at this point should also be negative.

Similarly any minima of φ must occur where φ is positive

$\Rightarrow$ φ cannot tend to 0 as $x\to\infty$, thus it cannot be normalized.

$\Rightarrow$ $E > V_{min}$

3 a The time-independent Schrödinger equation can be written as.

$$\frac{d^2\varphi}{dx^2} = \frac{2m}{\hbar^2}(V-E)\varphi$$

① Outside the well, $V\to\infty$. To satisfy the equation above. we must have

$$\varphi = 0$$

Or you can think that the probability of finding the particle there is zero

② Inside the well, $V=0$

$$\Rightarrow \frac{d^2\varphi}{dx^2} = -\frac{2m}{\hbar^2}E\varphi$$

From Q 2, we know that $E > V_{min} = 0$.

$\Rightarrow$ we can define $k^2 = \frac{2mE}{\hbar^2}$, $k = \frac{\sqrt{2mE}}{\hbar}$

$$\Rightarrow \frac{d^2\varphi}{dx^2} = -k^2\varphi$$

The general solution is

$$\varphi = A\sin kx + B\cos kx$$

Boundary conditions: $\varphi(0) = \varphi(L) = 0$

$$\Rightarrow \begin{cases} \varphi(0) = A\sin 0 + B\cos 0 = B = 0 \\ \varphi(L) = A\sin kL + B\cos kL = A\sin kL = 0 \end{cases}$$

$$\Rightarrow \begin{cases} B = 0 \\ A = 0 \text{ or } \sin kL = 0 \end{cases} \Rightarrow \begin{cases} B = 0 \\ kL = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \cdots \end{cases}$$

while $kL = 0 \Rightarrow k = 0 \Rightarrow \varphi = A\sin 0 = 0$ is no good

and since $\sin(-\theta) = -\sin\theta$. we can absorb the minus sign into A.

$$\Rightarrow \quad k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \cdots$$

$$\Rightarrow \begin{cases} E_n = \dfrac{\hbar^2 k_n^2}{2m} = \dfrac{n^2\pi^2\hbar^2}{2mL^2} \\ \varphi_n = A\sin k_n x \end{cases}$$

where A can be determined by the normalization $\int_0^L |\varphi|^2 dx = 1$

$$\Rightarrow \int_0^L |A|^2 \sin^2 kx\, dx = |A|^2 \frac{L}{2} = 1$$

$$\Rightarrow |A|^2 = \frac{2}{L} \Rightarrow A = \sqrt{\frac{2}{L}}$$

$$\Rightarrow \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

For any $m \neq n$, we have

$$\int \varphi_m^* \varphi_n(x)\, dx = \frac{2}{L}\int_0^L \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{1}{L}\int_0^L \left[\cos\left(\frac{m-n}{L}\pi x\right) - \cos\left(\frac{m+n}{L}\pi x\right)\right] dx$$

Hint: you can use $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$

$$\Rightarrow \int \psi_m^* \psi_n \, dx = \frac{1}{(m-n)\pi} \sin\left(\frac{m-n}{L}\pi x\right) - \frac{1}{(m+n)\pi} \sin\left(\frac{m+n}{L}\pi x\right)\Big|_0^L$$

$$= \frac{1}{\pi}\left\{ \frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n} \right\} = 0$$

$\Rightarrow$ The eigen functions for different n are orthogonal

b. Since $\psi = \varphi(x) e^{-iEt/\hbar}$

$$\Rightarrow \psi_n = \varphi_n(x) e^{-iE_nt/\hbar} = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{in^2\pi^2\hbar}{2mL^2}t}$$

$\Rightarrow$ The most general solution is $\psi = \sum_n \psi_n = \sum_{n=1}^{\infty} C_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{in^2\pi^2\hbar}{2mL^2}t}$

where C_n can be determined by the initial condition

$$\psi(x,0) = \sum_{n=1}^{\infty} C_n \varphi_n(x)$$

since ψ_n and ψ_m are orthogonal to each other $(m \neq n)$

$$\Rightarrow \int \psi(x,0)\varphi_m(x)\,dx = \int \sum_{n=1}^{\infty} C_n \varphi_n(x)\varphi_m(x)\,dx$$

$$= \sum_{n=1}^{\infty} C_n \int \varphi_n(x)\varphi_m(x)\,dx = C_m \cancelto{1}{\int |\varphi_m(x)|^2 dx} + \sum_{n\neq m} C_n \cancelto{0}{\int \varphi_n(x)\varphi_m(x)\,dx}$$

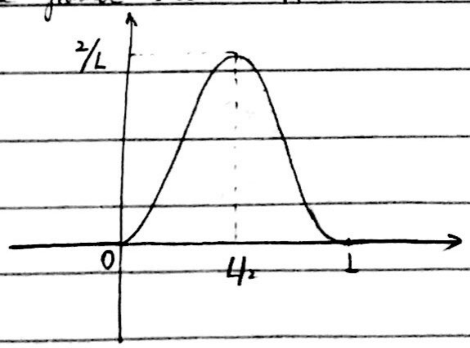
$$= C_m$$

$$\Rightarrow C_m = \underbrace{\int \psi(x,0)\varphi_m(x)\,dx}_{\text{the inner product of } \psi \text{ and the } \varphi_m} = \sqrt{\frac{2}{L}} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \psi(x,0)\,dx$$

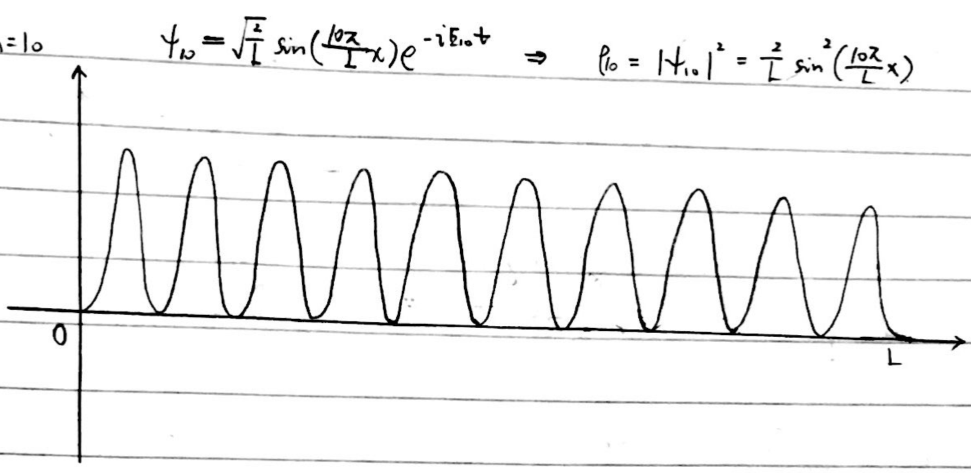
c The ground state $n=1$ $\quad \psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}x\right) e^{-iE_1t}$ $\quad \Rightarrow \rho_1 = |\psi_1|^2 = \frac{2}{L}\sin^2\left(\frac{\pi}{L}x\right)$

$$= \frac{2}{L}\cdot\frac{1}{2}\left(1 - \cos\frac{2\pi}{L}x\right)$$

$$= \frac{1}{L}\left(1 - \cos\frac{2\pi}{L}x\right)$$



$n=10 \qquad \psi_{10} = \sqrt{\frac{2}{L}} \sin\left(\frac{10\pi}{L}x\right) e^{-iE_{10}t} \Rightarrow \rho_{10} = |\psi_{10}|^2 = \frac{2}{L}\sin^2\left(\frac{10\pi}{L}x\right)$



The second one is more like the classical probability

4.

$$\psi(x,0) = A\left[\varphi_m(x) + \varphi_{m+1}(x)\right]$$

$$\Rightarrow \int |\psi(x,0)|^2 dx = A^2 \int (\varphi_m^* + \varphi_{m+1}^*)(\varphi_m + \varphi_{m+1})\, dx$$

Since φ_m & φ_{m+1} are orthogonal to each other and have been normalized

$$\Rightarrow \int |\psi(x,0)|^2 dx = A^2 \int (|\varphi_m|^2 + |\varphi_{m+1}|^2)\,dx$$

$$= A^2\left(\int |\varphi_m|^2 dx + \int |\varphi_{m+1}|^2 dx\right) = 2A^2 = 1$$

$$\Rightarrow A^2 = \frac{1}{2} \Rightarrow A = \frac{\sqrt{2}}{2} \Rightarrow \psi(x,0) = \frac{\sqrt{2}}{2}(\varphi_m + \varphi_{m+1})$$

$t=0$. $$\langle x\rangle = \int x |\psi(x,0)|^2 dx = \int_0^L x \frac{1}{2}(\varphi_m^* + \varphi_{m+1}^*)(\varphi_m + \varphi_{m+1})\, dx$$

$$= \frac{1}{2}\int_0^L x\left(|\varphi_m|^2 + |\varphi_{m+1}|^2 + \varphi_m^*\varphi_{m+1} + \varphi_{m+1}^*\varphi_m\right) dx$$

Since $\varphi_m = \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{L}x\right)$, $\varphi_{m+1} = \sqrt{\frac{2}{L}}\sin\left(\frac{(m+1)\pi}{L}x\right)$

$$\Rightarrow \langle x\rangle = \frac{2}{L}\frac{1}{2}\int_0^L x\left[\sin^2\left(\frac{m\pi}{L}x\right) + \sin^2\left(\frac{(m+1)\pi}{L}x\right) + 2\sin\left(\frac{m\pi x}{L}\right)\sin\frac{(m+1)\pi}{L}x\right] dx$$

$$= \frac{1}{L}\left[\int_0^L x\sin^2\frac{m\pi x}{L}dx + \int_0^L x\sin^2\frac{(m+1)\pi x}{L}dx + 2\int_0^L x\sin\frac{m\pi x}{L}\sin\frac{(m+1)\pi x}{L}dx\right]$$

$$= \frac{L\left(1+2m\left(3+m\left(3-32m(1+m)^3+2\pi^2+2m(3+2m)(2+m(3+2m))\pi^2\right)\right)-(1+6m(1+m))\right)}{8m^2(1+m)^2(1+2m)^2\pi^2}$$

$$= \frac{L}{2} - \frac{8Lm(1+m)}{(\pi+2m\pi)^2}$$

Since m is large, even integer

$$\Rightarrow \langle x\rangle \approx \frac{L}{2} - \frac{2\cancel{8}L\cancel{m^2}}{\cancel{4m^2}\pi^2} = \frac{L}{2} - \frac{2L}{\pi^2}$$

similarly $\langle x^2 \rangle = \frac{2}{L}\frac{1}{2}\int x^2 \left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi}{L}x \right)^2 dx$

$$= \frac{L^2\left(-\frac{6(1+6m(1+m))\pi}{m^2(1+m)^2(1+2m)^2} + 8\pi(-6+\pi^2) \right)}{24\pi^3}$$

$$\approx L^2 \frac{\cancel{\pi}(\pi^2-6)}{3\pi^2} = L^2\frac{\pi^2-6}{3\pi^2}$$

$$\langle p \rangle = -i\hbar\int \psi^* \frac{\partial}{\partial x}\psi\, dx$$

$$= -i\hbar\int \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\varphi_m + \varphi_{m+1})\frac{\partial}{\partial x}(\varphi_m + \varphi_{m+1})\, dx$$

$$= -i\hbar\,\frac{1}{2}\cdot\frac{2}{L}\int dx \left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi}{L}x\right)\frac{\partial}{\partial x}\left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi}{L}x\right)$$

$$= -i\hbar\frac{1}{L}\int_0^L dx \left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi}{L}x\right)\left(\frac{m\pi}{L}\cos\frac{m\pi}{L}x + \frac{(m+1)\pi}{L}\cos\frac{(m+1)\pi}{L}x\right)$$

$$= -i\hbar\left(-\frac{\sin(m\pi)^2}{L}\right) = 0$$

$$\langle p^2 \rangle = -\hbar^2\int \psi^* \frac{\partial^2}{\partial x^2}\psi\, dx$$

$$= -\hbar^2\int \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}(\varphi_m + \varphi_{m+1})\frac{\partial^2}{\partial x^2}(\varphi_m + \varphi_{m+1})\, dx$$

$$= +\hbar^2\int \frac{1}{2}\left(\sqrt{\frac{2}{L}}\right)^2 dx\left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi x}{L}\right)\left(\left(\frac{m\pi}{L}\right)^2\sin\frac{m\pi}{L}x + \frac{(m+1)^2\pi^2}{L^2}\sin\frac{(m+1)\pi}{L}x\right)$$

$$= \frac{\hbar^2}{L}\int dx\left(\sin\frac{m\pi}{L}x + \sin\frac{(m+1)\pi x}{L}\right)\left[\left(\frac{m\pi}{L}\right)^2\sin\frac{m\pi}{L}x + \frac{(m+1)^2\pi^2}{L^2}\sin\frac{(m+1)\pi}{L}x\right]$$

$$= \frac{\hbar^2}{L}\left[\int dx\left(\frac{m\pi}{L}\right)^2\sin^2\frac{m\pi}{L}x + \int dx \sin^2\frac{(m+1)\pi x}{L}\,\frac{(m+1)^2\pi^2}{L^2}\right]$$

$$= \frac{\hbar^2}{L}\left(\frac{m^2\pi^2}{L^2} + \frac{(m+1)^2\pi^2}{L^2}\right) = \frac{\hbar^2\pi^2}{L^3}\left(m^2 + (m+1)^2\right) \approx \frac{\hbar^2\pi^2}{L^3}(2m^2)$$

$$\Rightarrow \sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{L^2\frac{\pi^2-6}{3\pi^2} - \left(\frac{L}{2} - \frac{2L}{\pi^2}\right)^2} = L\sqrt{\frac{1}{12}\left(1 - \frac{48}{\pi^4}\right)}$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle} = \frac{\hbar\pi}{L^{3/2}}\sqrt{2}m$$

?

b. At time t, $\Psi(x,t) = \frac{\sqrt{2}}{2}\left[\varphi_m e^{-iE_m t} + \varphi_{m+1} e^{-iE_{m+1}t}\right]$

$\Rightarrow \quad \langle x\rangle = \frac{1}{L}\int_0^L x\left(\sin\frac{m\pi}{L}x \cdot e^{-iE_m t/\hbar} + \sin\frac{(m+1)\pi}{L}x \cdot e^{-iE_{m+1}t/\hbar}\right)^*\left(\sin\frac{m\pi}{L}x e^{-iE_m t/\hbar} + \sin\frac{(m+1)\pi x}{L} e^{-\frac{iE_{m+1}t}{\hbar}}\right)$

$= \frac{1}{L}\int_0^L x\left(\sin^2\frac{m\pi}{L}x + \sin^2\frac{(m+1)\pi}{L}x + 2\sin\frac{m\pi}{L}x\sin\frac{(m+1)\pi}{L}x\cos(E_{m+1}-E_m)\frac{t}{\hbar}\right)dx$

Note: $\cos\theta \pm i\sin\theta = e^{\pm i\theta}$, $\cos\theta = \cos(\theta)$

Since time dependence is not influenced by integrals of x

$\Rightarrow$ The oscillation frequency ω_m of $\langle x\rangle$ is

$$\omega_m = \frac{E_{m+1}-E_m}{\hbar} = \frac{\hbar^2(m+1)^2\pi^2 - \hbar^2\pi^2 m^2}{2m_e L^2\hbar} = \frac{(2m+1)\hbar\pi^2}{2m_e L^2}$$

c $\rho = |\Psi(t)|^2 = \sin^2\frac{m\pi}{L}x + \sin^2\frac{(m+1)\pi x}{L} + 2\sin\frac{m\pi}{L}x\sin\frac{(m+1)\pi}{L}x\cos\omega_m t$

d $\frac{1}{2}m_e v^2 = E \Rightarrow v = \sqrt{\frac{2E}{m_e}}$

in the box $T = \frac{2L}{v} = \frac{2L}{\sqrt{2E/m_e}} = \sqrt{\frac{m_e}{2E}}\,2L = \sqrt{\frac{2m_e L^2}{E}}$

$$\Rightarrow \omega_c = \frac{2\pi}{T} = 2\pi\sqrt{\frac{E}{2m_e L^2}} = \sqrt{\frac{2E\pi^2}{m_e L^2}} = \sqrt{\frac{2\pi^2}{m_e L^2}\cdot\frac{\hbar^2 m^2\pi^2}{2m_e L^2}}$$

$$= \frac{\hbar m\pi^2}{m_e L^2}$$

when m is large, $2m+1 \to 2m$, thus ω_m can be written as

$\omega_m \to \frac{2m\hbar\pi^2}{2m_e L^2} = \frac{m\hbar\pi^2}{m_e L^2}$, which is the same as the classical frequency

e $\Psi(x,t)$ should be $\varphi_m e^{-\frac{iE_m t}{\hbar}}$ for later times.

Specifically, since the measurement at $t = t_0$ gives E_m, then $\Psi(x,t_0) = \varphi_m e^{\frac{-iE_m t}{\hbar}} = \sqrt{\frac{2}{L}}\sin\frac{m\pi}{L}x\, e^{-iE_m t/\hbar}$

while $\varphi_m e^{-iE_m t/\hbar}$ is an eigenfunction of the system,

$\Rightarrow$ ψ will stay in this state

$\Rightarrow$ $\psi(x,t) = \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi}{L}x\right) e^{-iE_m t/\hbar}$ at later times

5.a. $\psi = \begin{cases} C & (0 < x < L/4) \\ 0 & (\text{others}) \end{cases}$ ~~but ψ is inconsistent now!~~

$$\int_{-\infty}^{\infty} |\psi|^2 dx = C^2 \frac{L}{4} = 1 \Rightarrow C = \frac{2}{\sqrt{L}}$$

$$\Rightarrow \psi(x,0) = \begin{cases} \frac{2}{\sqrt{L}} & (0 < x < L/4) \\ 0 & (\text{others}) \end{cases}$$

b. For E_1, $\psi_1 = \sqrt{\frac{2}{L}} \sin(\frac{\pi}{L}x)$

The overlap is $\int_{-\infty}^{\infty} \psi(x,0)\psi_1 dx = \frac{2}{\sqrt{L}} \cdot \frac{\sqrt{2}}{\sqrt{L}} \int_0^{L/4} \sin(\frac{\pi}{L}x) dx$

$$= -\frac{2\sqrt{2}}{L} \frac{L}{\pi} \cos\frac{\pi}{L}x \Big|_0^{L/4}$$

$$= -\frac{2\sqrt{2}}{L} \frac{L}{\pi} \left(\cos\frac{\pi}{4} - 1\right)$$

$$= -\frac{2\sqrt{2}}{\pi}\left(\frac{\sqrt{2}}{2} - 1\right) = \left(1 - \frac{\sqrt{2}}{2}\right)\frac{2\sqrt{2}}{\pi}$$

The probability is $|\text{overlap}|^2 = \left(1 - \frac{\sqrt{2}}{2}\right)^2 \frac{8}{\pi^2}$

c. The first measurement yields result $E_1 \Rightarrow \psi(x,t_1) = \varphi_1 = \sqrt{\frac{2}{L}} \sin\frac{\pi}{L}x$

$\Rightarrow$ The overlap between $\psi(x,t_1)$ & φ_2 is

$\int_{-\infty}^{\infty} \varphi_1 \varphi_2 dx$ since φ_1 & φ_2 are orthogonal,

$$\Rightarrow \int_{-\infty}^{\infty} \varphi_1 \varphi_2 dx = 0$$

$\Rightarrow$ There's no chance that we could get E_2 in the subsequent measurement

Or you can say that since at the first measurement, $\psi = \varphi_1$, which is an eigenwave of the system. thus it will remain as φ_1. Thus there's no chance to get E_2 at later times